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PEDRO FELIX DA SILVA JÚNIOR

ON THE EMERGENCE OF FRACTAL COSMIC SPACE FROM FRACTIONAL QUANTUM GRAVITY

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PEDRO FELIX DA SILVA JÚNIOR

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Dissertação apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Física.

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Para Sebastiana

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“Along the road ahead lies something you need. However, in order to claim it, you must lose something that is dear to you”

Mysterious man, Chain of Memories (2004, p. 23) [1]

ABSTRACT

This dissertation investigates a cosmological model that explains the observational data on the matter content of the Universe using Padmanabhan's theory of emergent cosmology and insights from fractional quantum gravity applied to the Schwarzschild black hole. Two main directions lead to this model. On the one hand, we start with the Hamiltonian formalism of general relativity and the canonical quantization of the theory leading to the Wheeler-DeWitt equation. A spherically symmetric spacetime then simplifies the application of the Wheeler-DeWitt equation and we can investigate the quantization of the Schwarzschild black hole, its mass spectrum, and thermodynamics, in the semi-classical limit. The study of fractals and the use of the Riesz fractional derivative via fractional quantum gravity show that the surface area of the event horizon of the Schwarzschild black hole has a random fractal structure, whose description is possible by fractional quantities. On the other hand, we show that the apparent cosmological horizon provides both a Hawking temperature associated with the horizon of an FLRW spacetime and is the most suitable horizon for obtaining the Friedmann equations with Padmanabhan's theory in which cosmic space and its expansion emerge due to the tendency to satisfy the holographic principle. Finally, due to the results indicated by fractional quantum cosmology, we argue the following proposition: the cosmological apparent horizon of the Universe has the same structure of a random fractal as the event horizon of the Schwarzschild black hole. This leads to modified Friedmann equations that reveal an effect of fractal geometry that amplifies the content of baryonic matter already existing in the Universe and thus simulates the additional content of matter that we currently call dark matter.

Keywords: emergent cosmology; fractional quantum gravity; Schwarzschild black hole; dark matter.

RESUMO

Esta dissertação investiga um modelo cosmológico que explica os dados observacionais sobre o conteúdo de matéria do Universo usando a teoria de Padmanabhan da cosmologia emergente e insights da gravidade quântica fracionária aplicada ao buraco negro de Schwarzschild. Duas direções principais levam a este modelo. Por um lado, começamos com o formalismo Hamiltoniano da relatividade geral e a quantização canônica da teoria que leva à equação de Wheeler-DeWitt. Um espaço-tempo esfericamente simétrico simplifica então a aplicação da equação de Wheeler-DeWitt e podemos investigar a quantização do buraco negro de Schwarzschild, seu espectro de massa, e sua termodinâmica, no limite semi-clássico. O estudo de fractais e o uso da derivada fracionária de Riesz através da gravidade quântica fracionária mostram que a área da superfície do horizonte de eventos do buraco negro de Schwarzschild tem a estrutura de um fractal aleatório, cuja descrição é possível por quantidades fracionárias. Por outro lado, mostramos que o horizonte aparente cosmológico fornece tanto uma temperatura Hawking associada ao horizonte de um espaço-tempo FLRW, como é o mais adequado horizonte para obtermos as equações de Friedmann com a teoria de Padmanabhan em que o espaço cósmico e sua expansão emergem devido à tendência de satisfazer o princípio holográfico. Finalmente, devido aos resultados indicados pela cosmologia quântica fracionária, defendemos a seguinte proposição: o horizonte aparente cosmológico do Universo tem a mesma estrutura de um fractal aleatório que o horizonte de eventos do buraco negro de Schwarzschild. Isto leva a equações de Friedmann modificadas que revelam um efeito de geometria fractal que amplifica o conteúdo de matéria bariônica já existente no Universo e assim simula o conteúdo adicional de matéria que atualmente chamamos de matéria escura.

Palavras-chave: cosmologia emergente; gravidade quântica fracionária; buraco negro de Schwarzschild; matéria escura.

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LIST OF ABBREVIATIONS AND ACRONYMS

GR	General relativity
CMB	Cosmic microwave background
FQC	Fractional quantum cosmology
FQM	Fractional quantum mechanics
CQG	Canonical quantum gravity
WDW	Wheeler-DeWitt
SBH	Schwarzschild black hole
FQG	Fractional quantum gravity
FLRW	Friedmann-Lemaître-Robertson-Walker
ΛCDM	Lambda-Cold Dark Matter
ADM	Arnowitt, Deser, and Misner
IR	Infrared
UV	Ultraviolet
MOND	Modified Newtonian dynamics
FSE	Fractional Schrödinger equation
EH	Event horizon
AH	Apparent horizon
KH	Killing horizon
HH	Hubble horizon
AdS	Anti de Sitter
CFT	Conformal field theory
CDM	Cold dark matter
ΛCBM	Lambda-Cold Baryonic Matter

CONTENTS

1 INTRODUCTION

Remember what the dormouse said: feed your head.

Jefferson Airplane - White Rabbit

In the book *Conceptions of Cosmos* [2], H. Kragh emphasizes that although modern cosmology has its roots in philosophical investigations aimed at a presumed order and intelligible rationality of the functioning of the Universe (referred to as the whole, nature in its entirety) according to Greek natural philosophy, this science was not well structured until the development of general relativity (GR) in the 20th century. Cosmology as a current solid scientific discipline can be traced back to the 1917 paper *Cosmological Considerations in the General Theory of Relativity* [3] by A. Einstein. After that, names like W. de Sitter, K. Schwarzschild, and A. Friedmann, built up a body of ideas that obtained the most important observational corroborations in astronomy and astrophysics in the last years. Major examples of this rapid advance include the observational confirmations of the cosmic microwave background (CMB) radiation, black holes, gravitational waves, the formation and evolution of the large-scale structures of the Universe, and many others.

Despite the advances made in modern cosmology, mysteries persist that cannot be immediately explained by our current model of the Universe, which means that we are unaware around 95% of the content of the Universe today. This unknown portion is divided into two parts which, in honor of our ignorance, are referred to as "dark matter" and "dark energy". Dark energy is generally attributed to the cosmological term called the cosmological constant, Λ, which provides for the accelerated expansion of the Universe, as confirmed by the distant type Ia supernovae in the 1990s. In turn, dark matter is responsible for the anomalous gravitational effects of gravitational lenses and for the observation of galaxy rotation curves that diverge from what is theoretically expected. If GR is correct on these scales, it is therefore suggested that there is more matter in the Universe than we can observe in galaxies and galaxy clusters.

In this sense, both these gaps in our current knowledge are often interpreted as reminders of the absence of a consistent theory of quantum gravity. In other words, dark energy and dark matter may be as yet ununderstood manifestations of a solid interface between gravitation and quantum mechanics [4]. With familiar proposals to deal with this issue such as string theory, loop quantum gravity, and asymptotically safe quantum gravity, the problem of quantum gravity has recently been explored from the pragmatic but non-fundamental point of view of fractional quantum cosmology (FQC) [5]. In this approach, Lévy processes are used to parameterize the trajectories of quantum particles in the formalism of path integrals according to Laskin's fractional quantum mechanics (FQM) [6], where fractional calculus is introduced using the Riesz fractional derivative. FQC occurs when we extend FQM to the context of canonical quantum gravity (CQG) theory and the canonical quantization on the cosmological minisuperspace perspective, hence obtaining the fractional generalization of the Wheeler-DeWitt (WDW) equation.

By transcending the cosmological scenario and applying it to the vacuum solution of a spacetime with a spherical symmetry, namely, the well-known Schwarzschild solution of GR, the FQC paradigm produces the fractional version of the WDW equation for the Schwarzschild black hole (SBH), and when we start dealing with practical results of this method in gravity we call it fractional quantum gravity (FQG). Using FQG, the authors of a recent work [7] showed that the surface area of an SBH has the structure of a random fractal which leads to correspondents fractional changes in the thermodynamic quantities of the black hole. Such effects of quantum gravity origin, now in fractional form, change the temperature and entropy of the black hole from the conventional Hawking temperature and Bekenstein-Hawking entropy in the semi-classical limit, respectively. Specifically, the entropy of the black hole becomes

$$\mathcal{S}_{\text{fractal}} = \mathcal{S}_{\text{B-H}}^{d/2}, \tag{1.1}$$

with $\mathcal{S}_{\text{B-H}}$ the Bekenstein-Hawking entropy of the SBH and d the fractal dimension associated with the surface area of the black hole.

Extending the connection already under investigation between the laws of thermodynamics and GR [8–10], Padmanabhan [11] proposed that both cosmic space and its expansion behave as an emergent phenomenon stemming from the tendency of the Universe to satisfy a form of the holographic principle[1], which had been called holographic equipartition. Since is suggested by the observations, our Universe is asymptotically a de

[1]As Susskind suggests as a definition [12]: the physical information of a higher order spacetime is contained in a lower order spacetime (in the boundary of the first one).

Sitter spacetime, and the idea is that at this stage there is equality between the degrees of freedom of the surface of the cosmological horizon, N_{sur}, and the cosmic volume, N_{bulk}, contained by this horizon. The Universe then emerges and expands due to the difference between these degrees of freedom according to the equation

$$\frac{dV}{dt} = L_{\text{P}}^2(N_{\text{sur}} - N_{\text{bulk}}), \tag{1.2}$$

where $L_{\text{P}} := G^{-1/2}$ is the Planck length (the natural units that $\hbar = c = k_{\text{B}} = 1$ will be used through this dissertation), V and t the cosmic volume and cosmic time (measured by a comoving observer), respectively. Using as degrees of freedom, N_{sur}, a measure of the entropy of the cosmological horizon and for the degrees of freedom, N_{bulk}, a form of energy equipartition law in gravity from the Komar energy and the temperature associated with the cosmological horizon, equation (1.2) returns the Friedmann equations.

Therefore, by providing a method that leads directly to the Friedmann equations from the holographic principle and the thermodynamics of cosmological horizons, equation (1.2) opens a natural window of theoretical approaches that describe alternative cosmological models from modifications in the cosmological horizons. Taking into account the FQC, the authors of another recent work [13] showed that the effective area of the cosmological apparent horizon of the de Sitter spacetime has the same random fractal structure as the equation (1.1), obtained from the surface geometry of the fractional SBH. Being the cosmological apparent horizon the horizon that effectively possesses a thermodynamics for the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and adequately reflects the laws of thermodynamics from equation (1.2) [14], this dissertation has the following main objective: choosing the cosmological apparent horizon as the holographic screen, inspired by the SBH geometry given in equation (1.1), we use the modification to a horizon whose structure is of a random fractal and we apply the thermodynamic quantities resulting in equation (1.2) to obtain the modified Friedmann equations and study the consequences in the standard Lambda-Cold Dark Matter (ΛCDM) model.

This dissertation follows closely the reference [15], and has two auxiliary objectives: (i) lead to a detail that helps in the reading and assimilation of ideas present in [15], guiding the reader who wants more information on each of the topics addressed and (ii) while various ideas and contents are connected in the model proposed in [15] we intend to explore certain topics in slightly alternative ways to strengthen the understanding of some key points. Thus, this dissertation can be understood, but not limited to, as a guide to reference [15].

The rest of the dissertation is organized as follows:

Chapter 2: The theoretical foundations of CQG theory are revised by beginning with the Hamiltonian formalism of GR. Canonical quantization is then applied to the appropriate collection of variables and the WDW equation is obtained and discussed;

Chapter 3: The results of chapter 2 are applied to the Schwarzschild solution of GR, where the maximal analytic extension of this solution is considered and the SBH is quantized. The WDW equation for the SBH is analyzed in the semi-classical limit and the thermodynamics of the SBH is discussed;

Chapter 4: The general concept of fractals, their main characteristics, common properties in nature, and their particular occurrence as Brownian motion trajectories are presented. Then, the connection with fractional calculus and its interpretation in the context of FQM and FQC for the SBH is established. The event horizon of the SBH is shown to have a surface area of random fractal geometry;

Chapter 5: Horizons in GR and cosmology are studied with emphasis on the apparent horizon, which is shown to be the horizon in which a Hawking temperature can be associated with an FLRW spacetime. After that, emergent cosmology and its relation to the thermodynamics of cosmological horizons is discussed. The ΛCDM model is briefly reviewed;

Chapter 6: It is proposed to combine the results reached in the previous chapters: the fractal geometry of the SBH, and the use of emergent cosmology equipped with the thermodynamics of cosmological horizons to describe a cosmological model. As a result, a fractal structure is proposed for the apparent horizon of the Universe, and the modified Friedmann equations are obtained. Initial consequences to the ΛCDM model are investigated;

Chapter 7: The conclusions are presented in retrospect, and it is emphasized that in the model proposed in this dissertation dark matter is not necessary to justify the measurements of the cosmological parameters currently observed for the content of cold matter in the Universe.

2 CANONICAL QUANTUM GRAVITY

Saiba que ainda estão rolando os dados, porque o tempo, o tempo não para.

Cazuza - O Tempo Não Para

2.1 Prologue

The spotlight of this first chapter is to review and discuss some results of CQG theory. To do that is important to assume a dynamical description for the gravitational field in a Hamiltonian formalism of GR. This can be done by extracting the time coordinate with a splitting of the spacetime manifold in a manifold composed by a set Σ of 3-dimensional spacelike hypersurfaces Σ_t, joined with a time parameter t in an open subset of $\mathbb{R}$. The Arnowitt, Deser, and Misner (ADM) canonical variables are then naturally implemented for the canonical quantization [16]. After that, canonical quantization is performed and the resulting Hamiltonian constraints are studied, before and after the quantization. Finally, the WDW equation is identified and briefly analyzed.

2.2 Hamiltonian formalism and the ADM variables

The spacetime is modeled by a 4-dimensional Lorentzian manifold $(\mathscr{M}, \mathbf{g})$ with local coordinates x^μ in the Minkowski spacetime M^4 such that elevates space and time at the same physical level. The metric tensor $g_{\mu\nu}$ signature adopted is $(-, +, +, +)$. This scenario provides a way to write the laws of physics in a covariant form, which reflects no privileged coordinate system in nature. In GR, Einstein field equations describe the spacetime geometry due to the mass and energy distribution in a region of spacetime. But these equations do not have a "time" evolution in the gravitational field, which is contained by the metric tensor, $g_{\mu\nu}$. Considering a certain mass and energy distribution in a region of spacetime, $g_{\mu\nu}$ describes the correspondent geometry by field equations, but these equations alone do not prescribe how this curvature evolves over an inherent time flow. In this fashion, spacetime is said to be frozen, which means that a time dynamics of the spacetime geometry is not determined solely by the distribution of mass and energy (as described by the stress-energy tensor) and the field equations.

The Hamiltonian formalism of GR starts with a foliation (3+1 decomposition) of the spacetime manifold $(\mathscr{M}, \mathbf{g})$ in a family of 3-dimensional spacelike hypersurfaces, parameterized by a single, real, parameter that acts as time coordinate. Considering the spacetime manifold $(\mathscr{M}, \mathbf{g})$, a foliation $\mathscr{F}$ of $\mathscr{M}$ is a map $\mathscr{F} : \mathscr{M} \to \Sigma \times I \subset \mathbb{R}$, where I is a open subset of $\mathbb{R}$ and Σ is the set of all spacelike hypersurfaces which cover $(\mathscr{M}, \mathbf{g})$. That is, a foliation can be thought as a family $\Sigma = \{\Sigma_t\}$ of embedded 3-dimensional spacelike hypersurfaces Σ_t covering $\mathscr{M}$, where t is a real parameter which labels the hypersurfaces, interpreted as an instant of time, t = constant. Since GR disagrees with a privileged notion of time, this process seems to be problematic given the principle of general covariance. However, it is not, if we assume the spacelike hypersurfaces to be Cauchy surfaces [17]. A Cauchy surface is a spacelike hypersurface that intersects every timelike curve exactly once [18]. In other words, the physical future[1] of every event in spacetime can be uniquely specified by the event projection in a Cauchy surface. A spacetime $(\mathscr{M}, \mathbf{g})$ is said to be globally hyperbolic if it admits a Cauchy surface [18]. And, by assuming there exists a global "time function" t such that we can take a foliation with Cauchy surfaces as the spacelike hypersurfaces Σ_t with t = constant, as proved possible by Hawking and Ellis [19], one can shown that $\mathscr{F}$ is a diffeomorphism and GR has a diffeomorphism invariance by the foliation in this sense; i.e., $\mathscr{M} \cong \Sigma \times I$ [17, 20].

2.2.1 3+1 decomposition

One starts with the Einstein-Hilbert action, whose stationary condition generates the field equations and it is a functional of the spacetime metric tensor $g_{\mu\nu}$, with $c = 1$

$$S_{\text{E-H}} = S_{\text{E-H}}[g_{\mu\nu}] = \int_{\mathscr{M}} d^4x\, \mathscr{L}_{\text{E-H}} = \frac{1}{16\pi G} \int_{\mathscr{M}} d^4x\, \sqrt{-g}\, R. \tag{2.1}$$

Henceforth, the natural system of units will be used: $c = \hbar = k_B = 1$. An additional assumption through the foliation $\mathscr{F}$ of the spacetime manifold is that $(\mathscr{M}, \mathbf{g})$ (henceforth, just $\mathscr{M}$) to be a spatially closed manifold[2], since, in general, FLRW cosmological models are covered by this description [17]. The goal hereafter is to rewrite the Einstein-Hilbert action in the context of spacetime foliation. Defining the set of the spacelike hypersurfaces Σ_t such that each of which is parametrized by a value of t, the foliation $\mathscr{F}$ can be formally

[1]Formally, the causal future. Please, see section 5.1.3 for a reminder.

[2]A compact manifold without boundary [18].

implemented by splitting $\mathscr{M}$ as the set

$$\Sigma_t := \{u^\mu(x^i, t)\}, \tag{2.2}$$

with u^μ arbitrary spacetime coordinates given by parametric functions for each t which preserve diffeomorphism invariance, then being properly invertible, continuous, and differentiable. One can consider a set of vectors $\{\mathbf{e}_\mu\}$ which form vector basis (each one associated with the coordinates u^μ, t = constant) over $\mathscr{M}$. A vector basis $\mathbf{v}_i$ of tangent spacelike vectors to a generic hypersurface Σ_t, can be identified by a change of basis

$$\mathbf{v}_i = \big(\partial_i u^\mu\big)\mathbf{e}_\mu := v_i^\mu \mathbf{e}_\mu. \tag{2.3}$$

A vector field of unit normal timelike vectors $\mathbf{n} = n^\mu \mathbf{e}_\mu$ to this hypersurface naturally satisfies the condition $\mathbf{v}_i \cdot \mathbf{n} = 0$. Being $\mathbf{n}$ timelike, $n^\mu n_\mu = -1$. Then, sets as $\{\mathbf{n}, \mathbf{v}_i\}$ in each Σ_t can be taken as a complete vector basis over Σ_t. Being a vector $\mathbf{t} = t^\mu \mathbf{e}_\mu$ in $\mathscr{M}$ viewed as connecting a point (x^i, t) on such a hypersurface Σ_t to a point $(x^i, t+dt)$ on a neighboring hypersurface Σ_{t+dt} (please, see figure 1), we have by equation (2.3)

$$\begin{aligned} \mathbf{t} = t^\mu \mathbf{e}_\mu =& N\mathbf{n} + N^i \mathbf{v}_i \\ =& N(n^\mu \mathbf{e}_\mu) + N^i(v_i^\mu \mathbf{e}_\mu) \\ =& (Nn^\mu + N^i v_i^\mu)\,\mathbf{e}_\mu, \end{aligned} \tag{2.4}$$

where N and N^i are called lapse function and shift vector, respectively. The metric tensor $g_{\mu\nu}$ can be written in the new basis $\{\mathbf{t}, \mathbf{v}_i\}$ over $\mathscr{M}$. At the hypersurface, the spatial metric tensor given the spatial part of the metric tensor $g_{\mu\nu}$

$$g_{ij} = h_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j. \tag{2.5}$$

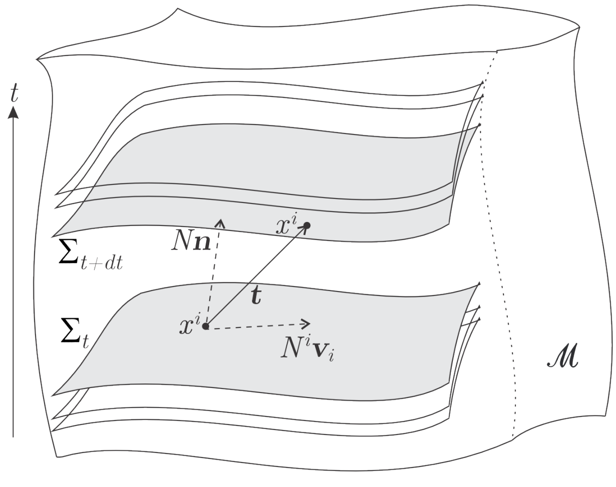
The other components of the metric tensor are given by

$$g_{0i} = \mathbf{t} \cdot \mathbf{v}_i = (N\mathbf{n} + N^k \mathbf{v}_k) \cdot \mathbf{v}_i = N^k h_{ki}, \tag{2.6}$$

and

$$g_{00} = \mathbf{t} \cdot \mathbf{t} = (N\mathbf{n} + N^i \mathbf{v}_i) \cdot (N\mathbf{n} + N^j \mathbf{v}_j) = -N^2 + N^i N^j h_{ij}, \tag{2.7}$$

Figure 1: Two neighboring spacelike hypersurfaces, in representation of the time flow defined by a timelike displacement vector **t** decomposed into its normal and tangent components on the hypersurface Σ_t.



Source: The author (2024).

where the condition $\mathbf{v}_i \cdot \mathbf{n} = 0$ was used. The general line element (henceforth, metric) after $\mathscr{F}$ now can be expressed by N, N^i and h_{ij} [17] as

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= g_{00}dx^0dx^0 + 2g_{0i}dx^0dx^i + g_{ij}dx^idx^j \\ &= \big(-N^2 + N^iN^jh_{ij}\big)dx^0dx^0 + 2N^kh_{ki}dx^0dx^i + h_{ij}dx^idx^j \\ &= -N^2dt^2 + h_{ij}(dx^i + N^idt)(dx^j + N^jdt), \end{aligned} \tag{2.8}$$

with $dx^0 = dt$. Noting the equation (2.8), we can interpret physically the lapse function and the shift vector. On the timelike curve along $\mathbf{n}$, that is $N^i = N^j = 0$, the rate of variation of the proper time τ is modulated by the lapse function N. Evaluated the flow rate of the proper time τ under these conditions, the rate of change in the local spatial coordinates from Σ_t to Σ_{t+dt} is given by the shift vector N^i, which thus describes how spacetime points move from one hypersurface to the next as time evolves. It is clear that all three quantities, N, N^i, and h_{ij}, are functions of the collection of coordinates (x^k, t). To use equation (2.1) one needs the determinant of metric tensor, $\det(\mathbf{g}) := g$, in terms of equation (2.8). Writing the metric with the diagonalized form of h_{ij}, the determinant can be easily identified if a new collection of coordinates (z^k, t) is employed

$$-N^2dt^2 + h_{ii}dz^idz^i = g_{00}dt^2 + g_{ii}dx^idx^i, \tag{2.9}$$

then,

$$g = \det(\mathbf{g}) = -N^2\det(\mathbf{h}) = -N^2 h, \tag{2.10}$$

such that $dz^k = dx^k + N^k dt$. To account for the change of coordinates performed it is necessary to obtain the Jacobian determinant J of the transformation $(x^k, t) \to (z^k, t)$. To explicitly denote the coordinate transformation in 4-dimensional spacetime, we introduce the variables χ^α and ζ^β such that $d\chi^0 = d\zeta^0 = dt$ and $d\zeta^k = d\chi^k + N^k dt$. The components of the Jacobian matrix are $J_{\alpha,\beta} = \partial\zeta^\beta/\partial\chi^\alpha$, then

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ N^1 & 1 & 0 & 0 \\ N^2 & 0 & 1 & 0 \\ N^3 & 0 & 0 & 1 \end{pmatrix}, \tag{2.11}$$

and the Jacobian determinant is $J = 1$. We can then write equation (2.10) in the coordinates (x^k, t) without a correction

$$\sqrt{-g} = N\sqrt{h}. \tag{2.12}$$

To go further, equation (2.1) also asks us to express the Ricci scalar R in terms of quantities related to the hypersurface. For that, one defines projection tensor $q_{\mu\nu}$ as

$$q_{\mu\nu} := g_{\mu\nu} - \epsilon\, n_\mu n_\nu, \tag{2.13}$$

with ϵ carries information about the signature of the metric tensor since $\epsilon = n_\mu n^\mu = -1$. This $q_{\mu\nu}$ tensor acts, for example, by projecting a generic vector V^μ in $\mathscr{M}$ onto the hypersurface Σ_t, i.e., with n^ν orthogonal to Σ_t we have $q_{\mu\nu}V^\mu n^\nu = 0$. In fact,

$$\begin{aligned} q_{\mu\nu}V^\mu n^\nu &= (g_{\mu\nu} - \epsilon\, n_\mu n_\nu)V^\mu n^\nu \\ &= g_{\mu\nu}V^\mu n^\nu - \epsilon\, n_\mu n_\nu V^\mu n^\nu \\ &= V^\mu n_\mu - \epsilon^2\, V^\mu n_\mu = 0. \end{aligned} \tag{2.14}$$

Next, we need to address a different notion of curvature to $\mathscr{M}$, from the one given by the intrinsic curvature carried by the Riemann tensor. Such a notion arises for the existence of curvature in $\mathscr{M}$ that is not evaluated locally, but globally. A simplified illustration is a 2-dimensional cylindrical surface, where a cylinder can be thought of as a developable surface, meaning it can be flattened without stretching or distorting its shape. When you flatten a cylinder, it becomes a flat surface, which has zero intrinsic curvature. However, the cylindrical surface looks curved when is embedded in a high-dimensional flat space, as the 3-dimensional Euclidean space. This perception of curvature is called extrinsic curvature. In our case, the embedded space is the hypersurface, and the high-dimensional space is $\mathscr{M}$. The extrinsic curvature tensor is then a tensor defined as

$$K_{\mu\nu} := q^{\rho}_{\mu} q^{\sigma}_{\nu} \nabla_{\rho} n_{\sigma}. \tag{2.15}$$

