A CLASS OF GENERALIZED BETA DISTRIBUTIONS, PARETO POWER SERIES AND WEIBULL POWER SERIES

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Resumo

Nesta dissertação trabalhamos com três classes de distribuições de probabilidade, sendo uma já conhecida na literatura, a Classe de Distribuições Generalizadas Beta (Beta-G) e duas outras novas classes introduzidas nesta tese, baseadas na composição das distribuições Pareto e Weibull com a classe de distribuições discretas power series. Fazemos uma revisão geral da classe Beta-G e introduzimos um caso especial, a distribuição beta logística generalizada do tipo IV (BGL(IV)). Introduzimos distribuições relacionadas à BGL(IV) que também pertencem à classe Beta-G, como a beta-beta prime e a beta-F. Introduzimos a classe Pareto power series (PPS), que é uma mistura de distribuições Pareto com pesos definidos pela distribuição power series, e apresentamos algumas de suas propriedades. Introduzimos a classe Weibull power series (WPS), cujo processo de construção é similar ao da classe PPS. Apresentamos algumas de suas propriedades e aplicação a um banco de dados reais. Distribuições nesta classe têm aplicação interessante a dados de tempo de vida devido à variedade de formas da função de risco. Para as classes PPS e WPS, fizemos uma simulação para avaliar métodos de seleção de modelo. A distribuição pareto é um caso especial limite da distribuição PPS, assim como a distribuição Weibull é um caso especial limite da distribuição WPS.

Palavras-chave: Distribuições generalizadas; Distribuição Beta; Distribuição Weibull; Distribuição de Pareto; Power series; Algoritmo EM.

Abstract

In this thesis we work with three classes of probability distributions, one already known in the literature, the Class of Generalized Beta Distributions (Beta-G) and two new classes, based on the composition of the Pareto and Weibull distributions and the class of discrete distributions power series. We make a general review of this class and introduced a special case, the beta generalized logistics of type IV (BGL(IV)) distribution. We introduce some distributions related to the BGL(IV) that are in the Beta-G class, such as beta-beta prime and beta-F distributions. We introduce the Pareto power series (PPS) class of distributions, which is a mixture of the Pareto distribution with weights defined by power series, and present some of their properties. We introduce the Weibull power series (WPS) class of distributions, whose construction process is similar to the PPS class. We present some of its properties and application to a real data set. Distributions in this class have interesting application to lifetime data due to the variety of the shapes of the hazard function. The Pareto distribution is a limiting special case of the PPS distribution and the Weibull distribution is a special limit case of the WPS distribution. For the PPS and WPS classes we made a simulation to evaluate methods for model selection.

Palavras-chave: Generalized distributions; Beta distribution; Weibull distribution; Pareto distribution; Power series; EM algorithm.

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Chapter 1

Introduction

1.1 Apresentação Inicial

Nesta dissertação estudamos três classes de distribuições de probabilidade, sendo uma já conhecida na literatura, a Classe de Distribuições Generalizadas Beta (Beta-G) e duas outras novas classes introduzidas nesta tese, baseadas na composição das distribuições Pareto e Weibull com a classe de distribuições discretas power series.

Eugene et al. (2002) [7] introduziram a distribuição beta-normal, baseada em uma composição da distribuição beta e a distribuição normal. A distribuição beta-normal generaliza a normal e tem formas mais flexíveis, dando a ela maior aplicabilidade. Desde então, muitos autores generalizaram outras distribuições de forma similar à beta-normal. No Capítulo 2 fazemos uma revisão da classe Beta-G e introduzimos algumas propriedades adicionais. No Capítulo 3 apresentamos alguns casos particulares da Beta-G estudados por outros autores, como a beta exponencial, beta Weibull, beta secante hiperbólica, beta gamma, beta Gumbel e beta Fréchet. São feitas neste capítulo aplicações a banco de dados reais e em alguns casos apresentamos algumas propriedades ainda não discutidas por outros autores. No Capítulo 4 introduzimos a distribuição beta logística generalizada do tipo IV, alguns de seus casos particulares que pertencem também à Beta-G, como a beta generalizada dos tipos I, II e III e algumas distribuições relacionadas como a beta-beta prime e a beta-F.

No Capítulo 5 introduzimos uma classe de distribuições contínuas baseada na composição da distribuição Pareto com a classe de distribuições discretas power series. Distribuições nesta classe são misturas da distribuição Pareto, que é um caso especial limite, com pesos definidos pela distribuição power series. Apresentamos expressões para os momentos, densidade e momentos da estatística de ordem, tempo médio de vida residual, função de risco, quantis, entropia de Rényi e geração de números aleatórios. Introduzimos três casos especiais, a distribuição Pareto Poisson, Pareto geométrica e Pareto logarítmica.

No Capítulo 6 introduzimos uma classe de distribuições contínuas similar à apresentada no capítulo anterior, baseada na composição da distribuição Weibull com a classe de distribuições discretas power series. Apresentamos expressões para os momentos, densidade e momentos da estatística de ordem, função de risco, quantis e geração de números aleatórios.

Introduzimos três casos especiais, a distribuição Weibull Poisson, Weibull geométrica e Weibull logarítmica. No final do capítulo, ajustamos os três casos especiais a um banco de dados reais.

As classes Pareto Power Series e Weibull Power Series foram criadas em parceria com Wagner Barreto de Souza.

Nos Apêndices A, B e C podem ser encontradas algumas funções implementadas no software R usadas ao longo da dissertação. No apêndice D deixamos disponíveis os bancos de dados usados nas aplicações.

1.2 Initial presentation

In this master's thesis, we study three classes of probability distributions, one already known in the literature, the Class of Generalized Beta Distributions (Beta-G) and two new classes introduced in this thesis, based on the composition of the Pareto and Weibull distributions and the class of discrete distributions power series.

Eugene et al. (2002) [7] introduced the beta-normal distribution, based on a composition of the beta distribution and the normal distribution. The beta-normal distribution generalizes the normal distribution and has flexible shapes, giving it greater applicability. Since then, many authors generalized other distributions similar to the beta-normal. In Chapter 2 we review the class Beta-G and introduce some additional property. In Chapter 3 we present some particular cases of Beta-G studied by other authors, such as beta exponential, beta Weibull, beta-hyperbolic secant, beta gamma, beta Gumbel and beta Fréchet. In this chapter are made some applications to real data set and in some cases we present some properties not discussed by other authors. In Chapter 4 we introduce the beta generalized logistic of type IV distribution, some of its special cases that also belong to the Beta-G, as the beta generalized logistic of types I, II and III and some related distributions, the beta-beta prime and the beta-F.

In Chapter 5 we introduce a class of continuous distributions based on the composition of the Pareto distribution with the class of discrete distributions power series, called Pareto power series (PPS). Distributions in this class are mixtures of Pareto distribution, which is a limiting special case, with weights defined by the power series distribuition. We present expressions for the moments, density and moments of order statistics, mean residual lifetime, hazard function, quantiles, Rényi entropy and generation of random numbers. We introduce three special cases, the Pareto Poisson, the Pareto geometric and the Pareto logarithmic distributions.

In Chapter 6 we introduce a class of continuous distributions similar to that presented in the previous chapter, based on the composition of the Weibull distribution and the class of discrete distributions power series, called Weibull power series (WPS). We present expressions for the moments, density and moments of the order statistics, hazard function, quantiles and generation of random numbers. We introduce three special cases, the Weibull Poisson, Weibull geometric and Weibull logarithmic distributions. At the end of this chapter, we adjust these three special cases to a real data set.

The Pareto Power Series and the Weibull Power Series classes were created in partnership with Wagner Barreto de Souza.

In Appendix A, B and C, some functions that are used throughout this thesis are implemented in the R software. In Appendix D we make available the data sets used in applications.

Chapter 2

General Definition of the Class of Generalized Beta Distributions

2.1 Resumo

A classe de distribuições Beta-G tem o objetivo de generalizar distribuições, adicionando dois parâmetros de forma. Estes parâmetros adicionais dão maior flexibilidade às distribuições, favorecendo uma melhor modelagem dos dados. Neste capítulo, definições e propriedades gerais da Beta-G já mostradas na literatura serão apresentadas.

2.2 Introduction

Generalized distributions have been widely studied recently. Amoroso (1925) [1] was the precursor of generalizing continuous distributions, discussing the generalized gamma distribution to fit observed distribution of income rate. Since then, numerous other authors have developed various classes of generalized distributions. In this chapter, the main focus is to study the class of generalized beta distributions.

This method of generalizing distributions was introduced by Eugene et al. (2002) [7] who defined the beta-normal (BN) distribution. Since then, several authors have been studying particular cases of this class of distributions. An advantage of the BN distribution over the normal distribution is that the BN can be unimodal and bimodal. The bimodal region for the BN distribution was studied by Famoye et al. (2003) [8] and some expressions for the moments were derived by Gupta and Nadarajah (2004) [12]. Nadarajah and Kotz (2004) [20] defined the beta-Gumbel distribution, which has greater tail flexibility than the Gumbel distribution, also known as the tipe I extreme value distribution. Nadarajah and Gupta (2004) [19] defined the beta-Fréchet distribution and Barreto-Souza et al. (2008) [2] present some additional mathematical properties. Nadarajah and Kotz (2006) [21] defined the beta-exponential distribution, whose hazard function can be increasing and decreasing. Famoye et al. (2005) [9] defined the beta-Weibull distribution and Lee et al. (2007) [16] made some applications to censored data. Kong et al. (2007) [15] proposed the beta-gamma ditribution.

Fischer and Vaughan (2007) [10] introduced the beta-hiperbolic secant distribution. Cordeiro and Barreto-Souza (2009) [4] derived some general results for this class.

In this chapter, we present some general properties of the class of generalized beta distributions. In the following chapter, some special cases of the Beta-G distribution will be present with their properties.

In this work we introduce a particular case of the class of generalized beta distributions, named the beta generalized logistic distribution of type IV.

2.3 Definition

The class of generalized beta distribution was first introduced by Eugene et al. (2002) [7] through its cumulative distribution function (cdf). The cdf of the class of generalized beta distributions is defined by

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt, \quad a > 0,\ b > 0, \tag{2.1}$$

where $G(x)$ is the cdf of a parent random variable and

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

This class of distributions, called Beta-G, generalizes the distribution G of a random variable with cdf $G(x)$. In fact, the cdf of the Beta-G coincides with $G(x)$ when $a = b = 1$. The role of the additional parameters is to introduce skewness and to vary tail weights. They provide greater flexibility in the form of the distribution and consequently in modeling observed data. The distribution with cdf $G(x)$ will be denoted by primitive distribution.

The cdf of Beta-G can be rewritten as

$$F(x) = I_{G(x)}(a,b) = \frac{B_{G(x)}(a,b)}{B(a,b)},$$

where $B_{G(x)}(a,b)$ denotes the incomplete beta function given by

$$B_{G(x)}(a,b) = \int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt, \quad a > 0,\ b > 0 \tag{2.2}$$

and $I_{G(x)}(a,b)$ is the incomplete beta ratio function.

2.4 Density function

The class of generalized beta distributions has been studied to generalize continuous distributions although it can also be used to generalize discrete distributions. The Beta-G distribution will have density or probability mass function if the G distribution is absolutely continous or discrete, respectively. The approach of this work is to generalize continuous distributions, so it makes sense to define the density function of the Beta-G distribution as

$$f(x) \equiv F'(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}[1 - G(x)]^{b-1}, \quad a > 0,\ b > 0, \tag{2.3}$$

where $g(x)$ is the density of the primitive distribution.

Let $\theta \in \Theta$ be the parameter vector of the primitive distribution. If there is a restriction Θ' of Θ such that when $\theta \in \Theta'$ the primitive distribution is symmetric around μ, then the Beta-G distribution will be symmetric around μ when $\theta \in \Theta'$ and $a = b$. To verify this, let X be a random variable following a Beta-G distribution and consider the restriction Θ' of the parametric space. Then,

$$\begin{aligned} f(\mu - x) &= \frac{g(\mu - x)}{B(a,b)} G(\mu - x)^{a-1} \left[1 - G(\mu - x)\right]^{b-1} \\ &= \frac{g(x + \mu)}{B(a,b)} \left[1 - G(x + \mu)\right]^{a-1} G(x + \mu)^{b-1} \\ &= f(\mu + x). \qquad (2.4) \\ & \qquad (2.5) \end{aligned}$$

Cordeiro and Barreto-Souza (2009) [4] gave an important expansion for the density of the Beta-G distribution to derive some general properties of this class. Let b be a non-integer real number and $|x| < 1$. Consider the expansion of the power series $(1 - x)^{b-1}$

$$(1 - x)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b - j) j!} x^j. \tag{2.6}$$

Then the incomplete beta function can be expressed as

$$B_x(a,b) = x^a \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) x^j}{\Gamma(b - j) j! (j + a)}. \tag{2.7}$$

If b is an integer, the index j stops at $b - 1$.

Define the constant

$$w_j(a,b) = \frac{(-1)^j \Gamma(b)}{\Gamma(b - j) j! (a + j)}. \tag{2.8}$$

From (2.7) and (2.8), the cdf of the Beta-G distribution can be expressed as

$$F(x) = \frac{1}{B(a,b)} \sum_{r=0}^{\infty} w_r(a,b) G(x)^{a+r}. \tag{2.9}$$

From a simple differentiation of (2.9), an expansion of $f(x)$ follows as

$$f(x) = \frac{1}{B(a,b)} g(x) \sum_{r=0}^{\infty} (a+r) w_r(a,b) G(x)^{a+r-1}. \tag{2.10}$$

Using (2.6) twice, consider the expansion of $G(x)^\alpha$ for α non-integer

$$\begin{aligned}
G(x)^\alpha = [1 - \{1 - G(x)\}]^\alpha &= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\alpha+1)}{\Gamma(\alpha-j+1) j!} \{1 - G(x)\}^j \\
&= \sum_{j=0}^{\infty} \sum_{r=0}^{j} \frac{(-1)^{j+r} \Gamma(\alpha+1)}{\Gamma(\alpha-j+1)(j-r)! r!} G(x)^r \\
&= \sum_{r=0}^{\infty} \sum_{j=r}^{\infty} \frac{(-1)^{j+r} \Gamma(\alpha+1)}{\Gamma(\alpha-j+1)(j-r)! r!} G(x)^r.
\end{aligned} \tag{2.11}$$

Let

$$s_r(\alpha) = \sum_{j=r}^{\infty} \frac{(-1)^{j+r} \Gamma(\alpha+1)}{\Gamma(\alpha-j+1)(j-r)! r!}. \tag{2.12}$$

Then, using (2.12), the expansion (2.11) can be expressed as

$$G(x)^\alpha = \sum_{r=0}^{\infty} s_r(\alpha) G(x)^r. \tag{2.13}$$

If a is not an integer, from equation (2.13), the cdf (2.9) can be given by

$$F(x) = \frac{1}{B(a,b)} \sum_{r=0}^{\infty} t_r(a,b) G(x)^r, \tag{2.14}$$

where $t_r(a,b) = \sum_{l=0}^{\infty} w_l(a,b) s_r(a+l)$. From a simple differentiation of (2.14), an expansion of $f(x)$ for a non-integer is

$$f(x) = \frac{1}{B(a,b)} g(x) \sum_{r=0}^{\infty} (r+1) t_{r+1}(a,b) G(x)^r. \tag{2.15}$$

The objective of this alternative form to the expansion of the density when a is a non-integer is to take the power series with $G(x)$ raised only to integer powers.

If $G(x)$ has no closed form expression, suppose that it admits the expansion

$$G(x) = \sum_{k=0}^{\infty} a_k x^{k+c}, \tag{2.16}$$

where $\{a_k\}_{k\geq 0}$ is a sequence of real numbers. Hence, for any α positive integer,

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{\alpha} = \sum_{k=0}^{\infty} c_{\alpha,k} x^k, \tag{2.17}$$

where the coeficients $c_{\alpha,k}$ for $k = 1, 2, \ldots$ are obtained from the recurrence equation

$$c_{\alpha,k} = (ka_0)^{-1} \sum_{m=1}^{i} (\alpha m - k + m) a_m c_{\alpha,k-m} \tag{2.18}$$

and $c_{\alpha,0} = a_0^{\alpha}$ (see Gradshteyn and Ryzhik, 2000).

If α is an integer, it comes immediately from (2.17)

$$G(x)^{\alpha} = \sum_{k=0}^{\infty} c_{\alpha,k} x^{k+c\alpha}. \tag{2.19}$$

By using equation (2.19), equations (2.10) and (2.15) can be rewritten, respectively, as

$$f(x) = \frac{1}{B(a,b)} g(x) \sum_{r,k=0}^{\infty} (a+r) w_r(a,b) c_{a+r-1,k} x^{k+c(a+r-1)} \tag{2.20}$$

and

$$f(x) = \frac{1}{B(a,b)} g(x) \sum_{r,k=0}^{\infty} (r+1) t_{r+1}(a,b) c_{r,k} x^{k+cr}. \tag{2.21}$$

The expansions (2.20) and (2.21) for the density function, derived by Cordeiro and Barreto-Souza (2009) [4], are fundamental to obtain some mathematical properties of the class of generalized beta distributions. These equations provide the density function $f(x)$ of the Beta-G distribution in terms of the density $g(x)$ of the parent distribution G multiplied by an infinite power series of x. In Section 2.6 the importance of these results will be clear.

2.5 Survival and hazard functions

Survival and hazard functions are generaly useful to reliability and survival studies. In these context, the support of the random variable of interest is $\mathbb{R}^+$.

An important property of the incomplete beta function is

$$B_x(a,b) = B(a,b) - B_{1-x}(b,a). \tag{2.22}$$

With this property, the survival function defined by $S(x) = 1 - F(x)$ of an distribution in the class of generalized beta distribution (if its support is $\mathbb{R}^+$) is given by

$$S(x) = 1 - \frac{B_{G(x)}(a,b)}{B(a,b)} = \frac{B_{1-G(x)}(b,a)}{B(a,b)} = \frac{B_{S^*(x)}(b,a)}{B(a,b)}, \tag{2.23}$$

where $S^*(x) = 1 - G(x)$ is the survival function of the G distribution. The survival function is given by a normalized incomplete beta function and depends on x only through $S^*(x)$.

The hazard function is defined by the ratio $\frac{f(x)}{S(x)}$, where f and S are the density and the survival functions, respectively. The hazard function of any distribution in the Beta-G family is given by

$$h(x) = \frac{(1 - G(x))^{b-1} G(x)^{a-1}}{B_{1-G(x)}(b, a)} g(x).$$

2.6 Moments

Cordeiro and Barreto-Souza (2009) [4] derived a general expression for the sth moment of the Beta-G by using the expansions of the density function in (2.20) and (2.21).

Let X be a Beta-G random variable and Y a random variable with cdf $G(x)$. For a integer, it follows from expansion (2.20) that the sth moment of the Beta-G can be expressed by infinite sums of the fractional moments of the G distribution as

$$E(X^s) = \frac{1}{B(a,b)} \sum_{j,k=0}^{\infty} (a+j) w_j(a,b) c_{a+j-1,k} E(Y^{s+k+c(a+j-1)}) \tag{2.24}$$

and for a an integer, it comes from (2.21) that

$$E(X^s) = \frac{1}{B(a,b)} \sum_{r,k=0}^{\infty} (r+1) t_{r+1}(a,b) c_{r,k} E(Y^{s+k+cr}). \tag{2.25}$$

2.7 Random number generation

Let Y be a random variable with cdf $F(y)$. The cumulative technique of the random number generation consists to solve

$$Y = F^{-1}(U),$$

where U follows a standard uniform distribution. Considering that X follows a Beta-G distribution, the cdf of X can be expressed as $F(x) = F_1(G(x))$, where F_1 is the cdf of the Beta(a, b) distribution and G is the cdf of the parent distribution. Hence, by the cumulative technique we have that

$$\begin{aligned} F_1(G(X)) &= U \\ G(X) &= F^{-1}(U) = B \\ X &= G^{-1}(B), \end{aligned}$$

where B is a beta random variable with parameters a and b. The cdf $G(x)$ sometimes have no analytical inverse. In this case, note that $G^{-1}(B)$ is the quantile B of a random variable with cdf $G(x)$ and it is not hard to compute this quantity. When the support is not limited, this method cannot generate large values of x. This is an obvious restriction of the capacity of the computer.

For example, suppose X is a random variable following a beta normal distribution with parameters μ, σ^2, a and b. To generate X, first generate a beta random variable B, with parameters a and b and then X is the quantile B of a normal distribution with parameters μ and σ^2. If the generated value of B is close to 1, the quantile of the normal distribution possibly will not be computed.

An implementation of the algorithm to generate random numbers following a Beta-G distribution performed on software R is given in Appendix A.

Chapter 3

Some Special Distributions of the Class of Generalized Beta Distributions

3.1 Resumo

Neste capítulo apresentamos algumas distribuições especiais da classe de distribuições Beta-G existentes na literatura. Para cada caso, apresentamos a função de densidade, expressão para momentos, estimação pelo método de máxima verossimilhança, geração de números aleatórios e, quando coerente, a função de risco. Ao final de cada sub-seção uma aplicação a dados reais.

3.2 Beta normal

The beta normal distribution (BN) was introduced by Eugene et al. (2002) [7] and its importance is more than just generalize the normal distribution. From the BN distribution, Eugene et al. (2002) [7] idea the concept of the class of beta generalized distributions.

In this section, the properties of the BN distribution already studied in the literature will be reviewed. Additional properties are introduced in Section 3.2.1, where are verified that the BN distribution belongs to the location-scale family of distributions and in Section 3.2.3 an algorithm to generate random numbers is proposed .

3.2.1 The distribution

Let $\Phi(x)$ and $\phi(x)$ be the cumulative and the density functions of the standard normal distribution, respectively. By using (2.3), the density function of the BN distribution is given by

$$f(x) = \frac{1}{\sigma B(a,b)} \left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a-1} \left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{b-1} \phi\left(\frac{x-\mu}{\sigma}\right),$$

where $a > 0$, $b > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ and $x \in \mathbb{R}$. When X is a random variable following the BN distribution, it will be denoted by $X \sim BN(a, b, \mu, \sigma)$.

The shape parameters a and b characterize the skewness, kurtosis and bimodality of the distribution. The parameters μ and σ play the same role as in the normal distribution, μ is a location parameter and σ is a scale parameter that stretches out or shrinks the distribution. When $\mu = 0$ and $\sigma = 1$, the BN distribution is said to be a standard BN distribution.

Figure 3.1 plots some densities of the BN distribution with $\mu = 0$ and $\sigma = 1$.

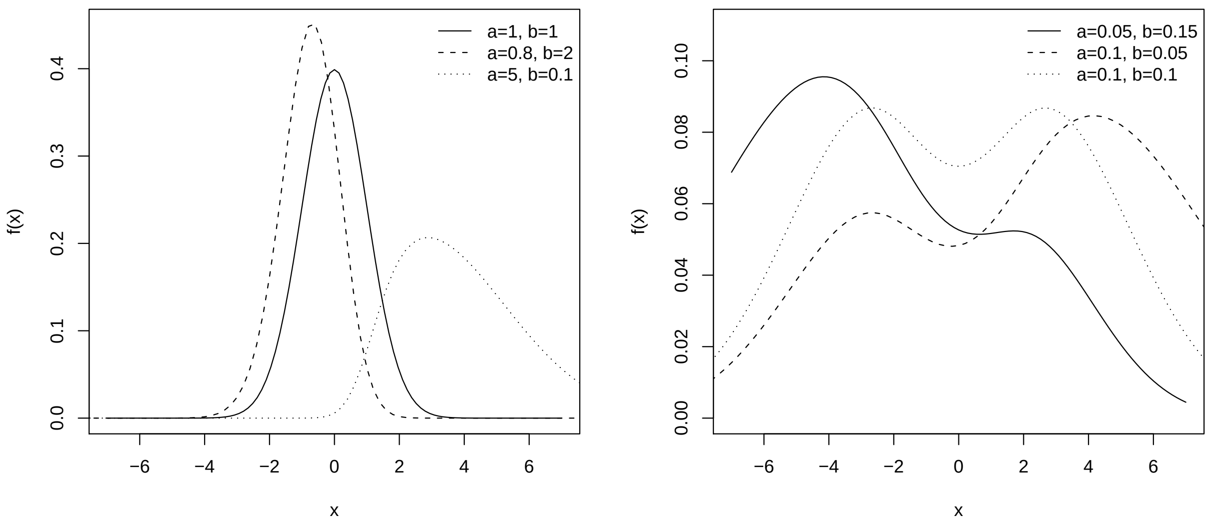

Figure 3.1: The beta normal pdf for some values of a and b with $\mu = 0$ and $\sigma = 1$.

The normal distribution is symmetric for all values of μ and σ. Hence, according to (2.4), the BN distribution is symmetric when $a = b$.

A class Ω of probability distributions is said to be a location-scale family of distributions if whenever F is the cdf of a member of Ω and α is any real number and $\beta > 0$, then $G(x) = F(\alpha + \beta x)$ is also the cdf of a member of Ω.

If $X \sim BN(a, b, \mu, \sigma)$, then $Y = \alpha + \beta X$ has density function

$$f_Y(y) = \frac{(\beta\sigma)^{-1}}{B(a,b)} \phi\left(\frac{y - (\alpha + \beta\mu)}{\beta\sigma}\right) \Phi\left(\frac{y - (\alpha + \beta\mu)}{\beta\sigma}\right)^{a-1} \left[1 - \Phi\left(\frac{y - (\alpha + \beta\mu)}{\beta\sigma}\right)\right]^{b-1},$$

which is the density function of the $BN(a, b, \alpha + \beta\mu, \beta\sigma)$. Hence, the BN distribution is a location-scale family. In other words, any random variable Y following the BN distribution can be expressed as a linear transformation of another BN random variable X, in particular when X follows the standard beta normal distribution ($X \sim BN(a,b,0,1)$).

3.2.2 Moments

Using the results obtained by Cordeiro and Barreto-Souza (2009) [4] (see Section 2.6), expressions for the sth moment of the BN(a, b,$\mu = 0$,$\sigma = 1$) distribution can be given using (2.25) for a integer and using (2.24) for a non-integer. The expansion for the cdf of the standard normal distribution follows as

$$
\begin{aligned}
G(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left\{-\frac{x^2}{2}\right\} dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!2^k} dx \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left\{\int_{0}^{x} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!2^k} I_{x\geq 0} dx - \int_{x}^{0} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!2^k} I_{x<0} dx\right\} \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left\{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!2^k(2k+1)} I_{x\geq 0} dx + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!2^k(2k+1)} I_{x<0} dx\right\} \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!2^k(2k+1)} \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{\forall k\in\mathbb{N}, j=2k+1} \frac{(-1)^k x^{j}}{k!2^k j}.
\end{aligned}
$$

Hence, for the standard normal distribution, the expansion (2.16) is given by the coefficients $a_0 = \frac{1}{2}$, $a_{2k} = 0$, $a_{2k+1} = \frac{(-1)^k 2^{-k}}{\sqrt{2\pi}(2k+1)k!}$ for $k \in \mathbb{N}$ and $c = 0$.

