UNIVERSIDADE FEDERAL DE PERNAMBUCO
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA

DIEGO ALEJANDRO MONROY ÁLVAREZ

Jacobi Polynomials Approach to the Random Search Problem in One Dimension

Recife

2023

DIEGO ALEJANDRO MONROY ÁLVAREZ

Jacobi Polynomials Approach to the Random Search Problem in One Dimension

Dissertação apresentada ao Programa de Pós-graduação em Física da Universidade Federal de Pernambuco como parte dos requisitos necessários para a obtenção do título de Mestre em Física

Área de Concentração: Física Teórica e Computacional

Orientador: Ernesto Carneiro Pessoa Raposo

Recife

2023

Catalogação na fonte
Bibliotecária Nataly Soares Leite Moro, CRB4-1722

M753j Monroy Álvarez, Diego Alejandro
Jacobi polynomials approach to the random search problem in one dimension / Diego Alejandro Monroy Álvarez. – 2023.
103 f.: il., fig.

Orientador: Ernesto Carneiro Pessoa Raposo.
Dissertação (Mestrado) – Universidade Federal de Pernambuco. CCEN, Física, Recife, 2023.
Inclui referências e apêndice.

1. Física teórica e computacional. 2. Buscas aleatórias. 3. Polinômios de Jacobi. I. Raposo, Ernesto Carneiro Pessoa (orientador). II. Título.

530.1 CDD (23. ed.) UFPE- CCEN 2023 - 103

DIEGO ALEJANDRO MONROY ALVAREZ

JACOBI POLYNOMIALS APPROACH TO THE RANDOM SEARCH PROBLEM IN ONE DIMENSION

Dissertação apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Física.

Aprovada em: 13/06/2023.

BANCA EXAMINADORA

____

Prof. Dr. Ernesto Carneiro Pessoa Raposo
Orientador
Universidade Federal de Pernambuco

____

Prof. Dr. Maurício Domingues Coutinho Filho
Examinador Interno
Universidade Federal de Pernambuco

____

Prof. Dr. Marcos Gomes Eleuterio da Luz
Examinador Externo
Universidade Federal do Paraná

To my beloved Tatiana.

ACKNOWLEDGEMENTS

I would like to thank my wife Tatiana, to whom this work is dedicated, for her immense love, her refreshing optimism and her unconditional support, which ultimately provided the drive to bring this project through, no matter the outcome.

I thank my parents, Germán and Flor Alba, my siblings, Silvia and Germán, and my nephew Dante. They brought calm and joy in times of despair and their support has always been unwavering.

I am deeply thankful to Ernesto for his guidance, his teachings, the great deal of patience and kindness with me not only during the last two years, but also during times of trouble back in 2014.

Speaking of 2014, I would like to thank Ronald Romero, Cesar Agudelo, Susana Patiño, Seu Paulo, Josefa and specially Dona Rita for their immense kindness and friendship.

A very special appreciation to my dear friend David, whose wisdom and patience is always reassuring. I would also like to express my gratitude to Andrea, Laura, María Ximena, Emily, Emperatriz, Magaly and Camille, who have been following up the snail pace progress of this project.

My thanks also go to my *alma mater*, Universidad Nacional de Colombia, specially because at some point in the recent past they decided that alumni should have access to academic resources.

I apologise if I committed an act of unfairness by not mentioning any other person or institution that might have been related to this endeavour. Please do not take offense of my short memory.

Last but not least, I am very grateful to FACEPE for providing financial support during my "first half" at the MSc program in Physics at UFPE during 2013 and early 2014.

ABSTRACT

Lévy processes, either flights or walks, have attracted a great deal of attention from diverse fields. They have been successfully applied to model anomalous transport phenomena in superconductors, turbulence, sunlight scattering in clouds, spectroscopy and random lasers. In ecology, there are numerous evidence that living organism often forage "non-gaussianly", a behaviour that, in theory, results in more efficient searches. Short-term deviations from normality have also been observed in financial assets prices and Lévy processes have been applied to analyse market microstructure and market friction. We address the problem of one-dimensional symmetric Lévy flights that take place in a finite interval with absorbing endpoints, i.e. the target sites. Pure Lévy flights are by no means easy to tackle analitically, hence the jump step length is sampled from a power-law (Pareto I) distribution with shape parameter $0 < \alpha < 2$ thus resembling the asymptotic heavy-tailed behaviour of the Lévy α-stable distribution. For such simplified system, closed-form expressions have been reported in the literature for the absorption probability at a specific target, the mean number of steps and the mean path length before a target is encountered, of which the last two quantities are of special interest since they are related to the mean first-passage time of Lévy flyers and walkers respectively. Those approximate closed-form expressions have been obtained by means of inversion formulae related to fractional integro-differential equations and perform reasonably well provided that the departure site is not too close to the targets and away from the Gaussian regime. This work not only intends to revisit the aforementioned approach but also to explore alternative methods, such as the spectral relationship method using classical Jacobi polynomials. This method allows the inclusion of correction terms that are difficult to handle with inversion formulae. The obtained solutions predict the simulated results more accurately and in broader ranges of the stability index and the departure site location than their inversion formulae counterparts. As a drawback, one must resort to numerical methods and regularization techniques to deal with the instability arising for the ill-conditioned nature of problem.

Keywords: random searches; Lévy α-stable distribution; classical Jacobi polynomials.

RESUMO

Os processos de Lévy, sejam voos ou caminhadas, têm atraído muita atenção de diversos campos. Eles foram aplicados com sucesso para modelar fenômenos de transporte anômalo em supercondutores, turbulência, dispersão da luz solar em nuvens, espectroscopia e lasers aleatórios. Em ecologia, existem inúmeras evidências de que organismos vivos costumam forragear "não gaussianamente", um comportamento que, em teoria, resulta em buscas mais eficientes. Desvios de curto prazo da normalidade também foram observados nos preços dos ativos financeiros e os processos de Lévy foram aplicados para analisar a microestrutura e o atrito do mercado. Abordamos o problema de voos de Lévy simétricos unidimensionais que ocorrem em um intervalo finito com extremidades absorventes, ou seja, os locais de destino. Os vôos Lévy puros não são fáceis de lidar analiticamente, portanto, o comprimento do passo do salto é amostrado a partir de uma distribuição de lei de potência (Pareto I) com parâmetro de forma $0 < \alpha < 2$, assemelhando-se assim ao comportamento assintótico de cauda pesada do Lévy Distribuição α-estável. Para tal sistema simplificado, expressões de forma fechada foram relatadas na literatura para a probabilidade de absorção em um alvo específico, o número médio de etapas e o comprimento médio do caminho antes de um alvo ser encontrado, dos quais as duas últimas quantidades são de interesse especial uma vez que estão relacionados com o tempo médio de primeira passagem dos voadores e caminhantes de Lévy, respectivamente. Essas expressões aproximadas de forma fechada foram obtidas por meio de fórmulas de inversão relacionadas a equações integrais-diferenciais fracionárias e funcionam razoavelmente bem, desde que o local de partida não esteja muito próximo dos alvos e longe do regime gaussiano. Este trabalho pretende não só revisitar a abordagem acima mencionada, mas também explorar métodos alternativos, como o método de relações espectrais usando polinômios clássicos de Jacobi. Este último permite a inclusão de termos de correção que são difíceis de lidar com fórmulas de inversão. As soluções obtidas prevêem os resultados simulados com mais precisão e em intervalos mais amplos do índice de estabilidade e da localização do local de partida do que suas contrapartes de fórmulas de inversão. Como desvantagem, deve-se recorrer a métodos numéricos e técnicas de regularização para lidar com a instabilidade decorrente da natureza mal condicionada do problema.

Palavras-chave: buscas aleatórias; distribuição α-estável de Lévy; polinômios de Jacobi.

LIST OF FIGURES

LIST OF SYMBOLS

$\mathbb{N}_0$	Set of natural numbers including zero.
$\mathbb{1}_A(x)$	Indicator function that takes value 1 if $x \in A$ and 0 otherwise.
$P_n^{(\gamma,\delta)}(t)$	Jacobi polynomial of degree n and parameters γ and δ.
$\widetilde{P}_n^{(\gamma,\delta)}(t)$	Weighted Jacobi polynomial of degree n and parameters γ and δ with weight $\rho^{(\gamma,\delta)}$.
$(x)_n$	Pochhammer symbol or rising factorial $(x)_n \equiv x(x+1)\cdots(x-n-1)$
$B(x_0, l_0)$	Open ball of radius l_0 centered at x_0.
$\mathrm{B}(w,z)$ / $\mathrm{B}(w,z;u)$	Euler Beta function / Incomplete Euler Beta function.
$\mathrm{b}(w,z)$	Reciprocal Euler Beta function.
$⨍$	Integral in Cauchy Principal Value sense.
$\mathcal{F}$ / $\mathcal{F}^{-1}$	Direct/inverse Fourier transform.
$\mathcal{M}$ / $\mathcal{M}^{-1}$	Direct/inverse Mellin transform.
r.v.	Random variable.
iid	Independent and identically distributed.
$X \sim F$	The r.v. X is distributed according to F.
f_X	Probability density function (pdf) of the r.v. X.
φ_X	Characteristic function of the r.v. X.
$X \overset{d}{=} Y$	R.v.s X and Y are equal in distribution.
$\mathcal{I}_{b-}^{\alpha}$ / $\mathcal{I}_{a+}^{\alpha}$	Right / left fractional integral of order $0 < \alpha < 1$.
$\mathcal{D}_{b-}^{\alpha}$ / $\mathcal{D}_{a+}^{\alpha}$	Right / left Riemann-Liouville fractional derivative of order $0 < \alpha < 1$.

CONTENTS

1 INTRODUCTION

Consider the problem addressed in (BULDYREV et al., 2001b) and (BULDYREV et al., 2001a): a Lévy random flyer or walker in a finite interval $[0, L]$ with *absorbing boundaries*, i.e. the process terminates whenever the flyer/walker reaches an endpoint of the interval for the first time. The key result of the aforementioned works is to derive closed-form *approximate* expressions for the absorption probability at a given endpoint, the mean number of steps $N(x_0)$ and the mean path length $S(x_0)$ traversed by an *asymptotic* Lévy flyer before absorption given that flight departs from x_0 - to see this, recall (2.34). $N(x_0)$ and $S(x_0)$ are relevant because they correspond to the mean first-passage time of Lévy flights and walks, respectively. Those expressions are presented below

$$N(x_0) = \frac{\sin\frac{\pi\alpha}{2}}{\frac{\pi\alpha}{2}} \left[\frac{(L - x_0)\, x_0}{l_0^2}\right]^{\frac{\alpha}{2}} \tag{1.1}$$

and

$$S(x_0) = l_0 \frac{\sin\frac{\pi\alpha}{2}}{\pi\frac{\alpha-1}{2}} \left[\frac{(L - x_0)\, x_0}{l_0^2}\right]^{\frac{\alpha}{2}} + \frac{L(2-\alpha)}{2(1-\alpha)} \left[1 - 4\frac{\psi_\alpha\left(\frac{x_0}{L}\right) + \psi_\alpha\left(1 - \frac{x_0}{L}\right)}{\alpha(\alpha+2)\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)}\right], \tag{1.2}$$

where $\mathrm{B}(x, y)$ stands for Euler Beta function and the function $\psi_\alpha(z)$ can be expressed in terms of the Gaussian hypergeometric function as $\psi_\alpha(z) = {}_2F_1(2 - \frac{\alpha}{2}, \frac{\alpha}{2}; \frac{\alpha}{2} + 2; z)z^{\frac{\alpha}{2}+1}$.