The tensor $K_{\mu\nu}$ gives the curvature of Σ_t from the visualization of the spacetime manifold, $\mathscr{M}$, and measure the change in the direction of $\mathbf{n}$ projected onto Σ_t. Physically, the extrinsic curvature will tell how the spatial part of spacetime bends along the evolution of time. Using the so-called fundamental theorem of submanifolds [17], we are allowed to express the 4-dimensional Ricci scalar R in terms of the 3-dimensional Ricci scalar $\bar{R}$ on Σ_t, the extrinsic curvature tensor $K_{\mu\nu}$, and the components of $\mathbf{n}$, and then write

$$R = \bar{R} + K_{\mu\nu} K^{\mu\nu} - K^2 - 2\nabla_{\mu}(n^{\nu}\nabla_{\nu} n^{\mu} - n^{\mu} K), \tag{2.16}$$

where K is the contraction of K^{ν}_{μ}. Equation (2.16) is called the Gauss-Codazzi equation [17]. We can express the extrinsic curvature of a particular hypersurface by writing the 4-dimensional $K_{\mu\nu}$ tensor in terms of 3-dimensional indices

$$K_{\mu\nu} = v^i_{\mu} v^j_{\nu} K_{ij}, \tag{2.17}$$

with $v^i_{\mu} := \partial x^i / \partial u^{\mu}$. Since $\{\mathbf{v}_i\}$ form a vector basis on Σ_t, the full contraction of its components v^{μ}_i is $v^{\mu}_i v^j_{\nu} = \delta^j_i \delta^{\mu}_{\nu}$, and then $K_{\mu\nu} K^{\mu\nu} = K_{ij} K^{ij}$. The equation (2.16) becomes

$$R = \bar{R} + K_{ij} K^{ij} - K^2 - 2\nabla_{\mu}(n^{\nu}\nabla_{\nu} n^{\mu} - n^{\mu} K). \tag{2.18}$$

We can then rewrite the Einstein-Hilbert Lagrangian density in the action (2.1) by substituting equations (2.12) and (2.18)

$$\mathscr{L}_{\text{E-H}} = \frac{1}{16\pi G} N\sqrt{h}\big[\bar{R} + K_{ij} K^{ij} - K^2 - 2\nabla_{\mu}(n^{\nu}\nabla_{\nu} n^{\mu} - n^{\mu} K)\big]. \tag{2.19}$$

To evaluate the action with this Lagrangian density we must redefine the integration domain of equation (2.1) according to the foliation $\mathscr{F}$ of spacetime, thus delimiting the integral on the hypersurfaces Σ between two time instants t_1 and t_2, which gives

$$S_{\text{E-H}} = \frac{1}{16\pi G}\int_{t_1}^{t_2} dt \int_{\Sigma} d^3x\, N\sqrt{h}\big[\bar{R} + K_{ij}K^{ij} - K^2 - 2\nabla_\mu(n^\nu\nabla_\nu n^\mu - n^\mu K)\big]. \tag{2.20}$$

Defining the brackets of the last term in the integral as $\eta^\mu := n^\nu\nabla_\nu n^\mu - n^\mu K$, we can rewrite equation (2.20) in terms of the covariant divergent

$$S_{\text{E-H}} = \frac{1}{16\pi G}\int_{t_1}^{t_2} dt \int_{\Sigma} d^3x\, N\sqrt{h}\big[\bar{R} + K_{ij}K^{ij} - K^2 - 2\nabla_\mu\eta^\mu\big]. \tag{2.21}$$

The divergence theorem says that the term with the covariant divergent in (2.21) gives rise to a boundary contribution in the action integral, $S_{\partial\mathscr{M}}$, and from what was previously said we will not consider boundary terms for physical reasons[3]. The remaining action we will refer to as the ADM action, which is expressed in the configuration variables called ADM (in honor of R. Arnowitt, S. Deser and C. Misner [16]), namely, N, N^i, h_{ij}. It is possible to verify the explicit dependency of K_{ij} with N^i by[4]

$$K_{ij} = \frac{1}{2N}\big(-2D_{(i}N_{j)} + \partial_t h_{ij}\big), \tag{2.22}$$

where D_i is the 3-dimensional covariant derivative. Thus, the ADM Lagrangian density $\mathscr{L}_{ADM}$ is:

$$\mathscr{L}_{ADM} = \frac{1}{16\pi G} N\sqrt{h}\big(\bar{R} + K_{ij}K^{ij} - K^2\big), \tag{2.23}$$

such that, at first, $\mathscr{L}_{ADM} = \mathscr{L}_{ADM}[N, N^i, h_{ij}, \partial_t N, \partial_t N^i, \partial_t h_{ij}]$. Lowering the index of the extrinsic curvature tensor, the symmetrization of the spatial metric tensor h_{ij} gives us

$$\mathscr{L}_{ADM} = \frac{1}{16\pi G} N\sqrt{h}\big[\bar{R} + \frac{1}{2}\big(h^{ik}h^{jl} + h^{il}h^{jk} - 2h^{ij}h^{kl}\big)K_{ij}K_{kl}\big], \tag{2.24}$$

with the identification

$$\mathscr{G}^{ijkl} := \frac{\sqrt{h}}{2}\big(h^{ik}h^{jl} + h^{il}h^{jk} - 2h^{ij}h^{kl}\big), \tag{2.25}$$

[3]For the reader interested in the unfolding of boundary terms to obtain a well-posed variational principle of the Einstein-Hilbert action, we recommend reading section 4.3 of [21].

[4]The deduction of the analytical dependence can be seen in section 3.2 of [17].

called the DeWitt metric tensor [17]. Thus, equation (2.23) becomes

$$\mathscr{L}_{ADM} = \frac{1}{16\pi G} N\sqrt{h}\big(\bar{R} + \frac{1}{\sqrt{h}}\mathscr{G}^{ijkl}K_{ij}K_{kl}\big). \tag{2.26}$$

2.2.2 Constraints

To proceed to the construction of the Hamiltonian density in the ADM variables, we need to calculate the conjugate momenta to these variables. For N, and N^i, we have,

$$\Pi_N = \frac{\partial \mathscr{L}_{ADM}}{\partial\big(\partial_t N\big)} = 0, \tag{2.27a}$$

$$\Pi_i = \frac{\partial \mathscr{L}_{ADM}}{\partial\big(\partial_t N^i\big)} = 0, \tag{2.27b}$$

that is, the canonical momenta associated with N and N^i are constrained to be zero, indicating that N and N^i have not a particular dynamics. Thus, more degrees of freedom than the actual physical degrees are considered. In the canonical quantization, we will carry the Poisson brackets of the variables, which generate the equations of motion, in the usual quantum commutators. Thus, the implementation of the conditions (2.27) should be performed only afterwards the calculation of the Poisson brackets. Such conditions then denote the so-called weak equality of Dirac [17], and for the conjugate momenta

$$\Pi_N \approx 0, \tag{2.28a}$$

$$\Pi_i \approx 0. \tag{2.28b}$$

The above expressions are the so-called primary constraints of the Hamiltonian formalism of GR. With the conjugate momentum of h_{ij} given by

$$\Pi^{ij} = \frac{\partial \mathscr{L}_{ADM}}{\partial\big(\partial_t h_{ij}\big)} = \frac{1}{16\pi G}\mathscr{G}^{ijkl}K_{kl}, \tag{2.29}$$

where it was used that $h_{kl} = \delta^i_k \delta^j_l h_{ij}$ and (2.22). The ADM Hamiltonian density will be a functional, $\mathscr{H}_{ADM} = \mathscr{H}_{ADM}[N, N^i, h_{ij}, \Pi_N, \Pi_i, \Pi^{ij}]$, and then, as usual, we will have to express our configuration variables in terms of the conjugate momenta. By defining the inverse of the DeWitt metric tensor as

$$\mathscr{G}_{ijkl} := \frac{1}{2\sqrt{h}}\big(h_{ik}h_{jl} + h_{il}h_{jk} - 2h_{ij}h_{kl}\big), \tag{2.30}$$

the equation (2.29) is easily inverted

$$K_{ij} = 16\pi G\, \mathscr{G}_{ijkl}\Pi^{kl}, \tag{2.31}$$

and then the second term of the right-hand side of the equation (2.26) becomes

$$\begin{aligned} \mathscr{G}^{ijkl}K_{ij}K_{kl} &= \mathscr{G}^{ijkl}\big(16\pi G\, \mathscr{G}_{ijkl}\Pi^{kl}\big)\big(16\pi G\, \mathscr{G}_{klij}\Pi^{ij}\big) \\ &= (16\pi G)^2\, \mathscr{G}^{ijkl}\mathscr{G}_{klij}\mathscr{G}_{ijkl}\Pi^{ij}\Pi^{kl} \\ &= (16\pi G)^2\, \mathscr{G}_{ijkl}\Pi^{ij}\Pi^{kl}. \end{aligned} \tag{2.32}$$

where we use the condition that $\mathscr{G}^{ijkl}\mathscr{G}_{klrs} = \delta^i_{(r}\delta^j_{s)}$, which naturally follows from the contraction of the product of the metric tensors h_{ij}. The time derivative of h_{ij} can also be rewritten by inverting equation (2.22) and using equation (2.31) as

$$\partial_t h_{ij} = 32\pi G N\, \mathscr{G}_{ijkl}\Pi^{kl} + 2D_{(i}N_{j)}. \tag{2.33}$$

The ADM action can now be rewritten defining the ADM Hamiltonian density

$$\begin{aligned} S_{\mathrm{ADM}} &= \int_{t_1}^{t_2} dt \int_\Sigma d^3x\, \mathscr{L}_{\mathrm{ADM}} \\ &= \frac{1}{16\pi G}\int_{t_1}^{t_2} dt \int_\Sigma d^3x\, N\sqrt{h}\left(\bar{R} + \frac{1}{\sqrt{h}}\mathscr{G}^{ijkl}K_{ij}K_{kl}\right) \\ &= \int_{t_1}^{t_2} dt \int_\Sigma d^3x \left[\frac{1}{16\pi G}\, N\sqrt{h}\bar{R} + 16\pi G N\, \mathscr{G}_{ijkl}\Pi^{ij}\Pi^{kl}\right] \\ &= \int_{t_1}^{t_2} dt \int_\Sigma d^3x \left[\big(\partial_t y_a \Pi^a\big)_{\mathrm{ADM}} - \lambda^N\Pi_N - \lambda^i\Pi_i - \mathscr{H}_{\mathrm{ADM}}\right], \end{aligned} \tag{2.34}$$

where $\{\partial_t y_a\}$ and $\{\Pi^a\}$ are the set of ADM variables derived in time and the set of corresponding conjugate momenta, respectively. The Lagrange multipliers λ^N and λ^i are introduced as new independent variables to preserve the primary constraints, equations (2.28), so that the stationary condition of the action for variations in the multipliers immediately recovers the constraints. Hence, using equation (2.33)

$$\begin{aligned} S_{\mathrm{ADM}} &= \int_{t_1}^{t_2} dt \int_\Sigma d^3x \left[\Pi^{ij}\partial_t h_{ij} - \lambda^N\Pi_N - \lambda^i\Pi_i - \mathscr{H}_{\mathrm{ADM}}\right] \\ &= \int_{t_1}^{t_2} dt \int_\Sigma d^3x \left[32\pi G N\, \mathscr{G}_{ijkl}\Pi^{ij}\Pi^{kl} + 2\Pi^{ij}D_iN_j - \lambda^N\Pi_N - \lambda^i\Pi_i - \mathscr{H}_{\mathrm{ADM}}\right]. \end{aligned} \tag{2.35}$$

Working only with the second term in the integral of equation (2.35) we have

$$\int_\Sigma d^3x\,\Pi^{ij}D_iN_j = -\int_\Sigma d^3x\,N_jD_i\Pi^{ij} + \int_\Sigma d^3x\,D_i\big(N_j\Pi^{ij}\big), \tag{2.36}$$

and defining $\chi^i := N_j\Pi^{ij}$, the divergence theorem assures us that the second term in the right-hand side of equation (2.36) gives rise to a boundary contribution and then vanishes. Equation (2.35) then becomes

$$S_{\text{ADM}} = \int_{t_1}^{t_2} dt \int_\Sigma d^3x\,\big[32\pi GN\,\mathscr{G}_{ijkl}\Pi^{ij}\Pi^{kl} - 2N_jD_i\Pi^{ij} - \lambda^N\Pi_N - \lambda^i\Pi_i - \mathscr{H}_{\text{ADM}}\big]. \tag{2.37}$$

By substituting in the equation (2.34) the ADM Hamiltonian density is identified as

$$\mathscr{H}_{\text{ADM}} = \lambda^N\Pi_N + \lambda^i\Pi_i + N\Big(16\pi G\,\mathscr{G}_{ijkl}\Pi^{ij}\Pi^{kl} - \frac{\sqrt{h}\bar{R}}{16\pi G}\Big) + N^i\Big(-2h_{ij}D_k\Pi^{kj}\Big), \tag{2.38}$$

and to the above objects under parentheses, we give special definitions

$$\mathcal{H} := 16\pi G\,\mathscr{G}_{ijkl}\Pi^{ij}\Pi^{kl} - \frac{\sqrt{h}\bar{R}}{16\pi G}, \tag{2.39a}$$

$$\mathcal{H}_i := -2h_{ij}D_k\Pi^{kj}, \tag{2.39b}$$

which are the so-called super-Hamiltonian and supermomentum, respectively. Thus, we can write the primary Hamiltonian H of the theory as

$$\begin{aligned} H &= \int_\Sigma d^3x\,\mathscr{H}_{\text{ADM}} \\ &= \int_\Sigma d^3x\,\Big(\lambda^N\Pi_N + \lambda^i\Pi_i + N^i\mathcal{H}_i + N\mathcal{H}\Big). \end{aligned} \tag{2.40}$$

To evaluate the situation of the primary constraints in this Hamiltonian formalism, we can investigate the equations of motion for Π_N and Π_i by calculating the respective Poisson brackets, $\{\Pi_N, \mathscr{H}_{\text{ADM}}\}$ and $\{\Pi_i, \mathscr{H}_{\text{ADM}}\}$. To do this we write the fundamental Poisson brackets at equal-time in the phase space of the ADM variables [17], then

$$\{N(\mathbf{x},t), \Pi_N(\mathbf{x}',t)\} = \delta(\mathbf{x}-\mathbf{x}'), \tag{2.41a}$$

$$\{N^i(\mathbf{x},t), \Pi_j(\mathbf{x}',t)\} = \delta^i_j\delta(\mathbf{x}-\mathbf{x}'), \tag{2.41b}$$

$$\{h_{ij}(\mathbf{x},t), \Pi^{kl}(\mathbf{x}',t)\} = \delta_i^{(k}\delta_j^{l)}\delta(\mathbf{x}-\mathbf{x}') \tag{2.41c}$$

and the other Poisson brackets vanish, as usual in the Hamiltonian formalism of a general classical field theory [22]. Hence, first, we note that the equations of motion for N and N^i are clearly, using the Poisson brackets of the variables (2.41), and equation (2.40)

$$\partial_t N = \{N, \mathscr{H}_{\text{ADM}}\} = \lambda^N, \tag{2.42a}$$

$$\partial_t N^i = \{N^i, \mathscr{H}_{\text{ADM}}\} = \lambda^i, \tag{2.42b}$$

and, as expected, N and N^i have arbitrary dynamics. Next, the equations of motion of the conjugate momenta, again with (2.41) and (2.40) are

$$\partial_t \Pi_N = \{\Pi_N, \mathscr{H}_{\text{ADM}}\} = \mathcal{H}, \tag{2.43a}$$

$$\partial_t \Pi_i = \{\Pi_i, \mathscr{H}_{\text{ADM}}\} = \mathcal{H}_i, \tag{2.43b}$$

Note that the equations (2.43) do not respect the primary constraints, equations (2.28), and then for consistency, we are asked to establish the so-called secondary constraints of the theory, that is

$$\mathcal{H} \approx 0, \tag{2.44a}$$

$$\mathcal{H}_i \approx 0. \tag{2.44b}$$

From equation (2.40), we write explicitly the ADM Hamiltonian density $\mathscr{H}_{\text{ADM}}$

$$\mathscr{H}_{\text{ADM}} = \lambda^N \Pi_N + \lambda^i \Pi_i + N^i \mathcal{H}_i + N\mathcal{H}, \tag{2.45}$$

and realized that it is a linear combination of the primary and secondary constraints, thus $\mathscr{H}_{\text{ADM}}$ itself weakly vanish

$$\mathscr{H}_{\text{ADM}} \approx 0. \tag{2.46}$$

The Poisson brackets of the set $\mathcal{H}$ and $\mathcal{H}_i$, forms the so-called Dirac algebra and are linear combinations of $\mathcal{H}$ and $\mathcal{H}_i$ [17]. This can be used to show that the equations of motion for $\mathcal{H}$ and $\mathcal{H}_i$, that is, $\{\mathcal{H}, \mathscr{H}_{\text{ADM}}\}$ and $\{\mathcal{H}_i, \mathscr{H}_{\text{ADM}}\}$, weakly vanishes and then do not justify the introduction of additional constraints in the theory[5].

2.2.3 Discussion on the constraints

As we saw, the identification of the four constraints, equations (2.28) and (2.44), in the Hamiltonian formalism of GR allows us to observe that there are degrees of freedom that are not physically relevant. The primary constraints (2.28) do not participate in the actual construction of the physical degrees of freedom of the theory, which only delimits our phase space to a reduced phase space of the canonical coordinates (h_{ij}, Π^{kl}) [21]. Therefore, we should concentrate on dealing with the secondary constraints (2.44), which will apply on each hypersurface[6] represented by the phase space of the canonical variables (h_{ij}, Π^{kl}).

Looking at the true degrees of freedom from the secondary constraints, we first look at the phase space and notice that we have six independent components in the spatial metric tensor h_{ij}, which varies for each point in the correspondent hypersurface (remember that here time is frozen, and a priori we are dealing with a given hypersurface), which we usually represent as the number of degrees of freedom in the phase space of $6 \times \infty^3$. In addition, the conjugate momenta Π_{kl} also have six independent components for each point in the hypersurface, then $6 \times \infty^3$ degrees of freedom in the phase space. Thus, the total degrees of freedom in the phase space is $12 \times \infty^3$. In components of the supermomentum, the secondary constraints, equations (2.44), with equations (2.39) that express dependency relationships among the variables, give rise to four constraints and make a reduction in the number of independent components at each point on the hypersurface to $12 - 4 = 8$, hence $8 \times \infty^3$ degrees of freedom. Furthermore, being $C^k \approx 0$ constraints (first-class constraints; see [17] and [20]) with $k = 0, 1, ..., n$, when calculating the equations of motion of any dynamical variable of interest A, we have $\dot{A} = \{A, \mathscr{H}_{\text{ADM}}\} + \lambda_k \{A, C^k\}$ and then the equation of motion of A is not uniquely determined due to gauge freedom caused by the

[5]The interested reader is invited to refer to section 4.1 of [17] for a little more detail and calculations.

[6]So far, all steps have implicitly assumed that the constraints apply to every possible hypersurface Σ_t of the foliation considered; this can be proved by tracing the ADM Hamiltonian formalism of GR as an initial value problem [18].

k Lagrange multipliers[7] [20].

In our case, there are four secondary constraints to be considered, then the hypersurface is restricted to having a loss in the number of degrees of freedom in phase space of the order of 4, and this redundancy is often called the gauge-invariance condition of GR [17]. Thus, the final number of degrees of freedom in the phase space is $4 \times \infty^3$. The configuration space, in turn, must have half the number of degrees of freedom of the phase space, which is in agreement if we note that the six independent components of the metric tensor per point on the hypersurface are reduced by the same four secondary constraints, that is, $2 \times \infty^3$ degrees of freedom in the configuration space. Now, the interpretation of the specific constraints (2.44) is that they represent the manifestation of the diffeomorphism-invariance in the realized spacetime foliation. In the phase space, the supermomentum constraint, equation (2.44b), denotes this invariance at each hypersurface, which is just the freedom of choice for any coordinate system [21]. In turn, the super-Hamiltonian constraint, equation (2.44a), denotes the diffeomorphism-invariance in the normal direction at each hypersurface in the phase space; that is, there is consistent freedom for the choice of the time parameter in the foliation using a reparameterization transformation of the hypersurfaces, $t \to \tau$ [20].

It is important to say that how the diffeomorphism-invariance of GR will manifest itself, through the constraints (2.44), when the canonical quantization takes place is a matter of deep attention. The supermomentum constraint can be trivially satisfied and interpreted by the quantization process, while the super-Hamiltonian constraint will give rise to profound physical implications (namely, the WDW equation), as will be seen in what follows.

2.3 Canonical quantization

The Hamiltonian formalism of GR from a foliation $\mathscr{F}$ process of spacetime $\mathscr{M}$ in a way that the diffeomorphism-invariance of the theory is preserved was obtained in the last section. The canonical variables used $N, N^i, h_{ij}, \Pi_N, \Pi_i, \Pi^{ij}$ were obtained from the so-called ADM variables N, N^i, h_{ij}. It is worth noting that there is another set of variables

[7]The natural condition to be imposed is the gauge-invariance of the dynamical variable, imposing then that $\{A, C^k\} \approx 0$. In GR, this leads to the notion of observable, which is to date still an open issue [17, 23].

notably used to construct an alternative Hamiltonian formalism of GR, which are the so-called loop variables and with which the well-known loop quantum gravity theory is generated [20, 24]. Following the formalism according to the ADM variables, the classical field theory produced is a constrained theory, where we have two primary (2.28) and two secondary (2.44) constraints.

2.3.1 Dirac method, superspace and minisuperspace

To realize the canonical quantization, in addition to the Poisson brackets of the classical variables simply reflecting the quantum commutators of the corresponding operators, it is also necessary to choose which method to use for the resolution, or application, of the constraints of the theory at the quantum level. That is, the constraints can be treated before quantization and thus at the classical level (ADM reduction method), or only after quantization (Dirac quantization method). In the first method, the constraints are applied classically and then the phase space of the theory is modified to a reduced physical phase space and quantized. In the second method the constraints are applied as quantum operators after quantization and then directly constrain the quantum state of some Hilbert space (we will be interested in its projection onto the space of the ADM variables, i.e., the corresponding space of functionals $\Psi = \Psi[N, N^i, h_{ij}]$) of the obtained theory [17].

Despite the ADM reduction method being able to describe analogues of a Schrödinger equation with time dependency in, for example, cosmological models using this method, such an equation describing the evolution of the wave function proves analytically intractable in its general form [21]. The Dirac quantization method, on the other hand, can describe the dynamics for the wave function as a direct consequence of the application of the super-Hamiltonian constraint (2.44a), although such dynamics needs to be carefully interpreted due to the so-called time problem. The quantization according to the Dirac method will be preferred in the CQG program.

A last procedure that needs to be defined refers to the composition of the space of the ADM variables that will be adopted for the quantization to generate the corresponding Hilbert space. Recall that spacetime $\mathscr{M}$ has been split by a foliation $\mathscr{F}$ into spacelike hypersurfaces Σ_t , the set of which we denote by Σ. The configuration space formed by all possible Riemannian geometries in Σ, namely, all possible 3-dimensional metric tensors, $h_{ij}(x^k,t)$, with $\{x^i\}\in\Sigma_t\subset\mathscr{M}$, is denoted by Riem($\Sigma$). However, the collection of all possible diffeomorphisms over Σ, denoted by Diff(Σ), naturally leads to redundant geometries and the physical configuration space that remains, denoted by $\mathcal{S}(\Sigma)$, is given by the quotient space of Riem(Σ), that is, $\mathcal{S}(\Sigma)$ = Riem(Σ)/Diff(Σ). The configuration space $\mathcal{S}(\Sigma)$ is called superspace [17] and is the space that canonical quantization in its general form is realized, being an infinite-dimensional space because we are considering all points of spacetime without any kind of symmetry or equivalence.

Another possible approach is to construct a configuration space, based on the superspace, that takes into account symmetries in spacetime resulting then in a reduced, finite-dimensional, space. This is particularly interesting in the context of cosmological models and black holes. This type of configuration space is called minisuperspace. Although the quantization in minisuperspace is in fact what we will deal with in the main application of CQG, presented in the next chapter of this work, the quantization in the superspace will be made below for a general obtaining and discussion of the properties of the WDW equation.

2.3.2 Dirac quantization

Canonical quantization according to Dirac [25] follows at the superspace. A classical constraint $C\approx 0$ becomes an quantum operator $\hat{C}$ that directly constrains the functional describing the quantum state $\Psi=\Psi[N,N^i,h_{ij}]$ that will be constructed

$$C\approx 0\rightarrow \hat{C}\,\Psi=0. \tag{2.47}$$

A conventional sequence of steps to perform during the program of such a canonical quantization can be retrieved from [17]. Here we will follow a more pragmatic approach. Initially the ADM variables as functions $N(x)$, $N^i(x)$ and $h_{ij}(x)$, of spacetime points x, and their associated momenta are raised to quantum operators

$$N(x)\rightarrow \hat{N}(x):=N(x),\qquad \Pi_N(x)\rightarrow\hat{\Pi}_N(x):=-i\hbar\frac{\delta}{\delta N(x)}, \tag{2.48a}$$

$$N^i(x)\rightarrow \hat{N}^i(x):=N^i(x),\qquad \Pi_i(x)\rightarrow\hat{\Pi}_i(x):=-i\hbar\frac{\delta}{\delta N^i(x)}, \tag{2.48b}$$

$$h_{ij}(x)\rightarrow \hat{h}_{ij}(x):=h_{ij}(x),\qquad \Pi^{ij}(x)\rightarrow\hat{\Pi}_{ij}(x):=-i\hbar\frac{\delta}{\delta h_{ij}(x)}, \tag{2.48c}$$

which act on functionals $\Psi = \Psi[N, N^i, h_{ij}]$ that it is natural to call wave functionals, and which in general do not inhabit a well-defined Hilbert space, endowed with an inner product between the objects Ψ. In equations (2.48) the "δs" compose what we call functional derivatives, i.e., given a functional, if we want to find the rate of change of the functional with respect to an infinitesimally small change in the function, we take its functional derivative. The operators need not yet be self-adjoint. Only with the application of the constraints the quantum states are defined physically and a usual Hilbert space can be generated [21]. Then, for the first constraints (2.28), one obtains

$$\hat{\Pi}_N \Psi[N, N^i, h_{ij}] = -i\hbar \frac{\delta}{\delta N} \Psi[N, N^i, h_{ij}] = 0, \tag{2.49a}$$

$$\hat{\Pi}_i \Psi[N, N^i, h_{ij}] = -i\hbar \frac{\delta}{\delta N^i} \Psi[N, N^i, h_{ij}] = 0. \tag{2.49b}$$

These conditions are satisfied if the wave functional does not depend on N and N^i, then $\Psi = \Psi[h_{ij}]$, and the chosen foliation does not change quantum state, which is consistent with the classical first constraints (2.28) in which N and N^i do not change the dynamics of the system. The supermomentum constraint (2.44b), with equation (2.39b), becomes

$$\hat{\mathcal{H}}_i \Psi[h_{ij}] = 0 \implies D_k \left(\frac{\delta}{\delta h_{kj}} \Psi \right) = 0, \tag{2.50}$$

and then changes of the metric tensor $h_{ij}(x)$ by the points of the spacetime do not alter Ψ. In other words, the functional Ψ is invariant under coordinate transformations of h_{ij}, which is consistent with the classical supermomentum constraint (2.44a) that translates the spatial diffeomorphism-invariance of GR. In turn, the super-Hamiltonian constraint (2.44a), with equation (2.39a), becomes

$$\hat{\mathcal{H}} \Psi[h_{ij}] = 0 \implies \left(16\pi G \mathscr{G}_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} - \frac{1}{16\pi G} \sqrt{h} \bar{R} \right) \Psi = 0, \tag{2.51}$$

which indeed seems to describe a dynamic behavior for Ψ, but with $\hat{\mathcal{H}}\Psi = 0$. While a natural interpretation in terms of the diffeomorphism-invariance of GR like $\hat{\mathcal{H}}_i \Psi = 0$ in (2.50) is not trivial, equation (2.51) is a Schrödinger-like equation for Ψ but without a time evolution. Note that the implementation of $\hat{\mathcal{H}}\Psi = 0$ is not unambiguous and depends on the choice of operator ordering in the equation after quantization. A natural choice for this operator ordering, which is justified by the construction of the theory itself, is that which preserves the Dirac algebra [17], responsible for ensuring that the theory is closed concerning the constraints already presented and has no tertiary constraints [26]; a more detailed explanation of this can be found in [21]. The equation (2.51) is the so-called WDW equation.