Let Y be a random variable with a standard normal distribution. Then, $E(Y^s) = 0$, if s is odd and $E(Y^s) = 2^{s/2}\gamma((s+1)/2)/\sqrt{\pi}$, if s is even. Using these moments and the above a_k's, an expansion for the sth moment of the standard BN distribution is obtained from equations (2.24) and (2.25). If $X \sim BN(a,b,\mu,\sigma)$, using the property that the BN is a location-scale family, expressions for the moments are obtained by making $E(X^r) = E[(\mu+\sigma Z)^r] = \sum_{t=0}^{r}\binom{r}{t}\mu^{r-t}\sigma^r E(Z^r)$, where $Z \sim$BN(a, b, 0, 1).

Gupta and Nadarajah (2004) [12] gave an expression for the sth moment of the BN distribution, which holds only when the parameter a is an integer.

3.2.3 Random number generation

The cdf of the normal distribution has no analytical inverse. Hence, to generate random numbers following the BN distribution by the proposed technique in Section 2.7, the quantile function of the normal distribution has to be evaluated numerically. The software R has many quantile functions already implemented, including the quantile function of the normal distribution. Figure 3.2 plots generated random samples of the BN distribution with the respective curve of the density function obtained using R.

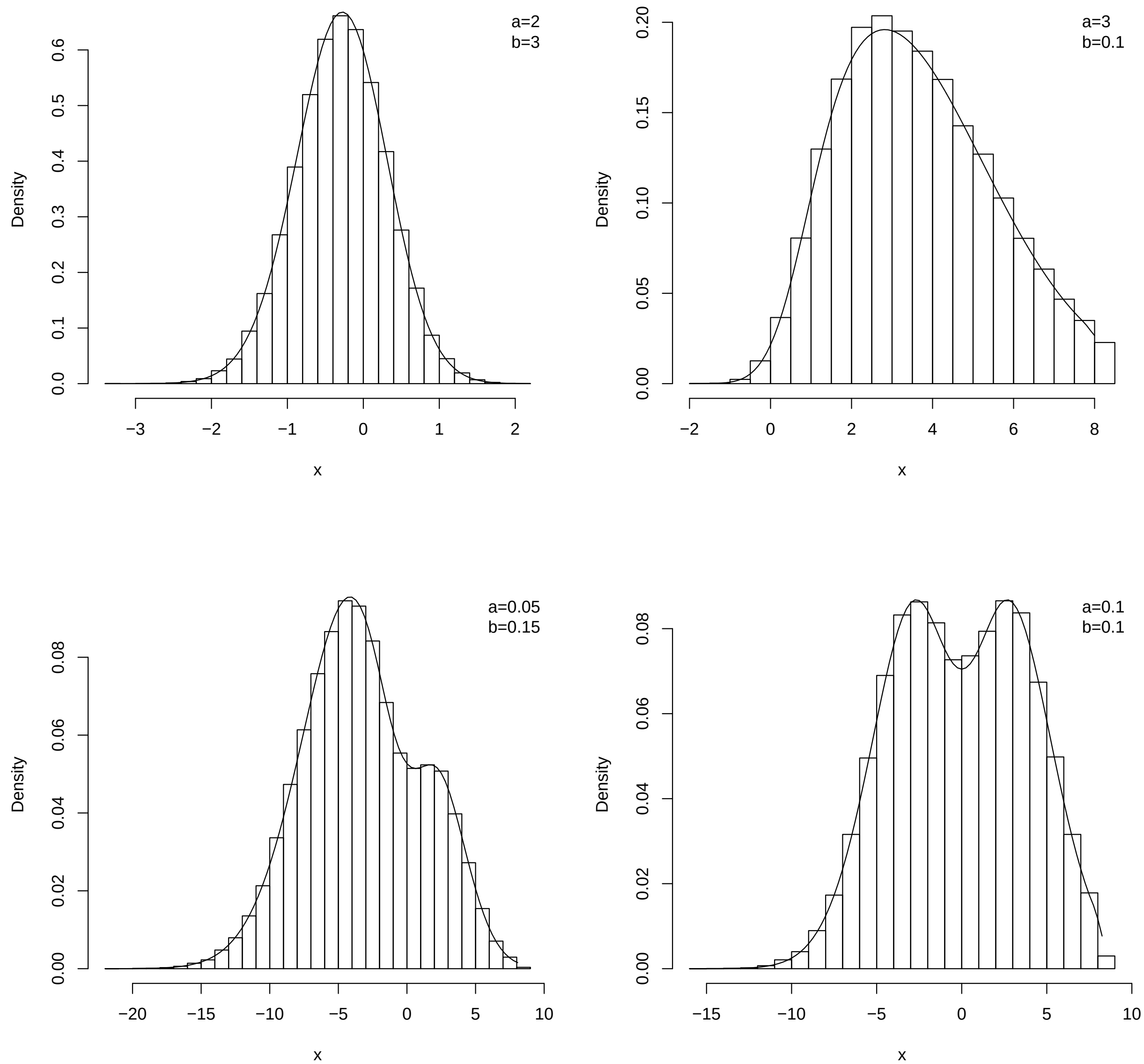

Figure 3.2: Generated samples of the BN distribution for some values of a and b with $\mu = 0$ and $\sigma = 1$.

3.2.4 Estimation

Let $x_1, \ldots, x_n$ be an independent random sample from the BN distribution. The total log-likelihood function is given by

$$\begin{aligned}\ell = \ell(a,b,\mu,\sigma) &= -n\log B(a,b) - \frac{n}{2}\log(2\pi) - n\log\sigma + \sum_{i=1}^{n}(a-1)\log\left(\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right) \\ &\quad + \sum_{i=1}^{n}(b-1)\log\left(1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right) - \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}\end{aligned}$$

The score function $\nabla\ell = \left(\frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\mu}, \frac{\partial\ell}{\partial\sigma}\right)$ has components

$$\begin{aligned}\frac{\partial\ell}{\partial a} &= n\psi(a+b) - n\psi(a) + \sum_{i=1}^{n}\log\left(\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right), \\ \frac{\partial\ell}{\partial b} &= n\psi(a+b) - n\psi(b) + \sum_{i=1}^{n}\log\left(1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right), \\ \frac{\partial\ell}{\partial\mu} &= \sum_{i=1}^{n}\left\{\frac{(1-a)\phi((x_i-\mu)/\sigma)}{\sigma(\Phi((x_i-\mu)/\sigma))} + \frac{(b-1)\phi((x_i-\mu)/\sigma)}{\sigma(1-\Phi((x_i-\mu)/\sigma))} + \frac{x_i-\mu}{\sigma^2}\right\}, \\ \frac{\partial\ell}{\partial\sigma} &= \sum_{i=1}^{n}\left\{\frac{(1-a)\phi((x_i-\mu)/\sigma)}{\Phi((x_i-\mu)/\sigma)}\frac{x_i-\mu}{\sigma^2}\right\} + \\ &\quad \sum_{i=1}^{n}\left\{\frac{(b-1)\phi((x_i-\mu)/\sigma)}{1-\Phi((x_i-\mu)/\sigma)}\frac{x_i-\mu}{\sigma^2} + \frac{(x_i-\mu)^2}{\sigma^3}\right\} - \frac{n}{\sigma},\end{aligned}$$

where $\psi(x)$ is the digamma function.

Solving the system of nonlinear equations $\nabla\ell = 0$, the maximum likelihood estimates (MLEs) of the model parameters are obtained.

3.2.5 Application

Fonseca and França (2007) [11] studied the soil fertility influence and the characterization of the biologic fixation of N_2 for the *Dimorphandra wilsonii rizz* growth. For 128 plants, they made measures of the phosphorus concentration in the leaves. We fitted the BN model for this data set. The estimated parameters using maximum likelihood were $\hat{a} = 66.558$, $\hat{b} = 0.125$, $\hat{\mu} = -0.020$ and $\hat{\sigma} = 0.038$ and the maximized log-likelihood were $\hat{l} = 198.054$. Figure 3.3 plots the fitted pdf and the estimated quantiles versus observed quantiles.

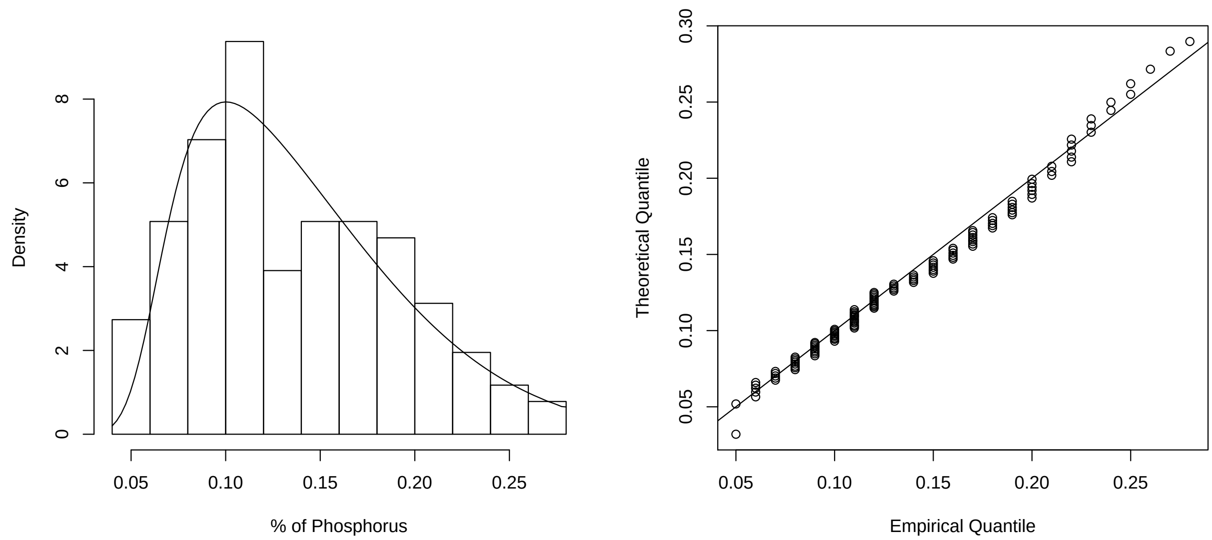

Figure 3.3: Plots of the fitted pdf and the estimated quantiles versus observed quantiles of the BN data set.

3.3 Beta-exponential

The beta-exponential (BE) distribution was introduced by Nadarajah and Kotz (2006) [21]. They discussed in their paper an expression for the sth moment, properties of the hazard function, results for the distribution of the sum of BE random variables, maximum likelihood estimation and some asymptotics results.

3.3.1 The distribution

Let $G(x) = 1 - e^{-\lambda x}$, with $\lambda > 0$ and $x > 0$, be the cdf of the exponential distribution. From (2.3), the density of the BE distribution is given by

$$f(x) = \frac{\lambda}{B(a,b)} \exp(-b\lambda x)\{1 - \exp(-\lambda x)\}^{a-1}, \quad a > 0,\, b > 0,\, \lambda > 0,\, x > 0.$$

This distribution contains the exponentiated exponential distribution (Gupta and Kundu, 2001 [14]) as a special case for $b = 1$. When $a = 1$, the BE distribution coincides with the exponential distribution with parameter $b\lambda$. A random variable following a beta exponential distribution will be denoted by $X \sim BE(a, b, \lambda)$.

Figure 3.4 plots some densities of the BE distribution with $b = \lambda = 1$.

3.3.2 Hazard function

The hazard function is given by

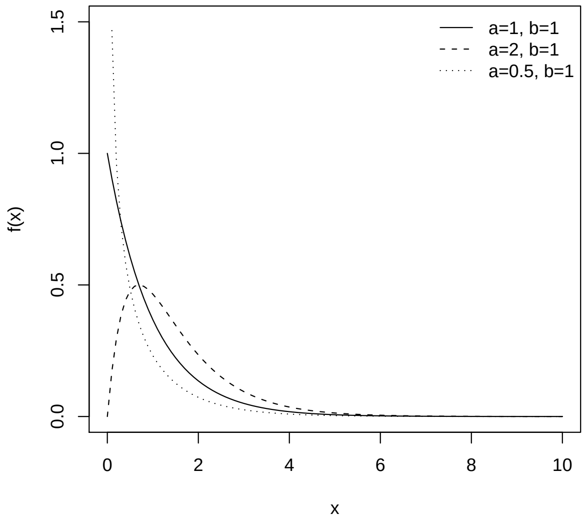

Figure 3.4: The BE pdf for some values of a with $b = \lambda = 1$.

$$h(x) = \frac{\lambda \exp(-b\lambda x)\{1 - \exp(-\lambda x)\}^{a-1}}{B_{\exp(-\lambda x)}(b, a)}, \quad a, b, \lambda, x > 0,$$

where $B_{\exp(-\lambda x)}(b, a)$ is the incomplete beta function defined in (2.2).

The shape of the hazard function depends only on the parameter a. When $a < 1$, $h(x)$ monotonically decreases with x and when $a > 1$, $h(x)$ is monotonically increasing. The hazard function is constant when $a = 1$. Figure 3.5 plots the hazard function for $b = \lambda = 1$ and $a = \{0.5, 1, 2\}$.

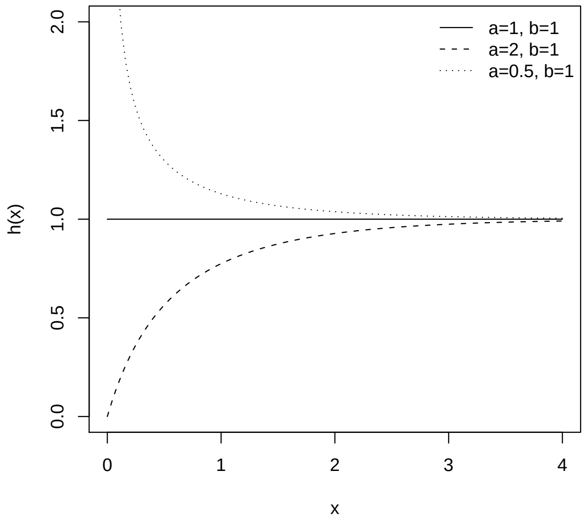

Figure 3.5: The BE hazard function for some values of a with $b = \lambda = 1$.

3.3.3 Moments

Let Y be a random variable following the exponential distribution with parameter $\lambda > 0$. Then, the sth moment of Y is given by $E(Y^s) = s!\lambda^s$. The cdf of the exponential distribution admits the expansion (2.16), with $a_m = \frac{(-1)^m \lambda^{m+1}}{(m+1)!}$ and $c = 1$. Then, the sth moment of a random variable X following a BE distribution can be obtained from the expressions (2.25) and (2.24) if the parameter a is an integer or a non-integer respectively.

The moment generating function, derived by Nadarajah and Kotz (2006) [21], is

$$M(t) = \frac{B(b - \frac{it}{\lambda}, a)}{B(a, b)}.$$

3.3.4 Estimation

Let $x_1, \ldots, x_n$ be an independent random sample from the BE distribution. The total log-likelihood function is given by

$$\ell = \ell(x; a, b, \lambda) = n \log \lambda - n \log B(a, b) - b\lambda \sum_{i=1}^{n} x_i + (a-1) \sum_{i=1}^{n} \log\{1 - \exp(-\lambda x_i)\}.$$

The score function $\nabla \ell = \left(\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \lambda}\right)$ has components

$$\frac{\partial \ell}{\partial a} = -n\psi(a) + n\psi(a+b) + \sum_{i=1}^{n} \log\{1 - \exp(-\lambda x_i)\},$$
$$\frac{\partial \ell}{\partial b} = -n\psi(b) + n\psi(a+b) - \lambda \sum_{i=1}^{n} x_i,$$
$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - b\sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n} \frac{x_i \exp(-\lambda x_i)}{1-\exp(-\lambda x_i)}.$$

Solving the system of nonlinear equations $\nabla \ell = 0$, the MLEs for the model parameters are obtained.

3.3.5 Random number generation

By using the cumulative technique for random number generation, a random number X following the BE distribution with parameters a, b and λ can be obtained from $X = \lambda^{-1} \log\left(\frac{\lambda}{B}\right)$, where B follows the standard beta distribution with parameters a e b.

Figure 3.6 plots some samples from the BE distribution with its respective density curve.

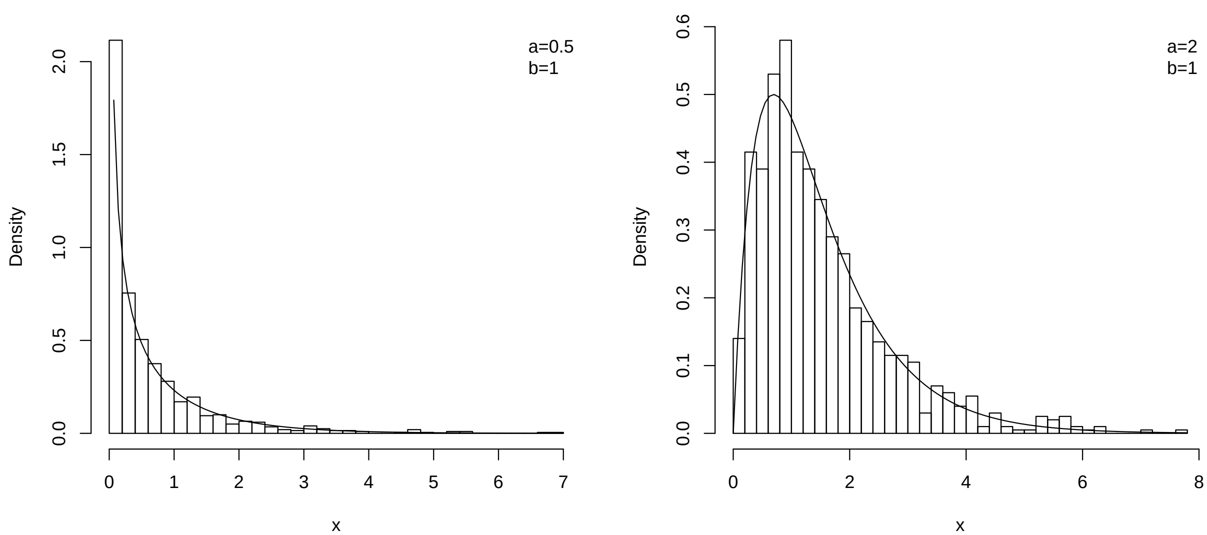

Figure 3.6: Generated samples from the BE distribution for some values of a with $b = \lambda = 1$.

3.3.6 Application

Gupta and Kundu (1999) [13] analyzed a real data set that arose in tests on the endurance of deep groove ball bearings. They fitted the generalized exponential model and made comparisons to other models. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test. For this data set, the estimates of the BE model were $\hat{a} = 5.203$, $\hat{b} = 1.044$ and $\hat{\lambda} = 0.0312$ and the maximized log-likelihood were $\hat{l} - 112.978$. Figure 3.7 shows the fitted pdf and the estimated quantiles versus observed quantiles.

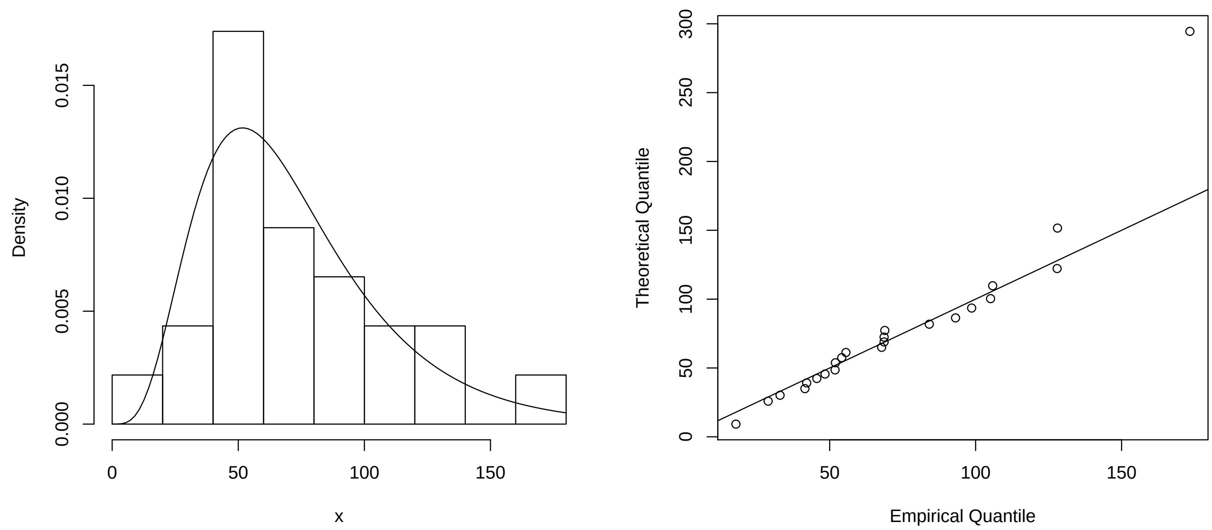

Figure 3.7: Plots of the fitted pdf and the estimated quantiles versus observed quantiles of the BE data set.

3.4 Beta Weibull

The beta Weibull (BW) distribution was introduced by Famoye et al. (2005) [9]. Lee et al. (2007) [16] gave some properties of the hazard function, entropies and an aplication to censored data. Additional mathematical properties of this distribution was derived by Cordeiro et al. (2008) [5], such as expressions for the moments.

3.4.1 The distribution

Let $G(x) = 1 - \exp\{-(x\lambda)^c\}$, with $\lambda > 0$, $c > 0$ and $x > 0$, be the cdf of the Weibull distribution. From (2.3), the density function of the BW distribution is given by

$$f(x) = \frac{c\lambda}{B(a,b)}\,(x\lambda)^{c-1}\left[1 - e^{-(x\lambda)^c}\right]^{a-1} e^{-b(x/\lambda)^c}, \quad a, b, c, \lambda > 0.$$

We denote a random variable X following a BW distribution by $X \sim \mathrm{BW}(a, b, c, \lambda)$. Figure 3.8 plots the density function of the BW distribution, for $c = \lambda = 1$.

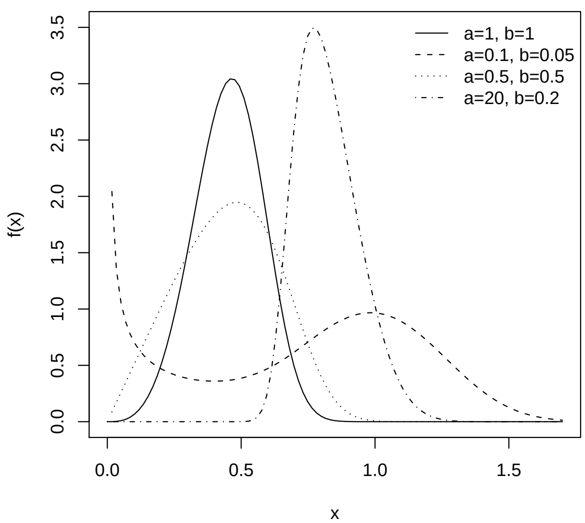

Figure 3.8: The BW pdf for some values of a and b with $c = \lambda = 1$.

3.4.2 Hazard function

The hazard function of the BW distribution is given by

$$h(x) = \frac{c\lambda^c x^{c-1} \exp\{-b(\lambda x)^c\}[1 - \exp\{-(\lambda x)^c\}^{a-1}}{B_{1-\exp\{-(\lambda x)^c\}}(a, b)}, \quad a, b, c, \lambda > 0.$$

Lee et al. (2007) showed that the hazard function is

- constant ($h(x) = b\lambda$) when $a = c = 1$,
- decreasing when $ac \leq 1$ and $c \leq 1$,
- increasing when $ac \geq 1$ and $c \geq 1$,
- a bathtub failure rate when $ac < 1$ and $c > 1$, and
- upside down bathtub (or unimodal) failure rate when $ac > 1$ and $c < 1$.

Figure 3.9 plots some shapes of the hazard function of the BW distribution, for $\lambda = 4$ and some values of a, b and c.

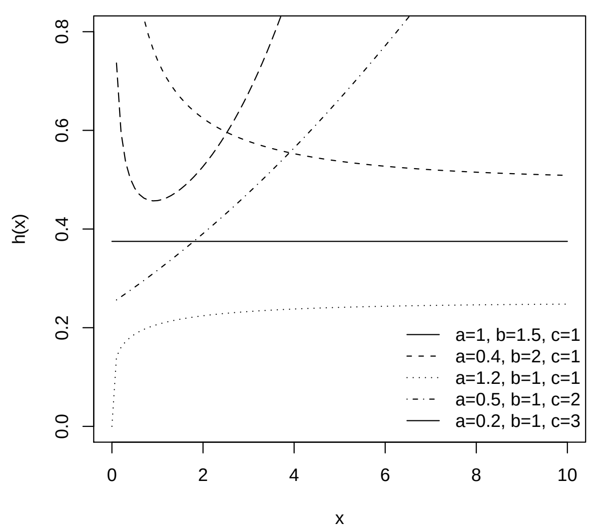

Figure 3.9: The BW hazard function for some values of a, b and c with $\lambda = 4$.

3.4.3 Moments

Cordeiro et al. (2008) [5] derive expressions for the sth moment of the BW distribution. When the parameter a is an integer,

$$E(X^r) = \frac{\Gamma(r/c+1)}{\lambda^r B(a,b)} \sum_{j=0}^{a-1} \binom{a-1}{j} \frac{(-1)^j}{(b+j)^{r/c+1}}$$

and when a is a non-integer,

$$E(X^r) = \frac{\Gamma(a)\Gamma(r/c+1)}{\lambda^r B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(a-j)j!(b+j)^{r/c+1}}.$$

3.4.4 Estimation

Let x_1, $x_2, \ldots$, x_n be an independent random sample from the BW distribution. The total log-likelihood for a single observation x of X is given by

$$\begin{aligned}\ell = \ell(x; a, b, \lambda, c) \quad &= \quad n\log(c) + cn\log(\lambda) + (c-1)\sum_{i=1}^{n}\log(x_i) - n\log\{B(a,b)\} \\ &\quad -b\lambda^c \sum_{i=1}^{n} x_i^c + (a-1)\sum_{i=1}^{n}\log[1-e^{-(\lambda x_i)^c}].\end{aligned}$$

The corresponding components of the score vector $\nabla\ell = \left(\frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial c}, \frac{\partial\ell}{\partial\lambda}\right)$ are:

$$\begin{aligned}\frac{\partial\ell}{\partial a} &= -n\{\psi(a)-\psi(a+b)\} + \sum_{i=1}^{n}\log\left\{1-e^{-(\lambda x_i)^c}\right\}, \\ \frac{\partial\ell}{\partial b} &= -n\{\psi(b)-\psi(a+b)\} - \lambda^c \sum_{i=1}^{n} x_i^c, \\ \frac{\partial\ell}{\partial c} &= \frac{n}{c} + n\log(\lambda) + \sum_{i=1}^{n}\log(\lambda x_i) - b\sum_{i=1}^{n}(\lambda x_i)^c\log(\lambda x_i) + (a-1)\lambda^c \sum_{i=1}^{n}\frac{x_i^c\log(\lambda x_i)e^{-(\lambda x_i)^c}}{1-e^{-(\lambda x_i)^c}}, \\ \frac{\partial\ell}{\partial\lambda} &= n\frac{c}{\lambda} - \frac{bc}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^c + c(a-1)\lambda^{c-1}\sum_{i=1}^{n}\frac{x_i^c e^{-(\lambda x_i)^c}}{\{1-e^{-(\lambda x_i)^c}\}}.\end{aligned}$$

Solving the system of nonlinear equations $\nabla\ell = 0$, the MLEs for the parameters of the BW distribution are obtained.