Despite the good agreement of (1.1) and (1.2) with numerical solutions and simulations, the authors disclosed that those approximate expressions fail in the vicinity of the boundaries and approaching normality $\alpha \to 2$. This work is intended to elaborate further on this issue: its origin and how to tackle it.

The manuscript is organised as follows. Chapter 2 presents the core ideas regarding random walk models are introduced using the Weierstrass random walk as "specimen". This "pathological" specimen will make way to the Generalised Central Limit Theorem (GCLT for short), which in this context concerns the asymptotic distribution of the position of the random walker, followed by a brief review of the main features of Lévy α-stable distribution. Finally, a brief review of (truncated) Lévy random flights and the results in (BULDYREV et al., 2001a) and (BULDYREV et al., 2001b) regarding the mean first passage time of Lévy flights/walks that motivate this work.

Chapter 3 is devoted to the mathematical tools on which the work relies. Among the topics discussed there, the characterisation of the classical Jacobi polynomials, the spectral relationships method for solving fractional integral equations and the proof of relevant spectral relations involving Jacobi polynomials stand out. In addition, a section is devoted to the closed-form solution of the generalized Abel integral equation by means of some results of the theory of singular integral equations.

It will be shown in Chapter 4 that a general class of observables recorded along the flight path are the solution of a Fredholm integral equation of the second kind. For a power law single step length distribution, such integral equation is proved to take the approximate form of a generalised Abel integral equation. With the aid of the closed-form inversion formulae introduced in Chapter 3, the latter integral equation is solved for quantities such as the the absorption probability at a boundary site and the mean number of steps taken before absorption.

Perhaps the most important results of the work are presented in Chapter 5, where the spectral relationship method is applied to the solution of the generalized Abel equation including first-order corrections that cannot be easily accounted for in closed-form inversion formulae approaches such as (BULDYREV et al., 2001b) and (BULDYREV et al., 2001a). In that sense, the solutions obtained by means of spectral relationship method perform better than closed-form solutions, at the expense of resorting to numerical methods and regularization techniques.

2 LÉVY RANDOM FLIGHTS/WALKS: CONCEPTUAL FRAMEWORK

2.1 WHY TO LÉVY FLY/WALK RANDOMLY?

One of the astonishing facts about random walk models is that they are so simple in their essence that one can model a vast range of systems with them. For instance, Lévy flight statistics have been observed in diverse context such as spectroscopy, where spectral random flights occur in energy space (ZUMOFEN; KLAFTER, 1994); optics, where intensity fluctuations observed in random lasers follow an α-stable distribution (GOMES et al., 2016); atmospheric sciences, where evidence of Lévy-like transport of solar photons in (fractal) clouds has been observed (PFEILSTICKER, 1999); mathematical finance, where stock price indexes exhibit non-normal temporal scaling behaviour (MANTEGNA; STANLEY, 1995).

A very interesting application of Lévy flights in ecology is the Lévy flight foraging hypothesis (VISWANATHAN; RAPOSO; LUZ, 2008). Let us suppose that a forager searches for randomly located target sites (for instance, food sources). A search strategy can be regarded as optimal if it minimizes the length of the flight paths taken by the forager between two encounters: the shorter the path, the lesser the "fuel" consumption (neglect jammed junctions for the sake of discussion) and it can be assign a stability index $0 < \alpha_{opt} \leq 2$, whose value depends roughly on the recovery time of a foraged site and the abundance of sites. For instance, in a regime of sparse (diluted) sites and non-destructive foraging conditions allowing to quick site regeneration, the optimal strategy is attained for $\alpha_{opt} \approx 1$. However, if food sources are widely available, there is no need to take unusual long jumps since there is no efficiency gain compared to a Brownian strategy. The Levy flight foraging hypothesis states that, if Lévy flight searches turn out to be optimal, living organisms might have learned to exploit those advantages. Although some issues have been raised regarding methodological shortcomings that may lead to false positives, there is empirical evidence in favor of the Lévy flight foraging hypothesis arising from data collected from species of diverse degree of complexity: dinoflagellates (BARTUMEUS et al., 2003), flies (COLE, 1995), albatrosses (HUMPHRIES et al., 2012), even extinct 50 My-old sea urchins (SIMS et al., 2014).

On the other hand, Lévy walks solve a particular shortcoming of Lévy flights that will be disclosed at the end of the introduction. In so doing, they have been applied to relevant problems in physics such as turbulent flow and chaotic dynamics observed in superconducting devices (SHLESINGER; KLAFTER; J. West, 1986), as well as in chaotic Hamiltonian systems

(SHLESINGER; ZASLAVSKY; KLAFTER, 1993) (ZASLAVSKY, 2005). In the context of biological physics, Lévy walks have recently been applied to study the anomalous diffusive behaviour of *Plasmodium* -the parasite that causes malaria- in the search for intravasation hotspots in the inner layers of the skin (FORMAGLIO et al., 2023).

2.2 WEIERSTRASS RANDOM WALK

This section is intended to present some fundamental ideas regarding random walks using the one-dimensional Weierstrass discrete random walk (REICHL, 1998) as example.

Let us consider a flyer initially located at 0 that takes instantaneous jumps of length

$$|\Delta X_i| = \lambda b^{J_i}, \tag{2.1}$$

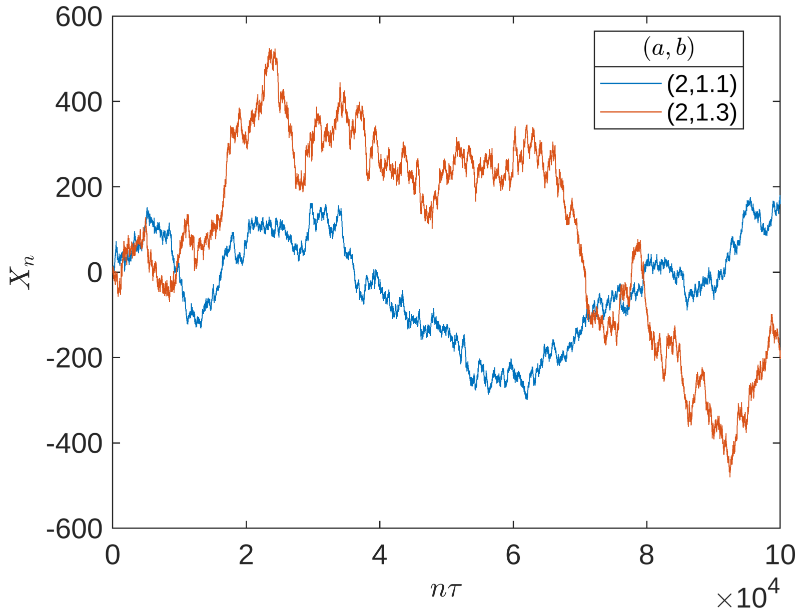
where J_i are independent identically distributed random variables sampled with probability mass function given by

$$p_J(j) = \left(1 - \frac{1}{a}\right)\left(\frac{1}{a}\right)^j. \tag{2.2}$$

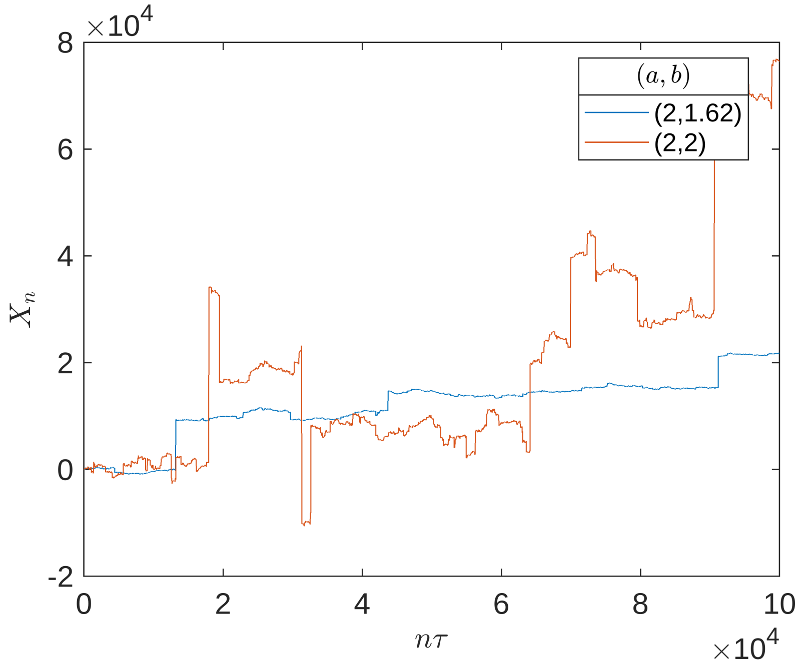
Each jump is directed either to the left or to the right with equal probability and remaining still is not an option. The real constants a, b are greater than 1, the length scale parameter λ is strictly positive and successive jumps occur at a regular time span denoted by τ. The position of walker after n jumps, X_n, is equivalent to the sum of the n iid random variables ΔX_i.

Let us set $a = 2$ and take b from $\{1.10, 1.30, 1.62, 2.00, 3.00\}$. A single sample of a flight trajectory for each value of b is presented in Figure 1. Note that the sample paths for $b \in \{1.62, 2.00\}$ (Exhibit b) consist of clusters of "localized" motion connected by unusual large jumps, a behaviour that is not evident for $b \in \{1.10, 1.30\}$ (Exhibit a).

Figure 1 – Sample path of a Weierstrass random walks with parameters $a = 2$ and $b \in \{1.10, 1.30, 1.62, 2.00\}$. In contrast to $b \in \{1.10, 1.30\}$ (Exhibit a), the sample paths for $b \in \{1.62, 2.00\}$ show localized motion (clusters) connected by infrequently large jumps (Exhibit b).