2.4 The Wheeler-DeWitt equation

As we have seen, the WDW equation (2.51) is the application of the super-Hamiltonian constraint (2.44a) of the Hamiltonian formalism of GR to canonical quantization via the Dirac quantization method: first quantize and then constrain. We rewrite equation (2.51) below

$$\left(16\pi G\,\mathcal{G}_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - \frac{1}{16\pi G}\sqrt{h}\bar{R}\right)\Psi = 0. \tag{2.52}$$

A few technical comments are needed on this equation:

(I) The wave functional $\Psi[h_{ij}]$ space

Since the WDW equation (2.52) is a non-linear equation, due to the functional derivatives it contains, the definition of a basis as in a usual vector space (presumably here a kind of Hilbert space) becomes non-trivial. In addition, we would need to provide a notion of inner product between the elements of the space; DeWitt proposed an attempt [27], but it is not shown to be positive-defined [21], thus negative probability problems must arise. So far, a well-defined notion of inner product (and Hilbert space) for $\Psi[h_{ij}]$ is still under discussion [28].

(II) The matter coupling

To account for inflationary Universe models, a minimally coupled scalar field Φ with self-interacting potential $V(\Phi)$ is usually introduced into the 3+1 decomposition of the previously developed Hamiltonian formalism of GR, leading to new constraints that produce the WDW equation for gravity-scalar field systems [17] (a detailed and self-contained description of this case can be found at [17]).

(III) The problem of regularization

The presence of functional second derivatives in the WDW equation (2.52) refers to the presence of 3-dimensional Dirac deltas $\delta^3(x-y)$ at the points in space [20]. For the limit of short distances, $x \to y$, this will naturally lead to infinities $\delta^3(0)$. This difficulty indicates the domain of validity of the WDW equation in its (2.52) form (superspace form), which refers to a corresponding effective theory rather than a fundamental one. Thus, at the fundamental level, a regularization of the $\delta^3(0)$ terms is necessary. An example of regularization is that of DeWitt: $\delta^3(0) = 0$ [27].

(IV) The problem of time

The natural interpretation of the absence of an explicit time evolution term of $\Psi[h_{ij}]$ in the WDW equation (2.52), when compared to the Schrödinger equation, is that we are mixing two different notions of time [29]. In GR time is a coordinate, and it is a quantity intrinsic to the theory in the sense that different observers measure different times depending on their dynamics and the gravitational field. In quantum mechanics, time is absolute, and an external parameter to the theory. It is not observable, but it is fundamental for the probabilistic interpretation [20]. Also, realize that the implementation of the constraints, (2.50) and (2.51), is equivalent to applying the ADM Hamiltonian density operator $\hat{\mathscr{H}}\Psi$ to the wave functional, then that it annihilates the quantum state and motivates us to visualize the result as a time-independent Schrödinger equation. The question of which of these two forms of time (or another) is consistent with CQG is called the time problem [20].

The conventional classification for its solution comes from Kuchař [30] and Isham [31]: i) the notion of time must come before quantization, ii) after quantization, or iii) is possibly absent at the most fundamental level. The first two methods revealed a succession of shortcomings [20], which seems to indicate iii) the right choice. One option is to divide the Hamiltonian density of the system into two parts

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_{\mathrm{G}} + \hat{\mathscr{H}}_{\mathrm{N}}, \tag{2.53}$$

where $\hat{\mathscr{H}}_{\mathrm{G}}$ carries the gravitational degrees of freedom of the theory, and $\hat{\mathscr{H}}_{\mathrm{N}}$ carries the non-gravitational ones. Next, a semi-classical WKB-like approximation is performed on the portion contained in $\hat{\mathscr{H}}_{\mathrm{G}}$, while the portion $\hat{\mathscr{H}}_{\mathrm{N}}$ remains entirely at the quantum level. In this way, it is possible to show that a (time-dependent) Schrödinger equation emerges for $\Psi[h_{ij}]$ (see [20] for more details).

3 SCHWARZSCHILD BLACK HOLE

A lua, tal qual a dona do bordel, pedia a cada estrela fria um brilho de aluguel.

Elis Regina - O bêbado e a equilibrista

3.1 Schwarzschild solution

The initial aim of this chapter is to apply the QCG results of sections 2.2 and 2.3 in the SBH. This results in the minisuperspace WDW equation for the SBH. We begin by reviewing the fundamental features of the Schwarzschild solution. The Schwarzschild solution of the vacuum field equations deals with a spherically symmetric and static spacetime[1], whose metric is [33]

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right)dt^2 + \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.1}$$

as that which applies to the exterior of a spherical, non-rotating, massive object. In equation (3.1), m is identified as the mass of the object, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric of a unit 2-sphere. The set (t, r, θ, ϕ) are called the Schwarzschild coordinates. This coordinate system describes, in addition to the true singularity at $r = 0$, an apparent singularity (or coordinate singularity) at $r = 2Gm$. Despite this limitation of the Schwarzschild coordinates, we can use them to analyze the corresponding spacetime diagram, as in figure 2.

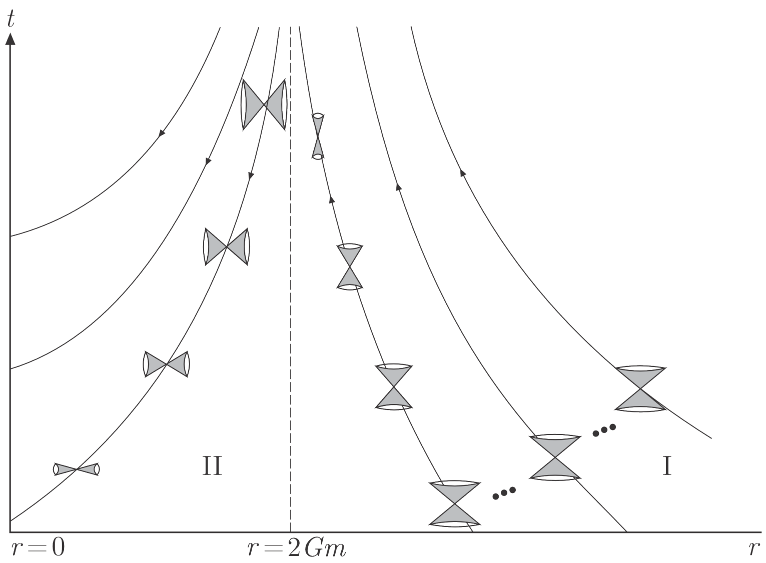
The slope of the light cones in these coordinates are given by the radial lightlike curves, setting θ and ϕ as constants, and $ds^2 = 0$ in equation (3.1)

$$\frac{dt}{dr} = \pm\left(1 - \frac{2Gm}{r}\right)^{-1}. \tag{3.2}$$

From equation (3.2) we note that with $r \to \infty$, $dt/dr \to \pm 1$, and the light cones of Minkowski spacetime appears. As a light signal (or timelike observer) approaches $r \to 2Gm$ (the so-called Schwarzschild radius, that is, $r = 2Gm$), the light cone squeezes until its shape diverges as $dt/dr \to \pm\infty$; i.e., in the Schwarzschild time coordinate it takes a infinite time

[1] Actually, the Birkhoff's theorem guarantees that a vacuum solution of the field equations with spherical symmetry is necessarily a static spacetime [32].

Figure 2: Schwarzschild spacetime diagram in Schwarzschild coordinates showing the ingoing causal curves.



Source: The author (2024).

Δt for something to cross the surface $r = 2Gm$ from region I, $2Gm < r < \infty$, to region II, $0 < r < 2Gm$.

Finally, in region II, looking at the ingoing lightlike, or timelike, curves (causal curves[2]) in the diagram 2, we see that with $r < 2Gm$ then $dt/dr < |1|$. Thus in region II, the Schwarzschild coordinates t and r reverse their character, t becomes spacelike, r becomes timelike, and the light cones axis rotate $\pi/2$ counterclockwise. Inverting equation (3.2) to obtain the velocity of the radial lightlike curves, we see that with $r < 2Gm$ the outgoing causal curves have negative velocity. That is, only future-directed causal curves can cross the surface $r < 2Gm$ and an observer in region II cannot stay at rest but is forced to move in towards the singularity at $r = 0$. Hence, no event in region II can be accessed by an external observer in region I and the surface $r = 2Gm$ is called the event horizon[3] [33]. Objects which are described by the Schwarzschild solution and whose size is close to their Schwarzschild radius are called SBH.

[2]A differentiable curve $\lambda(u)$ in spacetime, with u a time parameter, that for each event p on $\lambda(u)$, the tangent vector at p, t^μ, is either timelike or lightlike [18].

[3]This concept will be addressed more rigorously in section 5.1 and 5.2.

3.1.1 *The Kruskal extension*

To make the Schwarzschild metric regular on the event horizon and naturally note that an observer who crosses it does not take infinite time to do so in his proper time, we use the Kruskal-Szekres coordinates, namely (t', x', θ, ϕ). The Schwarzschild metric in the Kruskal-Szekres coordinate is [33]

$$ds^2 = \frac{32G^3m^3}{r}e^{-r/2Gm}(-dt'^2 + dx'^2) + r^2 d\Omega^2, \tag{3.3}$$

where t' is the timelike coordinate and x' is the spacelike non-angular coordinate, and r relates to such coordinates as [33]

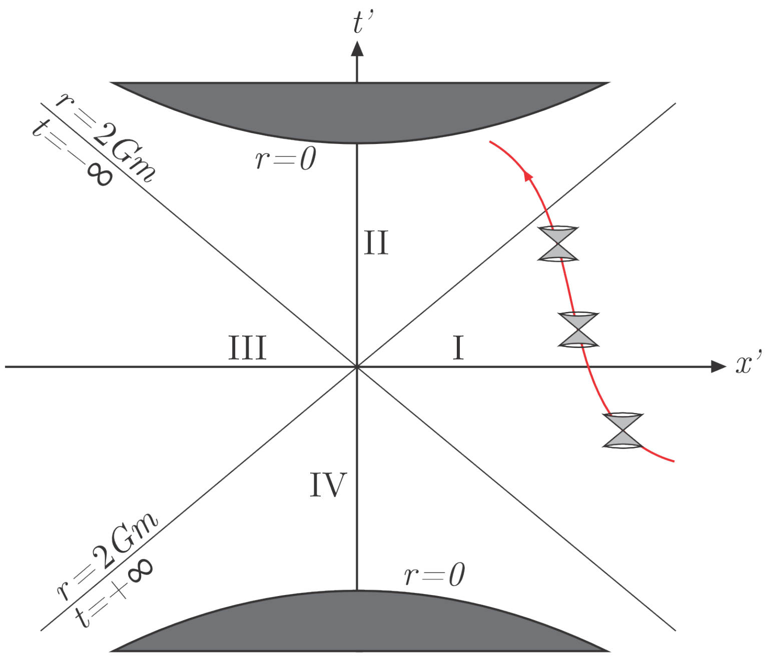
$$t'^2 - x'^2 = \left(1 - \frac{r}{2Gm}\right)e^{r/2Gm}. \tag{3.4}$$

Clearly from (3.3) the radial lightlike curves give the new slope of the light cones at these coordinates, as

$$\frac{dt'}{dx'} = \pm 1, \tag{3.5}$$

which are equal to the light cones of Minkowski spacetime. The spacetime diagram in Kruskal-Szekres coordinates is shown in figure 3, in which two new regions, III and IV, are revealed. Region I and II are the same as those obtained by the Schwarzschild coordinates, but since the Kruskal-Szekres coordinates allow us to cover the entire $-\infty < t' < \infty$ interval so that the metric is regular, region IV extends the causal curves to $t \to -\infty$ and these curves now cross a $r = 2Gm$ surface in region IV only in the past-directed direction [33]. In this sense, region IV looks like a time reversal of region II and is called a white hole, since no causal curve can follow towards the singularity. The straight line $r = 2Gm$ and the singularity $r = 0$ in region IV are called the past event horizon and the past singularity, respectively. Equivalently, we rename $r = 2Gm$ and $r = 0$ in region II as the future event horizon and the future singularity, respectively. The extension of spacelike curves ($t' = \text{constant}$) to the $-\infty < x' < \infty$ interval encompasses region III, which is asymptotically flat as region I, acting as a mirror of the latter.

Figure 3: Schwarzschild spacetime diagram in Kruskal-Szekres coordinates.



Source: The author (2024).

3.1.2 The Einstein-Rosen bridge

In this scenario, something interesting can be seen when we analyze the geometry of the spacetime manifold covered by the Kruskal-Szekres coordinates with the hypersurfaces $t' =$ constant. With equation (3.4), three cases can be examined: $t' = 0$, $t' = \pm 1$, and $t' < 1$ or $t' > 1$. For $t' = 0$, we have

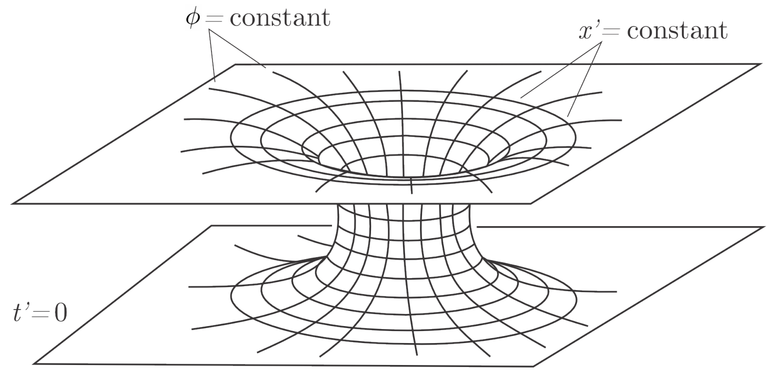
$$\begin{cases} x' \to +\infty \quad \Rightarrow \quad r \to +\infty; \\ x' \to 0 \quad \Rightarrow \quad r \to 2Gm; \\ x' \to -\infty \quad \Rightarrow \quad r \to +\infty. \end{cases} \tag{3.6}$$

For $t' = \pm 1$

$$\begin{cases} x' \to +\infty \quad \Rightarrow \quad r \to +\infty; \\ x' \to 0 \quad \Rightarrow \quad r \to 0; \\ x' \to -\infty \quad \Rightarrow \quad r \to +\infty. \end{cases} \tag{3.7}$$

For $t' < 1$ or $t' > 1$, besides $x' \to \pm\infty \Rightarrow r \to +\infty$, when $x' \to 0$, r does not converge to any value. Remembering that the interval $-\infty < x' < 0$ describes region III, and the interval

Figure 4: The Einstein-Rosen bridge, where $t' = 0$, and the "throat" of the wormhole has a radius $r = 2Gm$.



Source: The author (2024).

$0 < x' < +\infty$ describes region I, by (3.6) and (3.7), we can conceive of the following. For almost the entire time interval measured by t', the asymptotically flat regions I and III are disconnected and have, completely independently, an event horizon that delimits a black hole and white hole region, respectively. Then, in a short interval of time symmetrical to the origin $t' = 0$, both regions share the same point in spacetime, given by $r = 0$, and then they also share the same surface at $r = 2Gm$. The process then reverses, these regions intersect again at $r = 0$ and finally disconnect once more.

A view from the equatorial plane $\theta = \pi/2$ of the structure formed in this process, called a wormhole, or Einstein-Rosen bridge [33], can be seen in figure 4 at the moment $t' = 0$, of greatest connection between regions I and III. It is important to note that this construction of extending Schwarzschild solution to cover a maximal manifold, called maximal analytic extension, although mathematically consistent, lacks a definitive answer as to whether it has physical reality [34]. For black holes formed by gravitational collapse, the Kruskal diagram must have a cut-off at a timelike boundary that represents the surface of the collapsed body, and then regions III and IV disappear.

3.2 Canonical quantization of the SBH

The canonical quantization of the SBH now follows, so that the condition of a spherically symmetric and static spacetime $\mathscr{M}$ must now be translated to the set of hypersurfaces, Σ. Then the metric must be rewritten through the foliation that selects the corresponding ADM variables for such conditions. Of course, the quantization will be done in a reduced configuration space, which represents a minisuperspace. According to Kuchař [35], the metric for a spherical symmetric and static hypersurface Σ_t, in the coordinates $x^i = (r, \theta, \phi)$, is

$$ds^2{}_{\Sigma_t} = \Lambda(r)^2 dr^2 + R(r)^2 d\Omega^2, \tag{3.8}$$

where $\Lambda(r) > 0$, $R(r) > 0$, and again $d\Omega^2$ is the metric of a unit 2-sphere. By equation (2.8), the spherical symmetric ADM metric is

$$ds^2 = -N(r,t)^2 dt^2 + h_{rr}(r,t)(dr + N^r(r,t)dt)^2 + h_{\theta\theta}(r,t)d\theta^2 + h_{\phi\phi}(r,t)d\phi^2, \tag{3.9}$$

due to spherical symmetry: $N = N(r,t)$, $N^i = N^i(r,t)$, and $h_{ij} = h_{ij}(r,t)$, with $N^i = (N^r, 0, 0)$. Noting that the spatial metric tensor h_{ij} in matrix notation

$$h_{ij} \doteq \begin{pmatrix} \Lambda(r)^2 & 0 & 0 \\ 0 & R(r)^2 & 0 \\ 0 & 0 & R(r)^2 \sin^2\theta \end{pmatrix}, \tag{3.10}$$

and using the metric for Σ_t (3.8) in (3.9), we have

$$ds^2 = -N(r,t)^2 dt^2 + \Lambda(r,t)^2 (dr + N^r(r,t)dt)^2 + R(r,t)^2 d\Omega^2. \tag{3.11}$$

To correctly describe the whole Schwarzschild spacetime, we need to adopt two extra conditions: i) the coordinates used must vary in such a way as to cover the entire Kruskal's maximal analytic extension of the Schwarzschild spacetime, i.e., the Kruskal diagram (figure 3), and ii) that the spacetime after the foliation remains asymptotically flat. The first condition can be guaranteed if we take $-\infty < r < \infty$ and $-\infty < t < \infty$ [35], as well as the second condition if the functions Λ, R, N, and N^r (and the conjugated momenta to Λ and R) admit certain fall-off conditions [35]. The ADM lagrangian density of this spacetime, namely, in terms of Λ, R, N, and N^r, is [35]

$$\begin{aligned} \mathscr{L}_{\text{ADM}} = & \frac{1}{4\pi G} \sin\theta N \left[-\lambda^{-1} R\, \partial_r^2 R + \Lambda^{-2} R\, \partial_r R\, \partial_r \Lambda - \frac{1}{2}\Lambda^{-1}(\partial_r R)^2 + \frac{1}{2}\Lambda \right] \\ & - \frac{1}{4\pi G} \frac{\sin\theta}{N} \left[R\, (\partial_r(\Lambda N^r) - \partial_t \Lambda)(\partial_r R\, N^r - \partial_t R) + \frac{1}{2}\Lambda\, (\partial_t R - \partial_r R\, N^r)^2 \right], \end{aligned} \tag{3.12}$$

and then the ADM action reads

$$
\begin{aligned}
S_{\rm ADM} &= \int_{-\infty}^{\infty} dt \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta \int_{-\infty}^{\infty} dr \mathscr{L}_{\rm ADM} \\
&= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dr \, \frac{1}{G} \Bigg\{ N \Bigg[-\lambda^{-1} R \, \partial_r^2 R + \Lambda^{-2} R \, \partial_r R \, \partial_r \Lambda - \frac{1}{2}\Lambda^{-1} (\partial_r R)^2 + \frac{1}{2}\Lambda \Bigg] \\
&\quad - \frac{1}{N} \Bigg[R \, (\partial_r(\Lambda N^r) - \partial_t \Lambda)(\partial_r R \, N^r - \partial_t R) + \frac{1}{2}\Lambda \, (\partial_t R - \partial_r R \, N^r)^2 \Bigg] \Bigg\}.
\end{aligned}
\tag{3.13}
$$

The conjugated momenta to Λ and R are, respectively

$$
\Pi_\Lambda = \frac{\delta S_{\rm ADM}}{\delta(\partial_t \Lambda)} = -M_{\rm P}^2 \frac{R}{N} \big(\partial_t R - \partial_r R \, N^r\big), \tag{3.14a}
$$

$$
\Pi_R = \frac{\delta S_{\rm ADM}}{\delta(\partial_t R)} = -M_{\rm P}^2 \frac{1}{N} \big[R \, (\partial_t \Lambda - \partial_r(\Lambda N^r)) + \Lambda(\partial_t R - \partial_r R \, N^r) \big], \tag{3.14b}
$$

with $M_{\rm P} := G^{-1/2}$ the Planck mass. As we already know, from equations (2.27), the conjugated momenta to N and N^r weakly vanish. Before going any further, we must assess the existence of boundary terms in the ADM action (3.13). Unlike the general ADM action (2.21) that we constructed earlier, here we need to preserve the fall-off conditions of the ADM variables of interest in the regions of right and left spatial infinity (respectively relative to regions I and III in the Kruskal diagram 3). In other words, we need to specify the boundary of the Schwarzschild spacetime in the ADM action (3.13) in such a way as to guarantee that this spacetime is consistently asymptotically flat [35, 36]. Then we consider the following boundary term action [17, 35]

$$
S_{\partial \mathscr{M}} = -\int_{-\infty}^{\infty} dt (N_+(t) M_+(t) + N_-(t) M_-(t)), \tag{3.15}
$$

where $N_\pm(t) M_\pm(t)$ is the product of the lapse function and the SBH mass evaluated at the right (denoted by $+$) and left (denoted by $-$) spatial infinity, that is

$$
M_\pm(t) := \lim_{r \to \pm\infty} M(r,t), \tag{3.16a}
$$

$$
N_\pm(t) := \lim_{r \to \pm\infty} N(r,t). \tag{3.16b}
$$

The total action, denoted $S_{\mathscr{M}}$, in terms of the Hamiltonian density is the ADM action S_{ADM} given by (3.13), as in equation (2.35), with the boundary term action $S_{\partial\mathscr{M}}$ at (3.15)

$$S_{\mathscr{M}} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dr \big[\Pi_\Lambda \partial_t \Lambda + \Pi_R\, \partial_t R - N^r \mathcal{H}_r - N\mathcal{H}\big] - \int_{-\infty}^{\infty} dt (N_+ M_+ + N_- M_-), \quad (3.17)$$

where equation (2.40) was used. By comparing equations (3.13) and (3.17), the supermomentum and the super-Hamiltonian, $\mathcal{H}_r$ and $\mathcal{H}$, can be identified as

$$\mathcal{H}_r = \frac{1}{M_{\text{P}}^2}(\Pi_R\, \partial_r R - \Lambda\, \partial_r \Pi_\Lambda), \quad (3.18a)$$

$$\mathcal{H} = -\frac{1}{R\, M_{\text{P}}^4}\Pi_R\, \Pi_\Lambda + \frac{1}{2R^2 M_{\text{P}}^2}{\Pi_\Lambda}^2 + R\, \partial_r^2 R - \frac{R}{\Lambda^2}\partial_r R\, \partial_r \Lambda - \frac{1}{2\Lambda}(\partial_r R)^2 - \frac{\Lambda}{2}. \quad (3.18b)$$

In order to prescribe canonical transformations on the variables Λ and R that simplify the study of the boundary term action (3.15), we make the following pair of canonical transformations [35], first (Λ, Π_Λ) to (M, Π_M)

$$M = \frac{1}{2RM_{\text{P}}^4}{\Pi_\Lambda}^2 - \frac{R}{2\Lambda^2}(\partial_r R)^2 + \frac{R}{2}, \quad (3.19a)$$

$$\Pi_M = \frac{\Lambda}{M_{\text{P}}^2}\Pi_\Lambda \left[\left(\frac{\partial_r R}{\Lambda}\right)^2 - \frac{1}{M_{\text{P}}^4}\left(\frac{\Pi_\Lambda}{R}\right)^2\right]^{-1}, \quad (3.19b)$$

with M the SBH mass, and second (R, Π_R) to $(\mathcal{R}, \Pi_{\mathcal{R}})$

$$\mathcal{R} = R, \quad (3.20a)$$

$$\Pi_{\mathcal{R}} = \left(\frac{\mathcal{H}}{R\, M_{\text{P}}^2}\Pi_\Lambda + \frac{\mathcal{H}_r}{\Lambda^2}\partial_r R\right)\left[\left(\frac{\partial_r R}{\Lambda}\right)^2 - \frac{1}{M_{\text{P}}^4}\left(\frac{\Pi_\Lambda}{R}\right)^2\right]^{-1}. \quad (3.20b)$$

