UNIVERSIDADE FEDERAL DE PERNAMBUCO
DEPARTAMENTO DE MATEMÁTICA

Estimativas a Priori para Sistemas de Equações de Advecção-Difusão

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Recife, fevereiro de 2011

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Tese apresentada ao Departamento de Matemática da Universidade Federal de Pernambuco para obtenção do título de Doutor em Matemática

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Durante o desenvolvimento deste trabalho o autor recebeu auxílio financeiro da CNPq

Recife, fevereiro de 2011

Catalogação na fonte
Bibliotecária Joana D'Arc L. Salvador, CRB 4-572

Melo, Wilberclay Gonçalves.
Estimativas a priori para sistemas de equações de advecção-difusão / Wilberclay Gonçalves Melo. - Recife: O Autor, 2011.
ix, 84 folhas.

Orientador: Pablo Gustavo A. Braz e Silva
Tese (Doutorado) - Universidade Federal de Pernambuco. CCEN. Matemática, 2011.

Inclui bibliografia.

1.Equações diferenciais. 2.Equações diferenciais parciais. I.Silva, Pablo Gustavo A. Braz e (orientador). II. Título.

515.3 (22.ed.) MEI 2011-011

Tese submetida ao Corpo Docente do Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco como parte dos requisitos necessários para a obtenção do Grau de Doutorado em Matemática.

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ESTIMATIVAS A PRIORI PARA SISTEMAS DE EQUAÇÕES DE ADVECÇÃO-DIFUSÃO

Por

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Fevereiro – 2011

Dedicatória

A minha Família

Agradecimentos

A minha mãe Edilma, a qual é, resumidamente, tudo que preciso e precisarei para ser alguém; Ao meu pai Wilton, sem o qual não possuiria princípios e não conseguiria diferenciar o honesto do desonesto; a minha família: Evani, Ezilda, Emíla, Enilda, Erivaldo, Edna que rezaram por mim em todos os momentos; a minha tia e madrinha Eneide que lutou por minha estadia em PE e esteve presente em todos os momentos da minha vida, aos professores Pablo Braz, Paulo Zingano e Miguel Loayza por suas orientações, dedicação e disponibilidade; à CNPQ pelo auxílio financeiro; aos professores César Castilho, Lineia Schütz, Marko Rojas, Uberlândio Batista, Joaquim Tavares, por participarem da banca examinadora; a minha namorada Jamile pelo companheirismo e compreensão; a meus irmãos Gardênia e Alex, que sempre torceram por mim; a todos meus colegas da UFS e da UFPE que contribuíram, direta ou indiretamente, para realização deste trabalho; aos meus amigos que compartilharam seus conhecimentos e suas amizades; a todos os professores e funcionários da UFPE, da UNIT e da UFS que tiveram um papel importante na minha formação; por todos os momentos da minha vida acadêmica, agradeço aos professores Genaro e Danilo por seus conselhos, suas amizades, pela confiança e pelo respeito.

Resumo

Estudamos sistemas de equações de advecção-difusão com o objetivo de obter estimativas a priori para normas L^p de suas soluções.

Palavras-chave: Advecção-difusão, estimativas a priori, estimativas de energia.

Abstract

We study systems of advection-diffusion equations, with the aim of deriving a priori estimates for various L^p norms of its solutions.

Keywords: Advection-diffusion, a priori estimates, energy estimates.

Sumário

Capítulo 1

Introdução

Estamos interessados em estudar sistemas de equações de advecção-difusão. Procuramos em cada sistema, como principal objetivo, encontrar uma estimativa a priori para a norma do sup de soluções para cada problema trabalhado. Vejamos, capítulo por capítulo, os principais resultados obtidos.

No capítulo 3, estudaremos algumas propriedades para soluções $u(\cdot, t)$ da equação de advecção-difusão

$$\begin{aligned} u_t(x,t) + [b(x,t,u(x,t))f(x,t,u(x,t))]_x &= \mu(t)u_{xx}(x,t) + c(x,t,u(x,t))u(x,t), \\ u(\cdot,0) &= u_0 \in L^p(\mathbb{R}), \end{aligned} \tag{1.1}$$

onde $x \in \mathbb{R}$, $t > 0$, supondo

$$\begin{aligned} |b(x,t,u)| &\leq B(t), \\ |f(x,t,u)| &\leq D|u|, \\ |c(x,t,u)| &\leq C(t). \end{aligned} \tag{1.2}$$

A função $\mu \in C^0([0,\infty))$ é positiva, f, b são de classe C^1, as funções $B, C \in C^0([0,\infty))$. Aplicaremos as técnicas utilizadas em [3, 29] para provar que as soluções do problema (1.1), juntamente com as hipóteses (1.2), satisfazem as estimativas

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(t)\|u_0\|_{L^p(\mathbb{R})}, \forall\ t \geq 0,$$

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\kappa \exp\left\{\frac{3pt}{2}E(t)\right\} D(t)^{\frac{1}{p}} t^{-\frac{2}{p}},\ \forall\ t > 0,$$

onde $K(t) = K(t,p,B,C,\mu)$, $D(t) = D(t,B,C,\mu)$, $E(t) = E(t,B,C,\mu)$ e $\kappa = \{p, \|u_0\|_{L^p(\mathbb{R})}\}$ são os parâmetros.

No capítulo 4, estudaremos o problema rotacionalmente invariante

$$\begin{aligned} \mathbf{u}_t(x,t) + \left[|\mathbf{u}(x,t)|^2\mathbf{u}(x,t)\right]_x &= \mathbf{u}_{xx}(x,t), \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0; \end{aligned} \tag{1.3}$$

onde $x \in \mathbb{R}$, $t \in [0,T]$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, $\mathbf{u}_0 \in L^p(\mathbb{R})$, com $1 \leq p < \infty$, e $|\cdot|$ indica a norma Euclidiana em $\mathbb{R}^n$, supondo que

$$|\mathbf{u}(\cdot,t)|^2 \leq B(T), \text{ q.t.p. em } \mathbb{R}. \tag{1.4}$$

Provaremos que as soluções de (1.3)-(1.4) satifazem as taxas de decaimento

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq C_\gamma t^{-\frac{3}{4p}}, \forall\, t \in (0,T];$$

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\kappa t^{-\frac{1}{2p}}, \ \forall\, t \ \in (0,T],$$

onde $\gamma, \kappa = \left\{n, T, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\right\}$. O problema (1.3) foi considerado inicialmente em 1979 por Keyfitz e Kranzer [21]. Outros trabalhos sobre este problema podem ser encontrados em [14, 15].

No capítulo 5, mostraremos alguns resultados para soluções $\mathbf{u}(\cdot,t)$ do sistema

$$\begin{aligned} \mathbf{u}_t(x,t) + [b(x,t,\mathbf{u}(x,t))\mathbf{u}(x,t)]_x &= [A(x,t,\mathbf{u}(x,t))\mathbf{u}_x(x,t)]_x + c(x,t,\mathbf{u}(x,t))\mathbf{u}(x,t), \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0, \end{aligned} \tag{1.5}$$

onde $x \in \mathbb{R}$ e $t > 0$. Supondo,

$$\begin{aligned} a_{ii}(x,t,\mathbf{u}(x,t)) &\geq \mu(t), \\ |b(x,t,\mathbf{u}(x,t))| &\leq B(t), \\ |c(x,t,\mathbf{u}(x,t))| &\leq C(t), \end{aligned} \tag{1.6}$$

onde $A(x,t,\mathbf{u}(x,t)) = \operatorname{diag}(a_{ii}(x,t,\mathbf{u}(x,t)))_{1\leq i\leq n}$ é uma matriz diagonal constituída de funções suaves reais, a função $\mu \in C^0([0,\infty))$ é positiva, as funções $B, C \in C^0([0,\infty))$. O sistema (1.5)-(1.6) possui soluções que satisfazem as desigualdades

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \forall\, t \geq 0,$$

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\kappa \exp\left\{\frac{3pt}{2}E(t)\right\} D(t)^{\frac{1}{p}} t^{-\frac{2}{p}}, \ \forall\, t > 0,$$

onde $K(t) = K(t,p,B,C,\mu)$, $D(t) = D(t,B,C,\mu)$, $E(t) = E(t,B,C,\mu)$ e $\kappa = \{n, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\}$.

No capítulo 6, trabalharemos no sistema

$$\begin{aligned} \mathbf{u}_t(\mathbf{x},t) + \sum_{k=1}^m \frac{\partial}{\partial x_k}\left[|\mathbf{u}(\mathbf{x},t)|^2\mathbf{u}(\mathbf{x},t)\right] &= \sum_{k=1}^m \frac{\partial^2}{\partial x_k^2}\mathbf{u}(\mathbf{x},t), \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0, \end{aligned} \tag{1.7}$$

onde $\mathbf{x} \in \mathbb{R}^m$, $t \in (0,T]$. Supondo que

$$|\mathbf{u}(\mathbf{x},t)|^2 \leq B(T), \text{ a.e. em } \mathbb{R}^m, \tag{1.8}$$

Estabeleceremos que uma solução do problema (1.7)-(1.8) satisfaz as seguintes estimativas

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, \forall\, t \in [0,T],$$

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)} \leq C(t)t^{-\frac{m+2}{4p}}, \forall\, t \in (0,T],$$

$$\|\mathbf{u}(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \leq E(t)t^{-\frac{m+2}{p}},\ \forall\, t \in (0,T],$$

onde

$$K(t) = K(t,m,p,T,B(T)), C(t) = C(t,m,n,p,\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, K, B(T))$$

e

$$E(t) = E(t,m,n,p,\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, B(T)).$$

No capítulo 7, consideraremos o sistema de equações

$$\begin{aligned}
\mathbf{u}_t(\mathbf{x},t) + \sum_{k=1}^{m} \frac{\partial}{\partial x_k}[b_k(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\mathbf{u}(\mathbf{x},t)] &= \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left[A^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\frac{\partial \mathbf{u}}{\partial x_k}(\mathbf{x},t)\right] \\
&+ c(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\mathbf{u}(\mathbf{x},t), \qquad (1.9) \\
\mathbf{u}(\cdot,0) &= \mathbf{u}_0,
\end{aligned}$$

onde $\mathbf{x} \in \mathbb{R}^m$, $t > 0$. Supondo que

$$\begin{aligned}
a_{ii}^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t)) &\geq \mu(t), \\
|b_k(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))| &\leq B(t), \qquad (1.10) \\
|c(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))| &\leq C(t),
\end{aligned}$$

onde $A^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t)) = \operatorname{diag}(a_{ii}^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t)))_{1\leq i\leq n}$ é uma matriz diagonal constituída de funções suaves reais, a função $\mu \in C^0([0,\infty))$ é positiva, as funções $B, C \in C^0([0,\infty))$. [3, 29] nos auxiliam a concluir que as soluções do sistema (1.9)-(1.10) satisfazem as estimativas a priori

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, \forall\, t \geq 0,$$

$$\|\mathbf{u}(\cdot,t)\|_{L^{\infty}(\mathbb{R})} \leq C_{\kappa} \exp\left\{\frac{3pt}{2}E(t)\right\} D(t)^{\frac{1}{p}} t^{-\frac{m+2}{2p}},\ \forall\, t > 0,$$

onde $K(t) = K(t,m,p,B,C,\mu)$, $D(t) = D(t,m,B,C,\mu)$, $E(t) = E(t,m,B,C,\mu)$ e $\kappa = \{m,p,\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}\}$.

No capítulo 8, estudaremos o problema

$$\begin{aligned}
\mathbf{u}_t(x,t) + \mathbf{f}(x,t,\mathbf{u}(x,t))_x &= \mu(t)\mathbf{u}_{xx}(x,t), \qquad (1.11) \\
\mathbf{u}(\cdot,0) &= \mathbf{u}_0,
\end{aligned}$$

onde $x \in \mathbb{R}$, $t > 0$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, $\mathbf{u}_0 \in L^p(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$. Admitindo que

$$|\mathbf{f}(x,t,\mathbf{u}(x,t))| \leq B(t)|\mathbf{u}(x,t)|, \qquad (1.12)$$

onde $\mathbf{f}(x,t,\mathbf{u}(x,t)) = (f_1(x,t,\mathbf{u}(x,t)), f_2(x,t,\mathbf{u}(x,t)), ..., f_n(x,t,\mathbf{u}(x,t)))$ é constituída de funções de classe C^1 e a função $B \in C^0([0,\infty))$. Provaremos que, quando $\left.\frac{d}{dt}\right|_{t=s}\|\mathbf{u}(\cdot,t)\|_{L^q(\mathbb{R})}^q \geq 0$, tem-se

$$\|\mathbf{u}(\cdot,s)\|_{L^q(\mathbb{R})} \leq C_{\gamma}\left(\frac{B(s)}{\mu(s)}\right)^{\frac{1}{q}} \|\mathbf{u}(\cdot,s)\|_{L^{\frac{q}{2}}(\mathbb{R})},$$

onde $2 \leq q < \infty$ e $\gamma = \{n, q\}$ são parâmetros. Por outro lado, mostraremos que, se

$$\int^{\infty} \mu(t)dt = \infty,$$

então

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \quad \leq \quad C_\kappa \left(\limsup_{t\to\infty} \frac{B(t)}{\mu(t)} \right)^{\frac{1}{p}} \limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})},$$

onde $1 \leq p < \infty$ e C_κ é uma constante positiva que depende dos parâmetros $\kappa = \{n, p\}$.

Para obter os resultados citados acima, tomamos como principal inspiração os artigos [2, 3, 32] e a Tese de doutorado [29]. Vejamos que resultados podemos encontrar em algumas destas referências.

No artigo [32], é apresentado um estudo ao sistema de equações

$$\mathbf{u}_t(x,t) + [\varphi(|\mathbf{u}(x,t)|)\mathbf{u}(x,t)]_x \quad = \quad (B(\mathbf{u})\mathbf{u}_x(x,t))_x, \tag{1.13}$$

onde φ é uma função real suave, $x \in \mathbb{R}$, $t > 0$, e $B(\mathbf{u}(x,t)) = (b_i(\mathbf{u}(x,t)))_{n\times n}$ é uma matriz $n \times n$ diagonal com funções-entrada reais e suaves tais que $b_i(\mathbf{u}(x,t)) \geq \mu$, $\forall\ i = 1, 2, ..., n$, onde μ é uma constante positiva. O resultado lá obtido é representado pela desigualdade

$$\|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R})} \leq K_\Lambda (1+t)^{-\frac{1}{4}},\ \forall\ t > 0,$$

onde $\mathbf{u}(\cdot, 0) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, K_Λ é uma constante que depende apenas dos parâmetros

$$\Lambda = \{\|\mathbf{u}(\cdot,0)\|_{L^1(\mathbb{R})}, \|\mathbf{u}(\cdot,0)\|_{L^2(\mathbb{R})}, n, \mu\}.$$

Em [29], estuda-se a equação multidimensional

$$u_t(\mathbf{x},t) + \mathrm{div}[b(\mathbf{x},t,u(\mathbf{x},t))u(\mathbf{x},t)] = \mathrm{div}[A(\mathbf{x},t,u(\mathbf{x},t))\nabla u(\mathbf{x},t)] + c(\mathbf{x},t,u(\mathbf{x},t))u(\mathbf{x},t),$$

onde $x \in \mathbb{R}^m$, $t \in (0, T]$ e b, c são funções reais suaves tais que

$$|b(\mathbf{x},t,u)| \leq B(T),\ \ |c(\mathbf{x},t,u)| \leq C(T)$$

e $A(\mathbf{x},t,u) = (a_{ij}(\mathbf{x},t,u))_{n\times n}$ é uma matriz $n \times n$ com funções-entrada reais a_{ij} suaves tais que

$$\sum_{i,j=1}^{n} a_{ij}(x,t,u)\xi_i\xi_j \geq \mu \sum_{i=1}^{n} \xi_i^2,$$

onde $\mu, B(T), C(T)$ são constantes positivas e $\xi_i \in \mathbb{R}$, $\forall\ i = 1, 2, ..., n$. Os principais resultados encontrados para uma solução $u(\mathbf{x}, t)$ são

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K_\rho \|u_0\|_{L^p(\mathbb{R}^m)},\ \forall\ t \in [0,T],$$

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} \leq K_\tau t^{-\frac{m}{2p}},\ \forall\ t \in (0,T],$$

onde $\rho = \{p, T, \mu\}$ e $\tau = \{m, p, T, \mu\}$ são parâmetros.

Para um estudo de equações de advecção-difusão utilizando técnicas distintas das utilizadas aqui, ver artigos [1, 7, 8, 10, 11, 4, 12, 13, 14, 15, 20, 25, 31]. Para pesquisar outros artigos que auxiliam o estudo destes ver [5, 17, 18, 19, 23, 24, 26, 27, 28, 30].

Capítulo 2

Preliminares

2.1 Notações e Normas para o capítulo 4

No decorrer do capítulo 4, usaremos as normas

$$\|\mathbf{u}(x,t)\|_{L^p(\mathbb{R})}^p = \sum_{i=1}^{n} \|u_i(x,t)\|_{L^p(\mathbb{R})}^p, \tag{2.1}$$

onde $1 \leq p < \infty$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, $t \in [0,T]$ e $x \in \mathbb{R}$, e,

$$\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R})} = \max_{i=1,\ldots,n} \left\{\|u_i(x,t)\|_{L^\infty(\mathbb{R})}\right\}, \tag{2.2}$$

onde $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, $t \in [0,T]$ e $x \in \mathbb{R}$.

2.2 Notações e Normas para o capítulo 5

No decorrer do capítulo 5, usaremos as normas

$$\|\mathbf{u}(x,t)\|_{L^p(\mathbb{R})}^p = \sum_{i=1}^{n} \|u_i(x,t)\|_{L^p(\mathbb{R})}^p, \tag{2.3}$$

onde $1 \leq p < \infty$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, $t \geq 0$ e $x \in \mathbb{R}$, e,

$$\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R})} = \max_{i=1,\ldots,n} \left\{\|u_i(x,t)\|_{L^\infty(\mathbb{R})}\right\}, \tag{2.4}$$

onde $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, $t \geq 0$ e $x \in \mathbb{R}$.

2.3 Notações e Normas para o capítulo 6

No decorrer do capítulo 6, usaremos as normas

$$\|\mathbf{u}(\mathbf{x},t)\|^p_{L^p(\mathbb{R}^m)} = \sum_{i=1}^{n} \|u_i(\mathbf{x},t)\|^p_{L^p(\mathbb{R}^m)}, \tag{2.5}$$

onde $1 \le p < \infty$, $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), ..., u_n(\mathbf{x},t))$, $t \in [0,T]$ e $\mathbf{x} \in \mathbb{R}^m$, e,

$$\|\mathbf{u}(\mathbf{x},t)\|_{L^\infty(\mathbb{R}^m)} = \max_{i=1,\dots,n} \left\{\|u_i(\mathbf{x},t)\|_{L^\infty(\mathbb{R}^m)}\right\}, \tag{2.6}$$

onde $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), ..., u_n(\mathbf{x},t))$, $t \in [0,T]$ e $\mathbf{x} \in \mathbb{R}^m$.

$$\|Du_i(\mathbf{x},t)\|^2 = \sum_{k=1}^{n} \left(\frac{\partial u_i}{\partial x_k}(\mathbf{x},t)\right)^2, \tag{2.7}$$

onde $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), ..., u_n(\mathbf{x},t))$, $t \in [0,T]$ e $\mathbf{x} \in \mathbb{R}^m$.

2.4 Notações e Normas para o capítulo 7

No decorrer do capítulo 7, usaremos as normas

$$\|\mathbf{u}(\mathbf{x},t)\|^p_{L^p(\mathbb{R}^m)} = \sum_{i=1}^{n} \|u_i(\mathbf{x},t)\|^p_{L^p(\mathbb{R}^m)}, \tag{2.8}$$

onde $1 \le p < \infty$, $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), ..., u_n(\mathbf{x},t))$, $t \ge 0$ e $\mathbf{x} \in \mathbb{R}^m$, e,

$$\|\mathbf{u}(\mathbf{x},t)\|_{L^\infty(\mathbb{R}^m)} = \max_{i=1,\dots,n} \left\{\|u_i(\mathbf{x},t)\|_{L^\infty(\mathbb{R}^m)}\right\}, \tag{2.9}$$

onde $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), ..., u_n(\mathbf{x},t))$, $t \ge 0$ e $\mathbf{x} \in \mathbb{R}^m$.

$$\|Du_i(\mathbf{x},t)\|^2 = \sum_{k=1}^{n} \left(\frac{\partial u_i}{\partial x_k}(\mathbf{x},t)\right)^2, \tag{2.10}$$

onde $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t), ..., u_n(\mathbf{x},t))$, $t \ge 0$ e $\mathbf{x} \in \mathbb{R}^m$.

2.5 Notações e Normas para o capítulo 8

No decorrer do capítulo 8, usaremos as normas

$$\|\mathbf{u}(x,t)\|^p_{L^p(\mathbb{R})} = \sum_{i=1}^{n} \|u_i(x,t)\|^p_{L^p(\mathbb{R})}, \tag{2.11}$$

onde $1 \le p < \infty$, $\mathbf{u}(\mathbf{x},t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, $t \ge 0$ e $x \in \mathbb{R}$, e,

$$\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R})} = \max_{i=1,\dots,n} \left\{\|u_i(x,t)\|_{L^\infty(\mathbb{R})}\right\}, \tag{2.12}$$

onde $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$, $t \ge 0$ e $x \in \mathbb{R}$.

2.6 Funções Sinal Regularizadas

Precisaremos, nesta tese, de funções auxiliares que chamaremos de funções sinal regularizadas. Estas são de importante ajuda quando se faz necessário derivar a função modular. Vamos definir e estabelecer alguns resultados destas funções. Primeiramente, defina $f : \mathbb{R} \to \mathbb{R}$, suave, por

$$f(x) = \begin{cases} 1, & \text{se } x \geq 1; \\ g(x), & \text{se } x \in [-1, 1]; \\ 0, & \text{se } x = 0; \\ -1, & \text{se } x \leq -1. \end{cases},$$

onde g é uma função suave real crescente. É fácil ver que, para cada $\delta > 0$, $f_\delta(x) = f(x/\delta)$ é uma função suave que pode ser vista da seguinte forma

$$f_\delta(x) = \begin{cases} 1, & \text{se } x \geq \delta; \\ g(x/\delta), & \text{se } x \in [-\delta, \delta]; \\ 0, & \text{se } x = 0; \\ -1, & \text{se } x \leq -\delta. \end{cases}.$$

Defina $L_\delta(v) = \displaystyle\int_0^v f_\delta(x)dx$, $\forall\ v \in \mathbb{R}$. Esta função é denominada função sinal regularizada. Usaremos os seguintes resultados para este tipo de função

1. $\lim_{\delta \to 0} L_\delta(v) = |v|$, uniformemente em $\mathbb{R}$;
2. $\lim_{\delta \to 0} L'_\delta(v) = \lim_{\delta \to 0} f_\delta(v) = \text{sgn}(v)$, para cada $v \in \mathbb{R}$;
3. $L''_\delta(v) = \dfrac{1}{\delta} f'(v/\delta) \geq 0$, pois f é crescente em $[-1, 1]$ e f é constante em $(-\infty, -1]$ e $[1, \infty)$. Além disso, $|vL''_\delta(v)| \leq c$, $\forall\ \delta > 0$, $v \in \mathbb{R}$ e c é uma constante real.

Para o leitor mais interessado no estudo de funções, recomendamos a leitura do artigo [9] onde um bom estudo de aplicações não-lineares é realizado.

2.7 Funções de Corte

Implicitamente, usaremos funções de corte antes de realizarmos uma integração por partes. Para melhor compreensão do que será feito, considere $f : \mathbb{R} \to \mathbb{R}$ uma função em $C_0^\infty(\mathbb{R})$ tal que $0 < f(x) < 1$, se $-2 < x < -1$ ou $1 < x < 2$ e

$$f(x) = \begin{cases} 1, & x \in (-1, 1); \\ 0, & |x| \geq 2. \end{cases}$$

Assim sendo, para cada $r > 0$, definimos $f_r(x) = f\left(\dfrac{x}{r}\right)$, $\forall\ x \in \mathbb{R}$. Consequentemente, $f_r \in C_0^\infty$, $0 < f_r(x) < 1$, se $-2r < x < -r$ ou $r < x < 2r$ e

$$f_r(x) = \begin{cases} 1, & x \in (-r, r); \\ 0, & |x| \geq 2r. \end{cases}$$

Vejamos um exemplo de como utilizaremos, implicitamente, f_r na tese.

Exemplo 2.1. Seja $u = u(x,t)$ solução da equação $u_t + b(u)_x = u_{xx}$. Seja $\Phi_\delta = L_\delta^p$, onde L_δ é uma função sinal regularizada (ver seção 2.6) e $1 \leq p < \infty$. Assim sendo, multiplicando esta equação por $f_r(x)\Phi'_\delta(u)$, encontramos $f_r(x)\Phi'_\delta(u)u_t + f_r(x)\Phi'_\delta(u)b(u)_x = f_r(x)\Phi'_\delta(u)u_{xx}$,. Dessa forma, integrando sobre $\mathbb{R} \times [t_0, t]$, obtemos

$$\int_{t_0}^{t}\int_{\mathbb{R}} f_r(x)\Phi'_\delta(u)u_\tau dxd\tau + \int_{t_0}^{t}\int_{\mathbb{R}} f_r(x)\Phi'_\delta(u)b(u)_x dxd\tau = \int_{t_0}^{t}\int_{\mathbb{R}} f_r(x)\Phi'_\delta(u)u_{xx}dxd\tau. \quad (2.13)$$

Logo,

$$\int_{t_0}^{t}\int_{-2r}^{2r} f_r(x)\Phi'_\delta(u)u_\tau dxd\tau + \int_{t_0}^{t}\int_{-2r}^{2r} f_r(x)\Phi'_\delta(u)b(u)_x dxd\tau = \int_{t_0}^{t}\int_{-2r}^{2r} f_r(x)\Phi'_\delta(u)u_{xx}dxd\tau.$$

Consequentemente, analisando cada uma das parcelas acima, obtemos

$$\int_{-2r}^{2r} f_r(x)\int_{t_0}^{t}\frac{d}{d\tau}\left[\Phi_\delta(u)\right]d\tau dx = \int_{-2r}^{2r} f(x/r)\Phi_\delta(u(\cdot,t))dx - \int_{-2r}^{2r} f(x/r)\Phi_\delta(u(\cdot,t_0))dx.$$

Passando ao limite quando $r \to \infty$, encontramos

$$\int_{t_0}^{t}\int_{\mathbb{R}} f(0)\Phi'_\delta(u)u_\tau dxd\tau = \int_{\mathbb{R}} f(0)\Phi_\delta(u(\cdot,t))dx - \int_{\mathbb{R}} f(0)\Phi_\delta(u(\cdot,t_0))dx,$$

ou seja,

$$\int_{t_0}^{t}\int_{\mathbb{R}} \Phi'_\delta(u)u_\tau dxd\tau = \int_{\mathbb{R}} \Phi_\delta(u(\cdot,t))dx - \int_{\mathbb{R}} \Phi_\delta(u(\cdot,t_0))dx,$$

já que $f(0) = 1$. Analogamente,

$$\int_{t_0}^{t}\int_{\mathbb{R}} f_r(x)\Phi'_\delta(u)b(u)_x dxd\tau = \int_{t_0}^{t}\int_{-2r}^{2r} f_r(x)\Phi'_\delta(u)b(u)_x dxd\tau.$$

Integrando por partes, chegamos a

$$\begin{aligned}\int_{t_0}^{t}\int_{-2r}^{2r} f_r(x)\Phi'_\delta(u)b(u)_x dxd\tau &= \int_{t_0}^{t}\Big[f_r(x)\Phi'_\delta(u)b(u)|_{-2r}^{2r} - \frac{1}{r}\int_{-2r}^{2r} f'(x/r)\Phi'_\delta(u)b(u)dx \\ &\quad - \int_{-2r}^{2r} f(x/r)\Phi''_\delta(u)b(u)u_x dx\Big]d\tau.\end{aligned}$$

Como $f_r(-2r) = f_r(2r) = 0$, então

$$\int_{t_0}^{t}\int_{-2r}^{2r} f_r(x)\Phi'_\delta(u)b(u)_x dxd\tau = \int_{t_0}^{t}\Big[-\frac{1}{r}\int_{-2r}^{2r} f'(x/r)\Phi'_\delta(u)b(u)dx - \int_{-2r}^{2r} f(x/r)\Phi''_\delta(u)b(u)u_x dx\Big]d\tau.$$

Passando ao limite quando $r \to \infty$, temos

$$\int_{t_0}^{t}\int_{\mathbb{R}} f(0)\Phi'_\delta(u)b(u)_x dxd\tau = \int_{t_0}^{t}\Big[0\int_{\mathbb{R}} f'(0)\Phi'_\delta(u)b(u)dx - \int_{\mathbb{R}} f(0)\Phi''_\delta(u)b(u)u_x dx\Big]d\tau,$$

isto é,

$$\int_{t_0}^{t}\int_{\mathbb{R}} \Phi_\delta'(u)b(u)_x dxd\tau = -\int_{t_0}^{t}\int_{\mathbb{R}} \Phi_\delta''(u)b(u)u_x dxd\tau,$$

pois $f(0) = 1$ e $f'(0) = 0$. Por fim,

$$\int_{t_0}^{t}\int_{\mathbb{R}} f_r(x)\Phi_\delta'(u)u_{xx} dxd\tau = \int_{t_0}^{t}\int_{-2r}^{2r} f(x/r)\Phi_\delta'(u)u_{xx} dxd\tau.$$

Usando integração por partes, obtemos

$$\int_{t_0}^{t}\int_{\mathbb{R}} f_r(x)\Phi_\delta'(u)u_{xx} dxd\tau = \int_{t_0}^{t}\Big[f(x/r)\Phi_\delta'(u)u_x|_{-2r}^{2r} - \frac{1}{r}\int_{-2r}^{2r} f'(x/r)\Phi_\delta'(u)u_x dx$$
$$-\int_{-2r}^{2r} f(x/r)\Phi_\delta''(u)u_x^2 dx\Big]d\tau.$$

Como $f(2) = f(-2) = 0$, então

$$\int_{t_0}^{t}\int_{\mathbb{R}} f_r(x)\Phi_\delta'(u)u_{xx} dxd\tau = -\frac{1}{r}\int_{t_0}^{t}\int_{-2r}^{2r} f'(x/r)\Phi_\delta'(u)u_x dxd\tau - \int_{t_0}^{t}\int_{-2r}^{2r} f(x/r)\Phi_\delta''(u)u_x^2 dxd\tau.$$

Passando ao limite quando $r \to \infty$, encontramos

$$\int_{t_0}^{t}\int_{\mathbb{R}} f(0)\Phi_\delta'(u)u_{xx} dxd\tau = 0\int_{t_0}^{t}\int_{\mathbb{R}} f'(0)\Phi_\delta'(u)u_x dxd\tau - \int_{t_0}^{t}\int_{\mathbb{R}} f(0)\Phi_\delta''(u)u_x^2 dxd\tau.$$

Como $f(0) = 1$ e $f'(0) = 0$, então

$$\int_{t_0}^{t}\int_{\mathbb{R}} \Phi_\delta'(u)u_{xx} dxd\tau = -\int_{t_0}^{t}\int_{\mathbb{R}} \Phi_\delta''(u)u_x^2 dxd\tau.$$

Portanto, a equação (2.13) nos permite concluir que

$$\int_{\mathbb{R}} \Phi_\delta(u(\cdot,t))dx = \int_{\mathbb{R}} \Phi_\delta(u(\cdot,t_0))dx - \int_{t_0}^{t}\int_{\mathbb{R}} \Phi_\delta''(u)b(u)u_x dxd\tau - \int_{t_0}^{t}\int_{\mathbb{R}} \Phi_\delta''(u)u_x^2 dxd\tau.$$

Faremos uso, em toda a tese, de resultados análogos ao obtido neste exemplo, sem maiores explicações.

2.8 Algumas Desigualdades de Sobolev Básicas em $\mathbb{R}$

No decorrer desta tese utilizaremos algumas Desigualdades de Sobolev. Sendo assim, nada mais justo que mostrar a veracidade de tais desigualdades. Iniciaremos provando o seguinte

Teorema 2.2. *Para todo* $0 < p \leq \infty$ *e* $1 \leq q \leq \infty$, *tem-se*

$$\|u\|_{L^\infty(\mathbb{R})} \leq C(p,q)\|u\|_{L^p(\mathbb{R})}^{1-\theta}\|u_x\|_{L^q(\mathbb{R})}^{\theta}, \ \forall\ u \in C_0^1(\mathbb{R}),$$

onde $\theta = \dfrac{1}{1 + p(1 - \frac{1}{q})}$ *e* $C(p,q) = (2\theta)^{-\theta}$.