(a)

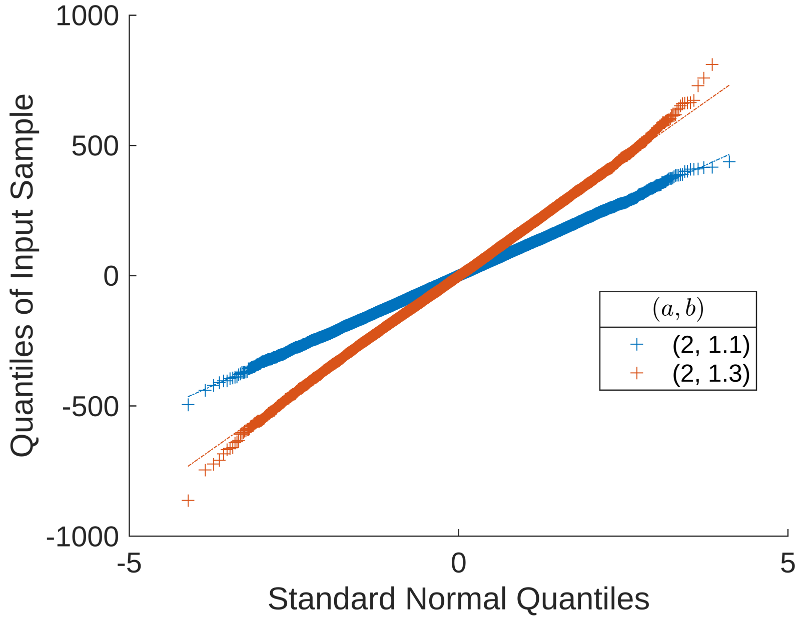


(b)

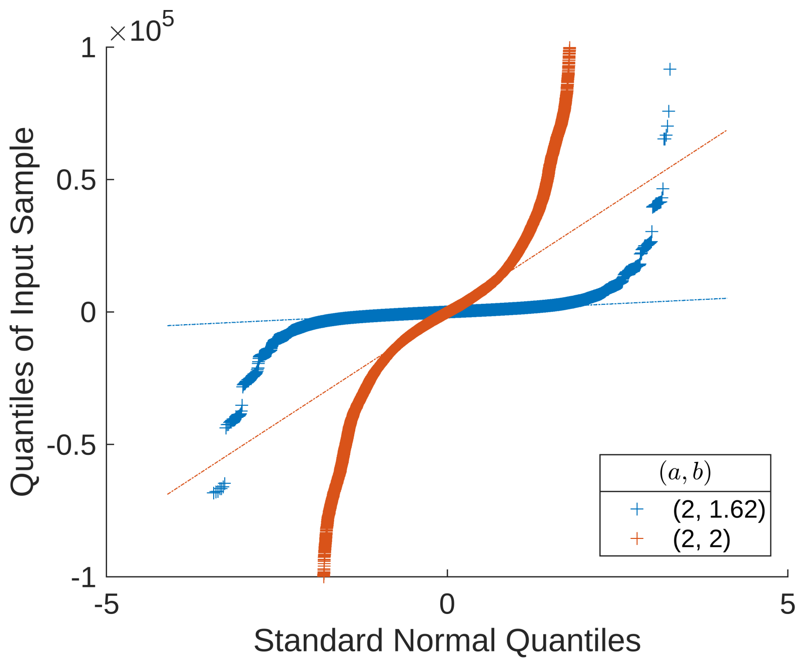
Source: The author (2023)

As for the asymptotic distribution of the position of the random walker, a sample of 25000 paths was simulated for each pair of parameters (a, b) and the quantile-quantile plot (hereafter, QQ plot) of the empirical distribution of the position after 10000 jumps, X_{10000}, against the standard normal is depicted in Figure 2. The normality of X_{10000} is evident for the chosen value of a and $b \in \{1.10, 1.30\}$ (Exhibit a). Nevertheless, it is also evident that the distribution of X_{10000} strongly deviates from normal behaviour (Exhibit b). In that case, the shape of the QQ plot clearly points at a heavy-tailed distribution.

Figure 2 – Quantile-quantile plot of the distribution of the position of the random walker after 10000 jumps, X_{10000}, for $a = 2$ and $b \in \{1.10, 1.30, 1.62, 2.00\}$ versus the standard normal distribution. The QQ plots in (a) supports the normality of X_{10000} for $b \in \{1.10, 1.30\}$. Nevertheless, this is not the case for $b \in \{1.62, 2.00\}$, for which the QQ plots in (b) reveal heavy-tailed limiting distributions. A sample of 25.000 paths was generated for each pair of parameters.



(a)



(b)

Source: The author (2023)

For the time being, let us leave the "abnormal" cases in Figure 2 (b) aside. The characteristic function $\varphi_{\Delta X}(k)$ of the single jump r.v. ΔX, defined as $\varphi_{\Delta X}(k) \equiv \mathcal{F}[f_{\Delta X}] = \mathbb{E}[e^{ikX}]$, is given by

$$\varphi_{\Delta X}(k) = \mathcal{F}\left[\frac{1}{2}\left(1-\frac{1}{a}\right)\sum_{j\geq 0}\left(\frac{1}{a}\right)^j\Big(\delta(x-\lambda b^j)+\delta(x+\lambda b^j)\Big)\right] \tag{2.3}$$

$$= \left(1-\frac{1}{a}\right)\sum_{j\geq 0}\left(\frac{1}{a}\right)^j\cos\Big(k\lambda b^j\Big) = \Big(1-b^{-\alpha}\Big)\sum_{j\geq 0} b^{-\alpha j}\cos\Big(k\lambda b^j\Big), \tag{2.4}$$

with α such that $b^\alpha \equiv a$. It is known as *Weierstrass function* and has very special features. If $0 < \alpha < 1$, i.e. $b > a$, then $\varphi_{\Delta X}(k)$ is uniformly continuous everywhere, yet nowhere differentiable. In addition, it exhibits self-similarity (REICHL, 1998). Indeed, it follows from (2.4) that

$$\varphi_{\Delta X}(bk) = b^{-\alpha}\varphi_{\Delta X}(k) + (1-b^{-\alpha})\cos\lambda k, \tag{2.5}$$

i.e. the magnified $\varphi_{\Delta X}(bk)$ corresponds to the rescaled $\varphi_{\Delta X}(k)$ up to an additive "level" term.

The reason to deal with the characteristic function is that the iterated convolution of n distributions in Euclidean space translates into a very convenient product of characteristic functions in Fourier space, hence the characteristic function of the position X_n after n idd jumps ΔX corresponds to $(\varphi_{\Delta X}(k))^n$. For small positive λ, one has by retaining the first two terms from the cosine power series expansion in (2.4) that

$$\varphi_{\Delta X}(k) = (1-b^{-\alpha})\sum_{j=0}^{\infty}\left(b^{-\alpha j} - \frac{1}{2}k^2\lambda^2 b^{(2-\alpha)j} + O(\lambda^4)\right). \tag{2.6}$$

If $\alpha > 2$, which is equivalent to $b < \sqrt{a}$, then the geometric series arising from the second term is convergent and one has that

$$\log\varphi_{\Delta X}(k) \approx \log\left[(1-b^{-\alpha})\left(\frac{1}{1-b^{-\alpha}} - \frac{1}{2}\frac{\lambda^2}{1-b^{2-\alpha}}k^2\right)\right] \tag{2.7}$$

$$\approx \log\left[1-\frac{1}{2}\frac{\lambda^2}{b^2}\frac{b^\alpha-1}{b^{\alpha-2}-1}k^2\right] \tag{2.8}$$

$$\approx -\frac{1}{2}\frac{\lambda^2}{b^2}\frac{b^\alpha-1}{b^{\alpha-2}-1}k^2. \tag{2.9}$$

This means that the characteristic function of X_n is $\varphi_{X_n}(k) \approx \exp\left(-\frac{1}{2}\frac{n\lambda^2}{b^2}\frac{b^\alpha-1}{b^{\alpha-2}-1}k^2\right)$, hence the limiting distribution is normal with mean 0 and variance $\frac{n\lambda^2}{b^2}\frac{b^\alpha-1}{b^{\alpha-2}-1}$. This is a particular case of a more general result: the Central Limit Theorem (hereafter CLT). Observe that $\sigma \sim \sqrt{n}$ roughly corresponds to the relation $\sqrt{\langle x^2\rangle} \sim \sqrt{t}$ valid for Brownian motion (EINSTEIN, 1905).

Now let us include a time scale parameter τ such that the n-th jump taking place in the instant $t = n\tau$ and take the asymptotic $n \to \infty$ continuous time limit in which both the time span between successive jumps $\tau \to 0$ and the scale parameter of a single jump $\lambda \to 0$, while keeping both $\frac{\lambda^2}{\tau b^2} \frac{b^\alpha - 1}{b^{\alpha-2} - 1} \to D_\alpha$ and t finite. For that purpose, consider

$$\lim_{\tau \to 0} \lim_{\lambda \to 0} \frac{\varphi_{X_{n+1}} - \varphi_{X_n}}{\tau} = \lim_{\tau \to 0} \lim_{\lambda \to 0} \frac{\varphi_{\Delta X} - 1}{\tau} \varphi_{X_n}. \tag{2.10}$$

The left hand side of (2.10) corresponds to $\frac{\partial \varphi_X(k,t)}{\partial t}$, while the right hand side is equivalent to $-D_\alpha k^2 \varphi_{X_n} = D_\alpha \mathcal{F}\left[\frac{\partial^2 f_X(x,t)}{\partial x^2}\right]$, where f_X is the probability density function corresponding to φ_X, in virtue of (2.8) and the Fourier transform property $\mathcal{F}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right](k) = -k^2 F\left[f\right](k)$. Therefore, one has in Euclidean space that

$$\frac{\partial f_X(x,t)}{\partial t} = D_\alpha \frac{\partial^2 f_X(x,t)}{\partial x^2}. \tag{2.11}$$

This is an extremely important result: a discrete random walk subject to the CLT corresponds, in the continuous limit, to a diffusion process.

2.3 GENERALIZED CENTRAL LIMIT THEOREM

If the position of the Weierstrass walker in Figure 2 does not converge to a normal distribution according to the CLT, what is the asymptotic distribution then? As mentioned earlier, the answer to this question lies in the Generalised Central Limit Theorem (GCLT for short).

More than a formal proof of the GCLT, a compelling argument from Renormalisation Group theory, originally published by (CALVO et al., 2010), is presented below. Before that, let us define what a (strictly) stable distribution is.

Definition 2.1. *Let X_1, X_2 be independent identically-distributed (iid) random variables such that X_i is distributed according to F, i.e. $X_i \sim F$. The distribution F is said* ***stable*** *iff for any pair of positive constants c_1 and c_2, there are constants c and d such that $cX + d \stackrel{d}{=} c_1X_1 + c_2X_2 \sim F$. If $d = 0$ for all choices of c_1 and c_2, i.e. there is no constant shift, the distribution F is said* ***strictly stable****.*

Hereafter, the discussion will be restricted to strictly stable distributions, for which a "classical" example is the normal distribution $N(0, \sigma^2)$. To see that the normal distribution is strictly stable, let $X_i \sim N(0, \sigma^2)$ and c_1 and c_2 be arbitrary positive constants. The distribution of the random variable $Y_i = c_i X_i$ follows a normal distribution $N(0, c_i^2 \sigma^2)$ and the sum of two iid random variables Y_1 and Y_2 follows the distribution $N\left(0,\left(c_1^2+c_2^2\right)\sigma^2\right)$. The latter assertion can be easily seen in Fourier space as the product of the characteristic functions of each r.v. is $\exp\left(-\frac{1}{2}c_1^2\sigma^2k^2\right)\exp\left(-\frac{1}{2}c_2^2\sigma^2k^2\right)=\exp\left(-\frac{1}{2}\left(c_1^2+c_2^2\right)\sigma^2k^2\right)$. Thus, one has that $cX=c_1X_1+c_2X_2$ with $X\sim N(0,\sigma^2)$ and $c=\sqrt{c_1^2+c_2^2}$.