The new supermomentum and super-Hamiltonian become

$$\mathcal{H}_r = \frac{1}{M_{\text{P}}^2}(\Pi_M\, \partial_r M + \Pi_{\mathcal{R}}\, \partial_r \mathcal{R}), \quad (3.21a)$$

$$\mathcal{H} = -\frac{\left(1 - \frac{2M}{R\, M_{\text{P}}^2}\right)^{-1}\partial_r M\, \partial_r \mathcal{R} + M_{\text{P}}^{-4}\left(1 - \frac{2M}{R\, M_{\text{P}}^2}\right)\Pi_M\, \Pi_{\mathcal{R}}}{\left[\left(1 - \frac{2M}{R\, M_{\text{P}}^2}\right)^{-1}\partial_r \mathcal{R}^2 - M_{\text{P}}^{-4}\left(1 - \frac{2M}{R\, M_{\text{P}}^2}\right){\Pi_M}^2\right]^{1/2}}. \quad (3.21b)$$

The second constraints (2.44), from equations (3.21), are clearly manifested as

$$\partial_r M \approx 0, \tag{3.22a}$$

$$\Pi_{\mathcal{R}} \approx 0. \tag{3.22b}$$

The total action (3.17) can be rewritten as a consequence of the constraint given by (3.22a); i.e., $M = M(t)$, and defining a new conjugated momentum to M which

$$P(t) := \int_{-\infty}^{\infty} dr\, \Pi_M, \tag{3.23}$$

then

$$\begin{aligned} S_{\mathscr{M}} &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dr \big[\Pi_M \partial_t M + \Pi_{\mathcal{R}}\, \partial_t \mathcal{R} - N^r \mathcal{H}_r - N\mathcal{H}\big] - \int_{-\infty}^{\infty} dt (N_+ M_+ + N_- M_-) \\ &= \int_{-\infty}^{\infty} dt \Bigg[P\, \partial_t M - (N_+ + N_-) M \Bigg], \end{aligned} \tag{3.24}$$

such that, from equation (3.22a), $M_\pm = M(t)$. Furthermore, to avoid the non-physical solution in which $M(t) = 0$, the lapse function at infinities must have a fixed value [35]. A consistent choice of values is such that an observer at rest in the right spatial infinity (our region I of the Kruskal diagram 3) measures his proper time as measured by the time coordinate of the Minkowski spacetime, then $N_+ = 1$ and $N_- = 0$ [37]. Then, the total action becomes

$$S_{\mathscr{M}} = \int_{-\infty}^{\infty} dt (P\, \partial_t M - M). \tag{3.25}$$

From the equation above we can recognize the action in its canonical form and then identify the Hamiltonian as $H = M$. Note that from Hamilton-Jacobi equations in the action (3.25), the equations of motion are $M =$ constant and $P = -t$. Since we are admitting the maximal analytic extension represented by the Kruskal diagram in the construction of the action, then coordinates for the wormhole solution can parameterize the SBH solution. To do this, the canonical transformation introduced by Louko and Mäkelä [37], where (M, P) goes to (x, p), is used

$$|P(x,p)| = \int_x^{2GM} dy \left(\frac{2GM}{y} - 1 \right)^{-1/2}, \tag{3.26a}$$

$$M(x,p) = \frac{p^2}{2M_{\rm P}^2 x} + \frac{M_{\rm P}^2 x}{2}, \tag{3.26b}$$

where $x \geqslant 0$ is interpretead as the wormhole throat and $-\infty < p < \infty$ is just its conjugate momentum. Naturally, the SBH Hamiltonian becomes

$$H(x,p) = \frac{p^2}{2M_{\rm P}^2 x} + \frac{M_{\rm P}^2 x}{2}. \tag{3.27}$$

As seen previously, applying the Hamiltonian density operator to the wave functional leads to the quantization of the constraints and hence to the WDW equation. In the case of the SBH, classically, the spherical symmetry leads to the constraints (3.22) that allow us to identify the canonical reduced action (3.25) without the spatial integral over r. Thus, at the canonical quantization level, the Hamiltonian operator (3.27) is equivalent to the Hamiltonian density operator that leads to the WDW equation (2.52). This consistently reflects quantization in minisuperspace where the number of degrees of freedom is reduced due to a spacetime symmetry. Proceeding with canonical quantization, in the coordinate representation taken with x, we have $x \to \hat{x} := x$ and $p \to \hat{p} := -i\hbar\,(d/dx)$. The WDW equation is then

$$\hat{H}\left(x, -i\hbar\frac{d}{dx}\right)\psi(x) = M\psi(x), \tag{3.28}$$

and by substituting equation (3.27)

$$-\frac{1}{2M_{\rm P}^2}\frac{d^2}{dx^2}\psi(x) + \frac{M_{\rm P}^2}{2}x^2\,\psi(x) = Mx\;\psi(x). \tag{3.29}$$

Completing the square and factoring $M_{\rm P}^3/2$ in the second term of the left-hand side of the equation (3.29), we have

$$-\frac{1}{2M_{\rm P}}\frac{d^2}{dx^2}\psi(x) + \frac{M_{\rm P}\,{\omega_{\rm P}}^2}{2}\left(x - \frac{M}{M_{\rm P}^2}\right)^2\psi(x) = \frac{M^2}{2M_{\rm P}}\psi(x), \tag{3.30}$$

where $\omega_{\rm P} := 1/t_{\rm P}$ is the Planck angular frequency, and the Planck time is $t_{\rm P} = 1/M_{\rm P}$. Equation (3.30) is a Schrödinger-like equation for a harmonic oscillator and represents the

WDW equation of the quantized SBH, which is mathematically simpler than the general WDW equation (2.52) in superspace.

3.3 Thermodynamics of the quantized SBH

The suggestion that black holes should have a discrete mass spectrum due to quantum effects goes back to Bekenstein [38]. Analysis of the solutions of the WDW equation for the SBH[4], equation (3.30), shows that for a massive black hole, $M/M_{\mathrm{P}} \gg 1$, the semi-classical limit $n \gg 1$, with the quantum states levels n, provides the mass spectrum given by [37]

$$M(n) = M_{\mathrm{P}}\sqrt{2n+1}, \qquad n \gg 1. \tag{3.31}$$

We can try to articulate this result with the mechanism of black hole mass loss through the creation of particle-antiparticle pairs on the surface of this black hole event horizon, called Hawking radiation [39, 40]. The direction will then be to obtain the temperature associated with the event horizon, an idea first proposed by Bekenstein [41]. Hawking radiation takes the form of black-body radiation, whose well-known emission temperature for an SBH is [33]

$$T_{\mathrm{H}} = \frac{1}{8\pi}\frac{M_{\mathrm{P}}^2}{M}. \tag{3.32}$$

In agreement with Mukhanov [42] and Xiang [43], we will assume that for the conditions henceforth assumed of a massive black hole, $M/M_{\mathrm{P}} \gg 1$, in the semi-classical limit, $n \gg 1$, the emission frequency ω_0 of the thermal radiation from the transition $n+1$ to n of the quantum states (due to the Hawking radiation) is given by the mass loss of the black hole between these states. Then,

$$\begin{aligned} \omega_0 &:= M(n+1) - M(n) \\ &= M_{\mathrm{P}}\sqrt{2n+3} - M_{\mathrm{P}}\sqrt{2n+1} \\ &= M_{\mathrm{P}}\sqrt{2n}\left(\sqrt{1+\frac{3}{2n}} - \sqrt{1+\frac{1}{2n}}\right), \end{aligned} \tag{3.33}$$

[4] A full discussion of these solutions is beyond the scope of this text and, given the above, we ask the interested reader to follow the details in reference [17].

where equation (3.31) was used, and with the polynomial approximation $(1+|x|)^\alpha \approx 1 + \alpha|x|$, for $|x| \ll 1$, we have

$$\omega_0 = \frac{M_\text{P}}{\sqrt{2n}}. \tag{3.34}$$

Solving equation (3.31) for n returns

$$n = \frac{1}{2}\left[\left(\frac{M_\text{P}}{M}\right)^2 - 1\right], \tag{3.35}$$

and then substituting in equation (3.34)

$$\omega_0 \approx \frac{M_\text{P}^2}{M}\left[1 + \frac{1}{2}\left(\frac{M_\text{P}}{M}\right)^2\right], \tag{3.36}$$

again with the polynomial approximation. The characteristic time τ_n for the quantized SBH in state $n+1$ to decay into state n is defined as [43]

$$\tau_n^{-1} := \frac{\dot{M}}{\omega_0} \approx \dot{M}\frac{M}{M_\text{P}^2}\left[1 - \frac{1}{2}\left(\frac{M_\text{P}}{M}\right)^2\right], \tag{3.37}$$

with equation (3.36), and $\dot{M} \equiv dM/dt$ being the mass loss rate of the transition. In turn, following Mukhanov [42], the width W_n between the quantum states of the quantized SBH is proportional to the mass loss between the consecutive states

$$W_n = \beta(M(n+1) - M(n)) = \beta\,\omega_0, \tag{3.38}$$

with $\beta \ll 1$ a dimensionless constant. In the case of a quantum system such as the SBH, in which the mass difference between consecutive states ΔM varies with $n^{-1/2}$, it can be seen that

$$\lim_{n\to N} \frac{\Delta M}{M} = 0, \qquad N \gg 1. \tag{3.39}$$

Therefore, since $\tau_n = \omega_0/\dot{M}$, and with equation (3.34), it is reasonable to assume that τ_n must decrease when n increases and, therefore, the frequency of the radiation ω_0 emitted by the black hole must also decrease. However, the width between the quantum states W_n coming from the transition during the radiation emitted by the black hole and the values of the characteristic time in which this transition takes place τ_n must not be simultaneously arbitrary small quantities. Otherwise, the energy (equivalently, the mass) of the transition between arbitrarily close states cannot be properly evaluated during a characteristic time that is also arbitrarily small. Such a situation would mimic a continuum of states. To avoid this, we set an uncertainty ratio between τ_n and W_n

$$\tau_n W_n \approx 1. \tag{3.40}$$

One can think of this relationship as a kind of boundary condition for the transitions between the quantum states of the black hole. Hence, joining equation (3.36) with $\tau_n = \omega_0/\dot{M}$, we have

$$\begin{aligned} \dot{M} = \beta\,{\omega_0}^2 &= \beta\,\frac{M_{\rm P}^4}{M^2}\left[1+\frac{1}{2}\left(\frac{M_{\rm P}}{M}\right)^2\right]^2 \\ &\approx \beta\,\frac{M_{\rm P}^4}{M^2}\left[1+\left(\frac{M_{\rm P}}{M}\right)^2\right]. \end{aligned} \tag{3.41}$$

Considering the Hawking radiation from the black hole as a black-body radiation (remembering the assumed semi-classical limit, $n \gg 1$), we can associate $\dot{M}$ with the temperature of the black hole event horizon T by the Stefan-Boltzmann law [44]

$$\dot{M} = \sigma_{\rm S} A T^4, \tag{3.42}$$

where $\sigma_{\rm S} = \pi^2/60$ is the Stefan-Boltzmann constant. By using the area of the SBH event horizon $A = 16\pi M^2/M_{\rm P}^4$ in equation (3.42), and joining with equation (3.41), the desired temperature is obtained

$$T \approx \left(\frac{\beta}{16\pi\sigma_{\rm S}}\right)^{1/4}\frac{M_{\rm P}^2}{M}\left[1+\frac{1}{4}\left(\frac{M_{\rm P}}{M}\right)^2\right], \tag{3.43}$$

where the polynomial approximation was used once again. Realize that this temperature agrees with the Hawking temperature for the SBH, equation (3.32), by choosing the constant $\beta = 1/15360\pi$, in a first approximation. One way of obtaining the associated entropy is to verify that the horizon area of black holes is an adiabatic invariant and then according to the Bohr-Sommerfeld quantization rule has an associated quantum spectrum [38]. We can use the following adiabatic invariant $I_{\rm adia}$ as the black hole entropy, corresponding to the Bekenstein-Hawking entropy [40], $\mathcal{S}_{\text{B-H}}$, for the SBH [45]

$$I_{\rm adia} := 2\pi \int_0^H \frac{dH'}{\kappa}, \tag{3.44}$$

where H is the SBH Hamiltonian, and κ is the SBH surface gravity (for a specialized discussion of this topic, see section 5.1.4). Substituting these quantities for the SBH, $H = M$ and $\kappa = M_{\mathrm{P}}^2/4M = \omega_0/4$ with equation (3.36), we have

$$8\pi \int_0^M \frac{dM'}{\omega_0} = 4\pi G M^2 \equiv \left(\frac{A_{\mathrm{S}}}{4G} \right) = \left(\mathcal{S}_{\text{B-H}} \right)_{\text{SBH}}, \tag{3.45}$$

with $A_{\mathrm{S}} = 4\pi r_{\mathrm{S}}^2$ in equation (3.45) the area of the SBH, and $r_{\mathrm{S}} = 2GM$ the Schwarzschild radius.

4 FRACTALS AND FRACTIONAL QUANTUM GRAVITY

No, I don't think life is quite that simple.

Utada Hikaru - Simple and Clean

4.1 Fractal geometry

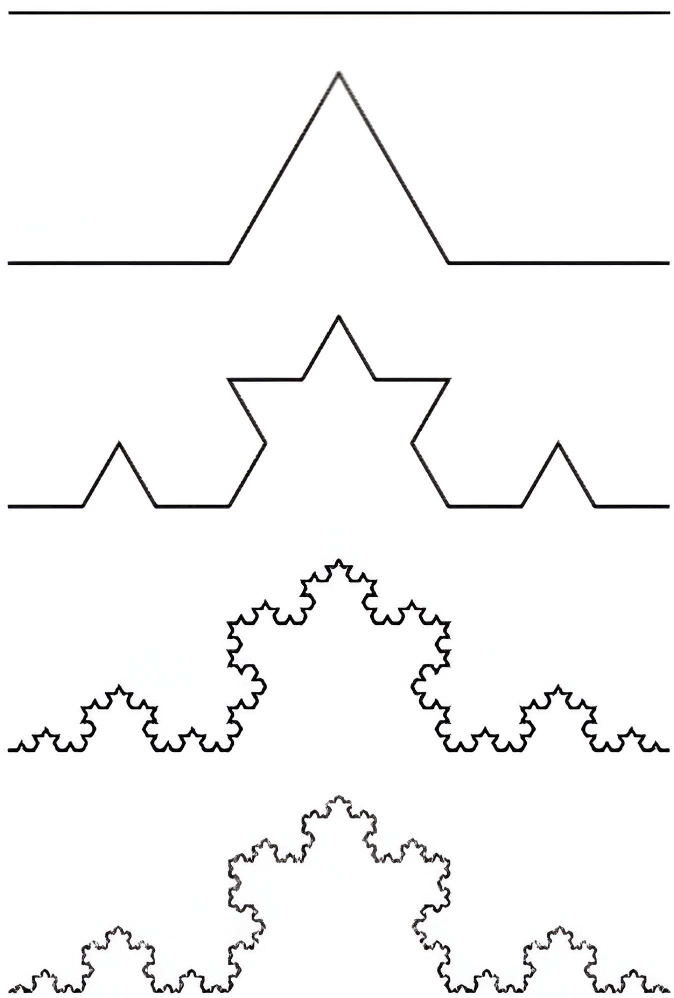
Fractal geometry describes the shape and structure of objects with a non-simple geometry whose associated dimension (fractal dimension) is usually a non-integer number, which we call fractals. Often, fractals have their fractal dimension greater than the topological dimension [46]. This was one of the first definitions given in Mandelbrot's seminal book [46]. Another common property for defining fractals is self-similarity; that is, the parts of a fractal are exactly or approximately similar to a part of itself on various scales [47]. However, a precise and general definition for a fractal is difficult to obtain [47], so this section will focus on properties and motivating the physics of these objects.

Many physical phenomena have properties that define a fractal system, such as the flow of fluids and the study of turbulence, the use of high-frequency radio antennas, trajectories of Brownian motion, and various systems whose temporal evolution refers to fractals, phenomena called fractal growth [47]. For our discussion, it is sufficient to present an overview of the subject from the perspective of the anomalous (non-topological) dimension that these objects possess. Before that, we will give some motivation by constructing the fractal dimension of a famous fractal: the Koch curve [48].

4.1.1 A motivation: the Koch curve

The Koch curve is our archetype of a fractal [48]. Consider a line segment of unit length. The Koch curve now goes through an iteration procedure: divide the segment into three equal line segments, the middle segment of which is transformed into an equilateral triangle. The second iteration repeats the same protocol for each line segment previously generated. The Koch curve is the limit to infinity of these successive iterations on the original line segment [47]. The representation of the iterations that build the Koch curve can be seen in figure 5.

Figure 5: Schematic view of the iterations leading to the Koch curve.



Source: The author (2024).

As a way of studying the geometry of the Koch curve about the geometry of simpler objects, one can ask what is the dimension of the Koch curve. To answer this question, we need to establish a formal definition of dimension, consistent with the intuitive value of topological dimension for objects in ordinary geometry.

An initial simple idea is to use the notion of measure[1] (Hausdorff measure [47]) m of a subset of $\mathbb{R}^n$, as follows: when we scale an interval $I := [a, b] \subset \mathbb{R}$ by a factor $K > 0$, the measure of I which was initially equal to the length of I, $m(I) = b - a$, becomes

[1] A measure can be summarized as a non-negative function of a set, whose value for a countable union of disjoint sets is given by the sum of the values in each set [49]; the measure generalizes the common notions of length on the line, area on the plane, and volume in space, to other subsets of $\mathbb{R}^n$.

given by $m(K(I)) = K(b-a) = Km(I)$ [49], where $K(I)$ is the scaled interval. If we start with a rectangle $R := [a,b] \times [c,d] \subset \mathbb{R}^2$ and scaling by K, the measure goes from $m(R) = (b-a)(d-c)$, to $m(K(R)) = K^2 m(R)$. In turn, taking a parallelepiped $P := [a,b] \times [c,d] \times [e,f] \subset \mathbb{R}^3$ and scaling by K, the measure goes from $m(P) = (b-a)(d-c)(f-e)$, to $m(K(P)) = K^3 m(P)$. This suggests that our intuitive idea of dimension for objects in $\mathbb{R}^n$ can be obtained from this procedure. More generally, the Hausdorff measure satisfies the so-called scaling property [47], that is

$$m(K(A)) = K^d m(A), \tag{4.1}$$

where $A \subset \mathbb{R}^n$, K is a scale factor, and d is initially a non-rigorous notion of dimension for A. The scaling property holds for fractals [47] and can be used to first motivate a fractal dimension concept that extends the topological dimension.

Looking at the Koch curve (K) (please, do not confuse it with the previous scale factor), we can split it into four parts (K/4) of equal measure, as shown in figure **??**. Of course, in terms of the set, K = 4(K/4), and the measure of K is given by the sum of the measures of the parts that compose it, so

$$m(\mathrm{K}) = 4m(\mathrm{K}/4). \tag{4.2}$$

Using equation (4.1), and looking at figure 6 once again, we can see that for K we have

$$m(\mathrm{K}) = 3^d m(\mathrm{K}/4), \tag{4.3}$$

and the measure of K is obtained by scaling the one-quarter of K by a factor of 3. Now, joining equations (4.2) and (4.3)

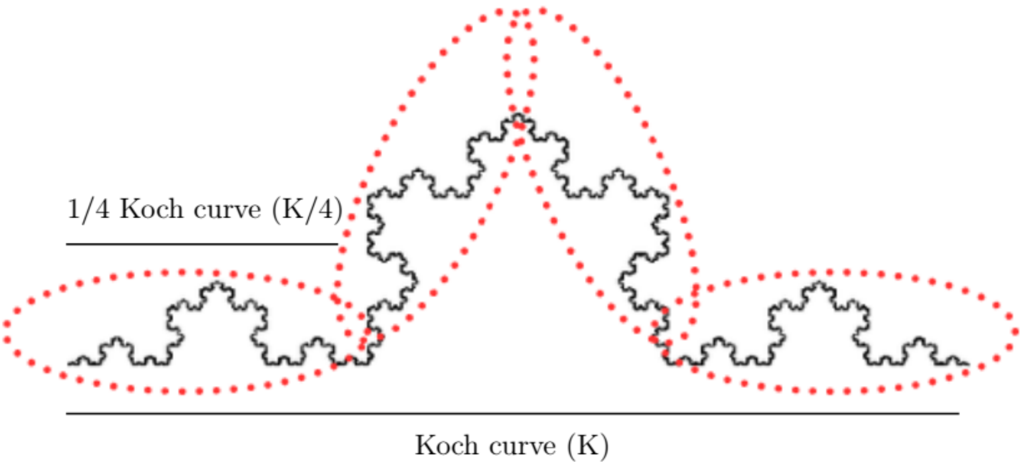
$$4 = 3^d, \tag{4.4}$$

such that K is an object with a generalized dimension, or fractal dimension d_{fractal}, given by

$$d_{\text{fractal}} = \frac{\log 4}{\log 3} \approx 1.26. \tag{4.5}$$

The scaling property is an important condition for defining fractals [47], and a precise

Figure 6: Schematic splitting of the Koch curve into four parts of equal measure.



Source: The author (2024).

notion of dimension, the Hausdorff dimension[2], d_{H}, can be taken as the fractal dimension, d_{fractal}. Before we discuss the notion of dimension we will use to define fractals, comments on equation (4.5) and the Koch curve itself are necessary. Note that the fractal dimension (Hausdorff dimension) of K is greater than 1 and less than 2, which is greater than the topological dimension, D, of this set, $D = 1$. Self-similarity, although not commented on, is masked in the scaling property: the object maintains its appearance on various scales by a relationship such as (4.1).

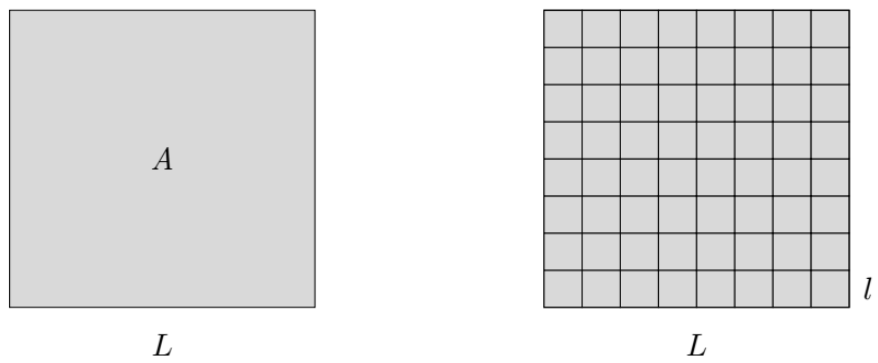
Next, we will analyze a more operational and physical definition of fractal dimension, the box-counting dimension, or Minkowski–Bouligand dimension, $d_{\text{M-B}}$, which conveniently is often equivalent to the Hausdorff dimension [47] (we will assume that both coincide).

4.1.2 Fractal dimension

A visual and practical approach to recognizing fractals is by observing the area-to-volume quotient of candidate objects (formally, the quotient formed by the measure of the area by the measure of the volume, of the set that constitutes the object). It is easy to see that for simple macroscopic objects such as a parallelepiped or a sphere, this quantity

[2]Using the formal definition of the Hausdorff dimension is beyond the scope of this work (see, for example, [47]), and equation 4.1 can be used as a method of calculating Hausdorff dimension for fractals.

Figure 7: A set A that forms a cube of linear size L covered by boxes of linear size l.



Source: The author (2024).

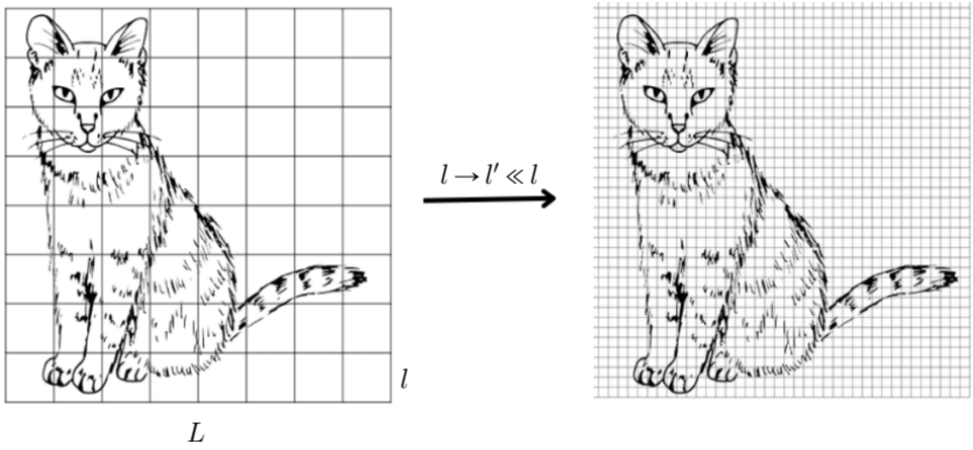
is inversely proportional to a characteristic linear size of the object. Also, such quotient is generally small in that case. Fractals in turn often have a high area-to-volume ratio, which indicates their intricate and complex geometric structure [46]. With this addition to the common properties of fractals, a notion of fractal dimension can be intuited. For visual simplicity, we will be dealing with sets in $\mathbb{R}^3$, so the extension to $\mathbb{R}^n$ is immediate. Consider a set A that forms a cube of linear size L and is contained in a grid of cubes (often referred to as boxes) of linear size l, as shown in figure 7.

The relative volume between the cube and the boxes that cover it is $\Omega(L)/\Omega(l) = 64$. From figure 7, the number $N(l, L)$ of boxes needed to completely cover the cube is also 64, and by defining $\varepsilon := l/L$ we have that $N(l, L) = (1/\varepsilon)^D$, with D equals to the (topological) dimension of the cube, $D = 3$.

Is expected that for the case of a simple object like a cube, regardless of ε, i.e. the number of boxes to cover the set, the dimension of the cube should not change: repeating the previous exercise with a much larger number of boxes should not capture any additional detail of the cube, being an object with unlimited smooth boundaries.

Now, this simple construction allows us to precisely define a generalized form of dimension to include fractals. Consider a set that forms an object of more complicated geometry, again associated with a linear size L, and repeat the method of covering it with boxes of linear size l. See the representation in figure 8. Should make the number of boxes smaller (with a linear size $l' \ll l$), suggested by capturing more detail of the geometry, change the value of D?

Figure 8: An object with a more complicated geometry (a cat), which suggests a greater number of boxes to capture the details of the structure.



Source: The author (2024).