Demonstração. Considere, primeiramente, que $1 < q < \infty$. Sejam $t_0 > 1$ (a ser determinado) e $x_0 \in \mathbb{R}$. Seja L_δ uma função sinal regularizada (ver seção 2.6). Pelo Teorema Fundamental do Cálculo, temos

$$[L_\delta(u(x_0))]^{t_0} = \int_{-\infty}^{x_0} \frac{d}{dx} [L_\delta(u(x))]^{t_0}\, dx.$$

Por outro lado, derivando em relação a x, obtemos

$$\begin{aligned}\int_{-\infty}^{x_0} \frac{d}{dx} [L_\delta(u(x))]^{t_0}\, dx &= \int_{-\infty}^{x_0} t_0 [L_\delta(u(x))]^{t_0-1} L'_\delta(u(x)) u_x dx \\ &\leq t_0 \int_{-\infty}^{x_0} [L_\delta(u(x))]^{t_0-1} |L'_\delta(u(x))||u_x| dx.\end{aligned}$$

Com isso,

$$[L_\delta(u(x_0))]^{t_0} \leq t_0 \int_{-\infty}^{x_0} [L_\delta(u(x))]^{t_0-1} |L'_\delta(u(x))||u_x| dx. \tag{2.14}$$

Analogamente,

$$-\int_{x_0}^{\infty} \frac{d}{dx} [L_\delta(u(x))]^{t_0}\, dx = -\int_{x_0}^{\infty} t_0 [L_\delta(u(x))]^{t_0-1} L'_\delta(u(x)) u_x dx \leq t_0 \int_{x_0}^{\infty} [L_\delta(u(x))]^{t_0-1} |L'_\delta(u(x))||u_x| dx.$$

Pelo Teorema Fundamental do Cálculo, encontramos

$$[L_\delta(u(x_0))]^{t_0} \leq t_0 \int_{x_0}^{\infty} [L_\delta(u(x))]^{t_0-1} |L'_\delta(u(x))||u_x| dx. \tag{2.15}$$

Assim, Passando ao limite quando $\delta \to 0$, obtemos

$$\begin{aligned}|u(x_0)|^{t_0} &\leq t_0 \int_{-\infty}^{x_0} |u(x)|^{t_0-1} |\mathrm{sgn}(u(x))||u_x| dx \\ &= t_0 \int_{-\infty}^{x_0} |u(x)|^{t_0-1} |u_x| dx\end{aligned}$$

e

$$\begin{aligned}|u(x_0)|^{t_0} &\leq t_0 \int_{x_0}^{\infty} |u(x)|^{t_0-1} |\mathrm{sgn}(u(x))||u_x| dx \\ &= t_0 \int_{x_0}^{\infty} |u(x)|^{t_0-1} |u_x| dx.\end{aligned}$$

Somando as duas desigualdades acima, chegamos a

$$\begin{aligned}2|u(x_0)|^{t_0} &\leq t_0 \int_{-\infty}^{x_0} |u(x)|^{t_0-1} |u_x| dx + t_0 \int_{x_0}^{\infty} |u(x)|^{t_0-1} |u_x| dx \\ &= t_0 \int_{-\infty}^{\infty} |u(x)|^{t_0-1} |u_x| dx,\end{aligned}$$

isto é,

$$|u(x_0)|^{t_0} \leq \frac{t_0}{2} \int_{\mathbb{R}} |u(x)|^{t_0-1} |u_x| dx. \tag{2.16}$$

Por outro lado, pela desigualdade de Hölder, obtemos

$$\int_{\mathbb{R}} |u(x)|^{t_0-1}|u_x|dx \leq \|u\|^{t_0-1}_{L^{(t_0-1)\frac{q}{q-1}}(\mathbb{R})}\|u_x\|_{L^q(\mathbb{R})}.$$

Consequentemente,

$$|u(x_0)|^{t_0} \quad \leq \quad \frac{t_0}{2}\|u\|^{t_0-1}_{L^{(t_0-1)\frac{q}{q-1}}(\mathbb{R})}\|u_x\|_{L^q(\mathbb{R})}.$$

Portanto,

$$|u(x_0)| \quad \leq \quad \left(\frac{t_0}{2}\right)^{\frac{1}{t_0}}\|u\|^{1-\frac{1}{t_0}}_{L^{(t_0-1)\frac{q}{q-1}}(\mathbb{R})}\|u_x\|^{\frac{1}{t_0}}_{L^q(\mathbb{R})}$$

e, por fim,

$$\|u\|_{L^\infty(\mathbb{R})} \quad \leq \quad \left(\frac{t_0}{2}\right)^{\frac{1}{t_0}}\|u\|^{1-\frac{1}{t_0}}_{L^{(t_0-1)\frac{q}{q-1}}(\mathbb{R})}\|u_x\|^{\frac{1}{t_0}}_{L^q(\mathbb{R})}.$$

Faça $t_0 = 1 + p\left(1 - \dfrac{1}{q}\right)$. Assim, para $0 < \theta = \dfrac{1}{t_0} < 1$, encontramos

$$\|u\|_{L^\infty(\mathbb{R})} \quad \leq \quad (2\theta)^{-\theta}\,\|u\|^{1-\theta}_{L^p(\mathbb{R})}\|u_x\|^{\theta}_{L^q(\mathbb{R})}.$$

Para o caso $q = 1$, basta fazer $t_0 \to 1$, a desigualdade (2.16). Para o caso $q = \infty$, faça $t_0 = 1+p > 1$. Assim sendo, pela desigualdade (2.16), temos que

$$|u(x_0)|^{1+p} \leq \frac{1+p}{2}\int_{\mathbb{R}}|u(x)|^p|u_x|dx \leq \frac{1+p}{2}\|u_x\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|u(x)|^p dx = \frac{1+p}{2}\|u_x\|_{L^\infty(\mathbb{R})}\|u\|^p_{L^p(\mathbb{R})}.$$

Portanto,

$$\|u\|_{L^\infty(\mathbb{R})} \leq \left(\frac{1+p}{2}\right)^{\frac{1}{1+p}}\|u_x\|^{\frac{1}{1+p}}_{L^\infty(\mathbb{R})}\|u\|^{\frac{p}{1+p}}_{L^p(\mathbb{R})}.$$

Isto estabelece o resultado para $q = \infty$. □

Teorema 2.3 (Desigualdade de Sobolev)**.** *Para todo* $0 < p \leq r \leq \infty$ *e* $1 \leq q \leq \infty$, *tem-se*

$$\|u\|_{L^r(\mathbb{R})} \quad \leq \quad K(p,q,r)\|u\|^{1-\lambda}_{L^p(\mathbb{R})}\|u_x\|^{\lambda}_{L^q(\mathbb{R})},\ \forall\ u \in C^1_0(\mathbb{R}),$$

onde $\lambda = \dfrac{1-\frac{p}{r}}{1+p\left(1-\frac{1}{q}\right)}$, $K(p,q,r) = (2\theta)^{-\lambda}$ *e* $\theta = \dfrac{1}{1+p\left(1-\frac{1}{q}\right)}$.

Demonstração. Por interpolação, sabemos que

$$\|u\|_{L^r(\mathbb{R})} \leq \|u\|^{\frac{p}{r}}_{L^p(\mathbb{R})}\|u_x\|^{1-\frac{p}{r}}_{L^\infty(\mathbb{R})},\ \forall\ u \in C^1_0(\mathbb{R}).$$

Utilizando o Teorema 2.2, encontramos

$$\|u\|_{L^r(\mathbb{R})} \leq \|u\|^{\frac{p}{r}}_{L^p(\mathbb{R})}\|u_x\|^{1-\frac{p}{r}}_{L^\infty(\mathbb{R})} \leq \|u\|^{\frac{p}{r}}_{L^p(\mathbb{R})}\left((2\theta)^{-\theta}\,\|u\|^{1-\theta}_{L^p(\mathbb{R})}\|u_x\|^{\theta}_{L^q(\mathbb{R})}\right)^{1-\frac{p}{r}},\ \forall\ u \in C^1_0(\mathbb{R}),$$

ou seja,

$$\|u\|_{L^r(\mathbb{R})} \leq (2\theta)^{-\theta\left(1-\frac{p}{r}\right)} \|u\|_{L^p(\mathbb{R})}^{1-\theta\left(1-\frac{p}{r}\right)} \|u_x\|_{L^q(\mathbb{R})}^{\theta\left(1-\frac{p}{r}\right)}, \ \forall\, u \in C_0^1(\mathbb{R}).$$

Portanto,

$$\|u\|_{L^r(\mathbb{R})} \leq (2\theta)^{-\lambda} \|u\|_{L^p(\mathbb{R})}^{1-\lambda} \|u_x\|_{L^q(\mathbb{R})}^{\lambda}, \ \forall\, u \in C_0^1(\mathbb{R}).$$

□

Na tese utilizaremos dois casos particulares do Teorema 2.3. Estes são os seguintes

Exemplo 2.4. Tome $r = 2, p = 1$ e $q = 2$ no Teorema 2.3 para obter

$$\|u\|_{L^2(\mathbb{R})} \leq \left(\frac{3}{4}\right)^{\frac{1}{3}} \|u\|_{L^1(\mathbb{R})}^{\frac{2}{3}} \|u_x\|_{L^2(\mathbb{R})}^{\frac{1}{3}}, \ \forall\, u \in C_0^1(\mathbb{R}).$$

Exemplo 2.5. Tome $r = \infty, p = 1$ e $q = 2$ no Teorema 2.3 para obter

$$\|u\|_{L^\infty(\mathbb{R})} \leq \left(\frac{3}{4}\right)^{\frac{2}{3}} \|u\|_{L^1(\mathbb{R})}^{\frac{1}{3}} \|u_x\|_{L^2(\mathbb{R})}^{\frac{2}{3}}, \ \forall\, u \in C_0^1(\mathbb{R}).$$

Capítulo 3

Equação de Advecção-Difusão Unidimensional

Provaremos algumas propriedades para soluções $u(\cdot,t)$ da equação unidimensional

$$u_t + [b(u)f(u)]_x = \mu(t)u_{xx} + c(u)u.$$

Mais precisamente, neste capítulo, obteremos algumas estimativas a priori para soluções $\mathbf{u}(\cdot,t)$ do problema

$$\begin{aligned} u_t(x,t) + [b(x,t,u(x,t))f(x,t,u(x,t))]_x &= \mu(t)u_{xx}(x,t) + c(x,t,u(x,t))u(x,t), \\ u(\cdot,0) &= u_0, \end{aligned} \tag{3.1}$$

onde $x \in \mathbb{R}$, $t > 0$, $\mu \in C^0\left([0,\infty)\right), b, c$ são funções tais que

$$\begin{aligned} |b(x,t,u)| &\leq B(t), \\ |c(x,t,u)| &\leq C(t), \\ |f(x,t,u)| &\leq D|u|, \\ \mu(t) &> 0, \end{aligned} \tag{3.2}$$

onde as funções $B, C \in C^0([0,\infty))$, f, b são de classe C^1 e $u(\cdot,0) = u_0 \in L^p(\mathbb{R})$, com $1 \leq p < \infty$, é satisfeita no sentido L^p, ou seja, $\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^p(\mathbb{R})} = 0$.

3.1 Estimativa para a Norma L^p

Sob as hipóteses acima, mostraremos que existe uma constante $C_\gamma > 0$, dependendo somente dos parâmetros $\left\{p, \|u_0\|_{L^p(\mathbb{R})}\right\}$, tal que

$$\|u(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq C_\gamma F(t)^{\frac{3}{4p}} t^{-\frac{1}{p}}, \forall\ t > 0, \tag{3.3}$$

onde

$$F(t) = 2\left(\int_0^t K(\tau)^{2p}\tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}}$$
$$+\left(\int_0^t \left[\frac{2p(2p-1)(DB(\tau))^2}{2\mu(\tau)} + 2pC(\tau)\right]^{\frac{3}{2}} K(\tau)^{2p}\tau^2\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}}.$$

Para isso, mostraremos, primeiramente, que uma solução de (3.1) – (3.2) é L^p-limitado. Este fato será enunciado no seguinte

Teorema 3.1 (Estimativa para a Norma L^p)**.** *Seja $u(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ solução da equação (3.1) sob as mesmas hipóteses em (3.2). Então,*

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(t)\|u_0\|_{L^p(\mathbb{R})}, \forall\ t \geq 0, \tag{3.4}$$

onde $K(t) = K(t,p) = \exp\left\{\int_0^t \left[\frac{(p-1)(DB(\tau))^2}{2\mu(\tau)} + C(\tau)\right]d\tau\right\}.$

Demonstração. Seja L_δ uma função sinal regularizada (ver seção 2.6). Seja $\Phi_\delta(\cdot) = L_\delta(\cdot)^p$. Multiplique a equação

$$u_t + [b(u)f(u)]_x = \mu(t)u_{xx} + c(u)u$$

por $\Phi'_\delta(u)$ e integre sobre $\mathbb{R}\times[t_0,t]$, onde $t_0 \in (0,t)$ e $t > 0$. Dessa forma,

$$\int_{t_0}^t\int_{\mathbb{R}} \Phi'_\delta(u)u_\tau dxd\tau + \int_{t_0}^t\int_{\mathbb{R}} \Phi'_\delta(u)[b(u)f(u)]_x dxd\tau = \int_{t_0}^t\int_{\mathbb{R}} \Phi'_\delta(u)\mu(\tau)u_{xx}dxd\tau$$
$$+\int_{t_0}^t\int_{\mathbb{R}} \Phi'_\delta(u)c(u)udxd\tau.$$

Vamos estudar separadamente cada uma das parcelas obtidas na equação acima. Pelo Teorema Fundamental do Cálculo,

$$\begin{aligned}\int_{t_0}^t \Phi'_\delta(u)u_\tau d\tau &= \int_{t_0}^t \frac{d}{d\tau}[\Phi_\delta(u)]d\tau \\ &= \Phi_\delta(u(\cdot,t)) - \Phi_\delta(u(\cdot,t_0)).\end{aligned}$$

Integrando por partes, usando (3.2) e que $\Phi''_\delta(\cdot)$ é não-negativa, obtemos

$$\begin{aligned}-\int_{\mathbb{R}} \Phi'_\delta(u)[b(u)f(u)]_x dx &= \int_{\mathbb{R}} \Phi''_\delta(u)b(u)f(u)u_x dx \\ &\leq \int_{\mathbb{R}} \Phi''_\delta(u)|b(u)|D|u||u_x|dx. \\ &\leq \int_{\mathbb{R}} \Phi''_\delta(u)B(\tau)D|u||u_x|dx \\ &\leq \int_{\mathbb{R}} \Phi''_\delta(u)\frac{(DB(\tau))^2u^2}{2\mu(\tau)}dx + \int_{\mathbb{R}} \Phi''_\delta(u)\frac{\mu(\tau)u_x^2}{2}dx.\end{aligned}$$

Utilizando (3.2), concluímos

$$\begin{aligned}\int_{\mathbb{R}} \Phi'_\delta(u)c(u)u dx &\leq \int_{\mathbb{R}} |\Phi'_\delta(u)||c(u)||u|dx \\ &\leq \int_{\mathbb{R}} |\Phi'_\delta(u)|C(\tau)|u|dx.\end{aligned}$$

Além disso, através de integração por partes, chegamos a

$$\int_{\mathbb{R}} \Phi'_\delta(u)\mu(\tau)u_{xx}dx = -\mu(\tau)\int_{\mathbb{R}} \Phi''_\delta(u)u_x^2 dx.$$

Como $\mu(\cdot)$, $\Phi''_\delta(\cdot)$ são não-negativos, então

$$\begin{aligned}\int_{\mathbb{R}} \Phi_\delta(u(\cdot,t))dx - \int_{\mathbb{R}} \Phi_\delta(u(\cdot,t_0))dx &\leq \int_{t_0}^t \int_{\mathbb{R}} \Phi''_\delta(u)\frac{(DB(\tau))^2u^2}{2\mu(\tau)}dxd\tau + \int_{t_0}^t \int_{\mathbb{R}} \Phi''_\delta(u)\frac{\mu(\tau)u_x^2}{2}dxd\tau \\ &\quad + \int_{t_0}^t \int_{\mathbb{R}} |\Phi'_\delta(u)|C(\tau)|u|dxd\tau - \int_{t_0}^t \int_{\mathbb{R}} \Phi''_\delta(u)\mu(\tau)u_x^2 dxd\tau \\ &\leq \int_{t_0}^t \frac{(DB(\tau))^2}{2\mu(\tau)} \int_{\mathbb{R}} \Phi''_\delta(u)u^2 dxd\tau - \int_{t_0}^t \frac{\mu(\tau)}{2}\int_{\mathbb{R}} \Phi''_\delta(u)u_x^2 dxd\tau \\ &\quad + \int_{t_0}^t C(\tau)\int_{\mathbb{R}} |\Phi'_\delta(u)||u|dxd\tau \\ &\leq \int_{t_0}^t \frac{(DB(\tau))^2}{2\mu(\tau)} \int_{\mathbb{R}} \Phi''_\delta(u)u^2 dxd\tau + \int_{t_0}^t C(\tau)\int_{\mathbb{R}} |\Phi'_\delta(u)||u|dxd\tau.\end{aligned}$$

Passando ao limite quando $\delta \to 0$, temos que

$\Phi_\delta(u(\cdot,t)) \to |u(\cdot,t)|^p, \Phi'_\delta(u(\cdot,t)) \to p|u(\cdot,t)|^{p-1}\text{sgn}(u(\cdot,t)), \Phi''_\delta(u(\cdot,t)) \to p(p-1)|u(\cdot,t)|^{p-2}, \ \forall\, t > 0.$

Consequentemente, pelo Teorema da Convergência Dominada, obtemos

$$\begin{aligned}\|u(\cdot,t)\|^p_{L^p(\mathbb{R})} &\leq \|u(\cdot,t_0)\|^p_{L^p(\mathbb{R})} + \int_{t_0}^t \frac{(DB(\tau))^2}{2\mu(\tau)}\int_{\mathbb{R}} p(p-1)|u|^p dxd\tau + \int_{t_0}^t C(\tau)\int_{\mathbb{R}} p|u|^p dxd\tau \\ &\leq \|u(\cdot,t_0)\|^p_{L^p(\mathbb{R})} + p(p-1)\int_{t_0}^t \frac{(DB(\tau))^2}{2\mu(\tau)}\|u\|^p_{L^p(\mathbb{R})}d\tau + p\int_{t_0}^t C(\tau)\|u\|^p_{L^p(\mathbb{R})}d\tau \\ &\leq \|u(\cdot,t_0)\|^p_{L^p(\mathbb{R})} + \int_{t_0}^t \left[\frac{(DB(\tau))^2p(p-1)}{2\mu(\tau)} + pC(\tau)\right]\|u\|^p_{L^p(\mathbb{R})}d\tau, \ 0 < t_0 < t.\end{aligned}$$

Como $u(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$, então passando ao limite quando $t_0 \to 0$, encontramos

$$\|u(\cdot,t)\|^p_{L^p(\mathbb{R})} \leq \|u_0\|^p_{L^p(\mathbb{R})} + \int_0^t \left[\frac{(DB(\tau))^2p(p-1)}{2\mu(\tau)} + pC(\tau)\right]\|u\|^p_{L^p(\mathbb{R})}d\tau,$$

Pelo Lema de Gronwall, concluímos que

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})}^p \quad \leq \quad K(t)^p\|u_0\|_{L^p(\mathbb{R})}^p, \forall\, t \geq 0,$$

onde $K(t) = K(t,p) = \exp\left\{\int_0^t \left[\frac{(p-1)(DB(\tau))^2}{2\mu(\tau)} + C(\tau)\right] d\tau\right\}$. Portanto,

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})} \quad \leq \quad K(t)\|u_0\|_{L^p(\mathbb{R})}, \forall\, t \geq 0. \tag{3.5}$$

□

3.2 Estimativa de Energia

O Teorema a seguir nos permite obter diretamente a estimativa (3.3), descrita na seção 3.3.

Teorema 3.2 (Estimativa de energia). *Seja $u(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ solução da equação (3.1) sob as mesmas hipóteses em (3.2). Então,*

$$\|u(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \quad \leq \quad C\|u_0\|_{L^p(\mathbb{R})}^{2p}\left[\frac{2p(2p-1)}{2p^2}\right]^{-\frac{1}{2}} F(t)^{\frac{3}{2}}t^{-2}, \forall\, t > 0,$$

onde $F(t) = 2\left(\int_0^t K(\tau)^{2p}\tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \left[\frac{2p(2p-1)(DB(\tau))^2}{2\mu(\tau)} + 2pC(\tau)\right]^{\frac{3}{2}} K(\tau)^{2p}\tau^2\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}}$,
$F(t) = F(t,p)$ e C é uma constante positiva.

Demonstração. Seja $\Phi_\delta(\cdot) = L_\delta(\cdot)^{2p}$. Multiplique a equação

$$u_t + [b(u)f(u)]_x \quad = \quad \mu(t)u_{xx} + c(u)u$$

por $(t-t_0)^2\Phi'_\delta(u)$ e integre sobre $\mathbb{R}\times[t_0,t]$, onde $t_0 \in (0,t)$ e $t > 0$. Dessa forma,

$$\begin{aligned}
&\int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^2\Phi'_\delta(u)u_\tau dxd\tau + \int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^2\Phi'_\delta(u)[b(u)f(u)]_x dxd\tau \\
&= \int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^2\Phi'_\delta(u)\mu(\tau)u_{xx}dxd\tau + \int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^2\Phi'_\delta(u)c(u)udxd\tau.
\end{aligned}$$

Integrando por partes a primeira parcela da equação acima, obtemos

$$\begin{aligned}
\int_{t_0}^t(\tau-t_0)^2\Phi'_\delta(u)u_\tau d\tau &= \int_{t_0}^t(\tau-t_0)^2\frac{d}{d\tau}[\Phi_\delta(u)]d\tau \\
&= (\tau-t_0)^2\Phi_\delta(u(\cdot,\tau))|_{t_0}^t - 2\int_{t_0}^t(\tau-t_0)\Phi_\delta(u)d\tau \\
&= (t-t_0)^2\Phi_\delta(u(\cdot,t)) - 2\int_{t_0}^t(\tau-t_0)\Phi_\delta(u)d\tau.
\end{aligned}$$

Vimos na seção 3.1 que

$$-\int_{\mathbb{R}} \Phi'_\delta(u)[b(u)f(u)]_x dx \leq \int_{\mathbb{R}} \Phi''_\delta(u)\frac{(DB(\tau))^2u^2}{2\mu(\tau)}dx + \int_{\mathbb{R}} \Phi''_\delta(u)\frac{\mu(\tau)u_x^2}{2}dx,$$

$$\int_{\mathbb{R}} \Phi'_\delta(u)c(u)udx \leq C(\tau)\int_{\mathbb{R}} |\Phi'_\delta(u)||u|dx$$

e

$$\int_{\mathbb{R}} \Phi'_\delta(u)\mu(\tau)u_{xx}dx = -\mu(\tau)\int_{\mathbb{R}} \Phi''_\delta(u)u_x^2dx.$$

Portanto,

$$(t-t_0)^2\int_{\mathbb{R}} \Phi_\delta(u(\cdot,t))dx - 2\int_{t_0}^t(\tau-t_0)\int_{\mathbb{R}} \Phi_\delta(u)dxd\tau \leq \int_{t_0}^t(\tau-t_0)^2\frac{(DB(\tau))^2}{2\mu(\tau)}\int_{\mathbb{R}} \Phi''_\delta(u)u^2dxd\tau$$
$$-\int_{t_0}^t(\tau-t_0)^2\frac{\mu(\tau)}{2}\int_{\mathbb{R}} \Phi''_\delta(u)u_x^2dxd\tau + \int_{t_0}^t(\tau-t_0)^2C(\tau)\int_{\mathbb{R}} |\Phi'_\delta(u)||u|dxd\tau.$$

Passando ao limite quando $\delta \to 0$, temos que

$\Phi_\delta(u(\cdot,t)) \to |u(\cdot,t)|^{2p}, \Phi'_\delta(u(\cdot,t)) \to 2p|u(\cdot,t)|^{2p-1}\text{sgn}(u(\cdot,t)), \Phi''_\delta(u(\cdot,t)) \to 2p(2p-1)|u(\cdot,t)|^{2p-2},$

para todo $t > 0$. Consequentemente,

$$(t-t_0)^2\|u(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} - 2\int_{t_0}^t(\tau-t_0)\|u\|^{2p}_{L^{2p}(\mathbb{R})}d\tau \leq \int_{t_0}^t\frac{(\tau-t_0)^2(DB(\tau))^2}{2\mu(\tau)}2p(2p-1)\|u\|^{2p}_{L^{2p}(\mathbb{R})}d\tau$$
$$+2p\int_{t_0}^t(\tau-t_0)^2C(\tau)\|u\|^{2p}_{L^{2p}(\mathbb{R})}d\tau - \int_{t_0}^t\frac{(\tau-t_0)^2\mu(\tau)2p(2p-1)}{2}\int_{\mathbb{R}} |u|^{2p-2}u_x^2dxd\tau, \ \forall\ t > 0.$$

Por conseguinte,

$$(t-t_0)^2\|u(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} + \frac{2p(2p-1)}{2}\int_{t_0}^t(\tau-t_0)^2\mu(\tau)\int_{\mathbb{R}} |u|^{2p-2}u_x^2dxd\tau$$
$$\leq 2\int_{t_0}^t(\tau-t_0)\|u\|^{2p}_{L^{2p}(\mathbb{R})}d\tau + \int_{t_0}^t(\tau-t_0)^2\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]\|u\|^{2p}_{L^{2p}(\mathbb{R})}d\tau.$$

Passando ao limite quando $t_0 \to 0$, concluímos, através do Teorema da Convergência Dominada, que

$$t^2\|u(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} + \frac{2p(2p-1)}{2}\int_0^t\tau^2\mu(\tau)\int_{\mathbb{R}} |u|^{2p-2}u_x^2dxd\tau$$
$$\leq 2\int_0^t\tau\|u\|^{2p}_{L^{2p}(\mathbb{R})}d\tau + \int_0^t\tau^2\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]\|u\|^{2p}_{L^{2p}(\mathbb{R})}d\tau.$$

Defina a função

$$w(\cdot,t) = \begin{cases} u(\cdot,t), & \text{se } p = 1; \\ \\ |u(\cdot,t)|^p, & \text{se } p > 1. \end{cases}$$

Observe que $w_x^2 = p^2|u|^{2p-2}u_x^2$ e $\|w\|^2_{L^2(\mathbb{R})} = \ \|u\|^{2p}_{L^{2p}(\mathbb{R})}$. Seja

$$\begin{aligned} X(t) &= t^2\|u(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} + \frac{2p(2p-1)}{2}\int_0^t \tau^2\mu(\tau)\int_{\mathbb{R}}|u|^{2p-2}u_x^2 dx d\tau \\ &= t^2\|w(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{2p(2p-1)}{2p^2}\int_0^t \tau^2\mu(\tau)\|w_x\|^2_{L^2(\mathbb{R})}d\tau. \end{aligned} \tag{3.6}$$

Note que $X(t) = X(t,p)$. Além disso,

$$X(t) \leq 2\int_0^t \tau\|w\|^2_{L^2(\mathbb{R})}d\tau + \int_0^t \tau^2\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1) + 2pC(\tau)\right]\|w\|^2_{L^2(\mathbb{R})}d\tau.$$

Vamos utilizar a Desigualdade de Sobolev

$$\|v\|_{L^2(\mathbb{R})} \leq C\|v\|^{\frac{2}{3}}_{L^1(\mathbb{R})}\|v_x\|^{\frac{1}{3}}_{L^2(\mathbb{R})}, \ \forall\ v \in C_0^1(\mathbb{R}). \tag{3.7}$$

Usando as desigualdades (3.5), (3.7) e a Desigualdade de Hölder, concluímos

$$\begin{aligned} X(t) &\leq 2\int_0^t \tau C^2\|w\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\|w_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\quad + \int_0^t \tau^2\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1) + 2pC(\tau)\right]C^2\|w\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\|w_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\leq 2\int_0^t \tau C^2\|u\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\|w_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\quad + \int_0^t \tau^2\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1) + 2pC(\tau)\right]C^2\|u\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\|w_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\leq 2C^2\|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\int_0^t \tau K(\tau)^{\frac{4p}{3}}\|w_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\quad + C^2\|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\int_0^t \tau^2 K(\tau)^{\frac{4p}{3}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1) + 2pC(\tau)\right]\|w_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\leq 2C^2\|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\int_0^t \tau^{\frac{1}{3}}K(\tau)^{\frac{4p}{3}}\mu(\tau)^{-\frac{1}{3}}\mu(\tau)^{\frac{1}{3}}\tau^{\frac{2}{3}}\|w_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\quad + C^2\|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\int_0^t \tau^{\frac{4}{3}}K(\tau)^{\frac{4p}{3}}\mu(\tau)^{-\frac{1}{3}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1) + 2pC(\tau)\right]\tau^{\frac{2}{3}}\mu(\tau)^{\frac{1}{3}}\|w_x\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\leq 2C^2\|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\left(\int_0^t \tau^{\frac{1}{2}}K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}}\left(\int_0^t \mu(\tau)\tau^2\|w_x\|^2_{L^2(\mathbb{R})}d\tau\right)^{\frac{1}{3}} \\ &\quad + C^2\|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\left(\int_0^t \tau^2 K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1) + 2pC(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}} \end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\int_0^t \tau^2 \mu(\tau) \|w_x\|^2_{L^2(\mathbb{R})} d\tau \right)^{\frac{1}{3}} \\
= \quad & C^2 \|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})} \left(\int_0^t \mu(\tau) \tau^2 \|w_x\|^2_{L^2(\mathbb{R})} d\tau \right)^{\frac{1}{3}} F(t), \ \forall \ t > 0,
\end{aligned}
$$

onde $F(t) = F(t,p)$ é dada por

$$
\begin{aligned}
F(t) \quad = \quad & 2 \left(\int_0^t \tau^{\frac{1}{2}} K(\tau)^{2p} \mu(\tau)^{-\frac{1}{2}} d\tau \right)^{\frac{2}{3}} \\
& + \left(\int_0^t \tau^2 K(\tau)^{2p} \mu(\tau)^{-\frac{1}{2}} \left[\frac{(DB(\tau))^2}{2\mu(\tau)} 2p(2p-1) + 2pC(\tau) \right]^{\frac{3}{2}} d\tau \right)^{\frac{2}{3}}. \qquad (3.8)
\end{aligned}
$$

Pela definição de $X(t)$ (ver (3.6)), obtemos

$$
X(t) \quad \leq \quad C^2 \|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{3}} X(t)^{\frac{1}{3}} F(t).
$$

Dessa forma,

$$
X(t)^{\frac{2}{3}} \quad \leq \quad C^2 \|u_0\|^{\frac{4p}{3}}_{L^p(\mathbb{R})} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{3}} F(t).
$$

Consequentemente,

$$
X(t) \quad \leq \quad C^3 \|u_0\|^{2p}_{L^p(\mathbb{R})} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}} F(t)^{\frac{3}{2}}.
$$

Utilizando a definição (3.6), concluímos que

$$
t^2 \|u(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} + \frac{2p(2p-1)}{2} \int_0^t \tau^2 \mu(\tau) \int_{\mathbb{R}} |u|^{2p-2} u_x^2 dx d\tau \quad \leq \quad C^3 \|u_0\|^{2p}_{L^p(\mathbb{R})} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}} F(t)^{\frac{3}{2}},
$$

para todo $t > 0$. Portanto,

$$
\|u(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} \quad \leq \quad C^3 \|u_0\|^{2p}_{L^p(\mathbb{R})} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}} F(t)^{\frac{3}{2}} t^{-2}, \ \forall \ t > 0. \qquad (3.9)
$$

□

3.3 Estimativa para a Norma do Sup

Nesta seção, estudaremos o comportamento da norma do sup para soluções de (3.1)−(3.2). Provaremos que

$$
\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \quad \leq \quad C_\kappa \exp\left\{ \frac{3pt}{2} E(t) \right\} D(t)^{\frac{1}{p}} t^{-\frac{2}{p}}, \ \forall \ t > 0,
$$

onde $D(t) = D(t,B,C,\mu)$, $E(t) = E(t,B,C,\mu)$ e C_κ é uma constante positiva que depende dos parâmetros $\kappa = \{p, \|u_0\|_{L^p(\mathbb{R})}\}$.

Teorema 3.3 (Estimativa para a Norma do Sup). *Seja $u(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ solução do sistema (3.1) sob as mesmas hipóteses em (3.2). Então,*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C^{\frac{3}{p}}\|u_0\|_{L^p(\mathbb{R})}2^{\frac{14}{p}}3^{-\frac{2}{p}}p^{\frac{3}{p}}\exp\left\{\frac{3pt}{2}E(t)\right\}D(t)^{\frac{1}{p}}t^{-\frac{2}{p}}, \forall\ t>0,$$

onde $C>0$ e $D(t) = \left[\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)} + C(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}$.