Let $f_X(x)$ be any probability density function on $\mathbb{R}$ and T_a the RG transformation defined as

$$T_a f_X(x)=|a|\cdot\left(f_X * f_X\right)(ax)=|a|\int_{-\infty}^{\infty}\mathrm{d}s\, f_X(s)f_X(ax-s), \tag{2.12}$$

with scalar a.

Take into account that the distribution of the sum of two random variables can be expressed as the convolution of their distributions (REICHL, 1998). In addition, recall that for $Z=g(Y)$ with invertible function $g(x)$, the preservation of measure $\mathrm{d}F_Z=\mathrm{d}F_Y$ implies that $f_Z(z)=|g'\left[g^{-1}(z)\right]|^{-1}f_Y\left[g^{-1}(z)\right]$. Therefore, T_a is an operator acting on probability measures that represents *aggregation* and *scaling* in the sense that two iid random variables are aggregated into another r.v. $Y=X_1+X_2$, which is rescaled into $Z=\frac{1}{a}Y$ in a similar fashion to Kadanoff's spin block and rescaling. For the present work, let us focus on *proper* dilations, that is $a\in\mathbb{R}^+$.

The fact that a convolution is involved in (2.12) is a strong indication to have it mapped into Fourier space in terms of the characteristic function $\varphi_X(k)\equiv\mathcal{F}[f_X](k)$ as

$$\widetilde{T}_a\varphi_X(k)=\left[\varphi_X\left(\tfrac{k}{a}\right)\right]^2. \tag{2.13}$$

The characteristic function has certain properties arising from Kolmogorov axioms, in particular that the pdf is real-valued, non-negative and normalised to the unity: $\varphi_X(0)=1$ and $\varphi_X(-k)=\varphi_X^*(k)$. Therefore, $\varphi_X(k)$ takes the general form $\varphi_X(k)=\exp\left[g(k)\right]$ with $g(k)$ a complex-valued function such that $g(0)=0$ and $g(-k)=g^*(k)$.

Let us determine the fixed points of $\widetilde{T}_a\varphi_X(k)$, that is functions $\varphi_X^{(0)}(k)$ for which $\widetilde{T}_a\varphi_X^{(0)}(k)=\varphi_X^{(0)}(k)$ and thus one has that $\left[\varphi_X^{(0)}\left(\frac{k}{a}\right)\right]^2=\varphi_X^{(0)}(k)$. In terms of $g(k)$, the latter implies that

$$g^{(0)}\left(\tfrac{k}{a}\right)=\tfrac{1}{2}g^{(0)}(k). \tag{2.14}$$

The simplest form of $g^{(0)}(k)$ that satisfies (2.14) is

$$g^{(0)}(k)=0 \tag{2.15}$$

for all k. In this case, the characteristic function is $\varphi_X^{(0)}(k)=1$, whose corresponding distribution is a Dirac delta $f_X^{(0)}(x)=\mathcal{F}^{-1}[1](x)=\delta(x)$. Furthermore, suppose that $g^{(0)}(k)$ is a homogeneous function of degree α that does not depend on the phase of the argument, i.e. $g^{(0)}(tk) = g^{(0)}(|tk|) = t^\alpha g^{(0)}(|k|)$ for any (real) scalar t. Therefore, (2.14) implies that $a^\alpha = 2$ which in turn leads to

$$\alpha = \frac{\log 2}{\log a}. \tag{2.16}$$

Observe that the aforementioned requirements on $g^{(0)}(k)$, in order to have a proper characteristic function, are satisfied by the assumption

$$g^{(0)}(k) = A|k|^\alpha \mathbb{1}_{(0,\infty)}(k) + A^*|k|^\alpha \mathbb{1}_{(0,\infty)}(-k), \tag{2.17}$$

with $A \in \mathbb{C}$ and assuming certain restrictions on the phase that are too specific to be described here.

At this point, note that the stable points of the RG flow (2.15) and (2.17) correspond to stable distributions in the sense of Definition 2.1.

Regarding the stability of the RG flow in the vicinity of the fixed points $\varphi_X^{(0)}$ corresponding to (2.15) and (2.17), let us consider the linearised flow in Fourier space around them as

$$\delta\left[\widetilde{T}_a^{(0)}\varphi_X\right] \equiv \widetilde{T}_a\varphi_X - \varphi_X^{(0)} \approx \left.\frac{\delta\widetilde{T}_a}{\delta\varphi_X}\right|_{\varphi_X^{(0)}} \delta\varphi_X \tag{2.18}$$

$$\delta\left[\widetilde{T}_a^{(0)}\varphi_X\right] \approx (\widetilde{DT}_a)_{\varphi_X^{(0)}}\delta\varphi_X. \tag{2.19}$$

With regard to $(\widetilde{DT}_a)_{\varphi_X^{(0)}}$, whose effect on ζ is given by $(\widetilde{DT}_a)_{\varphi_X^{(0)}}\zeta(k) = 2\varphi_X^{(0)}\left(\frac{k}{a}\right)\zeta\left(\frac{k}{a}\right)$ as a result of (2.13), let us consider the eigenvalue problem given by

$$2\exp\left[g\left(\tfrac{k}{a}\right)\right]\zeta_s\left(\tfrac{k}{a}\right) = \lambda_s\zeta_s(k), \tag{2.20}$$

with eigenfunction $\zeta_s(k)$ and the corresponding eigenvalue λ_s.

The following proposition regarding the point spectrum of $(\widetilde{DT}_a)_{\varphi_X^{(0)}}$ holds. It is stated without proof for the sake of brevity but the reader can find further details in (CALVO et al., 2010).

Proposition 2.1. *Let $\boldsymbol{\sigma}_p$ be the point spectrum of $(\widetilde{DT}_a)_{\varphi_X^{(0)}}$ and $\varphi_X^{(0)} = \exp(g(k))$ with $g(k)$ in* (2.17)*.*

1. *If $\mathrm{Re}(A) > 0$, then $\boldsymbol{\sigma}_p = \{\lambda_s : 0 < |\lambda_s| \le 2\}$.*

2. *If $\mathrm{Re}(A) = 0$ or $A = 0$, then $\boldsymbol{\sigma}_p = \{2\}$.*

Assume that $\zeta_s(k) = \exp g(k) \cdot \xi_s(k)$. Recall that $\varphi_X(k)$ being a fixed point of T_a implies that (2.14) holds, so the exponential functions on both sides of (2.20) cancel out, leading to

$$2\xi_s\left(\frac{k}{a}\right) = \lambda_s \xi_s(k). \tag{2.21}$$

Let us suppose that $\xi_s(k)$ is a homogeneous function of degree s, that is $\xi_s(tk) = t^s\xi_s(k)$ for any real t, (2.21) leads to $2a^{-s} = \lambda_s$. Recalling that $a = 2^{\frac{1}{\alpha}}$, one has that

$$\lambda_s = 2^{1-\frac{s}{\alpha}}. \tag{2.22}$$

Observe that $s > \alpha$ corresponds to $|\lambda_s| < 1$, i.e. the fixed point $\varphi_X^{(0)}$ is stable in the directions determined by ζ_s. Conversely, if $s < \alpha$, then $1 < |\lambda_s| < 2$ and $\varphi_X^{(0)}$ is unstable in the directions ζ_s, which is the case for $f_X(x) = \delta(x)$ as a result of item 2 in Proposition 2.1.

Now comes the statement of the GCLT. In few words, it means that if the asymptotic distribution of the sum of iid random variables exists, it has to be a stable distribution with characteristic function $\varphi_X(k) = \exp[g(k)]$ corresponding to (2.17) or (trivially) (2.15).

Theorem 2.1. *[B. Gnedenko P. Lévy] (METZLER; KLAFTER, 2000) Let X_i be iid random variables and $Y_n = \sum_{i=1}^n X_i$. If Y_n converges in distribution to Y, i.e. $F_{Y_n}(y) \overset{n\to\infty}{\longrightarrow} F_Y(y)$ for all y, then F_Y is stable.*

Given a parameter $a = 2^{\frac{1}{\alpha}} > 1$ (i.e. $\alpha > 0$) and an initial pdf f_X, let us apply T_a^∞ iteratively and consider the asymptotic distribution $T_a^\infty f_X = \lim_{n\to\infty} T_a^n f_X$. In general, the characteristic function $\varphi_X(k)$ corresponding to f_X admits the following expansion

$$\varphi_X(k) = 1 + B|k|^\nu \mathbb{1}_{(0,\infty)}(k) + B^*|k|^\nu \mathbb{1}_{(0,\infty)}(-k) + o(|k|^\nu), \tag{2.23}$$

with $0 < \nu \leq 2$. Moreover, taking into account that $\mathcal{F}[f_X^{*k}] = (\varphi_X)^k$ with $k \in \mathbb{Z}^+$, (2.13) can be directly generalised to n iterations as

$$\widetilde{T}_a^n \varphi_X(k) = \left[\varphi_X\left(\frac{k}{a}\right)\right]^{2^n} \tag{2.24}$$

$$= \left[1 - a^{-\nu n}\left(B|k|^\nu \mathbb{1}_{(0,\infty)}(k) + B^*|k|^\nu \mathbb{1}_{(0,\infty)}(-k)\right) + o(a^{-\nu n}|k|^\nu)\right]^{2^n}$$

$$\approx \left[1 - (2^n)^{-\frac{\nu}{\alpha}}\left(B|k|^\nu \mathbb{1}_{(0,\infty)}(k) + B^*|k|^\nu \mathbb{1}_{(0,\infty)}(-k)\right)\right]^{2^n}. \tag{2.25}$$

Let us consider the case $\nu = \alpha$. As $\exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$ by virtue of the binomial theorem, one has that

$$\widetilde{T}_a^\infty \varphi_X(k) \approx \exp\left[-B|k|^\alpha \mathbb{1}_{(0,\infty)}(k) - B^*|k|^\alpha \mathbb{1}_{(0,\infty)}(-k)\right] \tag{2.26}$$

and the asymptotic distribution corresponds to (2.17), hence it is a stable distribution with stability index α. It should be noted that pointwise convergence of a sequence of characteristic functions $\varphi_{(n)} \to \varphi$ implies convergence in distribution $F_n \to F$ as a result of Lévy continuity theorem (FRISTEDT; LAWRENCE, 1996).

On the other hand, if $\nu > \alpha$, then $\widetilde{T}_a^{\infty}\varphi_X(k) \approx 1 - (2^n)^{\left(1-\frac{\nu}{\alpha}\right)} = 1$ and the limiting distribution corresponds to (2.15), i.e. to $\delta(x)$.