We can treat the problem rigorously and then take the limit of $\varepsilon \to 0$ in the expression[3] $N(l, L) = N(\varepsilon) = C(1/\varepsilon)^D$, with D replaced by the fractal dimension $d_{\text{M-B}}$, defining the Minkowski-Bouligand dimension, or box-counting dimension [47], as

$$d_{\text{M-B}} := C \lim_{\varepsilon \to 0} \frac{\log (N(\varepsilon))}{\log (1/\varepsilon)}. \tag{4.6}$$

The interpretation is that an increase in the complexity of the geometry of an object, such as porous, hairy, or crooked one, suggests more information when we increase the number of boxes. In practice, as the box count is physically limited by the atomic scale, the limit on (4.6) is such that ε must decrease until the dimension $d_{\text{M-B}}$ no longer changes. Fractals change their dimension as more boxes are added and more detail is captured of their structure. Once we reach the $\varepsilon \ll 1$ regime, at the available observational limit (this is the situation where the area-to-volume ratio is large), fractals tend to have their dimension, $d_{\text{M-B}}$, well defined. As a consequence of this behavior, fractals generally have a non-integer dimension.

Note that in the expression

$$N(\varepsilon) = C(1/\varepsilon)^{d_{\text{M-B}}}, \tag{4.7}$$

[3]The constant dependent on the object, and independent of the resolution, is added so that different objects can characteristically vary their fractal dimension.

we must take $\varepsilon \ll 1$ to operationally work with the limit of equation (4.6). If $d_{\text{M-B}}$ does not change with $\varepsilon \ll 1$ then we can only say that l/L is simply a macroscopic quantity n of parts that split the linear size of the set equally and the number of boxes is $N = n^D$, with D the topological dimension. By consequence, $n^D = \varepsilon^{d_{\text{M-B}}}$, and we have ordinary geometry, $D = d_{\text{M-B}}$. Otherwise, as the linear size of the boxes cannot be greater than the linear size of the set, $l \leqslant L$, then the fractal dimension is necessarily $d_{\text{M-B}} \geqslant D$. The self-similarity property of fractals can be included in equation (4.7) by scaling the set by a factor $K > 0$, to obtain

$$N(K\varepsilon) = CK^{-d_{\text{M-B}}}(1/\varepsilon)^{d_{\text{M-B}}} = K^{-d_{\text{M-B}}}N(\varepsilon). \tag{4.8}$$

,

4.1.3 Random fractals

There is a class of fractals in which random or probabilistic realization rules are used in its construction. They are called random fractals [47]. On the one hand, the Koch curve is a so-called deterministic fractal, and the realization steps that construct it are iterations that carry self-similarity. On the other hand, random fractals are characterized by randomness, and trajectories of Brownian motion are an example. In quantum mechanics, the paths of massive free particles are non-differentiable curves with self-similarity [50, 51]. It has been proven that we can associate a fractal dimension (Hausdorff dimension) $d_{\text{H}} = 2$ to these curves [50–52], i.e. a fractal curve that carries the geometric information of a more complicated object [51].

In 2020, inspired by the complex geometrical structure of the surface of the COVID-19 virus, and influenced by the Wheeler [53] proposal of a quantum modification in the smoothness of spacetime in very small scales[4], Barrow [55] proposed that the very surface of the black hole should have the characteristics of a fractal on the quantum scale. With this effect of quantum gravity on the black hole as a premise, Barrow constructed the SBH surface as a kind of Koch snowflake[5], i.e. as a deterministic fractal.

[4]Very brief quantum fluctuations on the Planck scale that should alter the smooth geometry of spacetime and are called spacetime foam [54].

[5]Instead of starting with a line segment, start with an equilateral triangle and apply the Koch curve algorithm. In the limit of the iterations going to infinity, you get the Koch snowflake.

However, due to Barrow's [55] premise that the fractal effect comes from the spacetime foam, which is generated by quantum fluctuations of spacetime on very small scales [54], driven by the fractal randomness of the particle paths [51], it is argued here that the surface of the black hole must be a random fractal and not a deterministic one. With this, we will analyze the physics of the SBH as a random fractal, and investigate the use of fractional calculus in quantum mechanics and quantum cosmology to describe such a black hole structure.

4.2 Fractional quantum gravity

Quantum gravity, nowadays, is the great puzzle of theoretical physics. It is hoped that the experiments currently possible, or in the future, will lead to insights and clues about the convergence between quantum physics and gravitational interaction. So far, several theories of quantum gravity have been proposed, and are under continuous refinement and investigation, as examples: string theory [56], loop quantum gravity [57], and asymptotically safe quantum gravity [58]. A difficult task for self-consistent analyses of a quantum gravity theory is to obtain non-divergent predictions of the theory for both the infrared (IR) (lower energy effects) and ultraviolet (UV) (high energy effects) limits when transiting from the gravitational scale to the Planck scale. A useful and often desirable approach is to study the quantum UV corrections to the gravitational scale arising from an effective field theory, even if one does not know the underlying fundamental theory of quantum gravity [59] (i.e., an operationally semi-classical analysis of the physics involved).

It should be mentioned that there are numerous cases in gravity theories where fractional calculus and fractal models has been employed, showing the wide and significant applications of these mathematical areas. For example, fractional calculus finds application in the study of fractional generalization of field equations based on the Riemann-Liouville derivative [60, 61], fractional gravity for spacetimes with non–integer dimensions and the Caputo derivative in fractional manifolds [62, 63], modified Newtonian dynamics (MOND)[6] from a fractional version of Newton's theory based on the fractional Poisson equation [65], model of gravity based on the theory of D-dimension metric spaces and its

[6] At low accelerations, MOND proposes that the gravitational force should deviate from the standard $1/r^2$ law to a relation as $1/r$. This modification was introduced by Milgrom [64] to account for the observed discrepancies in the dynamics of galaxies and galaxy clusters without invoking dark matter.

applications to Newtonian gravity [66–70], its relativistic extension [71], the gravitational potential associated to dark matter as determined by a modified Poisson equation including fractional derivatives [72–74], perturbative theories of quantum gravity based on fractional operators [75], modified field equations from generalized fractional Bekenstein-Hawking entropy [76], and to justify that fractional corrections in gravity is generated by quantum stochastic fluctuations of spacetime [77]. Also, fractal models are used to investigate a power-counting renormalizable field theory living in a fractal spacetime [78, 79], a multi-fractional scenario inspired by multi-fractal geometry, where the spacetime dimension changes with the scale [75, 80], models of higher derivative quantum gravity to associate a fractal dimension to spacetime at very small distances [81], using Caputo fractional derivative to determine the spacetime geometry of a fractional cosmic string [82], the fractal structure of spacetime in loop quantum gravity [83], and in asymptotically safe quantum gravity [84].

In cosmology, fractional calculus is also often found in proposals of a generalization of the Friedman equation with a fractional time derivative based in Riemann-Liouville derivative [85], cosmological models derived from the Einstein-Hilbert action of fractional order [86–90], modified Friedmann equations with Caputo's fractional derivative to explain a late cosmic acceleration without introducing a dark energy component [91], and to study the Hubble tension [92]. Because of the applications of fractals as a model of the distribution of galaxies and galaxies clusters in the Universe [93, 94], fractals are also used in cosmology to study the cosmological principle [95], and models of a inhomogeneous Universe [96].

As suggested by almost all the dozens of references above, several quantum gravity scenarios predict that spacetime has a fractal behavior at a very small scale, or an apparent fractional (non-integer) order regime in derivatives and operators, often implying that the dimension of spacetime changes in different scales [97]. The following objective is to take the consequences of fractional calculus in quantum mechanics, namely FQM [6, 98, 99], and mainly in quantum cosmology, namely FQC [5, 7, 13, 15, 17, 100–102], as an effect of quantum gravity into the fractal SBH. Such a framework, clearly not fundamental[7], but at the level of an effective theory composes which we call FQG. First, we separate the

[7]It is important to emphasize that, as with its root in FQC [5], this does not prescribe a fundamental theory, but rather a heuristic methodology in the quantum effective setting of GR solutions.

concepts and weave the relationships between fractional calculus and fractals.

4.2.1 Fractional calculus and fractals

It is generally assumed that the origin of the concept of fractional derivative follows from a question from L'Hôpital to Leibniz about the meaning of a half-order derivative; i.e. something like $d^{1/2}y/dx^{1/2}$. Since then, the problem has undergone various treatments and proposals for a general formulation of the calculus for non-integer orders of derivative, in particular, it has received attention from names such as Liouville, Riemann, Caputo, Riesz, Grünwald, Letnikov, and others [103]. Over time, fractional calculus has gone from being a theoretical extension of ordinary calculus to having the potential to adequately describe various phenomena in nature, such as rheology, quantitative biology, electrochemistry, scattering theory, diffusion and transport theory in complex media, probability, potential theory, elasticity [103], many other applications in physics [104], and as we shall see later, particularly in gravitation and cosmology.

A crucial point to the applicability of a fractional derivative of a function is that its definition often includes an integral of a certain order, such as a fractional order, which requires information about the function over a range of values. Let us elaborate and then consider a single-valued analytic function $f:\mathbb{R}\rightarrow\mathbb{R}$ where its indefinite integral is represented by an operator ${}_aI$ such that [104]

$$({}_aIf)(x) := \int_a^x f(\xi)d\xi. \tag{4.9}$$

Since it is natural to define fractional integrals via Cauchy's formula of repeated integration [104], one can realize that the integration of a function can be considered as the inverse operation of differentiation, and then

$$\left(\frac{d}{dx}\right)({}_aIf)(x) = f(x). \tag{4.10}$$

This means that the route to a fractional derivative is not only by reversing the integration operation for a fractional integral but also that the derivative calculation will be just as challenging as the integration [104]. In other words, the value of the fractional derivative at a particular point depends on the behavior of the function over a whole interval, making it non-local in this sense [104] (as a well-known example, of a non-local operator there is the Fourier transform [104]). Taking the description of anomalous diffusion as an application example, the non-locality of fractional operators allows for an adequate description of the spatial and temporal evolution of diffusion processes which have broad spatial jump and waiting time probability distributions [105] (also called memory effect). Furthermore, the concept of non-locality and the memory effects provided by non-local operators has a long historical tradition in physics [104].

As we discussed in section 4.1, fractals are objects of non-simple geometry, with a complex structure at different scales, often endowed with self-similarity and whose associated (non-integer) fractal dimension differs from the topological dimension. In turn, fractional calculus, with its non-local nature and ability to capture long-range dependencies, such as big jumps and memory [105], provides a mathematical framework to represent and analyze fractal phenomena. In physical terms, it is not a new idea that the evolution of fractal systems has fractional derivatives, and fractional equations of motion, as a natural tool to be described [106, 107], or that fractional calculus can be used to change the fractal dimension of any random or deterministic fractal [108]. Indeed, it will be seen below that the fractal structure of the SBH emerges from fractional calculus when justifying its use in quantum mechanics and quantum cosmology.

4.2.2 Fractional quantum mechanics

Before we can discuss the FQC developed by Jalalzadeh and Moniz [5], we must clarify the motivation and scenario behind the FQM created by Laskin [6]. As mentioned in the previous paragraphs, it was realized by Feynman and Hibbs [50] that the paths of a massive quantum particle are non-differentiable curves with self-similarity, i.e. zigzag curves of similar shape at different scales [51]. Furthermore, these paths bear a great resemblance to the paths of Brownian motion, since both have a fractal dimension (Hausdorff) $d_{\mathrm{H}} = 2$ [52], and the Brownian motion diffusion equation can be heuristically mapped onto the Schrödinger equation for a massive quantum particle [109]. Using these facts and in the search for a formulation via path integrals for (non-relativistic) quantum mechanics, Feynman's formulation was proposed [50], so that the integrals are constructed from the Wiener process, i.e. a stochastic process that models Brownian motion, and whose trajectory increments (steps realizations through the trajectory) follow a Gaussian distribution in their direction [5, 6].

The conventional form of the quantum Hamiltonian for a particle with mass m

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\hat{\mathbf{r}}), \tag{4.11}$$

with $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ the momentum and position operators, respectively, and $\hat{V}(\hat{\mathbf{r}})$ the potential operator, Feynman's formulation was used by Laskin [6] to answer the following question: are there non-quadratic orders of exponent in the kinetic term of the Hamiltonian (4.11), without violating the laws of physics? To answer this question, Laskin [6] generalized the formulation of path integrals by using Lévy's process instead of integrals built on Wiener's process. Lévy's processes are stochastic processes that generalize the description of Brownian motion to non-continuous trajectories, endowed with jumps and a non-Gaussian distribution for the direction of the increments [99]. Another property of these processes is that they indeed generalize Brownian motion (Wiener process) in the sense that a quantity α, $1 < \alpha \leqslant 2$, called Lévy's index, emerges to denote the fractal dimension of the trajectory, generally, $d_{\text{fractal}} = \alpha$ [99].

With this, FQM was inaugurated and Laskin was able to generalize Feynman's formulation of path integrals, in which the quantum Hamiltonian (4.11) becomes [6]

$$\hat{H}_\alpha = D_\alpha |\hat{\mathbf{p}}|^\alpha + \hat{V}(\hat{\mathbf{r}}), \qquad 1 < \alpha \leqslant 2, \tag{4.12}$$

with D_α a generalized coefficient with (cgs) dimension $[D_\alpha] = \text{erg}^{1-\alpha}\cdot \text{cm}^\alpha \cdot \text{sec}^{-\alpha}$ [6]. Note that when $\alpha = 2$ and $D_\alpha = 1/2m$ equation (4.12) recover the Hamiltonian (4.11). Furthermore, this can be implemented to produce Laskin's version of the time-dependent Schrödinger equation with conventional choice for the space representation of operators ($\hat{\mathbf{r}} := \mathbf{r}$. $\hat{\mathbf{p}} := -i\nabla$, with $\nabla \equiv \partial/\partial\mathbf{r}$)[8] as [98]

$$i\frac{\partial \Psi(\mathbf{r},t)}{\partial t} = D_\alpha\big(-\Delta\big)^{\alpha/2}\Psi(\mathbf{r},t) + \hat{V}(\mathbf{r},t)\Psi(\mathbf{r},t), \qquad 1 < \alpha \leqslant 2, \tag{4.13}$$

where $\Delta := \nabla^2$ the Laplacian operator, and the operator $\left(-\Delta\right)^{\alpha/2}$ is introduced as the 3-dimensional generalization of the Riesz fractional derivative[9], following Laskin [98] (the variation range of α, $1 < \alpha \leqslant 2$, is henceforth implied)

[8]Remember that we are using the units which $\hbar = 1$.

[9]An in-depth investigation into the possible representations found in the literature of Riesz's fractional derivative can be found at [110].

$$
\begin{aligned}
(-\Delta)^{\alpha/2}\Psi(\mathbf{r},t) &= \left(\mathscr{F}^{-1}|\hat{\mathbf{p}}|^{\alpha}\mathscr{F}\right)\Psi(\mathbf{r},t) \\
&= \frac{1}{(2\pi)^3}\int d^3p \exp\{i(\hat{\mathbf{p}}\cdot\mathbf{r})\}|\hat{\mathbf{p}}|^{\alpha}\int d^3r' \exp\{-i(\hat{\mathbf{p}}\cdot\mathbf{r}')\}\Psi(\mathbf{r}',t),
\end{aligned}
\tag{4.14}
$$

where $\mathscr{F}$ denotes the Fourier transform of $\Psi(\mathbf{r},t)$.

Note that when $\alpha = 2$ we have the full recovery of the usual results of quantum mechanics, and more than that we recover Feynman's formulation of path integrals [99]. On the other hand, with $\alpha \neq 2$ there must be a new physics. Equation (4.14), by introducing a generalization of Riesz's fractional derivative, embeds the fractional calculus in the Schrödinger equation and Lévy's index becomes the indicator of the fractional derivative order (for this reason, α it will henceforth be called Lévy's fractional parameter). Equation (4.13) is the so-called fractional Schrödinger equation (FSE) [98]. Before going any further, a few comments on the FQM are necessary.

Despite the years that have passed since its inception and the inherent difficulty in its experimental verification due to the need to find a quantum regime known to be endowed with a Lévy process, FQM has only recently been explored and tested via experimental and simulation analyses [111]. Successfully, FQM has been used to model optical media with properties governed by the Lévy fractional parameter [111], in systems with electrical screening effects [112], and to study the role of disorder in the vibration spectra of molecules and atoms in solids [113]. Also, given the random and unpredictable nature of the paths of quantum particles [50], a generalization of path integrals based on Lévy processes that admit big jumps and discontinuity in the trajectory has a reasonable physical justification. Given the context of quantum gravity, the natural question now would be: does the conjecture that extends quantum particle paths (and then the very quantum mechanics) to Lévy processes have consequences on the scale of quantum effects in spacetime?

4.2.3 Fractional SBH

As mentioned in chapter 2 and demonstrated in chapter 3, running the CQG program from spacetime symmetries considerations which limit the degrees of freedom considered in the quantization, and thus simplify the study of the WDW equation produced, is called a quantization in minisuperspace. The quantized SBH is the archetype of this scenario, as we see by comparing the general WDW equation (2.52) in superspace (with functional derivatives that return divergence problems) and that in minisuperspace (3.30) for the SBH (with ordinary derivatives and no divergence). Halliwell [114] showed that obtaining the WDW equation in configuration spaces of a minisuperspace from a quantum formulation of path integrals is possible. Given the generalization of the notion of path integrals in quantum mechanics to the fractional case via Lévy processes following Laskin [6], Jalalzadeh and Moniz [5] proposed that the same reasoning should be extended to quantum cosmology (i.e., minisuperspace models in CQG) and then to the very WDW equation. This approach to quantum cosmology is the so-called FQC [17].

On the one hand, instead of repeating Laskin [6] and then working on the difficulties shown by Halliwell [114] in obtaining the general WDW equation in superspace at the level of path integrals modified by the inclusion of Lévy processes, Jalalzadeh and Moniz [5] were motivated by the form of the FSE (4.13) to heuristically obtain a fractional extension of the WDW equation, focusing on minisuperspace models which quantum cosmology takes place. The method is then to induce the modification to fractional differential operators directly in the WDW equation for the minisuperspace models [5].

On the other hand, considering FQM [6, 98, 99] and the non-smooth structure of the spacetime foam suggested on the quantum scale [53, 54], it is to be expected that the virtual particles of the quantum fluctuations that produce the spacetime foam have their paths influenced by the admission of Lévy processes [7]. In other words, the alteration of the quantum paths described by the FSE (4.13) must produce quantum gravity effects on a certain scale. Furthermore, the intrinsic non-locality of fractional calculus operators via FQC suggests new quantum gravity effects with a non-local behaviour in spacetime on certain scales [101].

Details about the implementation of the fractional operator in the WDW equation for an arbitrary minisuperspace (which was not discussed in this work), that is the d'Alembertian operator in its fractional version, can be found in the original works by Jalalzadeh and Moniz [5, 17]. Our particular interest at this point is to analyze the fractional extension of the WDW equation of the SBH (3.30), and then investigate the fractal nature of the SBH. We define a new coordinate z in the minisuperspace of the quantized SBH (please, see equation (3.30)) as [7]

$$z := x - \frac{M}{M_{\mathrm{P}}^2}, \tag{4.15}$$

and, inspired by the FSE (4.13), the quantum Riesz fractional derivative operator (1-dimensional) becomes [7]

$$\left(-\frac{d^2}{dz^2}\right) \quad \rightarrow \quad \frac{1}{M_{\mathrm{P}}^{\alpha-2}}\left(-\frac{d^2}{dz^2}\right)^{\alpha/2}. \tag{4.16}$$

The fractional WDW equation of the SBH is then

$$\frac{1}{2}M_{\mathrm{P}}^{1-\alpha}\big(-\Delta\big)^{\alpha/2}\psi(z)+\frac{1}{2}M_{\mathrm{P}}\,{\omega_{\mathrm{P}}}^2z^2\,\psi(z)=\frac{M^2}{2M_{\mathrm{P}}}\psi(z). \tag{4.17}$$

Since this equation does not have a trivial solution [7], to study the thermodynamics of the fractional SBH, the approach is to use a semi-classical analysis and obtain the emission frequency. Therefore, we recover $|\hat{p}|^\alpha$ (1-dimensional) from the quantum Riesz fractional derivative operator $\big(-\Delta\big)^{\alpha/2}$ in equation (4.17) as constructed in equation (4.12), then

$$\frac{1}{2}M_{\mathrm{P}}^{1-\alpha}|\hat{p}|^\alpha+\frac{1}{2}M_{\mathrm{P}}\,{\omega_{\mathrm{P}}}^2z^2=\frac{M^2}{2M_{\mathrm{P}}}. \tag{4.18}$$

Note that the equation above is the fractional extension of the equation (3.27), with the change of coordinate shown in equation (4.15) and applied canonical quantization ($z\rightarrow\hat{z}:=z, p\rightarrow\hat{p}$). Before canonical quantization, we then have the fractional Hamiltonian equation for the SBH as

$$\frac{1}{2}M_{\mathrm{P}}^{1-\alpha}|p|^\alpha+\frac{1}{2}M_{\mathrm{P}}\,{\omega_{\mathrm{P}}}^2z^2=\frac{M^2}{2M_{\mathrm{P}}}, \tag{4.19}$$

and now solving for $|p|$, we obtain

$$|p|=\frac{M^{2/\alpha}}{M_{\mathrm{P}}^{2/\alpha-1}}\left(1-\frac{M_{\mathrm{P}}^4}{M^2}z^2\right)^{1/\alpha}. \tag{4.20}$$

Looking at the phase space of the system, we have orbits in which the turning points are given by $|p|=0$ in equation (4.19), and then $z=\pm M/M_{\mathrm{P}}^2$. We now invoke the Bohr-Sommerfeld quantization rule to find out the emission frequency of the black hole from the mass spectrum; note that in the minisuperspace considered, p is the conjugate momentum to z, and we have a dynamics like the harmonic oscillator in equation (4.20). Hence, the Bohr-Sommerfeld quantization rule reads [7]

$$2\pi\left(n+\frac{1}{2}\right)=\oint p\,dz. \tag{4.21}$$

Using equation (4.20), we have

$$2\pi\left(n+\frac{1}{2}\right)=\frac{M^{2/\alpha}}{M_{\rm P}^{2/\alpha-1}}\oint dz\left(1-\frac{M_{\rm P}^4}{M^2}z^2\right)^{1/\alpha}. \tag{4.22}$$

The function in the integral is an even function, so the limits can be evaluated by symmetry resulting in

$$2\pi\left(n+\frac{1}{2}\right)=4\frac{M^{2/\alpha}}{M_{\rm P}^{2/\alpha-1}}\int_0^{M/M_{\rm P}^2} dz\left(1-\frac{M_{\rm P}^4}{M^2}z^2\right)^{1/\alpha}, \tag{4.23}$$

for a complete orbit in the phase space. Moreover, we define a suitable variable for integration $y := (M_{\rm P}^4/M^2)^{1/2}z$, such that $y^2 = (M_{\rm P}^4/M^2)z^2$. Equation (4.23) now becomes

$$2\pi\left(n+\frac{1}{2}\right)=4\left(\frac{M}{M_{\rm P}}\right)^{2/\alpha+1}\int_0^1 dy\big(1-y^2\big)^{1/\alpha}, \tag{4.24}$$

where $dz = (M/M_{\rm P}^2)dy$, and $z = M/M_{\rm P}^2$ implies $y = 1$. The integral in equation (4.24) can be identified as the definition of the beta function [115],

$$B(z_1,z_2)=\int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt, \tag{4.25}$$

with z_1, z_2 complex numbers. Comparing with the integral in (4.24) we find that $t = y^2$ and $dt = 2ydy$, as well as, $z_1 = 1/2$ and $z_2 = 1/\alpha + 1$. Then, we write

$$\int_0^1 dy\big(1-y^2\big)^{1/\alpha}=\frac{1}{2}B\left(\frac{1}{2},\frac{1}{\alpha}+1\right). \tag{4.26}$$

The beta function relates to the gamma function as [115]

$$B(z_1,z_2)=\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}, \tag{4.27}$$

and then

$$B\left(\frac{1}{2},\frac{1}{\alpha}+1\right)=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{\alpha}+1)}{\Gamma(\frac{1}{\alpha}+\frac{3}{2})}=\sqrt{\pi}\,\frac{\Gamma(\frac{1}{\alpha}+1)}{\Gamma(\frac{1}{\alpha}+\frac{3}{2})}. \tag{4.28}$$

Now, a suggested fractal dimension d is defined in terms of the Lévy parameter α, as follows [7]

$$d=\frac{2}{\alpha}+1, \qquad 2\leqslant d<3, \tag{4.29}$$

so the validity of this definition as a fractal dimension will be checked below. Substituting equations (4.26), 4.28), and (4.29), in equation (4.24), one gets

$$\pi\left(n+\frac{1}{2}\right)=\sqrt{\pi}\left(\frac{M}{M_{\mathrm{P}}}\right)^{d}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+2}{2})}. \tag{4.30}$$

It is well known that the volume V_d of a d-dimensional unit sphere is given by

$$V_d=\frac{\pi^{d/2}}{\Gamma(\frac{d+2}{2})}, \tag{4.31}$$

and with the introduction of d as a fractal dimension, we can write the gamma function in equation (4.30) concerning the volume V_d, that is

$$\pi\left(n+\frac{1}{2}\right)=\frac{V_d}{V_{d-1}}\left(\frac{M}{M_{\mathrm{P}}}\right)^{d}, \tag{4.32}$$

or

$$M=M_{\mathrm{P}}\left[\pi\left(n+\frac{1}{2}\right)\frac{V_{d-1}}{V_d}\right]^{1/d}. \tag{4.33}$$

Note that, when $d=2$ ($\alpha=2$) equation (4.33) becomes equation (3.31), which validates the semi-classical analysis (implicitly considering the limit $n\gg 1$) via the Bohr-Sommerfeld quantization rule. With the mass spectrum (4.33), we can follow the same steps used in equation (3.36) to obtain ω_0, and calculate its fractional version, $\omega_0(d)$, as

$$\begin{aligned}\omega_0(d)=M(n+1)-M(n)&=M_{\mathrm{P}}\left[\pi\left(n+\frac{3}{2}\right)\frac{V_{d-1}}{V_d}\right]^{1/d}-M_{\mathrm{P}}\left[\pi\left(n+\frac{1}{2}\right)\frac{V_{d-1}}{V_d}\right]^{1/d}\\&=M_{\mathrm{P}}\left(\pi\frac{V_{d-1}}{V_d}\right)^{1/d}\left[\left(n+\frac{3}{2}\right)^{1/d}-\left(n+\frac{1}{2}\right)^{1/d}\right]\\&=M_{\mathrm{P}}\left(n\pi\frac{V_{d-1}}{V_d}\right)^{1/d}\left[\left(1+\frac{3}{2n}\right)^{1/d}-\left(1+\frac{1}{2n}\right)^{1/d}\right].\end{aligned} \tag{4.34}$$

Using the polynomial approximation as in equation (3.33), we have

$$\begin{aligned}\omega_0(d)&=M_{\mathrm{P}}\left(n\pi\frac{V_{d-1}}{V_d}\right)^{1/d}\left[\left(1+\frac{1}{d}\frac{3}{2n}\right)^{1/d}-\left(1+\frac{1}{d}\frac{1}{2n}\right)^{1/d}\right]\\&=\frac{M_{\mathrm{P}}}{nd}\left(n\pi\frac{V_{d-1}}{V_d}\right)^{1/d}.\end{aligned} \tag{4.35}$$

We will be interested only in the first approximation of the emission frequency (please, see equation (3.36), so we solve for n in equation (4.32) such that

$$n = \frac{1}{\pi}\frac{V_d}{V_{d-1}}\left(\frac{M}{M_{\rm P}}\right)^{d}. \tag{4.36}$$

Substituting the equation above in equation (4.35), we obtain

$$\omega_0(d) = \frac{\pi}{d}\frac{V_{d-1}}{V_d}\left(\frac{M}{M_{\rm P}}\right)^{1-d} M_{\rm P}. \tag{4.37}$$

The next step is to generalize the adiabatic invariant used to obtain the entropy of the fractional SBH, according to the equation (3.45), as follows

$$8\pi\int_0^M \frac{dM'}{\omega_0(d)} = 8\,d\,M_{\rm P}^{-d}\frac{V_d}{V_{d-1}}\int_0^M dM'M'^{d-1} = 8\,\frac{V_d}{V_{d-1}}\left(\frac{M}{M_{\rm P}}\right)^{d}. \tag{4.38}$$

Looking at the Bekenstein-Hawking entropy $\mathcal{S}_{\text{B-H}}$ of ordinary SBH in equation (3.45), one finds (remember, $G = M_{\rm P}^{-2}$)

$$\begin{aligned} 8\pi\int_0^M \frac{dM'}{\omega_0(d)} &= 8\,\frac{V_d}{V_{d-1}}\frac{1}{(4\pi)^{d/2}}\left(4\pi G M^2\right)^{d/2} \\ &= 8\,\frac{V_d}{V_{d-1}}\frac{1}{(4\pi)^{d/2}}\,\mathcal{S}_{\text{B-H}}^{d/2}. \end{aligned} \tag{4.39}$$

The fractional entropy of the fractional SBH is then defined in terms of the Bekenstein-Hawking entropy of the SBH [7]

$$\mathcal{S}_{\rm fractional} = \mathcal{S}_{\text{B-H}}^{d/2}. \tag{4.40}$$

Using such a definition, to obtain the correspondent fractional temperature of the fractional SBH the equation (4.39) is taken in the differential form

$$dM = \frac{\omega_0(d)}{\pi}\frac{V_d}{V_{d-1}}\frac{1}{(4\pi)^{d/2}}\,d\mathcal{S}_{\rm fractional}, \tag{4.41}$$

and by the first law of black hole thermodynamics [18] with $dM = dE$ we write

$$dM = T\,d\mathcal{S}_{\rm fractional}, \tag{4.42}$$

the fractional temperature can, finally, be identified as

$$T_{\text{fractional}} = \frac{\omega_0(d)}{\pi}\frac{V_d}{V_{d-1}}\frac{1}{(4\pi)^{d/2}} = \frac{1}{d(4\pi)^{d/2}}\left(\frac{M}{M_{\text{P}}}\right)^{1-d} M_{\text{P}}, \tag{4.43}$$

where $V_d/V_{d-1} = \pi/2$ and equation (4.37) was used. Once again, when $d = 2$ the first approximation of the temperature in the equation (3.43) can be recovered from (4.43).