Demonstração. Primeiramente, observe que

$$\begin{aligned}
F(t) &= 2\left(\int_0^t \tau^{\frac{1}{2}}K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\\
&\leq K(t)^{\frac{4p}{3}}\left[2\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\right]\\
&\leq K(t)^{\frac{4p}{3}}\left[2\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}(2p)^2+(2p)^2C(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\right]\\
&\leq K(t)^{\frac{4p}{3}}\left[2\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + (2p)^2\left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\right]\\
&\leq K(t)^{\frac{4p}{3}}\left[(2p)^2\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + (2p)^2\left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\right]\\
&\leq (2p)^2K(t)^{\frac{4p}{3}}\left[\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\right],
\end{aligned}$$

para todo $t>0$. Por conseguinte,

$$F(t)^{\frac{3}{2}} \leq (2p)^3K(t)^{2p}\left[\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\right]^{\frac{3}{2}},$$

para todo $t>0$. Seja

$$D(t) = \left[\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{(DB(\tau))^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\right]^{\frac{3}{2}},\ \forall\ t>0.$$

Com isso,

$$F(t)^{\frac{3}{2}} \leq (2p)^3K(t)^{2p}D(t)$$

$$
\begin{aligned}
&\leq (2p)^3 \exp\left\{2p\int_0^t \left[\frac{(p-1)(DB(\tau))^2}{2\mu(\tau)} + C(\tau)\right] d\tau\right\} D(t)\\
&\leq (2p)^3 \exp\left\{2p\int_0^t \left[\frac{p(DB(\tau))^2}{2\mu(\tau)} + pC(\tau)\right] d\tau\right\} D(t)\\
&\leq (2p)^3 \exp\left\{2p^2\int_0^t \left[\frac{(DB(\tau))^2}{2\mu(\tau)} + C(\tau)\right] d\tau\right\} D(t)\\
&\leq (2p)^3 \exp\left\{2p^2 E(t)t\right\} D(t), \ \forall\ t > 0
\end{aligned}
$$

e $\left[\frac{(DB(\tau))^2}{2\mu(\tau)} + C(\tau)\right] \leq E(t)$, para todo $\tau \in [0,t]$. Por fim,

$$
F(t)^{\frac{3}{2}} \leq (2p)^3 \exp\left\{2p^2 E(t)t\right\} D(t), \ \forall\ t > 0.
$$

Note que $D(t)$ e $E(t)$ não dependem de p. Por outro lado, analogamente ao que foi feito para encontrar a desigualdade (3.9), é possível estabelecer a seguinte desigualdade

$$
\|u(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \leq C^3\|u(\cdot,s)\|_{L^p(\mathbb{R})}^{2p}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}}(2p)^3\exp\left\{2p^2E(t)(t-s)\right\}D(t)(t-s)^{-2},
$$

onde $0 < s < t$. Sejam $k \in \mathbb{N}$ arbitrário e $t_0 = \frac{t}{4^k}$. Defina $t_j = t_{j-1} + \frac{3t}{4^j}$, para $j \in \mathbb{N}$ e $1 \leq j \leq k$. Portanto, $t_k = t$. Utilizando a desigualdade acima, concluímos

$$
\begin{aligned}
\|u(\cdot,t_j)\|_{L^{2^jp}(\mathbb{R})}^{2^jp} &\leq C^3\|u(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^jp}\left(\frac{2^jp(2^jp-1)}{2^{2j-1}p^2}\right)^{-\frac{1}{2}}(2^jp)^3\exp\left\{2^{2j-1}p^2E(t)\frac{3t}{4^j}\right\}D(t)\\
&\quad\cdot\left(\frac{3t}{4^j}\right)^{-2},
\end{aligned}
$$

Faça $j = k, k-1, ..., 1$ para obter

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^{2^kp}(\mathbb{R})} &\leq C^{\frac{3}{2^kp}}\|u(\cdot,t_{k-1})\|_{L^{2^{k-1}p}(\mathbb{R})}\left(\frac{2^kp(2^kp-1)}{2^{2k-1}p^2}\right)^{-\frac{1}{2^{k+1}p}}(2^kp)^{\frac{3}{2^kp}}\exp\left\{\frac{2^{2k-1}p^2}{2^kp}E(t)\frac{3t}{4^k}\right\}\\
&\quad\cdot D(t)^{\frac{1}{2^kp}}\left(\frac{3t}{4^k}\right)^{-\frac{2}{2^kp}}\\
&\leq \prod_{j=1}^k C^{\frac{3}{2^jp}}\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}\prod_{j=1}^k\left(\frac{2^jp(2^jp-1)}{2^{2j-1}p^2}\right)^{-\frac{1}{2^{j+1}p}}\prod_{j=1}^k(2^jp)^{\frac{3}{2^jp}}\\
&\quad\cdot\prod_{j=1}^k\exp\left\{\frac{2^{2j-1}p^2}{2^jp}E(t)\frac{3t}{4^j}\right\}\prod_{j=1}^kD(t)^{\frac{1}{2^jp}}\prod_{j=1}^k\left(\frac{3t}{4^j}\right)^{-\frac{2}{2^jp}}\\
&\leq C^{\frac{3}{p}\sum_{j=1}^k\frac{1}{2^j}}\|u(\cdot,t_0)\|_{L^p(\mathbb{R})}\prod_{j=1}^k\left(\frac{2(2^jp-1)}{2^jp}\right)^{-\frac{1}{2^{j+1}p}}2^{\frac{3}{p}\sum_{j=1}^k\frac{j}{2^j}}p^{\frac{3}{p}\sum_{j=1}^k\frac{1}{2^j}}\\
&\quad\cdot\exp\left\{\frac{3pt}{2}E(t)\sum_{j=1}^k\frac{1}{2^j}\right\}D(t)^{\frac{1}{p}\sum_{j=1}^k\frac{1}{2^j}}(3t)^{-\frac{2}{p}\sum_{j=1}^k\frac{1}{2^j}}2^{\frac{4}{p}\sum_{j=1}^k\frac{j}{2^j}}
\end{aligned}
$$

$$
\leq \; C^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \|u(\cdot,t_0)\|_{L^p(\mathbb{R})} 2^{\frac{7}{p}\sum_{j=1}^{k}\frac{j}{2^j}} p^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \exp\left\{\frac{3pt}{2}E(t)\sum_{j=1}^{k}\frac{1}{2^j}\right\} D(t)^{\frac{1}{p}\sum_{j=1}^{k}\frac{1}{2^j}}
$$

$$
\cdot\, (3t)^{-\frac{2}{p}\sum_{j=1}^{k}\frac{1}{2^j}}. \tag{3.10}
$$

Na desigualde (3.10) usamos que

$$
\frac{2^j p - 1}{2^j p} \;\geq\; \frac{1}{2}, \;\forall\; j = 1, 2, ..., k.
$$

Deste modo, obtemos

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^{2^k p}(\mathbb{R})} \;\leq\; & C^{\frac{3}{p}\left(1-\frac{1}{2^k}\right)} \|u(\cdot,t_0)\|_{L^p(\mathbb{R})} 2^{\frac{7}{p}\sum_{j=1}^{k}\frac{j}{2^j}} p^{\frac{3}{p}\left(1-\frac{1}{2^k}\right)} \exp\left\{\frac{3pt}{2}E(t)\left(1-\frac{1}{2^k}\right)\right\} \\
& \cdot D(t)^{\frac{1}{p}\left(1-\frac{1}{2^k}\right)} (3t)^{-\frac{2}{p}\left(1-\frac{1}{2^k}\right)}.
\end{aligned}
$$

Passando ao limite quando $k \to \infty$, concluímos que

$$
t_0 = \frac{t}{2^k} \to 0, \; \frac{1}{2^k} \to 0, \; \|u(\cdot,t)\|_{L^{2^k p}(\mathbb{R})} \to \; \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \text{ e } \sum_{j=1}^{k}\frac{j}{2^j} \to 2.
$$

Como, por hipótese, $u(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$, temos que

$$
\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \;\leq\; C^{\frac{3}{p}} \|u_0\|_{L^p(\mathbb{R})} 2^{\frac{14}{p}} p^{\frac{3}{p}} \exp\left\{\frac{3pt}{2}E(t)\right\} D(t)^{\frac{1}{p}} 3^{-\frac{2}{p}} t^{-\frac{2}{p}}, \;\forall\; t > 0 \tag{3.11}
$$

onde C é uma constante positiva, como queríamos demonstrar. □

Capítulo 4

Sistema Rotacionalmente Invariante Unidimensional

No capítulo 3, estudamos a equação

$$u_t(x,t) + [b(x,t,u(x,t))f(x,t,u(x,t))]_x = \mu(t)u_{xx}(x,t) + c(x,t,u(x,t))u(x,t).$$

Neste e nos próximos capítulos, estenderemos as estimativas encontradas para esta equação à estimativas de um sistema de equações de advecção-difusão. Inicialmente consideraremos o problema

$$\mathbf{u}_t(x,t) + [b(x,t,\mathbf{u}(x,t))\mathbf{u}(x,t)]_x = \mu(t)\mathbf{u}_{xx}(x,t) + c(x,t,\mathbf{u}(x,t))\mathbf{u}(x,t),$$

onde $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$. Para este capítulo, supomos que

$$b(x,t,\mathbf{u}(x,t)) = |\mathbf{u}(x,t)|^2, f(x,t,\mathbf{u}(x,t)) = \mathbf{u}(x,t), \mu(t) = 1 \text{ e } c(x,t,\mathbf{u}(x,t)) = 0,\ \forall\ x \in \mathbb{R},\ t > 0.$$

Provaremos alguns resultados para soluções $\mathbf{u}(\cdot,t)$ do sistema, chamado sistema rotacionalmente invariante, $\mathbf{u}_t + \left[|\mathbf{u}|^2\mathbf{u}\right]_x = \mathbf{u}_{xx}$, supondo que o dado inicial $\mathbf{u}(\cdot,0) = \mathbf{u}_0 \in L^p(\mathbb{R})$, para $1 \le p < \infty$. Esta condição inicial é satisfeita no sentido L^p. Aqui $|\mathbf{u}|$ denotará a norma Euclidiana da função $\mathbf{u}$. Mais precisamente, estudaremos o problema

$$\begin{aligned} \mathbf{u}_t(x,t) + \left[|\mathbf{u}(x,t)|^2\mathbf{u}(x,t)\right]_x &= \mathbf{u}_{xx}(x,t), \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0, \end{aligned} \tag{4.1}$$

onde $t \in [0,T]$. Supondo que

$$|\mathbf{u}(\cdot,t)|^2 \le B(T), \text{ q.t.p. em } \mathbb{R}, \tag{4.2}$$

4.1 Estimativa para a Norma L^p

Na seção 4.2, provaremos que existe uma constante $C_\gamma > 0$, dependendo somente dos parâmetros $\gamma = \left\{n, T, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\right\}$, que satisfaz

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \le C_\gamma t^{-\frac{3}{4p}}, \forall\ t \in (0,T], \tag{4.3}$$

onde $\mathbf{u}(\cdot,t)$ uma solução de (4.1) – (4.2). Primeiramente, provaremos o seguinte

Teorema 4.1 (Estimativa para a Norma L^p)**.** *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,T],L^p(\mathbb{R}))$ *solução do sistema (4.1) satisfazendo a hipótese em (4.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \quad \leq \quad K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \forall\ t \in [0,T], \tag{4.4}$$

para T *constante positiva e* $K(t) = K(t,T,p) = \exp\left\{\dfrac{B(T)^2(p-1)}{2}t\right\}$.

Demonstração. Consideremos $\Phi_\delta(\cdot) = L_\delta(\cdot)^p$ e $T > 0$, onde L_δ é uma função sinal regularizada (ver seção 2.6). Inicialmente, provaremos o resultado para cada componente de uma solução de (4.1) – (4.2), i.e.,

$$\|u_i(\cdot,t)\|_{L^p(\mathbb{R})} \quad \leq \quad K(t)\|u_{i0}\|_{L^p(\mathbb{R})}, \forall\ t \in [0,T].$$

Com efeito, a i-ésima equação do sistema em (4.1) é

$$u_{it} + \left[|\mathbf{u}|^2 u_i\right]_x \quad = \quad u_{ixx}.$$

Consequentemente

$$\int_{t_0}^{t}\int_{\mathbb{R}} \Phi'_\delta(u_i)u_{i\tau}dxd\tau + \int_{t_0}^{t}\int_{\mathbb{R}} \Phi'_\delta(u_i)\left[|\mathbf{u}|^2 u_i\right]_x dxd\tau \quad = \quad \int_{t_0}^{t}\int_{\mathbb{R}} \Phi'_\delta(u_i)u_{ixx}dxd\tau. \tag{4.5}$$

Utilizando o Teorema Fundamental do Cálculo, encontramos

$$\begin{aligned}\int_{t_0}^{t} \Phi'_\delta(u_i)u_{i\tau}d\tau &= \int_{t_0}^{t} \frac{d}{d\tau}[\Phi_\delta(u_i)]d\tau \\ &= \Phi_\delta(u_i(\cdot,t)) - \Phi_\delta(u_i(\cdot,t_0)).\end{aligned}$$

Integrando por partes e utilizando (4.2), obtemos

$$\begin{aligned}\int_{\mathbb{R}} \Phi'_\delta(u_i)\left[|\mathbf{u}|^2 u_i\right]_x dx &= -\int_{\mathbb{R}} \Phi''_\delta(u_i)|\mathbf{u}|^2 u_i u_{ix}dx \\ &\geq -\int_{\mathbb{R}} \Phi''_\delta(u_i)|\mathbf{u}|^2|u_i||u_{ix}|dx \\ &\geq -\frac{1}{2}\int_{\mathbb{R}} \Phi''_\delta(u_i)|\mathbf{u}|^4 u_i^2dx - \frac{1}{2}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2dx \\ &\geq -\frac{B(T)^2}{2}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2dx - \frac{1}{2}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2dx.\end{aligned}$$

Integrando por partes novamente, concluímos que

$$\int_{\mathbb{R}} \Phi'_\delta(u_i)u_{ixx}dx \quad = \quad -\int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2dx.$$

Por conseguinte, usando (4.5), inferimos que

$$\begin{aligned}\int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t))dx &\leq \int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t_0))dx + \frac{B(T)^2}{2}\int_{t_0}^{t}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2dxd\tau - \frac{1}{2}\int_{t_0}^{t}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2dxd\tau \\ &\leq \int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t_0))dx + \frac{B(T)^2}{2}\int_{t_0}^{t}\int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2dxd\tau.\end{aligned}$$

Portanto,

$$\int_{\mathbb{R}} |u_i(\cdot,t)|^p dx \quad \leq \quad \int_{\mathbb{R}} |u_i(\cdot,t_0)|^p dx + \frac{B(T)^2 p(p-1)}{2} \int_{t_0}^{t} \int_{\mathbb{R}} |u_i|^p dx d\tau.$$

Equivalentemente,

$$\|u_i(\cdot,t)\|^p_{L^p(\mathbb{R})} \quad \leq \quad \|u_i(\cdot,t_0)\|^p_{L^p(\mathbb{R})} + \frac{B(T)^2 p(p-1)}{2} \int_{t_0}^{t} \|u_i\|^p_{L^p(\mathbb{R})} d\tau.$$

Pelo Lema de Gronwall, concluímos que

$$\|u_i(\cdot,t)\|^p_{L^p(\mathbb{R})} \quad \leq \quad \|u_i(\cdot,t_0)\|^p_{L^p(\mathbb{R})} \exp\left\{\frac{B(T)^2 p(p-1)}{2}(t-t_0)\right\}, \ \forall\ t \in [0,T].$$

Por fim, passando ao limite quando $t_0 \to 0$, e usando que $\mathbf{u}(\cdot,t) \in C^0\left([0,T], L^p(\mathbb{R})\right)$, chegamos a

$$\|u_i(\cdot,t)\|_{L^p(\mathbb{R})} \quad \leq \quad \|u_{i0}\|_{L^p(\mathbb{R})} \exp\left\{\frac{B(T)^2 (p-1)}{2} t\right\}, \ \forall\ t \in [0,T]. \tag{4.6}$$

Consequentemente, somando i de 1 a n (ver (2.1)), obtemos

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \quad \leq \quad K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \forall\ t \in [0,T],$$

onde $\mathbf{u}_0 = (u_{10}, u_{20}, ..., u_{i0}, ..., u_{n0})$ e $K(t) = K(t,T,p) = \exp\left\{\frac{B(T)^2(p-1)}{2} t\right\}$. □

4.2 Estimativa de Energia

O Teorema a seguir verifica a estimativa (4.3), descrita na seção 4.1.

Teorema 4.2 (Estimativa de Energia)**.** *Seja $\mathbf{u}(\cdot,t) \in C^0([0,T], L^p(\mathbb{R}))$ solução do sistema (4.1) sob a hipótese em (4.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} \quad \leq \quad C^3 \|\mathbf{u}_0\|^{2p}_{L^p(\mathbb{R})} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} t^{-\frac{3}{2}}, \forall\ t \in (0,T], \tag{4.7}$$

onde $C = C(n)$, T são constantes positivas e

$$F(t) = F(t,T,p) = \frac{3}{2}\left(\int_0^t K(\tau)^{2p} d\tau\right)^{\frac{2}{3}} + \frac{2p(2p-1)B(T)^2}{2}\left(\int_0^t \tau^{\frac{3}{2}} K(\tau)^{2p} d\tau\right)^{\frac{2}{3}}.$$

.

Demonstração. Como na demonstração do Teorema 4.1, sejam $\Phi_\delta(\cdot) = L_\delta(\cdot)^{2p}$ e $T > 0$. Mostraremos, primeiramente, uma estimativa análoga à estabelecida em (4.7) para cada componente de uma solução de (4.1). De fato, multiplicando a equação

$$u_{it} + \left[|\mathbf{u}|^2 u_i\right]_x \quad = \quad u_{ixx}$$

por $(t-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)$ e integrando sobre $\mathbb{R}\times[t_0,t]$, onde $t_0\in(0,t)$ e $t\in(0,T]$, obtemos

$$\int_{t_0}^{t}\int_{\mathbb{R}}(\tau-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)u_{i\tau}dxd\tau+\int_{t_0}^{t}\int_{\mathbb{R}}(\tau-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)\left[|\mathbf{u}|^2u_i\right]_x dxd\tau$$
$$=\int_{t_0}^{t}\int_{\mathbb{R}}(\tau-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)u_{ixx}dxd\tau.$$

Integrando por partes, encontramos

$$\begin{aligned}\int_{t_0}^{t}(\tau-t_0)^{\frac{3}{2}}\Phi'_\delta(u_i)u_{i\tau}d\tau &= \int_{t_0}^{t}(\tau-t_0)^{\frac{3}{2}}\frac{d}{d\tau}[\Phi_\delta(u_i)]d\tau\\ &= (\tau-t_0)^{\frac{3}{2}}\Phi_\delta(u_i(\cdot,\tau))|_{t_0}^{t}-\frac{3}{2}\int_{t_0}^{t}(\tau-t_0)^{\frac{1}{2}}\Phi_\delta(u_i)d\tau\\ &= (t-t_0)^{\frac{3}{2}}\Phi_\delta(u_i(\cdot,t))-\frac{3}{2}\int_{t_0}^{t}(\tau-t_0)^{\frac{1}{2}}\Phi_\delta(u_i)d\tau.\end{aligned}$$

Os resultados

$$\int_{\mathbb{R}}\Phi'_\delta(u_i)\left[|\mathbf{u}|^2u_i\right]_x dx \geq -\frac{B(T)^2}{2}\int_{\mathbb{R}}\Phi''_\delta(u_i)u_i^2dx-\frac{1}{2}\int_{\mathbb{R}}\Phi''_\delta(u_i)u_{ix}^2dx$$

e

$$\int_{\mathbb{R}}\Phi'_\delta(u_i)u_{ixx}dx = -\int_{\mathbb{R}}\Phi''_\delta(u_i)u_{ix}^2dx,$$

foram encontrados na seção 4.1. Portanto, passando ao limite quando $\delta\to0$, encontramos

$$(t-t_0)^{\frac{3}{2}}\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p}+\frac{2p(2p-1)}{2}\int_{t_0}^{t}(\tau-t_0)^{\frac{3}{2}}\int_{\mathbb{R}}|u_i|^{2p-2}u_{ix}^2dxd\tau\leq\frac{3}{2}\int_{t_0}^{t}(\tau-t_0)^{\frac{1}{2}}\|u_i\|_{L^{2p}(\mathbb{R})}^{2p}d\tau$$
$$+\frac{2p(2p-1)B(T)^2}{2}\int_{t_0}^{t}(\tau-t_0)^{\frac{3}{2}}\|u_i\|_{L^{2p}(\mathbb{R})}^{2p}d\tau.$$

Dessa forma, passando ao limite quando $t_0\to0$, concluímos que

$$t^{\frac{3}{2}}\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p}+\frac{2p(2p-1)}{2}\int_0^t\tau^{\frac{3}{2}}\int_{\mathbb{R}}|u_i|^{2p-2}u_{ix}^2dxd\tau\leq\frac{3}{2}\int_0^t\tau^{\frac{1}{2}}\|u_i\|_{L^{2p}(\mathbb{R})}^{2p}d\tau$$
$$+\frac{2p(2p-1)B(T)^2}{2}\int_0^t\tau^{\frac{3}{2}}\|u_i\|_{L^{2p}(\mathbb{R})}^{2p}d\tau.$$

Defina a função

$$w_i(x,t)=\begin{cases}u_i(x,t), & \text{se } p=1;\\ \\ |u_i(x,t)|^p, & \text{se } p>1,\end{cases}$$

onde $x \in \mathbb{R}$ e $t \in [0,T]$. Relembre que da seção 3.1, concluímos que $w_{ix}^2 = p^2|u_i|^{2p-2}u_{ix}^2$ e $\|w_i\|_{L^2(\mathbb{R})}^2 = \|u_i\|_{L^{2p}(\mathbb{R})}^{2p}$. Denote

$$\begin{aligned} X(t) &= t^{\frac{3}{2}}\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} + \frac{2p(2p-1)}{2}\int_0^t \tau^{\frac{3}{2}}\int_{\mathbb{R}}|u_i|^{2p-2}u_{ix}^2 dx d\tau \\ &= t^{\frac{3}{2}}\|w_i(\cdot,t)\|_{L^2(\mathbb{R})}^2 + \frac{2p(2p-1)}{2p^2}\int_0^t \tau^{\frac{3}{2}}\int_{\mathbb{R}} w_{ix}^2 dx d\tau \\ &= t^{\frac{3}{2}}\|w_i(\cdot,t)\|_{L^2(\mathbb{R})}^2 + \frac{2p(2p-1)}{2p^2}\int_0^t \tau^{\frac{3}{2}}\|w_{ix}\|_{L^2(\mathbb{R})}^2 d\tau. \end{aligned} \tag{4.8}$$

Observe que $X(t) = X(t,p)$. Com isso,

$$\begin{aligned} X(t) &\leq \frac{3}{2}\int_0^t \tau^{\frac{1}{2}}\|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau + \frac{2p(2p-1)B(T)^2}{2}\int_0^t \tau^{\frac{3}{2}}\|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau \\ &= \frac{3}{2}\int_0^t \tau^{\frac{1}{2}}\|w_i\|_{L^2(\mathbb{R})}^2 d\tau + \frac{2p(2p-1)B(T)^2}{2}\int_0^t \tau^{\frac{3}{2}}\|w_i\|_{L^2(\mathbb{R})}^2 d\tau. \end{aligned}$$

Utilizando as desigualdades (4.6), (3.7) e a Desigualdade de Hölder, concluímos que

$$\begin{aligned} X(t) &\leq \frac{3}{2}\int_0^t \tau^{\frac{1}{2}} C_i^2\|w_i\|_{L^1(\mathbb{R})}^{\frac{4}{3}}\|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} d\tau + \frac{2p(2p-1)B(T)^2}{2}\int_0^t \tau^{\frac{3}{2}} C_i^2\|w_i\|_{L^1(\mathbb{R})}^{\frac{4}{3}}\|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} d\tau \\ &\leq \frac{3}{2}C_i^2\int_0^t \tau^{\frac{1}{2}}\|u_i\|_{L^p(\mathbb{R})}^{\frac{4p}{3}}\|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} d\tau + \frac{2p(2p-1)B(T)^2}{2}C_i^2\int_0^t \tau^{\frac{3}{2}}\|u_i\|_{L^p(\mathbb{R})}^{\frac{4p}{3}}\|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} d\tau \\ &\leq \frac{3}{2}C_i^2\|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}}\int_0^t \tau^{\frac{1}{2}}K(\tau)^{\frac{4p}{3}}\|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} d\tau + \frac{2p(2p-1)B(T)^2}{2}C_i^2\|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \\ &\quad \cdot \int_0^t \tau^{\frac{3}{2}}K(\tau)^{\frac{4p}{3}}\|w_{ix}\|_{L^2(\mathbb{R})}^{\frac{2}{3}} d\tau \\ &\leq \frac{3}{2}C_i^2\|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}}\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{3}}\left(\int_0^t \tau^{\frac{3}{2}}\|w_{ix}\|_{L^2(\mathbb{R})}^2 d\tau\right)^{\frac{1}{3}} + \frac{2p(2p-1)B(T)^2}{2}C_i^2 \\ &\quad \cdot \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}}\left(\int_0^t \tau^{\frac{3}{2}}K(\tau)^{2p}d\tau\right)^{\frac{2}{3}}\left(\int_0^t \tau^{\frac{3}{2}}\|w_{ix}\|_{L^2(\mathbb{R})}^2 d\tau\right)^{\frac{1}{3}} \\ &\leq C_i^2\|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}}F(t)\left(\int_0^t \tau^{\frac{3}{2}}\|w_{ix}\|_{L^2(\mathbb{R})}^2 d\tau\right)^{\frac{1}{3}}, \ \forall\ t > 0, \end{aligned}$$

onde $F(t) = F(t,T,p) = \left[\frac{3}{2}\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{3}} + \frac{2p(2p-1)B(T)^2}{2}\left(\int_0^t \tau^{\frac{3}{2}}K(\tau)^{2p}d\tau\right)^{\frac{2}{3}}\right]$. Dessa forma, da definição de $X(t)$ (ver (4.8)), segue que

$$X(t) \leq C_i^2\|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}}F(t)\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{3}}X(t)^{\frac{1}{3}}.$$

Assim sendo,

$$X(t)^{\frac{2}{3}} \leq C_i^2\|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}}F(t)\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{3}},$$

e, consequentemente,

$$X(t) \quad \leq \quad C_i^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}}.$$

Com a definição (4.8), temos que

$$t^{\frac{3}{2}} \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} + \frac{2p(2p-1)}{2} \int_0^t \tau^{\frac{3}{2}} \int_{\mathbb{R}} |u_i|^{2p-2} u_{ix}^2 dx d\tau \quad \leq \quad C_i^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}},$$

para todo $t \in [0,T]$. Finalmente,

$$\begin{aligned} \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \quad &\leq \quad C_i^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}} t^{-\frac{3}{2}} \\ &\leq \quad C_i^3 \|\mathbf{u}_0\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}} t^{-\frac{3}{2}}, \forall\, t \in (0,T]. \end{aligned} \tag{4.9}$$

Portanto, somando i de 1 a n (ver (2.1)), chegamos ao resultado desejado, i.e.,

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \quad \leq \quad C^3 \|\mathbf{u}_0\|_{L^p(\mathbb{R})}^{2p} F(t)^{\frac{3}{2}} \left(\frac{2p(2p-1)}{2p^2} \right)^{-\frac{1}{2}} t^{-\frac{3}{2}},\ \forall\, t \in (0,T],$$

$C = C(n)$ é uma constante positiva. □

4.3 Estimativa para a Norma do Sup

Nesta seção, para a norma do sup de uma solução de (4.1) – (4.2), obteremos uma estimativa da forma

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \quad \leq \quad C_\kappa t^{-\frac{1}{2p}},\ \forall\, t\ \in (0,T],$$

onde C_κ é uma constante positiva que depende dos parâmetros $\kappa = \{n, T, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\}$. Mais precisamente,

Teorema 4.3 (Estimativa para a Norma do Sup)**.** *Seja $\mathbf{u}(\cdot,t) \in C^0([0,T], L^p(\mathbb{R}))$ solução do sistema (4.1) sob a mesma hipótese em (4.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \quad \leq \quad C^{\frac{3}{p}} \|\mathbf{u}_0\|_{L^p(\mathbb{R})} 2^{\frac{8}{p}} 3^{-\frac{1}{2p}} p^{\frac{3}{p}} C(T)^{\frac{3}{2p}} \exp\left\{ \frac{B(T)^2}{4} 3pt \right\} t^{-\frac{1}{2p}},\ \forall\, t \in (0,T], \tag{4.10}$$

onde $C = C(n)$ é uma constante positiva e $C(T) = \left[\frac{1}{2} + \frac{B(T)^2 T}{2} \right]$.

Demonstração. Observe que

$$\begin{aligned}
F(t) &= \frac{3}{2}\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{3}} + \frac{2p(2p-1)B(T)^2}{2}\left(\int_0^t \tau^{\frac{3}{2}}K(\tau)^{2p}d\tau\right)^{\frac{2}{3}} \\
&\leq \frac{3}{2}\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{3}} + \frac{2p(2p-1)B(T)^2T}{2}\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{3}} \\
&\leq \left[\frac{3}{2} + \frac{2p(2p-1)B(T)^2T}{2}\right]\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{3}} \\
&\leq (2p)^2\left[\frac{1}{2} + \frac{B(T)^2T}{2}\right]\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{3}} \\
&\leq (2p)^2C(T)\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{3}} \\
&\leq (2p)^2C(T)K(t)^{2p\frac{2}{3}}t^{\frac{2}{3}},\ \forall\ t\in(0,T]
\end{aligned}$$

e $C(T) = \left[\frac{1}{2} + \frac{B(T)^2T}{2}\right]$, isto é,

$$F(t)^{\frac{3}{2}} \leq (2p)^3C(T)^{\frac{3}{2}}K(t)^{2p}t,\ \forall\ t\in(0,T].$$

É possível estabelecer a desigualdade

$$\begin{aligned}
\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} &\leq C_i^3\|u_i(\cdot,s)\|_{L^p(\mathbb{R})}^{2p}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}}\left[(2p)^3C(T)^{\frac{3}{2}}\exp\left\{\frac{B(T)^2}{2}2p(p-1)(t-s)\right\}\right] \\
&\quad\cdot(t-s)^{-\frac{1}{2}}, 0\leq s<t\leq T,
\end{aligned}$$

da mesma maneira que provamos (4.9). Seja k um número natural qualquer e seja $t_0 = \frac{t}{4^k}$. Defina $t_j = t_{j-1} + \frac{3t}{4^j}$, para $j\in\mathbb{N}$ e $1\leq j\leq k$. É fácil verificar que $t_k = t$. Utilizando a desigualdade acima, obtemos

$$\begin{aligned}
\|u_i(\cdot,t_j)\|_{L^{2^jp}(\mathbb{R})}^{2^jp} &\leq C_i^3\|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^jp}\left(\frac{2^jp(2^jp-1)}{2^{2j-1}p^2}\right)^{-\frac{1}{2}}(2^jp)^3C(T)^{\frac{3}{2}} \\
&\quad\cdot\exp\left\{\frac{B(T)^2}{2}2^jp(2^{j-1}p-1)\left(\frac{3t}{4^j}\right)\right\}\left(\frac{3t}{4^j}\right)^{-\frac{1}{2}} \\
&\leq C_i^3\|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^jp}\left(\frac{2(2^jp-1)}{2^jp}\right)^{-\frac{1}{2}}(2^jp)^3C(T)^{\frac{3}{2}} \\
&\quad\cdot\exp\left\{\frac{B(T)^2}{2}p(2^{j-1}p-1)\left(\frac{3t}{2^j}\right)\right\}\left(\frac{3t}{4^j}\right)^{-\frac{1}{2}} \\
&\leq C_i^3\|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^jp}2^{-\frac{1}{2}}\left(\frac{2^jp-1}{2^jp}\right)^{-\frac{1}{2}}(2^jp)^3C(T)^{\frac{3}{2}} \\
&\quad\cdot\exp\left\{\frac{B(T)^2}{2}p(2^{j-1}p)\left(\frac{3t}{2^j}\right)\right\}\left(\frac{3t}{4^j}\right)^{-\frac{1}{2}} \\
&\leq C_i^3\|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^jp}(2^jp)^3C(T)^{\frac{3}{2}}\exp\left\{\frac{B(T)^2}{4}p^23t\right\}\left(\frac{3t}{4^j}\right)^{-\frac{1}{2}}.
\end{aligned}$$

Consequentemente,

$$\begin{aligned}
\|u_i(\cdot,t)\|_{L^{2^k p}(\mathbb{R})} &\leq C_i^{\frac{3}{2^k p}} \|u_i(\cdot,t_{k-1})\|_{L^{2^{k-1}p}(\mathbb{R})} (2^k p)^{\frac{3}{2^k p}} C(T)^{\frac{3}{2^{k+1}p}} \exp\left\{\frac{B(T)^2}{4} p3t \frac{1}{2^k}\right\} \left(\frac{3t}{4^k}\right)^{-\frac{1}{2^{k+1}p}} \\
&\leq \cdots\cdots\cdots \\
&\leq \prod_{j=1}^{k} C_i^{\frac{3}{2^j p}} \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})} \prod_{j=1}^{k} (2^j p)^{\frac{3}{2^j p}} \prod_{j=1}^{k} C(T)^{\frac{3}{2^{j+1}p}} \prod_{j=1}^{k} \exp\left\{\frac{B(T)^2}{4} p3t \frac{1}{2^j}\right\} \\
&\quad \cdot \prod_{j=1}^{k} \left(\frac{3t}{4^j}\right)^{-\frac{1}{2^{j+1}p}} \\
&\leq C_i^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})} \prod_{j=1}^{k} (2^j p)^{\frac{3}{2^j p}} C(T)^{\frac{3}{2p}\sum_{j=1}^{k}\frac{1}{2^j}} \exp\left\{\frac{B(T)^2}{4} p3t \sum_{j=1}^{k}\frac{1}{2^j}\right\} \\
&\quad \cdot \prod_{j=1}^{k} \left(\frac{3t}{4^j}\right)^{-\frac{1}{2^{j+1}p}} \\
&\leq C_i^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})} 2^{\frac{3}{p}\sum_{j=1}^{k}\frac{j}{2^j}} p^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} C(T)^{\frac{3}{2p}\sum_{j=1}^{k}\frac{1}{2^j}} \exp\left\{\frac{B(T)^2}{4} p3t \sum_{j=1}^{k}\frac{1}{2^j}\right\} \\
&\quad \cdot (3t)^{-\frac{1}{2p}\sum_{j=1}^{k}\frac{1}{2^j}} 2^{\frac{1}{p}\sum_{j=1}^{k}\frac{j}{2^j}} \\
&\leq C_i^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})} 2^{\frac{4}{p}\sum_{j=1}^{k}\frac{j}{2^j}} p^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} C(T)^{\frac{3}{2p}\sum_{j=1}^{k}\frac{1}{2^j}} \exp\left\{\frac{B(T)^2}{4} p3t \sum_{j=1}^{k}\frac{1}{2^j}\right\} \\
&\quad \cdot (3t)^{-\frac{1}{2p}\sum_{j=1}^{k}\frac{1}{2^j}}.
\end{aligned}$$

Dessa forma, passando ao limite quando $k \to \infty$, concluímos que

$$\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_i^{\frac{3}{p}} \|u_{i0}\|_{L^p(\mathbb{R})} 2^{\frac{8}{p}} 3^{-\frac{1}{2p}} p^{\frac{3}{p}} C(T)^{\frac{3}{2p}} \exp\left\{\frac{B(T)^2}{4} 3pt\right\} t^{-\frac{1}{2p}}. \tag{4.11}$$

Utilizando a definição (2.2), obtemos

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C^{\frac{3}{p}} \|\mathbf{u}_0\|_{L^p(\mathbb{R})} 2^{\frac{8}{p}} 3^{-\frac{1}{2p}} p^{\frac{3}{p}} C(T)^{\frac{3}{2p}} \exp\left\{\frac{B(T)^2}{4} 3pt\right\} t^{-\frac{1}{2p}},\ \forall\ t \in (0,T],$$

onde $C = C(n)$ é uma constante positiva. □

Capítulo 5

Sistema de Equações de Advecção-difusão Unidimensional

Neste capítulo, estamos interessados em obter estimativas para um sistema de equações de advecção-difusão unidimensional que estenda os resultados obtidos no capítulo 3 para a equação

$$u_t(x,t) + [b(x,t,u(x,t))f(x,t,u(x,t))]_x = \mu(t)u_{xx}(x,t) + c(x,t,u(x,t))u(x,t).$$

Além da conversão a sistema, acrescentaremos uma matriz diagonal definida positiva $A(x,t,\mathbf{u})$, substituindo a função μ. Ou seja, provaremos algumas propriedades para soluções $\mathbf{u}(\cdot,t)$ do sistema $\mathbf{u}_t + [b(x,t,\mathbf{u})\mathbf{u}]_x = [A(x,t,\mathbf{u})\mathbf{u}_x]_x + c(x,t,\mathbf{u})\mathbf{u}$, supondo que o dado inicial $\mathbf{u}(\cdot,0) = \mathbf{u}_0 \in L^p(\mathbb{R})$, para $1 \leq p < \infty$. A condição inicial é satisfeita no sentido L^p. Mais precisamente, estudaremos estimativas para a norma do sup de uma solução $\mathbf{u}(\cdot,t)$ do problema

$$\begin{aligned} \mathbf{u}_t(x,t) + [b(x,t,\mathbf{u}(x,t))\mathbf{u}(x,t)]_x &= [A(x,t,\mathbf{u}(x,t))\mathbf{u}_x(x,t)]_x + c(x,t,\mathbf{u}(x,t))\mathbf{u}(x,t) \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0, \end{aligned} \tag{5.1}$$

onde $x \in \mathbb{R}$, $t > 0$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$ e $\mu \in C^0([0,\infty))$, b, c são funções tais que

$$\begin{aligned} a_{ii}(x,t,\mathbf{u}(x,t)) &\geq \mu(t), \forall\ i = 1,2,...,n, \\ |b(x,t,\mathbf{u}(x,t))| &\leq B(t), \\ |c(x,t,\mathbf{u}(x,t))| &\leq C(t), \\ \mu(t) &> 0, \end{aligned} \tag{5.2}$$

onde $A(x,t,\mathbf{u}(x,t)) = \text{diag}(a_{ii}(x,t,\mathbf{u}(x,t)))_{1\leq i\leq n}$ é uma matriz diagonal constituída de funções suaves reais, b é de classe C^1 e as funções $B, C \in C^0([0,\infty))$.