3.4.5 Random number generation

The quantile B of the Weibull distribution is given by

$$X = \lambda^{-1} \{-\log(1 - B)\}^{1/c}.$$

The cumulative technique to generate a random number X following the BW distribution with parameters a, b, c and λ consists of obtaining X from $B \sim Beta(a, b)$

Figure 3.10 plots some generated samples from a BW distribution with its respective density curve.

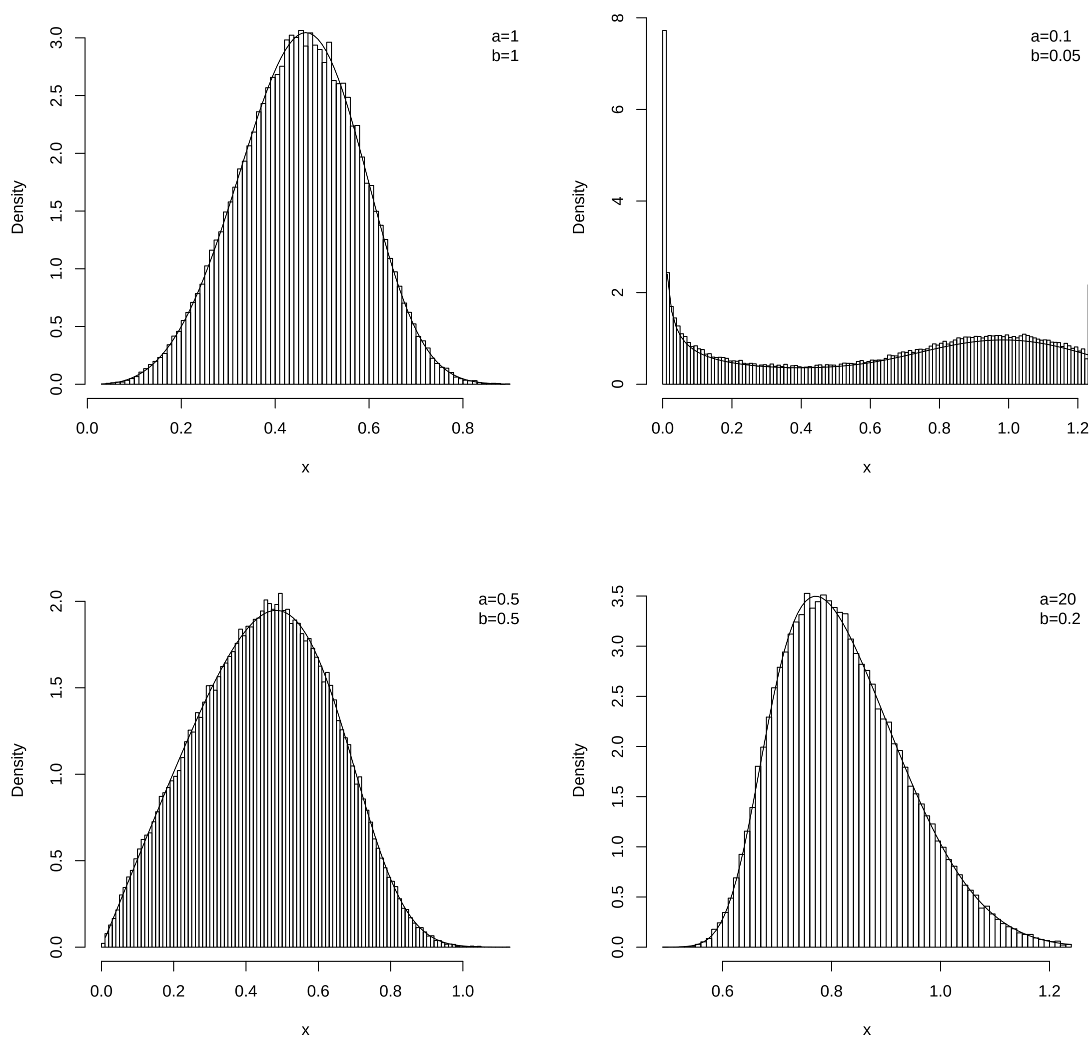

Figure 3.10: Generated samples of the BW distribution for some values of a and b with $c = \lambda = 1$.

3.4.6 Application

In this section, we fit BW model to a real data set already known in the literature. The data set (Linhart and Zucchini [17], 1986) consists of failure times of the air conditioning system of an airplane. The estimated values of the parameters were $\hat{a} = 3.087$, $\hat{b} = 0.132$, $\hat{c} = 0.667$ and $\hat{\lambda} = 1.798$ and the maximized log-likelihood were $\hat{l} = -151.076$. Figure 3.11 shows the fitted pdf and the estimated quantiles versus observed quantiles.

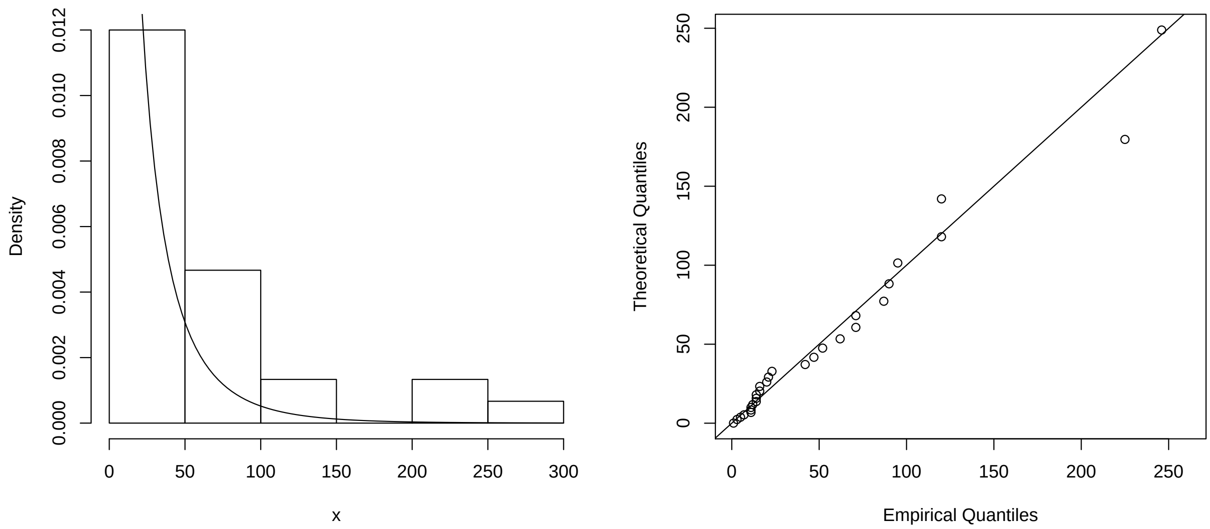

Figure 3.11: Plots of the fitted pdf and the estimated quantiles versus observed quantiles of the BW data set.

3.5 Beta-hyperbolic secant

The beta-hyperbolic secant (BHS) distribution was introduced by Fischer and Vaughan (2007) [10]. They gave some properties of this distribution, like the moments, special and limiting cases and compare it with others distributions using a real data set.

3.5.1 The distribution

Let $G(x) = \frac{2}{\pi}\arctan(e^x)$, with $x \in \mathbb{R}$, be the cdf of the hyperbolic secant distribution. From (2.3), the density of the BHS distribution is given by

$$f(x) = \frac{\left[\frac{2}{\pi}\arctan(e^x)\right]^{a-1}\left[1 - \frac{2}{\pi}\arctan(e^x)\right]^{b-1}}{B(a,b)\pi\cosh(x)}, \quad a > 0,\ b > 0,\ x \in \mathbb{R}.$$

We now introduce a location parameter $\mu \in \mathbb{R}$ and a scale parameter $\sigma > 0$. The BHS density generalizes to

$$f(x) = \frac{\left[\frac{2}{\pi}\arctan(e^{\frac{x-\mu}{\sigma}})\right]^{a-1}\left[1-\frac{2}{\pi}\arctan(e^{\frac{x-\mu}{\sigma}})\right]^{b-1}}{\pi\sigma B(a,b)\cosh(\frac{x-\mu}{\sigma})}.$$

When a random variable X follows a beta-hyperbolic secant distribution, it will be denoted by $X \sim BHS(a, b, \mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, the BHS distribution will be said to be a standard BHS distribution.

Figure 3.12 plots the density of the standard BHS distribution for some values of a and b.

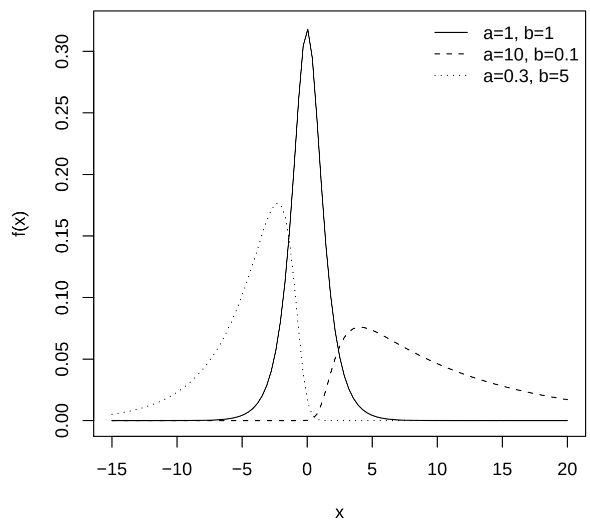

Figure 3.12: The BHS pdf for some values of a and b with $\mu = 0$ and $\sigma = 1$.

3.5.2 Estimation

Let $x_1, \ldots, x_n$ be an independent random sample from the BHS distribution. The total log-likelihood is given by

$$\begin{aligned}\ell = \ell(x;a,b,\lambda,c) &= -n\log B(a,b) - \sum_{i=1}^{n}\log\left[\cosh\left(\frac{x_i-\mu}{\sigma}\right)\right] - n\log(\sigma\pi)\\ &+(a-1)\sum_{i=1}^{n}\left\{\log\left(\frac{2}{\pi}\right) + \log\left[\arctan\left(e^{\frac{x_i-\mu}{\sigma}}\right)\right]\right\}\\ &+(b-1)\sum_{i=1}^{n}\log\left[1-\frac{2}{\pi}\arctan\left(e^{\frac{x_i-\mu}{\sigma}}\right)\right].\end{aligned}$$

The corresponding components of the score function $\nabla\ell = \left(\frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\mu}, \frac{\partial\ell}{\partial\sigma}\right)$ are:

$$\begin{aligned}\frac{\partial\ell}{\partial a} &= -n\psi(a) + n\psi(a+b) + \sum_{i=1}^{n}\log\frac{2}{\pi} + \log\left[\arctan\left(e^{\frac{x_i-\mu}{\sigma}}\right)\right],\\ \frac{\partial\ell}{\partial b} &= -n\{\psi(b)+\psi(a+b)\} + \sum_{i=1}^{n}\log\left[1-\frac{2}{\pi}\arctan\left(e^{\frac{x_i-\mu}{\sigma}}\right)\right],\\ \frac{\partial\ell}{\partial\mu} &= -\frac{1}{\sigma}\sum_{i=1}^{n}\tanh\left(\frac{x_i-\mu}{\sigma}\right)\\ &+\sum_{i=1}^{n}\frac{1-a}{\sigma}\frac{\arctan^{-1}\left(e^{\frac{x_i-\mu}{\sigma}}\right)}{1+e^{2\frac{x_i-\mu}{\sigma}}}e^{\frac{x_i-\mu}{\sigma}}\\ &+\sum_{i=1}^{n}\frac{2(b-1)}{\sigma\pi}\frac{\left[1-\frac{2}{\pi}\arctan\left(e^{\frac{x_i-\mu}{\sigma}}\right)\right]^{-1}}{1+e^{2\frac{x_i-\mu}{\sigma}}}e^{\frac{x_i-\mu}{\sigma}},\\ \frac{\partial\ell}{\partial\sigma} &= -\sum_{i=1}^{n}\frac{x_i-\mu}{\sigma^2}\tanh\left(\frac{x_i-\mu}{\sigma}\right) - n\sigma^{-1}\\ &+\sum_{i=1}^{n}\frac{(1-a)(x_i-\mu)}{\sigma^{-2}}\frac{\arctan^{-1}\left(e^{\frac{x_i-\mu}{\sigma}}\right)}{1+e^{2\frac{x_i-\mu}{\sigma}}}e^{\frac{x_i-\mu}{\sigma}}\\ &+\sum_{i=1}^{n}\frac{2(b-1)(x_i-\mu)}{\sigma^{-2}\pi}\frac{\left[1-\frac{2}{\pi}\arctan\left(e^{\frac{x_i-\mu}{\sigma}}\right)\right]^{-1}}{1+e^{2\frac{x_i-\mu}{\sigma}}}e^{\frac{x_i-\mu}{\sigma}}.\end{aligned}$$

Solving the system of nonlinear equations $\nabla\ell = 0$, the MLEs for the model parameters of the BHS distribution are obtained.

3.5.3 Random number generation

Using the cumulative technique, a random number X following the BHS distribution with parameters a, b, μ and σ can be obtained from

$$X = \log\tan\left(\frac{\pi B}{2}\right)\sigma + \mu,$$

where B follows the standard beta distribution with parameters a e b.

Figure 3.13 plots some generated samples from the BHS distribution with its respective density curve.

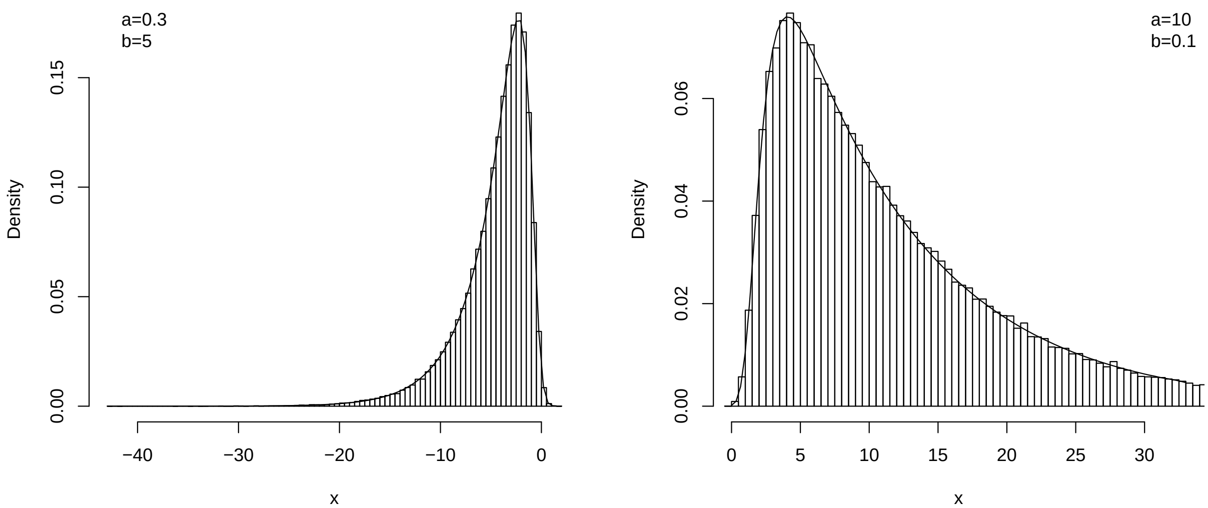

Figure 3.13: Generated samples from the BHS distribution for some values of a and b with $\mu = 0$ and $\sigma = 1$.

3.5.4 Application

In this section, we fit the BHS model to a real data set. The data consists of fracture toughness from the silicon nitride. The data taken from the web-site *http://www.ceramics.nist.gov/srd/summary/ftmain.htm* was already studied by Nadarajah and Kotz (2007) [22]. The estimated values of the parameters were $\hat{a} = 2.072$, $\hat{b} = 4.385$, $\hat{\mu} = 5.278$ and $\hat{\sigma} = 1.297$ and the maximized log-likelihood were $\hat{l} = -32.340$. Figure 3.14 shows the fitted pdf and the estimated quantiles versus observed quantiles.

3.6 Beta gamma

The beta gamma distribution (BG) was introduced by Kong et al. (2007) [15]. In their paper, they derived some properties of the limit of the density function and of the hazard

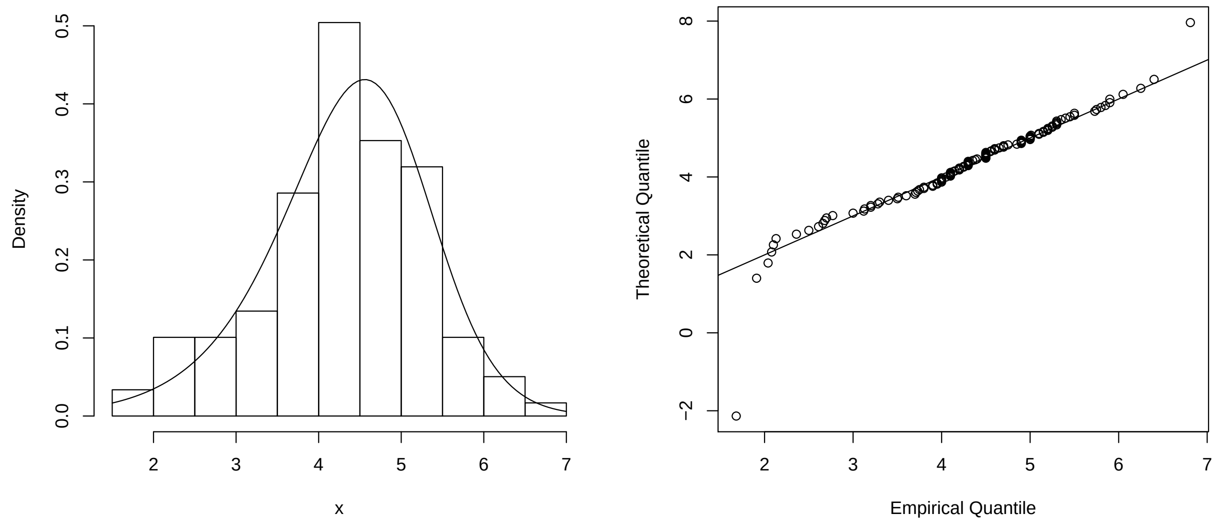

Figure 3.14: Plots of the fitted pdf and the estimated quantiles versus observed quantiles of the BHS data set.

function, gave an expression for the moments when the shape parameter a is an integer and made an application.

We now present a closed form expression for the sth moment when a is not an integer, due to a general result derived by Cordeiro and Barreto-Souza (2009) [4] and an algorithm to generate random numbers following this distribution.

3.6.1 The distribution

The cdf of the gamma distribution is given by

$$G(x) = \frac{\Gamma_{x/\lambda}(\rho)}{\Gamma(\rho)}, \quad \rho,\, \lambda,\, x > 0,$$

where $\Gamma_x(\rho) = \int_0^x y^{\rho-1} e^{-y} dy$ is the incomplete gamma function.

The BG distribution defined by Kong et al. (2007) [15] has density

$$f(x) = \frac{x^{\rho-1} e^{-x/\lambda}}{B(a,b)\Gamma(\rho)^a \lambda^\rho} \Gamma_{x/\lambda}(\rho)^{a-1} \left\{1 - \frac{\Gamma_{x/\lambda}(\rho)}{\Gamma(\rho)}\right\}^{b-1}, \quad a,\, b,\, \rho,\, \lambda,\, x > 0.$$

Figure 3.15 plots the density function of the BG distribution for $\lambda = \rho = 1$ and some values of a and b.

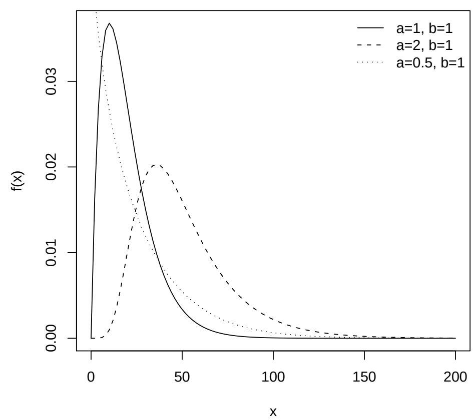

Figure 3.15: The BG pdf for some values of a and b with $\lambda = \rho = 1$.

3.6.2 Hazard function

The hazard function of the BG distribution is

$$f(x) = \frac{x^{\rho-1}e^{-x/\lambda}}{B_{1-\frac{\Gamma_{x/\lambda}(\rho)}{\Gamma(\rho)}}(b,a)\Gamma(\rho)^a\lambda^\rho}\Gamma_{x/\lambda}(\rho)^{a-1}\left\{1-\frac{\Gamma_{x/\lambda}(\rho)}{\Gamma(\rho)}\right\}^{b-1}.$$

Figure 3.16 plots some shapes for the hazard function of the BG distribution for $\lambda = 1$ and some values of a, b and ρ.

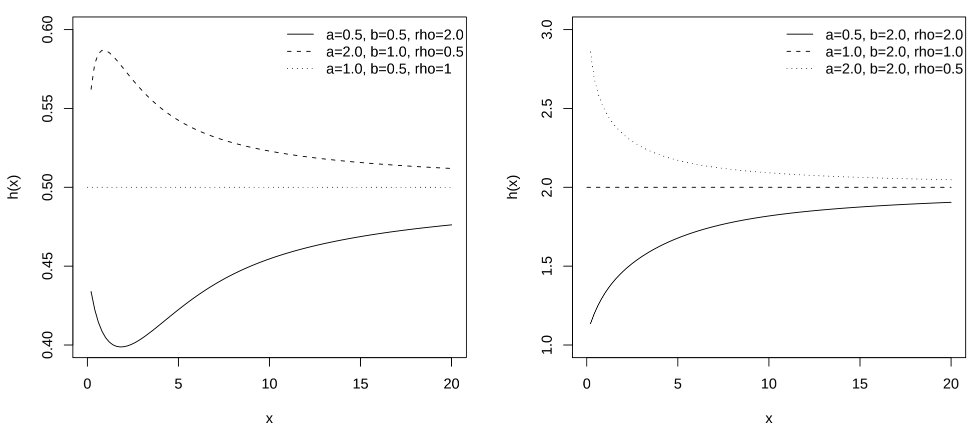

Figure 3.16: The BG hazard function for some values of a, b and ρ with $\lambda = 1$.

3.6.3 Moments

Let Y be a random variable following the gamma distribution with parameters $\lambda > 0$ and $\rho > 0$. Then, the sth moment of Y is given by $E(Y^s) = \frac{\Gamma(s+\rho)\lambda^s}{\Gamma(\rho)}$. The cdf of the gamma distribution admits the expansion (2.16), with $a_n = \frac{(-1)^n \lambda^{-(n+\rho)}}{\Gamma(\rho) n! (n+\rho)}$ and $c = \rho$. Then, the sth moment of a random variable X following a BG distribution can be given by the expressions (2.25) and (2.24) when the parameter a is an integer or a non-integer, respectively.

3.6.4 Estimation

Let $x_1, \ldots, x_n$ be an independent random sample from the BG distribution. The total log-likelihood is given by

$$\begin{aligned}\ell = \ell(x; a, b, \rho, \lambda) &= (\rho - 1)\sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \frac{x_i}{\lambda} - n \log B(a,b) - an \log \Gamma(\rho) - n\rho \log \lambda \\ &\quad + (a-1)\sum_{i=1}^{n} \log \Gamma_{x_i/\lambda}(\rho) + (b-1)\sum_{i=1}^{n} \log\left[1 - \frac{\Gamma_{x_i/\lambda}(\rho)}{\Gamma(\rho)}\right]\end{aligned}$$

The corresponding components of the score vector $\nabla\ell = \left(\frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\rho}, \frac{\partial\ell}{\partial\gamma}\right)$ are:

$$\begin{aligned}\frac{\partial\ell}{\partial a} &= n\psi(a+b) - n\psi(a) - n\log\Gamma(\rho) + \sum_{i=1}^{n} \log\Gamma_{x_i/\lambda}(\rho), \\ \frac{\partial\ell}{\partial b} &= n\psi(a+b) - n\psi(a) + \sum_{i=1}^{n} \log\left[1 - \frac{\Gamma_{x_i/\lambda}(\rho)}{\Gamma(\rho)}\right], \\ \frac{\partial\ell}{\partial\lambda} &= \sum_{i=1}^{n} \frac{x_i}{\lambda^2} - n\frac{\rho}{\lambda} + \frac{1-a}{\lambda^\rho}\sum_{i=1}^{n} (\Gamma_{x_i/\lambda}(\rho))^{-1} x_i^{\rho-1} e^{-x_i/\lambda} \\ &\quad - \frac{b-1}{\Gamma(\rho)\lambda^\rho}\sum_{i=1}^{n}\left[1 - \frac{\Gamma_{x_i/\lambda}(\rho)}{\Gamma(\rho)}\right]^{-1} x_i^{\rho-1} e^{-x_i/\lambda}, \\ \frac{\partial\ell}{\partial\rho} &= \sum_{i=1}^{n} \log x_i - na\psi(\rho) - n\log\lambda + (a-1)\sum_{i=1}^{n} \psi_{x_i/\lambda}(\rho) \\ &\quad + (b-1)\sum_{i=1}^{n} \frac{\Gamma_{x_i/\lambda}(\rho)\psi(\rho) - \psi_{x_i/\lambda}(\rho)\Gamma_{x_i/\lambda}(\rho)}{\Gamma(\rho) - \Gamma_{x_i/\lambda}},\end{aligned}$$

where $\psi(\rho) = \frac{\partial}{\partial\rho}\log\Gamma(\rho)$ is the digamma function and $\psi_x(\rho) = \frac{\partial}{\partial\rho}\log\Gamma_x(\rho)$. Solving the system of nonlinear equations $\nabla\ell = 0$, the MLEs for the parameters of the BG distribution are obtained.