4.3 Fractional-fractal SBH

As we have seen, non-locality effects are supported by using fractional operators [104] and fractional calculus is a tool for modeling fractal geometry systems [106–108]. On the one hand, in section 4.1 we constructed the reasoning that the SBH must have an area whose surface has the geometry of a random fractal. On the other hand, in the previous section 4.2 we saw that the application of the FQC arising from the fractional generalization of quantum effects with the inclusion of a dynamics of Lévy processes, leads to the description of a fractional SBH [7], in particular, provided with a temperature (4.43) and a fractional entropy (4.40). We have now demonstrated that the fractional SBH produced by FQC does capture the fractal structure of the SBH as a random fractal. The fractional entropy of the fractional SBH, by equation (4.40), is

$$\mathcal{S}_{\text{fractional}} = \mathcal{S}_{\text{B-H}}^{d/2} = \left(\frac{A_{\text{S}}}{A_{\text{P}}}\right)^{d/2}. \tag{4.44}$$

In equation (4.44), $A_{\text{S}} = 4\pi r_{\text{S}}^2 = 16\pi G^2 M^2$ is the area of the SBH, and $A_{\text{P}} = 4L_{\text{P}}^2 = 4G$ is the Planck area. Following the equation (4.44), we define the fractional area of the SBH by

$$\mathcal{S}_{\text{fractional}} := \left(\frac{A_{\text{fractional}}}{A_{\text{P}}}\right). \tag{4.45}$$

Considering equations (4.44) and (4.45) together, we can express $A_{\text{fractional}}$ as

$$A_{\text{fractional}} = A_{\text{P}}\left(\frac{A_{\text{S}}}{A_{\text{P}}}\right)^{d/2} = 4L_{\text{P}}^2\left(\frac{4\pi r_{\text{S}}^2}{4L_{\text{P}}^2}\right)^{d/2} = 4\pi^{d/2}\left(\frac{r_{\text{S}}}{L_{\text{P}}}\right)^{d} L_{\text{P}}^2. \tag{4.46}$$

Revisiting equation (4.7), we have a description of the number $N(\varepsilon)$ of boxes that cover the set of fractal geometry. Considering the case of the fractional SBH, $N(\varepsilon)$ is defined by the quotient between the fractional area of the SBH and a fundamental unit of area, which we take to be the Planck area. Hence, by equations (4.44) and (4.45), we can write for the SBH

$$N(\varepsilon) := \left(\frac{A_{\text{fractional}}}{A_{\text{P}}}\right) = \left(\frac{A_{\text{S}}}{A_{\text{P}}}\right)^{d/2} = \left(\frac{4\pi r_{\text{S}}^2}{4L_{\text{P}}^2}\right)^{d/2} = \pi^{d/2}\left(\frac{r_{\text{S}}}{L_{\text{P}}}\right)^{d}. \tag{4.47}$$

Two important points: notice that the number of boxes to cover the area of the fractional SBH is equal to the fractional entropy given in equation (4.44). Furthermore, equation (4.7) shows that the structure of fractal geometry for the area of the SBH emerges from the fractional SBH (the suggested fractal dimension d coincides with the fractal Minkowski-Bouligand dimension $d_{\text{M-B}}$ in equation (4.7)), where $\varepsilon = L_{\text{P}}/r_{\text{S}}$ and $C = \pi^{d/2}$ given equation (4.7).

Therefore, the SBH can be identified as a system of fractal geometry, described by fractional calculus from the perspective of FQC, which has a fractal dimension $2 \leqslant d < 3$ and henceforth we will call fractional-fractal SBH. From this result, we draw consequences for the effective quantities that describe the thermodynamics of the fractional-fractal SBH, where the fractional label will be replaced by a fractal to emphasize the physical nature of the fractal geometry of the SBH. We start by defining the effective Schwarzschild radius r_{eff} with equation (4.46) such that

$$A_{\text{fractal}} = 4\pi r_{\text{eff}}^2 = 4\pi^{d/2}\left(\frac{r_{\text{S}}}{L_{\text{P}}}\right)^{d} L_{\text{P}}^2, \tag{4.48}$$

and solving for r_{eff}

$$r_{\text{eff}} = \pi^{(d-2)/4}\left(\frac{r_{\text{S}}}{L_{\text{P}}}\right)^{d/2} L_{\text{P}}. \tag{4.49}$$

From equation (4.49) and equation (4.45), we have

$$\mathcal{S}_{\text{fractal}} = \frac{A_{\text{fractal}}}{4G} = \frac{\pi}{G}\, r_{\text{eff}}^2. \tag{4.50}$$

The effective mass of the fractional-fractal SBH is

$$M_{\text{eff}} = \frac{r_{\text{eff}}}{2G} = \frac{r_{\text{eff}}}{2L_{\text{P}}^2} = 2^{(d/2)-1}\pi^{(d-2)/4}\left(\frac{M}{M_{\text{P}}}\right)^{d/2} M_{\text{P}}. \tag{4.51}$$

With the first law of black hole thermodynamics, equation (4.42), we obtain the effective temperature of the fractional-fractal SBH using equations (4.50) and (4.51)

$$T_{\text{eff}} = \frac{1}{4\pi r_{\text{eff}}}. \tag{4.52}$$

It is worth noting that when $d = 2$, the fractional-fractal will be reduced to their ordinary values. Our next step is to explore possible developments in thermodynamics of horizons with the advent of fractional-fractal SBH in cosmology. The link between these ideas will be provided by Padmanabhan's theory [11] of the emergence of cosmic space. To do this, we begin by studying the general notions of horizons.

5 HORIZONS AND COSMOLOGY

With so many light-years to go, and things to be found [...]

Europe - The Final Countdown

5.1 General horizons

The event horizon (EH), as presented in chapter 3, is the boundary of a region of spacetime $\mathscr{M}$ which is characteristic of black holes. In this chapter, we will discuss this concept a little more formally, as well as define other types of horizons in GR, so that we can broaden this discussion in section 5.2 to the cosmological horizons. In section 5.3, we will apply this discussion to Padmanabhan's theory of emergent cosmology [11] to produce the Friedmann equations and analyze the ΛCDM model in section 5.4.

As will be explained below, operationally we prioritize the study of quasi-local horizons in GR, i.e., horizons as regions of spacetime that can be identified via the measurements of an observer in a finite-time experiment [116, 117]. The quasi-local horizon usually taken in GR is the apparent horizon (AH), which can be well defined by the behavior of the congruences of radial lightlike geodesics ingoing and outgoing through the horizon. The dynamics of these congruences on the horizon are given by the sign of the so-called expansion parameter [32], or just expansion [116].

5.1.1 Lightlike geodesic congruences and expansion

Consider first the equation of a lightlike geodesic that admits an affine parameter t, as

$$l^{\nu}\nabla_{\nu}\, l^{\mu} = 0, \tag{5.1}$$

where $l^{\mu}l_{\mu} = 0$. Furthermore, we know that a well-behaved vector field in a submanifold $\mathscr{O}$ of the spacetime $\mathscr{M}$ generates a local[1] congruence of curves so that the tangent vectors to the curves are the vector field itself

[1] A local congruence of curves is a set of curves such that, locally, through every point of $\mathscr{O}$ passes one and only one curve of the set [118].

$$l^\mu = \frac{dx^\mu}{dt}. \tag{5.2}$$

Any parameter can be used to parameterize a congruence of curves. We then choose another parameter s so that now every point in the region $\mathscr{O}$ that admits the congruence will be parameterized by $x^\mu = x^\mu(t,s)$. This defines a 2-surface in $\mathscr{O}$. If we choose to define the geodesic deviation vector η^μ, by

$$\eta^\mu := \frac{dx^\mu}{ds}, \tag{5.3}$$

with $\eta^\mu l_\mu = 0$, then such a condition will not guarantee that η^μ is transversal to l^μ. Since l^μ is lightlike, a choice as $\eta^\mu = c\,l^\mu$ identically leads to $\eta^\mu l_\mu = c\,l^\mu l_\mu = 0$, with c a constant. This suggests that we won't be able to decompose the spacetime metric tensor $g_{\mu\nu}$ of $\mathscr{O}$ into its longitudinal part, given by $-l_\mu l_\nu$, and transverse part, given by $h_{\mu\nu}$, as usual: $g_{\mu\nu} = h_{\mu\nu} - l_\mu l_\nu$. Indeed, $h_{\mu\nu}$ is not transverse to l_μ, because $h_{\mu\nu}l^\nu = g_{\mu\nu}l^\nu + l_\mu l_\nu l^\nu = l_\mu$ do not vanish [32]. The strategy then is to make a non-univocal choice of an auxiliary lightlike vector field, n^μ such that we adopt the normalization $l^\mu n_\mu = -1$ without loss of generality, as we shall see. So the following metric tensor

$$h'_{\mu\nu} := g_{\mu\nu} + l_\mu n_\nu + n_\mu l_\nu, \tag{5.4}$$

is properly transverse to l^μ, since

$$\begin{aligned} h'_{\mu\nu}l^\nu &= g_{\mu\nu}l^\nu + l_\mu n_\nu l^\nu + n_\mu l_\nu l^\nu \\ &= l_\mu + g_{\mu\nu}l^\nu n_\nu g^{\nu\mu} l_\mu \\ &= l_\mu + (-1) g_{\mu\nu} g^{\nu\mu} l_\mu \\ &= 0, \end{aligned} \tag{5.5}$$

and,

$$\begin{aligned} h'_{\mu\nu}n^\nu &= g_{\mu\nu}n^\nu + l_\mu n_\nu n^\nu + n_\mu l_\nu n^\nu \\ &= n_\mu + n_\mu g_{\mu\nu} l^\mu g^{\nu\mu} n_\mu \\ &= n_\mu + (-1) g_{\mu\nu} g^{\nu\mu} n_\mu \\ &= 0, \end{aligned} \tag{5.6}$$

as well as, $h'^{\mu}_{\mu} = 2$ and $h'^{\mu}_{\sigma} h'^{\sigma}_{\nu} = h'^{\mu}_{\nu}$ [32]. Now using the partial derivative, we recover that $l^{\mu} = \partial x^{\mu}/\partial t$ and $\eta^{\mu} = \partial x^{\mu}/\partial s$, then

$$\frac{\partial l^{\mu}}{\partial s} = \frac{\partial \eta^{\mu}}{\partial t}. \tag{5.7}$$

Recall that the Lie derivative of a contravariant vector field v^{μ} in the direction of the curves given by the contravariant vector field u^{μ}, denoted $\mathcal{L}_u v^{\mu}$, is [118]

$$\begin{aligned} \mathcal{L}_u v^{\mu} &= u^{\nu}\partial_{\nu} v^{\mu} - v^{\nu}\partial_{\nu} u^{\mu} \\ &= u^{\nu}\nabla_{\nu} v^{\mu} - v^{\nu}\nabla_{\nu} u^{\mu}. \end{aligned} \tag{5.8}$$

Using equation (5.7) we obtain $\mathcal{L}_{\eta} l^{\mu} = \mathcal{L}_l \, \eta^{\mu}$, or

$$\eta^{\nu}\nabla_{\nu} l^{\mu} = l^{\nu}\nabla_{\nu}\eta^{\mu}. \tag{5.9}$$

Introducing the tensor field [32, 116]

$$B_{\mu\nu} := \nabla_{\nu} l_{\mu}, \tag{5.10}$$

and with the equation (5.9), we obtain

$$\begin{aligned} l^{\nu}\nabla_{\nu}\eta^{\mu} &= \eta^{\nu}\nabla_{\nu} l^{\mu} = \eta^{\nu} h'^{\mu\sigma}\nabla_{\nu} l_{\sigma} \\ &= \eta^{\nu} h'^{\mu\sigma} B_{\sigma\nu} = \eta^{\nu} B^{\mu}_{\nu} = B^{\mu}_{\nu}\eta^{\nu}. \end{aligned} \tag{5.11}$$

Hence B^{μ}_{ν} can be seen as a measure of the difference from η^{μ} for a parallel transported vector of the lightlike geodesics given by l^{μ}. Moreover, the following quantity can be defined [116]

$$\begin{aligned} \theta &:= g^{\mu\nu} B_{\mu}\nu \\ &= g^{\nu\mu}\nabla_{\nu} l_{\mu} = \nabla_{\alpha} l^{\alpha}, \end{aligned} \tag{5.12}$$

where equation (5.10) was used, and θ is called the expansion parameter [32], or just expansion [116]. Realize that this quantity measures the covariant divergence of the vector field l^{μ} and does not depend on the choice of the auxiliary vector field n^{μ}. An evaluation of the behavior of θ along the affine parameter t of the lightlike geodesics given by l^{μ} can determine the behavior of the covariant divergence along the geodesics. The equation that describes the evolution of θ in relation to t, $d\theta/dt$, is the famous Raychaudhuri equation for expansion [116]

$$\frac{d\theta}{dt} = -\frac{\theta^2}{2} - \sigma^2 + \omega^2 - R_{\mu\nu}l^\mu l^\nu. \tag{5.13}$$

In Raychaudhuri equation above, $R_{\mu\nu}$ is the Ricci tensor, and the quantities $\sigma^2 := \sigma_{\mu\nu}\sigma^{\mu\nu}$, called the shear scalar, and $\omega^2 := \omega_{\mu\nu}\omega^{\mu\nu}$, called the vorticity scalar, are respectively defined by the so-called shear, $\sigma_{\mu\nu}$, and vorticity, $\omega_{\mu\nu}$, tensors[2]. Note that both these tensors and the Raychaudhuri equation itself do not depend on the choice of the auxiliary vector field n^μ. In turn, if $d\theta/dt < 0$ the lightlike geodesics are focused[3], and if $d\theta/dt > 0$ are defocused [116].

5.1.2 Trapped and marginal surfaces

The most general case for lightlike geodesics that do not admit an affine parameter has the geodesic equation as

$$l^\nu \nabla_\nu l^\mu = \kappa l^\mu, \tag{5.14}$$

where we will see later that the quantity κ can be interpreted as a definition of surface gravity in spacetimes that admit a Killing horizon (KH). To define horizons in GR we must also give a better interpretation to the 2-surface characterized by the metric tensor $h'_{\mu\nu}$ which is simultaneously transversal to l^μ and n^μ. We will then assume that such a 2-surface, denoted henceforth by Σ, is compact (it is a compact set), and orientable [116]. From here we already have the suggestion of two natural orthogonal directions to Σ, i.e. outgoing and incoming through Σ. Next, the lightlike vector field n^μ can be defined as lightlike geodesics, which generally do not admit an affine parameter either, and thus naturally define the two possible orientations of Σ together with l^μ.

In chapter 3 we have already seen that the EH for the SBH is a spherical surface (2-surface) given by $R = 2GM$. It is common to define horizons in GR as having spherical symmetry, so we assume that Σ has spherical symmetry and lightlike geodesics can now be seen as radial lightlike geodesics. By convention, the direction of the ingoing radial

[2]The shear and vorticity tensors are also formally defined solely from the tensor field $B_{\mu\nu}$ [116].

[3]The focusing theorem ensures us that, for vanishing vorticity lightlike geodesics congruence, we have $d\theta/dt < 0$ [32].

lightlike geodesics (henceforth radial light rays) through Σ is determined by the vector field n^μ, and the direction of the outgoing radial light rays through Σ is determined by the vector field l^μ. Once in the general case of light rays that are non-affine parameterized geodesics, the expansion of the light rays outgoing, θ_l, and ingoing, θ_n, through Σ are defined by, respectively [117]

$$\theta_l := h^{\mu\nu}\nabla_\mu l_\nu; \tag{5.15a}$$

$$\theta_n := h^{\mu\nu}\nabla_\mu n_\nu. \tag{5.15b}$$

With that, we can define the three most common classes of 2-surfaces Σ, naturally closed, which are used to define quasi-local horizons in GR [116]:

Normal surface. $\theta_l > 0$ and $\theta_n < 0$ through Σ; i.e. refers to the case where outgoing radial light rays of Σ effectively diverge, and ingoing light rays of Σ effectively converge (the expected for a 2-sphere in Minkowski spacetime, that is, in absence of gravity);

Trapped surface. $\theta_l < 0$ and $\theta_n < 0$ through Σ; i.e. refers to the case where outgoing radial light rays of Σ effectively converge, as well as ingoing light rays of Σ (intuitively, the expected effect of a strong gravitational field that captures light rays trying to get out Σ);

Marginal surface. $\theta_l = 0$ and $\theta_n < 0$ through Σ; i.e. refers to the case where outgoing radial light rays of Σ transit between converging and diverging regimes, and ingoing light rays of Σ effectively converge (intuitively, this can be interpreted as the outgoing radial light rays of Σ reversing its direction).

Horizons such as the EH and the KH are usually not defined by the behavior of the radial light rays on the horizon, unlike the AH which can be located in this way. We will now show why we have chosen to use AH as the horizon of interest in general spacetimes.

5.1.3 Event, Killing, and apparent horizons

The general concept of EH, despite its intuitive character presented with the simple example of the static Schwarzschild spacetime, as described in the chapter 3, becomes inoperable for non-static spacetimes. This type of horizon is formally defined from the causal structure of spacetime. Recalling that the region of spacetime in which all existing lightlike geodesics have reached their end point, when cosmic (or comoving) time $t \to \infty$, is called the future null (lightlike) infinity, denoted $\mathscr{I}^+$ [19]. Also, the causal past of a submanifold $\mathcal{S}$ of spacetime $\mathscr{M}$, denoted $J^-(\mathcal{S})$, is defined as the union of all causal pasts[4] of the spacetime points (events) x in $\mathcal{S}$, denoted $J^-(x)$; i.e. $J^-(\mathcal{S}) := \cup_{x\in\mathcal{S}} J^-(x)$ [19].

Event horizon. The EH is defined as the boundary of the causal past of the future null infinite [18, 19]: $\partial J^-(\mathscr{I}^+)$.

The EH then means boundary that delimits how far physical signals can be emitted to reach the future null infinity, i.e. an infinitely spatially distant observer in the future. It fits with the intuitive expectations of the EH for the SBH. Looking at it operationally, for non-static spacetimes this definition proves problematic. To locate the EH, one needs to know all the light rays in the future null infinity and then trace them back to the limit position where they could be emitted, which characterizes the EH as a lightlike hypersurface. In dynamic, or non-static, spacetimes (here, can be a synonym for non-stationary due to Birkhoff's theorem [18]), clearly we have a problem due to temporal evolution, which requires us to know the entire future history of spacetime and thus know *a priori* information outside our future light cone [116]. Because its location depends on knowledge of the future history of spacetime, we call it a global horizon [116].

The KH is a naturally defined horizon for spacetimes that initially admit a timelike killing vector field, and are then stationary spacetimes [118].

Killing horizon. The KH is a lightlike hypersurface in a spacetime that admits a timelike Killing vector field, k^μ, which is everywhere tangent to it and becomes lightlike over the hypersurface [116].

Note that generally in static spacetimes, the KH coincides with the EH, if they exist [119]. One of the biggest interests in KH is the idea of surface gravity which they produce as mentioned with equation (5.14), but for general, non-stationary spacetimes, KH is no longer useful. In these cases, as we shall see, we introduce a vector that produces a certain generalization of the Killing vector, the so-called Kodama vector [120], K^μ.

[4] The causal past(future) of an event p in spacetime is defined as the set of all events q such that there exists a past(future)-directed causal curve from q to p(from p to q) [18].

Finally, the notion of the AH is given in terms of the expansion of the radial light rays ingoing and outgoing through the horizon.

Apparent horizon. The AH is the closure (union of the interior and boundary of a region) of a hypersurface which is foliated by marginal 2-surfaces ($\theta_l = 0$ and $\theta_n < 0$)[116]; or, the AH is the boundary of a portion of a 2-surface Σ that this region contains trapped surfaces [32].

Not immediately equivalent, since the first definition conceives a hypersurface while the second a 2-surface as AH, note that in reality, the true condition implemented to locate and properly define an AH is only the condition of the expansion of light rays to a marginal 2-surface: $\theta_l = 0$, $\theta_n < 0$ [32, 116]. In this sense, although it is more physically a hypersurface, we only use the definition of a marginal 2-surface to identify an AH. Note the important point that if we can calculate the expansion of the radial light rays through the horizon, regardless of whether we know about the causal structure and future history of spacetime as for the EH, we can apply the condition $\theta_l = 0$ and $\theta_n < 0$ to thus locate the AH. This is why we say that the AH has a quasi-local definition [116].

Although the AH and the EH coincide in stationary spacetimes [32], both differ even in the evolution of the formation of a real black hole by gravitational collapse of a star, such that the EH forms first and the AH then tends to the EH as the static regime is achieved [19]. Before extending the concept of horizon to the cosmological scale, in section 5.2, we will briefly discuss the role of surface gravity in the study of black holes and how this quantity is related to the thermodynamics of these objects.

5.1.4 Surface gravity

In the Newtonian context, surface gravity is simply the gravitational acceleration that a test particle undergoes on the surface of a massive body due to the gravitational attraction that this body generates around it. In the relativistic regime, obtaining such an acceleration for a particle on the surface of a black hole is non-trivial; the particle acceleration diverges at the limit where it approaches the black hole EH. A regularization for the surface gravity of a black hole is however possible for static black holes, where the Newtonian interpretation is recovered [18, 116]. For stationary black holes, the surface gravity defined is the Killing surface gravity, denoted κ_{Killing}, which comes from the study

of the KH. On the other hand, for dynamical black holes, whose spacetime in general does not admit a timelike Killing vector field, we deal better with the AH, and a generalization of κ_{Killing} is necessary.
The Killing surface gravity, κ_{Killing}, can be defined by [121]

$$k^{\nu}\nabla_{\nu}k^{\mu} = \kappa_{\text{Killing}}\, k^{\mu}, \tag{5.16}$$

then, κ_{Killing} is defined by the fact that the Killing vector is a non-affinely parameterized geodesic on the Killing horizon[5] (where the Killing vector becomes lightlike, $k^{\mu}k_{\mu} = 0$). It can be shown that the quantity κ_{Killing} appears in black hole thermodynamics as the analog of a temperature associated with the black hole EH points, satisfying the so-called zeroth law for stationary black holes [18, 19]. The general form of the Hawking temperature, T_{H}, for the black hole thermal radiation supports this verification, because [18]

$$T_{\text{H}} = \frac{\kappa_{\text{Killing}}}{2\pi}, \tag{5.17}$$

that is, as shown by Hawking [40], due to quantum fluctuations when studying quantum field theory in curved spacetimes, particle-antiparticle pairs are created in the neighborhood of the EH of a black hole, resulting in the effective emission of particles by the black hole. The effect is then black-body thermal radiation, which is emitted at the temperature given by equation (5.17). The Killing surface gravity then actually works as a measure of the temperature of the black hole EH points [18].