5.1 Estimativa para a Norma L^p

Sob as hipóteses estabelecidas acima, mostraremos que existe uma constante $C_\gamma > 0$, dependendo somente dos parâmetros $\gamma = \{n, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\}$ tal que

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq C_\gamma F(t)^{\frac{3}{4p}} t^{-\frac{1}{p}}, \forall\ t > 0, \tag{5.3}$$

onde

$$F(t) = 2\left(\int_0^t \tau^{\frac{1}{2}} K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2 K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}} \left[\frac{B(\tau)^2}{2\mu(\tau)} 2p(2p-1) + 2pC(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}.$$

Primeiramente, provemos o

Teorema 5.1 (Estimativa para a Norma L^p)**.** *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (5.1) sob as mesmas hipóteses em (5.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \forall\ t \geq 0, \tag{5.4}$$

onde $K(t) = K(t,p) = \exp\left\{\int_0^t \left[\frac{(p-1)B(\tau)^2}{2\mu(\tau)} + C(\tau)\right] d\tau\right\}$.

Demonstração. Seja L_δ uma função sinal regularizada (ver seção 2.6). Seja $\Phi_\delta(\cdot) = L_\delta(\cdot)^p$. A i-ésima equação do sistema (5.1) é dada por

$$u_{it} + [b(\mathbf{u})u_i]_x = [a_{ii}(\mathbf{u})u_{ix}]_x + c(\mathbf{u})u_i.$$

Então

$$\int_{t_0}^t \int_{\mathbb{R}} \Phi_\delta'(u_i)u_{i\tau} dx d\tau + \int_{t_0}^t \int_{\mathbb{R}} \Phi_\delta'(u_i)[b(\mathbf{u})u_i]_x dx d\tau = \int_{t_0}^t \int_{\mathbb{R}} \Phi_\delta'(u_i)\left[a_{ii}(\mathbf{u})u_{ix}\right]_x dx d\tau$$
$$+ \int_{t_0}^t \int_{\mathbb{R}} \Phi_\delta'(u_i)c(\mathbf{u})u_i dx d\tau.$$

Vimos na seção 4.1 que

$$\int_{t_0}^t \Phi_\delta'(u_i)u_{i\tau} d\tau = \Phi_\delta(u_i(\cdot,t)) - \Phi_\delta(u_i(\cdot,t_0)).$$

Integrando por partes, obtemos

$$\begin{aligned} -\int_{\mathbb{R}} \Phi_\delta'(u_i)[b(\mathbf{u})u_i]_x dx &= \int_{\mathbb{R}} \Phi_\delta''(u_i)b(\mathbf{u})u_i u_{ix} dx \\ &\leq \int_{\mathbb{R}} \Phi_\delta''(u_i)|b(\mathbf{u})||u_i||u_{ix}|dx. \\ &\leq \int_{\mathbb{R}} \Phi_\delta''(u_i)B(\tau)|u_i||u_{ix}|dx \\ &\leq \int_{\mathbb{R}} \Phi_\delta''(u_i)\frac{B(\tau)^2 u_i^2}{2\mu(\tau)} dx + \int_{\mathbb{R}} \Phi_\delta''(u_i)\frac{\mu(\tau)u_{ix}^2}{2} dx. \end{aligned}$$

Utilizando (5.2), concluímos que

$$\begin{aligned}\int_{\mathbb{R}} \Phi'_\delta(u_i)c(\mathbf{u})u_i dx &\leq \int_{\mathbb{R}} |\Phi'_\delta(u_i)||c(\mathbf{u})||u_i|dx \\ &\leq \int_{\mathbb{R}} |\Phi'_\delta(u_i)|C(\tau)|u_i|dx.\end{aligned}$$

Integrando por partes novamente, chegamos a

$$\begin{aligned}\int_{\mathbb{R}} \Phi'_\delta(u_i)\left[a_{ii}(\mathbf{u})u_{ix}\right]_x dx &= -\int_{\mathbb{R}} \Phi''_\delta(u_i)a_{ii}(\mathbf{u})u_{ix}u_{ix}dx \\ &\leq -\int_{\mathbb{R}} \Phi''_\delta(u_i)\mu(\tau)u_{ix}^2 dx.\end{aligned}$$

Consequentemente,

$$\begin{aligned}\int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t))dx - \int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t_0))dx &\leq \int_{t_0}^t \int_{\mathbb{R}} \Phi''_\delta(u_i)\frac{B(\tau)^2u_i^2}{2\mu(\tau)}dxd\tau + \int_{t_0}^t \int_{\mathbb{R}} \Phi''_\delta(u_i)\frac{\mu(\tau)u_{ix}^2}{2}dxd\tau \\ &\quad + \int_{t_0}^t \int_{\mathbb{R}} |\Phi'_\delta(u_i)|C(\tau)|u_i|dxd\tau - \int_{t_0}^t \int_{\mathbb{R}} \Phi''_\delta(u_i)\mu(\tau)u_{ix}^2 dxd\tau \\ &\leq \int_{t_0}^t \frac{B(\tau)^2}{2\mu(\tau)} \int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2 dxd\tau - \int_{t_0}^t \frac{\mu(\tau)}{2} \int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2 dxd\tau \\ &\quad + \int_{t_0}^t C(\tau) \int_{\mathbb{R}} |\Phi'_\delta(u_i)||u_i|dxd\tau \\ &\leq \int_{t_0}^t \frac{B(\tau)^2}{2\mu(\tau)} \int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2 dxd\tau + \int_{t_0}^t C(\tau) \int_{\mathbb{R}} |\Phi'_\delta(u_i)||u_i|dxd\tau.\end{aligned}$$

Passando ao limite quando $\delta \to 0$ e usando o Teorema da Convergência Dominada, obtemos

$$\begin{aligned}\|u_i(\cdot,t)\|_{L^p(\mathbb{R})}^p &\leq \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})}^p + \int_{t_0}^t \frac{B(\tau)^2}{2\mu(\tau)} \int_{\mathbb{R}} p(p-1)|u_i|^p dxd\tau + \int_{t_0}^t C(\tau) \int_{\mathbb{R}} p|u_i|^p dxd\tau \\ &\leq \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})}^p + p(p-1)\int_{t_0}^t \frac{B(\tau)^2}{2\mu(\tau)}\|u_i\|_{L^p(\mathbb{R})}^p d\tau + p\int_{t_0}^t C(\tau)\|u_i\|_{L^p(\mathbb{R})}^p d\tau \\ &\leq \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})}^p + \int_{t_0}^t \left[\frac{B(\tau)^2p(p-1)}{2\mu(\tau)} + pC(\tau)\right]\|u_i\|_{L^p(\mathbb{R})}^p d\tau.\end{aligned}$$

Agora, passando ao limite quando $t_0 \to 0$, encontramos

$$\|u_i(\cdot,t)\|_{L^p(\mathbb{R})}^p \leq \|u_{i0}\|_{L^p(\mathbb{R})}^p + \int_0^t \left[\frac{B(\tau)^2p(p-1)}{2\mu(\tau)} + pC(\tau)\right]\|u_i\|_{L^p(\mathbb{R})}^p d\tau.$$

Pelo Lema de Gronwall, concluímos que

$$\|u_i(\cdot,t)\|_{L^p(\mathbb{R})}^p \leq K(t)^p\|u_{i0}\|_{L^p(\mathbb{R})}^p, \forall\ t \geq 0,$$

onde $K(t) = K(t,p) = \exp\left\{\int_0^t \left[\frac{(p-1)B(\tau)^2}{2\mu(\tau)} + C(\tau)\right] d\tau\right\}$. Portanto,

$$\|u_i(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(t)\|u_{i0}\|_{L^p(\mathbb{R})}, \forall\ t \geq 0. \tag{5.5}$$

Somando i de 1 a n (ver (2.3)), obtemos

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \forall\ t \geq 0.$$

□

5.2 Estimativa de Energia

O Teorema a seguir nos permite concluir diretamente a estimativa (5.3).

Teorema 5.2 (Estimativa de Energia)**.** *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (5.1) sob as mesmas hipóteses em (5.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} \leq C\|\mathbf{u}_0\|^{2p}_{L^p(\mathbb{R})}\left[\frac{2p(2p-1)}{2p^2}\right]^{-\frac{1}{2}} F(t)^{\frac{3}{2}}t^{-2}, \forall\ t > 0,$$

onde $F(t) = 2\left(\int_0^t K(\tau)^{2p}\tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \left[\frac{2p(2p-1)B(\tau)^2}{2\mu(\tau)} + 2pC(\tau)\right]^{\frac{3}{2}} K(\tau)^{2p}\tau^2\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}}$, $F(t) = F(t,p)$ *e* $C = C(n)$ *é uma constante positiva.*

Demonstração. Seja $\Phi_\delta(\cdot) = L_\delta(\cdot)^{2p}$. Multiplique a equação

$$u_{it} + [b(\mathbf{u})u_i]_x = [a_{ii}(\mathbf{u})u_{ix}]_x + c(\mathbf{u})u_i$$

por $(t-t_0)^2\Phi'_\delta(u_i)$ e integre sobre $\mathbb{R}\times[t_0,t]$, onde $t_0 \in (0,t)$ e $t > 0$. Dessa forma,

$$\begin{aligned}
&\int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^2\Phi'_\delta(u_i)u_{i\tau}dxd\tau + \int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^2\Phi'_\delta(u_i)[b(\mathbf{u})u_i]_x dxd\tau \\
&= \int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^2\Phi'_\delta(u_i)\left[a_{ii}(\mathbf{u})u_{ix}\right]_x dxd\tau + \int_{t_0}^t\int_{\mathbb{R}}(\tau-t_0)^2\Phi'_\delta(u_i)c(\mathbf{u})u_i dxd\tau.
\end{aligned}$$

Note que

$$\begin{aligned}
\int_{t_0}^t(\tau-t_0)^2\Phi'_\delta(u_i)u_{i\tau}d\tau &= \int_{t_0}^t(\tau-t_0)^2\frac{d}{d\tau}[\Phi_\delta(u_i)]d\tau \\
&= (\tau-t_0)^2\Phi_\delta(u_i(\cdot,\tau))|_{t_0}^t - 2\int_{t_0}^t(\tau-t_0)\Phi_\delta(u_i)d\tau \\
&= (t-t_0)^2\Phi_\delta(u_i(\cdot,t)) - 2\int_{t_0}^t(\tau-t_0)\Phi_\delta(u_i)d\tau.
\end{aligned}$$

Na seção 5.1, vimos que

$$-\int_{\mathbb{R}} \Phi'_\delta(u_i)[b(\mathbf{u})u_i]_x dx \quad \leq \quad \frac{B(\tau)^2}{2\mu(\tau)} \int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2 dx + \frac{\mu(\tau)}{2} \int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2 dx,$$

$$\int_{\mathbb{R}} \Phi'_\delta(u_i)c(\mathbf{u})u_i dx \quad \leq \quad C(\tau) \int_{\mathbb{R}} |\Phi'_\delta(u_i)||u_i| dx$$

e

$$\int_{\mathbb{R}} \Phi'_\delta(u_i)\left[a_{ii}(\mathbf{u})u_{ix}\right]_x dx \quad \leq \quad -\mu(\tau) \int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2 dx.$$

Por conseguinte,

$$(t-t_0)^2 \int_{\mathbb{R}} \Phi_\delta(u_i(\cdot,t))dx - 2\int_{t_0}^t (\tau-t_0)\int_{\mathbb{R}} \Phi_\delta(u_i)dxd\tau \leq \int_{t_0}^t (\tau-t_0)^2 \frac{B(\tau)^2}{2\mu(\tau)} \int_{\mathbb{R}} \Phi''_\delta(u_i)u_i^2 dxd\tau$$
$$-\int_{t_0}^t (\tau-t_0)^2 \frac{\mu(\tau)}{2} \int_{\mathbb{R}} \Phi''_\delta(u_i)u_{ix}^2 dxd\tau + \int_{t_0}^t (\tau-t_0)^2 C(\tau) \int_{\mathbb{R}} |\Phi'_\delta(u_i)||u_i| dxd\tau.$$

Passando ao limite quando $\delta \to 0$, temos que

$$(t-t_0)^2 \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} - 2\int_{t_0}^t (\tau-t_0)\|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau \leq \int_{t_0}^t \frac{(\tau-t_0)^2 B(\tau)^2}{2\mu(\tau)} 2p(2p-1)\|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau$$
$$+ 2p\int_{t_0}^t (\tau-t_0)^2 C(\tau)\|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau - \int_{t_0}^t \frac{(\tau-t_0)^2 \mu(\tau) 2p(2p-1)}{2} \int_{\mathbb{R}} |u_i|^{2p-2} u_{ix}^2 dxd\tau, \ \forall\ t > 0.$$

Consequentemente,

$$(t-t_0)^2 \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} + \frac{2p(2p-1)}{2} \int_{t_0}^t (\tau-t_0)^2 \mu(\tau) \int_{\mathbb{R}} |u_i|^{2p-2} u_{ix}^2 dxd\tau$$
$$\leq 2\int_{t_0}^t (\tau-t_0)\|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau + \int_{t_0}^t (\tau-t_0)^2 \left[\frac{B(\tau)^2}{2\mu(\tau)} 2p(2p-1) + 2pC(\tau)\right] \|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau.$$

Passando ao limite quando $t_0 \to 0$, concluímos

$$t^2 \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} + \frac{2p(2p-1)}{2} \int_0^t \tau^2 \mu(\tau) \int_{\mathbb{R}} |u_i|^{2p-2} u_{ix}^2 dxd\tau \leq 2\int_0^t \tau \|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau$$
$$+ \int_0^t \tau^2 \left[\frac{B(\tau)^2}{2\mu(\tau)} 2p(2p-1) + 2pC(\tau)\right] \|u_i\|_{L^{2p}(\mathbb{R})}^{2p} d\tau.$$

Relembre a definição da função

$$w_i(\cdot,t) = \begin{cases} u_i(\cdot,t), & \text{se } p = 1; \\ |u_i(\cdot,t)|^p, & \text{se } p > 1. \end{cases}$$

Recorde que $w_{ix}^2 = p^2|u_i|^{2p-2}u_{ix}^2$ e $\|w_i\|^2_{L^2(\mathbb{R})} = \|u_i\|^{2p}_{L^{2p}(\mathbb{R})}$. Seja

$$\begin{aligned} X(t) &= t^2\|u_i(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} + \frac{2p(2p-1)}{2}\int_0^t \tau^2\mu(\tau)\int_{\mathbb{R}}|u_i|^{2p-2}u_{ix}^2 dx d\tau \\ &= t^2\|w_i(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{2p(2p-1)}{2p^2}\int_0^t \tau^2\mu(\tau)\|w_{ix}\|^2_{L^2(\mathbb{R})}d\tau. \end{aligned} \tag{5.6}$$

Note que $X(t) = X(t,p)$. Além disso,

$$X(t) \leq 2\int_0^t \tau\|w_i\|^2_{L^2(\mathbb{R})}d\tau + \int_0^t \tau^2\left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1) + 2pC(\tau)\right]\|w_i\|^2_{L^2(\mathbb{R})}d\tau.$$

Através das desigualdades (5.5), (3.7) e Desigualdade de Hölder, obtemos

$$\begin{aligned} X(t) &\leq 2\int_0^t \tau C_i^2\|w_i\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau + \int_0^t \tau^2\left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]C_i^2\|w_i\|^{\frac{4}{3}}_{L^1(\mathbb{R})} \\ &\quad \cdot \|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\leq 2\int_0^t \tau C_i^2\|u_i\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau + \int_0^t \tau^2\left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]C_i^2\|u_i\|^{\frac{4p}{3}}_{L^p(\mathbb{R})} \\ &\quad \cdot \|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\leq 2C_i^2\|u_{i0}\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\int_0^t \tau K(\tau)^{\frac{4p}{3}}\|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau + C_i^2\|u_{i0}\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\int_0^t \tau^2 K(\tau)^{\frac{4p}{3}} \\ &\quad \cdot \left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]\|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\leq 2C_i^2\|u_{i0}\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\int_0^t \tau^{\frac{1}{3}}K(\tau)^{\frac{4p}{3}}\mu(\tau)^{-\frac{1}{3}}\mu(\tau)^{\frac{1}{3}}\tau^{\frac{2}{3}}\|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau + C_i^2\|u_{i0}\|^{\frac{4p}{3}}_{L^p(\mathbb{R})} \\ &\quad \cdot \int_0^t \tau^{\frac{4}{3}}K(\tau)^{\frac{4p}{3}}\mu(\tau)^{-\frac{1}{3}}\left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]\tau^{\frac{2}{3}}\mu(\tau)^{\frac{1}{3}}\|w_{ix}\|^{\frac{2}{3}}_{L^2(\mathbb{R})}d\tau \\ &\leq 2C_i^2\|u_{i0}\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\left(\int_0^t \tau^{\frac{1}{2}}K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}}\left(\int_0^t \mu(\tau)\tau^2\|w_{ix}\|^2_{L^2(\mathbb{R})}d\tau\right)^{\frac{1}{3}} + C_i^2\|u_{i0}\|^{\frac{4p}{3}}_{L^p(\mathbb{R})} \\ &\quad \cdot \left(\int_0^t \tau^2K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\left(\int_0^t \tau^2\mu(\tau)\|w_{ix}\|^2_{L^2(\mathbb{R})}d\tau\right)^{\frac{1}{3}} \\ &= C_i^2\|u_{i0}\|^{\frac{4p}{3}}_{L^p(\mathbb{R})}\left(\int_0^t \mu(\tau)\tau^2\|w_{ix}\|^2_{L^2(\mathbb{R})}d\tau\right)^{\frac{1}{3}}F(t),\ \forall\ t>0, \end{aligned}$$

onde $F(t) = F(t,p)$ é dada por

$$\begin{aligned} F(t) &= 2\left(\int_0^t \tau^{\frac{1}{2}}K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{2}{3}} \\ &\quad + \left(\int_0^t \tau^2K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}. \end{aligned} \tag{5.7}$$

Logo, pela definição de $X(t)$ (ver (5.6)), obtemos

$$X(t) \leq C_i^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{3}} X(t)^{\frac{1}{3}} F(t).$$

Dessa forma,

$$X(t)^{\frac{2}{3}} \leq C_i^2 \|u_{i0}\|_{L^p(\mathbb{R})}^{\frac{4p}{3}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{3}} F(t)$$

e então,

$$X(t) \leq C_i^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} F(t)^{\frac{3}{2}}.$$

Utilizando a definição (5.6), concluímos que

$$t^2 \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} + \frac{2p(2p-1)}{2} \int_0^t \tau^2 \mu(\tau) \int_{\mathbb{R}} |u_i|^{2p-2} u_{ix}^2 dx d\tau \leq C_i^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} F(t)^{\frac{3}{2}},$$

para todo $t > 0$. Portanto,

$$\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \leq C_i^3 \|u_{i0}\|_{L^p(\mathbb{R})}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} F(t)^{\frac{3}{2}} t^{-2}, \ \forall\, t > 0. \qquad (5.8)$$

Por fim, somando i de 1 a n (ver (2.3)) chegamos ao resultado desejado, i.e.,

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \leq C \|\mathbf{u}_0\|_{L^p(\mathbb{R})}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} F(t)^{\frac{3}{2}} t^{-2}, \ \forall\, t > 0,$$

onde $C = C(n)$ é uma constante positiva. □

5.3 Estimativa para a Norma do Sup

Nesta seção, provaremos a seguinte estimativa a priori para a norma do sup de uma solução de (5.1) – (5.2) :

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\kappa \exp\left\{\frac{3pt}{2} E(t)\right\} D(t)^{\frac{1}{p}} t^{-\frac{2}{p}}, \ \forall\, t > 0,$$

onde $D(t) = D(t,B,C,\mu)$, $E(t) = E(t,B,C,\mu)$ e C_κ é uma constante positiva que depende dos parâmetros $\kappa = \{n, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R})}\}$.

Teorema 5.3 (Estimativa para a Norma do Sup)**.** *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (5.1) sob as mesmas hipóteses em (5.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C^{\frac{3}{p}} \|\mathbf{u}_0\|_{L^p(\mathbb{R})} 2^{\frac{14}{p}} 3^{-\frac{2}{p}} p^{\frac{3}{p}} \exp\left\{\frac{3pt}{2} E(t)\right\} D(t)^{\frac{1}{p}} t^{-\frac{2}{p}}, \forall\, t > 0,$$

onde $C = C(n) > 0$ *e* $D(t) = \left[\left(\int_0^t \tau^{\frac{1}{2}} \mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2 \mu(\tau)^{-\frac{1}{2}} \left[\frac{B(\tau)^2}{2\mu(\tau)} + C(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}.$

Demonstração. Note que

$$
\begin{aligned}
F(t) &= 2\left(\int_0^t \tau^{\frac{1}{2}} K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2 K(\tau)^{2p}\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}} \\
&\leq K(t)^{\frac{4p}{3}}\left[2\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}2p(2p-1)+2pC(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}\right] \\
&\leq K(t)^{\frac{4p}{3}}\left[2\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}(2p)^2+(2p)^2C(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}\right] \\
&\leq K(t)^{\frac{4p}{3}}\left[2\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + (2p)^2\left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}\right] \\
&\leq K(t)^{\frac{4p}{3}}\left[(2p)^2\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + (2p)^2\left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}\right] \\
&\leq (2p)^2 K(t)^{\frac{4p}{3}}\left[\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}\right], \ \forall\, t>0.
\end{aligned}
$$

Por conseguinte,

$$
F(t)^{\frac{3}{2}} \leq (2p)^3 K(t)^{2p}\left[\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}\right]^{\frac{3}{2}},
$$

para todo $t > 0$. Seja

$$
D(t) = \left[\left(\int_0^t \tau^{\frac{1}{2}}\mu(\tau)^{-\frac{1}{2}} d\tau\right)^{\frac{2}{3}} + \left(\int_0^t \tau^2\mu(\tau)^{-\frac{1}{2}}\left[\frac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right]^{\frac{3}{2}} d\tau\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}, \ \forall\, t>0.
$$

Com isso,

$$
\begin{aligned}
F(t)^{\frac{3}{2}} &\leq (2p)^3 K(t)^{2p} D(t) \\
&\leq (2p)^3 \exp\left\{2p\int_0^t \left[\frac{(p-1)B(\tau)^2}{2\mu(\tau)}+C(\tau)\right] d\tau\right\} D(t) \\
&\leq (2p)^3 \exp\left\{2p\int_0^t \left[\frac{pB(\tau)^2}{2\mu(\tau)}+pC(\tau)\right] d\tau\right\} D(t) \\
&\leq (2p)^3 \exp\left\{2p^2\int_0^t \left[\frac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right] d\tau\right\} D(t) \\
&\leq (2p)^3 \exp\left\{2p^2 E(t)t\right\} D(t), \ \forall\, t>0,
\end{aligned}
$$

onde $\left[\dfrac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right] \leq E(t)$, para todo $\tau \in [0,t]$. Por fim,

$$
F(t)^{\frac{3}{2}} \leq (2p)^3 \exp\left\{2p^2 E(t)t\right\} D(t), \ \forall\, t>0.
$$

Portanto,

$$\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R})}^{2p} \leq C_i^3 \|u_i(\cdot,s)\|_{L^p(\mathbb{R})}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{1}{2}} (2p)^3 \exp\left\{2p^2 E(t)(t-s)\right\} D(t)(t-s)^{-2},$$

onde $0 < s < t$, basta utilizar uma desigualdade análoga a (5.8). Sejam $k \in \mathbb{N}$ arbitrário e $t_0 = \frac{t}{4^k}$. Recorde a definição $t_j = t_{j-1} + \frac{3t}{4^j}$, para $j \in \mathbb{N}$ e $1 \leq j \leq k$. Logo,

$$\|u_i(\cdot,t_j)\|_{L^{2^j p}(\mathbb{R})}^{2^j p} \leq C_i^3 \|u_i(\cdot,t_{j-1})\|_{L^{2^{j-1}p}(\mathbb{R})}^{2^j p} \left(\frac{2^j p(2^j p-1)}{2^{2j-1}p^2}\right)^{-\frac{1}{2}} (2^j p)^3 \exp\left\{2^{2j-1}p^2 E(t)\frac{3t}{4^j}\right\} D(t) \cdot \left(\frac{3t}{4^j}\right)^{-2}.$$

Assim,

$$\begin{aligned}
\|u_i(\cdot,t)\|_{L^{2^k p}(\mathbb{R})} &\leq C_i^{\frac{3}{2^k p}} \|u_i(\cdot,t_{k-1})\|_{L^{2^{k-1}p}(\mathbb{R})} \left(\frac{2^k p(2^k p-1)}{2^{2k-1}p^2}\right)^{-\frac{1}{2^{k+1}p}} (2^k p)^{\frac{3}{2^k p}} \exp\left\{\frac{2^{2k-1}p^2}{2^k p} E(t)\frac{3t}{4^k}\right\} \\
&\quad \cdot D(t)^{\frac{1}{2^k p}} \left(\frac{3t}{4^k}\right)^{-\frac{2}{2^k p}} \\
&\leq \prod_{j=1}^{k} C_i^{\frac{3}{2^j p}} \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})} \prod_{j=1}^{k} \left(\frac{2^j p(2^j p-1)}{2^{2j-1}p^2}\right)^{-\frac{1}{2^{j+1}p}} \prod_{j=1}^{k} (2^j p)^{\frac{3}{2^j p}} \\
&\quad \cdot \prod_{j=1}^{k} \exp\left\{\frac{2^{2j-1}p^2}{2^j p} E(t)\frac{3t}{4^j}\right\} \prod_{j=1}^{k} D(t)^{\frac{1}{2^j p}} \prod_{j=1}^{k} \left(\frac{3t}{4^j}\right)^{-\frac{2}{2^j p}} \\
&\leq C_i^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})} \prod_{j=1}^{k} \left(\frac{2(2^j p-1)}{2^j p}\right)^{-\frac{1}{2^{j+1}p}} 2^{\frac{3}{p}\sum_{j=1}^{k}\frac{j}{2^j}} p^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \\
&\quad \cdot \exp\left\{\frac{3pt}{2} E(t) \sum_{j=1}^{k}\frac{1}{2^j}\right\} D(t)^{\frac{1}{p}\sum_{j=1}^{k}\frac{1}{2^j}} (3t)^{-\frac{2}{p}\sum_{j=1}^{k}\frac{1}{2^j}} 2^{\frac{4}{p}\sum_{j=1}^{k}\frac{j}{2^j}} \\
&\leq C_i^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})} 2^{\frac{7}{p}\sum_{j=1}^{k}\frac{j}{2^j}} p^{\frac{3}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \exp\left\{\frac{3pt}{2} E(t) \sum_{j=1}^{k}\frac{1}{2^j}\right\} D(t)^{\frac{1}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \\
&\quad \cdot (3t)^{-\frac{2}{p}\sum_{j=1}^{k}\frac{1}{2^j}}.
\end{aligned}$$

Passando ao limite quando $k \to \infty$, concluímos que

$$\begin{aligned}
\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R})} &\leq C_i^{\frac{3}{p}} \|u_{i0}\|_{L^p(\mathbb{R})} 2^{\frac{14}{p}} p^{\frac{3}{p}} \exp\left\{\frac{3pt}{2} E(t)\right\} D(t)^{\frac{1}{p}} 3^{-\frac{2}{p}} t^{-\frac{2}{p}} \qquad (5.9) \\
&\leq C^{\frac{3}{p}} \|\mathbf{u}_0\|_{L^p(\mathbb{R})} 2^{\frac{14}{p}} p^{\frac{3}{p}} \exp\left\{\frac{3pt}{2} E(t)\right\} D(t)^{\frac{1}{p}} 3^{-\frac{2}{p}} t^{-\frac{2}{p}}, \ \forall\, t > 0,
\end{aligned}$$

onde $C = C(n)$ é uma constante positiva. Portanto, usando a definição (2.4), concluímos que

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C^{\frac{3}{p}}\|\mathbf{u}_0\|_{L^p(\mathbb{R})}2^{\frac{14}{p}}3^{-\frac{2}{p}}p^{\frac{3}{p}}\exp\left\{\frac{3pt}{2}E(t)\right\}D(t)^{\frac{1}{p}}t^{-\frac{2}{p}},\ \forall\, t>0,$$

onde $C = C(n)$ é uma constante positiva. □

É importante ressaltar que, quando as funções b, c, expostas no sistema de equações de advecção-difusão (5.1), satisfazem as hipóteses

$$\begin{aligned} \int_{\mathbb{R}} |u_i(x,t)|^p[b(x,t,\mathbf{u}(x,t))]_x dx &\geq 0, \forall\, i = 1,2,...,n, \\ c(x,t,\mathbf{u}(x,t)) &\leq 0, \end{aligned}$$

podemos, analogamente, obter as estimativas

$$\begin{aligned} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} &\leq \|\mathbf{u}_0\|_{L^p(\mathbb{R})}, \\ \|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} &\leq \left(\frac{3}{2}\right)^{\frac{3}{2}} C_2^3\|\mathbf{u}_0\|^{2p}_{L^p(\mathbb{R})}\left(\frac{2p(2p-1)}{p^2}\right)^{-\frac{1}{2}}\left(\int_0^t \mu(\tau)^{-\frac{1}{2}}d\tau\right)t^{-\frac{3}{2}}, \\ \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} &\leq 3^{\frac{3}{2p}}2^{\frac{-1}{2p}}C_2^{\frac{3}{p}}\|\mathbf{u}_0\|_{L^p(\mathbb{R})}\left(\int_0^t \mu(\tau)^{-\frac{1}{2}}d\tau\right)^{\frac{1}{p}}t^{-\frac{3}{2p}},\ \forall\, t>0. \end{aligned}$$

As mudanças ocorrem nas integrais

$$\begin{aligned} \int_{\mathbb{R}} \Phi'_\delta(u_i)[b(\mathbf{u})u_i]_x dx &= \int_{\mathbb{R}} \left[\Phi'_\delta(u_i)b(\mathbf{u})_x u_i + \Phi'_\delta(u_i)b(\mathbf{u})u_{ix}\right]dx \\ &= \int_{\mathbb{R}} \Phi'_\delta(u_i)b(\mathbf{u})_x u_i dx - \int_{\mathbb{R}} \Phi_\delta(u_i)b(\mathbf{u})_x dx \end{aligned}$$

e

$$\int_{\mathbb{R}} \Phi'_\delta(u_i)c(\mathbf{u})u_i dx.$$

Passando ao limite quando $\delta \to 0$, estas não interferem nas estimativas, pois

$$\begin{aligned} \int_{\mathbb{R}} \Phi'_\delta(u_i)[b(\mathbf{u})u_i]_x dx &\to (p-1)\int_{\mathbb{R}} |u_i|^p b(\mathbf{u})_x dx \\ &\geq 0 \end{aligned}$$

e

$$\begin{aligned} \int_{\mathbb{R}} \Phi'_\delta(u_i)c(\mathbf{u})u_i dx &\to p\int_{\mathbb{R}} |u_i|^p c(\mathbf{u}) dx \\ &\leq 0. \end{aligned}$$

Capítulo 6

Sistema Rotacionalmente Invariante Multidimensional

Neste capítulo, estudaremos o sistema rotacionalmente invariante, estudado no capítulo 4, no caso em que uma solução deste possua variável espacial em $\mathbb{R}^m$. Apresentaremos algumas propriedades para soluções $\mathbf{u}(\cdot,t)$ do sistema

$$\mathbf{u}_t(\mathbf{x},t)+\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left[|\mathbf{u}(\mathbf{x},t)|^2\mathbf{u}(\mathbf{x},t)\right]=\sum_{k=1}^{m}\frac{\partial^2}{\partial x_k^2}\mathbf{u}(\mathbf{x},t),$$

supondo que o dado inicial $\mathbf{u}(\cdot,0)=\mathbf{u}_0\in L^p(\mathbb{R}^m)$, para $1\leq p<\infty$. Esta condição inicial é satisfeita no sentido L^p. Mais precisamente, neste capítulo, estudaremos soluções $\mathbf{u}(\cdot,t)$ para o problema

$$\begin{aligned}\mathbf{u}_t(\mathbf{x},t)+\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left[|\mathbf{u}(\mathbf{x},t)|^2\mathbf{u}(\mathbf{x},t)\right] &= \sum_{k=1}^{m}\frac{\partial^2}{\partial x_k^2}\mathbf{u}(\mathbf{x},t) \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0,\end{aligned}\tag{6.1}$$

onde $\mathbf{x}\in\mathbb{R}^m$, $t\in(0,T]$ e

$$|\mathbf{u}(\mathbf{x},t)|^2 \leq B(T),\ \text{q.t.p. em } \mathbb{R}^m.\tag{6.2}$$

6.1 Estimativa para a Norma L^p

Sob as hipóteses acima, provaremos que

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)} \leq C_\gamma F(t)^{\frac{m+2}{4p}}t^{-\frac{m+2}{4p}}, \forall\, t\in(0,T],\tag{6.3}$$

onde C_γ é uma constante positiva que depende dos parâmetros $\gamma=\left\{m,n,p,\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}\right\}$ e

$$F(t)=\left(1+\frac{m}{2}\right)\left(\int_0^t K(\tau)^{2p}d\tau\right)^{\frac{2}{m+2}}+\frac{B(T)^2 2p(2p-1)m}{2}\left(\int_0^t K(\tau)^{2p}\tau^{1+\frac{m}{2}}d\tau\right)^{\frac{2}{m+2}}.$$

Teorema 6.1 (Estimativa para a Norma L^p). *Seja $\mathbf{u}(\cdot,t) \in C^0([0,T], L^p(\mathbb{R}^m))$ solução do sistema (6.1) sob a mesma hipótese em (6.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, \forall\ t \in [0,T], \tag{6.4}$$

onde $K(t) = K(t,T,p,m) = \exp\left\{\dfrac{(p-1)B(T)^2 m}{2}t\right\}$.