3.6.5 Random number generation

The cdf of the gamma distribution has no analytical inverse, just like the normal distribution. Hence, we generate random numbers following the BG distribution using the proposed technique in Section 2.7. However, the quantile function of the gamma distribution has to be numerically evaluated. Figure 3.17 plots generated random samples from the BG distribution with its respective density curve.

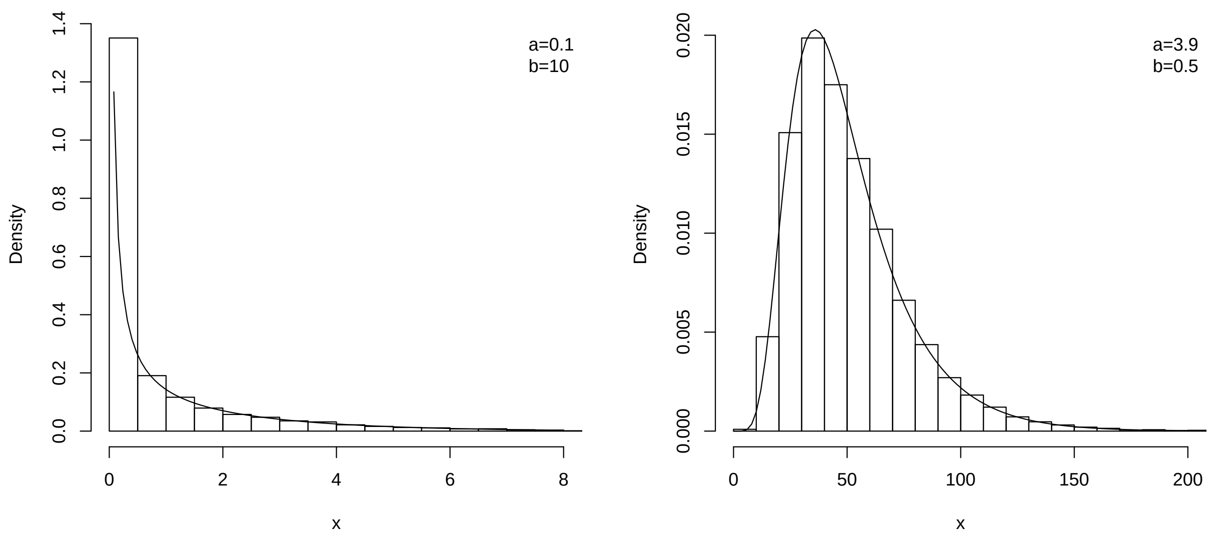

Figure 3.17: Generated samples from the beta gamma pdf for some values of a and b with $\lambda = \rho = 1$.

3.6.6 Application

In this section, we fit the BG model to a real data set taken from the R base package. It is located in the *Indometh* object. The data consists of plasma concentrations of indomethicin (mcg/ml). The estimated values of the parameters were $\hat{a} = 0.977$, $\hat{b} = 9.181$, $\hat{\rho} = 1.055$ and $\hat{\lambda} = 5.553$ and the maximized log-likelihood were $\hat{l} = -31.368$. Figure 3.18 shows the fitted pdf and the estimated quantiles versus observed quantiles.

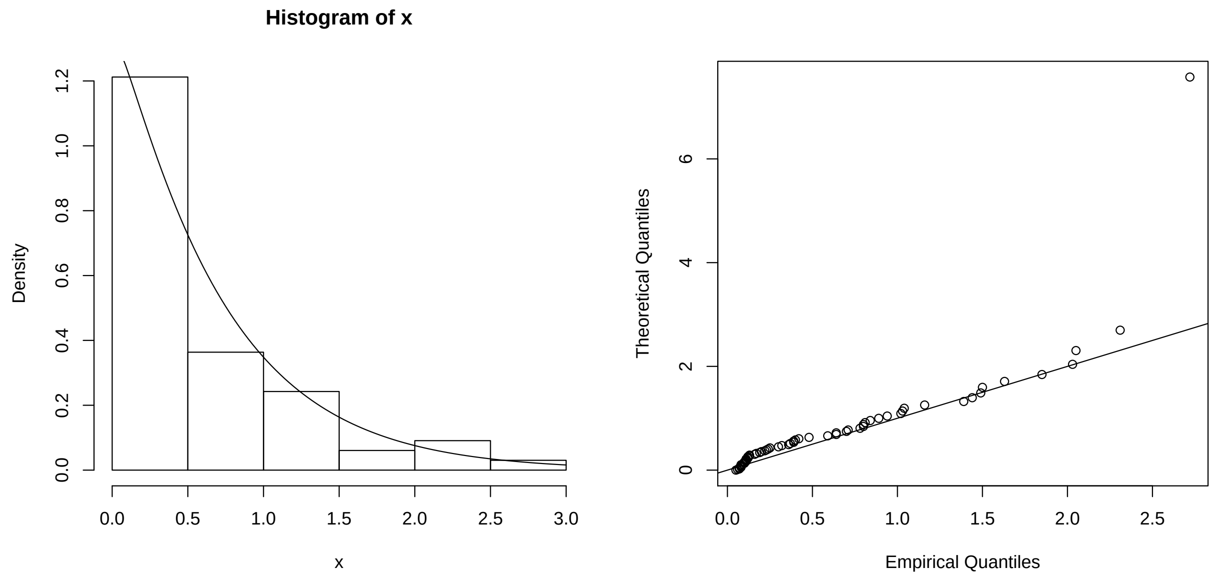

Figure 3.18: Plots of the fitted pdf and the estimated quantiles versus observed quantiles of the BG data set.

3.7 Beta Gumbel

The beta Gumbel (BGU) distribution, introduced by Nadarajah and Kotz (2004) [20], generalizes a distribution largely used on engineering problems. They calculated expressions for the sth moment, gave some particular cases and studied the density function.

3.7.1 The distribution

Let $G(x) = \exp\{-\exp\{-(x-\mu)/\sigma\}\}$, with $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $\sigma > 0$, be the density of the cdf of the Gumbel distribution. From (2.3), the density of the BGU distribution is given by

$$f(x) = \frac{u}{\sigma B(a,b)} e^{-au}(1-e^{-u})^{b-1}, \quad a > 0,\, b > 0,$$

where $u = \exp\{-(x-\mu)/\sigma\}$, $\mu \in \mathbb{R}$, $\sigma > 0$ and $x \in \mathbb{R}$. Figure 3.19 plots some shapes of the standard BGU density for some values of a and b.

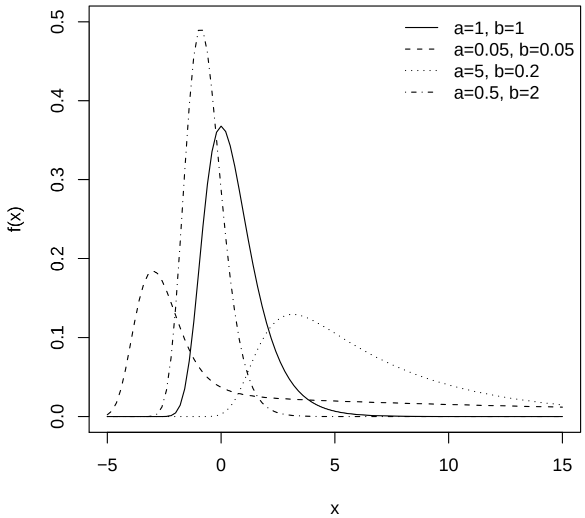

Figure 3.19: The BGU pdf for some values of a and b with $\mu = 0$ and $\sigma = 1$.

3.7.2 Estimation

Let x_1,x_2, ..., x_n be an independent random sample from the BGU distribution. The total log-likelihood function is given by

$$\begin{aligned} \ell = \ell(x; a, b, \mu, \sigma) &= -n \log \sigma + (b-1) \sum_{i=1}^{n} \log \left[1 - \exp \left\{ - \exp \left(- \frac{x_i - \mu}{\sigma} \right) \right\} \right] \\ &\quad - \sum_{i=1}^{n} \frac{x_i - \mu}{\sigma} - a \sum_{i=1}^{n} \exp \left(- \frac{x_i - \mu}{\sigma} \right) - n \log B(a, b). \end{aligned}$$

The score function $\nabla \ell = \left(\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \mu}, \frac{\partial \ell}{\partial \sigma} \right)$ has components

$$
\begin{aligned}
\frac{\partial \ell}{\partial a} &= n\psi(a+b) - n\psi(a) - \sum_{i=1}^{n} \exp\left(-\frac{x_i-\mu}{\sigma}\right), \\
\frac{\partial \ell}{\partial b} &= n\psi(a+b) - n\psi(a) + \sum_{i=1}^{n} \log\left[1 - \exp\left\{-\exp\left(-\frac{x_i-\mu}{\sigma}\right)\right\}\right], \\
\frac{\partial \ell}{\partial \mu} &= \frac{n}{\sigma} - \frac{a}{\sigma}\sum_{i=1}^{n} \exp\left(-\frac{x_i-\mu}{\sigma}\right) \\
&\quad + \frac{b-1}{\sigma}\sum_{i=1}^{n} \frac{\exp(-(x_i-\mu)/\sigma)\exp\{-\exp(-(x_i-\mu)/\sigma)\}}{1-\exp\{-\exp(-(x_i-\mu)/\sigma)\}}, \\
\frac{\partial \ell}{\partial \sigma} &= -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{x_i-\mu}{\sigma^2}\left\{1 - a\exp\left(-\frac{x_i-\mu}{\sigma}\right)\right\} \\
&\quad + \frac{b-1}{\sigma^2}\sum_{i=1}^{n} \frac{(x_i-\mu)\exp(-(x_i-\mu)/\sigma)\exp\{-\exp(-(x_i-\mu)/\sigma)\}}{1-\exp\{-\exp(-(x_i-\mu)/\sigma)\}}.
\end{aligned}
$$

Solving the system of nonlinear equations $\nabla\ell = 0$, the MLEs of the BGU parameters are obtained.

3.7.3 Random number generation

Using the cumulative technique, a random number X following the BGU distribution with parameters a, b, μ and σ can be obtained by

$$
X = \mu - \log(-\log B)\sigma,
$$

where B follows the standard beta distribution with parameters a e b.

Figure 3.20 plots some generated samples from the BGU distribution with its respective density curve.

3.7.4 Application

Children of the *Programa Hormonal de Crescimento da Secretaria da Saúde de Minas Gerais* were diagnosed with growth hormone deficiency. The data consists of the estimated time since the growth hormone medication until the children reached the target height. We fit the BGU model for this data set. The estimated values of the parameters were $\hat{a} = 1.438$, $\hat{b} = 0.110$, $\hat{\mu} = 2.183$ and $\hat{\sigma} = 0.328$ and the maximized log-likelihood were $\hat{l} = -76.441$. Figure 3.21 shows the fitted pdf and the estimated quantiles versus observed quantiles.

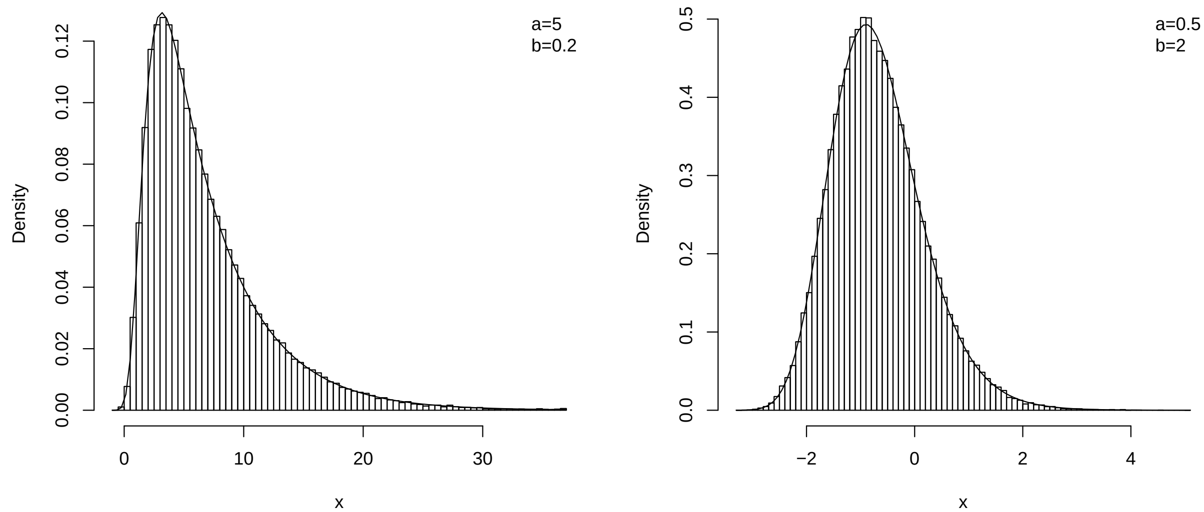

Figure 3.20: Generated values from the BGU distribution for some values of a and b with $\mu = 0$ and $\sigma = 1$.

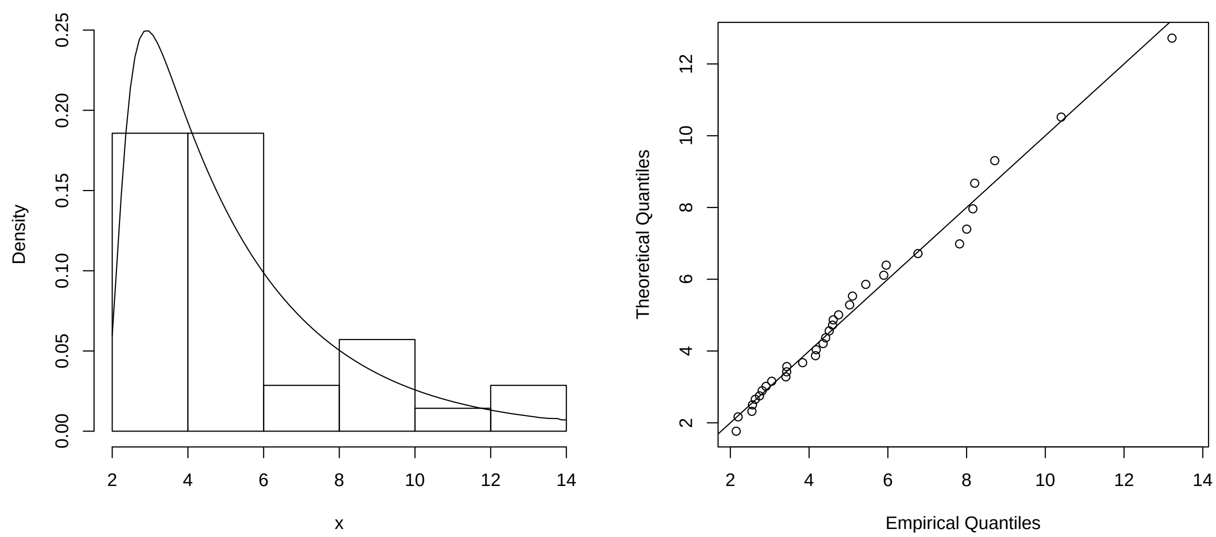

Figure 3.21: Plots of the fitted pdf and the estimated quantiles versus observed quantiles of the BGU data set.

3.8 Beta Fréchet

The beta Fréchet (BF) distribution was introduced by Nadarajah and Gupta (2004) [19]. They derived some properties of the density and hazard functions and gave an expression for the sth moment. Barreto-Souza et al. (2008) [2] gave another expression for the moments and derived some additional mathematical properties.

3.8.1 The distribution

Let $G(x) = \exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}$, with $x > 0$, $\sigma > 0$ and $\lambda > 0$, be the cdf of the standard Fréchet distribution. From (2.3), the density of the BF distribution is given by

$$f(x) = \frac{\lambda\sigma^{\lambda}\exp\left\{-a\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\left[1-\exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\right]^{b-1}}{x^{1+\lambda}B(a,b)}, \quad a,\, b,\, \sigma,\, \lambda,\, x > 0.$$

Figure 3.22 plots some shapes of the hazard function of the BF distribution for $\lambda = \sigma = 1$.

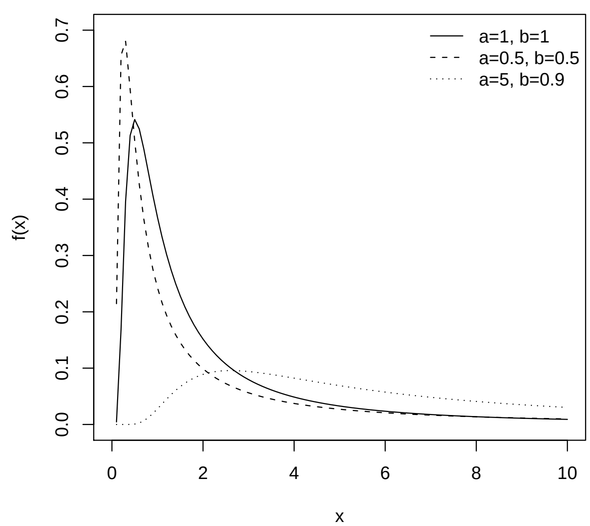

Figure 3.22: The BF pdf for some values of a and b with $\lambda = \sigma = 1$.

3.8.2 Hazard function

The hazard function of the BF distribution is

$$f(x) = \frac{\lambda\sigma^{\lambda}\exp\left\{-a\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\left[1-\exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\right]^{b-1}}{x^{1+\lambda}B_{\exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}}(b,a)}, \quad a,\, b,\, \sigma,\, \lambda,\, x > 0.$$

Figure 3.23 plots some shapes for the hazard function of the BF distribution for $\sigma = \lambda = 1$ and some values of a and b.

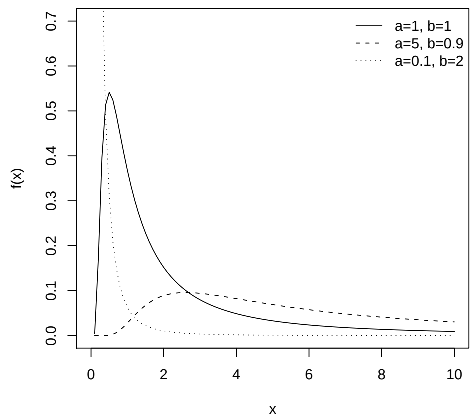

Figure 3.23: The BF hazard function for some values of a and b with $\lambda = \sigma = 1$.

3.8.3 Estimation

Let x_1,x_2, ..., x_n be an independent random sample from the BF distribution. The total log-likelihood function is given by

$$\begin{aligned}\ell = \ell(x; a, b, \sigma, \lambda) &= n\lambda \log \sigma - n \log B(a, b) + (b-1) \sum_{i=1}^{n} \log \left[1 - \exp\left\{-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right\}\right] \\ & n \log \lambda - (1+\lambda) \sum_{i=1}^{n} \log x_i - a\sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}.\end{aligned}$$

The score function $\nabla \ell = \left(\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \sigma}\right)$ has components

$$\begin{aligned}\frac{\partial \ell}{\partial a} &= n\psi(a+b) - n\psi(a) - \sigma^{\lambda} \sum_{i=1}^{n} x_i^{-\lambda}, \\ \frac{\partial \ell}{\partial b} &= n\psi(a+b) - n\psi(b) - \sum_{i=1}^{n} \log \left[1 - \exp\left\{-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right\}\right], \\ \frac{\partial \ell}{\partial \sigma} &= \frac{n\lambda}{\sigma} + (b-1)\lambda\sigma^{\lambda-1} \sum_{i=1}^{n} \frac{\exp\{-(\sigma/x_i)^{\lambda}\}}{x_i^{\lambda}\left[1 - \exp\{-(\sigma/x_i)^{\lambda}\}\right]} - a\sigma^{\lambda-1} \sum_{i=1}^{n} x_i^{-\lambda},\end{aligned}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + n\log\sigma + (b-1)\sigma^{\lambda}\sum_{i=1}^{n}\frac{\log(\sigma/x_i)\exp\{-(\sigma/x_i)^{\lambda}\}}{x_i^{\lambda}\left[1-\exp\{-(\sigma/x_i)^{\lambda}\}\right]} - \sum_{i=1}^{n}\log x_i$$
$$-a\sum_{i=1}^{n}\log\left(\frac{\sigma}{x_i}\right)\left(\frac{\sigma}{x_i}\right)^{\lambda}.$$

Solving the system of nonlinear equations $\nabla \ell = 0$, the MLEs for the BF parameters are obtained.

3.8.4 Random number generation

Using the cumulative technique, a random number X following the BF distribution with parameters a, b, λ and σ can be obtained by

$$X = \frac{\sigma}{(-log(B))^{\lambda}},$$

where B follows the standard beta distribution with parameters a and b.

Figure 3.24 plots a generated sample from the BF distribution with its respective density curve.

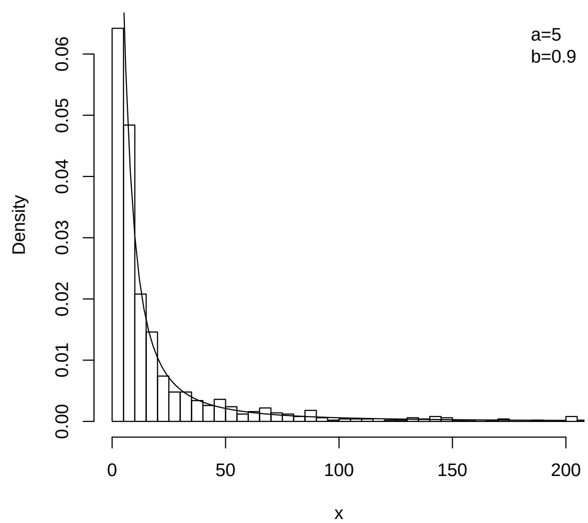

Figure 3.24: Generated values from the BF distribution with $a = 5$, $b = 0.9$ and $\lambda = \sigma = 1$.

3.8.5 Application

In this section, we fit the BF model to a real data set obtained from Smith and Naylor (1987) [24]. The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. The estimated values of the parameters were $\hat{a} = 0.407$, $\hat{b} = 238.956$, $\hat{\lambda} = 1.277$ and $\hat{\sigma} = 7.087$ and the maximized log-likelihood were $\hat{l} = -19.519$. Figure 3.25 shows the fitted pdf and the estimated quantiles versus observed quantiles.

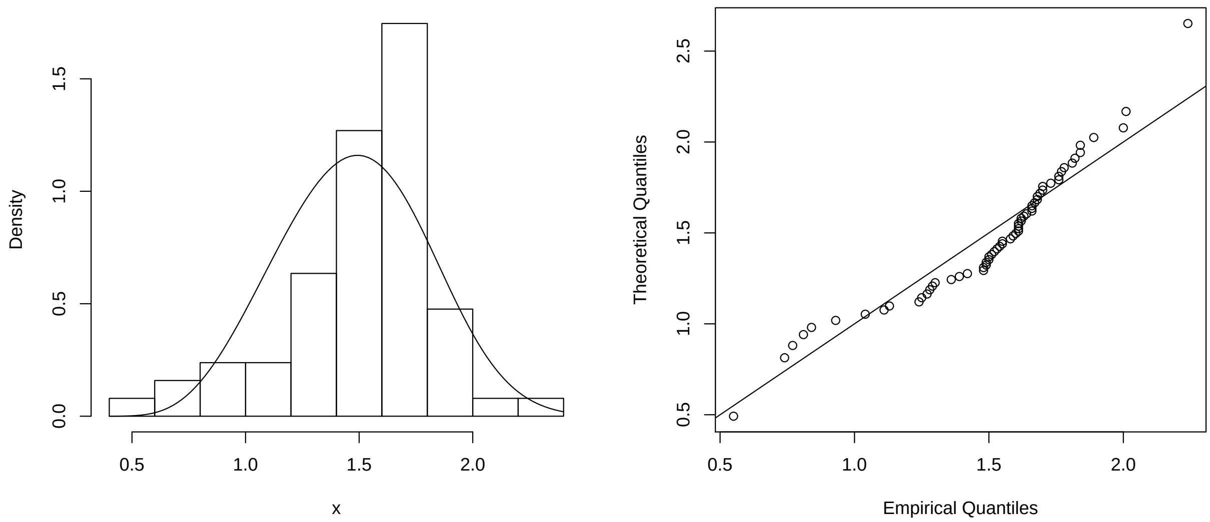

Figure 3.25: Plots of the fitted pdf and the estimated quantiles versus observed quantiles of the BF data set.

Chapter 4

Beta Generalized Logistic of Type IV

4.1 Resumo

Neste capítulo introduziremos a distribuição Beta Logística Generalizada do tipo IV, baseada na composição da distribuição beta padrão e a distribuição logística generalizada do tipo IV, proposta por Prentice (1976) [23]. Apresentaremos expressões para momentos, estimação e geração de números aleatórios. Introduziremos alguns de seus casos especiais que pertencem, também, à classe Beta-G, como as distribuições beta logística generalizada dos tipos I, II e III e algumas distribuições relacionadas, como a beta-beta prime e a beta-F.

4.2 Introduction

The generalized logistic of type IV (GL(IV)) distribution was proposed by Prentice (1976) [23] as an alternative to modeling binary response data with the usual symetric logistic distribution. The density of the GL(IV) distribution, say GL(IV)(p,q), is given by

$$g(x) = \frac{1}{B(p,q)} \frac{e^{-qx}}{(1+e^{-x})^{p+q}}, \quad x \in \mathbb{R},\, p > 0,\, q > 0.$$

The cdf of the GL(IV) distribution can be given by a normalized incomplete beta function as follows

$$G(x) = \frac{B_{\frac{1}{1+e^{-x}}}(p,q)}{B(p,q)}, \quad x \in \mathbb{R},\, p > 0,\, q > 0.$$

Note that the argument of the incomplete beta function is the cdf of the standard logistic distribution given by $G^*(x) = (1+e^{-x})^{-1}$. Hence, by the definition (2.1) of the class of generalized beta distribution, the GL(IV) distribution is already in this class.

In this chapter we introduce the beta generalized logistic of type IV and its special cases such as the beta generalized logistic of types I, II and III. Two other related distributions are also introduced and some of their properties are discussed. They are the beta-beta prime distribution and the beta F distribution.

4.3 The distribution

The beta generalized logistic of type IV (BGL(IV)) distribution has density function

$$f(x) = \frac{B(p,q)^{1-a-b}}{B(a,b)} \frac{e^{-qx}}{(1+e^{-x})^{p+q}} \left[B_{\frac{1}{1+e^{-x}}}(p,q)\right]^{a-1} \left[B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{b-1}, \; x \in \mathbb{R}, \; a, \; b, \; p, \; q > 0.$$

Figure 4.1 plots some shapes of the BGL(IV) distribution for $p = 0.2$ and $q = 1.5$ and some values of a and b.