2.3.1 Lévy α-stable distribution

In general, the Lévy α-stable distribution family admits the parameterisation $S(\alpha, \beta, \lambda, \delta)$ with the stability index or characteristic exponent $\alpha \in (0, 2]$, skewness parameter $\beta \in [-1, 1]$, scale parameter $\lambda > 0$ and location parameter δ and the characteristic function given by

$$\varphi_Z(k) = \exp\left(i\delta k - \lambda^{\alpha}|k|^{\alpha}\left[1 + i\beta\,\mathrm{sgn}(k)\Phi\right]\right) \tag{2.27}$$

with

$$\Phi = \tfrac{2}{\pi}\log(\lambda|k|)\mathbb{1}_{\{1\}}(\alpha) + \left(|\lambda k|^{1-\alpha} - 1\right)\tan\tfrac{\pi\alpha}{2}\mathbb{1}_{(0,1)\cup(1,2]}(\alpha). \tag{2.28}$$

Hereafter, we will only be interested in *symmetric* distribution, i.e. $\beta = 0$, located at $\delta = 0$. In such case, one has that

$$\varphi_Z(k) = \exp\left(-\lambda^{\alpha}|k|^{\alpha}\right). \tag{2.29}$$

Note that, as $\varphi_Z'(k)\Big|_{k=0} \sim -\lambda^{\alpha}|k|^{\alpha-1}$, the expectation value of Z given (2.29) is finite iff $\alpha > 1$. With regard to the variance of Z, it is not finite in the parameter region of interest as $\varphi_Z''(k)\Big|_{k=0} \sim -\lambda^{\alpha}|k|^{\alpha-2}$.

As for the probability density function $f_Z(\alpha, \beta, \lambda, \delta; z)$, closed-forms expressions in terms of standard functions are not always possible. Notable exceptions are the Gaussian distribution $f_Z(2, 0, \lambda, 0; z) = (2\pi\lambda^2)^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left(\frac{z}{\lambda}\right)^2\right]$; the Cauchy-Lorentz distribution corresponding to $\alpha = 1$, for which one has that $f_Z(1, 0, \lambda, 0; z) = \left[\pi\lambda\left(1 + \left(\frac{z}{\lambda}\right)^2\right)\right]^{-1}$; and the Lévy distribution $f_Z(\frac{1}{2}, 1, \lambda, 0; z) = (2\pi\lambda^2)^{-\frac{1}{2}}\exp\left[-\frac{\lambda}{2z}\right]\left(\frac{z}{\lambda}\right)^{-\frac{3}{2}}$.

Nevertheless, $f_Z(\alpha, \beta, \lambda, \delta; z)$ admits representations in terms of Fox H-functions (SCHNEIDER, 1986) (RATHIE; OZELIM; OTINIANO, 2016), defined as

$$H_{p,q}^{m,n}\left[\begin{matrix}(a_1, A_1)\cdots(a_p, A_p)\\(b_1, B_1)\cdots(b_q, B_q)\end{matrix}; z\right] = \frac{1}{2\pi i}\int_L \mathrm{d}s\, z^{-s}\,\Xi_{p,q}^{m,n}(s) \tag{2.30}$$

with

$$\Xi_{p,q}^{m,n}(s) = \frac{\prod_{i=1}^{m} \Gamma\left(b_i + B_i s\right)}{\prod_{j=n+1}^{p} \Gamma\left(a_j + A_j s\right)} \frac{\prod_{j=1}^{n} \Gamma\left(1 - a_j - A_j s\right)}{\prod_{i=m+1}^{q} \Gamma\left(1 - b_i - B_i s\right)} \tag{2.31}$$

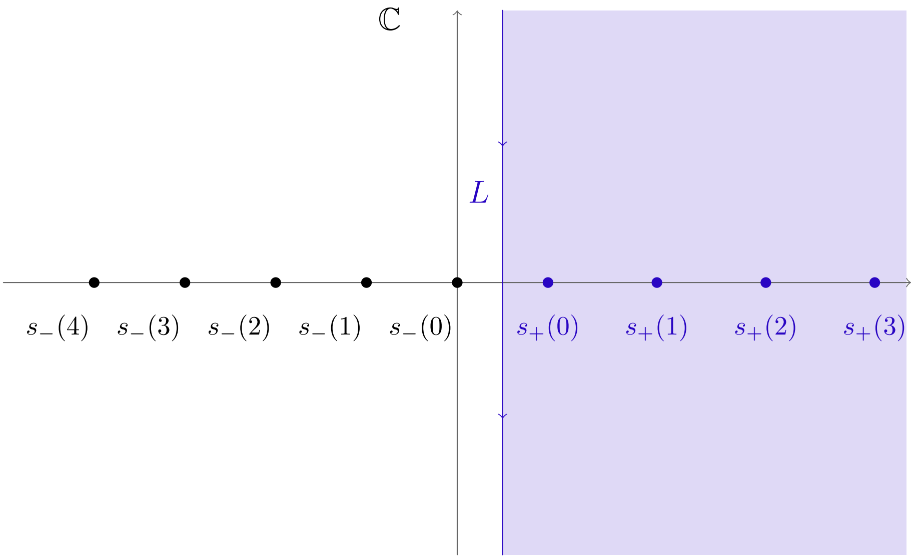
and the integration path L on the complex plane is such that it separates the poles of $\Gamma\left(b_i + B_i s\right)$ with $1 \leq i \leq m$ from the poles of $\Gamma\left(1 - a_j + A_j s\right)$ with $1 \leq j \leq n$.

In the case of our interest, one has that the representation of $f_Z(\alpha, 0, \lambda, 0; z)$ in terms of H-functions for $0 < \alpha < 2$ and $\alpha \neq 1$ is

$$f_Z(\alpha, 0, \lambda, 0; z) = \frac{1}{\alpha\lambda} H_{2,2}^{1,1}\left[\begin{matrix} \left(1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \\ (0, 1), \left(\frac{1}{2}, \frac{1}{2}\right) \end{matrix} ; \frac{z}{\lambda}\right], \tag{2.32}$$

for which one needs to consider the contour integral of $\Xi_{2,2}^{1,1}(s) = \dfrac{\Gamma\left(s\right)\Gamma\left(\frac{1}{\alpha}\left(1-s\right)\right)}{\Gamma\left(\frac{1}{2}\left(1+s\right)\right)\Gamma\left(\frac{1}{2}\left(1-s\right)\right)}$. Since the poles of $\Gamma(s)$ are located at $s_-(k) = -k$ and the poles of $\Gamma\left(\frac{1}{\alpha}\left(1-s\right)\right)$ are located at $s_+(k) = 1 + \alpha k$, for any non-negative integer k, let the integration path L be any line with $\mathrm{Re}\{s\} = c$ with $0 < c < 1$, as depicted in Figure 3.

Figure 3 – Poles of the Fox function $H_{2,2}^{1,1}$ in the Lévy α-stable pdf (2.32) for $\alpha = 1.2$. The integration path L encloses the poles s_+ where the residues are computed.



Source: The author (2023)

The choice of the path L separates s_- from s_+, and enclose s_+ such that the contribution of the arc at $s \to +\infty$ is negligible. The residue of $\Gamma\left(\frac{1}{\alpha}\left(1-s\right)\right)$ at each s_+ is $\dfrac{(-1)^k}{k!}$ and, by the residue theorem, one has for $x \geq 0$ that

$$\begin{aligned} f_Z(\alpha, 0, \lambda, 0; z) &= \frac{1}{\alpha\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(1 + \alpha k\right)}{\Gamma\left(1 + \frac{\alpha}{2} k\right)\Gamma\left(-\frac{\alpha}{2} k\right)} \left(\frac{x}{\lambda}\right)^{-(1+\alpha k)} \\ &= \frac{1}{\pi\alpha\lambda} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \Gamma\left(1 + \alpha k\right) \sin\frac{\alpha\pi k}{2} \left(\frac{x}{\lambda}\right)^{-(1+\alpha k)}, \end{aligned} \tag{2.33}$$

where the last step follows from the reflection formulae (3.4) and (3.6). In the asymptotic limit $x \to \infty$, the leading term is clearly

$$f_Z(\alpha, 0, \lambda, 0; z) \sim \frac{1}{\pi\alpha\lambda}\Gamma\left(1+\alpha\right)\sin\frac{\alpha\pi}{2}\left(\frac{\lambda}{x}\right)^{1+\alpha} \tag{2.34}$$

2.4 REVISITING WEIERSTRASS RANDOM WALKS

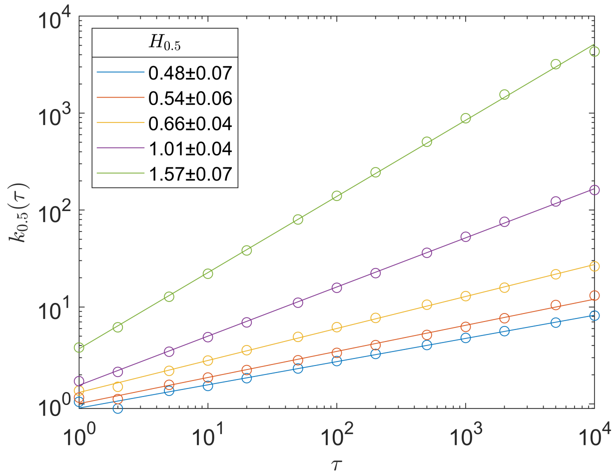
With regard to the asymptotic distribution of the Weiertrass random walk when $b > \sqrt{a}$, let us briefly introduce a new tool: the Generalised Hurst Exponent $H(q)$. It was originally proposed by (BARABÁSI; VICSEK, 1991) for the characterisation of multifractality of self-affine fractals as an extension of the ordinary Hurst exponent H such that $\langle (X_{t+\tau} - X_t)^2 \rangle \sim \tau^{2H}$, with $H = \frac{1}{2}$ in the realm of the CLT. The driving idea is that a single scaling exponent H, instead of a "spectrum" of H's, does not provide enough information if, for instance, small and large fluctuations scale differently, different parts of data exhibit different scaling behaviour or fractals subsets are interlocked (KANTELHARDT et al., 2002). It was shown in (BARABÁSI; VICSEK, 1991) that, for a certain class of self-affine functions, the qth-order height-height correlation function with "lag" τ, namely $k_q(\tau) = E\left[\,|X_{t+\tau} - X_t|^q\,\right]$, scales with τ as

$$k_q(\tau) \sim \tau^{qH(q)}. \tag{2.35}$$

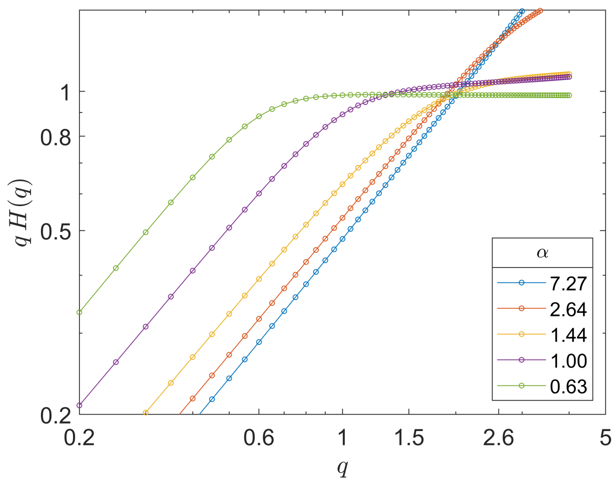
The non-linear dependence of $qH(q)$ on q points at a multi-affine structure of the increments $X_{t+\tau} - X_t$. For the purpose of this section, it suffices to say that, although general Lévy processes exhibit multiscaling, by neglecting the Brownian component and considering a stable process, one has that $H(q) = \frac{1}{\alpha}$ for $q < \alpha$ with α being the stability index (NAKAO, 2000) (JAFFARD, 1999).