Demonstração. Seja L_δ uma função sinal regularizada (ver seção 2.6). Seja $\Phi_\delta(\cdot) = L_\delta(\cdot)^p$. Neste caso, a equação que contém a i-ésima componente de uma solução $\mathbf{u}$ é dada por

$$u_{it} + \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left[|\mathbf{u}|^2 u_i\right] = \sum_{k=1}^{m} \frac{\partial^2}{\partial x_k^2} u_i.$$

Logo,

$$\int_{t_0}^{t}\int_{\mathbb{R}^m} \Phi'_\delta(u_i) u_{i\tau} d\mathbf{x} d\tau + \int_{t_0}^{t}\int_{\mathbb{R}^m} \Phi'_\delta(u_i) \sum_{k=1}^{m} \frac{\partial}{\partial x_k}[|\mathbf{u}|^2 u_i] d\mathbf{x} d\tau = \int_{t_0}^{t}\int_{\mathbb{R}^m} \Phi'_\delta(u_i) \sum_{k=1}^{m} \frac{\partial^2}{\partial x_k^2} u_i d\mathbf{x} d\tau.$$

Vimos na seção 4.1 que

$$\int_{t_0}^{t} \Phi'_\delta(u_i) u_{i\tau} d\tau = \Phi_\delta(u_i(\cdot,t)) - \Phi_\delta(u_i(\cdot,t_0)).$$

Integrando por partes e usando (6.2), obtemos

$$\begin{aligned}
-\int_{\mathbb{R}^m} \Phi'_\delta(u_i) \sum_{k=1}^{m} \frac{\partial}{\partial x_k}[|\mathbf{u}|^2 u_i] d\mathbf{x} &= \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \sum_{k=1}^{m} |\mathbf{u}|^2 u_i \frac{\partial u_i}{\partial x_k} d\mathbf{x} \\
&\leq \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \left| \sum_{k=1}^{m} |\mathbf{u}|^2 u_i \frac{\partial u_i}{\partial x_k} \right| d\mathbf{x} \\
&\leq \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \sum_{k=1}^{m} B(T)|u_i| \left| \frac{\partial u_i}{\partial x_k} \right| d\mathbf{x} \\
&\leq \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \frac{B(T)^2 m u_i^2}{2} d\mathbf{x} + \int_{\mathbb{R}^m} \frac{\Phi''_\delta(u_i)}{2} \sum_{k=1}^{m} \left(\frac{\partial u_i}{\partial x_k}\right)^2 d\mathbf{x} \\
&= \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \frac{B(T)^2 m u_i^2}{2} d\mathbf{x} + \int_{\mathbb{R}^m} \frac{\Phi''_\delta(u_i)}{2} \|Du_i\|^2 d\mathbf{x},
\end{aligned}$$

ver definição (2.7). Além disso,

$$\begin{aligned}
\int_{\mathbb{R}^m} \Phi'_\delta(u_i) \sum_{k=1}^{m} \frac{\partial^2}{\partial x_k^2} u_i d\mathbf{x} &= -\int_{\mathbb{R}^m} \Phi''_\delta(u_i) \sum_{k=1}^{m} \left[\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}\right] d\mathbf{x} \\
&\leq -\int_{\mathbb{R}^m} \Phi''_\delta(u_i) \sum_{k=1}^{m} \left(\frac{\partial u_i}{\partial x_k}\right)^2 d\mathbf{x} \\
&= -\int_{\mathbb{R}^m} \Phi''_\delta(u_i) \|Du_i\|^2 d\mathbf{x}.
\end{aligned}$$

Consequentemente,

$$\int_{\mathbb{R}^m} \Phi_\delta(u_i(\cdot,t))d\mathbf{x} - \int_{\mathbb{R}^m} \Phi_\delta(u_i(\cdot,t_0))d\mathbf{x} \leq \frac{B(T)^2 m}{2} \int_{t_0}^{t} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) u_i^2 d\mathbf{x} d\tau$$
$$- \frac{1}{2} \int_{t_0}^{t} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \|Du_i\|^2 d\mathbf{x} d\tau \leq \frac{B(T)^2 m}{2} \int_{t_0}^{t} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) u_i^2 d\mathbf{x} d\tau.$$

Passando ao limite quando $\delta \to 0$, temos que

$$\begin{aligned} \|u_i(\cdot,t)\|^p_{L^p(\mathbb{R}^m)} &\leq \|u_i(\cdot,t_0)\|^p_{L^p(\mathbb{R}^m)} + \frac{B(T)^2 m}{2} \int_{t_0}^{t} \int_{\mathbb{R}^m} p(p-1)|u_i|^p d\mathbf{x} d\tau \\ &\leq \|u_i(\cdot,t_0)\|^p_{L^p(\mathbb{R}^m)} + p(p-1)\frac{B(T)^2 m}{2} \int_{t_0}^{t} \|u_i\|^p_{L^p(\mathbb{R}^m)} d\tau, \ \forall\ t \in [0,T]. \end{aligned}$$

Pelo Lema de Gronwall, concluímos

$$\|u_i(\cdot,t)\|^p_{L^p(\mathbb{R}^m)} \leq \exp\left\{ \frac{p(p-1)B(T)^2 m}{2}(t-t_0) \right\} \|u_i(\cdot,t_0)\|^p_{L^p(\mathbb{R}^m)}, \forall\ t \in [0,T].$$

Portanto, passando ao limite quando $t_0 \to 0$, encontramos

$$\|u_i(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K(t)\|u_{i0}\|_{L^p(\mathbb{R}^m)}, \forall\ t \in [0,T], \tag{6.5}$$

onde $K(t) = K(t,T,p,m) = \exp\left\{ \frac{(p-1)B(T)^2 m}{2} t \right\}$. Por fim, somando i de 1 a n (ver (2.5)), chegamos ao resultado desejado, isto é,

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, \forall\ t \in [0,T].$$

□

6.2 Estimativa de Energia

O Teorema a seguir nos permite concluir a estimativa (6.3).

Teorema 6.2 (Estimativa de Energia). *Seja $\mathbf{u}(\cdot,t) \in C^0([0,T], L^p(\mathbb{R}^m))$ solução do sistema (6.1) sob a mesma hipótese em (6.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^m)} \leq C_m^{m+2} \|\mathbf{u}_0\|^{2p}_{L^p(\mathbb{R}^m)} \left[\frac{2p(2p-1)}{2p^2} \right]^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}} t^{-(1+\frac{m}{2})}, \forall\ t \in (0,T],$$

onde $C_m^{m+2} = C_m^{m+2}(m,n)$ é uma constante positiva e $F(t) = F(t,T,p,m)$ é dada por

$$F(t) = \left(1 + \frac{m}{2}\right) \left(\int_0^t K(\tau)^{2p} d\tau \right)^{\frac{2}{m+2}} + \frac{B(T)^2 2p(2p-1)m}{2} \left(\int_0^t K(\tau)^{2p} \tau^{1+\frac{m}{2}} d\tau \right)^{\frac{2}{m+2}}.$$

Demonstração. Seja $\Phi_\delta(\cdot) = L_\delta(\cdot)^{2p}$. Multiplique a equação

$$u_{it} + \sum_{k=1}^{m} \frac{\partial}{\partial x_k}[|\mathbf{u}|^2 u_i] \quad = \quad \sum_{k=1}^{m} \frac{\partial^2}{\partial x_k^2} u_i$$

por $(t-t_0)^{1+\frac{m}{2}}\Phi'_\delta(u_i)$ e integre sobre $\mathbb{R}^m \times [t_0, t]$, onde $t_0 \in (0,t)$ e $t \in (0,T]$. Dessa forma,

$$\int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi'_\delta(u_i)u_{i\tau}d\mathbf{x}d\tau + \int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi'_\delta(u_i)\sum_{k=1}^{m}\frac{\partial}{\partial x_k}[|\mathbf{u}|^2u_i]d\mathbf{x}d\tau$$
$$= \int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi'_\delta(u_i)\sum_{k=1}^{m}\frac{\partial^2}{\partial x_k^2}u_i d\mathbf{x}d\tau.$$

Integrando por partes, temos que

$$\begin{aligned}
\int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\Phi'_\delta(u_i)u_{i\tau}d\tau &= \int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\frac{d}{d\tau}[\Phi_\delta(u_i)]d\tau \\
&= (\tau-t_0)^{1+\frac{m}{2}}\Phi_\delta(u_i(\cdot,\tau))|_{t_0}^{t} - \left(1+\frac{m}{2}\right)\int_{t_0}^{t}(\tau-t_0)^{\frac{m}{2}}\Phi_\delta(u_i)d\tau \\
&= (t-t_0)^{1+\frac{m}{2}}\Phi_\delta(u_i(\cdot,t)) - \left(1+\frac{m}{2}\right)\int_{t_0}^{t}(\tau-t_0)^{\frac{m}{2}}\Phi_\delta(u_i)d\tau.
\end{aligned}$$

Vimos na seção 6.1 que

$$-\int_{\mathbb{R}^m}\Phi'_\delta(u_i)\sum_{k=1}^{m}\frac{\partial}{\partial x_k}[|\mathbf{u}|^2u_i]d\mathbf{x} \quad \leq \quad \frac{B(T)^2m}{2}\int_{\mathbb{R}^m}\Phi''_\delta(u_i)u_i^2d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^m}\Phi''_\delta(u_i)\|Du_i\|^2d\mathbf{x}$$

e

$$\int_{\mathbb{R}^m}\Phi'_\delta(u_i)\sum_{k=1}^{m}\frac{\partial^2}{\partial x_k^2}u_i d\mathbf{x} \quad \leq \quad -\int_{\mathbb{R}^m}\Phi''_\delta(u_i)\|Du_i\|^2d\mathbf{x}.$$

Por conseguinte

$$(t-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi_\delta(u_i(\cdot,t))d\mathbf{x} - \left(1+\frac{m}{2}\right)\int_{t_0}^{t}(\tau-t_0)^{\frac{m}{2}}\int_{\mathbb{R}^m}\Phi_\delta(u_i)d\mathbf{x}d\tau$$
$$\leq \frac{B(T)^2m}{2}\int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi''_\delta(u_i)u_i^2d\mathbf{x}d\tau - \frac{1}{2}\int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi''_\delta(u_i)\|Du_i\|^2d\mathbf{x}d\tau.$$

Passando ao limite quando $\delta \to 0$, temos que

$$(t-t_0)^{1+\frac{m}{2}}\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} + \frac{2p(2p-1)}{2}\int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}|u_i|^{2p-2}\|Du_i\|^2d\mathbf{x}d\tau \leq \left(1+\frac{m}{2}\right)$$
$$\cdot\int_{t_0}^{t}(\tau-t_0)^{\frac{m}{2}}\|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p}d\tau + \frac{B(T)^2m}{2}2p(2p-1)\int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p}d\tau.$$

Passando ao limite quando $t_0 \to 0$, concluímos que

$$t^{1+\frac{m}{2}}\|u_i(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^m)} + \frac{2p(2p-1)}{2}\int_0^t \tau^{1+\frac{m}{2}}\int_{\mathbb{R}^m}|u_i|^{2p-2}\|Du_i\|^2 d\mathbf{x}d\tau$$
$$\leq \left(1+\frac{m}{2}\right)\int_0^t \tau^{\frac{m}{2}}\|u_i\|^{2p}_{L^{2p}(\mathbb{R}^m)}d\tau + \frac{B(T)^2 m}{2}2p(2p-1)\int_0^t \tau^{1+\frac{m}{2}}\|u_i\|^{2p}_{L^{2p}(\mathbb{R}^m)}d\tau.$$

Recorde a definição da função

$$w_i(\cdot,t) = \begin{cases} u_i(\cdot,t), & \text{se } p=1, \\ |u_i(\cdot,t)|^p, & \text{se } p>1. \end{cases}$$

É fácil ver que $\|Dw_i\|^2 = \sum_{k=1}^m \left(\frac{\partial w_i}{\partial x_k}\right)^2 = p^2|u_i|^{2p-2}\sum_{k=1}^m\left(\frac{\partial u_i}{\partial x_k}\right)^2 = p^2|u_i|^{2p-2}\|Du_i\|^2$ e, como visto anteriormente, $\|w_i\|^2_{L^2(\mathbb{R}^m)} = \|u_i\|^{2p}_{L^{2p}(\mathbb{R}^m)}$. Seja

$$\begin{aligned} X(t) &= t^{1+\frac{m}{2}}\|u_i(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^m)} + \frac{2p(2p-1)}{2}\int_0^t \tau^{1+\frac{m}{2}}\int_{\mathbb{R}^m}|u_i|^{2p-2}\|Du_i\|^2 d\mathbf{x}d\tau \\ &= t^{1+\frac{m}{2}}\|w_i(\cdot,t)\|^2_{L^2(\mathbb{R}^m)} + \frac{2p(2p-1)}{2p^2}\int_0^t \tau^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}d\tau. \end{aligned} \tag{6.6}$$

Dessa forma,

$$X(t) \leq \left(1+\frac{m}{2}\right)\int_0^t \tau^{\frac{m}{2}}\|w_i\|^2_{L^2(\mathbb{R}^m)}d\tau + \frac{B(T)^2 m}{2}2p(2p-1)\int_0^t \tau^{1+\frac{m}{2}}\|w_i\|^2_{L^2(\mathbb{R}^m)}d\tau.$$

Considere a Desigualdade de Sobolev

$$\|v\|_{L^2(\mathbb{R}^m)} \leq C_m\|v\|^{\frac{2}{m+2}}_{L^1(\mathbb{R}^m)}\|\nabla v\|^{\frac{m}{m+2}}_{L^2(\mathbb{R}^m)},\ \forall\ v\in C^1_0(\mathbb{R}^m). \tag{6.7}$$

Usando as desigualdades (6.5), (6.7) e a Desigualdade de Hölder, concluímos

$$\begin{aligned} X(t) &\leq \left(1+\frac{m}{2}\right)\int_0^t \tau^{\frac{m}{2}}C_m^2\|w_i\|^{\frac{4}{m+2}}_{L^1(\mathbb{R}^m)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau + \frac{B(T)^2 m2p(2p-1)}{2} \\ &\quad\cdot\int_0^t \tau^{\left(1+\frac{m}{2}\right)}C_m^2\|w_i\|^{\frac{4}{m+2}}_{L^1(\mathbb{R}^m)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \\ &\leq \left(1+\frac{m}{2}\right)C_m^2\int_0^t \tau^{\frac{m}{2}}\|u_i\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau + C_m^2\frac{B(T)^2 m2p(2p-1)}{2} \\ &\quad\cdot\int_0^t \tau^{\left(1+\frac{m}{2}\right)}\|u_i\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \\ &\leq \left(1+\frac{m}{2}\right)C_m^2\|u_{i0}\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\int_0^t K(\tau)^{\frac{4p}{m+2}}\tau^{\frac{m}{2}}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau + C_m^2\|u_{i0}\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)} \\ &\quad\cdot\frac{B(T)^2 m2p(2p-1)}{2}\int_0^t K(\tau)^{\frac{4p}{m+2}}\tau^{\left(1+\frac{m}{2}\right)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq \left(1+\frac{m}{2}\right) C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \int_0^t K(\tau)^{\frac{4p}{m+2}} \tau^{\frac{m}{2}} \left(\int_{\mathbb{R}^m} \|Dw_i\|^2 d\mathbf{x}\right)^{\frac{m}{m+2}} d\tau + C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \\
&\quad \cdot \frac{B(T)^2 m 2p(2p-1)}{2} \int_0^t K(\tau)^{\frac{4p}{m+2}} \tau^{\left(1+\frac{m}{2}\right)} \left(\int_{\mathbb{R}^m} \|Dw_i\|^2 d\mathbf{x}\right)^{\frac{m}{m+2}} d\tau \\
&\leq \left(1+\frac{m}{2}\right) C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\int_0^t K(\tau)^{2p} d\tau\right)^{\frac{2}{m+2}} \left(\int_0^t \tau^{1+\frac{m}{2}} \int_{\mathbb{R}^m} \|Dw_i\|^2 d\mathbf{x} d\tau\right)^{\frac{m}{m+2}} \\
&\quad + C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \frac{B(T)^2 m 2p(2p-1)}{2} \left(\int_0^t K(\tau)^{2p} \tau^{1+\frac{m}{2}} d\tau\right)^{\frac{2}{m+2}} \\
&\quad \cdot \left(\int_0^t \tau^{1+\frac{m}{2}} \int_{\mathbb{R}^m} \|Dw_i\|^2 d\mathbf{x} d\tau\right)^{\frac{m}{m+2}} \\
&\leq C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\int_0^t \tau^{1+\frac{m}{2}} \int_{\mathbb{R}^m} \|Dw_i\|^2 d\mathbf{x} d\tau\right)^{\frac{m}{m+2}} F(t), \ \forall\, t \in (0,T],
\end{aligned}
$$

onde $F(t) = F(t,T,p,m)$ é dada por

$$
F(t) = \left(1+\frac{m}{2}\right) \left(\int_0^t K(\tau)^{2p} d\tau\right)^{\frac{2}{m+2}} + \frac{B(T)^2 2p(2p-1)m}{2} \left(\int_0^t K(\tau)^{2p} \tau^{1+\frac{m}{2}} d\tau\right)^{\frac{2}{m+2}}.
$$

Pela definição de $X(t)$, ver (6.6), temos que

$$
X(t) \leq C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{m+2}} X(t)^{\frac{m}{m+2}} F(t).
$$

Dessa forma,

$$
X(t)^{\frac{2}{m+2}} \leq C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{m+2}} F(t).
$$

Consequentemente,

$$
X(t) \leq C_m^{m+2} \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}}, \ \forall\, t \in [0,T].
$$

Utilizando a definição (6.6), concluímos que

$$
t^{1+\frac{m}{2}} \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} + \frac{2p(2p-1)}{2} \int_0^t \tau^{1+\frac{m}{2}} \mu(\tau) \int_{\mathbb{R}^m} |u_i|^{2p-2} \|Du_i\|^2 d\mathbf{x} d\tau \leq C_m^{m+2} \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{2p}
\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}}, \ \forall\, t \in [0,T].
$$

Portanto,

$$
\begin{aligned}
\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} &\leq C_m^{m+2} \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}} t^{-\left(1+\frac{m}{2}\right)} \qquad (6.8) \\
&\leq C_m^{m+2} \|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}} t^{-\left(1+\frac{m}{2}\right)}, \ \forall\, t \in (0,T].
\end{aligned}
$$

Por fim, somando i de 1 a n (ver (2.5)), chegamos a

$$\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^m)} \leq C_m^{m+2}\|\mathbf{u}_0\|^{2p}_{L^p(\mathbb{R}^m)}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}}t^{-\left(1+\frac{m}{2}\right)},\ \forall\ t\in(0,T],$$

onde $C_m = C_m(m,n)$ é uma constante positiva. □

6.3 Estimativa para a Norma do Sup

Nesta seção, demonstraremos que a norma do sup de uma solução de (6.1) – (6.2) satisfaz a taxa de decaimento

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\kappa \exp\left\{\frac{3pt}{4}mB(T)^2\right\} D(t)^{\frac{1}{p}}t^{-\frac{m+2}{p}},\ \forall\ t\in(0,T],$$

onde $D(t) = D(t,m,B(T))$ e C_κ é uma constante positiva que depende dos parâmetros $\kappa = \{m,n,p,\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}\}$.

Teorema 6.3 (Estimativa para a Norma do Sup)**.** *Seja* $\mathbf{u}(\cdot,t)\in C^0([0,T],L^p(\mathbb{R}^m))$ *solução do sistema* (*6.1*) *sob a mesma hipóteses em* (*6.2*)*. Então,*

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} \leq C_m^{\frac{m+2}{p}}\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}2^{\frac{4(m+2)}{p}}p^{\frac{m+2}{p}}3^{-\frac{m+2}{p}}\exp\left\{\frac{3pt}{4}mB(T)^2\right\}D(t)^{\frac{1}{p}}t^{-\frac{m+2}{p}}, \quad (6.9)$$

para todo $t\in[0,T]$*,* $C_m = C_m(m,n)$ *é constante positiva e* $D(t) = D(t,T,m)$ *é dada por*

$$D(t) = \left[\left(1+\frac{m}{2}\right)t^{\frac{2}{m+2}} + m\frac{B(T)^2}{2}\left(\int_0^t \tau^{1+\frac{m}{2}}d\tau\right)^{\frac{2}{m+2}}\right]^{\frac{m+2}{2}}.$$

Demonstração. Perceba que

$$F(t)\leq (2p)^2K(t)^{2p\frac{2}{m+2}}\left[\left(1+\frac{m}{2}\right)t^{\frac{2}{m+2}} + m\frac{B(T)^2}{2}\left(\int_0^t \tau^{1+\frac{m}{2}}d\tau\right)^{\frac{2}{m+2}}\right],\ \forall\ t\in[0,T].$$

Seja $D(t) = D(t,T,m)$ dada por

$$D(t) = \left[\left(1+\frac{m}{2}\right)t^{\frac{2}{m+2}} + m\frac{B(T)^2}{2}\left(\int_0^t \tau^{1+\frac{m}{2}}d\tau\right)^{\frac{2}{m+2}}\right]^{\frac{m+2}{2}},\ \forall\ t\in[0,T].$$

Por conseguinte,

$$F(t)^{1+\frac{m}{2}} \leq (2p)^{m+2}K(t)^{2p}D(t),\ \forall\ t\in[0,T].$$

Logo,

$$\begin{aligned} F(t)^{1+\frac{m}{2}} &\leq (2p)^{m+2}K(t)^{2p}D(t) \\ &\leq (2p)^{m+2}\exp\left\{2p\frac{m(p-1)B(T)^2}{2}t\right\}D(t) \\ &\leq (2p)^{m+2}\exp\left\{2p^2\frac{mB(T)^2}{2}t\right\}D(t),\ \forall\ t\in[0,T]. \end{aligned}$$

Por fim,

$$F(t)^{1+\frac{m}{2}} \leq (2p)^{m+2}\exp\left\{2p^2\frac{mB(T)^2}{2}t\right\}D(t),\ \forall\ t\in[0,T].$$

Através da desigualdade (6.8), podemos concluir

$$\|u_i(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^m)} \leq C_m^{m+2}\|u_i(\cdot,s)\|^{2p}_{L^p(\mathbb{R}^m)}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}}(2p)^{m+2}\exp\left\{2pp\frac{mB(T)^2}{2}(t-s)\right\}$$
$$\cdot D(t)(t-s)^{-\frac{m+2}{2}},$$

onde $0<s<t\leq T$. Sejam $k\in\mathbb{N}$ e seja $t_0=\frac{t}{4^k}$. Relembre a definição $t_j=t_{j-1}+\frac{3t}{4^j}$, para $j\in\mathbb{N}$ e $1\leq j\leq k$. Vimos que, $t_k=t$. Portanto, utilizando a desigualdade acima, concluímos

$$\|u_i(\cdot,t_j)\|^{2^jp}_{L^{2^jp}(\mathbb{R}^m)} \leq C_m^{m+2}\|u_i(\cdot,t_{j-1})\|^{2^jp}_{L^{2^{j-1}p}(\mathbb{R}^m)}\left(\frac{2^jp(2^jp-1)}{2^{2j-1}p^2}\right)^{-\frac{m}{2}}(2^jp)^{m+2}\exp\left\{2^{2j-1}p^2\frac{mB(T)^2}{2}\frac{3t}{4^j}\right\}$$
$$D(t)\left(\frac{3t}{4^j}\right)^{-\frac{m+2}{2}}.$$

Logo,

$$\begin{aligned} \|u_i(\cdot,t)\|_{L^{2^kp}(\mathbb{R}^m)} &\leq C_m^{\frac{m+2}{2^kp}}\|u_i(\cdot,t_{k-1})\|_{L^{2^{k-1}p}(\mathbb{R}^m)}\left(\frac{2^kp(2^kp-1)}{2^{2k-1}p^2}\right)^{-\frac{m}{2^{k+1}p}}(2^kp)^{\frac{m+2}{2^kp}} \\ &\quad\cdot\exp\left\{\frac{2^{2k-1}p^2}{2^kp}\frac{mB(T)^2}{2}\frac{3t}{4^k}\right\}D(t)^{\frac{1}{2^kp}}\left(\frac{3t}{4^k}\right)^{-\frac{m+2}{2^{k+1}p}} \\ &\leq \prod_{j=1}^{k}C_m^{\frac{m+2}{2^jp}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R}^m)}\prod_{j=1}^{k}\left(\frac{2^jp(2^jp-1)}{2^{2j-1}p^2}\right)^{-\frac{m}{2^{j+1}p}}\prod_{j=1}^{k}(2^jp)^{\frac{m+2}{2^jp}} \\ &\quad\cdot\prod_{j=1}^{k}\exp\left\{\frac{2^{2j-1}p^2}{2^jp}\frac{mB(T)^2}{2}\frac{3t}{4^j}\right\}\prod_{j=1}^{k}D(t)^{\frac{1}{2^jp}}\prod_{j=1}^{k}\left(\frac{3t}{4^j}\right)^{-\frac{m+2}{2^{j+1}p}} \\ &\leq C_m^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{1}{2^j}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})^m}\prod_{j=1}^{k}\left(\frac{2(2^jp-1)}{2^jp}\right)^{-\frac{m}{2^{j+1}p}}2^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{j}{2^j}}p^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \\ &\quad\cdot\exp\left\{\frac{3pt}{2}\frac{mB(T)^2}{2}\sum_{j=1}^{k}\frac{1}{2^j}\right\}D(t)^{\frac{1}{p}\sum_{j=1}^{k}\frac{1}{2^j}}(3t)^{-\frac{m+2}{2p}\sum_{j=1}^{k}\frac{1}{2^j}}2^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{j}{2^j}} \end{aligned}$$

$$\leq \quad C_m^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \|u_i(\cdot,t_0)\|_{L^p(\mathbb{R}^m)} 2^{\frac{2(m+2)}{p}\sum_{j=1}^{k}\frac{j}{2^j}} p^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{1}{2^j}} \exp\left\{\frac{3pt}{2}\frac{mB(T)^2}{2}\sum_{j=1}^{k}\frac{1}{2^j}\right\}$$

$$\cdot D(t)^{\frac{1}{p}\sum_{j=1}^{k}\frac{1}{2^j}} (3t)^{-\frac{m+2}{p}\sum_{j=1}^{k}\frac{1}{2^j}}. \tag{6.10}$$

Passando ao limite quando $k \to \infty$, obtemos

$$\begin{aligned} \|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} &\leq C_m^{\frac{m+2}{p}} \|u_{i0}\|_{L^p(\mathbb{R}^m)} 2^{\frac{4(m+2)}{p}} p^{\frac{m+2}{p}} 3^{-\frac{m+2}{p}} \exp\left\{\frac{3pt}{2}\frac{mB(T)^2}{2}\right\} D(t)^{\frac{1}{p}} \\ &\quad \cdot t^{-\frac{m+2}{p}} \\ &\leq C_m^{\frac{m+2}{p}} \|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)} 2^{\frac{4(m+2)}{p}} p^{\frac{m+2}{p}} 3^{-\frac{m+2}{p}} \exp\left\{\frac{3pt}{2}\frac{mB(T)^2}{2}\right\} D(t)^{\frac{1}{p}} t^{-\frac{m+2}{p}}, \end{aligned} \tag{6.11}$$

onde $C_m = C_m(m,n)$ é uma constante positiva. Por fim, usando a definição (2.6), encontramos

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} \leq C_m^{\frac{m+2}{p}} \|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)} 2^{\frac{4(m+2)}{p}} p^{\frac{m+2}{p}} 3^{-\frac{m+2}{p}} \exp\left\{\frac{3pt}{4}mB(T)^2\right\} D(t)^{\frac{1}{p}} t^{-\frac{m+2}{p}}, \ \forall\, t \in (0,T].$$

□

Capítulo 7

Sistema de Equações de Advecção-difusão Multidimensional

Neste capítulo, estudaremos o sistema visto no capítulo 5, no caso em que uma solução deste possua variável espacial em $\mathbb{R}^m$. Encontraremos estimativas para soluções $\mathbf{u}(\cdot,t)$ do sistema

$$\mathbf{u}_t(\mathbf{x},t)+\sum_{k=1}^{m}\frac{\partial}{\partial x_k}[b_k(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\mathbf{u}(\mathbf{x},t)] = \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left[A^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\frac{\partial \mathbf{u}}{\partial x_k}(\mathbf{x},t)\right]+c(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\mathbf{u}(\mathbf{x},t),$$

supondo que o dado inicial $\mathbf{u}(\cdot,0)=\mathbf{u}_0\in L^p(\mathbb{R}^m)$, para $1\le p<\infty$. Esta condição inicial é satisfeita no sentido L^p. Mais especificamente, neste capítulo, estudaremos estimativas a priori para a norma do sup das soluções $\mathbf{u}(\cdot,t)$ do problema

$$\begin{aligned}\mathbf{u}_t(\mathbf{x},t)+\sum_{k=1}^{m}\frac{\partial}{\partial x_k}[b_k(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\mathbf{u}(\mathbf{x},t)] &= \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left[A^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\frac{\partial \mathbf{u}}{\partial x_k}(\mathbf{x},t)\right]\\ &\quad + c(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))\mathbf{u}(\mathbf{x},t) \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0,\end{aligned} \tag{7.1}$$

onde $\mathbf{x}\in\mathbb{R}^m$, $t>0$, b_k, c são funções tais que

$$\begin{aligned} a_{ii}^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t)) &\ge \mu(t), \forall\ i=1,2,...,n \text{ e } k=1,2,...,m,\\ |b_k(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))| &\le B(t), \forall\ k=1,2,...,m,\\ |c(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))| &\le C(t),\end{aligned} \tag{7.2}$$

onde $A^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t))=\text{diag}(a_{ii}^{[k]}(\mathbf{x},t,\mathbf{u}(\mathbf{x},t)))_{1\le i\le n}$ é uma matriz diagonal constituída de funções suaves reais, b_k é de classe C^1, a função $\mu\in C^0([0,\infty))$ é positiva e as funções $B,C\in C^0([0,\infty))$.