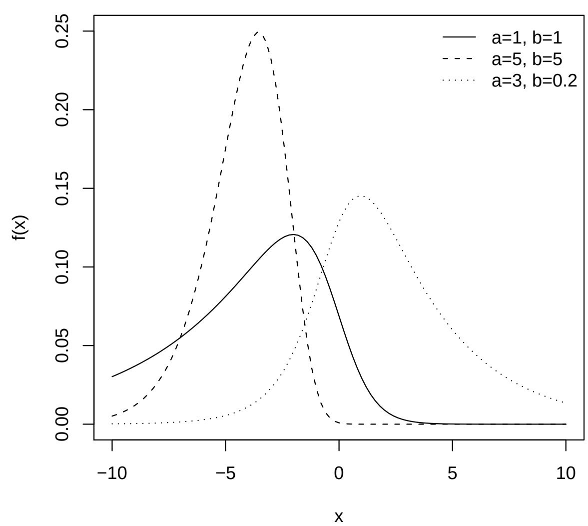

Figure 4.1: The beta generalized logistic pdf for some values of a and b with $p = 0.2$ and $q = 1.5$.

A random variable X following the BGL(IV) distribution with parameters a, b, p and q will be denoted by $X \sim BGL(IV)(a, b, p, q)$. Location and scale parameters can be included by making the transformation $Y = \frac{X-\mu}{\sigma}$, but they will not be considered in this work.

The GL(IV) distribution is symmetric when $p = q$. Hence, according to (2.4), the BGL(IV) distribution is symmetric when $p = q$ and $a = b$.

4.4 Moments

By using the expansion (2.6) and the property of the incomplete beta function (2.22), the rth moment of a random variable X following the BGL(IV)(a,b,p,q) is given by

$$
\begin{aligned}
E(X^r) &= \frac{B(p,q)^{-1}}{B(a,b)} \int_{-\infty}^{\infty} \frac{x^r e^{-qx}}{(1+e^{-x})^{p+q}} \left[I_{\frac{1}{1+e^{-x}}}(p,q)\right]^{a-1} \left[I_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{b-1} dx \\
&= \frac{B(p,q)^{-1}}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a)}{\Gamma(a-j) j!} \int_{-\infty}^{\infty} \frac{x^r e^{-qx}}{(1+e^{-x})^{p+q}} \left[I_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{b+j-1} dx \\
&= \frac{B(p,q)^{-1}}{B(a,b)} \sum_{j=0}^{\infty} w_j(1-j,a) \int_{-\infty}^{\infty} \frac{x^r e^{-qx}}{(1+e^{-x})^{p+q}} \left[I_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{b+j-1} dx,
\end{aligned}
\tag{4.1}
$$

where the index j stops at $a-1$ if a is an integer and the function $w_j(a,b)$ was defined by (2.8). If b is an integer, we consider the expansion of the incomplete beta function (2.7) and the expansion of a power series given in (2.17). Hence,

$$
\begin{aligned}
\left[I_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{b+j-1} &= \left[\frac{1}{B(p,q)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(p)}{\Gamma(p-k) k! (q+k)} \left(\frac{e^{-x}}{1+e^{-x}}\right)^{k+q}\right]^{b+j-1} \\
&= \left(\frac{e^{-x}}{1+e^{-x}}\right)^{q(b+j-1)} \sum_{k=0}^{\infty} c_{b+j-1,k} \left(\frac{e^{-x}}{1+e^{-x}}\right)^{k},
\end{aligned}
$$

where $c_{b+j-1,k} = (ka_0)^{-1} \sum_{m=1}^{k} [(b+j-1)m - m + k] a_m c_{b+j-1,k-m}$, for $k = 1, \ldots, \infty$, $c_{b+j-1,0} = a_0^{b+j-1}$ and $a_k = B(p,q)^{-1} \frac{(-1)^k \Gamma(p)}{\Gamma(p-k) k! (q+k)}$. Hence, for b an integer, the sth moment is given by

$$
\begin{aligned}
E(X^r) &= \frac{B(p,q)^{-1}}{B(a,b)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} w_j(1-j,a) c_{b+j-1,k} \int_{-\infty}^{\infty} \frac{x^r e^{-(q(b+j)+k)x}}{(1+e^{-x})^{p+k+q(b+j)}} dx \\
&= \frac{B(p,q)^{-1}}{B(a,b)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} w_j(1-j,a) c_{b+j-1,k} B(p, q(b+j)+k) E(Y^r_{p,q(b+j)+k}),
\end{aligned}
$$

where $Y_{p,q(b+j)+k} \sim GL(p, q(b+j)+k)$. Note that the order of the expected value of $Y_{p,q(b+j)+k}$ does not depend on any index and is of the same order of the expected value of X. The moment generating function of a random variable Y following the $GL(p,q)$ distribution is

$$
M(t) = \frac{\Gamma(p+t)\Gamma(q-t)}{\Gamma(p)\Gamma(q)}.
$$

If the parameter b is not an integer, it will be necessary one more expansion to use equation (2.17). Then, using (2.6) on $\left[I_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{b+j-1}$, the expected value (4.1) is given by

$$
\begin{aligned}
E(X^r) &= \frac{B(p,q)^{-1}}{B(a,b)}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} w_j(1-j,a)\frac{(-1)^i\Gamma(b+j)}{\Gamma(b+j-i)i!}\int_{-\infty}^{\infty}\frac{x^r e^{-qx}}{(1+e^{-x})^{p+q}}\left[I_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^i dx\\
&= \frac{B(p,q)^{-1}}{B(a,b)}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} w_j(1-j,a)w_i(1-i,b+j)c_{i,k}\int_{-\infty}^{\infty}\frac{x^r e^{-qx}}{(1+e^{-x})^{p(1+i)+q+k}}dx\\
&= \frac{B(p,q)^{-1}}{B(a,b)}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} J_i(a,b)c_{i,k}B(p(1+i)+k,q)E(Y^r_{p(1+i)+k,q}),
\end{aligned}
$$

where $Y_{p(1+i)+k,q} \sim GL(p(1+i)+k,q)$ and $J_i(a,b) = \sum_{j=0}^{\infty} w_j(1-j,a)w_i(1-i,b+j)$.

4.5 Estimation

Let $x_1, x_2, \ldots, x_n$ be an independent random sample from the BGL(IV) distribution. The total log-likelihood is given by

$$
\begin{aligned}
\ell = \ell(a,b,p,q;x) &= -n\log B(a,b) + (1-a-b)n\log B(p,q) - (p+q)\sum_{i=1}^{n}\log(1+e^{-x_i})\\
&\quad +(a-1)\sum_{i=1}^{n}\log B_{\frac{1}{1+e^{-x_i}}}(p,q) + (b-1)\sum_{i=1}^{n}\log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p) - q\sum_{i=1}^{n}x_i.
\end{aligned}
$$

The score function has the following components

$$
\begin{aligned}
\frac{\partial\ell}{\partial a} &= n\psi(a+b) - n\psi(a) - n\log B(p,q) + \sum_{i=1}^{n}\log B_{\frac{1}{1+e^{-x_i}}}(p,q),\\
\frac{\partial\ell}{\partial b} &= n\psi(a+b) - n\psi(b) - n\log B(p,q) + \sum_{i=1}^{n}\log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p),\\
\frac{\partial\ell}{\partial p} &= (1-a-b)n\{\psi(p)-\psi(p+q)\} - \sum_{i=1}^{n}\log(1+e^{-x_i})\\
&\quad +(a-1)\frac{\partial}{\partial p}\sum_{i=1}^{n}\log B_{\frac{1}{1+e^{-x_i}}}(p,q) + (b-1)\frac{\partial}{\partial p}\sum_{i=1}^{n}\log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p),\\
\frac{\partial\ell}{\partial q} &= (1-a-b)n\{\psi(q)-\psi(p+q)\} - \sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\log(1+e^{-x_i})\\
&\quad +(a-1)\frac{\partial}{\partial q}\sum_{i=1}^{n}\log B_{\frac{1}{1+e^{-x_i}}}(p,q) + (b-1)\frac{\partial}{\partial q}\sum_{i=1}^{n}\log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p).
\end{aligned}
$$

Solving the system of nonlinear equations $\nabla \ell = 0$, the MLEs of the parameters are obtained. The Hessian matrix for making interval inference and used in the estimation equations has elements given by

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial a^2} &= n\psi'(a+b) - n\psi'(a), \\
\frac{\partial^2 \ell}{\partial a \partial b} &= n\psi'(a+b), \\
\frac{\partial^2 \ell}{\partial a \partial p} &= n\psi(p+q) - n\psi(p) + \frac{\partial}{\partial p} \sum_{i=1}^{n} \log B_{\frac{1}{1+e^{-x_i}}}(p,q), \\
\frac{\partial^2 \ell}{\partial a \partial q} &= n\psi(p+q) - n\psi(q) + \frac{\partial}{\partial q} \sum_{i=1}^{n} \log B_{\frac{1}{1+e^{-x_i}}}(p,q), \\
\frac{\partial^2 \ell}{\partial b^2} &= n\psi'(a+b) - n\psi'(b), \\
\frac{\partial^2 \ell}{\partial b \partial p} &= n\psi(p+q) - n\psi(p) + \frac{\partial}{\partial p} \sum_{i=1}^{n} \log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p), \\
\frac{\partial^2 \ell}{\partial b \partial q} &= n\psi(p+q) - n\psi(q) + \frac{\partial}{\partial q} \sum_{i=1}^{n} \log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p),
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial p^2} &= (1-a-b)n\{\psi'(p)-\psi'(p+q)\} + (a-1)\frac{\partial^2}{\partial p^2}\sum_{i=1}^{n}\log B_{\frac{1}{1+e^{-x_i}}}(p,q), \\
&+(b-1)\frac{\partial^2}{\partial p^2}\sum_{i=1}^{n}\log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p) \\
\frac{\partial^2 \ell}{\partial p \partial q} &= -(1-a-b)n\psi'(p+q) + (a-1)\frac{\partial^2}{\partial p \partial q}\sum_{i=1}^{n}\log B_{\frac{1}{1+e^{-x_i}}}(p,q) \\
&+(b-1)\frac{\partial^2}{\partial p \partial q}\sum_{i=1}^{n}\log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p), \\
\frac{\partial^2 \ell}{\partial q^2} &= (1-a-b)n\{\psi'(q)-\psi'(p+q)\} + (a-1)\frac{\partial^2}{\partial q^2}\sum_{i=1}^{n}\log B_{\frac{1}{1+e^{-x_i}}}(p,q) \\
&+(b-1)\frac{\partial^2}{\partial q^2}\sum_{i=1}^{n}\log B_{\frac{e^{-x_i}}{1+e^{-x_i}}}(q,p).
\end{aligned}
$$

4.6 Random number generation

To generate a random number following the BGL(IV) distribution, the cumulative technique presented in Section 2.7 is used.

Let X be a BGL(IV)(a,b,p,q) random variable. By the definition 2.1 of the class Beta-G distribution, the cdf of X is the composition $F(x) = F_1(G(x))$, where F_1 is the cdf of the Beta(a,b) distribution and G is the cdf of the parent distribution, GL(IV)(p,q). The cdf G can also be expressed as a composition $G(x) = F_2(Q(x))$, where F_2 is the cdf of the Beta(p,q) distribution and $Q(x) = (1+e^{-x})^{-1}$ is the cdf of the standard logistic distribution. Hende, the cdf of X can be expressed as $F(x) = F_1(F_2(Q(x)))$ and by the cumulative technique of a random generation, we have

$$
\begin{aligned}
F(X) &= F_1(F_2(Q(X)) = U, \;\; U \sim \text{Uniform}(0,1) \\
F_2(Q(X)) &= F_1^{-1}(U) = B, \;\; B \sim \text{Beta}(a,b) \\
Q(X) &= F_2^{-1}(B) = C \\
X &= -\log(C^{-1}-1),
\end{aligned}
$$

where C is the quantile B of a Beta(p,q).

Figure 4.6 plots some samples of the BGL(IV) distribution with its respective density curve.

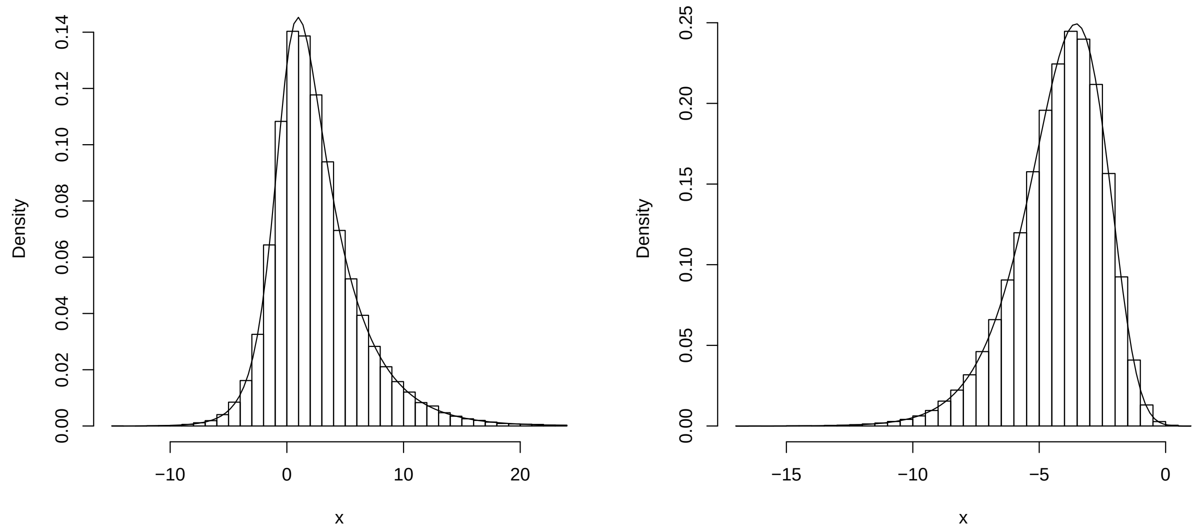

Figure 4.2: Generated samples of the BGL(IV) distribution for some values of a and b with $p = 0.2$ and $q = 1.5$.

4.7 Related distributions

4.7.1 Beta Generalized Logistic of Type I

The generalized logistic of type I is a special case of the GL(IV) distribution when $q = 1$. This property extends to the class of beta generalized distribution, i.e. the beta generalized logistic of type I (BGL(I)) distribution is a special case of the BGL(IV) distribution when $q = 1$. Then, the density of the BGL(I) is given by

$$f(x) = \frac{pe^{-x}}{B(a,b)} \frac{[(1+e^{-ax})^p - 1]^{b-1}}{(1+e^{-x})^{a+pb}},\ x \in \mathbb{R},\ a, b, p > 0.$$

Figure 4.3 plots some special shapes of the BGL(I) distribution, for $p = 0.2$ and some values of a and b.

A random variable X following a BGL(I) distribution with parameters a, b and p will be denoted by $X \sim BGL(I)(a, b, p)$. We can verify that if the parameter $p = 1$, the BGL(I)(a,b,p) distribution coincides with the GL(IV)(a,b) distribution. In fact, when $p = 1$, the primitive distribution of the BGL(I), the GL(I) distribution, reduces to the standard logistic which is the primitive distribution of the GL(IV) distribution.

To generate a random number X following the BGL(I)(a,b,p) distribution, we have to generate a random number $B \sim \text{Beta}(a, b)$ and then $X = -\log(B^{-1/p} - 1)$. Others properties of this distribution will be omitted here because they can be obtained from the BGL(IV) distribution by making $q = 1$.

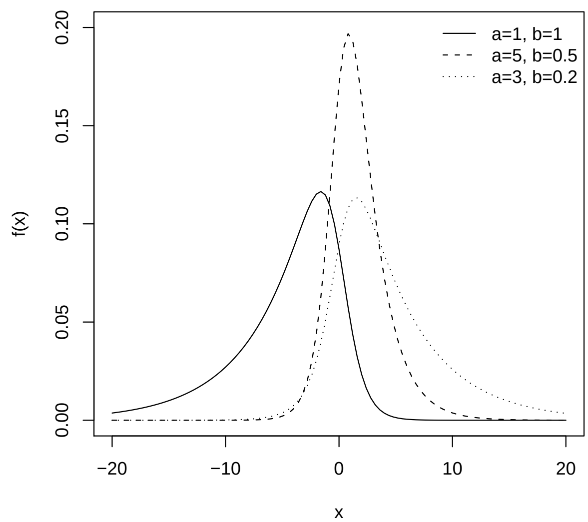

Figure 4.3: The BGL(I) pdf for some values of a and b with $p = 0.2$.

4.7.2 Beta Generalized Logistic of Type II

The beta generalized logistic of type II is a special case of the BGL(IV) distribution when $p = 1$. This distribution can also be obtained through the transformation $X = -Y$ where Y follows the BGL(I) distribution. The density is given by

$$f(x) = \frac{qe^{-bqx}}{B(a,b)(1+e^{-x})^{qb+1}}\left[1 - \frac{e^{-qx}}{(1+e^{-x})^q}\right], \; x \in \mathbb{R}, \; a, b, q > 0.$$

Figure 4.4 plots some special shapes of the BGL(II) distribution for $q = 0.5$ and some values of a and b.

A random variable X following the BGL(II) distribution with parameters a, b and q will be denoted by $X \sim \text{BGL(II)}(a, b, q)$.

4.7.3 Beta Generalized Logistic of Type III

The beta generalized logistic of type III is a special case of the BGL(IV) when $p = q$. This distribution is symmetric when $a = b$. The BGL(III) density is given by

$$f(x) = \frac{B(p,p)^{1-a-b}}{B(a,b)} \frac{e^{-px}}{(1+e^{-x})^{2p}} \left[B_{\frac{1}{1+e^{-x}}}(p,p)\right]^{a-1} \left[B_{\frac{e^{-x}}{1+e^{-x}}}(p,p)\right]^{b-1}, \quad x \in \mathbb{R}, \; a, \; b, \; p > 0.$$

Figure 4.5 plots some special shapes of the BGL(III) distribution for $p = 0.5$ and some values of a and b.

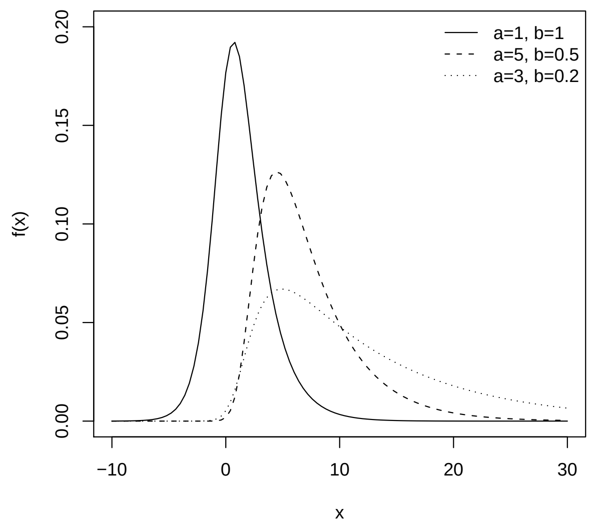

Figure 4.4: The BGL(II) pdf for some values of a and b with $p = 0.2$.

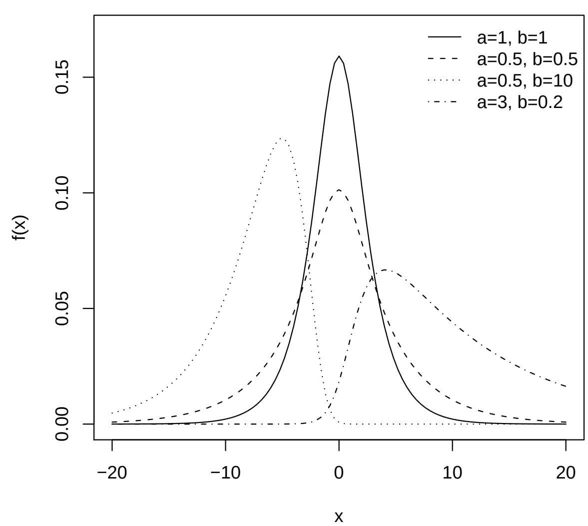

Figure 4.5: The BGL(III) pdf for some values of a and b with $p = 0.5$.

A random variable X following the BGL(III) distribution with parameters a, b and p will be denoted by $X \sim \text{BGL(III)}(a, b, p)$.

4.7.4 Beta-Beta Prime

Let Y be a random variable following the BGL(IV)(a,b,p,q) distribution. Then, $X = e^{-Y}$ follows the beta-beta prime distribution with parameters b, a, p and q. The density is given by

$$f(x) = \frac{B(p,q)^{1-a-b}}{B(a,b)} \frac{x^{q-1}}{(1+x)^{p+q}} \left[B_{\frac{x}{1+x}}(q,p)\right]^{(a-1)} \left[B_{\frac{1}{1+x}}(p,q)\right]^{(b-1)}, \ a,b,p,q,x > 0.$$

Figure 4.6 plots some shapes of the beta-beta prime (BBP) distribution for some values of b, a, p and q.

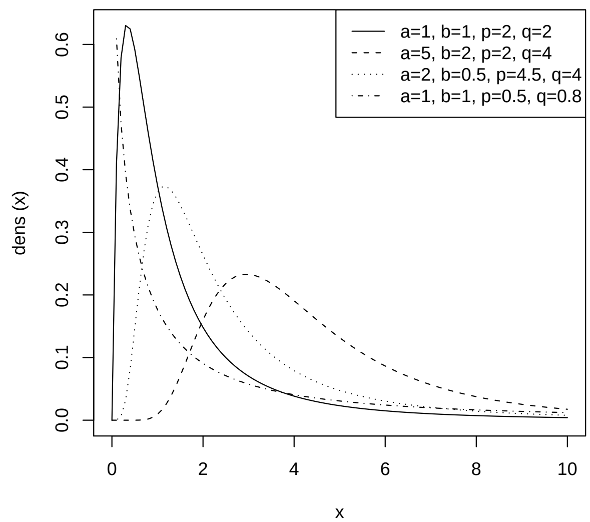

Figure 4.6: The BBP pdf for some values of b, a, p and q.

A random variable X following the BBP distribution with parameters a, b, p and q will be denoted by $X \sim \text{BBP}(b,a,p,q)$.

For this distribution the support is $(0,\infty)$. Thus, it makes sense to study the hazard function which is given by

$$h(x) = \frac{B(p,q)^{1-a-b}}{B_{I_{\frac{x}{1+x}}(q,p)}(a,b)} \frac{x^{q-1}}{(1+x)^{p+q}} \left[B_{\frac{x}{1+x}}(q,p)\right]^{(a-1)} \left[B_{\frac{1}{1+x}}(p,q)\right]^{(b-1)}, \ a,b,p,q,x > 0.$$

4.7.5 Beta-F

The density of the F distribution is given by

$$g(x) = \frac{(q/p)^{(q/2)}}{B(p/2,q/2)} \frac{x^{q/2-1}}{(1+(q/p)x)^{(p+q)/2}}, \quad p,q,x > 0$$

and its cdf can be written as

$$G(x) = \frac{1}{B(p/2,q/2)} B_{\frac{(q/p)x}{1+(q/p)x}}(q/2,p/2).$$

The Beta-F (B-F) distribution is given by

$$f(x) = \frac{B(a,b)^{-1}}{B(p/2,q/2)} \frac{(q/p)^{q/2} x^{q/2-1}}{(1+(q/p)x)^{(p+q)/2}} \left[I_{\frac{(q/p)x}{1+(q/p)x}}(q/2,p/2) \right]^{a-1} \left[I_{\frac{1}{1+(q/p)x}}(p/2,q/2) \right]^{b-1},$$

with a, b, p, q, $x > 0$. A random variable X following a B-F distribution with parameters a, b, p and q will be denoted by $X \sim$ B-F(a,b,p,q). It is easy to verify that if $Y \sim$ BGL(IV)(a,b,p,q), then $F = \frac{p}{q} e^{-Y} \sim$ B-F$(a,b,2p,2q)$.

Figure 4.7 plots some shapes of the B-F distribution for $p = 2$, $q = 10$ and some values of a and b.

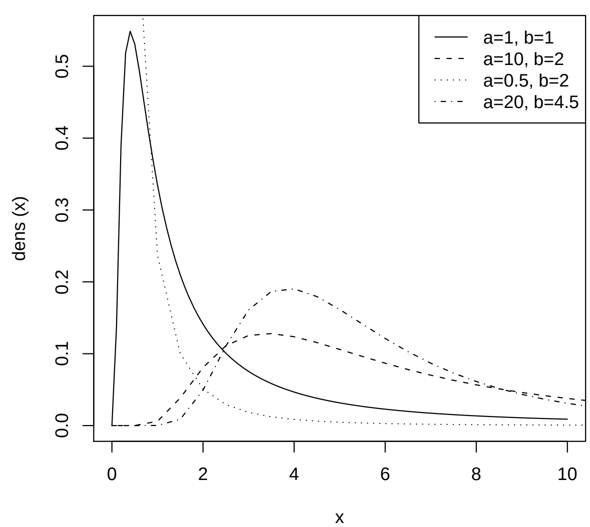

Figure 4.7: The B-F pdf for some values of b and a and $p = 2$ and $q = 10$.

4.8 Application

The INPC is a national index of consumer prices of Brazil, produced by the IBGE since 1979. The period of collection extends from the day 01 to 30 of the reference month. The

INPC measures the cost of living of households with heads employees. The search is done in the metropolitan regions of Rio de Janeiro, Porto Alegre, Belo Horizonte, Recife, São Paulo, Belém, Fortaleza, Salvador and Curitiba, in addition to Brasília and the city of Goiânia. This index can be found on *http://www.ibge.gov.br/series_estatisticas/exibedados.php?idnivel=BR &idserie=PRECO101*. For this data set, the estimated parameters are $\hat{a} = 12.95$, $\hat{b} = 0.37$, $\hat{p} = 4.09$ and $\hat{q} = 8.74$ and the maximized log-likelihood were $\hat{l} = -120.26$. Figure 4.8 shows a good fit of the BGL(IV) distribution for this data set.