Let us analyse the spectrum of $H(q)$ for the Weierstrass random walks considered in Section 2.2. Figure 4 was obtained from a single sample path simulated for each pair of parameters a, b. Figure 4(a) supports that (2.35) holds for the Weierstrass random walks considered. However, it is Figure 4(b) and (c) what clearly exhibit the signature of an asymptotic Lévy α-stable distribution: for each pair a, b, there is a threshold value q^* such that $H(q)$ is a constant H for $q < q^*$ and $H(q)$ decays as a power of q, for $q > q^*$. Also note that $q^* \approx \alpha$ and that $H \approx \alpha^{-1}$ for $\alpha < 2$.

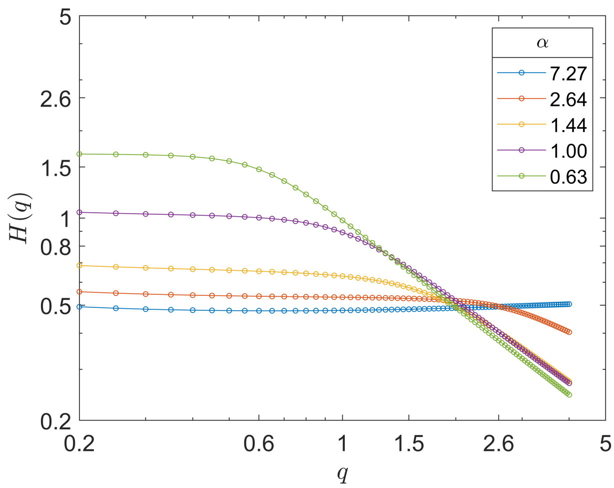
Figure 4 – Generalized Hurst exponent $H(q)$ for the position of the Weierstrass random walker with $a = 2$ and $b \in \{1.1, 1.3, 1.62, 2, 3\}$, i.e. $\alpha \in \{7.27, 2.64, 1.44, 1.00, 0.63\}$ respectively.(a) Sample of a log-log plot of $k_{0.5}(\tau)$ against τ that support the power-law relation (2.35). (b) For the pairs of parameters a, b with $b > \sqrt{a}$ $(0 < \alpha < 2)$ for which departures from normality were observed earlier, $qH(q)$ is linear up to a threshold q^*. From q^* on, the curve flattens to $qH(q) \approx 1$. (c) The plot of $H(q)$ vs. q suggests that $H(q) = \frac{1}{\alpha}$ for q smaller than $q^* \approx \alpha$, which is a signature of a Lévy α-stable distribution.



(a)



(b)



(c)

Source: The author (2023)

Having provided some empirical evidence, some analytical results regarding the asymptotic α-stable distribution of the Weierstrass random walk when $b > \sqrt{a}$ comes next. Let us express $\cos\left(k\lambda b^j\right)$ in terms of Mellin transforms (ZASLAVSKY, 2005), that is

$$\cos\left(k\lambda b^j\right) = \mathcal{M}^{-1}\{\mathcal{M}\{\cos\left(k\lambda b^j\right)\}\}, \tag{2.36}$$

where the direct Mellin transform $\mathcal{M}$ is defined as

$$\mathcal{M}\{f(k)\}(s) \equiv \int_0^\infty \mathrm{d}k\, f(k) k^{s-1} \tag{2.37}$$

and the inverse transform $\mathcal{M}^{-1}$ as

$$\mathcal{M}^{-1}\{\phi(s)\}(k) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{d}s\, \phi(s) k^{-s} \tag{2.38}$$

with real c in the fundamental strip of $\phi(s)$. It can be proved that $\mathcal{M}\{\cos k\}(s) = \Gamma(s)\cos\frac{\pi s}{2}$ in the fundamental strip $0 < \mathrm{Re}\{s\} < 1$, hence $\mathcal{M}\{\cos\left(k\lambda b^j\right)\}(s) = \left(\lambda b^j\right)^{-s}\Gamma(s)\cos\frac{\pi s}{2}$. Therefore, $\varphi_{\Delta X}(k)$ in (2.4) admits the representation

$$\varphi_{\Delta X}(k) = \left(1 - b^{-\alpha}\right)\sum_{j\geq 0} b^{-\alpha j}\int_{c-i\infty}^{c+i\infty} \mathrm{d}s\left(\lambda b^j\right)^{-s}\Gamma(s)\cos\tfrac{\pi s}{2} k^{-s} \tag{2.39}$$

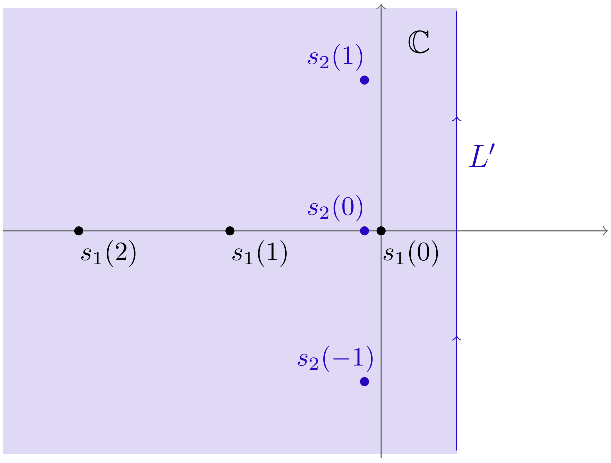
$$= \frac{1}{2\pi i}\left(1 - b^{-\alpha}\right)\int_{c-i\infty}^{c+i\infty} \mathrm{d}s\,\Gamma(s)\cos\tfrac{\pi s}{2}\left(\lambda|k|\right)^{-s}\sum_{j\geq 0} b^{-(\alpha+s)j}. \tag{2.40}$$

The geometric series converges for $|b^{-(\alpha+s)}| < 1$, which is valid for $0 < \alpha < 2$ and s in the region $\mathrm{Re}\{s\} > -\alpha$, leading to

$$\varphi_{\Delta X}(k) = \frac{1}{2\pi i}\left(1 - b^{-\alpha}\right)\int_{c-i\infty}^{c+i\infty} \mathrm{d}s\,\Gamma(s)\cos\tfrac{\pi s}{2}\left(\lambda|k|\right)^{-s}\frac{1}{1 - b^{-(\alpha+s)}}. \tag{2.41}$$

The function $\Gamma(s)\cos\frac{\pi s}{2}$ has poles at $s_1(j) = -2j$ with residue $\frac{(-1)^j}{(2j)!}$ for any non-negative integer j, whereas $\left(1 - b^{-(\alpha+s)}\right)^{-1}$ has poles at $b^{s_2} = b^{-\alpha}e^{i2\pi l}$ for any integer l with residue $\frac{1}{\log b}$, which corresponds to $s_2(l) = -\alpha + \frac{i2\pi l}{\log b}$, as seen in Figure 5.

Figure 5 – Poles of the Mellin transform of the characteristic function of the Weierstrass random walk with parameters $a = 2$ and $\alpha = 0.22$. The integration path L' encloses the region that contains every pole (shaded area).



Source: The author (2023)

Enclosing the poles in the halfplane $\mathrm{Re}\{s\} < 0$ with the countour L' in Figure 5 and applying the residue formula lead to

$$\varphi_{\Delta X}(k) = 1 + \left(1 - b^{-\alpha}\right) \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j)!} \frac{(\lambda|k|)^{2\alpha j}}{1 - b^{-\alpha} b^{2j}} + \lambda^\alpha |k|^\alpha \frac{(1 - b^{-\alpha})}{\log b} \sum_{l=-\infty}^{\infty} \Gamma\left(s_2(l)\right) \cos \frac{\pi s_2(l)}{2} \exp\left(-\frac{i2\pi l \log\left(\lambda|k|\right)}{\log b}\right), \tag{2.42}$$

where the first term corresponds to $s_k^{(1)}$ and the third term come from $s_l^{(2)}$. For the asymptotic distribution, the leading terms of $\log \varphi_{X_n}(k)$ corresponds to $k = 0$ and $l = 0$, namely

$$\log \varphi_{X_n}(k) \approx \log\left(1 + \lambda^\alpha |k|^\alpha \frac{(1 - b^{-\alpha})}{\log b} \Gamma\left(-\alpha\right) \cos \frac{\pi\alpha}{2}\right) \approx 1 - \lambda^\alpha |k|^\alpha \frac{(1 - b^{-\alpha})}{\log b} \left|\Gamma\left(-\alpha\right) \cos \frac{\pi\alpha}{2}\right|. \tag{2.43}$$

In this way, we have found that the asymptotic distribution corresponds to an α-stable Lévy distribution with stability index α. Observe that in the region of interest $0 < \alpha < 2$, $\Gamma\left(-\alpha\right) \cos \frac{\pi\alpha}{2} < 0$ because $\cos \frac{\pi\alpha}{2} < 0$ for $1 < \alpha < 2$ and $\Gamma\left(-\alpha\right) < 0$ for $0 < \alpha < 1$.

With regard to the continuous time and space limit $\tau \to 0$ and the scale parameter of a single jump $\lambda \to 0$, while keeping finite $\frac{\lambda^\alpha}{\tau} \to \delta$ and $t \equiv n\tau$, one can prove for the right hand side of (2.10) that (REICHL, 1998) (MONTROLL; WEST, 1979)

$$\lim_{\tau \to 0} \lim_{\lambda \to 0} \frac{\varphi_{\Delta X} - 1}{\tau} = -\alpha\delta \left|\Gamma\left(-\alpha\right) \cos \tfrac{\pi\alpha}{2}\right| |k|^\alpha. \tag{2.44}$$

One can see that from (2.43), where the factor $\frac{(1-b^{-\alpha})}{\log b}$ in (2.43) admits the following expansion with respect to b around 1

$$\frac{(1-b^{-\alpha})}{\log b} = \alpha - \frac{1}{2}(b-1)\alpha^2 + \frac{1}{12}(b-1)^2\alpha^2(2\alpha+3) + O\left((b-1)^3\right). \tag{2.45}$$

Hence, for $b = 1+\epsilon$ with small $\epsilon > 0$, (2.44) corresponds to (2.44).