Now, if the black hole is non-stationary, as we have already seen, a natural horizon to study is the AH. We can ask if it is possible to extend the idea of surface gravity and Hawking temperature to these black holes. To solve this question, We introduce the Kodama vector [120], which is defined only to spherical symmetric spacetimes, as

$$K^{\mu} := \epsilon^{\mu\nu}\nabla_{\nu}R, \tag{5.18}$$

where $R = a(t)\, r$ is the areal radius of a spherically symmetric metric,

$$ds^2 = h_{ij}dx^i dx^j + R^2 d\Omega^2, \tag{5.19}$$

with $i, j = 0, 1$, and $\epsilon^{\mu\nu}$ is the inverse of the volume form induced by the metric tensor $h_{\mu\nu}$, namely, $\epsilon_{\mu\nu} = \sqrt{|h|}\,\varepsilon_{\mu\nu}$ with $\varepsilon_{\mu\nu}$ the generalized Levi-Civitá symbol [122]. This vector

[5] A simple proof of equation (5.16) can be consulted [116, 121].

has the property that, even in the absence of a Killing vector field, it indicates a locally conserved vector field J^μ, which $\nabla_\mu J^\mu = 0$, called the Kodama energy current [116]. It can be shown that because of the spherical symmetry the Kodama vector mimics the Killing vector also to characterize a horizon as the KH, so that $K^\mu K_\mu < 0$ outside the horizon, and $K^\mu K_\mu = 0$ on the horizon [116].

More significantly for what follows, the Kodama vector can be used to generalize the definition of surface gravity for dynamic black holes. As proposed by Hayward [123], the surface gravity defined by the Kodama vector, and then called the Kodama-Hayward surface gravity, denoted κ_{Kodama}, is given by [123]

$$\kappa_{\text{Kodama}} := \frac{1}{2\sqrt{-h}}\,\partial_\mu\Big(\sqrt{-h}\,h^{\mu\nu}\partial_\nu R\Big), \tag{5.20}$$

where h is the determinant of the metric tensor h_{ij} in the 2-space of (t, r). This definition of surface gravity recovers the result for the Reissner-Nordström black hole (Killing) surface gravity [124]. As an additional justification, the Hamilton-Jacobi approach to evaluating Hawking radiation and Hawking temperature for non-stationary black holes leads to the same expression (5.20) [125]. The preceding discussions will now be applied to the context of cosmology.

5.2 Cosmological horizons

As we have seen, stationary black holes have a natural notion of surface gravity through their KH, which in turn is shown to correspond to the temperature of the black hole EH due to Hawking radiation [40]. Considering the de Sitter cosmological spacetime, Gibbons and Hawking [126] showed that the respective cosmological EH is associated with a Hawking-like temperature: $T_{\text{H, de Sitter}} = \kappa_{\text{Killing}}/2\pi$. Note that despite being non-stationary (cosmological spacetime), the de Sitter spacetime has a Killing vector field revealed in its Schwarzschild-like coordinates [116]. In this sense, the de Sitter spacetime is an exception, and when we take into account a Universe that is homogeneous and isotropic on large scale[6], we should certainly focus on describing a non-stationary spacetime in the cosmological context. We will then study a FLRW spacetime, which metric is [33]

[6]For distances greater than 100 Mpc [127].

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{5.21}$$

with $a(t)$ the scale factor, k the spatial curvature, and in comoving coordinates (t, r, θ, ϕ), i.e., coordinates such that the spatial portion (r, θ, ϕ) is constant for observers moving following the expansion of the Universe, and t the cosmic time [128]. We can study which horizon concepts are available for an FLRW spacetime and therefore ask: can any horizon of this dynamic spacetime be consistently associated with a Hawking temperature, with an adequate notion of surface gravity? It will be shown that a horizon easy to work in FLRW spacetimes, as well as for general black holes, is the cosmological AH, which indeed has a Hawking temperature associated with it.

5.2.1 Event, Hubble and apparent horizons

Consider a comoving observer in $r = 0$ between the instants of cosmic time t (measured by a clock following the expansion of the Universe) $t = t_0$ and $t \to \infty$. The radial comoving distance which delimits the causally accessible region to the observer in $r = 0$ by a signal emitted in $t = t_0$ when a cosmic time interval that tends to infinite has elapsed is given by the integral [116]

$$\int_{t_0}^{\infty} \frac{dt'}{a(t')}. \tag{5.22}$$

If this integral converges, the boundary of the spacetime region delimited by the comoving distance specified by the integral is called the cosmological EH, then equation (5.22) is denoted by r_{EH}, and events beyond r_{EH} will never be communicated to the observer in $r = 0$. If the integral (5.22) diverges, it does not define an EH, and any events occurring at $t = t_0$ can be accessed to the observer at $r = 0$ when enough time is expected. The comoving distance is related to the proper distance, d_P, by [129]

$$d_P = a(t)\, r, \tag{5.23}$$

then, for r_{EH} the proper distance, $d_{P,\text{EH}}$, which characterize the cosmological EH is [116]

$$d_{P,\text{EH}} = a(t) \int_{t_0}^{\infty} \frac{dt'}{a(t')}. \tag{5.24}$$

Note that the cosmological EH has the same nature of needing to know the entire future history of spacetime to locate it, as the EH for black holes. Particularly important is the fact that this horizon is not generally defined for FLRW spacetimes, except in the case where, modeling the source of matter by a perfect fluid of equation of state $P = \omega\rho$ [33], we have $P < -\rho/3$ [116], with ω a constant, P and ρ respectively the pressure and density of the matter source. This condition produces the de Sitter spacetime, and we conclude that only in this regime does an FLRW spacetime admit a cosmological EH.

In the cosmological context, another type of horizon often defined is the so-called Hubble horizon (HH) [130]. Due to Hubble's law, which expresses the recessional velocity recession v_r between galaxies as a function of their proper distance d_P, we have

$$v_r = H(t)\, d_P, \tag{5.25}$$

with $H(t) := \dot{a}(t)/a(t)$ the Hubble parameter[7]. At the limit of $v_r \to c$, the recessional velocity will define a region delimited by the corresponding proper distance in which galaxies (and other astronomical objects) will begin to move away from each other faster than light [130]. This distance is called the Hubble radius, r_{H}, which defines the HH as (remember that $c = 1$) [116]

$$r_{\mathrm{H}} = \frac{1}{H(t)}. \tag{5.26}$$

As with black holes, there is also a notion of AH in cosmology, and this will be the horizon of interest for a dynamic FLRW spacetime, to associate a temperature with the horizon. To construct the intuitive idea of a cosmological AH, we consider the radial lightlike geodesics ingoing and outgoing through a boundary (usually a hypersurface) of a certain region of spacetime, and their respective expansion parameters, θ_n and θ_l. Now, to try to define the boundary of this region as a cosmological horizon that limits the causal accessibility between two portions of spacetime, we must expect that inside the horizon: $\theta_n < 0$ and $\theta_l > 0$; as well as, outside the horizon: $\theta_n > 0$ and $\theta_l > 0$. Thus, the cosmological horizon in question must be defined by the condition: $\theta_n = 0$ and $\theta_l > 0$. This is precisely the definition of the cosmological AH [116]. One can easily calculate the tangent vectors, l^μ and n^μ, that define the radial lightlike geodesics for an FLRW

[7]Henceforth, as usual in cosmology, a dot upwards the quantity will refer to the derivative concerning time (cosmic time).

spacetime

$$l^{\mu} = \left(1, +\frac{\sqrt{1-kr^2}}{a(t)}, 0, 0\right); \tag{5.27a}$$

$$n^{\mu} = \left(1, -\frac{\sqrt{1-kr^2}}{a(t)}, 0, 0\right), \tag{5.27b}$$

since $ds^2 = d\theta = d\phi = 0$ in equation (5.22) produces directly the two radial vector directions $dr/dt = \pm\sqrt{1-kr^2}/a(t)$ [116]. With some calculations, substituting equations (5.27) in equations (5.15) give us [116]

$$\theta_l = 2\left(H(t) + \frac{1}{R}\sqrt{1 - \frac{kR^2}{a^2(t)}}\right); \tag{5.28a}$$

$$\theta_n = 2\left(H(t) - \frac{1}{R}\sqrt{1 + \frac{kR^2}{a^2(t)}}\right), \tag{5.28b}$$

where R is the areal radius. Applying the condition $\theta_n = 0$ and $\theta_l > 0$ to obtain the cosmological AH location, from equation (5.28b)

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{1}{R^2}\left(1 - \frac{kR^2}{a^2(t)}\right), \tag{5.29}$$

and, solving for R,

$$R \equiv r_{\mathrm{AH}} = \frac{1}{\sqrt{H^2(t) + k/a^2(t)}}, \tag{5.30}$$

with r_{AH} as the definition of the cosmological AH radius. Equation (5.30) shows that for a closed ($k > 0$), open ($k < 0$), and flat ($k = 0$) Universe, the radius of the cosmological AH is respectively smaller, larger, and equal to the Hubble radius, r_{H}. Unlike the cosmological EH, the cosmological AH exists for all regimes of an FLRW spacetime [116], and has the advantage common to the case of black holes of being quasi-local, not depending on the causal structure of spacetime.

5.2.2 Temperature of the apparent horizon

We are now in a position to better answer the question posed at the beginning of this section. In fact, Cai et al. [131] showed that the cosmological horizon that contains an associated notion of Hawking temperature in FLRW spacetime is the cosmological AH. Furthermore, the surface gravity that appears in the expression for this Hawking temperature is the Kodama-Hayward surface gravity (5.20) [131]. To obtain the expression for the Hawking temperature of the cosmological AH, first the Kodama-Hayward surface gravity, κ_{Kodama}, for a FLRW spacetime must be obtained from equation (5.20) and equation (5.21) decomposed in the form of (5.19). The 2-space metric tensor has determinant $-a^2(t)/1-kr^2$, then

$$\begin{aligned}
\kappa_{\text{Kodama}} =& \frac{1}{2}\frac{\sqrt{1-kr^2}}{a(t)}\,\partial_\mu\left[\frac{a(t)}{\sqrt{1-kr^2}}\left(h^{\mu 0}\partial_0 R + h^{\mu 1}\partial_1 R\right)\right] \\
=& \frac{1}{2}\frac{\sqrt{1-kr^2}}{a(t)}\,\partial_\mu\left[\frac{a(t)}{\sqrt{1-kr^2}}\left(h^{\mu 0}\dot{a}(t)\,r + h^{\mu 1}a(t)\right)\right] \\
=& \frac{1}{2}\frac{\sqrt{1-kr^2}}{a(t)}\,\partial_\mu\left[\frac{a(t)}{\sqrt{1-kr^2}}\left(-\dot{a}(t)\,r\delta^{\mu 0} + \frac{1-kr^2}{a^2(t)}\delta^{\mu 1}\right)\right] \\
=& \frac{1}{2}\frac{\sqrt{1-kr^2}}{a(t)}\left[-\partial_0\left(\frac{a(t)\dot{a}(t)\,r}{\sqrt{1-kr^2}}\right) + \partial_1\left(\sqrt{1-kr^2}\right)\right] \\
=& -\frac{1}{2}ra(t)\left[\left(\frac{\dot{a}^2(t)}{a^2(t)} + \frac{\ddot{a}(t)}{a(t)}\right) + \frac{k}{a^2(t)}\right] \\
=& -\frac{1}{2}R\left(2H^2(t) + \dot{H}(t) + \frac{k}{a^2(t)}\right),
\end{aligned} \tag{5.31}$$

where in the last line we used $R = a(t)\,r$ and $H(t) = \dot{a}(t)/a(t)$. The calculated surface gravity can be adequately expressed in terms of the cosmological AH radius, r_{AH}, if we calculate the rate of change in cosmic time of this quantity from equation (5.30), which is

$$\begin{aligned}
\dot{r}_{\text{AH}} = \frac{d}{dt}\left(H^2(t) + \frac{k}{a^2(t)}\right)^{-1/2} &= -\frac{\left(H(t)\dot{H}(t) - k\,\dot{a}(t)/a^3(t)\right)}{\left(H^2(t) + k/a^2(t)\right)^{3/2}} \\
= -\frac{H(t)\left(\dot{H}(t) - k/a^2(t)\right)}{\left(H^2(t) + k/a^2(t)\right)^{3/2}} &= H(t)\,r_{\text{AH}}^3\left(\frac{k}{a^2(t)} - \dot{H}(t)\right),
\end{aligned} \tag{5.32}$$

where equation (5.30) was used once again. Finally, using equation (5.32) to eliminate $\dot{H}(t)$ in equation (5.31), and setting $R \equiv r_{\text{AH}}$, we have

$$\begin{aligned}
\kappa_{\text{Kodama}} &= -\frac{1}{2}\, r_{\text{AH}} \left(2H^2(t) + \frac{k}{a^2(t)} - \frac{\dot{r}_{\text{AH}}}{H(t)\, r_{\text{AH}}^3} + \frac{k}{a^2(t)} \right) \\
&= -\frac{1}{2}\, r_{\text{AH}} \left[2\left(H^2(t) + \frac{k}{a^2(t)} \right) - \frac{\dot{r}_{\text{AH}}}{H(t)\, r_{\text{AH}}^3} \right] \\
&= -\frac{1}{2}\, r_{\text{AH}} \left(\frac{2}{r_{\text{AH}}^2} - \frac{\dot{r}_{\text{AH}}}{H(t)\, r_{\text{AH}}^3} \right) \\
&= -\frac{1}{r_{\text{AH}}} \left(1 - \frac{\dot{r}_{\text{AH}}}{2H(t)\, r_{\text{AH}}} \right).
\end{aligned} \tag{5.33}$$

Then, the Hawking temperature of the cosmological AH in an FLRW spacetime will be $T_{\text{H}} = \kappa_{\text{Kodama}}/2\pi$. where the Kodama-Hayward surface gravity is given by equation (5.33). However, to obtain a physically reasonable result, we should write [14]

$$T_{\text{H}} = \frac{|\kappa_{\text{Kodama}}|}{2\pi} = \pm \frac{1}{2\pi\, r_{\text{AH}}} \left(1 - \frac{\dot{r}_{\text{AH}}}{2H(t)\, r_{\text{AH}}} \right), \tag{5.34}$$

since we selected the positive sign to ensure that the heat capacity of the Universe is positive, maintaining its thermodynamic stability [132]. A final comment on why we chose the cosmological AH and its associated Hawking temperature in cosmology is due to the suitability of this scenario with Padmanabhan's proposal for the emergence of cosmic space [11]. Padmanabhan's theory [11] is our next stop.

5.3 Emergent cosmic space

Recently, the quantum gravity paradigm has given way to the idea that gravitational interaction is just an emergent phenomenon from a more fundamental microstructure that arises from compliance with the laws of thermodynamics [8–10, 133]. In particular, the gravitational field equations can be seen as equations of state for spacetime as a thermodynamic system [8]. Also, nowadays is common to approach the gravitational field as the thermodynamic (macroscopic) limit of the microscopic structure of spacetime [9, 10, 133]. In this work, we endorse and reaffirm this perspective and adopt it as the basis for what follows. More than that, Padmanabhan [11] proposed that spacetime itself is in some sense an emergent phenomenon from some fundamental microscopic structure. Using a version of the holographic principle, Padmanabhan [11] was able to derive the Friedmann equations, that govern the dynamics of spacetime on a large scale.

We will then use Padmanabhan's theory [11] as a bridge between the study of cosmological horizons, seen in section 5.2, and the description of cosmological dynamics given by the Friedmann equations. In other words, a suitable analysis of the cosmological horizons considered will produce associated Friedmann equations. We begin by presenting the basis for understanding the emergence of cosmic space according to Padmanabhan [11] given the so-called holographic principle [12], and then justify the use of the cosmological apparent horizon.

5.3.1 Holographic principle

The holographic principle was inspired by the thermodynamics of black holes, in which the entropy of the black hole is dependent on the surface area of the black hole, as we have already seen in chapters 3 and 4, according to Bekenstein [41]. We know that the entropy of a physical system is often associated with the information content of that system, in the sense, for example, of the Von Neumann entropy [134]. The information content (entropy) of objects falling into the black hole would be given by the entropy of the black hole itself and then contained in the surface area of the EH. Using this concept, 't Hooft [135] and Susskind [12] developed what we now call the holographic principle, which we can define in the present context as [12]: the physical information of a D-dimensional spacetime is contained in the $(D-1)$-dimensional boundary of that spacetime[8].

The holographic principle is often cited as manifested in the so-called AdS/CFT correspondence[9], as well as having a great potential impact on several open problems in current physics [138]. Furthermore, Consistently applied to black holes, the holographic principle offers a possible solution to the black hole information paradox, where a few features such as mass, charge, and angular momentum characterize the black hole, regardless of their history and particular interactions with objects that cross the EH. Hence, the particular information content of the black hole is apparently deleted. The holographic principle then holds that the information of the bulk is entirely contained within the surface of the

[8]Any formal detail on this topic is beyond the scope of this work, and we suggest that the interested reader turns to the review literature [136].

[9]Suggested by Maldacena [137], AdS/CFT establishes a duality between a theory of gravity in a higher-dimensional Anti-de Sitter (AdS) space and a conformal field theory (CFT) that lives on the boundary of that space. In other words, the physics inside the bulk (the AdS space) could be equivalently described by the physics on the boundary.

black hole [139]. This idea can give rise to a more fundamental physical background, from which the very cosmic space and its expansion emerge.

5.3.2 Holographic equipartition

Padmanabhan [11] investigated the possibility of the very spacetime being an emergent structure from something more fundamental. Around finite gravitational systems, such as the Earth or the Milky Way, the conclusion is that it is difficult to conceive of the emergence of spacetime since physical observation does not point in this direction. Furthermore, in a covariant treatment where time does not play a special role, according to Padmanabhan [11], it is difficult to see time emerging from a more fundamental mechanism. Padmanabhan's insight was that both these difficulties do not remain for spacetime on a large scale, i.e., in the cosmological scenario [11]. For cosmology, the expansion of the Universe and the cosmological principle can be adequately established for an observer measuring the cosmic time (or comoving time) as a natural parameter in describing the evolution of cosmological dynamics (the Friedmann equations).

The holographic principle arises when we realize that our Universe obeys a de Sitter spacetime asymptotically. At this stage, the holographic principle is represented when we fix the equality between the degrees of freedom of the bulk $N_{\rm bulk}$ and the surface $N_{\rm sur}$ of the EH of the de Sitter spacetime (characterizing what we call the holographic screen) by maximizing the entropy of the EH [11, 136]. In other words, when the Universe reaches de Sitter spacetime, we will have [11]

$$N_{\rm sur} = N_{\rm bulk}, \tag{5.35}$$

where degrees of freedom mean the count of information content in the associated portion of space (cosmic time serves as a parameter). While in field theories we generally associate degrees of freedom with the components of the fields at each point in spacetime, in the present context the understanding of degrees of freedom is not so clear because it must take into account the smallest distances of the order of the Planck scale and thus effects attributed to quantum gravity [12]; in other words, the rules of quantum mechanics applied to the fundamental structure of spacetime. A complete overview of this fundamental structure is, of course, only possible with the advent of the very theory of quantum gravity [133]. A suitable guess is to take the degrees of freedom in the holographic screen as proportional to the ratio between the surface area of the screen, A_{screen}, and the Planck area A_{P}, as suggested by the Bekenstein-Hawking entropy for black holes [40, 41], that is

$$N_{\text{sur}} \propto \left(\frac{A_{\text{screen}}}{A_{\text{P}}}\right). \tag{5.36}$$

In addition, Padmanabhan proved that the number of degrees of freedom of the bulk for static spacetimes takes the form of an equipartition law in the form of [140]

$$N_{\text{bulk}} = 2\frac{|E_{\text{Komar}}|}{T}, \tag{5.37}$$

with E the gravitational energy of the bulk in the form of Komar energy [141], and T the Hawking temperature of a general horizon (for details, see [142]). Once more, emphasizing that regardless of whether one knows the fundamental structure of spacetime at the smallest scales, spacetime has complete agreement with the laws of thermodynamics.

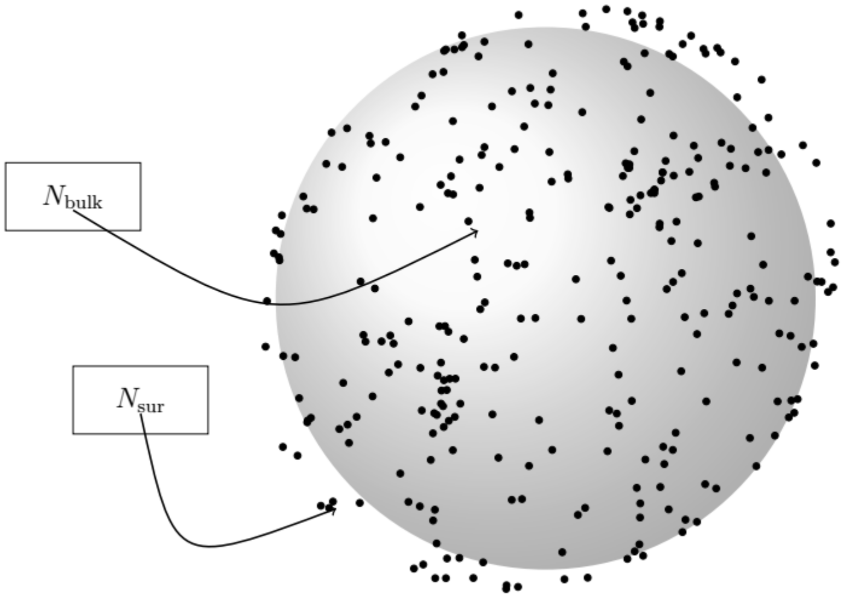
To extend these ideas to a cosmological spacetime, using $N_{\text{sur}} = 4(A_{\text{screen}}/A_{\text{P}})$, with A_{screen} the area of the holographic screen (horizon), and the Hubble horizon as the holographic screen, Padmanabhan was able to justify that space (not spacetime) in large scale (namely, cosmic space) does indeed emerge over the passage of cosmic time in the form of the expansion of the Universe. In other words, due to the tendency of the condition (5.36), called holographic equipartition [11], to be achieved, in the limit of an asymptotically de Sitter spacetime, the cosmic space and its expansion emerges. The law proposed to rule this dynamic is [11]

$$\frac{dV}{dt} = L_{\text{P}}^2(N_{\text{sur}} - N_{\text{bulk}}), \tag{5.38}$$

where $V = 4\pi r_{\text{H}}^3/3$ is the Hubble volume and t denotes the cosmic time. In this perspective, the expansion of the Universe is conceptually equivalent to the emergence of cosmic space, as represented in figure 9. Now, using d/dt denoted by a superscript dot, $N_{\text{sur}} = 4\pi r_{\text{H}}^2/L_{\text{P}}^2$, $T = 1/2\pi r_{\text{H}}$, and

$$|E_{\text{Komar}}| = \epsilon(\rho + 3p)V, \qquad \epsilon = \begin{cases} +1, & \text{if } (\rho + 3p) < 0 \quad (\Lambda - \text{dominated}) \\ -1, & \text{if } (\rho + 3p) > 0 \quad (\text{matter/radiation-dominated}), \end{cases} \tag{5.39}$$

Figure 9: Degrees of freedom that have already emerged in the cosmic bulk and degrees of freedom that have not yet emerged on the surface of the cosmic bulk.



Source: The author (2024).

being p and ρ the pressure and energy density, respectively, for the cosmological perfect fluid obeying the equation of state $p = \omega\rho$, and equations (5.36), (5.37), and (5.39), in equation (5.38) we have

$$\begin{aligned}\frac{dV}{dt} = 4\pi r_{\mathrm{H}}^2 \dot{r}_{\mathrm{H}} = L_{\mathrm{P}}^2 \left(\frac{4\pi r_{\mathrm{H}}^2}{L_{\mathrm{P}}^2} + \frac{(\rho + 3p)}{T} \frac{16\pi}{3} r_{\mathrm{H}}^3 \right) \\ = \left(4\pi r_{\mathrm{H}}^2 + \frac{16\pi^2}{3} L_{\mathrm{P}}^2 (\rho + 3p) r_{\mathrm{H}}^4 \right).\end{aligned} \tag{5.40}$$

Solving for r_{H}

$$r_{\mathrm{H}}^{-2} \dot{r}_{\mathrm{H}} - r_{\mathrm{H}}^{-2} = \frac{4\pi}{3} L_{\mathrm{P}}^2 (\rho + 3p). \tag{5.41}$$

Using that $H = r_{\mathrm{H}}^{-1} = \dot{a}(t)/a$, with $a(t)$ the scale factor, we obtain

$$\dot{H} + H^2 = \frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi}{3} L_{\mathrm{P}}^2 (\rho + 3p), \tag{5.42}$$

so, the second Friedmann equation (or Raychaudhuri equation). Although it does not explicitly contain a cosmological constant term (and consequently not in equation (5.42)), as Padmanabhan pointed out, equation (5.38) presupposes a dark energy component in the Universe to achieve holographic equipartition [11]; i.e. without a dark energy component is not expected to reach holographic equipartition. This premise is embedded in the definition (5.39). To obtain the first Friedmann equation we need to use the so-called continuity equation [33], which can be obtained from the Misner-Sharp-Hernandez mass[10] [144]

$$M_{\rm MSH} = \rho\, V_{\rm P}, \tag{5.43}$$

with $V_{\rm P} = 4\pi R^3/3$ the proper volume defined by the proper Hubble radius $R = r_{\rm H}\, a(t)$, in the following relation

$$\frac{dM_{\rm MSH}}{dt} = -p\frac{dV_{\rm P}}{dt}. \tag{5.44}$$

Substituting equation (5.43) in equation (5.44), the continuity equation arises

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}(t)}{a(t)}. \tag{5.45}$$

Now, solving the above equation to p, so $3p = -3\rho - \dot{\rho}/H$, and using in equation (5.42) we have

$$\frac{\ddot{a}(t)}{a(t)} = \frac{4\pi}{3}L_{\rm P}^2\left(2\rho + \frac{\dot{\rho}}{H}\right) = \frac{4\pi}{3}L_{\rm P}^2\left(2\rho + a(t)\frac{\dot{\rho}}{\dot{a}(t)}\right), \tag{5.46}$$

and multiplying the whole equation by $a(t)\dot{a}(t)$

$$\ddot{a}(t)\dot{a}(t) = \frac{4\pi}{3}L_{\rm P}^2(2\rho\, a(t)\dot{a}(t) + a^2(t)\dot{\rho}), \tag{5.47}$$

equivalently

$$\frac{d}{dt}\left(\frac{\dot{a}^2(t)}{2}\right) = \frac{4\pi}{3}L_{\rm P}^2\frac{d}{dt}(a^2(t)\rho), \tag{5.48}$$

or

$$\frac{\dot{a}^2(t)}{a^2(t)} = H^2 = \frac{8\pi}{3}L_{\rm P}^2\rho. \tag{5.49}$$

[10] A quantity used to study symmetrically spherical spacetimes [116, 143].