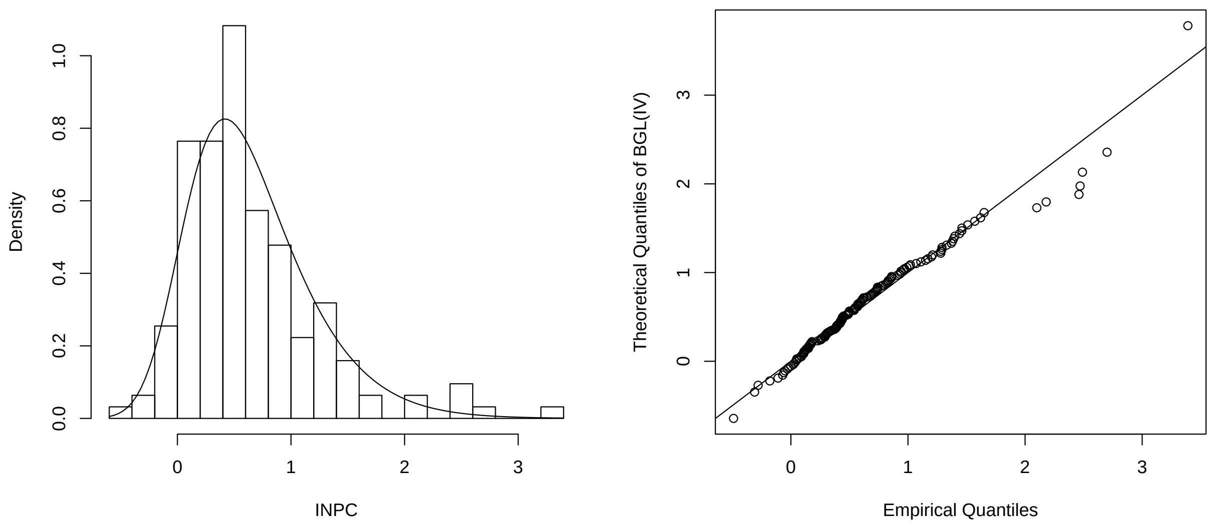

Figure 4.8: Plots of the fitted pdf and of the estimated quantiles versus theoretical quantiles of the BGL(IV) data set.

Chapter 5

Pareto Power Series

5.1 Resumo

Neste capítulo introduzimos uma nova classe de distribuições, Pareto power series, baseada em uma composição da distribuição de Pareto e a classe de distribuições power series. As distribuições nesta classe terão dois parâmetros desconhecidos e um parâmetro conhecido μ, suporte (μ, ∞), função de risco decrescente e a distribuição de Pareto como caso limite. São, também, misturas da Pareto com pesos definidos pela distribuição power series. Apresentaremos algumas propriedades tais como o s-ésimo momento, função de risco, entropia de Rényi, densidade e momentos da estatística de ordem, estimação, casos especiais e uma aplicação.

5.2 Introduction

We now introduce a new class of distributions based on a composition of the Pareto distribution and the power series class of distributions, called Pareto power series class of distributions. A distribution in this class, PPS distribution say, has two unknown parameters, a known parameter μ, suport (μ, ∞), decreasing failure rate and the Pareto distribution as a limiting case. The PPS distribution is a mixture of the Pareto distribution with weights defined by the power series distribution. We give the sth moment of a distribution in this class, its failure rate, Rényi entropy, density and moments of the order statistics, some special cases and an application.

5.3 Definition

Let X_1, X_2, $\ldots$, X_N be independent random variables following the Pareto distribution whose density is given by

$$g(x) = \alpha\mu^{\alpha}x^{-(\alpha+1)}, \quad x \geq \mu > 0,\, \alpha > 0.$$

The index N is a random variable following a distribution in the power series class of distributions whose the probability mass function is given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \ldots, \tag{5.1}$$

where a_n depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ and $\theta > 0$ such that $C(\theta)$ is finite. Just to remark, the random variable N can not follow any power series distribution, just those who support is the set of natural numbers $\mathbb{N}$.

Let $X_{(1)} = \min(X_1, \ldots, X_N)$. The cdf of $X_{(1)}|N = n$ is given by

$$G_{X_{(1)}|N=n}(x) = 1 - \left(\frac{\mu}{x}\right)^{\alpha n}, \quad x \geq \mu,$$

i.e., $X_{(1)}|N = n$ follows a Pareto distribution with known parameter μ and shape parameter αn. The joint probability function of $X_{(1)}, N$ is given by

$$g_{X,N}(x, n) = \frac{n a_n \alpha}{x C(\theta)} \left(\frac{\theta \mu^\alpha}{x^\alpha}\right)^n, \quad x \geq \mu.$$

The density of a Pareto power series (PPS) distribution is defined by the marginal density of $X_{(1)}$

$$f(x) = \frac{\alpha}{x C(\theta)} \sum_{n=1}^{\infty} n a_n \left(\frac{\theta \mu^\alpha}{x^\alpha}\right)^n, \quad x \geq \mu. \tag{5.2}$$

Note that the function $C(\theta)$ is decreasing on θ and $\frac{\mu^\alpha}{x^\alpha} < 1$. So, $C\left(\frac{\theta\mu^\alpha}{x^\alpha}\right) = \sum_{n=1}^{\infty} a_n \left(\frac{\theta\mu^\alpha}{x^\alpha}\right)^n$ is convergent and

$$E(Y) = \sum_{n=1}^{\infty} n \frac{a_n \left(\frac{\theta\mu^\alpha}{x^\alpha}\right)^n}{C\left(\frac{\theta\mu^\alpha}{x^\alpha}\right)},$$

where Y follows a power series distribution with parameter $\theta \frac{\mu^\alpha}{x^\alpha}$. The generating moment function of a power series distribution is given by

$$M(t) = \frac{C(\theta e^t)}{C(\theta)}.$$

Hence,

$$E(Y) = M'(0) = \frac{C'(\theta)}{C(\theta)} \theta$$

and the density in (5.2) can be rewritten as

$$f(x) = \frac{\alpha \theta \mu^\alpha}{x^{\alpha+1}} \frac{C'\left(\theta \frac{\mu^\alpha}{x^\alpha}\right)}{C(\theta)}, \quad x \geq \mu.$$

The density of the PPS distribution is a mixture of the Pareto distribution and it can be rewritten as

$$f(x)=\sum_{n=1}^{\infty}P(N=n)f_n(x),\quad x\geq\mu, \tag{5.3}$$

where $f_n(x)$ is the density function of a Pareto distribution with unknown parameter $n\alpha$. The Pareto distribution is a limiting special case of the PPS distribution when $\theta\to 0^+$. In fact, let N be a power series random variable. By using the Lopital rule once, for $n=2,3,\ldots$,

$$\lim_{\theta\to 0^+}P(N=n)=\lim_{\theta\to 0^+}\frac{a_n\theta^n}{\sum_{i=1}^{\infty}a_i\theta^i}=\lim_{\theta\to 0^+}\frac{na_n\theta^{n-1}}{a_1+\sum_{i=2}^{\infty}ia_i\theta^{i-1}}=0$$

and for $n=1$, $\lim_{\theta\to 0^+}P(N=n)=\lim_{\theta\to 0^+}\frac{a_1}{a_1+\sum_{i=2}^{\infty}ia_i\theta^{i-1}}=1$. Thus, when $\theta\to 0^+$ the power series converges to a distribution degenerated at one and then from (5.3) the PPS distribution leads to the Pareto distribution.

The cdf of a Pareto power series distribution is given by

$$F(x)=1-\frac{C\left(\theta\left(\frac{\mu}{x}\right)^{\alpha}\right)}{C(\theta)},\quad x\geq\mu.$$

5.4 Hazard function

The hazard function is defined by the ratio $h(x)=\frac{f(x)}{S(x)}$, where $S(x)=1-F(x)$ is the survival function. Hence, a Pareto power series distribution has hazard function

$$h(x)=\frac{\alpha\theta\mu^{\alpha}}{x^{\alpha+1}}\frac{C'\left(\theta\left(\frac{\mu}{x}\right)^{\alpha}\right)}{C\left(\theta\left(\frac{\mu}{x}\right)^{\alpha}\right)},\quad x\geq\mu,.$$

Diferentiating $h(x)$ once, we can conclude that the hazard function of a Pareto power series is increasing on x for all values of μ, α and θ.

5.5 Order statistics

Let X_1, …, X_n be random variables iid such that $X_i\sim\text{PPS}(\mu,\alpha,\theta)$ for $i=1,\ldots,n$. The pdf of the ith order statistic, $X_{i:n}$ say, is given by

$$f_{i:n}(x)\quad=\quad\frac{n!f(x)}{(n-i)!(i-1)!}\left[1-\frac{C\left(\theta\left(\frac{\mu}{x}\right)^{\alpha}\right)}{C(\theta)}\right]^{i-1}\left[\frac{C\left(\theta\left(\frac{\mu}{x}\right)^{\alpha}\right)}{C(\theta)}\right]^{n-i}$$

To obtain some mathematical properties of the order statistic, using (2.6) and (2.17), the density of $X_{i:n}$ can be rewritten as

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!f(x)}{(n-i)!(i-1)!}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\frac{C\left(\theta\left(\frac{\mu}{x}\right)^{\alpha}\right)^{j+n-i}}{C(\theta)^{j+n-i}} \\
&= \frac{n!f(x)}{(n-i)!(i-1)!}\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}C_{j+n-i,k}C(\theta)^{i-j-n}\theta^{k+1}\left(\frac{\mu}{x}\right)^{\alpha k+1} \\
&= \frac{n!f(x)}{(n-i)!(i-1)!}\sum_{k=0}^{\infty}T_k(i,n,\mu,\alpha,\theta)x^{\alpha k+1},
\end{aligned} \tag{5.4}$$

where $c_{j+n-i,k} = (ka_1)^{-1}\sum_{r=1}^{k}((j+n-i)r-k+r)a_{r+1}C_{j+n-i,k-r}$, $C_{j+n-i,0} = a_1^{j+n-i}$, a_r is defined on(5.1) and $T_k(i,n,\mu,\alpha,\theta) = \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}C_{j+n-i,k}C(\theta)^{i-j-n}\theta^{k+1}\mu^{\alpha k+1}$.

5.6 Quantiles and Moments

The quantile γ of a PPS distribution is given by

$$x_\gamma = \mu\left[\theta C^{-1}\left\{C(\theta)(1-\gamma)\right\}\right]^{-1/\alpha},$$

where $C^{-1}(\cdot)$ is the inverse function of $C(\cdot)$.

Let X be a random variable following a PPS distribution. To obtain the rth moment of X, consider the first derivative of $C(\theta)$

$$C'(\theta) = \frac{d}{d\theta}\sum_{n=1}^{\infty}a_n\theta^n = \sum_{n=1}^{\infty}na_n\theta^{n-1}.$$

Then, the rth moment is given by

$$\begin{aligned}
E(X^r) &= \int_{\mu}^{\infty}x^r\frac{\alpha\theta\mu^\alpha}{x^{\alpha+1}}\frac{C'\left(\frac{\theta\mu^\alpha}{x^\alpha}\right)}{C(\theta)}dx \\
&= \frac{\alpha\theta\mu^\alpha}{C(\theta)}\sum_{n=1}^{\infty}na_n\theta^{n-1}\mu^{\alpha(n-1)}\int_{\mu}^{\infty}x^{r-\alpha n-1}dx \\
&= \frac{\alpha\mu^r}{C(\theta)}\sum_{n=1}^{\infty}\frac{na_n\theta^n}{\alpha n-r}, \quad \text{if } r<\alpha.
\end{aligned}$$

From (5.4), the rth moment of the ith order statistic $X_{i:n}$ is given by

$$E(X_{i:n}^r) = \frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{\infty} T_k(i,n,\mu,\alpha,\theta)E(Y^{r-\alpha k-1}),$$

where $Y \sim \text{PPS}(\mu,\alpha,\theta)$.

The mean residual lifetime (MRL) of a random variable X is the conditional expected value of $X|X>t$. If $\alpha > 1$, by using the expansion of $C(\theta)$, the MRL of the PPS distribution is given by

$$\begin{aligned}\text{MRL}(t) &\equiv \frac{\int_t^\infty S(x)dx}{S(t)} \\ &= \frac{\int_t^\infty C\left(\theta\left(\frac{\mu}{x}\right)^\alpha\right)dx}{C\left(\theta\left(\frac{\mu}{t}\right)^\alpha\right)} \\ &= \frac{\sum_{n=1}^\infty \frac{a_n}{\alpha n-1}\theta^n\mu^{\alpha n}t^{1-\alpha n}}{C\left(\theta\left(\frac{\mu}{t}\right)^\alpha\right)}.\end{aligned}$$

5.7 Rényi Entropy

An entropy of a random variable X is a measure of variation of the uncertainty. Rényi entropy is defined by $I_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\int_{\mathbb{R}} f^\gamma(x)dx\right\}$, where $\gamma > 0$ and $\gamma \neq 1$.

By using the expansion of $C'(\theta)$ and the expansion (2.17), if $1-\alpha-(\alpha+1)\gamma < 0$ the Rényi entropy of the PPS distribution is given by

$$\begin{aligned}I_R(\gamma) &= \frac{1}{1-\gamma}\log\left\{\int_\mu^\infty \left(\frac{\alpha\theta\mu^\alpha}{C(\theta)}\right)^\gamma \frac{C'\left(\theta\left(\frac{\mu}{x}\right)^\alpha\right)^\gamma}{x^{\gamma(\alpha+1)}}\right\} \\ &= \frac{1}{1-\gamma}\log\left\{\sum_{n=0}^\infty \frac{\alpha^\gamma\theta^{n+\gamma}\mu^{1-\gamma}c_{\gamma,n}}{C(\theta)(\alpha n+(\alpha+1)\gamma+1)}\right\},\end{aligned}$$

where $c_{\gamma,n} = (na_1)^{(-1)}\sum_{m=1}^i(\gamma m - k + m)(m+1)a_{m+1}c_{\gamma,n-m}$ for $n = 1,2,\ldots$ and $c_{\gamma,0} = a_1^\gamma$.

5.8 Random Number Generation

From the cumulative technique, a random number X following the PPS distribution is obtained from a standard uniform random variable U by

$$X = \mu\left[\theta^{-1}C^{-1}\{C(\theta)U\}\right]^{-1/\alpha},$$

where $C^{-1}(\cdot)$ is the inverse function of $C(\cdot)$.

5.9 Estimation

Let $x_1, \ldots, x_n$ be an independent random sample from the PPS distribution. The total log-likelihood function is given by

$$\ell = \ell(x; \alpha, \theta) = +n \log \alpha + n \log \theta + n\alpha \log \mu - n \log C(\theta) + \sum_{i=1}^{n} \log C' \left\{ \theta \left(\frac{\mu}{x_i} \right)^{\alpha} \right\} - \sum_{i=1}^{n} (\alpha + 1) \log x_i.$$

The components of the score function are

$$\frac{\partial \ell}{\partial \alpha} = -\sum_{i=1}^{n} \log(x_i) + n\frac{1}{\alpha} + n\log(\mu) + \sum_{i=1}^{n} \theta \left(\frac{\mu}{x_i} \right)^{\alpha} \log \left(\frac{\mu}{x_i} \right) C' \left(\frac{\theta\mu^{\alpha}}{x_i^{\alpha}} \right)^{-1} C'' \left(\frac{\theta\mu^{\alpha}}{x_i^{\alpha}} \right),$$

$$\frac{\partial \ell}{\partial \theta} = n\frac{1}{\theta} - n\frac{C'(\theta)}{C(\theta)} + \sum_{i=1}^{n} \left(\frac{\mu}{x_i} \right)^{\alpha} C' \left(\frac{\theta\mu^{\alpha}}{x_i^{\alpha}} \right)^{-1} C'' \left(\frac{\theta\mu^{\alpha}}{x_i^{\alpha}} \right).$$

5.9.1 EM algorithm

To obtain the estimates of the parameters we propose the EM algorithm (Dempster et al. [6], 1977; McLachlan and Krishnan [18], 1997). This is a recurrent method such that each step consists of estimating the expected value of a hypothetical random variable and later maximize the log-likelihood of the complete data. Let the complete data be $x_1, \ldots, x_n$ and the hypothetical random variables, $z_1, \ldots, z_n$. The joint probability function is such that the marginal joint density of $x_1, \ldots, x_n$ is the likelihood of interest. Then, consider the joint probability function

$$g_{X,Z}(x, z) = \frac{z a_z \alpha}{x C(\theta)} \left(\frac{\theta\mu^{\alpha}}{x^{\alpha}} \right)^{z}, \quad x \geq \mu, \; Z = 1, 2, \ldots.$$

The marginal distribution of Z is given by

$$P(Z = z | X = x) = z a_z \theta^{z-1} \mu^{\alpha(z-1)} x^{-\alpha(z-1)} C' \left(\frac{\theta\mu^{\alpha}}{x^{\alpha}} \right)^{-1},$$

and then its expected value is

$$\begin{aligned} E(Z|X = x) &= \frac{\theta^{-1}\mu^{-\alpha}x^{\alpha}}{C'\left(\frac{\theta\mu^{\alpha}}{x^{\alpha}}\right)} \sum_{z=1}^{\infty} z^2 a_z \left(\frac{\theta\mu^{\alpha}}{x^{\alpha}} \right)^{z} \\ &= \frac{\theta\mu^{\alpha}}{x^{\alpha}} \frac{C''\left(\frac{\theta\mu^{\alpha}}{x^{\alpha}}\right)}{C'\left(\frac{\theta\mu^{\alpha}}{x^{\alpha}}\right)} + 1. \end{aligned}$$

The log-likelihood of the model parameters for the complete data set is

$$\ell(\alpha, \theta; x, z) \propto n \log \alpha - n \log C(\theta) + \sum_{i=1}^{n} z_i \log\left(\frac{\theta \mu^\alpha}{x_i^\alpha}\right)$$

and the components of the score vector are

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} z_i \log\left(\frac{\mu}{x_i}\right), \\
\frac{\partial \ell}{\partial \theta} &= -n\frac{C'(\theta)}{C(\theta)} + \frac{1}{\theta}\sum_{i=1}^{\infty} z_i.
\end{aligned}$$

The iterative algorithm to obtain the estimates of the parameters is given by

$$\begin{aligned}
\hat{\alpha}^{(t+1)} &= \frac{n}{\sum_{i=1}^{\infty} z_i^{(t)} \log\left(\frac{x_i}{\mu}\right)}, \\
\hat{\theta}^{(t+1)} &= \frac{C(\hat{\theta}^{(t+1)})}{nC'(\hat{\theta}^{(t+1)})}\sum_{i=1}^{n} z_i^{(t)},
\end{aligned}$$

where

$$z_i^{(t)} = \frac{\hat{\theta}^{(t)} \mu^{\hat{\alpha}^{(t)}}}{x_i^{\hat{\alpha}^{(t)}}} \frac{C''\left(\frac{\hat{\theta}^{(t)} \mu^{\hat{\alpha}^{(t)}}}{x_i^{\hat{\alpha}^{(t)}}}\right)}{C'\left(\frac{\hat{\theta}^{(t)} \mu^{\hat{\alpha}^{(t)}}}{x_i^{\hat{\alpha}^{(t)}}}\right)} + 1.$$

5.10 Some special distributions

We will now introduce some special cases of the PPS class of distributions. The Poisson, logarithmic and geometric distributions are power series distributions with suport $\mathbb{N}$. The following table summarizes these distributions according to (5.1).

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(u)$	Θ
Poisson	$n!^{-1}$	$e^\theta - 1$	e^θ	e^θ	$\log(u+1)$	$\theta \in (0, \infty)$
Logarithmic	n^{-1}	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1 - e^{-u}$	$\theta \in (0, 1)$
Geometric	1	$\frac{\theta}{1-\theta}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\frac{u}{u+1}$	$\theta \in (0, 1)$

Table 5.1: Summary of the Poisson, logarithmic and geometric distributions.

5.10.1 Pareto Poisson distribution

The density of the Pareto Poisson (PP) distribution is given by

$$f(x) = \frac{\alpha\theta\mu^{\alpha}}{x^{\alpha+1}} \frac{e^{\theta\left(\frac{\mu}{x}\right)^{\alpha}}}{e^{\theta}-1},\ \theta, \alpha, x > 0$$

and the cdf is

$$F(x) = \frac{e^{\theta} - e^{\theta\left(\frac{\mu}{x}\right)^{\alpha}}}{e^{\theta}-1}.$$

The hazard function of the PP distribution is given by

$$h(x) = \frac{\alpha\theta\mu^{\alpha}}{x^{\alpha+1}} \left\{1 - e^{-\theta\left(\frac{\mu}{x}\right)^{\alpha}}\right\}^{-1},\ \theta, \alpha, x > 0.$$

Figure 5.1 plots some PP densities and hazard functions, with $\mu = 1$ and some values of α and θ.

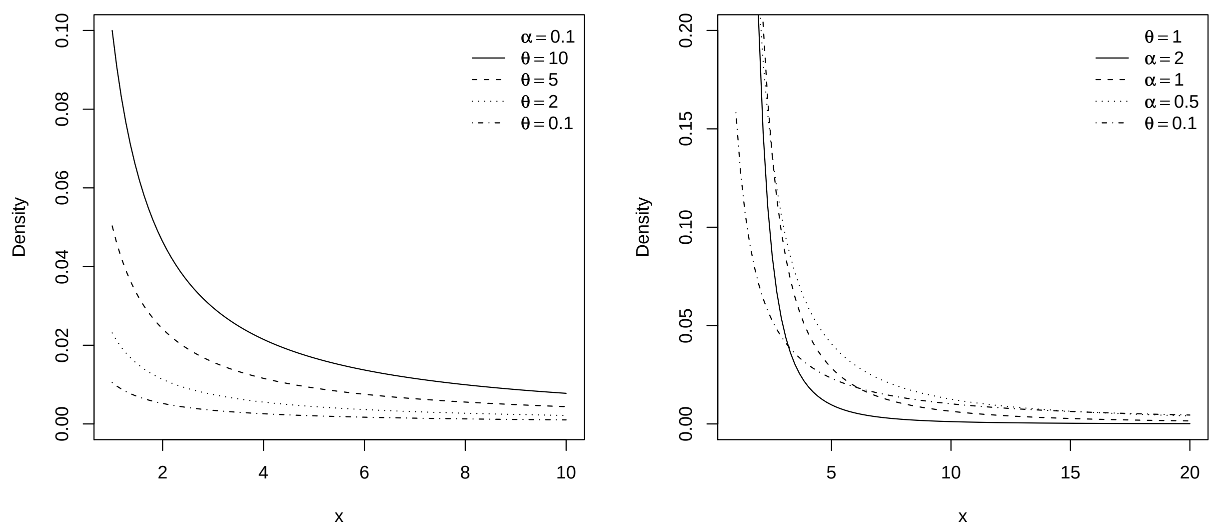

Figure 5.1: The PP pdf for some values of α and θ with $\mu = 1$.

5.10.2 Pareto logarithmic distribution

The density of the Pareto logarithmic (PL) distribution is given by

$$f(x) = \frac{\alpha\theta\mu^{\alpha}}{x^{\alpha+1}} \frac{\left\{\theta\left(\frac{\mu}{x}\right)^{\alpha} - 1\right\}^{-1}}{\log(1-\theta)},\ \theta, \alpha, x > 0$$

and the cdf is

$$F(x) = 1 - \frac{\log\left(1 - \theta\left(\frac{\mu}{x}\right)^{\alpha}\right)}{\log(1-\theta)}.$$

The hazard function of the PL distribution is given by

$$h(x) = \frac{\alpha\theta\mu^{\alpha}}{x^{\alpha+1}} \frac{\left\{\theta\left(\frac{\mu}{x}\right)^{\alpha} - 1\right\}^{-1}}{\log(1-\theta\left(\frac{\mu}{x}\right)^{\alpha})},\ \theta, \alpha, x > 0$$

Figure 5.2 plots some PP densities and hazard functions, with $\mu = 1$ and some values of α and θ.

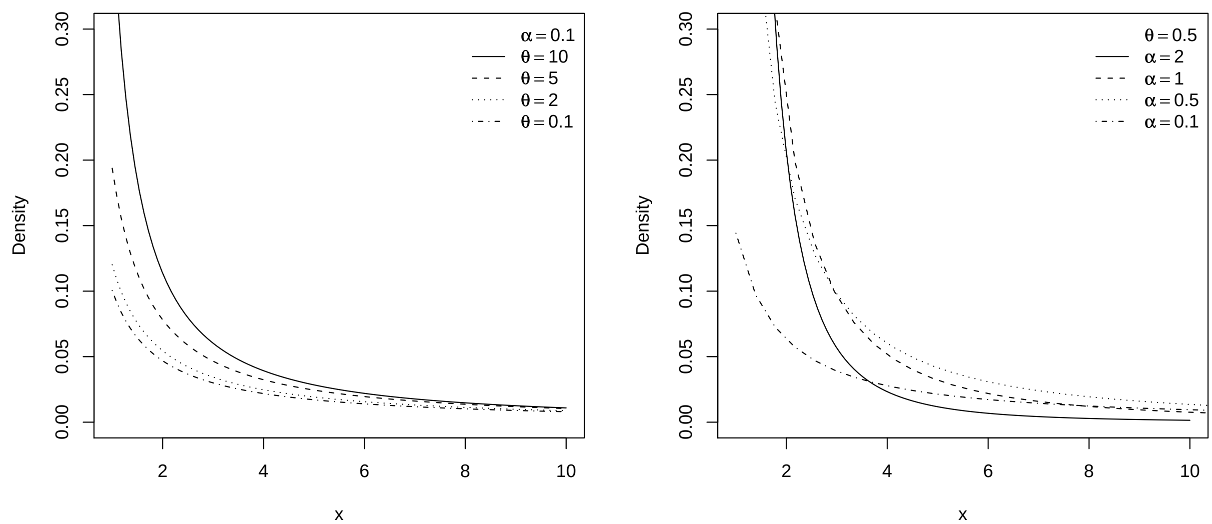

Figure 5.2: The PL pdf for some values of α and θ with $\mu = 1$.

5.10.3 Pareto geometric distribution

The density of the Pareto geometric (PG) distribution is given by

$$f(x) = \frac{\alpha\mu^{\alpha}}{x^{\alpha+1}} \frac{1-\theta}{\left(1 - \theta\left(\frac{\mu}{x}\right)^{\alpha}\right)^{2}},\ \theta, \alpha, x > 0$$

and the cdf is

$$F(x) = \frac{1 - \left(\frac{\mu}{x}\right)^{\alpha}}{1 - \theta\left(\frac{\mu}{x}\right)^{\alpha}}.$$

The hazard function of the PG distribution reduces to

$$f(x) = \frac{\alpha x^{-1}}{1-\theta\left(\frac{\mu}{x}\right)^{\alpha}},\ \theta, \alpha, x > 0.$$

Figure 5.3 plots some PG densities and hazard functions, with $\mu = 1$ and some values of α and θ.