One has from (2.44), with $D_\alpha \equiv \alpha\delta\left|\Gamma\left(-\alpha\right)\cos\frac{\pi\alpha}{2}\right|$, that (2.10) becomes

$$\frac{\partial\varphi_X\left(k,t\right)}{\partial t} = -D_\alpha|k|^\alpha\varphi_X(k,t). \tag{2.46}$$

The left-hand side of (2.46) is proportional to the *Riesz derivative* of the probability density function in the space coordinate, defined as $\frac{\partial^\alpha f_X(x,t)}{\partial x^\alpha} \equiv \mathcal{F}^{-1}\left[-|k|^\alpha\varphi_X(k,t)\right]$. Therefore, (2.46) corresponds to the one-dimensional fractional diffusion equation with Riesz derivative of order α

$$\frac{\partial f_X(x,t)}{\partial t} = D_\alpha\frac{\partial^\alpha f_X(x,t)}{\partial x^\alpha}. \tag{2.47}$$

In this way, a bridge is established between continuous-time random walks (CTRWs) subject to the GCLT and *anomalous* diffusion. It is relevant to disclose that anomalous diffusion and, in general, anomalous transport are clearly much more rich and complex subject than the previous result. For instance, CTRWs not only allow for a step length distribution but also for a distribution of the time elapsed between successive jumps (the *waiting time*) and both distributions determine the diffusion regime (METZLER; KLAFTER, 2000). For example, a heavy-tailed waiting time distribution with non-finite expectation value such as the probability density function $f_T(t) \sim t^{-(\xi+1)}$ with $\xi < 1$ and a step-length distribution with finite variance (a sort of "procrastinating" walker) result in a sub-diffusive regime in which $\langle x^2\rangle \sim t^\xi$, i.e. the mean squared displacement grows sub-linearly with time (METZLER; KLAFTER, 2000). Nevertheless, the present work is not focused on dynamical aspects, so the simplified example presented above fits our purpose here.

Although the work is focused on one-dimensional random walks, it is worth mentioning that Pólya (PóLYA, 1921) proved that, provided that the variance of the jump distribution is finite, random walkers in 1D and 2D are persistent, that is the probability of returning to the origin is equal to 1. However, that is not the case in 3D, let alone higher dimensions, where random walkers are *transient*, i.e. there is a finite probability that they will never return to the starting point, a result that has been humorously stated as "a drunk man will find his way home, but a drunk bird may get lost forever" [1]. On the other hand, if the variance of

[1] Quote attributed to Shizuo Kakutani

the jump distribution is not finite, as in isotropic Rayleigh-Pearson random walks in 2D and 3D with Weierstrass *step length* distribution (2.3), the random walk is transient if $0 < \alpha < 2$ (HUGHES; SHLESINGER; MONTROLL, 1981) and exhibits clustering in all scales above λ.

2.5 LÉVY FLIGHTS

One can argue that Lévy flights have already been introduced in the previous section, without being called as such. Let us consider the case in which a one-dimensional random "flyer" takes *instantaneous* jumps with step length sampled from a Lévy α-stable distribution with stability index $0 < \alpha < 2$, that is individual jumps with non-finite variance. As done in the previous sections, let us leave aside the waiting time distribution by considering $\delta(t-\tau)$, i.e. jumps take place at regular time τ. This finite (null) waiting time distribution implies that the stochastic process is a Markov process (CHECHKIN et al., 2008), that is the conditional probability of a path $P(x_1, \cdots x_{n-1}|x_n)$ given the current state x_n depends solely on the state x_{n-1} $P(x_1, \cdots x_{n-1}|x_n) = P(x_{n-1}|x_n)$ (REICHL, 1998) or, in informal terms, the system has no memory beyond the previous state. If one considered the "procrastinating" walker mentioned before, Markovian property would no longer be valid as procrastination induces long-range time correlations (the walker is "trapped" in the same state).

The instantaneous nature of jumps in Lévy flights is problematic for physical objects. In contrast, Lévy *walks* are physically feasible as time and space are coupled by means of the finite velocity of the walker, although the time evolution and the analytical treatment become more difficult. In any case, Lévy walks are not the main subject of this work and the closest contact with them will be when dealing with the mean path length traversed by a flyer before absorption, as it corresponds to the mean first passage time of a walker.

In either case, flight or walk, there is a major caveat regarding pure Lévy processes: the non-finite single jump variable conflicts with the finite size of systems. Therefore, real world feasible models must take into account the fact that the size of the system imposes an upper cut-off to the jump length. An alternative is to consider *truncated* jumps whose pdf is given by

$$f_{\Delta X}^{(TS)} = c\, f_{\Delta X}(\alpha, 0, \lambda, 0; x)\mathbb{1}_{\left(-\frac{L}{2}, -\frac{L}{2}\right)}(x), \tag{2.48}$$

where L is the size of the system and c is a normalising constant (MANTEGNA; STANLEY, 1994). It is clear that such cut-off implies finite variance, which in turn bring us back to the realm of CLT. Interestingly enough, the rate of convergence of X_n to a normal distribution can be "ultralow", depending on the stability index α and L, so a truncated flight would not be distinguishable from the pure flight provided that the number of jumps n (i.e. the time frame), yet large, was less than a crossover $n_x \sim L^\alpha$ above which the CLT regime becomes apparent. In other words, if the duration of the process is shorter than the crossover time, one cannot tell the difference between a pure and a truncated process.

3 MATHEMATICAL BACKGROUND

This chapter is intended to provide a brief presentation of the mathematical tools and results that constitute the foundations of this dissertation.

3.1 SPECIAL FUNCTIONS

3.1.1 Gamma function

The Gamma function is generally defined by the integral (3.1)

$$\Gamma(z) = \int_0^\infty \mathrm{d}s\, s^{z-1} e^{-s}, \tag{3.1}$$

which is convergent for any $z \in \mathbb{C}, \mathrm{Re}(z) > 0$. Note that, for $z = n$ with $n \in \mathbb{N}$, the iterated partial integration ($n - 1$ times) with $\mathrm{d}v = e^{-s}\mathrm{d}s$ and $u_1 = s^{z-1}$, $u_2 = s^{z-2}$ up to $u_{n-1} = s$ respectively leads to

$$\Gamma(n) = (n-1)!\,. \tag{3.2}$$

Moreover, it can be proved by partial integration with $u = s^z$ and $\mathrm{d}v = e^{-s}\,\mathrm{d}s$ that, in the domain of convergence of (3.1),

$$\Gamma(z+1) = z\Gamma(z). \tag{3.3}$$

Despite the restrictions for the convergence of (3.1), $\Gamma(z)$ can be extended to the complex half plane $\mathrm{Re}(z) < 0$ excluding non-positive integer z, by means of analytic (rather meromorphic) continuation (LANG, 1999) through the iterative application of (3.3). For that purpose, consider the connected sequence $[(f_0, D_0), (f_1, D_1), \cdots (f_n, D_n), \cdots]$ with $f_k = [(z)_k]^{-1} \int_0^\infty \mathrm{d}s\, s^{z+k-1} e^{-s}$ and $D_k = \{z \in \mathbb{C}, (\mathrm{Re}(z) > -k) \wedge (-\mathrm{Re}(z) \notin \mathbb{Z}^+)\}$. The symbol $(a)_n$ denotes the Pochhammer symbol (or *rising* factorial) defined as $(a)_n = a(a+1)\cdots(a+n-1)$ with $(a)_0 = 1$.

Another remarkable property of $\Gamma(z)$ is Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{3.4}$$

which is valid for any non-integer $z \in \mathbb{C}$.

Here follows a sketch of a proof of (3.4). Let us take $\int_0^\infty \int_0^\infty \mathrm{d}s\, \mathrm{d}t\, t^{z-1} e^{-s(t+1)}$ as the starting point. Changing t by $u \equiv st$ and integrating in u leads to $\int_0^\infty \mathrm{d}s\, \Gamma(z) s^{-z} e^{-s}$. Provided that $\mathrm{Re}(z) < 1$, one has that the latter expression is equivalent to $\Gamma(z)\Gamma(1-z)$. On the other hand, let us integrate with respect to s first, leading to $\int_0^\infty \mathrm{d}t\, t^{z-1}\,(t+1)^{-1}\,\left[-e^{-s(t+1)}\right]\Big|_0^\infty$. By integrating with respect to t, one has that $\int_0^\infty \mathrm{d}t\, t^{z-1}\,(t+1)^{-1} = \pi \csc \pi z$. The latter follows, for instance, from the Mellin transform of the function $(t+1)^{-1}$ (ERDÉLYI et al., 1954).

Taking into account that $\sin(\pi z) = (-1)^{n-1}\sin(-\pi z + n\pi)$ for any $n \in \mathbb{N}$, one has from (3.4) that

$$\Gamma(z)\Gamma(1-z) = (-1)^{n-1}\frac{\pi}{\sin \pi(n-z)} = (-1)^{n-1}\Gamma(n-z)\Gamma(n+1-z), \tag{3.5}$$

which can be cast into the form

$$\frac{\Gamma(z-n)}{\Gamma(1+z)} = (-1)^{n-1}\frac{\Gamma(-z)}{\Gamma(n+1-z)}. \tag{3.6}$$

Finally, as $\Gamma(z)$ generalises the factorial for non-integer arguments by virtue of (3.3), the binomial coefficient can be extended to complex arguments z, w as follows (DAVIS, 1972)

$$\binom{z}{w} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}. \tag{3.7}$$

3.1.2 Euler Beta function

The incomplete Beta function is defined in (3.8) where $u \in [0, 1]$. The integral is convergent provided that w, z are complex numbers with strictly positive real part

$$\mathrm{B}(w, z; u) = \int_0^u \mathrm{d}s\, s^{w-1}(1-s)^{z-1}. \tag{3.8}$$

Moreover, the incomplete Beta function admits the representation (3.9) in terms of the Gaussian hypergeometric function ${}_2F_1$ (see 3.1.3)

$$\mathrm{B}(w, z; u) = \frac{u^w}{w}\,{}_2F_1(w, 1-z; z+1; u). \tag{3.9}$$

The Euler Beta function $\mathrm{B}(w, z)$ is a particular case of (3.8) with $u = 1$. Despite the conditions for the convergence of (3.8), namely that $\mathrm{Re}(w) > 0$ and $\mathrm{Re}(z) > 0$, $\mathrm{B}(w, z)$ admits continuation by means of

$$\mathrm{B}(w, z) = \frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)} \tag{3.10}$$

and following a similar procedure as in 3.1.1. To proof (3.10), one can begin with $\Gamma(z)\Gamma(w) = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}q\, e^{-(t+q)} t^{w-1} q^{z-1}$. The change of variables $y = t+q$ and $r = \frac{q}{t}$ leads to $\Gamma(z)\Gamma(w) =$

$\int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}r \, \frac{y}{(r+1)^2} e^{-y} y^{w+z-2} \left(\frac{1}{r+1}\right)^{w-1} \left(\frac{r}{r+1}\right)^{z-1}$. With the substitution $s = \frac{1}{r+1}$ one has that $\mathrm{d}s = -\frac{1}{(r+1)^2}\,\mathrm{d}r$, hence $\Gamma(z)\Gamma(w) = \int_0^\infty \mathrm{d}y \, e^{-y} y^{w+z-1} \int_0^1 \mathrm{d}s \, s^{w-1} \left(1-s\right)^{z-1}$. The latter expression clearly corresponds to $\Gamma(w+z)\mathrm{B}(w,z)$.