Equation (5.49) is the first Friedmann equation. We can now explore the cosmological implications of equation (5.49) through cosmological horizons in FLRW spacetimes and their thermodynamics, particularly for the cosmological AH.

5.3.3 The apparent horizon

As seen in section 5.2, the cosmological AH has the condition of being the horizon defined for non-stationary spacetimes, which is quasi-local and thus does not depend on the causal structure of spacetime, and is available for all regimes of a FLRW spacetime, endowed with an associated Hawking temperature whose surface gravity is Kodama-Hayward surface gravity, as in equation (5.34). In other words, this suggests that the cosmological AH is the most suitable horizon for cosmological scenarios, and for Padmanabhan's theory, which uses a Hawking temperature associated with the cosmological horizon. In fact, according to Hashemi et al. [14] the cosmological AH appears as a holographic screen that rewrites equation (5.38) only in terms of well-defined thermodynamic quantities, and the associated Hawking temperature is exact up to the second term of equation (5.34). Therefore, we justify the use of the cosmological AH as a holographic screen for the application of Padmanabhan's theory [11], according to [14].

5.4 *Lambda*-Cold Dark Matter model

In this section, we will review basic aspects of the ΛCDM model, the standard model of relativistic, cosmology to apply them to the modifications made in chapter 6. Data from TT, TE, EE+lowE+lensing CMB of the Planck 2018 collaboration [145] was used to constrain the cosmological parameters. This section follows the reference [15]. Roughly speaking, the ΛCDM assumes the scenario of three components in our Universe, namely, radiation, cold baryonic matter and cold dark matter (CDM) (which together are called cold matter), and a cosmological constant (attributed to the origin of dark energy and then the accelerated expansion of the Universe). The model then uses the GR field equations applied to the cosmological solution of FLRW (Friedmann equations) to parameterize the observations of the Big Bang theory and data, for example, from the cosmic microwave background (CMB) [118]. The Friedmann equation (5.49) for a spatially flat Universe ($k = 0$) with the three components: radiation, "rad", cold matter, "cm", and the cosmological constant, "Λ", can be written by

$$H^2 = H_0^2 \frac{8\pi L_{\rm P}^2}{3H_0^2}\left[\rho^{(\rm rad)}(t) + \rho^{(\rm cm)}(t) + \rho^{(\Lambda)}(t)\right] = H_0^2\left[\Omega_0^{(\rm rad)}\left(\frac{a_0}{a(t)}\right)^4 + \Omega_0^{(\rm cm)}\left(\frac{a_0}{a(t)}\right)^3 + \Omega_0^{\Lambda}\right], \tag{5.50}$$

with H_0 the Hubble parameter at the present epoch,

$$H_0 = 100h \text{ Km sec}^{-1} \text{ Mpc}^{-1} = 2.1332h \times 10^{-42} \text{ GeV}, \tag{5.51}$$

where h represents the uncertainty on the value H_0, and the observations of the Planck 2018 collaboration [145] constrain this value to be $h = 0.674 \pm 0.005$. In equation (5.49), $\Omega_0^{(i)}$ is the energy density parameter (or just density parameter) of the ith component of the Universe at the present epoch, defined as

$$\Omega_0^{(i)} := \frac{8\pi L_{\rm P}^2}{3H_0^2}\rho_0^{(i)}, \tag{5.52}$$

and, using the continuity equation (5.45), for radiation $\omega = 1/3$, for cold matter $\omega = 0$, and for the cosmological constant $\omega = -1$, in the state equation $p = \omega\rho$, we obtained

$$\rho^{(\rm rad)}(t) = \rho_0^{(\rm rad)}\left(\frac{a(t)}{a_0}\right)^{-4}; \tag{5.53a}$$

$$\rho^{(\rm cm)}(t) = \rho_0^{(\rm cm)}\left(\frac{a(t)}{a_0}\right)^{-3}; \tag{5.53b}$$

$$\rho^{(\Lambda)}(t) = \rho_0^{(\Lambda)}. \tag{5.53c}$$

Also, in equation (5.50) $a_0 \equiv a(t_0)$ is the scale factor at the present time, $t = t_0$. The current dark energy density parameter is [145]

$$\Omega_0^{(\Lambda)} = 0.685 \pm 0.007. \tag{5.54}$$

The density parameter of the cold matter is $\Omega_0^{(\rm cm)} = \Omega_0^{(\rm CDM)} + \Omega_0^{(\rm b)}$. where $\Omega_0^{(\rm b)}$ is the density parameter of the cold baryonic matter, and $\Omega_0^{(\rm CDM)}$ is the density parameter of CDM. Their current values are given by [145]

$$\begin{aligned} \Omega_0^{(\rm b)}h^2 &= 0.02237 \pm 0.00015, \\ \Omega_0^{(\rm CDM)}h^2 &= 0.1200 \pm 0.0012. \end{aligned} \tag{5.55}$$

Using $h = 0.674$, we have $\Omega_0^{(\mathrm{b})} = 0.04924319$, and $\Omega_0^{(\mathrm{CDM})} = 0.2641566$, for the central value. Hence, the density parameter of the cold matter is

$$\Omega_0^{(\mathrm{cm})} = 0.315. \tag{5.56}$$

The Planck 2018 collaboration [145] for TT,TE,EE+lowE+lensing+BAO data indicates that our Universe is virtually spatially flat with a curvature density $\Omega_0^{(k)} = 0.0007 \pm 0.0019$. The Friedmann equation (5.50) can be used to calculate the age of the Universe by the density parameters, writing $H = \dot{a}(t)/a(t)$, and so

$$t_0 = \int_0^{t_0} dt = \frac{1}{H_0} \int_0^{(a=a_0 \to x=1)} \frac{dx}{x \left[\Omega_0^{(\mathrm{rad})} x^{-4} + \Omega_0^{(\mathrm{cm})} x^{-3} + \Omega^{(\Lambda)} \right]^{\frac{1}{2}}}, \tag{5.57}$$

where $x = a/a_0$. The current density parameter of radiation is of the order of $10^{-5} - 10^{-4}$, then radiation becomes important only for extremely high redshifts, as $z \simeq 1000$. We therefore ignore the contribution of radiation and substitute the values of (5.54) and (5.56) in the integral (5.57), the age of the Universe is calculated to be $t_0 = 13.797 \pm 0.023$ Gyr [145]. Finally, the deceleration parameter q, which measures the acceleration (deceleration) of the expansion of the Universe can be expressed by [118] $q = -\Omega^{(\Lambda)} + \Omega_0^{(\mathrm{cm})}/2$, again neglecting the density parameter of radiation.

In the next chapter, we will combine the ideas presented in the previous chapters to devise an alternative cosmological model to ΛCDM.

6 EMERGENCE OF FRACTAL COSMIC SPACE

There is no dark side in the moon, really, matter of fact, it's all dark.

Gerry O'Driscoll in Pink Floyd - Eclipse

6.1 Fractal cosmological apparent horizon

This final chapter follows closely the reference [15] and aims to justify the connection between the results presented in the previous chapters of this dissertation.

On the one hand, the CQG applied to the SBH, together with the semi-classical analysis of its mass spectrum, has allowed a preliminary study of the thermodynamics of the SBH, where we obtain the Hawking temperature and the Bekenstein-Hawking entropy associated with its EH, given respectively by equations (3.43) and (3.45), as shown in chapter 3. Chapter 4 also introduces the concept of fractals for physical systems and their relationship with fractional calculus, thus motivating the hypothesis that the SBH has a surface whose geometry is that of a random fractal. In the same chapter, FQC [5] and its developments in gravity, namely FQG [7], are applied to the WDW equation of the SBH according to equation (4.17). With FQG, the fractional version of the semi-classical mass spectrum of the SBH is analyzed (quantum corrections to the spectrum are again disregarded, $\hbar \to 0$, but we keep the fractional-fractal feature carried by d), and we realize that the fractal strucute of the SBH can be obtained from FQG. Also, the fractional-fractal generalization of the quantities that describe the thermodynamics of the SBH, such as temperature (4.52) and entropy (4.44), can be obtained.

On the other hand, the study of cosmological horizons in the chapter 5 revealed that the cosmological AH is the natural choice for associating a Hawking temperature, in equation (5.34), in the cosmology scenario produced by a FLRW spacetime, such that the surface gravity in question must be that of Kodama-Hayward (5.20), consistent with non-stationary spacetimes. Padmanabhan's theory [11] of the emergence of cosmic space and its expansion has opened a new door to investigating the dynamics of the Universe by deriving the Friedmann equations, (5.42) and (5.49), from the thermodynamics of cosmological horizons (in particular, the cosmological AH) and a question of holographic equipartition, given in equation (5.35). Following [15], the hypothesis driven by this work is that the cosmological AH, used as a holographic screen, has a random fractal geometry on its surface similar to the surface of the EH of the SBH.

In turn, the quantities that describe the thermodynamics of the cosmological AH can be extended to the fractional-fractal case equal to the SBH, according to section 4.3. This approach finds support in the work of Jalalzadeh et al. [13], that the cosmological AH of the de Sitter spacetime has the structure of a random fractal mimicking the fractional-fractal SBH. Therefore, we extend this idea forward to the spatially flat ($k = 0$) FLRW spacetime with Padmanabhan theory [11], obtaining the modified Friedmann equations in section 6.2, and revealing the consequences for the ΛCDM model in section 6.3.

6.2 Fractional-fractal Friedmann equations

Inspired by Hashemi et al. [14] we adopt an equation for the emergent dynamics that rewrites equation (5.38) for the cosmological AH as the holographic screen and in terms of well-defined thermodynamic quantities only, that is

$$\frac{dV_{\mathrm{AH}}}{dt} = \frac{1}{T_{\mathrm{P}}^2}\frac{T_{\mathrm{AH}}}{T_{\mathrm{H}}}(N_{\mathrm{sur}} - N_{\mathrm{bulk}}), \tag{6.1}$$

with $V_{\mathrm{AH}} = 4\pi r_{\mathrm{AH}}^3/3$ the volume contained in the cosmological AH, T_{AH} and T_{H} the Hawking radiation respectively associated with the cosmological AH, given equation (5.34), and the HH, as $T_{\mathrm{H}} = 1/2\pi r_{\mathrm{AH}}^2 H$. In equation (6.1), $T_{\mathrm{P}} = 1/L_{\mathrm{P}}$ is the Planck temperature. For a spatially flat spacetime ($k = 0$), using equation (5.30) that expresses the relation between the cosmological AH radius and the Hubble radius, equation (6.1) recovers equation (5.38). To move on to the premise of the fractal nature of the surface of the cosmological AH, we rewrite equation (6.1) in a more appropriate form

$$\frac{dV_{\mathrm{AH}}}{dt} = L_{\mathrm{P}}^2\, r_{\mathrm{AH}} H (N_{\mathrm{sur}} - N_{\mathrm{bulk}}). \tag{6.2}$$

Now, we modify the above equation for the proposed scenario in which the cosmological AH is better described as a spacetime region of fractional-fractal properties just like the SBH considered in section 4.3, and therefore

$$\frac{dV_{\mathrm{eff}}}{dt} = \frac{d}{2} L_{\mathrm{P}}^2\, r_{\mathrm{eff}} H (N_{\mathrm{sur}} - N_{\mathrm{bulk}}), \tag{6.3}$$

where d, $2 \leqslant d < 3$, is the fractal dimension of the surface of the cosmological AH, $V_{\text{eff}} = 4\pi r_{\text{eff}}^3/3$, and r_{eff} the effective cosmological AH radius. In practical terms, the condition ($k = 0$) and the expressions developed in section 4.3 lead to express r_{eff} in terms of the Hubble radius r_{H}, as

$$r_{\text{eff}} = \pi^{(d-2)/4} \left(\frac{r_{\text{H}}}{L_{\text{P}}} \right)^{d/2} L_{\text{P}}. \tag{6.4}$$

The quantities required to obtain the modified Friedmann equations from equation (6.3) are then given their fractional-fractal extension, according to equation (6.4), by

$$N_{\text{sur}} = \frac{4\pi r_{\text{eff}}^2}{L_{\text{P}}^2}; \tag{6.5a}$$

$$N_{\text{bulk}} = 2\frac{|E_{\text{Komar}}|}{T} = \frac{2\epsilon \sum_i (\rho_i + 3p_i) V_{\text{eff}}}{T_{\text{eff}}}; \tag{6.5b}$$

$$T_{\text{eff}} = \frac{1}{2\pi r_{\text{eff}}}, \tag{6.5c}$$

considering the mixture of perfect fluids with state equations $p_i = \omega_i \rho_i$ for the components of the Universe. Substituting equations (6.5) in equation (6.3) gives us

$$r_{\text{eff}}^2 \, \dot{r}_{\text{eff}} = \frac{d}{2} r_{\text{eff}}^3 H + \frac{2\pi}{3} d L_{\text{P}}^2 \, r_{\text{eff}}^5 H \sum_i (\rho_i + 3p_i), \tag{6.6}$$

or

$$3\frac{\dot{r}_{\text{eff}}}{r_{\text{eff}}} = 3\frac{d}{2} H + 2\pi d L_{\text{P}}^2 \, r_{\text{eff}}^2 H \sum_i (\rho_i + 3p_i). \tag{6.7}$$

Using equation (6.4) in the left-hand side of equation (6.7), and with $r_{\text{H}} = H^{-1}$, leads to

$$\frac{3d}{2} \left(\frac{\dot{r}_{\text{H}}}{r_{\text{H}}} - H \right) = -\frac{3d}{2} \left(\frac{\dot{H}}{H} + H \right) = 2\pi d L_{\text{P}}^2 \, r_{\text{eff}}^2 H \sum_i (\rho_i + 3p_i), \tag{6.8}$$

and in a more familiar form

$$\dot{H} + H^2 = -\frac{4\pi}{3} L_{\text{P}}^2 \sum_i (\rho_i + 3p_i) \, r_{\text{eff}}^2 H^2. \tag{6.9}$$

This is the fractional-fractal Raychaudhuri equation, which recovers the ordinary Raychaudhuri equation (5.42) when $d = 2$ and then $r_{\text{eff}} = r_{\text{H}}$. The fractional-fractal version of the continuity equation can be obtained from the fractional-fractal modification of the Misner-Sharp-Hernandez mass, which is

$$M_{\mathrm{MSH,eff}} = \rho\, V_{\mathrm{P,eff}}. \tag{6.10}$$

where $V_{\mathrm{P,eff}} = 4\pi R_{\mathrm{eff}}^3/3$, with R_{eff} the effective Hubble radius in the sense of equation (6.4). For each considered component of the Universe, a relation similar to (5.44) produces a continuity equation for ρ_i, where the time derivative produces an additional $d/2$ factor and then

$$\dot{\rho}_i = -\frac{3d}{2}(\rho_i + p_i)H, \tag{6.11}$$

is the ith component fractional-fractal continuity equation. Using this equation, the following relation is established

$$\begin{aligned} -\frac{2}{d}\frac{1}{a^d(t)}\frac{d}{dt}(a^d(t)\rho_i) &= -\frac{2}{d}\dot{\rho}_i - 2\rho_i\frac{\dot{a}(t)}{a(t)} \\ &= -3(\rho_i + p_i)\frac{\dot{a}(t)}{a(t)} - 2\rho_i\frac{\dot{a}(t)}{a(t)} \\ &= (\rho_i + 3p_i)\frac{\dot{a}(t)}{a(t)}. \end{aligned} \tag{6.12}$$

The modified Friedmann equation can now be found. Using expression (6.12) in equation (6.8) we have

$$-\frac{3d}{2}\left(\frac{\dot{H}}{H} + H\right) = 2\pi d L_{\mathrm{P}}^2\, r_{\mathrm{eff}}^2 \sum_i \left[-\frac{2}{d}\frac{1}{a^d(t)}\frac{d}{dt}(a^d(t)\rho_i)\right], \tag{6.13}$$

and can be rewritten as

$$\frac{3d}{2}\left(\frac{\dot{H}}{H} + H\right)a^d(t) = 4\pi L_{\mathrm{P}}^2\, r_{\mathrm{eff}}^2 \frac{d}{dt}\sum_i (a^d(t)\rho_i). \tag{6.14}$$

Now, substituting equation (6.4) in the equation above

$$d\left(\frac{\dot{H}}{H} + H\right)a^d(t)H^d = \frac{8}{3}\pi^{d/2} L_{\mathrm{P}}^{4-d}\frac{d}{dt}\sum_i (a^d(t)\rho_i), \tag{6.15}$$

solving the left-hand side to identify a time derivative

$$d\left(\frac{\dot{H}}{H} + H\right)a^d(t)H^d = dH^{d-1}\dot{H}a^d(t) + da^{d-1}\dot{a}(t)H^d = \frac{d}{dt}(a^d(t)H^d), \tag{6.16}$$

we obtain

$$\frac{d}{dt}(a^d(t)H^d) = \frac{8}{3}\pi^{d/2}L_{\mathrm{P}}^{4-d}\,\frac{d}{dt}\sum_i (a^d(t)\rho_i), \tag{6.17}$$

or, equivalently

$$H^d = \frac{8}{3}\pi^{d/2}L_{\mathrm{P}}^{4-d}\sum_i \rho_i. \tag{6.18}$$

This is the fractional-fractal extension of the Friedmann equation. Equation (6.18) can be expressed in a form suitable for comparison with the equation (5.49), as

$$H^2 = \frac{8}{3}\pi L_{\mathrm{P}}^2 \sum_i \rho_i \left(\frac{8}{3}\frac{1}{\rho_{\mathrm{P}}}\sum_j \rho_j\right)^{2/d-1}, \tag{6.19}$$

where $\rho_{\mathrm{P}} = 1/L_{\mathrm{P}}^4$ is the Planck energy density. Note that equation (6.19) reduces to the ordinary case when $d = 2$, and differs from equation (5.49) only by a fractional-fractal factor which, as we shall see, has significant cosmological consequences when we examine modifications to the ΛCDM model.

6.3 *Lambda*-Cold Baryonic Matter model

We will see how the ΛCDM model responds to the fractional-fractal modifications of the Friedmann equations, i.e. in a similar fashion to section 5.4, one will investigate possible modifications to the standard cosmological model due to equation (6.19). Again, we consider a spatially flat ($k = 0$) Universe composed of three constituents for the energy density: radiation, cold matter, and the cosmological constant. The cosmological parameters are constrained with the data from TT, TE, EE+lowE+lensing CMB of the Planck 2018 collaboration [145] as in the section 5.4.

First, the continuity equation (6.11), for radiation $\omega = 1/3$, for cold matter $\omega = 0$, and for the cosmological constant $\omega = -1$, in the state equation $p = \omega\rho$, generates the solutions

$$\rho^{(\mathrm{rad})}(t) = \rho_0^{(\mathrm{rad})} \left(\frac{a(t)}{a_0} \right)^{-2d}; \tag{6.20a}$$

$$\rho^{(\mathrm{cm})}(t) = \rho_0^{(\mathrm{cm})} \left(\frac{a(t)}{a_0} \right)^{-3d/2}; \tag{6.20b}$$

$$\rho^{(\Lambda)}(t) = \rho_0^{(\Lambda)}. \tag{6.20c}$$

where again ρ_0 and a_0 stands for the present epoch of the energy density and the scale factor, respectively; "rad", "cm", and Λ denote radiation, cold matter, and the cosmological constant, respectively. Similarly to equation (5.50), these modifications applied to the fractional-fractal Friedmann equation (6.18) lead to

$$H^2 = H_0^2 \left[\Omega_0^{(\mathrm{rad,fractal})} \left(\frac{a_0}{a(t)} \right)^{2d} + \Omega_0^{(\mathrm{cm,fractal})} \left(\frac{a_0}{a(t)} \right)^{3d/2} + \Omega_0^{\Lambda,\mathrm{fractal}} \right]^{\frac{2}{d}}, \tag{6.21}$$

with H_0 the Hubble parameter at the present epoch as in expression (5.51), and $\Omega_0^{(i,\mathrm{fractal})}$ is the fractal density parameter of the ith component of the Universe at the present epoch, which can be identified from equation (6.18) as

$$\Omega_0^{(i,\mathrm{fractal})} := \frac{8\pi L_\mathrm{P}^2}{3H_0^2} \rho_0^{(i)} \left(\frac{L_\mathrm{P}\, H_0}{\sqrt{\pi}} \right)^{2-d}, \tag{6.22}$$

and in terms of the ordinary density parameter $\Omega_0^{(i)}$ defined in (5.51), we have

$$\Omega_0^{(i,\mathrm{fractal})} = \Omega_0^{(i)} \left(\frac{L_\mathrm{P}\, H_0}{\sqrt{\pi}} \right)^{2-d}. \tag{6.23}$$

Note that the fractal density parameters grow rapidly with small increases in the values of the fractal dimension d such that $d > 2$, compared to the value of the ordinary density parameter. Also, equation (6.21) can be used to calculate the correspondent age of the Universe by the density parameters

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{dx}{x \left[\Omega_0^{(\mathrm{rad,fractal})} x^{-2d} + \Omega_0^{(\mathrm{cm,fractal})} x^{-3d/2} + \Omega^{(\Lambda,\mathrm{fractal})} \right]^{\frac{1}{d}}}, \tag{6.24}$$

Table 1: The age of Universe, t_0, and the density parameter of cold matter for various values of fractal dimension d. Here we consider $h = 0.674$ and the Hubble time $t_H = 1/H_0 = 14.508$ Gyr.

d	t_0 (Gyr)	$\Omega_0^{(\mathrm{cm})}$
2	13.797	0.315
2.01321	13.776	0.049
2.1	13.648	2.2×10^{-7}
2.5	13.147	5.7×10^{-32}
2.7	12.941	2.9×10^{-44}
2.99	12.684	1.5×10^{-62}

Source: The author (2024).

where $x = a/a_0$ and we consider a spatially flat Universe; i.e. $\Omega_0^{(k)} \approx 0$. Again, we neglect the contribution of radiation to calculate t_0, and we will make the following assumption: the cold matter energy density content of the Universe that we measure is fractal, $\Omega_0^{(\mathrm{cm,fractal})} = 0.315$ (data from the reference [145]). The meaning of this will become clear below. Table 1 shows the age of the Universe for various values of the fractal dimension d in the range $2 \leqslant d < 3$.

It is worth noting that the age of the Universe varies from 12.684 Gyr to 13.797 Gyr, and the latter value holds for $d = 2$ in which ΛCDM stands. Furthermore, the variation in fractal dimension causes the effective value of the density parameter of cold matter to vary enormously, and when d goes from two to three, $\Omega_0^{(\mathrm{cm})}$ goes from 0.315 to 10^{-62}; see table 1. When $d = 2.01321$ the age of the Universe becomes 13.776 Gyr and the actual density parameter of the cold matter is $\Omega_0^{(\mathrm{cm})} = 0.049$, which is equal to the density parameter of baryonic matter $\Omega_0^{(\mathrm{b})}$. Therefore, if we conceive of the value of $\Omega_0^{(\mathrm{cm,fractal})}$ as just an amplification of the value of the cold matter density parameter of the Universe due to a small change in the fractal dimension d, then $\Omega_0^{(\mathrm{cm})} = \Omega_0^{(\mathrm{b})}$ can alone be responsible for the matter content of the Universe, which is only baryonic, and does not need a CDM constituent to validate the cosmological observations. In other words, by changing the value of d, the need for CDM no longer exists, the density parameter of the cold matter becomes only of baryonic matter origin and the ΛCDM model can be viewed as just a Lambda-Cold Baryonic Matter (ΛCBM) model.

Two final comments are necessary. Firstly, it must be emphasized that the above arguments are applied only in the cosmological context, i.e., on the scale of objects and structures such that the Universe obeys the cosmological principle, and obviously this is not an approach that studies CDM on galactic scales, in terms of the rotation curves or gravitational lenses of observational samples with data compared to the proposed model. However, there are several recent studies that link the absence of CDM on these scales to galaxies and galaxy clusters, using fractional modifications to the dynamics of rotation curves or the gravitational Poisson equation [146–148].

Secondly, of course, the fractal factor expressed by the equation (6.23) that relates the fractal and ordinary density parameters also applies to the other energy components of the Universe when considering the ΛCDM model. From relevant consequences applied to the radiation-dominated era, the recombination period, matter-radiation equality, to the late Universe dominated by Λ, equation (6.23) must be examined carefully in the face of current observational data. In particular, by modifying the effective radiation density parameter in the young Universe, we speculate that there will be important consequences for a non-trivial change in the dynamics of structure formation at this stage of the Universe, such as the formation rate of galaxies and primordial black holes. Further studies should be conducted to verify this possibility.

7 CONCLUSION

This dissertation considered a semi-classical analysis of the thermodynamics of the SBH in chapter 3, which had been quantized via CQG in its general results in chapter 2, where the WDW equation is the fundamental result. The SBH was found to satisfy the conditions for an object with random fractal geometry, and that FQG leads to a fractional semi-classical description of its mass spectrum and thermodynamics, as presented in chapter 4. Indications were given of the application of horizon thermodynamics to the study of cosmological models through emergent cosmology in chapter 5, and the cosmological AH in the de Sitter Universe was shown to be similar to the EH of the SBH in terms of fractal structure [13]. This conclusion led us to extend the fractional-fractal description of the SBH to the cosmological AH of the FLRW Universe in chapter 6. As a result, we obtained the modified Friedmann equations. We were able to analyze the implications of the fractional-fractal picture constructed in the ΛCDM model in section 6.3. In particular, the constitution of the density parameters of the cold matter component of the Universe and the necessity of CDM for the ΛCDM model were investigated.

The ΛCBM model presented in this work, which follows close to the reference [15], gives an alternate view of the cosmological paradigm of the CDM, suggesting that fractional-fractal features may have had an impact on the measurable characteristics of the matter content of the Universe, by modifying the density parameter of cold matter. That is, by changing the value of the fractal dimension d of the model, the need for CDM no longer exists, and the density parameter of cold matter becomes only of baryonic matter origin. This may justify previous results that show fractal and fractional modifications to spacetime geometry which avoid the assumption of CDM at various scales, namely, in galaxies and galaxy clusters [146–148].

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