7.1 Estimativa para a Norma L^p

Sob as hipóteses estabelecidas acima, mostraremos que existe uma constante C_γ positiva, dependendo somente dos parâmetros $\gamma = \big\{n, m, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}\big\}$, tal que

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)} \;\; \leq \;\; C_\gamma F(t)^{\frac{m+2}{4p}} t^{-\frac{m+2}{4p}}, \forall\, t > 0, \tag{7.3}$$

onde

$$\begin{aligned} F(t) \;\; &= \;\; \Big(1+\frac{m}{2}\Big)\Big(\int_0^t K(\tau)^{2p}\mu(\tau)^{-\frac{m}{2}}\,d\tau\Big)^{\frac{2}{m+2}} \\ &\quad + \Big(\int_0^t K(\tau)^{2p}\left[\frac{2p(2p-1)B(\tau)^2 m}{2\mu(\tau)} + 2pC(\tau)\right]^{\frac{m+2}{2}} \tau^{1+\frac{m}{2}}\mu(\tau)^{-\frac{m}{2}}\,d\tau\Big)^{\frac{2}{m+2}}. \end{aligned}$$

Para isso, provaremos, primeiramente, o seguinte

Teorema 7.1 (Estimativa para a Norma L^p)**.** *Seja $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}^m))$ solução do sistema (7.1) sob as mesmas hipóteses em (7.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^m)} \;\; \leq \;\; K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, \forall\, t \geq 0, \tag{7.4}$$

onde $K(t) = K(t,p,m) = \exp\left\{\int_0^t \left[\frac{(p-1)B(\tau)^2 m}{2\mu(\tau)} + C(\tau)\right] d\tau\right\}.$

Demonstração. Seja L_δ uma função sinal regularizada (ver seção 2.6). Seja $\Phi_\delta(\cdot) = L_\delta(\cdot)^p$. Note que, a i-ésima equação do sistema (7.1) é dada por

$$u_{it} + \sum_{k=1}^m \frac{\partial}{\partial x_k}[b_k(\mathbf{u})u_i] \;\; = \;\; \sum_{k=1}^m \frac{\partial}{\partial x_k}\left[a_{ii}^{[k]}(\mathbf{u})\frac{\partial u_i}{\partial x_k}\right] + c(\mathbf{u})u_i.$$

Dessa forma,

$$\begin{aligned} &\int_{t_0}^t \int_{\mathbb{R}^m} \Phi_\delta'(u_i)u_{i\tau}\,d\mathbf{x}d\tau + \int_{t_0}^t \int_{\mathbb{R}^m} \Phi_\delta'(u_i)\sum_{k=1}^m \frac{\partial}{\partial x_k}[b_k(\mathbf{u})u_i]\,d\mathbf{x}d\tau \\ &= \int_{t_0}^t \int_{\mathbb{R}^m} \Phi_\delta'(u_i)\sum_{k=1}^m \frac{\partial}{\partial x_k}\left[a_{ii}^{[k]}(\mathbf{u})\frac{\partial u_i}{\partial x_k}\right] d\mathbf{x}d\tau + \int_{t_0}^t \int_{\mathbb{R}^m} \Phi_\delta'(u_i)c(\mathbf{u})u_i\,d\mathbf{x}d\tau. \end{aligned}$$

Vimos na seção 4.1 que

$$\int_{t_0}^t \Phi_\delta'(u_i)u_{i\tau}\,d\tau \;\; = \;\; \Phi_\delta(u_i(\cdot,t)) - \Phi_\delta(u_i(\cdot,t_0)).$$

Através de integração por partes e usando (7.2), obtemos

$$-\int_{\mathbb{R}^m} \Phi_\delta'(u_i)\sum_{k=1}^m \frac{\partial}{\partial x_k}[b_k(\mathbf{u})u_i]\,d\mathbf{x} \;\; = \;\; \int_{\mathbb{R}^m} \Phi_\delta''(u_i)\sum_{k=1}^m b_k(\mathbf{u})u_i\frac{\partial u_i}{\partial x_k}\,d\mathbf{x}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \left| \sum_{k=1}^m b_k(\mathbf{u}) u_i \frac{\partial u_i}{\partial x_k} \right| d\mathbf{x} \\
&\leq \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \sum_{k=1}^m B(\tau)|u_i| \left| \frac{\partial u_i}{\partial x_k} \right| d\mathbf{x} \\
&\leq \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \frac{B(\tau)^2 m u_i^2}{2\mu(\tau)} d\mathbf{x} + \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \frac{\mu(\tau)}{2} \sum_{k=1}^m \left(\frac{\partial u_i}{\partial x_k} \right)^2 d\mathbf{x} \\
&= \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \frac{B(\tau)^2 m u_i^2}{2\mu(\tau)} d\mathbf{x} + \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \frac{\mu(\tau)}{2} \|Du_i\|^2 d\mathbf{x},
\end{aligned}
$$

ver definição (2.10). É fácil ver que

$$
\int_{\mathbb{R}^m} \Phi'_\delta(u_i) c(\mathbf{u}) u_i d\mathbf{x} \leq \int_{\mathbb{R}^m} |\Phi'_\delta(u_i)| C(\tau) |u_i| d\mathbf{x}.
$$

Além disso, integrando por partes e usando (7.2), inferimos

$$
\begin{aligned}
\int_{\mathbb{R}^m} \Phi'_\delta(u_i) \sum_{k=1}^m \frac{\partial}{\partial x_k} \left[a_{ii}^{[k]}(\mathbf{u}) \frac{\partial u_i}{\partial x_k} \right] d\mathbf{x} &= -\int_{\mathbb{R}^m} \Phi''_\delta(u_i) \sum_{k=1}^m \left[a_{ii}^{[k]}(\mathbf{u}) \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \right] d\mathbf{x} \\
&\leq -\int_{\mathbb{R}^m} \Phi''_\delta(u_i) \mu(\tau) \sum_{k=1}^m \left(\frac{\partial u_i}{\partial x_k} \right)^2 d\mathbf{x} \\
&= -\int_{\mathbb{R}^m} \Phi''_\delta(u_i) \mu(\tau) \|Du_i\|^2 d\mathbf{x}.
\end{aligned}
$$

Consequentemente,

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \Phi_\delta(u_i(\cdot, t)) d\mathbf{x} - \int_{\mathbb{R}^m} \Phi_\delta(u_i(\cdot, t_0)) d\mathbf{x} \leq \int_{t_0}^t \frac{B(\tau)^2 m}{2\mu(\tau)} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) u_i^2 d\mathbf{x} d\tau \\
&- \frac{1}{2} \int_{t_0}^t \mu(\tau) \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \|Du_i\|^2 d\mathbf{x} d\tau + \int_{t_0}^t C(\tau) \int_{\mathbb{R}^m} |\Phi'_\delta(u_i)| |u_i| d\mathbf{x} d\tau \\
&\leq \int_{t_0}^t \frac{B(\tau)^2 m}{2\mu(\tau)} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) u_i^2 d\mathbf{x} d\tau + \int_{t_0}^t C(\tau) \int_{\mathbb{R}^m} |\Phi'_\delta(u_i)| |u_i| d\mathbf{x} d\tau.
\end{aligned}
$$

Passando ao limite quando $\delta \to 0$, temos

$$
\begin{aligned}
\|u_i(\cdot, t)\|_{L^p(\mathbb{R}^m)}^p &\leq \|u_i(\cdot, t_0)\|_{L^p(\mathbb{R}^m)}^p + \int_{t_0}^t \frac{B(\tau)^2 m}{2\mu(\tau)} \int_{\mathbb{R}^m} p(p-1)|u_i|^p d\mathbf{x} d\tau + \int_{t_0}^t C(\tau) \int_{\mathbb{R}^m} p|u_i|^p d\mathbf{x} d\tau \\
&\leq \|u_i(\cdot, t_0)\|_{L^p(\mathbb{R}^m)}^p + p(p-1) \int_{t_0}^t \frac{B(\tau)^2 m}{2\mu(\tau)} \|u_i\|_{L^p(\mathbb{R}^m)}^p d\tau + p \int_{t_0}^t C(\tau) \|u_i\|_{L^p(\mathbb{R}^m)}^p d\tau \\
&\leq \|u_i(\cdot, t_0)\|_{L^p(\mathbb{R}^m)}^p + \int_{t_0}^t \left[\frac{B(\tau)^2 p(p-1) m}{2\mu(\tau)} + pC(\tau) \right] \|u_i\|_{L^p(\mathbb{R}^m)}^p d\tau, \ \forall\, t > 0.
\end{aligned}
$$

Pelo Lema de Gronwall, concluímos que

$$
\|u_i(\cdot, t)\|_{L^p(\mathbb{R}^m)}^p \leq K(t)^p \|u_i(\cdot, t_0)\|_{L^p(\mathbb{R}^m)}^p, \ \forall\, t \geq 0,
$$

onde $K(t) = \exp\left\{\int_{t_0}^{t}\left[\frac{(p-1)B(\tau)^2 m}{2\mu(\tau)} + C(\tau)\right]d\tau\right\}$. Portanto, passando ao limite quando $t_0 \to 0$, encontramos

$$\|u_i(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K(t)\|u_{i0}\|_{L^p(\mathbb{R}^m)}, \forall\ t \geq 0. \tag{7.5}$$

Somando i de 1 a n (ver (2.8)), chegamos a

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, \forall\ t \geq 0.$$

□

7.2 Estimativa de Energia

A estimativa (7.3) pode ser concluída a partir do

Teorema 7.2 (Estimativa de energia)**.** *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}^m))$ *solução do sistema (7.1) sob as mesmas hipóteses em (7.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^m)} \leq C_m^{m+2}\|\mathbf{u}_0\|^{2p}_{L^p(\mathbb{R}^m)}\left[\frac{2p(2p-1)}{2p^2}\right]^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}} t^{-(1+\frac{m}{2})}, \forall\ t > 0,$$

onde $C_m = C_m(m,n)$ *é uma constante positiva e* $F(t) = F(t,p,m)$ *é dada por*

$$\begin{aligned} F(t) &= \left(1+\frac{m}{2}\right)\left(\int_0^t K(\tau)^{2p}\mu(\tau)^{-\frac{m}{2}}d\tau\right)^{\frac{2}{m+2}} \\ &\quad + \left(\int_0^t K(\tau)^{2p}\left[\frac{2p(2p-1)B(\tau)^2 m}{2\mu(\tau)} + 2pC(\tau)\right]^{\frac{m+2}{2}} \tau^{1+\frac{m}{2}}\mu(\tau)^{-\frac{m}{2}}d\tau\right)^{\frac{2}{m+2}}. \end{aligned}$$

Demonstração. Seja $\Phi_\delta(\cdot) = L_\delta(\cdot)^{2p}$. Multiplique a equação

$$u_{it} + \sum_{k=1}^{m}\frac{\partial}{\partial x_k}[b_k(\mathbf{u})u_i] = \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left[a_{ii}^{[k]}(\mathbf{u})\frac{\partial u_i}{\partial x_k}\right] + c(\mathbf{u})u_i$$

por $(t-t_0)^{1+\frac{m}{2}}\Phi'_\delta(u_i)$ e integre sobre $\mathbb{R}^m \times [t_0,t]$, onde $t_0 \in (0,t)$ e $t > 0$. Dessa forma,

$$\begin{aligned} &\int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi'_\delta(u_i)u_{i\tau}d\mathbf{x}d\tau + \int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi'_\delta(u_i)\sum_{k=1}^{m}\frac{\partial}{\partial x_k}[b_k(\mathbf{u})u_i]d\mathbf{x}d\tau \\ &= \int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi'_\delta(u_i)\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left[a_{ii}^{[k]}(\mathbf{u})\frac{\partial u_i}{\partial x_k}\right]d\mathbf{x}d\tau + \int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\Phi'_\delta(u_i)c(\mathbf{u})u_i d\mathbf{x}d\tau. \end{aligned}$$

Vimos na seção 6.2 que

$$\int_{t_0}^{t}(\tau-t_0)^{1+\frac{m}{2}}\Phi'_\delta(u_i)u_{i\tau}d\tau = (t-t_0)^{1+\frac{m}{2}}\Phi_\delta(u_i(\cdot,t)) - \left(1+\frac{m}{2}\right)\int_{t_0}^{t}(\tau-t_0)^{\frac{m}{2}}\Phi_\delta(u_i)d\tau.$$

Pela seção 7.1, obtemos

$$-\int_{\mathbb{R}^m} \Phi'_\delta(u_i) \sum_{k=1}^{m} \frac{\partial}{\partial x_k}[b_k(\mathbf{u})u_i] d\mathbf{x} \;\leq\; \frac{B(\tau)^2 m}{2\mu(\tau)} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) u_i^2 d\mathbf{x} + \frac{\mu(\tau)}{2} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \|Du_i\|^2 d\mathbf{x},$$

$$\int_{\mathbb{R}^m} \Phi'_\delta(u_i) c(\mathbf{u}) u_i d\mathbf{x} \;\leq\; C(\tau) \int_{\mathbb{R}^m} |\Phi'_\delta(u_i)||u_i| d\mathbf{x}$$

e

$$\int_{\mathbb{R}^m} \Phi'_\delta(u_i) \sum_{k=1}^{m} \frac{\partial}{\partial x_k} \left[a_{ii}^{[k]}(\mathbf{u}) \frac{\partial u_i}{\partial x_k} \right] d\mathbf{x} \;\leq\; -\mu(\tau) \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \|Du_i\|^2 d\mathbf{x}.$$

Por conseguinte

$$\begin{aligned}
&(t-t_0)^{1+\frac{m}{2}} \int_{\mathbb{R}^m} \Phi_\delta(u_i(\cdot,t)) d\mathbf{x} - \left(1+\frac{m}{2}\right) \int_{t_0}^{t} (\tau - t_0)^{\frac{m}{2}} \int_{\mathbb{R}^m} \Phi_\delta(u_i) d\mathbf{x} d\tau \\
&\leq \int_{t_0}^{t} (\tau - t_0)^{1+\frac{m}{2}} \frac{B(\tau)^2 m}{2\mu(\tau)} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) u_i^2 d\mathbf{x} d\tau - \int_{t_0}^{t} (\tau - t_0)^{1+\frac{m}{2}} \frac{\mu(\tau)}{2} \int_{\mathbb{R}^m} \Phi''_\delta(u_i) \|Du_i\|^2 d\mathbf{x} d\tau \\
&+ \int_{t_0}^{t} (\tau - t_0)^{1+\frac{m}{2}} C(\tau) \int_{\mathbb{R}^m} |\Phi'_\delta(u_i)||u_i| d\mathbf{x} d\tau.
\end{aligned}$$

Passando ao limite quando $\delta \to 0$, temos que

$$\begin{aligned}
&(t-t_0)^{1+\frac{m}{2}} \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} - \left(1+\frac{m}{2}\right) \int_{t_0}^{t} (\tau - t_0)^{\frac{m}{2}} \|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p} d\tau \\
&\leq \int_{t_0}^{t} \frac{(\tau - t_0)^{1+\frac{m}{2}} B(\tau)^2 m}{2\mu(\tau)} 2p(2p-1) \|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p} d\tau + 2p \int_{t_0}^{t} (\tau - t_0)^{1+\frac{m}{2}} C(\tau) \|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p} d\tau \\
&- \int_{t_0}^{t} \frac{(\tau - t_0)^{1+\frac{m}{2}} \mu(\tau) 2p(2p-1)}{2} \int_{\mathbb{R}^m} |u_i|^{2p-2} \|Du_i\|^2 d\mathbf{x} d\tau, \ \forall\, t > 0.
\end{aligned}$$

Portanto,

$$\begin{aligned}
&(t-t_0)^{1+\frac{m}{2}} \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} + \frac{2p(2p-1)}{2} \int_{t_0}^{t} (\tau - t_0)^{1+\frac{m}{2}} \mu(\tau) \int_{\mathbb{R}^m} |u_i|^{2p-2} \|Du_i\|^2 d\mathbf{x} d\tau \leq \left(1+\frac{m}{2}\right) \\
&\cdot \int_{t_0}^{t} (\tau - t_0)^{\frac{m}{2}} \|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p} d\tau + \int_{t_0}^{t} (\tau - t_0)^{1+\frac{m}{2}} \left[\frac{B(\tau)^2 m}{2\mu(\tau)} 2p(2p-1) + 2pC(\tau) \right] \|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p} d\tau.
\end{aligned}$$

Passando ao limite quando $t_0 \to 0$, concluímos

$$\begin{aligned}
&t^{1+\frac{m}{2}} \|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} + \frac{2p(2p-1)}{2} \int_{0}^{t} \tau^{1+\frac{m}{2}} \mu(\tau) \int_{\mathbb{R}^m} |u_i|^{2p-2} \|Du_i\|^2 d\mathbf{x} d\tau \leq \\
&\left(1+\frac{m}{2}\right) \int_{0}^{t} \tau^{\frac{m}{2}} \|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p} d\tau + \int_{0}^{t} \tau^{1+\frac{m}{2}} \left[\frac{B(\tau)^2 m}{2\mu(\tau)} 2p(2p-1) + 2pC(\tau) \right] \|u_i\|_{L^{2p}(\mathbb{R}^m)}^{2p} d\tau.
\end{aligned}$$

Relembre a definição da função

$$w_i(\cdot,t) = \begin{cases} u_i(\cdot,t), & \text{se } p = 1, \\ |u_i(\cdot,t)|^p, & \text{se } p > 1, \end{cases}$$

para todo $t > 0$. Recorde que

$$\|Dw_i\|^2 = \sum_{k=1}^{m} \left(\frac{\partial w_i}{\partial x_k}\right)^2 = p^2|u_i|^{2p-2} \sum_{k=1}^{m} \left(\frac{\partial u_i}{\partial x_k}\right)^2 = p^2|u_i|^{2p-2}\|Du_i\|^2 \text{ e } \|w_i\|^2_{L^2(\mathbb{R}^m)} = \|u_i\|^{2p}_{L^{2p}(\mathbb{R}^m)}.$$

Seja

$$\begin{aligned} X(t) &= t^{1+\frac{m}{2}}\|u_i(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^m)} + \frac{2p(2p-1)}{2}\int_0^t \tau^{1+\frac{m}{2}}\mu(\tau)\int_{\mathbb{R}^m}|u_i|^{2p-2}\|Du_i\|^2 d\mathbf{x}d\tau \\ &= t^{1+\frac{m}{2}}\|w_i(\cdot,t)\|^2_{L^2(\mathbb{R}^m)} + \frac{2p(2p-1)}{2p^2}\int_0^t \tau^{1+\frac{m}{2}}\mu(\tau)\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}d\tau. \end{aligned} \tag{7.6}$$

Dessa forma,

$$X(t) \le \left(1+\frac{m}{2}\right)\int_0^t \tau^{\frac{m}{2}}\|w_i\|^2_{L^2(\mathbb{R}^m)}d\tau + \int_0^t \tau^{1+\frac{m}{2}}\left[\frac{B(\tau)^2 m}{2\mu(\tau)}2p(2p-1) + 2pC(\tau)\right]\|w_i\|^2_{L^2(\mathbb{R}^m)}d\tau.$$

Usando as desigualdades (7.5) e (6.7) juntamente com a Desigualdade de Hölder, obtemos

$$\begin{aligned} X(t) &\le \left(1+\frac{m}{2}\right)\int_0^t \tau^{\frac{m}{2}} C_m^2\|w_i\|^{\frac{4}{m+2}}_{L^1(\mathbb{R}^m)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \\ &\quad + \int_0^t \tau^{\left(1+\frac{m}{2}\right)}\left[\frac{B(\tau)^2 m 2p(2p-1)}{2\mu(\tau)} + 2pC(\tau)\right] C_m^2\|w_i\|^{\frac{4}{m+2}}_{L^1(\mathbb{R}^m)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \\ &\le \left(1+\frac{m}{2}\right)C_m^2\int_0^t \tau^{\frac{m}{2}}\|u_i\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \\ &\quad + C_m^2\int_0^t \tau^{\left(1+\frac{m}{2}\right)}\left[\frac{B(\tau)^2 m 2p(2p-1)}{2\mu(\tau)} + 2pC(\tau)\right]\|u_i\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \\ &\le \left(1+\frac{m}{2}\right)C_m^2\|u_{i0}\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\int_0^t K(\tau)^{\frac{4p}{m+2}}\tau^{\frac{m}{2}}\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \\ &\quad + C_m^2\|u_{i0}\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\int_0^t K(\tau)^{\frac{4p}{m+2}}\tau^{\left(1+\frac{m}{2}\right)}\left[\frac{B(\tau)^2 m 2p(2p-1)}{2\mu(\tau)} + 2pC(\tau)\right]\|\nabla w_i\|^{\frac{2m}{m+2}}_{L^2(\mathbb{R}^m)}d\tau \\ &= \left(1+\frac{m}{2}\right)C_m^2\|u_{i0}\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\int_0^t K(\tau)^{\frac{4p}{m+2}}\tau^{\frac{m}{2}}\left(\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}\right)^{\frac{m}{m+2}}d\tau \\ &\quad + C_m^2\|u_{i0}\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\int_0^t K(\tau)^{\frac{4p}{m+2}}\tau^{\left(1+\frac{m}{2}\right)}\left[\frac{B(\tau)^2 m 2p(2p-1)}{2\mu(\tau)} + 2pC(\tau)\right] \\ &\quad \cdot\left(\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}\right)^{\frac{m}{m+2}}d\tau \\ &= \left(1+\frac{m}{2}\right)C_m^2\|u_{i0}\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\int_0^t K(\tau)^{\frac{4p}{m+2}}\mu(\tau)^{-\frac{m}{m+2}}\mu(\tau)^{\frac{m}{m+2}}\tau^{\frac{m}{2}}\left(\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}\right)^{\frac{m}{m+2}}d\tau \\ &\quad + C_m^2\|u_{i0}\|^{\frac{4p}{m+2}}_{L^p(\mathbb{R}^m)}\int_0^t K(\tau)^{\frac{4p}{m+2}}\mu(\tau)^{-\frac{m}{m+2}}\tau\mu(\tau)^{\frac{m}{m+2}}\tau^{\frac{m}{2}}\left[\frac{B(\tau)^2 m 2p(2p-1)}{2\mu(\tau)} + 2pC(\tau)\right] \end{aligned}$$

$$
\begin{aligned}
&\cdot\left(\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}\right)^{\frac{m}{m+2}} d\tau\\
\leq\ & \left(1+\frac{m}{2}\right) C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\int_0^t K(\tau)^{2p}\mu(\tau)^{-\frac{m}{2}} d\tau\right)^{\frac{2}{m+2}}\\
&\cdot\left(\int_0^t \mu(\tau)\tau^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}d\tau\right)^{\frac{m}{m+2}}\\
&+C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\int_0^t K(\tau)^{2p}\mu(\tau)^{-\frac{m}{2}}\tau^{1+\frac{m}{2}}\left[\frac{B(\tau)^2 m2p(2p-1)}{2\mu(\tau)}+2pC(\tau)\right]^{\frac{m+2}{2}} d\tau\right)^{\frac{2}{m+2}}\\
&\cdot\left(\int_0^t \mu(\tau)\tau^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}d\tau\right)^{\frac{m}{m+2}}\\
=\ & C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\int_0^t \mu(\tau)\tau^{1+\frac{m}{2}}\int_{\mathbb{R}^m}\|Dw_i\|^2 d\mathbf{x}d\tau\right)^{\frac{m}{m+2}} F(t),\ \forall\ t>0,
\end{aligned}
$$

onde $F(t) = F(t,p,m)$ é dada por

$$
\begin{aligned}
F(t) \ =\ & \left(1+\frac{m}{2}\right)\left(\int_0^t K(\tau)^{2p}\mu(\tau)^{-\frac{m}{2}} d\tau\right)^{\frac{2}{m+2}}\\
&+\left(\int_0^t K(\tau)^{2p}\tau^{1+\frac{m}{2}}\mu(\tau)^{-\frac{m}{2}}\left[\frac{B(\tau)^2 2p(2p-1)m}{2\mu(\tau)}+2pC(\tau)\right]^{\frac{m+2}{2}} d\tau\right)^{\frac{2}{m+2}}.
\end{aligned}
$$

Logo, por (7.6), temos que

$$
X(t) \ \leq\ C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{m+2}} X(t)^{\frac{m}{m+2}} F(t).
$$

Dessa forma,

$$
X(t)^{\frac{2}{m+2}} \ \leq\ C_m^2 \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{\frac{4p}{m+2}} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{m+2}} F(t).
$$

Consequentemente,

$$
X(t) \ \leq\ C_m^{m+2} \|u_{i0}\|_{L^p(\mathbb{R}^m)}^{2p} \left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}},\ \forall\ t>0.
$$

Utilizando novamente, a definição (7.6), concluímos

$$
\begin{aligned}
& t^{1+\frac{m}{2}}\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} + \frac{2p(2p-1)}{2}\int_0^t \tau^{1+\frac{m}{2}}\mu(\tau)\int_{\mathbb{R}^m}|u_i|^{2p-2}\|Du_i\|^2 d\mathbf{x}d\tau\\
&\leq C_m^{m+2}\|u_{i0}\|_{L^p(\mathbb{R}^m)}^{2p}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}},\ \forall\ t>0.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
\|u_i(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} \ \leq\ & C_m^{m+2}\|u_{i0}\|_{L^p(\mathbb{R}^m)}^{2p}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}} t^{-\left(1+\frac{m}{2}\right)} \qquad (7.7)\\
\leq\ & C_m^{m+2}\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}^{2p}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}} t^{-\left(1+\frac{m}{2}\right)},\ \forall\ t>0.
\end{aligned}
$$

Por fim, somando i de 1 a n (ver (2.8)), chegamos a

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} \leq C_m^{m+2}\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}^{2p}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}} F(t)^{1+\frac{m}{2}} t^{-\left(1+\frac{m}{2}\right)}, \ \forall\, t>0,$$

onde $C_m = C_m(m,n)$ é uma constante positiva. □

7.3 Estimativa para a Norma do Sup

Nesta seção, provaremos que

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\kappa \exp\left\{\frac{3pt}{2}E(t)\right\} D(t)^{\frac{1}{p}} t^{-\frac{m+2}{2p}}, \ \forall\, t>0,$$

onde $D(t) = D(t,m,B,C,\mu)$, $E(t) = E(t,m,B,C,\mu)$ e C_κ é uma constante positiva que depende dos parâmetros $\kappa = \{m, p, \|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}\}$.

Teorema 7.3 (Estimativa para a Norma do Sup)**.** *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}^m))$ *solução do sistema (7.1) sob as mesmas hipóteses em (7.2). Então,*

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} \leq C_m^{\frac{m+2}{p}}\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)} 2^{\frac{4(m+2)}{p}} p^{\frac{m+2}{p}} 3^{-\frac{m+2}{2p}} \exp\left\{\frac{3pt}{2}E(t)\right\} D(t)^{\frac{1}{p}} t^{-\frac{m+2}{2p}}, \quad (7.8)$$

para todo $t>0$, $C_m = C_m(m)$ *é uma constante positiva e* $D(t) = D(t,m)$ *é dada por*

$$D(t) = \left[\left(1+\frac{m}{2}\right)\left(\int_0^t \mu(\tau)^{-\frac{m}{2}} d\tau\right)^{\frac{2}{m+2}} + \left(\int_0^t \tau^{1+\frac{m}{2}} \mu(\tau)^{-\frac{m}{2}} \left[m\frac{B(\tau)^2}{2\mu(\tau)} + 2C(\tau)\right]^{\frac{m+2}{2}} d\tau\right)^{\frac{2}{m+2}}\right]^{\frac{m+2}{2}}.$$

Demonstração. Primeiramente, observe que

$$\begin{aligned} F(t) \leq\ & (2p)^2 K(t)^{2p\frac{2}{m+2}}\Big[\left(1+\frac{m}{2}\right)\left(\int_0^t \mu(\tau)^{-\frac{m}{2}} d\tau\right)^{\frac{2}{m+2}} \\ & +\Big(\int_0^t \tau^{1+\frac{m}{2}} \mu(\tau)^{-\frac{m}{2}} \left[m\frac{B(\tau)^2}{2\mu(\tau)} + 2C(\tau)\right]^{\frac{m+2}{2}} d\tau\Big)^{\frac{2}{m+2}}\Big], \ \forall\, t>0. \end{aligned}$$

Seja

$$D(t) = \left[\left(1+\frac{m}{2}\right)\left(\int_0^t \mu(\tau)^{-\frac{m}{2}} d\tau\right)^{\frac{2}{m+2}} + \left(\int_0^t \tau^{1+\frac{m}{2}} \mu(\tau)^{-\frac{m}{2}} \left[m\frac{B(\tau)^2}{2\mu(\tau)} + 2C(\tau)\right]^{\frac{m+2}{2}} d\tau\right)^{\frac{2}{m+2}}\right]^{\frac{m+2}{2}},$$

para todo $t>0$. Por conseguinte,

$$F(t)^{1+\frac{m}{2}} \leq (2p)^{m+2} K(t)^{2p} D(t), \ \forall\, t>0.$$

Com isso,

$$\begin{aligned}
F(t)^{1+\frac{m}{2}} &\leq (2p)^{m+2}K(t)^{2p}D(t)\\
&\leq (2p)^{m+2}\exp\left\{2p\int_0^t\left[\frac{m(p-1)B(\tau)^2}{2\mu(\tau)}+C(\tau)\right]d\tau\right\}D(t)\\
&\leq (2p)^{m+2}\exp\left\{2p\int_0^t\left[\frac{mpB(\tau)^2}{2\mu(\tau)}+pC(\tau)\right]d\tau\right\}D(t)\\
&\leq (2p)^{m+2}\exp\left\{2p^2\int_0^t\left[m\frac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right]d\tau\right\}D(t)\\
&\leq (2p)^{m+2}\exp\left\{2p^2E(t)t\right\}D(t),\ \forall\ t>0,
\end{aligned}$$

onde $\left[m\dfrac{B(\tau)^2}{2\mu(\tau)}+C(\tau)\right]\leq E(t)=E(t,m)$, para todo $\tau\in[0,t]$. Por fim,

$$F(t)^{1+\frac{m}{2}} \leq (2p)^{m+2}\exp\left\{2p^2E(t)t\right\}D(t),\ \forall\ t>0.$$

Analogamente ao que foi feito para encontrar a desigualdade (7.7), concluímos que

$$\|u_i(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R}^m)}\leq C_m^{m+2}\|u_i(\cdot,s)\|^{2p}_{L^p(\mathbb{R}^m)}\left(\frac{2p(2p-1)}{2p^2}\right)^{-\frac{m}{2}}(2p)^{m+2}\exp\left\{2ppE(t)(t-s)\right\}D(t)(t-s)^{-\frac{m+2}{2}},$$

onde $0<s<t$. Seja $t_0=\frac{t}{4^k}$. Relembre a definição $t_j=t_{j-1}+\frac{3t}{4^j}$, para $j\in\mathbb{N}$ e $1\leq j\leq k$. Sabemos que, $t_k=t$. Portanto, utilizando a desigualdade acima, concluímos

$$\begin{aligned}
\|u_i(\cdot,t_j)\|^{2^jp}_{L^{2^jp}(\mathbb{R}^m)} &\leq C_m^{m+2}\|u_i(\cdot,t_{j-1})\|^{2^jp}_{L^{2^{j-1}p}(\mathbb{R}^m)}\left(\frac{2^jp(2^jp-1)}{2^{2j-1}p^2}\right)^{-\frac{m}{2}}(2^jp)^{m+2}\exp\left\{2^{2j-1}p^2E(t)\frac{3t}{4^j}\right\}\\
&\cdot D(t)\left(\frac{3t}{4^j}\right)^{-\frac{m+2}{2}}.
\end{aligned}$$

Logo,

$$\begin{aligned}
\|u_i(\cdot,t)\|_{L^{2^kp}(\mathbb{R}^m)} &\leq C_m^{\frac{m+2}{2^kp}}\|u_i(\cdot,t_{k-1})\|_{L^{2^{k-1}p}(\mathbb{R}^m)}\left(\frac{2^kp(2^kp-1)}{2^{2k-1}p^2}\right)^{-\frac{m}{2^{k+1}p}}(2^kp)^{\frac{m+2}{2^kp}}\\
&\cdot\exp\left\{\frac{2^{2k-1}p^2}{2^kp}E(t)\frac{3t}{4^k}\right\}D(t)^{\frac{1}{2^kp}}\left(\frac{3t}{4^k}\right)^{-\frac{m+2}{2^{k+1}p}}\\
&\leq \prod_{j=1}^k C_m^{\frac{m+2}{2^jp}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R}^m)}\prod_{j=1}^k\left(\frac{2^jp(2^jp-1)}{2^{2j-1}p^2}\right)^{-\frac{m}{2^{j+1}p}}\prod_{j=1}^k(2^jp)^{\frac{m+2}{2^jp}}\\
&\cdot\prod_{j=1}^k\exp\left\{\frac{2^{2j-1}p^2}{2^jp}E(t)\frac{3t}{4^j}\right\}\prod_{j=1}^k D(t)^{\frac{1}{2^jp}}\prod_{j=1}^k\left(\frac{3t}{4^j}\right)^{-\frac{m+2}{2^{j+1}p}}\\
&\leq C_m^{\frac{m+2}{p}\sum_{j=1}^k\frac{1}{2^j}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R})^m}\prod_{j=1}^k\left(\frac{2(2^jp-1)}{2^jp}\right)^{-\frac{m}{2^{j+1}p}}2^{\frac{m+2}{p}\sum_{j=1}^k\frac{j}{2^j}}p^{\frac{m+2}{p}\sum_{j=1}^k\frac{1}{2^j}}
\end{aligned}$$

$$\cdot \exp\left\{\frac{3pt}{2}E(t)\sum_{j=1}^{k}\frac{1}{2^j}\right\} D(t)^{\frac{1}{p}\sum_{j=1}^{k}\frac{1}{2^j}}(3t)^{-\frac{m+2}{2p}\sum_{j=1}^{k}\frac{1}{2^j}}2^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{j}{2^j}}$$

$$\leq C_m^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{1}{2^j}}\|u_i(\cdot,t_0)\|_{L^p(\mathbb{R}^m)}2^{\frac{2(m+2)}{p}\sum_{j=1}^{k}\frac{j}{2^j}}p^{\frac{m+2}{p}\sum_{j=1}^{k}\frac{1}{2^j}}\exp\left\{\frac{3pt}{2}E(t)\sum_{j=1}^{k}\frac{1}{2^j}\right\}$$

$$\cdot D(t)^{\frac{1}{p}\sum_{j=1}^{k}\frac{1}{2^j}}(3t)^{-\frac{m+2}{2p}\sum_{j=1}^{k}\frac{1}{2^j}}.$$