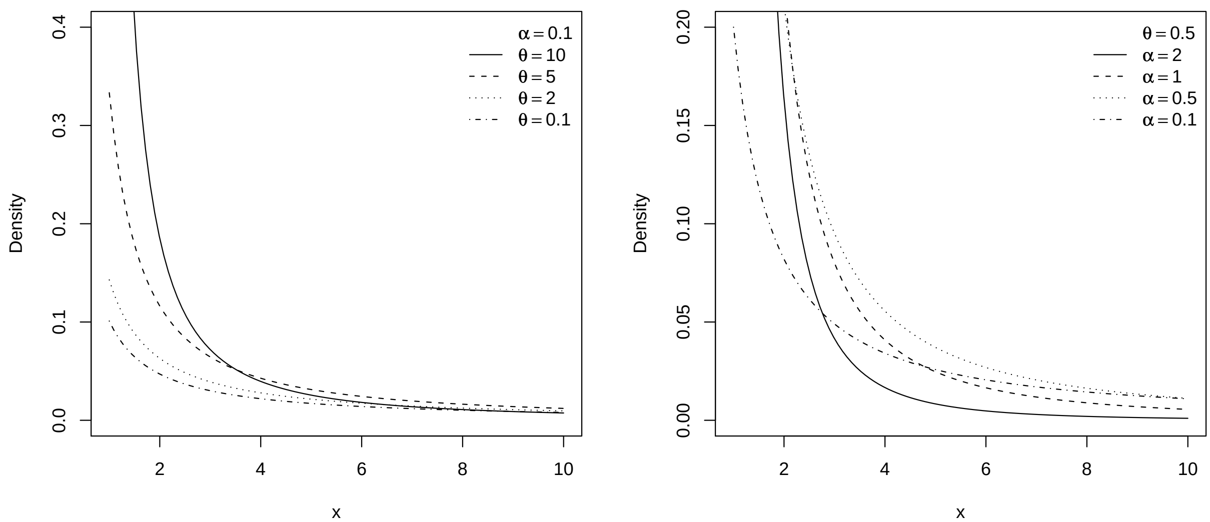

Figure 5.3: The PG pdf for some values of α and θ with $\mu = 1$.

Chapter 6

Weibull Power Series

6.1 Resumo

Neste capítulo introduzimos uma nova classe de distribuições, denominada Weibull power series, similar à definida no Capítulo 5, baseada em uma composição da distribuição de Weibull e a classe de distribuições power series. As distribuições nesta classe, as quais adotaremos com a sigla de distribuição WPS, têm três parâmetros desconhecidos e suporte $(0, \infty)$. A função de risco da distribuição WPS pode assumir forma crescente e decrescente, bem como forma de banheira, uma vantagem sobre a distribuição de Weibull. A distribuição WPS é uma mistura da Weibull com pesos definidos pela distribuição power series. Apresentaremos algumas propriedades tais como o s-ésimo momento, função de risco, densidade e momentos da estatística de ordem, estimação, casos especiais e uma aplicação.

6.2 Introduction

Similarly to the class introduced in Chapter 5, we now introduce a class of distributions based on a composition of the Weibull distribution and the power series class of distributions, which will be called the Weibull power series class of distributions. A distribution in this class, WPS distribution say, has three unknown parameters and suport $(0, \infty)$. The hazard function of the WPS distribution can take decreasing and increasing forms and also may have a bathtub shape. Unlike the Weibull distribution, the WPS distribution is useful for modelling unimodal failure rates. The WPS distribution is a mixture of the Weibull distribution with weights defined by the power series distribution. In some cases, the Weibull distribution can be obtained as a limiting case. We derive the sth moment of a distribution in this class, its failure rate, density and moments of the order statistics and some special cases.

6.3 Definition

Let $X_1, \ldots, X_N$ be independent random variables following the Weibull distribution whose density is given by

$$g(x) = \alpha\beta^\alpha x^{\alpha-1} e^{-(\beta x)^\alpha}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

The index N is a random variable following a distribution in the power series class of distributions for which the probability mass function is given by

$$P(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \ldots, \tag{6.1}$$

where a_n depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ and $\theta > 0$ such that $C(\theta)$ is finite. Let $X_{(1)} = \min(X_1, \ldots, X_N)$. The cdf of $X_{(1)}|N = n$ is given by

$$G_{X_{(1)}|N=n}(x) = 1 - e^{-n(\beta x)^\alpha},$$

i.e., $X_{(1)}|N = n$ follows a Weibull distribution with parameters α and $\beta n^{1/\alpha}$. The joint probability function of $X_{(1)}, N$ can be expressed as

$$g_{X,N}(x, n) = \frac{\alpha n a_n \theta^n}{C(\theta)} \beta^\alpha x^{\alpha-1} e^{-n(\beta x)^\alpha}.$$

The density of a distribution in the Weibull power series (WPS) class is defined by the marginal density of $X_{(1)}$, say

$$f(x) = \alpha\theta\beta^\alpha x^{\alpha-1} e^{-(\beta x)^\alpha} \frac{C'\left(\theta e^{-(\beta x)^\alpha}\right)}{C(\theta)}. \tag{6.2}$$

The density of the WPS distribution is a mixture of the Pareto distribution and it can be rewritten as

$$f(x) = \sum_{n=1}^{\infty} P(N = n) f_n(x), \tag{6.3}$$

where $f_n(x)$ is the density function of a Weibull distribution with parameters α and $n^{1/\alpha}\beta$.

The Weibull distribution is a limiting special case of the WPS ditribution when $\theta \to 0^+$. The proof is analogous to that one presented in Chapter 5. If $\theta \to 0^+$, the power series converges to a distribution degenerated at one and then from (6.3) the WPS distribution leads to the Weibull distribution.

The cdf of a WPS distribution is given by

$$F(x) = 1 - \frac{C\left(\theta e^{-(\beta x)^\alpha}\right)}{C(\theta)}.$$

6.4 Hazard function

The hazard function is defined by the ratio $\frac{f(x)}{S(x)}$, where $S(x) = 1 - F(x)$ is the survival function. Hence, a Weibull power series distribution has hazard function

$$h(x) = \alpha\theta\beta^\alpha x^{\alpha-1} e^{-(\beta x)^\alpha} \frac{C'\left(\theta e^{-(\beta x)^\alpha}\right)}{C\left(\theta e^{-(\beta x)^\alpha}\right)}.$$

Unlike the PPS class of distributions, the hazard function of the WPS class of distribution can assume different shapes.

6.5 Order statistics

Let $X_1, \ldots, X_n$ be random variables iid such that $X_i \sim \text{WPS}(\alpha, \beta, \theta)$ for $i = 1, \ldots, n$. The pdf of the ith order statistic, $X_{i:n}$ say, is given by

$$f_{i:n}(x) = \frac{n! f(x)}{(n-i)!(i-1)!} \left[1 - \frac{C\left(\theta e^{-(\beta x)^\alpha}\right)}{C(\theta)}\right]^{i-1} \left[\frac{C\left(\theta e^{-(\beta x)^\alpha}\right)}{C(\theta)}\right]^{n-i}.$$

6.6 Moments

The quantile γ of a WPS distribution is given by

$$x_\gamma = \beta^{-1}\left[-\log C^{-1}\left\{C(\theta)(1-\gamma)\right\} + \log\theta\right]^{1/\alpha},$$

where $C^{-1}(\cdot)$ is the inverse function of $C(\cdot)$.

Let X be a random variable following a WPS distribution. To obtain the rth moment of X, consider the first derivative of $C(\theta)$

$$C'(\theta) = \frac{d}{d\theta}\sum_{n=1}^{\infty} a_n\theta^n = \sum_{n=1}^{\infty} n a_n \theta^{n-1}.$$

Hence, the rth moment is given by

$$
\begin{aligned}
E(X^r) &= \int_0^\infty x^{r+\alpha-1}\alpha\theta\beta^\alpha e^{-(\beta x)^\alpha}\frac{C\left(\theta e^{-(\beta x)^\alpha}\right)}{C(\theta)}dx \\
&= \frac{\alpha\beta^\alpha}{C(\theta)}\sum_{n=1}^\infty n a_n\theta^n \int_0^\infty x^{r+\alpha-1}e^{-n(\beta x)^\alpha}dx \\
&= \frac{1}{C(\theta)}\sum_{n=1}^\infty a_n\theta^n E(Y^R), \quad \text{Y}\sim\text{Weibull}(\alpha, \beta n^{1/\alpha}).
\end{aligned}
$$

6.7 Estimation

Let $x_1, \ldots, x_n$ be an independent random sample from the WPS distribution. The total log-likelihood function is given by

$$
\begin{aligned}
\ell = \ell(x;\alpha,\beta,\theta) &= n\log\alpha + n\log\theta + n\alpha\log\beta - n\log C(\theta) + \\
&\sum_{i=1}^n \log C'\left\{\theta\exp^{-(\beta x_i)^\alpha}\right\} + (\alpha-1)\sum_{i=1}^n\log x_i - \sum_{i=1}^n(\beta x_i)^\alpha.
\end{aligned}
$$

The components of the score function are

$$
\begin{aligned}
\frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} + n\log(\beta) + \sum_{i=1}^n\log(x_i) - \sum_{i=1}^n(\beta x_i)^\alpha\log(\beta x_i) - \\
&\theta\sum_{i=1}^n\exp^{-(\beta x_i)^\alpha}(\beta x_i)^\alpha\log(\beta x_i)\log\left(\frac{\mu}{x_i}\right)\frac{C''\left(\theta\exp^{-(\beta x_i)^\alpha}\right)}{C'\left(\theta\exp^{-(\beta x_i)^\alpha}\right)}, \\
\frac{\partial\ell}{\partial\beta} &= \frac{n\alpha}{\beta} - \alpha\beta^{\alpha-1}\sum_{i=1}^n x_i^\alpha - \theta\alpha\beta^{\alpha-1}\sum_{i=1}^n\exp^{-(\beta x_i)^\alpha}x_i^\alpha\frac{C''\left(\theta\exp^{-(\beta x_i)^\alpha}\right)}{C'\left(\theta\exp^{-(\beta x_i)^\alpha}\right)}, \\
\frac{\partial\ell}{\partial\theta} &= \frac{n}{\theta} - n\frac{C'(\theta)}{C(\theta)} + \sum_{i=1}^n\exp^{-(\beta x_i)^\alpha}\frac{C''\left(\theta\exp^{-(\beta x_i)^\alpha}\right)}{C'\left(\theta\exp^{-(\beta x_i)^\alpha}\right)}.
\end{aligned}
$$

6.7.1 EM algorithm

To obtain the estimates of the parameters we propose the EM algorithm (Dempster et al., 1977; McLachlan and Krishnan, 1997) described in Section 5.9.1. Consider the joint probability function of the complete data set

$$
g_{X,Z}(x,z) = \frac{\alpha z a_z\theta^z}{C(\theta)}\beta^\alpha x^{\alpha-1}e^{-z(\beta x)^\alpha}.
$$

The marginal of Z is given by

$$P(Z=z|X=x) = za_z\theta^{z-1}e^{-(z-1)(\beta x)^\alpha}C'\left(\theta e^{-(\beta x)^\alpha}\right)^{-1},$$

and then its expected value reduces to

$$\begin{aligned} E(Z|X=x) &= \frac{\left(\theta e^{-(\beta x)^\alpha}\right)^{-1}}{C'\left(\theta e^{-(\beta x)^\alpha}\right)}\sum_{z=1}^{\infty} z^2{}_{-}z\left(\theta e^{-(\beta x)^\alpha}\right)^z \\ &= \theta e^{-(\beta x)^\alpha}\frac{C''\left(\theta e^{-(\beta x)^\alpha}\right)}{C'\left(\theta e^{-(\beta x)^\alpha}\right)}. \end{aligned}$$

The log-likelihood for the complete-data is

$$\begin{aligned} \ell(x_1,\ldots,x_n;z_1,\ldots,z_n;\alpha,\theta) \propto\ & n\log\alpha + n\alpha\log\beta + (\alpha-1)\sum_{i=1}^{n}\log x_i - \\ & \sum_{i=1}^{n} z_i(\beta x_i)^\alpha + \log\theta\sum_{i=1}^{n} z_i - n\log C(\theta) \end{aligned}$$

and the score components are

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} + n\log\beta - \sum_{i=1}^{n} z_i(\beta x_i)^\alpha \log\left(\beta x_i\right), \\ \frac{\partial\ell}{\partial\beta} &= n - \sum_{i=1}^{n} z_i(\beta x_i)^\alpha, \\ \frac{\partial\ell}{\partial\theta} &= -n\frac{C'(\theta)}{C(\theta)} + \frac{1}{\theta}\sum_{i=1}^{\infty} z_i. \end{aligned}$$

The maximum likelihood estimates can be obtained from the iterative algorithm given by

$$\begin{aligned} \hat{\alpha}^{(t+1)} &= \left(\frac{\sum_{i=1}^{n} z_i x_i \log(x_i)}{\sum_{i=1}^{n} z_i x_i} - \frac{1}{n}\sum_{i=1}^{n}\log(x_i)\right), \\ \hat{\beta}^{(t+1)} &= \left(\frac{n}{\sum_{i=1}^{n} z_i x_i^{\alpha^{(t)}}}\right)^{1/\alpha^{(t)}}, \\ \hat{\theta}^{(t+1)} &= \frac{C(\hat{\theta}^{(t+1)})}{nC'(\hat{\theta}^{(t+1)})}\sum_{i=1}^{n} z_i^{(t)}, \end{aligned}$$

where

$$z_i^{(t)} = \hat{\theta}^{(t)} \mu^{\hat{\alpha}^{(t)}} x_i^{\hat{\alpha}^{(t)}} \frac{C''\left(\frac{\hat{\theta}^{(t)}\mu^{\hat{\alpha}^{(t)}}}{x_i^{\hat{\alpha}^{(t)}}}\right)}{C'\left(\frac{\hat{\theta}^{(t)}\mu^{\hat{\alpha}^{(t)}}}{x_i^{\hat{\alpha}^{(t)}}}\right)}.$$

6.8 Some special distributions

We now introduce some special sub-models of the WPS class of distributions. The Weibull Geometric (WG) distribution was already defined by Barreto-Souza et al. (2008)

6.8.1 Weibull Poisson distribution

The density of the Weibull Poisson (WP) distribution is given by

$$f(x) = \alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}\theta\frac{\exp\{\theta e^{-(\beta x)^{\alpha}}\}}{e^{\theta}-1},\ \alpha,\beta,\theta,x>0$$

and the cdf is

$$F(x) = \frac{\exp\theta - \exp\{\theta e^{-(\beta x)^{\alpha}}\}}{e^{\theta}-1}.$$

Figure 6.1 plots some shapes of the WP distribution.

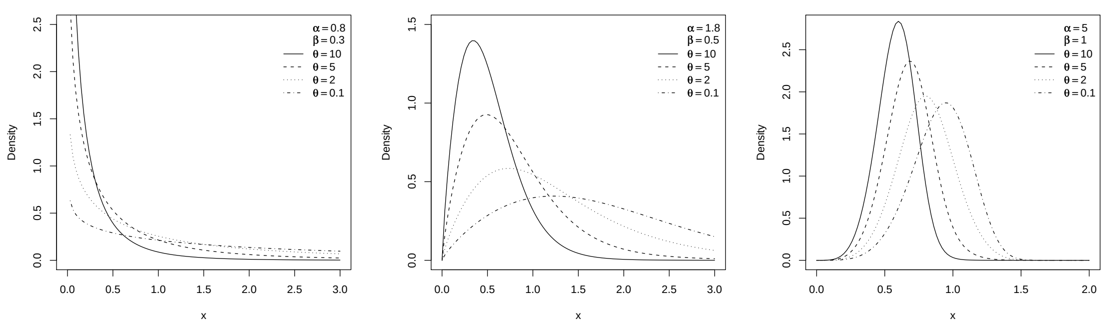

Figure 6.1: Pdf of the WP distribution for some values of the parameters.

The hazard function of the WP distribution reduces to

$$h(x) = \alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}\theta\frac{1}{1-\exp\{-\theta e^{-(\beta x)^{\alpha}}\}},\ \alpha,\beta,\theta,x>0.$$

Figure 6.2 plots some shapes of the WP hazard function.

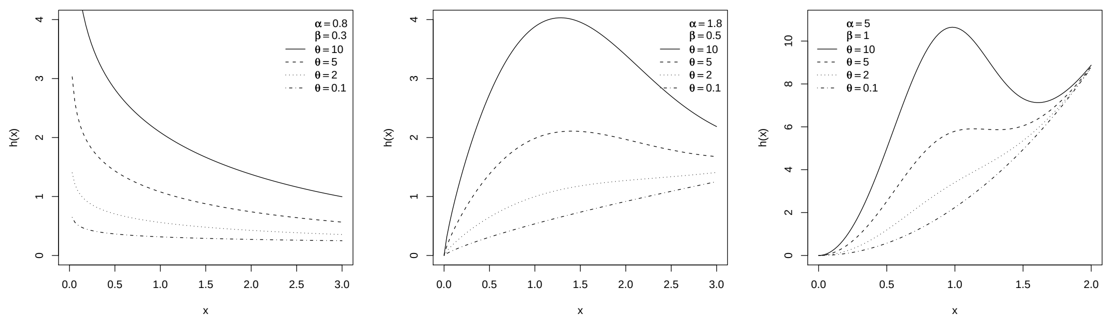

Figure 6.2: Hazard function of the WP distribution for selected values of the parameters.

6.8.2 Weibull logarithmic distribution

The density of the Weibull logarithmic (WL) distribution is given by

$$f(x) = \frac{\alpha\beta^{\alpha}x^{\alpha-1}\theta e^{-(\beta x)^{\alpha}}}{(\theta-1)\log(1-\theta)},\ \theta\in(0,1),\alpha,\beta,x>0$$

and the cdf is

$$F(x) = 1 - \frac{\log\left(1-\theta e^{-(\beta x)^{\alpha}}\right)}{\log(1-\theta)}$$

Figure 6.3 plots some shapes of the WL distribution.

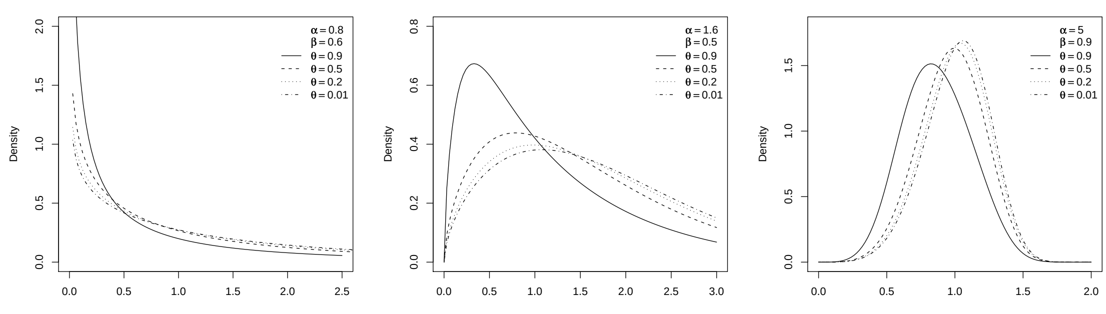

Figure 6.3: Pdf of the WL distribution for some values of the parameters.

The hazard function of the WL distribution reduces to

$$f(x) = \frac{\alpha\beta^{\alpha}x^{\alpha-1}\theta e^{-(\beta x)^{\alpha}}}{(\theta-1)\log(1-\theta e^{-(\beta x)^{\alpha}})},\ \theta\in(0,1),\alpha,\beta,x>0.$$

Figure 6.4 plots some shapes of the WL hazard function.

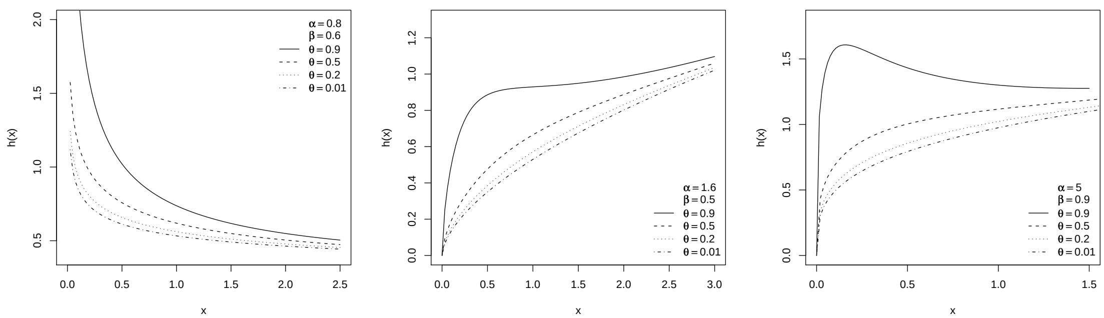

Figure 6.4: Hazard function of the WL distribution for selected values of the parameters.

6.8.3 Weibull geometric distribution

The Weibull geometric (WG) distribution, introduced by Barreto-Souza et al. (2008) [3], is given by

$$f(x) = \alpha\beta^{\alpha}(1-\theta)x^{\alpha-1}e^{-(\beta x)^{\alpha}}\{1-\theta e^{-(\beta x)^{\alpha}}\}^{-2},\ \theta \in (0,1), \alpha, \beta, x > 0$$

and the cdf is

$$F(x) = \frac{1-e^{-(\beta x)^{\alpha}}}{1-\theta e^{-(\beta x)^{\alpha}}}$$

Figure 6.5 plots some shapes of the WG distribution.

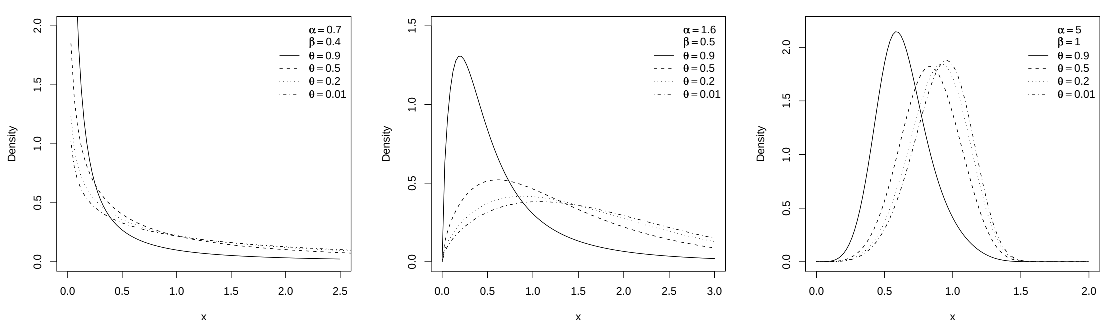

Figure 6.5: Pdf of the WG distribution for some values of the parameters.

The hazard function of the WG distribution reduces to

$$f(x) = \alpha\beta^{\alpha}x^{\alpha-1}\{1-\theta e^{-(\beta x)^{\alpha}}\}^{-1},\ \theta \in (0,1), \alpha, \beta, x > 0.$$

Figure 6.6 plots some shapes of the WG hazard function.

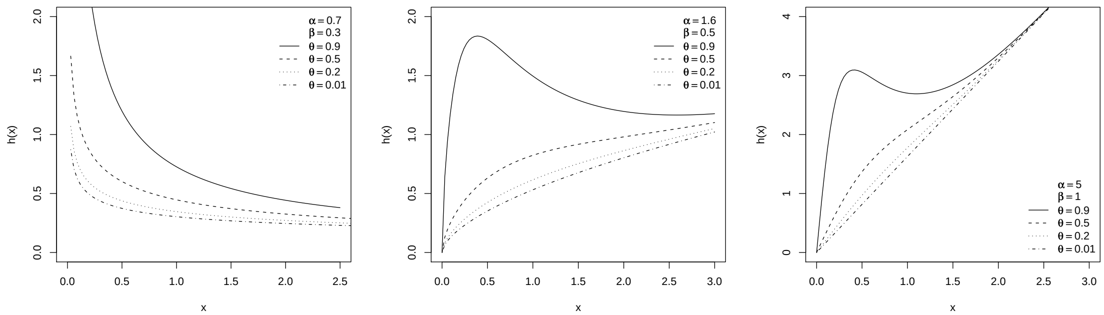

Figure 6.6: Hazard function of the WG distribution for selected values of the parameters.

When $\theta \to 1^-$, the WG distribution and the Weibull logarithmic (WL) distribution converges to a distribution degenerate at zero. The same occurs for the Weibull Poisson (WP) distribution when $\theta \to \infty$.

6.9 Applications

In this section, we fit the WP, WG and WL models to a real data set. The data set is an uncensored data from Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibres (in Gba).

Table 6.1 shows the estimated parameters and the maximized log-likelihood using the EM algorithm for the three models.

	WP	WG	WL
$\hat{\alpha}$	2.803	2.963	2.877
$\hat{\beta}$	0.338	0.320	0.330
$\hat{\theta}$	0.036	0.250	0.243
$\hat{l}$	-141.525	-141.485	-127.605

Table 6.1: Estimated parameters for the WPS data set.

According to the maximized log-likelihood, the WL distribution is the best model for this data set. Figure 6.7 plots the fitted pdf and the estimated quantiles versus observed quantiles.

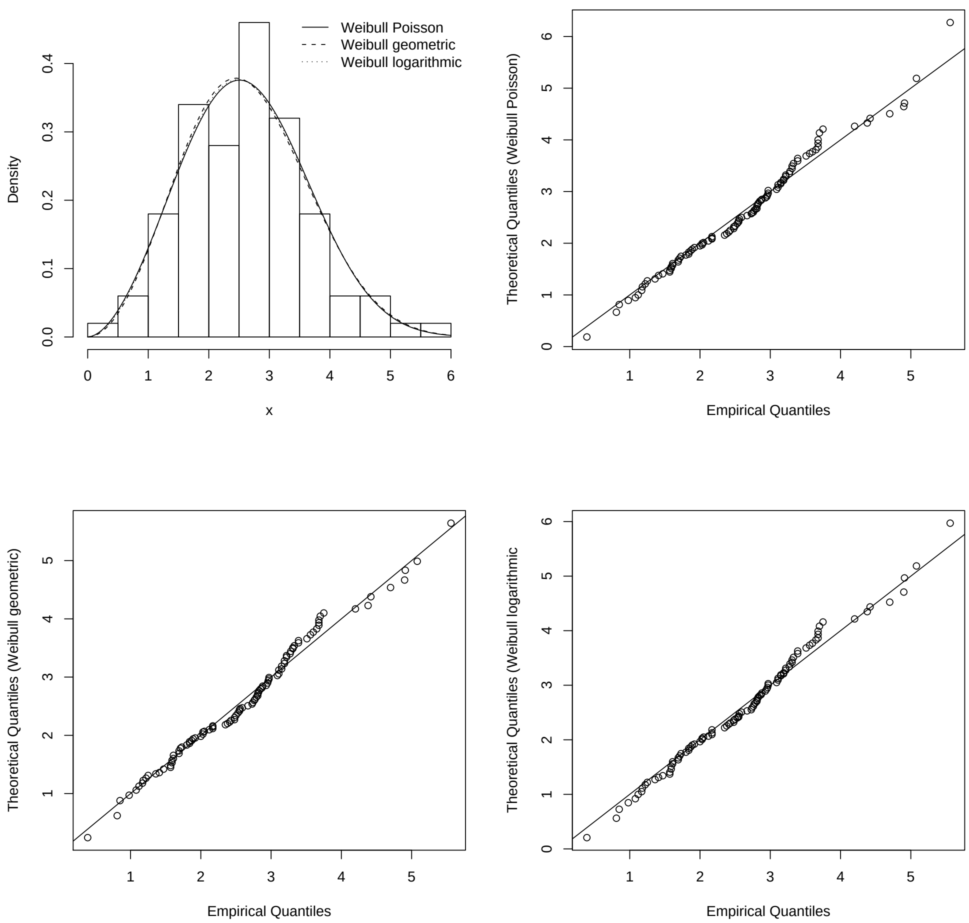

Figure 6.7: Estimated pdf for the WPS data set.