Passando ao limite quando $k \to \infty$, concluímos

$$\begin{aligned}
\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} &\leq C_m^{\frac{m+2}{p}}\|u_{i0}\|_{L^p(\mathbb{R}^m)}2^{\frac{4(m+2)}{p}}p^{\frac{m+2}{p}}3^{-\frac{m+2}{2p}}\exp\left\{\frac{3pt}{2}E(t)\right\}D(t)^{\frac{1}{p}}t^{-\frac{m+2}{2p}} \\
&\leq C_m^{\frac{m+2}{p}}\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}2^{\frac{4(m+2)}{p}}p^{\frac{m+2}{p}}3^{-\frac{m+2}{2p}}\exp\left\{\frac{3pt}{2}E(t)\right\}D(t)^{\frac{1}{p}}t^{-\frac{m+2}{2p}}, \ \forall\ t>0,
\end{aligned}$$

onde $C_m = C_m(m)$ é uma constante positiva. Por fim, usando a definição (2.9), encontramos

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} \leq C_m^{\frac{m+2}{p}}\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}2^{\frac{4(m+2)}{p}}p^{\frac{m+2}{p}}3^{-\frac{m+2}{2p}}\exp\left\{\frac{3pt}{2}E(t)\right\}D(t)^{\frac{1}{p}}t^{-\frac{m+2}{2p}}, \ \forall\ t>0.$$

□

É importante ressaltar que, quando substituimos no termo de advecção do sistema (7.1) as funções $b^{[k]}(\mathbf{u})$ e $\mathbf{u}(\mathbf{x},t)$ por funções de classe C^1

$$g^{[k]}(\mathbf{u}(\mathbf{x},t)) \text{ e } \mathbf{b}^{[k]}(t,\mathbf{x},\mathbf{u}(t,\mathbf{x})) = (b_1^{[k]}(t,\mathbf{x},\mathbf{u}(t,\mathbf{x})), b_2^{[k]}(t,\mathbf{x},\mathbf{u}(t,\mathbf{x})), ..., b_n^{[k]}(t,\mathbf{x},\mathbf{u}(t,\mathbf{x}))),$$

respectivamente, que satisfazem

$$|g^{[k]}(\mathbf{u})| \leq C\min\{|u_1|,|u_2|,...,|u_n|\},, \ \forall\ k=1,2,...,m,$$

$$|\mathbf{b}^{[k]}(t,\mathbf{x},\mathbf{u})| \leq B(t), \text{ q.t.p. em } \mathbb{R}^m, \ \forall\ k=1,2,...,m,$$

onde C é constante e $B \in C^0([0,\infty),\mathbb{R})$, é possível obter as estimativas

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^m)} \leq K(t)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}, \forall\ t\geq 0,$$

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R}^m)}^{2p} \leq C_m^{m+2}\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}^{2p}\left[\frac{2p(2p-1)}{2p^2}\right]^{-\frac{m}{2}}F(t)^{1+\frac{m}{2}}t^{-(1+\frac{m}{2})}, \forall\ t>0,$$

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} \leq C_m^{\frac{m+2}{p}}\|\mathbf{u}_0\|_{L^p(\mathbb{R}^m)}2^{\frac{4(m+2)}{p}}p^{\frac{m+2}{p}}3^{-\frac{m+2}{p}}\exp\left\{\frac{3pt}{2}E(t)\right\}D(t)^{\frac{1}{p}}t^{-\frac{m+2}{p}},$$

para todo $t > 0$, $K(t) = K(t,p,m) = \exp\left\{\int_0^t \left[\frac{(p-1)[CB(\tau)]^2 m}{2\mu(\tau)} + C(\tau)\right] d\tau\right\}$, $C_m = C_m(m,n)$ é uma constante positiva, $F(t) = F(t,p,m)$ é dada por

$$
\begin{aligned}
F(t) &= \left(1+\frac{m}{2}\right)\left(\int_0^t K(\tau)^{2p}\mu(\tau)^{-\frac{m}{2}}\,d\tau\right)^{\frac{2}{m+2}} \\
&\quad + \left(\int_0^t K(\tau)^{2p}\left[\frac{2p(2p-1)[CB(\tau)]^2 m}{2\mu(\tau)} + 2pC(\tau)\right]^{\frac{m+2}{2}} \tau^{1+\frac{m}{2}}\mu(\tau)^{-\frac{m}{2}}\,d\tau\right)^{\frac{2}{m+2}}
\end{aligned}
$$

e $D(t) = D(t,m)$ é dada por

$$
D(t) = \left[\left(1+\frac{m}{2}\right)\left(\int_0^t \mu(\tau)^{-\frac{m}{2}}\,d\tau\right)^{\frac{2}{m+2}} + \left(\int_0^t \tau^{1+\frac{m}{2}}\mu(\tau)^{-\frac{m}{2}}\left[m\frac{[CB(\tau)]^2}{2\mu(\tau)} + 2C(\tau)\right]^{\frac{m+2}{2}} d\tau\right)^{\frac{2}{m+2}}\right]^{\frac{m+2}{2}}.
$$

Capítulo 8

Sistema de Equações de Advecção-difusão Acopladas

Neste capítulo, modificamos o termo de advecção exposto no sistema rotacionalmente invariante, visto no capítulo 4, considerando agora o termo advectivo $\mathbf{f}(x,t,\mathbf{u})$, onde

$$\mathbf{f}(x,t,\mathbf{u}(x,t)) = (f_1(x,t,\mathbf{u}(x,t)), f_2(x,t,\mathbf{u}(x,t)), ..., f_n(x,t,\mathbf{u}(x,t)))$$

é constituída de funções de classe C^1. Aqui, impomos uma condição maior de acoplamento para o sistema. Além disso, consideramos uma função positiva $\mu(t)$ multiplicando o segundo membro em (4.1) e que o dado inicial está em L^p e L^∞ $(1 \leq p \leq \infty)$. Provaremos alguns resultados para uma solução $\mathbf{u}(\cdot,t)$ do sistema $\mathbf{u}_t + \mathbf{f}(x,t,\mathbf{u})_x = \mu(t)\mathbf{u}_{xx}$. Mais precisamente, trabalharemos no problema

$$\begin{aligned} \mathbf{u}_t(x,t) + \mathbf{f}(x,t,\mathbf{u}(x,t))_x &= \mu(t)\mathbf{u}_{xx}(x,t) \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_0, \end{aligned} \tag{8.1}$$

onde $x \in \mathbb{R}$, $t > 0$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_n(x,t))$. Supondo que, a função $\mu \in C^0([0,\infty))$ e

$$\begin{aligned} |\mathbf{f}(x,t,\mathbf{u}(x,t))| &\leq B(t)|\mathbf{u}(x,t)|, \\ \mu(t) &> 0, \end{aligned} \tag{8.2}$$

onde $B \in C^0([0,\infty))$.

Nossa meta para a seção 8.1 é provar que quando uma solução cresce, em um determinado instante, então, é possível obter estimativas para a norma L^q em função da norma $L^{\frac{q}{2}}$.

Na seção 8.2, mostraremos resultados sobre os limites superior e inferior das normas L^q e L^∞, respectivamente.

Por fim, na seção 8.3 encontramos estimativas a priori que nos possibilitam entender o comportamento da norma do sup de cada solução do sistema 8.1 através do limite superior de alguma norma L^p.

8.1 Estimativa de Energia

Nesta seção, verificaremos que quando $\frac{d}{dt}\Big|_{t=s}\|\mathbf{u}(\cdot,t)\|^q_{L^q(\mathbb{R})} \geq 0$ (para algum $s > 0$), existe uma constante C_γ positiva, dependendo somente dos parâmetros $\gamma = \{n, q\}$ que satisfaz

$$\|\mathbf{u}(\cdot,s)\|_{L^q(\mathbb{R})} \;\; \leq \;\; C_\gamma \left(\frac{B(s)}{\mu(s)}\right)^{\frac{1}{q}} \|\mathbf{u}(\cdot,s)\|_{L^{\frac{q}{2}}(\mathbb{R})}, \tag{8.3}$$

onde $2 \leq q < \infty$. Este fato está precisamente estabelecido no

Teorema 8.1. *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (8.1) satisfazendo as hipóteses em (8.2). Seja* $2 \leq q < \infty$*. Considere que* $\frac{d}{dt}\Big|_{t=s}\|\mathbf{u}(\cdot,t)\|^q_{L^q(\mathbb{R})} \geq 0$*, para algum* $s > 0$*. Então*

$$\|\mathbf{u}(\cdot,s)\|_{L^q(\mathbb{R})} \;\; \leq \;\; n^{\frac{2}{q}} \left(2^{-1} q C^{\frac{1}{2}} C_2^3 \frac{B(s)}{\mu(s)}\right)^{\frac{1}{q}} \|\mathbf{u}(\cdot,s)\|_{L^{\frac{q}{2}}(\mathbb{R})}.$$

Demonstração. Seja $2 \leq q < \infty$. Consideremos $\Phi_\delta(\cdot) = L_\delta(\cdot)^q$, onde L_δ é uma função sinal regularizada (ver seção 2.6). Multiplicando a equação

$$u_{it} + f_i(\mathbf{u})_x \;\; = \;\; \mu(t) u_{ixx}$$

por $\Phi'_\delta(u_i)$ e integrando sobre $\mathbb{R}$, obtemos

$$\int_{\mathbb{R}} \Phi'_\delta(u_i) u_{it} dx + \int_{\mathbb{R}} \Phi'_\delta(u_i) f_i(\mathbf{u})_x dx \;\; = \;\; \int_{\mathbb{R}} \Phi'_\delta(u_i) \mu(t) u_{ixx} dx. \tag{8.4}$$

Vimos na seção 4.1 que

$$\int_{\mathbb{R}} \Phi'_\delta(u_i) u_{it} dx \;\; = \;\; \frac{d}{dt} \int_{\mathbb{R}} \Phi_\delta(u_i) dx.$$

Integrando por partes, e usando (8.2), encontramos

$$\begin{aligned}
-\int_{\mathbb{R}} \Phi'_\delta(u_i) f_i(\mathbf{u})_x dx \;\; &= \;\; \int_{\mathbb{R}} \Phi''_\delta(u_i) f_i(\mathbf{u}) u_{ix} dx \\
&\leq \;\; \int_{\mathbb{R}} \Phi''_\delta(u_i) |f_i(\mathbf{u})| |u_{ix}| dx \\
&\leq \;\; \frac{1}{2\mu(t)} \int_{\mathbb{R}} \Phi''_\delta(u_i) f_i(\mathbf{u})^2 dx + \frac{\mu(t)}{2} \int_{\mathbb{R}} \Phi''_\delta(u_i) u_{ix}^2 dx \\
&\leq \;\; \frac{B(t)^2}{2\mu(t)} \int_{\mathbb{R}} \Phi''_\delta(u_i) |\mathbf{u}|^2 dx + \frac{\mu(t)}{2} \int_{\mathbb{R}} \Phi''_\delta(u_i) u_{ix}^2 dx.
\end{aligned}$$

Além disso,

$$\int_{\mathbb{R}} \Phi'_\delta(u_i) \mu(t) u_{ixx} dx \;\; = \;\; -\mu(t) \int_{\mathbb{R}} \Phi''_\delta(u_i) u_{ix}^2 dx.$$

Por conseguinte, usando (8.4), inferimos que

$$\frac{d}{dt}\int_{\mathbb{R}}\Phi_\delta(u_i)dx \;\leq\; \frac{B(t)^2}{2\mu(t)}\int_{\mathbb{R}}\Phi''_\delta(u_i)|\mathbf{u}|^2dx-\frac{\mu(t)}{2}\int_{\mathbb{R}}\Phi''_\delta(u_i)u_{ix}^2dx.$$

Consequentemente, passando ao limite quando $\delta\to 0$, encontramos

$$\frac{d}{dt}\int_{\mathbb{R}}|u_i(\cdot,t)|^qdx \;\leq\; \frac{B(t)^2q(q-1)}{2\mu(t)}\int_{\mathbb{R}}|u_i|^{q-2}|\mathbf{u}|^2dx-\frac{\mu(t)q(q-1)}{2}\int_{\mathbb{R}}|u_i|^{q-2}u_{ix}^2dx.$$

Equivalentemente,

$$\begin{aligned}\frac{d}{dt}\|u_i(\cdot,t)\|^q_{L^q(\mathbb{R})}+\frac{\mu(t)q(q-1)}{2}\int_{\mathbb{R}}|u_i|^{q-2}u_{ix}^2dx &\leq \frac{B(t)^2q(q-1)}{2\mu(t)}\int_{\mathbb{R}}|\mathbf{u}|^qdx\\ &\leq \frac{CB(t)^2q(q-1)}{2\mu(t)}\|\mathbf{u}\|^q_{L^q(\mathbb{R})},\end{aligned}$$

onde C é uma constante. Consequentemente, somando i de 1 a n (ver (2.11)), obtemos

$$\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|^q_{L^q(\mathbb{R})}+\frac{\mu(t)q(q-1)}{2}\sum_{i=1}^n\int_{\mathbb{R}}|u_i|^{q-2}u_{ix}^2dx \;\leq\; \frac{CB(t)^2q(q-1)n}{2\mu(t)}\|\mathbf{u}\|^q_{L^q(\mathbb{R})}.$$

Defina

$$w_i(\cdot,t)=\begin{cases}u_i(\cdot,t), & \text{se } q=2;\\ \\ |u_i(\cdot,t)|^{\frac{q}{2}}, & \text{se } q>2.\end{cases}$$

É fácil ver que $w_{ix}^2=\left(\frac{q}{2}\right)^2|u_i|^{q-2}u_{ix}^2$ e $\|w_i\|^2_{L^2(\mathbb{R})}=\|u_i\|^q_{L^q(\mathbb{R})}$. Seja $\mathbf{w}=(w_1,w_2,...,w_n)$. Assim sendo, $\|\mathbf{w}\|^2_{L^2(\mathbb{R})}=\|\mathbf{u}\|^q_{L^q(\mathbb{R})}$. Com isso,

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})}+\frac{\mu(t)q(q-1)}{2}\frac{4}{q^2}\sum_{i=1}^n\int_{\mathbb{R}}w_{ix}^2dx \;\leq\; \frac{CB(t)^2q(q-1)n}{2\mu(t)}\|\mathbf{w}\|^2_{L^2(\mathbb{R})},$$

ou seja,

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})}+\frac{4\mu(t)\left(1-\frac{1}{q}\right)}{2}\|\mathbf{w}_x\|^2_{L^2(\mathbb{R})} \;\leq\; \frac{CB(t)^2q(q-1)n}{2\mu(t)}\|\mathbf{w}\|^2_{L^2(\mathbb{R})}.$$

Usando a Desigualdade de Sobolev

$$\|v\|_{L^2(\mathbb{R})} \;\leq\; C_2\|v\|^{\frac{2}{3}}_{L^1(\mathbb{R})}\|v_x\|^{\frac{1}{3}}_{L^2(\mathbb{R})},\ \forall\, v\in C^1_0(\mathbb{R}), \tag{8.5}$$

onde $C_2=\dfrac{\sqrt{3}}{\sqrt[3]{4\pi}}$ (para prova de desigualdades deste tipo ver [6]), obtemos

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})}+\frac{4\mu(t)\left(1-\frac{1}{q}\right)}{2}\|\mathbf{w}_x\|^2_{L^2(\mathbb{R})} \;\leq\; \frac{CB(t)^2q(q-1)n^2}{2\mu(t)}C_2^2\|\mathbf{w}(\cdot,t)\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\|\mathbf{w}_x(\cdot,t)\|^{\frac{2}{3}}_{L^2(\mathbb{R})}.$$

Então,

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{4\mu(t)\left(1-\frac{1}{q}\right)}{2}\|\mathbf{w}_x\|^2_{L^2(\mathbb{R})} \leq q(q-1)\left(\frac{q^{\frac{2}{3}}CB(t)^2n^2}{2^{\frac{4}{3}}\mu(t)^{\frac{4}{3}}}C_2^2\|\mathbf{w}(\cdot,t)\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\right) \cdot\left(\frac{2^{\frac{1}{3}}\mu(t)^{\frac{1}{3}}}{q^{\frac{2}{3}}}\|\mathbf{w}_x(\cdot,t)\|^{\frac{2}{3}}_{L^2(\mathbb{R})}\right).$$

Através da Desigualdade de Young, chegamos a

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{4\mu(t)\left(1-\frac{1}{q}\right)}{2}\|\mathbf{w}_x\|^2_{L^2(\mathbb{R})} \leq q(q-1)\Big\{\left(\frac{1}{4}\frac{2}{3}\frac{qC^{\frac{3}{2}}C_2^3B(t)^3n^3}{\mu(t)^2}\|\mathbf{w}(\cdot,t)\|^2_{L^1(\mathbb{R})}\right) + \left(\frac{2}{3}\frac{\mu(t)}{q^2}\|\mathbf{w}_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\right)\Big\}.$$

Por fim,

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{4\mu(t)\left(1-\frac{1}{q}\right)}{3}\|\mathbf{w}_x\|^2_{L^2(\mathbb{R})} \leq q^3\left(1-\frac{1}{q}\right)\frac{C^{\frac{3}{2}}C_2^3B(t)^3n^3}{6\mu(t)^2}\|\mathbf{w}(\cdot,t)\|^2_{L^1(\mathbb{R})}. \quad (8.6)$$

Como $\frac{d}{dt}\Big|_{t=s}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})} \geq 0$, para algum $s > 0$, então

$$\frac{4\mu(s)\left(1-\frac{1}{q}\right)}{3}\|\mathbf{w}_x(\cdot,s)\|^2_{L^2(\mathbb{R})} \leq q^3\left(1-\frac{1}{q}\right)\frac{C^{\frac{3}{2}}C_2^3B(s)^3n^3}{6\mu(s)^2}\|\mathbf{w}(\cdot,s)\|^2_{L^1(\mathbb{R})},$$

isto é,

$$\|\mathbf{w}_x(\cdot,s)\|^2_{L^2(\mathbb{R})} \leq \frac{q^3C^{\frac{3}{2}}C_2^3n^3}{8}\frac{B(s)^3}{\mu(s)^3}\|\mathbf{w}(\cdot,s)\|^2_{L^1(\mathbb{R})}.$$

Por conseguinte,

$$\|\mathbf{w}_x(\cdot,s)\|_{L^2(\mathbb{R})} \leq \left(2^{-1}qC^{\frac{1}{2}}C_2n\frac{B(s)}{\mu(s)}\right)^{\frac{3}{2}}\|\mathbf{w}(\cdot,s)\|_{L^1(\mathbb{R})}.$$

Usando novamente, a Desigualdade de Sobolev (8.5), obtemos

$$\begin{aligned}\|\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R})} &\leq C_2^2n\|\mathbf{w}(\cdot,s)\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\|\mathbf{w}_x(\cdot,s)\|^{\frac{2}{3}}_{L^2(\mathbb{R})}\\ &\leq C_2^2n\|\mathbf{w}(\cdot,s)\|^{\frac{4}{3}}_{L^1(\mathbb{R})}\left(2^{-1}qC^{\frac{1}{2}}C_2n\frac{B(s)}{\mu(s)}\right)\|\mathbf{w}(\cdot,s)\|^{\frac{2}{3}}_{L^1(\mathbb{R})}.\end{aligned}$$

Portanto,

$$\|\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R})} \leq n\left(2^{-1}qC^{\frac{1}{2}}C_2^3\frac{B(s)}{\mu(s)}\right)^{\frac{1}{2}}\|\mathbf{w}(\cdot,s)\|_{L^1(\mathbb{R})}. \quad (8.7)$$

Utilizando a definição de $\mathbf{w}$, encontramos

$$\|\mathbf{u}(\cdot,s)\|_{L^q(\mathbb{R})} \leq n^{\frac{2}{q}}\left(2^{-1}qC^{\frac{1}{2}}C_2^3\frac{B(s)}{\mu(s)}\right)^{\frac{1}{q}}\|\mathbf{u}(\cdot,s)\|_{L^{\frac{q}{2}}(\mathbb{R})}. \quad (8.8)$$

□

8.2 Estimativas para o Limite inferior da norma do Sup e o Limite superior da norma L^q

Nesta seção, veremos que é possível estabelecer algumas estimativas para as normas do sup e L^q de uma solução do sistema (8.1) – (8.2). Para este fim, precisaremos da seguinte Desigualdade de Sobolev

$$\|v\|_{L^\infty(\mathbb{R})} \leq C_\infty \|v\|_{L^1(\mathbb{R})}^{\frac{1}{3}} \|v_x\|_{L^2(\mathbb{R})}^{\frac{2}{3}}, \ \forall\, v \in C_0^1(\mathbb{R}), \tag{8.9}$$

onde $C_\infty = \left(\dfrac{3}{4}\right)^{\frac{2}{3}}$.

Teorema 8.2. *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (8.1) sob as mesmas hipóteses em (8.2). Seja* $2 \leq q < \infty$. *Suponha que,*

$$\int^\infty \mu(t)dt = \infty \ \ e \ \lim_{t\to\infty} \sup \|\mathbf{u}(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{2}} < \infty.$$

Então,

$$\liminf_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq (n^2 2^{-1} C_2 C_\infty C^{\frac{1}{2}} q)^{\frac{2}{q}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{2}{q}} \|\mathbf{u}(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})} \right\},$$

onde C_∞ *é a constante exposta na desigualdade (8.9).*

Demonstração. Seja $Z = Z(q,n) = n^2 2^{-1} C_2 C^{\frac{1}{2}} q \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right) \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\}$. Vamos provar que $\liminf_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\infty Z$. Se $Z = \infty$ a desigualdade é óbvia. Considere, então, que $Z < \infty$. Observe que,

$$\liminf_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\infty Z \Leftrightarrow \forall\, \epsilon > 0, \liminf_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\infty(Z+\epsilon).$$

Vamos então provar que para todo $\epsilon > 0, \liminf_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\infty(Z+\epsilon)$. Suponha, por absurdo, que $\liminf_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})} > C_\infty(Z+\nu)$, para algum $\nu > 0$. Mas $Z < \infty$, então, para este $\nu > 0$, existe $s \gg 1$ tal que

$$n^2 2^{-1} C_2 C^{\frac{1}{2}} q \sup_{t\geq s} \left\{ \left(\frac{B(t)}{\mu(t)}\right) \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\} \leq Z + \frac{\nu}{2}.$$

Por conseguinte,

$$n^2 2^{-1} C_2 C^{\frac{1}{2}} q \left(\frac{B(t)}{\mu(t)}\right) \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \leq Z + \frac{\nu}{2}, \ \forall\, t \geq s.$$

Seja $T(t) = T(q,n,s,t) = n^2 2^{-1} C_2 C^{\frac{1}{2}} q \left\{ \left(\frac{B(t)}{\mu(t)}\right) \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\}$, $\forall\, t \geq s$. Assim sendo,

$$T(t) \leq Z + \frac{\nu}{2}, \ \forall\, t \geq s. \tag{8.10}$$

Por outro lado, $\liminf_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})} > C_\infty(Z+\nu)$, então existe $r>0$ tal que

$$\inf_{t\geq r}\{\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})}\} > C_\infty(Z+\nu).$$

Seja $l = \max\{s,r\} \gg 1$. Por fim,

$$\inf_{t\geq l}\{\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})}\} \;\; > \;\; C_\infty(Z+\nu).$$

Portanto,

$$\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})} \;\; > \;\; C_\infty(Z+\nu),\ \forall\ t\geq l. \tag{8.11}$$

Por conseguinte, pelas desigualdades (8.6), (8.9) e (8.11), obtemos

$$C_\infty^3(Z+\nu)^3 < \|\mathbf{w}(\cdot,t)\|^3_{L^\infty(\mathbb{R})} \leq n^3 C_\infty^3 \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}\|\mathbf{w}_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq n^3 C_\infty^3 \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}$$
$$\cdot\Big\{-\frac{3q}{4(q-1)\mu(t)}\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{3q}{4(q-1)\mu(t)}q^3\left(1-\frac{1}{q}\right)\frac{C^{\frac{3}{2}}C_2^3 B(t)^3 n^3}{6\mu(t)^2}\|\mathbf{w}(\cdot,t)\|^2_{L^1(\mathbb{R})}\Big\},$$

ou seja,

$$\begin{aligned} C_\infty^3(Z+\nu)^3 \;\; &< \;\; n^6 2^{-3} C_\infty^3 C_2^3 C^{\frac{3}{2}} q^3\left(\frac{B(t)}{\mu(t)}\right)^3\|\mathbf{w}(\cdot,t)\|^3_{L^1(\mathbb{R})} + \frac{3qn^3C_\infty^3}{4(q-1)\mu(t)}\|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \\ &\quad \cdot\left(-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})}\right) \\ &= \;\; C_\infty^3 T(t)^3 + \frac{3qn^3C_\infty^3}{4(q-1)\mu(t)}\|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}\left(-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})}\right). \end{aligned}$$

Logo, através da desigualdade (8.10), concluímos

$$(Z+\nu)^3 \;\; < \;\; \left(Z+\frac{\nu}{2}\right)^3 + \frac{3qn^3}{4(q-1)\mu(t)}\|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}\left(-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})}\right).$$

Portanto,

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})} < 0,\ \forall\ t\geq l.$$

Como, por hipótese, $L = \limsup_{t\to\infty}\|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} < \infty$, então existe $k>0$ tal que

$$\sup_{t\geq k}\|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \leq L+1 = S.$$

Consequentemente, $\|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \leq S,\ \forall\ t\geq k$. Seja $t_0 = \max\{k,l\}$. Então,

$$(Z+\nu)^3 < \left(Z+\frac{\nu}{2}\right)^3 + \frac{3n^3}{2\mu(t)}S\left(-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})}\right),\ \forall\ t\geq t_0,$$

pois $q\geq 2$. Dessa forma,

$$-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})} \geq \mu(t)E,\ \forall\ t\geq t_0,$$

onde $E = E(n, q, \nu, L)$ é uma constante positiva. Com isso, integrando de t_0 a t, obtemos

$$-\int_{t_0}^{t} \frac{d}{d\tau}\|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R})}^2 d\tau \geq E\int_{t_0}^{t} \mu(\tau)d\tau, \ \forall\ t \geq t_0,$$

ou equivalentemente,

$$\|\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R})}^2 - \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2 \geq E\int_{t_0}^{t} \mu(\tau)d\tau, \ \forall\ t \geq t_0.$$

Logo,

$$\|\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R})}^2 \geq E\int_{t_0}^{t} \mu(\tau)d\tau, \ \forall\ t \geq t_0.$$

Como, por hipótese, $\displaystyle\int^{\infty} \mu(\tau)d\tau = \infty$ e $\mu \in C^0([0,\infty))$, obtemos $\displaystyle\int_{t_0}^{\infty} \mu(\tau)d\tau = \infty$. Isto contradiz a desigualdade acima. Por fim,

$$\liminf_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R})} \quad \leq \quad n^2 2^{-1} C_\infty C_2 C^{\frac{1}{2}} q \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right) \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\}. \tag{8.12}$$

Usando a definição de $\mathbf{w}$, concluímos

$$\liminf_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \quad \leq \quad (n^2 2^{-1} C_\infty C_2 C^{\frac{1}{2}} q)^{\frac{2}{q}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{2}{q}} \|\mathbf{u}(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})} \right\}.$$

Assim sendo, se $q = 2p$, para algum $1 \leq p < \infty$, então

$$\liminf_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \quad \leq \quad (n^2 C_\infty C_2 C^{\frac{1}{2}} p)^{\frac{1}{p}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{p}} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \right\}. \tag{8.13}$$

□

Teorema 8.3. *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (8.1) sob as mesmas hipóteses em (8.2). Seja* $2 \leq q < \infty$. *Suponha que,*

$$\int^{\infty} \mu(t)dt = \infty \ \ e \ \ \lim_{t\to\infty} \sup \|\mathbf{u}(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{2}} < \infty.$$

Então,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^q(\mathbb{R})} \quad \leq \quad \left(n^3 2^{-1} q C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{q}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q}} \|\mathbf{u}(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})} \right\}.$$

Demonstração. Seja $X = X(q,n) = \left(n^3 2^{-1} q C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2}} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\}$. Vamos provar que $\limsup_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} \leq X$. Se $X = \infty$ o problema está resolvido. Considere, então, que $X < \infty$. Note que,

$$\limsup_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} \leq X \Leftrightarrow \forall\ \epsilon > 0, \limsup_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} \leq X + \epsilon.$$

Com esta equivalência em mãos, vamos provar que para todo $\epsilon > 0, \limsup_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} \leq X + \epsilon$. Suponha, por absurdo, que $\limsup_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} > X + \lambda$, para algum $\lambda > 0$. Como $X < \infty$, então, para este $\lambda > 0$, existe $s \gg 1$ tal que

$$\left(n^3 2^{-1} q C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2}} \sup_{t\geq s} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2}} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\} \leq X + \frac{\lambda}{2}.$$

Consequentemente,

$$\left(n^3 2^{-1} q C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2}} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2}} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\} \leq X + \frac{\lambda}{2},\ \forall\ t \geq s.$$

Seja $Y(t) = Y(q,n,s,t) = \left(n^3 2^{-1} q C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2}} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2}} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\},\ \forall\ t \geq s$. Com isso,

$$Y(t) \leq X + \frac{\lambda}{2},\ \forall\ t \geq s. \tag{8.14}$$

Por outro lado, $\limsup_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} > X + \lambda$, então

$$\sup_{t\geq r} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} > X + \lambda,\ \forall\ r > 0.$$

Assim sendo, existe uma sequência (t_m) tal que $t_m \geq s + m,\ \forall\ m \geq 0$ e $\|\mathbf{w}(\cdot,t_m)\|_{L^2(\mathbb{R})} > X + \lambda$. Veja que $t_m \to \infty$. Portanto,

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} > X + \lambda,\ \forall\ t \geq s. \tag{8.15}$$

De fato, suponha, por absurdo, que existe $t' \geq s$ tal que

$$\|\mathbf{w}(\cdot,t')\|_{L^2(\mathbb{R})} \leq X + \lambda.$$

Por outro lado, como $t_m \to \infty$ então existe $n \in \mathbb{N}$ (suficientemente grande), tal que $s \leq t' \leq t_n$. Pelo Teorema do Valor Intermediário, existe $t'' \in (t', t_n)$ tal que

$$\|\mathbf{w}(\cdot,t'')\|_{L^2(\mathbb{R})} = X + \lambda.$$

Considere que t'' é o primeiro valor entre t' e t_n tal que

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} \geq X + \lambda,\ \forall\ t \in [t'', t_n].$$

Com isso, existe $\theta \in [t'', t_n]$ tal que

$$\|\mathbf{w}(\cdot,\theta)\|_{L^2(\mathbb{R})} > X + \lambda \text{ e } \frac{d}{dt}\Big|_{t=\theta} \|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R})} \geq 0.$$

Utilizando o Teorema 8.1 e a desigualdade (8.14), obtemos

$$\begin{aligned} \|\mathbf{w}(\cdot,\theta)\|_{L^2(\mathbb{R})} &\leq n \left(2^{-1} q C^{\frac{1}{2}} C_2^3 \frac{B(\theta)}{\mu(\theta)}\right)^{\frac{1}{2}} \|\mathbf{w}(\cdot,\theta)\|_{L^1(\mathbb{R})} \\ &\leq n^{\frac{3}{2}} \left(2^{-1} q C^{\frac{1}{2}} C_2^3 \frac{B(\theta)}{\mu(\theta)}\right)^{\frac{1}{2}} \|\mathbf{w}(\cdot,\theta)\|_{L^1(\mathbb{R})} \\ &= Y(\theta) \\ &\leq X + \frac{\lambda}{2}. \end{aligned}$$

Com isso, $X+\lambda < X+\frac{\lambda}{2}$. Isto é um absurdo, pois $\lambda > 0$. Por fim,

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} \quad > \quad X+\lambda,\ \forall\ t \geq s.$$