Chapter 7

Conclusions

Resumo

Trabalhamos nesta dissertação com a Classe de Distribuições Generalizadas Beta, já conhecida na literatura. Reunimos resultados gerais desta classe em um capítulo de revisão, acrescentando algumas propriedades adicionais. Em um capítulo a parte, reunimos distribuições especiais desta classe introduzidas por outros autores, adicionando a elas aplicações a dados reais e algumas propriedades. Além dos capítulos de revisão, nossa contribuição para o tema é a distribuição beta logística generalizada do tipo IV. Esta distribuição pode ser simétrica, assimétrica à direita e à esquerda, com suporte $\mathbb{R}$, tem como casos especiais as distribuições beta generalizada do tipo I, tipo II e tipo III e está relacionada às distribuições beta-beta prime e beta-F.

Introduzimos uma nova classe de distribuições, denominada Pareto power series, baseada em uma composição da distribuição de Pareto com a classe de distribuições discretas de séries de potência. Apresentamos alguns resultados gerais da classe e três casos especiais, Pareto Poisson, Pareto geométrica e Pareto logarítmica. Mostramos que a distribuição de Pareto é um caso limite quando o parâmetro $\theta \to 0^+$.

Introduzimos uma nova classe de distribuições, denominada Weibull power series, similar à Pareto power series. Baseada na composição da distribuição de Weibull com a classe de distribuições discretas de séries de potência, a Weibull power series é uma classe interessante para estudos de confiabilidade e sobrevivência, pois tem função de risco com maior diversidade de formas, podendo ser unimodal, constante, crescente ou decrescente e podendo ainda assumir forma de banheira. Apresentamos algumas de suas propriedades gerais e introduzimos dois casos especiais, a distribuição Weibull Poisson e a distribuição Weibull logarítmica.

Conclusions

We work in this thesis with the Class of Generalized Beta distributions, introduced recently in the literature. In a review chapter, we collected some general results of this class, adding some additional properties. We presented some special models of this class proposed by other authors, adding some applications to real data and some properties. In addition to the review chapters, our contribution to the topic is the beta generalized logistic of type IV distribution. This distribution may be symmetric, asymmetric to the right and left, with support $\mathbb{R}$ and has as special cases the beta generalized logistic of type I, II and III and is related to distributions beta-beta prime and beta-F.

We introduced a new class of distributions, called Pareto power series, based on a composition of the Pareto distribution with the power-series class of discrete distribution. We presented some general results of this class and three special cases, the Pareto poisson, Pareto geometric and Pareto logarithmic distributions. We show that the Pareto distribution is a limiting special case of the Pareto power series when the parameter $\theta \to 0^+$.

We proposed another new class of distributions, called Weibull power series, similar to the Pareto power series. Based on the composition of the Weibull distribution with the power-series class, the Weibull power series is interesting to reliability and survival studies, due to its hazard function which can provide many shapes, such as unimodal, constant, increasing or decreasing and may take bathtube form. We presented some of their general properties and introduced two special models, the Weibull Poisson and the Weibull logarithmic distributions.

Appendix A

Function to generate random numbers from a Beta-G distribution.

```
rbeta.G<-function(n,shape1,shape2,qtl.f,...){

# n: an integer, number of observations to be generated;
# shape1, shape2: positive parameters of the Beta distribution;
# qtl.f: function to obtain the quantile of G distribution, following
#        the same structure of similar functions already implemented
#        on the base package of R;
# ...: optional arguments to qtl.f.

nbeta<-rbeta(n,shape1,shape2)

x<-qtl.f(nbeta,...)

while(max(x)==Inf){
aux<-which(x==Inf)
nbeta<-rbeta(length(aux),shape1,shape2)
x[aux]<-qtl.f(nbeta,...)
}
x
}
```

Example: Generate 10 random numbers from a standard beta normal distribution with parameters $a = 2$ and $b = 0.5$:

```
rbeta.G(n=10,shape1=2,shape2=0.5,qtl.f=qnorm,mean=0,sd=1)
```

Funtion to obtain the quantile of a Beta-G distribution.

```
qbeta.G<-function(p,shape1,shape2,qtl.f,...){

# p: vector of probabilities
# shape1, shape2: positive parameters of the Beta distribution
# qtl.f: function to obtain the quantile of the primitive distribution
# ...: optional arguments to pfunc.f


q.beta<-qbeta(p,shape1,shape2)

x<-qtl.f(q.beta,...)
x
}
```

Example: The quantiles 0.1,0.5,0.9 of a standard beta normal distribution with parameters a=2 and b=0.5:

```
qbeta.G(p=c(.1,.5,.9),shape1=2,shape2=0.5,qtl.f=qnorm,mean=0,sd=1)
```

Function to obtain the cdf of a Beta-G distribution.

```
pbeta.G<-function(x,shape1,shape2,pfunc.f,...){

# x: vector of quantiles
# shape1, shape2: positive parameters of the Beta distribution
# pfunc.f: function to obtain the probability of the primitive distribution
# ...: optional arguments to pfunc.f

p<-pbeta(pfunc.f(x,...),shape1,shape2)

p

}
```

Example: The cdf of the quantiles -2, 0, 2 of a standard beta normal distribution with parameters a=2 and b=0.5:

```
pbeta.G(x=c(-2,0,2),shape1=2,shape2=0.5,pfunc.f=pnorm,mean=0,sd=1)
```

Function of the density of a Beta-G distribution

```
dbeta.G<-function(x,shape1,shape2,dfunc.f,pfunc.f=NULL,lower=-Inf,...){

# x: vector of quantiles
# shape1, shape2: positive parameters of the Beta distribution
# dfunc.f: function to obtain the density of the primitive distribution
# pfunc.f: optional function to obtain the density of the G distribution.
#          If pfunc.f is not given, the cdf will be numerically evaluated
# lower: lower limit of integration, only used if pfunc.f=NULl
# ...: optional arguments to pfunc.f


if(!is.null(pfunc.f))
cdf<-pfunc.f(x,...) else
{
n<-length(x)
cdf<-vector(mode="numeric",length=n)
f1<-function(x) dfunc.f(x,...)
for(i in 1:n)
cdf[i]<-integrate(f1,lower,x[i])$value
}


dfunc.f(x,...)/beta(shape1,shape2)*cdf^(shape1-1)*(1-cdf)^(shape2-1)

}
```

Example: Curve of the density of the standard beta normal distribution with parameters a=0.1 and b=0.1:

```
f1<-function(x)
dbeta.G(x,shape1=.1,shape2=.1,dfunc.f=dnorm,pfunc.f=pnorm,mean=2,sd=1)
curve(f1,from=-10,to=10)
```

Appendix B

Function to generate random numbers from a PPS distribution.

```
rpps<-function(n,mu,alpha,theta,Ctheta=NULL,PS=NULL){
# n: number of observations to be generated;
# mu, alpha, theta: model parameters;
# Ctheta: a list() with the C function in the first position and the
#         inverse of C in the second position. It will be ignored if
#         PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithm".

if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(x){exp(x)-1},
                   invCtheta<-function(x)log(x+1)),
           Geometric=list(Ctheta<-function(x)x/(1-x),
                     invCtheta<-function(x)x/(1+x)),
           Logarithm=list(Ctheta<-function(x)-log(1-x),
                     diffCtheta<-function(x)1-exp(-x))
           )

alpha*(Ctheta[[2]]( Ctheta[[1]](theta)*runif(n) )/theta)^(-1/alpha)
}
```

Example: Generate 10 random numbers from a Pareto poisson distribution with parameters $\mu = 1$, $\alpha = 1$ and $\theta = 2$:

```
rpps(n=10,mu=1,alpha=1,theta=2,PS="Poisson")
```

Function to obtain the quantile of a PPS distribution.

```
qpps<-function(p,mu,alpha,theta,Ctheta=NULL,PS=NULL){
# x: vector of probabilities;
# mu, alpha, theta: model parameters;
# Ctheta: a list() with the C function in the first position and the inverse
#         of C in the second position. It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithm".


if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(x){exp(x)-1},
                   invCtheta<-function(x)log(x+1)),
           Geometric=list(Ctheta<-function(x)x/(1-x),
                     invCtheta<-function(x)x/(1+x)),
           Logarithm=list(Ctheta<-function(x)-log(1-x),
                     diffCtheta<-function(x)1-exp(-x))
           )

alpha*(Ctheta[[2]]( Ctheta[[1]](theta)*(1-p) )/theta)^(-1/alpha)
}
```

Example: The quantiles 0.1,0.5,0.9 of a Pareto geometric distribution with parameters $\mu = 1$, $\alpha = 2$ and $\theta = 0.5$:

```
qpps(x=c(.1,.5,.9),mu=1,alpha=2,theta=5,PS="Geometric")
```

Function to obtain the cdf of the PPS distribution.

```
ppps<-function(x,mu,alpha,theta,Ctheta=NULL,PS=NULL){
# x: vector of quantiles;
# mu, alpha, theta: model parameters;
# Ctheta: a list() with the C function in the first position.
#         It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithm".

if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(theta){exp(theta)-1},
```

```
                diffCtheta<-function(theta)exp(theta)),
           Geometric=list(Ctheta<-function(theta)theta/(1-theta),
                     diffCtheta<-function(theta)(1-theta)^(-2)),
           Logarithm=list(Ctheta<-function(theta)-log(1-theta),
                     diffCtheta<-function(theta)(1-theta)^(-1))
           )

1-Ctheta[[1]](theta*(mu/x)^alpha)/Ctheta[[1]](theta)
}
```

Example: The cdf of the quantiles 2, 5, 10 of the Pareto logarithmic distribution with parameters $\mu = 1$, $\alpha = 2$ and $\theta = 0.5$:

```
ppps(x=c(2,5,10),mu=1,alpha=2,theta=0.5,PS="Logarithmic")
```

Function of the density of the WPS distribution.

```
dpps<-function(x,mu,alpha,theta,Ctheta=NULL,PS=NULL){
# x: vector of quantiles;
# mu, alpha, theta: model parameters;
# Ctheta: a list() with the C function in the first position
#         and the first derivative of C in the second position.
#         It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithm".

if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(theta){exp(theta)-1},
                diffCtheta<-function(theta)exp(theta)),
           Geometric=list(Ctheta<-function(theta)theta/(1-theta),
                     diffCtheta<-function(theta)(1-theta)^(-2)),
           Logarithm=list(Ctheta<-function(theta)-log(1-theta),
                     diffCtheta<-function(theta)(1-theta)^(-1))
#           NULL=switch(Ctheta,f1
           )

alpha*theta*mu^alpha*x^(-alpha-1)*Ctheta[[2]](theta*(mu/x)^alpha)/
Ctheta[[1]](theta)

}
```

Example: Curve of the density of the Pareto poisson distribution with parameters $\mu = 1$, $\alpha = 1$ and $\theta = 2$:

```
f1<-function(x)
dpps(x,mu=1,alpha=1,theta=2,PS="Poisson")
curve(f1,from=1,to=10)
```

Function to fit the WPS model using the EM algorithm.

```
fit.PPS<-function(x,par,mu,Ctheta=NULL,PS,tol=1e-4,maxi=100){

# x: vector of observations;
# par: initial parameters values;
# mu: fixed parameter of the model;
# Ctheta: a list() with the C function in the first position, the first
#         derivative of C in the second position and the second
#         derivative of C on the third position.
#         It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithm";
# tol: tolerance of the algorithm;
# maxi: the upper end points of the interval to the parameter alpha
#       and in some cases the parameter theta be searched.



if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(x){exp(x)-1},
                   diffCtheta<-function(x)exp(x),
                   diff2Ctheta<-function(x)exp(x)),
           Geometric=list(Ctheta<-function(x)x/(1-x),
                     diffCtheta<-function(x)(1-x)^(-2),
                     diff2Ctheta<-function(x)2*(1-x)^(-3)),
           Logarithm=list(Ctheta<-function(x)-log(1-x),
                     diffCtheta<-function(x)(1-x)^(-1),
                     diff2Ctheta<-function(x)(1-x)^(-2))
           )


alpha<-par[1]
theta<-par[2]
```

```
n<-length(x)

z.temp<-function(){
theta*(mu/x)^alpha*Ctheta[[3]](theta*(mu/x)^alpha)/
   Ctheta[[2]](theta*(mu/x)^alpha)+1
}


theta.sc<-function(thet){
sum(z)/thet-n*Ctheta[[2]](thet)/Ctheta[[1]](thet)
}

test<-1
if(PS=="Logarithm") maxi.theta<-0.95
if(PS=="Poisson") maxi.theta<-maxi
while(test>tol){

z<-z.temp()
alpha.new<-n/sum(z*log(x/mu))

if(PS=="Geometric")
theta.new<-1-n/sum(z)
if(PS!="Geometric")
theta.new<-uniroot(theta.sc,interval=c(0,maxi.theta),
f.lower=theta.sc(1e-4),
f.upper=theta.sc(maxi.theta))$root


test<-max(abs(c(((alpha.new-alpha)),
                ((theta.new-theta)))))
alpha<-alpha.new
theta<-theta.new
}

c(alpha,theta)

}
```

Appendix C

Function to generate random numbers from a WPS distribution.

```
rwps<-function(n,shape1,shape2,theta,Ctheta=NULL,PS=NULL){
# n: number of observations to be generated;
# shape1, shape2, theta: model parameters;
# Ctheta: a list() with the C function in the first position and the inverse
#         of C in the second position. It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithm".

if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(x){exp(x)-1},
                   invCtheta<-function(x)log(x+1)),
           Geometric=list(Ctheta<-function(x)x/(1-x),
                     invCtheta<-function(x)x/(1+x)),
           Logarithm=list(Ctheta<-function(x)-log(1-x),
                     diffCtheta<-function(x)1-exp(-x))
           )

shape2^(-1)*( -log( Ctheta[[2]]( Ctheta[[1]](theta)*runif(n) ) /
theta ) )^(1/shape1)
}
```

Example: Generate 10 random numbers from a Weibull geometric distribution with parameters $\alpha = 2$, $\beta = 1$ and $\theta = 2$:

```
rpps(n=10,shape1=2,shape2=1,theta=2,PS="Geometric")
```

Funtion to obtain the quantile of a WPS distribution.

```
qwps<-function(x,shape1,shape2,theta,Ctheta=NULL,PS=NULL){
```

```
# x: vector of probabilities;
# shape1, shape2, theta: model parameters;
# Ctheta: a list() with the C function in the first position and the inverse
#         of C in the second position. It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithmic".

if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(x){exp(x)-1},
                   invCtheta<-function(x)log(x+1)),
           Geometric=list(Ctheta<-function(x)x/(1-x),
                     invCtheta<-function(x)x/(1+x)),
           Logarithmic=list(Ctheta<-function(x)-log(1-x),
                       diffCtheta<-function(x)1-exp(-x))
           )

shape1^(-1)*(-log(Ctheta[[2]](Ctheta[[1]](theta)*(1-x)))+
log(theta))^(1/shape1)
}
```

Example: The quantiles 0.1,0.5,0.9 of a Weibull poisson distribution with parameters $\alpha = 2$, $\beta = 1$ and $\theta = 5$:

```
qwps(x=c(.1,.5,.9),shape1=2,shape2=0.5,theta=5,PS="Poisson")
```

Function to obtain the cdf of the WPS distribution.

```
pwps<-function(x,shape1,shape2,theta,Ctheta=NULL,PS=NULL){
# x: vector of quantiles;
# shape1, shape2, theta: model parameters;
# Ctheta: a list() with the C function in the first position.
#         It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithmic".

if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(theta){exp(theta)-1},
                   diffCtheta<-function(theta)exp(theta)),
           Geometric=list(Ctheta<-function(theta)theta/(1-theta),
                     diffCtheta<-function(theta)(1-theta)^(-2)),
           Logarithmic=list(Ctheta<-function(theta)-log(1-theta),
```

```
                        diffCtheta<-function(theta)(1-theta)^(-1))
            )

1-Ctheta[[1]](theta*exp(-(shape2*x)^shape1))/Ctheta[[1]](theta)
}
```

Example: The cdf of the quantiles 1, 3, 5 of the Pareto logarithmic distribution with parameters $\alpha = 2$, $\beta = 1$, and $\theta = 0.5$:

```
pwps(x=c(1,3,5),shape1=2,shape2=1,theta=0.5,PS="Logarithmic")
```

Function of the density of the WPS distribution.

```
dwps<-function(x,shape1,shape2,theta,Ctheta=NULL,PS=NULL){
# x: vector of quantiles;
# shape1, shape2, theta: model parameters;
# Ctheta: a list() with the C function in the first position
#         and the first derivative of C in the second position.
#         It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithmic".

if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(theta){exp(theta)-1},
                   diffCtheta<-function(theta)exp(theta)),
           Geometric=list(Ctheta<-function(theta)theta/(1-theta),
                     diffCtheta<-function(theta)(1-theta)^(-2)),
           Logarithmic=list(Ctheta<-function(theta)-log(1-theta),
                        diffCtheta<-function(theta)(1-theta)^(-1))
           )

shape1*shape2^shape1*x^(shape1-1)*exp(-(shape2*x)^shape1)*
theta*Ctheta[[2]](theta*exp(-(shape2*x)^shape1))/Ctheta[[1]](theta)

}
```

Example: Curve of the density of the Weibull geometric distribution with parameters $\alpha = 1$, $\beta = 2$ and $\theta = 0.3$:

```
f1<-function(x)
dpps(x,shape1=1,shape2=2,theta=0.3,PS="Geometric")
curve(f1,from=0,to=10)
```

Function to fit the WPS model using the EM algorithm.

```
fit.WPS<-function(x,par,Ctheta=NULL,PS,tol=1e-4,maxi=100){

# x: vector of observations;
# par: initial parameters values;
# Ctheta: a list() with the C function in the first position, the first
#         derivative of C in the second position and the second
#         derivative of C on the third position.
#         It will be ignored if PS!=NULL;
# PS: could be "Poisson", "Geometric" or "Logarithmic";
# tol: tolerance of the algorithm;
# maxi: the upper end points of the interval to the parameter alpha
#       and in some cases the parameter theta be searched.

if(!is.null(PS))
Ctheta<-switch(PS,
           Poisson=list(Ctheta<-function(x){exp(x)-1},
                   diffCtheta<-function(x)exp(x),
                   diff2Ctheta<-function(x)exp(x)),
           Geometric=list(Ctheta<-function(x)x/(1-x),
                     diffCtheta<-function(x)(1-x)^(-2),
                     diff2Ctheta<-function(x)2*(1-x)^(-3)),
           Logarithmic=list(Ctheta<-function(x)-log(1-x),
                       diffCtheta<-function(x)(1-x)^(-1),
                       diff2Ctheta<-function(x)(1-x)^(-2))
           )

alpha<-par[1]
beta<-par[2]
theta<-par[3]
n<-length(x)

z.temp<-function(){
theta*exp(-(beta*x)^alpha)*Ctheta[[3]](theta*exp(-(beta*x)^alpha))/
   Ctheta[[2]](theta*exp(-(beta*x)^alpha))+1
}
```

```
alpha.sc<-function(alp){
n/alp+sum(log(x))-
sum(z*(beta.new*x)^alp*log(x))
}

theta.sc<-function(thet){
sum(z)/thet-n*Ctheta[[2]](thet)/Ctheta[[1]](thet)
}

test<-1
if(PS=="Logarithmic") maxi.theta<-0.99
if(PS=="Poisson")maxi.theta<-maxi
while(test>tol){

z<-z.temp()
beta.new<-(n/sum(x^alpha*z))^(1/alpha)
alpha.new<-uniroot(alpha.sc,interval=c(0,maxi),
f.upper=alpha.sc(maxi))$root


if(PS=="Geometric")
theta.new<-1-n/sum(z)
if(PS!="Geometric")
theta.new<-uniroot(theta.sc,interval=c(0,maxi.theta),
f.lower=theta.sc(1e-4),
f.upper=theta.sc(maxi.theta))$root


test<-max(abs(c(((alpha.new-alpha)),
                ((beta.new-beta)),
                ((theta.new-theta)))))
alpha<-alpha.new
beta<-beta.new
theta<-theta.new
}

c(alpha,beta,theta)

}
```

Appendix D

0.22	0.17	0.11	0.10	0.15	0.06	0.05	0.07	0.12	0.09	0.23	0.25	0.23
0.24	0.20	0.08	0.11	0.12	0.10	0.06	0.20	0.17	0.20	0.11	0.16	0.09
0.10	0.12	0.12	0.10	0.09	0.17	0.19	0.21	0.18	0.26	0.19	0.17	0.18
0.20	0.24	0.19	0.21	0.22	0.17	0.08	0.08	0.06	0.09	0.22	0.23	0.22
0.19	0.27	0.16	0.28	0.11	0.10	0.20	0.12	0.15	0.08	0.12	0.09	0.14
0.07	0.09	0.05	0.06	0.11	0.16	0.20	0.25	0.16	0.13	0.11	0.11	0.11
0.08	0.22	0.11	0.13	0.12	0.15	0.12	0.11	0.11	0.15	0.10	0.15	0.17
0.14	0.12	0.18	0.14	0.18	0.13	0.12	0.14	0.09	0.10	0.13	0.09	0.11
0.11	0.14	0.07	0.07	0.19	0.17	0.18	0.16	0.19	0.15	0.07	0.09	0.17
0.10	0.08	0.15	0.21	0.16	0.08	0.10	0.06	0.08	0.12	0.13		

Table 7.1: Phosphorus concentration in leaves data set

17.88	28.92	33.00	41.52	42.12	45.60	48.40	51.84	51.96
54.12	55.56	67.80	68.64	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40				

Table 7.2: Endurance of deep groove ball bearings data set

1	3	5	7	11	11	11	12	14	14	14	16	16	20	21
23	42	47	52	62	71	71	87	90	95	120	120	225	246	261

Table 7.3: Failure times of the air conditioning system of an airplane data set

5.50	5.00	4.90	6.40	5.10	5.20	5.20	5.00	4.70	4.00	4.50	4.20	4.10
4.56	5.01	4.70	3.13	3.12	2.68	2.77	2.70	2.36	4.38	5.73	4.35	6.81
1.91	2.66	2.61	1.68	2.04	2.08	2.13	3.80	3.73	3.71	3.28	3.90	4.00
3.80	4.10	3.90	4.05	4.00	3.95	4.00	4.50	4.50	4.20	4.55	4.65	4.10
4.25	4.30	4.50	4.70	5.15	4.30	4.50	4.90	5.00	5.35	5.15	5.25	5.80
5.85	5.90	5.75	6.25	6.05	5.90	3.60	4.10	4.50	5.30	4.85	5.30	5.45
5.10	5.30	5.20	5.30	5.25	4.75	4.50	4.20	4.00	4.15	4.25	4.30	3.75
3.95	3.51	4.13	5.40	5.00	2.10	4.60	3.20	2.50	4.10	3.50	3.20	3.30
4.60	4.30	4.30	4.50	5.50	4.60	4.90	4.30	3.00	3.40	3.70	4.40	4.90
4.90	5.00											

Table 7.4: Silicon nitride data set

1.50	0.94	0.78	0.48	0.37	0.19	0.12	0.11	0.08	0.07	0.05	2.03	1.63
0.71	0.70	0.64	0.36	0.32	0.20	0.25	0.12	0.08	2.72	1.49	1.16	0.80
0.80	0.39	0.22	0.12	0.11	0.08	0.08	1.85	1.39	1.02	0.89	0.59	0.40
0.16	0.11	0.10	0.07	0.07	2.05	1.04	0.81	0.39	0.30	0.23	0.13	0.11
0.08	0.10	0.06	2.31	1.44	1.03	0.84	0.64	0.42	0.24	0.17	0.13	0.10
0.09												

Table 7.5: Plasma concentrations of indomethicin data set

2.15	2.20	2.55	2.56	2.63	2.74	2.81	2.90	3.05	3.41	3.43
3.43	3.84	4.16	4.18	4.36	4.42	4.51	4.60	4.61	4.75	5.03
5.10	5.44	5.90	5.96	6.77	7.82	8.00	8.16	8.21	8.72	10.40
13.20	13.70									

Table 7.6: Growth hormone data set

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81	2.00
0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01
0.77	1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24
0.81	1.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66	1.70	1.77	1.84	0.84
1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89		

Table 7.7: Strengths of glass fibres data set

0.69	0.97	0.43	0.30	0.25	0.59	0.32	0.31	0.26	0.26
0.44	0.42	0.49	0.62	0.42	0.43	0.16	−0.02	0.11	−0.07
0.13	0.12	0.27	0.23	0.38	0.40	0.54	0.58	0.15	0.00
0.03	−0.11	0.70	0.91	0.73	0.44	0.57	0.86	0.44	0.17
0.17	0.50	0.73	0.50	0.40	0.41	0.57	0.39	0.83	0.54
0.37	0.39	0.82	0.18	0.04	−0.06	0.99	1.38	1.37	1.46
2.47	2.70	3.39	1.57	0.83	0.86	1.15	0.61	0.09	0.68
0.62	0.31	1.07	0.74	1.29	0.94	0.44	0.79	1.11	0.60
0.57	0.84	0.48	0.49	0.77	0.55	0.29	0.16	0.43	1.21
1.39	0.30	−0.05	0.09	0.13	0.05	0.61	0.74	0.94	0.96
0.39	0.55	0.74	0.07	0.05	0.47	1.28	1.29	0.65	0.42
−0.18	0.11	−0.31	−0.49	−0.28	0.15	0.72	0.45	0.49	0.54
0.85	0.57	0.15	0.29	0.10	−0.03	0.18	0.35	0.11	0.60
0.68	0.45	0.81	0.33	0.34	0.38	0.02	0.50	1.20	1.33
1.28	0.93	0.29	0.71	1.46	1.65	1.51	1.40	1.17	1.02
2.46	2.18	2.10	2.49	1.62	1.01	1.44			

Table 7.8: INPC data set

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	4.42	2.41	3.19
3.22	1.69	3.28	3.09	1.87	3.15	4.90	3.75	2.43	2.95	2.97	3.39	2.96
2.53	2.67	2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85	2.56
3.56	3.15	2.35	2.55	2.59	2.38	2.81	2.77	2.17	2.83	1.92	1.41	3.68
2.97	1.36	0.98	2.76	4.91	3.68	1.84	1.59	3.19	1.57	0.81	5.56	1.73
1.59	2.00	1.22	1.12	1.71	2.17	1.17	5.08	2.48	1.18	3.51	2.17	1.69
1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.80	1.57
1.08	2.03	1.61	2.12	1.89	2.88	2.82	2.05	3.65				

Table 7.9: Breaking stress of carbon fibres data set

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