Pelas desigualdades (8.5), (8.6) e (8.15), obtemos

$$\begin{aligned}
\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^6 &\leq n^6 C_2^6 \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}^4 \|\mathbf{w}_x(\cdot,t)\|_{L^2(\mathbb{R})}^2 \\
&\leq n^6 C_2^6 \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}^4 \Big\{ -\frac{3q}{4(q-1)\mu(t)} \frac{d}{dt} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2 \\
&\quad + \frac{3q}{4(q-1)\mu(t)} q^3 \left(1-\frac{1}{q}\right) \frac{C^{\frac{3}{2}} C_2^3 B(t)^3 n^3}{6\mu(t)^2} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}^2 \Big\},
\end{aligned}$$

ou seja,

$$\begin{aligned}
&(X+\lambda)^6 < n^9 2^{-3} C_2^9 C^{\frac{3}{2}} q^3 \left(\frac{B(t)}{\mu(t)}\right)^3 \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}^6 + \frac{3qn^6C_2^6}{4(q-1)\mu(t)} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}^4 \left(-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2\right) \\
&= Y(t)^6 + \frac{3qn^6C_2^6}{4(q-1)\mu(t)} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}^4 \left(-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2\right).
\end{aligned}$$

Então, através da desigualdade (8.14), concluímos

$$(X+\lambda)^6 \quad < \quad \left(X+\frac{\lambda}{2}\right)^6 + \frac{3qn^6C_2^6}{4(q-1)\mu(t)} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})}^4 \left(-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2\right).$$

Portanto,

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2 < 0,\ \forall\ t \geq s.$$

Como, por hipótese, $M = \limsup_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} < \infty$, então existe $l > 0$ tal que $\sup_{t\geq l} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \leq M+1 = N$. Consequentemente, $\|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \leq N,\ \forall\ t \geq l$. Seja $t_0 = \max\{s,l\}$. Daí,

$$(X+\lambda)^6 < \left(X+\frac{\lambda}{2}\right)^6 + \frac{3n^6C_2^6}{2\mu(t)} N^4 \left(-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2\right),$$

pois $q \geq 2$. Dessa forma,

$$-\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2 \geq \mu(t)D,\ \forall\ t \geq t_0,$$

onde $D = D(n,q,\lambda,M)$ é uma constante positiva. Com isso, integrando de t_0 a t, obtemos

$$-\int_{t_0}^t \frac{d}{d\tau}\|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R})}^2 d\tau \geq D\int_{t_0}^t \mu(\tau)d\tau,\ \forall\ t \geq t_0.$$

Equivalentemente,

$$\|\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R})}^2 - \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})}^2 \geq D\int_{t_0}^t \mu(\tau)d\tau,\ \forall\ t \geq t_0.$$

Logo,

$$\|\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R})}^2 \geq D\int_{t_0}^t \mu(\tau)d\tau,\ \forall\ t \geq t_0.$$

Então chegamos a um absurdo. Por fim,

$$\limsup_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R})} \leq \left(n^3 2^{-1} q C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2}} \|\mathbf{w}(\cdot,t)\|_{L^1(\mathbb{R})} \right\}. \tag{8.16}$$

Consequentemente

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^q(\mathbb{R})}^{\frac{q}{2}} \leq \left(n^3 2^{-1} q C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2}} \|\mathbf{u}(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}^{\frac{q}{2}} \right\}.$$

Logo,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^q(\mathbb{R})} \leq \left(n^3 2^{-1} q C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{q}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{q}} \|\mathbf{u}(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})} \right\}.$$

Assim sendo, se $q = 2p$, para algum $1 \leq p < \infty$, então

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq \left(n^3 p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2p}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2p}} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \right\}. \tag{8.17}$$

□

8.3 Estimativa para a Norma do Sup

Nesta seção, estabelecemos a estimativa para a norma do sup de soluções de (8.1) – (8.2)

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C_\kappa \left(\limsup_{t\to\infty} \frac{B(t)}{\mu(t)}\right)^{\frac{1}{p}} \limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})},$$

onde $1 \leq p < \infty$ e C_κ é uma constante positiva que depende dos parâmetros $\kappa = \{n, p\}$. Primeiramente, estudemos o seguinte

Lema 8.4. *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (8.1) sob as mesmas hipóteses em (8.2). Sejam* $t^0 > 0$, $\mathbb{B}_\mu(t^0) = \sup_{t\geq t^0} \left\{ \frac{B(t)}{\mu(t)} \right\}$ *e* $\mathbb{U}_p(t^0) = \sup_{t\geq t^0} \left\{ \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \right\}$. *Então são válidas as seguintes desigualdades*

i) $\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \leq \mathbb{U}_p(t^0), \ \forall\, t \geq t^0$*;*

ii) $\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}} \left(pC^{\frac{1}{2}} C_2^3 \mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}} \mathbb{U}_p(t^0)\}, \ \forall\, t \geq t^0$*;*

iii)

$$\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})}, n^{\frac{1}{2p}} \left(2pC^{\frac{1}{2}} C_2^3 \mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}} \|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})},$$
$$n^{\frac{1}{p}+\frac{1}{2p}} 2^{\frac{1}{4p}} \left(pC^{\frac{1}{2}} C_2^3 \mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}+\frac{1}{4p}} \mathbb{U}_p(t^0)\}, \ \forall\, t \geq t^0;$$

iv)

$$\|\mathbf{u}(\cdot,t)\|_{L^{2^k p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2^k p}(\mathbb{R})}, n^{\sum_{j=l}^{k}\frac{1}{2^{j-1}p}} 2^{\sum_{j=l}^{k}\frac{j-1}{2^j p}} \left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\sum_{j=l}^{k}\frac{1}{2^j p}}$$

$$\cdot \|\mathbf{u}(\cdot,t^0)\|_{L^{2^{l-1}p}(\mathbb{R})}(l=2,3,...,k), n^{\sum_{j=1}^{k}\frac{1}{2^j p}} 2^{\sum_{j=1}^{k}\frac{j-1}{2^j p}} \left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\sum_{j=1}^{k}\frac{1}{2^j p}} \mathbb{U}_p(t^0)\}, \ \forall\ t \geq t^0,$$

para todo $k \geq 2$.

Demonstração. **i)** Primeiramente, veja que

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \leq \sup_{t\geq t^0}\left\{\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})}\right\} = \mathbb{U}_p(t^0), \ \forall\ t \geq t^0,$$

ou seja, **i)** obviamente vale.

ii) Seja $t \geq t^0$. Vimos na desigualdade (8.8), para $q = 2p$ $(1 \leq p < \infty)$, que

$$\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} \geq 0 \Rightarrow \|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2p}}\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})}$$
$$\leq n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\sup_{t\geq t^0}\left\{\frac{B(t)}{\mu(t)}\right\}\right)^{\frac{1}{2p}}\sup_{t\geq t^0}\{\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})}\} = n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0).$$

Seja $U = U(n,p,t^0) = n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)$. Assim sendo,

$$\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} \geq 0 \Rightarrow \|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq U.$$

Equivalentemente,

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} > U \Rightarrow \frac{d}{dt}\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} < 0.$$

Vamos estudar três casos

1. Se $\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} > U$, para todo $t \geq t^0$, então $\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|^{2p}_{L^{2p}(\mathbb{R})} < 0$, para todo $t \geq t^0$, ou seja, $\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})}$ é decrescente, para todo $t \geq t^0$. Logo,

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq \|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, U\} = \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})},$$
$$n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\}, \ \forall\ t \geq t^0.$$

2. Se existe $t_1 \geq t^0$ tal que $\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} > U$, onde $t^0 \leq t < t_1$ e $\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq U$, onde $t \geq t_1$. Assim, $\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})}$ é decrescente, $t^0 \leq t < t_1$. Logo,

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq \|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t_0)\|_{L^{2p}(\mathbb{R})}, U\} = \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}}$$
$$\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\},$$

onde $t^0 \leq t < t_1$. Por outro lado,

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq U \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\}, \forall\ t \geq t_1.$$

Portanto,

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\},\ \forall\ t \geq t^0.$$

3. Se $\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq U$, para todo $t \geq t^0$, então

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq U \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\},\ \forall\ t \geq t^0.$$

De qualquer maneira,

$$\|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\},\ \forall\ t \geq t^0.$$

iii) Seja $t \geq t^0$. Vimos na desigualdade (8.8), para $q = 4p$ $(1 \leq p < \infty)$, que

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})}^{4p} \geq 0 \Rightarrow \|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} &\leq n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\frac{B(t)}{\mu(t)}\right)^{\frac{1}{4p}}\|\mathbf{u}(\cdot,s)\|_{L^{2p}(\mathbb{R})}\\ &\leq n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\}.\end{aligned}$$

Seja $U_1 = U_1(n,p,t^0) = n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\}$. Com isso,

$$\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})}^{4p} \geq 0 \Rightarrow \|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq U_1.$$

Equivalentemente,

$$\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} > U_1 \Rightarrow \frac{d}{dt}\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})}^{4p} < 0.$$

Novamente vamos estudar três casos

1. Se $\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} > U_1$, para todo $t \geq t^0$, então $\frac{d}{dt}\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})}^{4p} < 0$, para todo $t \geq t^0$, ou seja, $\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})}$ é decrescente, para todo $t \geq t^0$. Logo,

$$\begin{aligned}&\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq \|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})}, U_1\} = \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})},\\ &n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}}\mathbb{U}_p(t^0)\}\},\ \forall\ t \geq t^0,\end{aligned}$$

ou seja,

$$\begin{aligned}&\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})}, n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})},\\ &n^{\frac{1}{p}+\frac{1}{2p}}2^{\frac{1}{4p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}+\frac{1}{4p}}\mathbb{U}_p(t^0)\}\},\ \forall\ t \geq t^0.\end{aligned}$$

2. Se existe $t_2 \geq t^0$ tal que $\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} > U_1$, onde $t^0 \leq t < t_2$ e $\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq U_1$, para todo $t \geq t_2$. Assim, $\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})}$ é decrescente, $t^0 \leq t < t_2$. Logo,

$$\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq \|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})}, U_1\} = \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})},$$
$$n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})}, n^{\frac{1}{p}+\frac{1}{2p}}2^{\frac{1}{4p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}+\frac{1}{4p}}\mathbb{U}_p(t^0)\},\ t^0 \leq t < t_2.$$

Por outro lado,

$$\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq U_1 \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})}, n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})},$$
$$n^{\frac{1}{p}+\frac{1}{2p}}2^{\frac{1}{4p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}+\frac{1}{4p}}\mathbb{U}_p(t^0)\},\ \forall\ t \geq t_2.$$

Portanto,

$$\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})}, n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})},$$
$$n^{\frac{1}{p}+\frac{1}{2p}}2^{\frac{1}{4p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}+\frac{1}{4p}}\mathbb{U}_p(t^0)\},\ \forall\ t \geq t^0.$$

3. Se $\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq U_1$, para todo $t \geq t^0$, então

$$\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq U_1 \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})}, n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})},$$
$$n^{\frac{1}{p}+\frac{1}{2p}}2^{\frac{1}{4p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}+\frac{1}{4p}}\mathbb{U}_p(t^0)\},\ \forall\ t \geq t^0.$$

De qualquer maneira,

$$\|\mathbf{u}(\cdot,t)\|_{L^{4p}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{4p}(\mathbb{R})}, n^{\frac{1}{2p}}\left(2pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{4p}}\|\mathbf{u}(\cdot,t^0)\|_{L^{2p}(\mathbb{R})},$$
$$n^{\frac{1}{p}+\frac{1}{2p}}2^{\frac{1}{4p}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{2p}+\frac{1}{4p}}\mathbb{U}_p(t^0)\},\ \forall\ t \geq t^0.$$

iv) Indutivamente, concluímos que

$$\|\mathbf{u}(\cdot,t)\|_{L^{2^kp}(\mathbb{R})} \leq \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2^kp}(\mathbb{R})}, n^{\sum_{j=l}^{k}\frac{1}{2^{j-1}p}}2^{\sum_{j=l}^{k}\frac{j-1}{2^jp}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\sum_{j=l}^{k}\frac{1}{2^jp}}$$
$$\cdot\|\mathbf{u}(\cdot,t^0)\|_{L^{2^{l-1}p}(\mathbb{R})}(l = 2,3,...,k), n^{\sum_{j=1}^{k}\frac{1}{2^jp}}2^{\sum_{j=1}^{k}\frac{j-1}{2^jp}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\sum_{j=1}^{k}\frac{1}{2^jp}}\mathbb{U}_p(t^0)\},\ \forall\ t \geq t^0.$$

□

Teorema 8.5. *Seja $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ solução do sistema (8.1) sob as mesmas hipóteses em (8.2). Seja $1 \leq p < \infty$. Suponha que,*

$$\int^{\infty}\mu(t)dt = \infty \ \ e \ \lim_{t\to\infty}\sup\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})}^p < \infty.$$

Então,

$$\limsup_{t\to\infty}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})}\leq \max\{K\max\{1,B_\mu\}^{\frac{1}{2p}}(n^2C_2C_\infty C^{\frac{1}{2}}p)^{\frac{1}{p}}B_\mu^{\frac{1}{p}},K\max\{1,B_\mu\}^{\frac{1}{2p}},$$
$$\left(2npC^{\frac{1}{2}}C_2^3B_\mu\right)^{\frac{1}{p}}\}U_p,$$

onde $B_\mu=\limsup_{t\to\infty}\dfrac{B(t)}{\mu(t)}=\lim_{t^0\to\infty}\mathbb{B}_\mu(t^0)$ *e* $U_p=\limsup_{t\to\infty}\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})}=\lim_{t^0\to\infty}\mathbb{U}_p(t^0)$.

Demonstração. Primeiramente demonstremos duas afirmações.

Afirmação 8.6. *É verdade que*

$$n^{\sum_{j=l}^{k}\frac{1}{2^{j-1}p}}\,2^{\sum_{j=l}^{k}\frac{j-1}{2^jp}}\left(pC^{\frac{1}{2}}C_2^3\right)^{\sum_{j=l}^{k}\frac{1}{2^jp}}\leq 2^{\frac{1}{2p}}\max\left\{1,\left(2n^2pC^{\frac{1}{2}}C_2^3\right)\right\}^{\frac{1}{2p}}=K,$$

onde $K=K(n,p)$.

Demonstração. De fato,

$$\begin{aligned}
n^{\sum_{j=l}^{k}\frac{1}{2^{j-1}p}}\,2^{\sum_{j=l}^{k}\frac{j-1}{2^jp}}\left(pC^{\frac{1}{2}}C_2^3\right)^{\sum_{j=l}^{k}\frac{1}{2^jp}} &= 2^{\sum_{j=l}^{k}\frac{j-2}{2^jp}}\left(2n^2pC^{\frac{1}{2}}C_2^3\right)^{\sum_{j=l}^{k}\frac{1}{2^jp}}\\
&= 2^{\frac{1}{p}\sum_{j=l}^{k}\frac{j-2}{2^j}}\left(2n^2pC^{\frac{1}{2}}C_2^3\right)^{\frac{1}{p}\sum_{j=l}^{k}\frac{1}{2^j}}\\
&\leq 2^{\frac{1}{p}\sum_{j=l}^{k}\frac{j-2}{2^j}}\max\left\{1,\left(2n^2pC^{\frac{1}{2}}C_2^3\right)\right\}^{\frac{1}{p}\sum_{j=l}^{k}\frac{1}{2^j}}\\
&\leq 2^{\frac{1}{p}\sum_{j=2}^{\infty}\frac{j-2}{2^j}}\max\left\{1,\left(2n^2pC^{\frac{1}{2}}C_2^3\right)\right\}^{\frac{1}{p}\sum_{j=2}^{\infty}\frac{1}{2^j}}\\
&\leq 2^{\frac{1}{2p}}\max\left\{1,\left(2n^2pC^{\frac{1}{2}}C_2^3\right)\right\}^{\frac{1}{2p}}=K.
\end{aligned}$$

□

Afirmação 8.7. *É fato que*

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})}\leq\max\{K\max\{1,\mathbb{B}_\mu(t^0)\}^{\frac{1}{2p}}\|\mathbf{u}(\cdot,t^0)\|_{L^\infty(\mathbb{R})},K\max\{1,\mathbb{B}_\mu(t^0)\}^{\frac{1}{2p}}\mathbb{U}_p(t^0),$$
$$\left(2npC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{p}}\mathbb{U}_p(t^0)\},\ \forall\ t\geq t^0.$$

Demonstração. Usando o item **iv)** do Lema 8.4 e a Afirmação 8.6, obtemos

$$\|\mathbf{u}(\cdot,t)\|_{L^{2^kp}(\mathbb{R})} \le \max\{\|\mathbf{u}(\cdot,t^0)\|_{L^{2^kp}(\mathbb{R})}, K\mathbb{B}_\mu(t_0)^{\sum_{j=l}^{k}\frac{1}{2^jp}}\|\mathbf{u}(\cdot,t^0)\|_{L^{2^{l-1}p}(\mathbb{R})}(l=2,3,...,k),$$
$$n^{\sum_{j=1}^{k}\frac{1}{2^jp}}2^{\sum_{j=1}^{k}\frac{j-1}{2^jp}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\sum_{j=1}^{k}\frac{1}{2^jp}}\mathbb{U}_p(t^0)\},\ \forall\ t\ge t^0.$$

Observe que, por interpolação, temos que

$$\|\mathbf{u}(\cdot,t^0)\|_{L^q(\mathbb{R})} \le \|\mathbf{u}(\cdot,t^0)\|^{\frac{p}{q}}_{L^p(\mathbb{R})}\|\mathbf{u}(\cdot,t^0)\|^{1-\frac{p}{q}}_{L^\infty(\mathbb{R})},\ 1\le p\le q\le\infty.$$

Seja $E=E(t^0,p)=\max\{\|\mathbf{u}(\cdot,t^0)\|_{L^p(\mathbb{R})},\|\mathbf{u}(\cdot,t^0)\|_{L^\infty(\mathbb{R})}\}$. Com isso,

$$\|\mathbf{u}(\cdot,t^0)\|_{L^q(\mathbb{R})}\le E,\ 1\le p\le q\le\infty.$$

Então

$$\|\mathbf{u}(\cdot,t)\|_{L^{2^kp}(\mathbb{R})} \le \max\{E, K\mathbb{B}_\mu(t^0)^{\sum_{j=l}^{k}\frac{1}{2^jp}}E, n^{\sum_{j=1}^{k}\frac{1}{2^jp}}2^{\sum_{j=1}^{k}\frac{j-1}{2^jp}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\sum_{j=1}^{k}\frac{1}{2^jp}}\mathbb{U}_p(t^0)\},$$

onde $k\ge 2$, $l=2,3,...,k$ e $t\ge t^0$. Consequentemente,

$$\begin{aligned}\|\mathbf{u}(\cdot,t)\|_{L^{2^kp}(\mathbb{R})} &\le \max\{E, K\max\{1,\mathbb{B}_\mu(t^0)\}^{\sum_{j=2}^{k}\frac{1}{2^jp}}E, n^{\sum_{j=1}^{k}\frac{1}{2^jp}}2^{\sum_{j=1}^{k}\frac{j-1}{2^jp}}\left(pC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\sum_{j=1}^{k}\frac{1}{2^jp}}\\ &\quad\cdot\mathbb{U}_p(t^0)\},\ \forall\ t\ge t^0,\end{aligned}$$

onde $k\ge 2$. Passando ao limite quando $k\to\infty$, obtemos

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})}\le\max\{E, K\max\{1,\mathbb{B}_\mu(t^0)\}^{\frac{1}{2p}}E, \left(2npC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{p}}\mathbb{U}_p(t^0)\},\ \forall\ t\ge t^0.$$

Pela definição de K (ver Afirmação 8.6), obtemos

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})}\le\max\{K\max\{1,\mathbb{B}_\mu(t^0)\}^{\frac{1}{2p}}E, \left(2npC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{p}}\mathbb{U}_p(t^0)\},\ \forall\ t\ge t^0.$$

Com a definição de E, encontramos

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})}\le\max\{K\max\{1,\mathbb{B}_\mu(t^0)\}^{\frac{1}{2p}}\|\mathbf{u}(\cdot,t^0)\|_{L^\infty(\mathbb{R})}, K\max\{1,\mathbb{B}_\mu(t^0)\}^{\frac{1}{2p}}\|\mathbf{u}(\cdot,t^0)\|_{L^p(\mathbb{R})},$$
$$\left(2npC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\right)^{\frac{1}{p}}\mathbb{U}_p(t^0)\},\ \forall\ t\ge t^0.$$

Por fim,

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \max\{K\max\{1,\mathbb{B}_\mu(t^0)\}^{\frac{1}{2p}}\|\mathbf{u}(\cdot,t^0)\|_{L^\infty(\mathbb{R})}, K\max\{1,\mathbb{B}_\mu(t^0)\}^{\frac{1}{2p}}\mathbb{U}_p(t^0),$$
$$\Big(2npC^{\frac{1}{2}}C_2^3\mathbb{B}_\mu(t^0)\Big)^{\frac{1}{p}}\mathbb{U}_p(t^0)\}, \forall\ t \geq t^0.$$

□

Com isso,

$$\limsup_{t\to\infty}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \max\{K\max\{1,B_\mu\}^{\frac{1}{2p}}\liminf_{t\to\infty}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})}, K\max\{1,B_\mu\}^{\frac{1}{2p}}U_p,$$
$$\Big(2npC^{\frac{1}{2}}C_2^3B_\mu\Big)^{\frac{1}{p}}U_p\}.$$

Vimos no Teorema 8.2 que

$$\liminf_{t\to\infty}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq (n^2 2^{-1}C_2C_\infty C^{\frac{1}{2}}q)^{\frac{2}{q}}\limsup_{t\to\infty}\left\{\left(\frac{B(t)}{\mu(t)}\right)^{\frac{2}{q}}\|\mathbf{u}(\cdot,t)\|_{L^{\frac{q}{2}}(\mathbb{R})}\right\},$$

onde $2 \leq q < \infty$. Assim sendo,

$$\begin{aligned}\liminf_{t\to\infty}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} &\leq (n^2C_2C_\infty C^{\frac{1}{2}}p)^{\frac{1}{p}}\limsup_{t\to\infty}\left\{\left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{p}}\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})}\right\}\\ &= (n^2C_2C_\infty C^{\frac{1}{2}}p)^{\frac{1}{p}}B_\mu^{\frac{1}{p}}U_p.\end{aligned}$$

Por conseguinte,

$$\limsup_{t\to\infty}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \max\{K\max\{1,B_\mu\}^{\frac{1}{2p}}(n^2C_2C_\infty C^{\frac{1}{2}}p)^{\frac{1}{p}}B_\mu^{\frac{1}{p}}U_p, K\max\{1,B_\mu\}^{\frac{1}{2p}}U_p,$$
$$\Big(2npC^{\frac{1}{2}}C_2^3B_\mu\Big)^{\frac{1}{p}}U_p\},$$

ou seja,

$$\limsup_{t\to\infty}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \max\{K\max\{1,B_\mu\}^{\frac{1}{2p}}(n^2C_2C_\infty C^{\frac{1}{2}}p)^{\frac{1}{p}}B_\mu^{\frac{1}{p}}, K\max\{1,B_\mu\}^{\frac{1}{2p}},$$
$$\Big(2npC^{\frac{1}{2}}C_2^3B_\mu\Big)^{\frac{1}{p}}\}U_p.$$

□

Vejamos, agora, o seguinte resultado.

Teorema 8.8. *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (8.1) sob as mesmas hipóteses em (8.2). Seja* $1 \leq p < \infty$. *Suponha que,*

$$\int^{\infty} \mu(t)dt = \infty \ e \ \lim_{t\to\infty} \sup \|\mathbf{u}(\cdot,t)\|^p_{L^p(\mathbb{R})} < \infty.$$

Então,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K_1 B_\mu^{\frac{1}{p}} U_p,$$

onde $K_1 = K_1(n,p) = \max\{K(n^2 C_2 C_\infty C^{\frac{1}{2}} p)^{\frac{1}{p}}, K, \left(2npC^{\frac{1}{2}}C_2^3\right)^{\frac{1}{p}}\}$.

Demonstração. Este Teorema será demonstrado considerando os três casos possíveis

1. Considere que $B_\mu = 0$. Assim sendo, pelo Teorema 8.3, obtemos

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^{2p}(\mathbb{R})} \leq \left(n^3 p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2p}} \limsup_{t\to\infty} \left\{ \left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2p}} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} \right\}.$$

Consequentemente,

$$U_{2p} \leq \left(n^3 p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2p}} B_\mu^{\frac{1}{2p}} U_p = 0.$$

Logo, $U_{2p} = 0$. Usando o Teorema 8.5, temos que

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \max\{K \max\{1, B_\mu\}^{\dfrac{1}{2p}} (n^2 C_2 C_\infty C^{\frac{1}{2}} p)^{\frac{1}{p}} B_\mu^{\frac{1}{p}}, K \max\{1, B_\mu\}^{\dfrac{1}{2p}},$$
$$\left(2npC^{\frac{1}{2}}C_2^3 B_\mu\right)^{\dfrac{1}{p}}\} U_p.$$

E, portanto,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \max\{K(n,2p) \max\{1, B_\mu\}^{\dfrac{1}{4p}} (n^2 C_2 C_\infty C^{\frac{1}{2}} 2p)^{\frac{1}{2p}} B_\mu^{\frac{1}{2p}},$$
$$K(n,2p)\max\{1, B_\mu\}^{\dfrac{1}{4p}}, \left(4npC^{\frac{1}{2}}C_2^3 B_\mu\right)^{\dfrac{1}{2p}}\} U_{2p} = 0.$$

Por fim,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} = 0.$$

Nestas condições, obtemos

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K_1 B_\mu^{\frac{1}{p}} U_p.$$

2. Suponha que $B_\mu = 1$. Assim sendo, pelo Teorema 8.5, encontramos

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \max\{K(n^2 C_2 C_\infty C^{\frac{1}{2}} p)^{\frac{1}{p}}, K, \left(2npC^{\frac{1}{2}} C_2^3\right)^{\frac{1}{p}}\} U_p,$$

ou seja,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K_1 U_p = K_1 B_\mu^{\frac{1}{p}} U_p.$$

3. Por último, considere que $B_\mu > 0$ e $B_\mu \neq 1$. Sejam

$$\mathbf{v}(x,t) = \mathbf{u}(Lx,t), \mathbf{f_v}(x,t,\mathbf{z}) = \frac{1}{L}\mathbf{f}(Lx,t,\mathbf{z}), \mu_{\mathbf{v}}(t) = \frac{1}{L^2}\mu(t),$$

onde $\mathbf{z} \in \mathbb{R}^n$ e $L > 0$ é uma constante a ser determinada. É fácil ver que

$$\mathbf{v}_t + \mathbf{f_v}(\mathbf{v})_x = \mu_{\mathbf{v}}(t)\mathbf{v}_{xx},$$

onde $|\mathbf{f_v}(x,t,\mathbf{v})| \leq B_{\mathbf{v}}(t)|\mathbf{v}|$, com $B_{\mathbf{v}}(t) = \frac{1}{L}B(t)$, para todo $t > 0$. Vamos utilizar um argumento de escala sobre a função $\mathbf{v}$. Defina,

$$V_p = \limsup_{t\to\infty} \|\mathbf{v}(\cdot,t)\|_{L^p(\mathbb{R})} = \limsup_{t\to\infty} \left(\sum_{i=1}^n \int_{\mathbb{R}} |v_i(x,t)|^p dx\right)^{\frac{1}{p}} = \limsup_{t\to\infty} \left(\sum_{i=1}^n \int_{\mathbb{R}} |u_i(Lx,t)|^p dx\right)^{\frac{1}{p}}.$$

Pelo Teorema da Mudança de Variáveis, chegamos a

$$\begin{aligned} V_p &= \limsup_{t\to\infty} \left(\sum_{i=1}^n \frac{1}{L}\int_{\mathbb{R}} |u_i(y,t)|^p dy\right)^{\frac{1}{p}} = \frac{1}{L^{\frac{1}{p}}} \limsup_{t\to\infty} \left(\sum_{i=1}^n \|u_i(\cdot,t)\|_{L^p(\mathbb{R})}^p\right)^{\frac{1}{p}} \\ &= \frac{1}{L^{\frac{1}{p}}} \limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})} = L^{-\frac{1}{p}} U_p, \end{aligned}$$

ou seja, $V_p = L^{-\frac{1}{p}}U_p$. Além disso,

$$B_\mu^{\mathbf{v}} = \limsup_{t\to\infty} \frac{B_{\mathbf{v}}(t)}{\mu_{\mathbf{v}}(t)} = \limsup_{t\to\infty} \frac{L^2B(t)}{L\mu(t)} = L\limsup_{t\to\infty} \frac{B(t)}{\mu(t)} = LB_\mu,$$

isto é, $B_\mu^{\mathbf{v}} = LB_\mu$.

Para utilizar o caso 2 estabelecemos que $B_\mu^{\mathbf{v}} = 1$, ou seja, $L = B_\mu^{-1}$. Como K e K_1 só dependem de n, p, então, pelo caso anterior

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} = \limsup_{t\to\infty} \|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K_1(B_\mu^{\mathbf{v}})^{\frac{1}{p}}V_p = K_1V_p = K_1L^{-\frac{1}{p}}U_p = K_1B_\mu^{\frac{1}{p}}U_p.$$

□

Podemos melhorar a constante K_1. Este fato está estabelecido no

Teorema 8.9. *Seja* $\mathbf{u}(\cdot,t) \in C^0([0,\infty), L^p(\mathbb{R}))$ *solução do sistema (8.1) sob as mesmas hipóteses em (8.2). Seja* $1 \leq p < \infty$. *Suponha que,*

$$\int^\infty \mu(t)dt = \infty \ e \ \lim_{t\to\infty} \sup \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R})}^p < \infty.$$

Então,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K_2B_\mu^{\frac{1}{p}}U_p, \tag{8.18}$$

onde $K_2 = K_2(n,p) = \left(2n^3pC^{\frac{1}{2}}C_2^3\right)^{\frac{1}{p}} = \left(\frac{3\sqrt{3}n^3pC^{\frac{1}{2}}}{2\pi}\right)^{\frac{1}{p}}.$

Demonstração. Se $B_\mu = 0$, então a desigualdade (8.18) é verdadeira, basta rever o primeiro caso da demonstração do Teorema anterior. Com isso, considere que $B_\mu > 0$. Seja $1 \leq q < \infty$ fixo. Então para $1 \leq q \leq p < \infty$, temos pelo Teorema 8.8, que

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K_1(n, 2^kp)B_\mu^{\frac{1}{2^kp}}U_{2^kp}, \ \forall\, k \geq 1. \tag{8.19}$$

Vimos no Teorema 8.3 que

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^{2^kp}(\mathbb{R})} \leq \left(n^32^{-1}2^kpC^{\frac{1}{2}}C_2^3\right)^{\frac{1}{2^kp}} \limsup_{t\to\infty} \left\{\left(\frac{B(t)}{\mu(t)}\right)^{\frac{1}{2^kp}} \|\mathbf{u}(\cdot,t)\|_{L^{2^{k-1}p}(\mathbb{R})}\right\}, \ \forall\, k \geq 1,$$

ou seja,

$$\begin{aligned} U_{2^k p} &\le \left(n^3 2^{-1} 2^k p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2^k p}} B_\mu^{\frac{1}{2^k p}} U_{2^{k-1}p} \\ &\le \left(n^3 2^{-1} 2^k p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2^k p}} B_\mu^{\frac{1}{2^k p}} \left(n^3 2^{-1} 2^{k-1} p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2^{k-1} p}} B_\mu^{\frac{1}{2^{k-1} p}} U_{2^{k-2}p} \\ &\le \left(n^3 2^{-1} p C^{\frac{1}{2}} C_2^3\right)^{\sum_{j=1}^{k} \frac{1}{2^j p}} 2^{\sum_{j=1}^{k} \frac{j}{2^j p}} B_\mu^{\sum_{j=1}^{k} \frac{1}{2^j p}} U_p, \ \forall\, k \ge 1. \end{aligned}$$

Consequentemente, pela desigualdade (8.19), obtemos

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le K_1(n, 2^k p) B_\mu^{\frac{1}{2^k p}} \left(n^3 2^{-1} p C^{\frac{1}{2}} C_2^3\right)^{\sum_{j=1}^{k} \frac{1}{2^j p}} 2^{\sum_{j=1}^{k} \frac{j}{2^j p}} B_\mu^{\sum_{j=1}^{k} \frac{1}{2^j p}} U_p.$$

Passando ao limite quando $k \to \infty$, encontramos

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \left[\lim_{k\to\infty} K_1(n, 2^k p)\right] \left(n^3 2^{-1} p C^{\frac{1}{2}} C_2^3\right)^{\sum_{j=1}^{\infty} \frac{1}{2^j p}} 2^{\sum_{j=1}^{\infty} \frac{j}{2^j p}} B_\mu^{\sum_{j=1}^{\infty} \frac{1}{2^j p}} U_p.$$

Logo,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \left[\lim_{k\to\infty} K_1(n, 2^k p)\right] \left(n^3 2^{-1} p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{p}} 2^{\frac{2}{p}} B_\mu^{\frac{1}{p}} U_p,$$

isto é,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \left[\lim_{k\to\infty} K_1(n, 2^k p)\right] \left(2n^3 p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{p}} B_\mu^{\frac{1}{p}} U_p.$$

Lembre que

$$K_1 = K_1(n, 2^k p) = \max\{K(n^2 C_2 C_\infty C^{\frac{1}{2}} 2^k p)^{\frac{1}{2^k p}}, K, \left(2n 2^k p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2^k p}}\}.$$

Com isso,

$$\lim_{k\to\infty} K_1(n, 2^k p) = \lim_{k\to\infty} \left[\max\{K(n^2 C_2 C_\infty C^{\frac{1}{2}} 2^k p)^{\frac{1}{2^k p}}, K, \left(2^{k+1} n p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{2^k p}}\}\right] = 1.$$

Portanto,

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \left(2n^3 p C^{\frac{1}{2}} C_2^3\right)^{\frac{1}{p}} B_\mu^{\frac{1}{p}} U_p,$$

e então

$$\limsup_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K_2 B_\mu^{\frac{1}{p}} U_p.$$

□

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