Among the many relations that follow from (3.10) and (3.15), one has that both $\mathrm{B}(x,y)$ and $\mathrm{b}(x,y)$ is symmetric under the exchange of parameters

$$\mathrm{B}(w,z) = \mathrm{B}(z,w) \tag{3.11}$$

and the following recursion holds

$$\mathrm{B}(w+1,z) = \mathrm{B}(w,z)\frac{w}{w+z}. \tag{3.12}$$

Let $\mathrm{b}(w,z)$ denote the reciprocal Beta function defined as

$$\mathrm{b}(w,z) = \left[\mathrm{B}(w,z)\right]^{-1} = \frac{\Gamma(w+z)}{\Gamma(w)\Gamma(z)}. \tag{3.13}$$

With such definition, the generalisation of the binomial coefficients to any pair of complex numbers w and z is given by

$$\binom{w}{z} = \frac{\mathrm{b}(z+1, w-z+1)}{w+1}. \tag{3.14}$$

For example, if one of the parameters, say z, belongs to $\mathbb{N}$, one has the following representation in terms of the Pochhammer symbol

$$\mathrm{b}(w,n) = \frac{(w)_n}{(n-1)!}. \tag{3.15}$$

3.1.3 Gaussian hypergeometric function

The hypergeometric function ${}_2F_1(a,b;c;w)$ can be defined in terms of Gauss hypergeometric series (3.16) (OBERHETTINGER, 1972)

$${}_2F_1(a,b;c;w) = \sum_{\nu=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{w^\nu}{\nu!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{\nu=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{w^\nu}{\nu!}. \tag{3.16}$$

Recall from 3.1.1 that $(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)}$.

The series (3.16) is absolutely convergent when $\mathrm{Re}\{c-a-b\} > 0$, conditionally convergent when $-1 < \mathrm{Re}\{c-a-b\} \leq 0$ and divergent elsewhere. What is more important for the present work is that it becomes a finite sum whenever a or b is equal to a negative integer. Moreover, it is known that ${}_2F_1(a,b;c;w)$ is a particular solution to the hypergeometric differential equation (3.17) (OBERHETTINGER, 1972)

$$\left[w(1-w)\frac{\mathrm{d}^2}{\mathrm{d}w^2}+[c-(a+b+1)w]\frac{\mathrm{d}}{\mathrm{d}w}-ab\right]f(w)=0. \tag{3.17}$$

Among the myriad of properties and identities involving ${}_2F_1(a,b;c;w)$, a particular linear transformation formula will be of use later on. It states that

$$\begin{aligned}{}_2F_1(a,b;c;z)=&\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,1-c+b;1-a+b;z^{-1})\\&+\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,1-c+a;1-b+a;z^{-1})\end{aligned} \tag{3.18}$$

with $|\arg(-z)|<\pi$ (ERDÉLYI et al., 1953) (OBERHETTINGER, 1972). This comes from the integral representation

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{2\pi i\Gamma(a)\Gamma(b)}\int_{-i\infty}^{i\infty}\mathrm{d}s\,\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s. \tag{3.19}$$

3.2 A GENERAL OVERVIEW ON INTEGRAL EQUATIONS

3.2.1 Weighted L^2 spaces

Let $\rho(x)$ be a non-negative function on the interval $\Omega=(a,b)$, i.e. $\int_a^b \mathrm{d}x\,\rho(x)>0$, and define the inner product between two real-valued functions $f(x)$ and $g(x)$ on Ω with the *weight* function $\rho(x)$ as

$$\langle f|g\rangle_\rho=\int_a^b \mathrm{d}t\,f(t)\rho(t)g(t). \tag{3.20}$$

Let us consider the space $L^2_\rho[a,b]$ of equivalence classes f of real functions with (finite) norm induced by (3.20), namely

$$|f|_\rho=\langle f|f\rangle_\rho^{\frac{1}{2}}<\infty. \tag{3.21}$$

It is clear that the usual L^2 space is a particular case of $L^2_\rho[a,b]$ with uniform weight $\rho=1$. $L^2_\rho[a,b]$ are no more special than L^2 in the sense that $f\in L^2_\rho[a,b]$ if and only if $\sqrt{\rho}f\in L^2[a,b]$.

A very important fact about $L^2_\rho[a,b]$ is that every Cauchy sequence, i.e. a sequence of arbitrarily close elements such that for any $\epsilon>0$, there exists $N\in\mathbb{N}$ such that $\|f_n-f_m\|<\epsilon$ for every natural $m,n>N$, has a limit in $L^2_\rho[a,b]$. Hence, it is a *complete* normed space, i.e. a *Banach* space. Since it is also a vector space endowed with an inner product, it is also a *Hilbert space*. Paraphrasing (COSTIN, 2011), in $L^2_\rho[a,b]$ one can do linear algebra (it is a vector space), geometry (one has lengths and angles) and calculus (it is complete): *"and they are all combined when we write series expansions"*.

3.2.2 Sturm Liouville theory

The central idea of the theory of real second-order linear differential equations, the so called Sturm-Liouville theory, is synthesised in Theorem 3.1. For the present work, the key statement in Theorem 3.1 is that the sequence of eigenfunctions of the SL problem (3.22) and (3.23) forms an orthogonal basis of the space of functions $L^2_\rho(a,b)$.

Theorem 3.1. *Assuming $p', r, \rho \in C([a,b])$, and $p, \rho > 0$ on $[a,b]$, the Sturm-Liouville eigenvalue problem*

$$\hat{L}u + \lambda\rho u = \frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}}{\mathrm{d}x}\right]u + ru + \lambda\rho u = 0 \tag{3.22}$$

with boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) = 0, \qquad |\alpha_1| + |\alpha_2| > 0 \\ \beta_1 u(b) + \beta_2 u'(b) = 0, \qquad |\beta_1| + |\beta_2| > 0 \end{aligned} \tag{3.23}$$

has an infinite sequence of real eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ such that $\lambda_n \to \infty$.

To each eigenvalue λ_k corresponds a single eigenfunction ϕ_k, and the sequence of eigenfunctions $(\phi_k : n \in \mathbb{N}_0)$ forms an orthogonal basis of $L^2_\rho(a,b)$.

For further reference, see (ARFKEN; WEBER; HARRIS, 2013) or (AL-GWAIZ, 2008).

3.2.3 Fredholm integral equation theory

Given the functions $f(x)$ on $[a,b]$ and $k(x,x')$ on $[a,b]^2$, the integral equation of the form

$$\phi(x) = f(x) + \lambda \int_a^b \mathrm{d}x'\, k(x,x')\phi(x') \tag{3.24}$$

with unknown function $\phi(x)$ and scalar λ is called the Fredholm integral equation of the second kind with kernel $k(x,x')$. If the upper limit of the integral is the variable x itself, the resulting equation is called a Volterra equation of the second kind. It is arguably the most astonishing fact around (3.24) that every linear second-order ordinary differential with some boundary (initial) conditions can be cast into the form of a Fredholm (Volterra) integral equation of the second kind, whose kernel is the corresponding Green function of the problem (ARFKEN; WEBER; HARRIS, 2013). Nevertheless, the converse is not true since differential operators are essentially *local* whereas integral equations allow for non-local relations. Therefore, integral equations allow to tackle a broader range of problems, including diffusion and transport phenomena.

Let us cast (3.24) into a general form in terms of the linear operator $\hat{K}$ as follows

$$\hat{T}\phi(x) = \phi(x) - \lambda\hat{K}\phi(x) = f(x). \tag{3.25}$$

3.2.3.1 Preliminary definitions

Let X and Y be Banach spaces and let $A : X \to Y$ be a linear operator mapping X into Y. A is *bounded* if there is a constant $C > 0$ such that the norm of $\hat{A}x$ in Y is not greater than the norm of x amplified by C for any $x \in X$, i.e. $|\hat{A}x| \leq C|x|$. Provided boundness, one can define the norm of A as

$$\|\hat{A}\| \equiv \sup_{x\neq 0} \frac{|\hat{A}x|}{|x|} \leq C. \tag{3.26}$$

Let us consider for instance $X = L^2\left[\Omega\right]$ and integral operators of the form

$$\hat{K}\left[f\right] = \int\limits_{\Omega} \mathrm{d}\tau\, k(\tau, t) f(\tau) \tag{3.27}$$

with $k(\tau, t)$ in $L^2\left[\Omega^2\right]$. Such operators are clearly linear and it follows from Cauchy-Schwarz inequality that $|\hat{K}f| \leq |f||k|$, hence $\|\hat{K}\| \leq |k|$ and $\hat{K}$ is bounded (EIDELMAN; MILMAN; TSOLOMITIS, 2004). In fact, $\hat{K}$ is bounded even if the kernel is *weakly polar*, i.e. $k(\tau, t) = \frac{M(\tau,t)}{|\tau-t|^\alpha}$ with $0 < \alpha < 1$ and $M(\tau, t)$ a continuous function on Ω^2 (VLADIMIROV, 1984).

3.2.3.2 Neumann series

The Neumann series solution is the earliest successful approach to solving (3.25) in the context of potential theory, acknowledged by Fredholm himself (FREDHOLM, 1903). It is the result of successive approximations given by $\phi_{n+1} = f + \lambda\hat{K}\phi_n$ as $n \to \infty$, starting from $\phi_0 = f$ (KANWAL, 2013). The following theorem from (YOSHIDA, 1980) provides conditions under which $\hat{T}$ is invertible.

Theorem 3.2. *[C. Neumann] Let T be a bounded linear operator on a Banach space X (complete normed linear space) into X. Suppose that $\lambda\hat{K} \equiv 1 - \hat{T}$ is such that $\|\lambda\hat{K}\| < 1$. Then, $\hat{T}$ has a unique bounded linear inverse $\hat{T}^{-1}$ which is given by the Neumann series* (3.28)

$$\hat{T}^{-1}x = \left[\lim_{n\to\infty}\sum_{k=0}^{n}\left(\lambda\hat{K}\right)^k\right]x. \tag{3.28}$$

Proof: For any non-zero $x \in X$, one has that

$$\left|\left[\sum_{k=0}^{n}\left(\lambda\hat{K}\right)^k\right]x\right| \leq \sum_{k=0}^{n}\left|\left(\lambda\hat{K}\right)^k x\right| \leq \sum_{k=0}^{n}\left\|\left(\lambda\hat{K}\right)^k\right\|\left|x\right| \leq \sum_{k=0}^{n}\left\|\lambda\hat{K}\right\|^k\left|x\right|, \tag{3.29}$$

where the definition (3.26) of the norm of $\hat{K}$ was used.

Observe that $\sum_{k=0}^{n}\left\|\lambda\hat{K}\right\|^k|x| \leq \sum_{k=0}^{\infty}\left\|\lambda\hat{K}\right\|^k|x|$ for any $n \in \mathbb{N}$ and that the series $\sum_{k=0}^{\infty}\left\|\lambda\hat{K}\right\|^k$ is convergent as $\left\|\lambda\hat{K}\right\| < 1$ since it is a geometric series.

Because X is complete, every Cauchy sequence has limit in X. Hence, the operator $\lim_{n\to\infty}\sum_{k=0}^{n}\left(\lambda\hat{K}\right)^k$ is well-defined and bounded. If one applies $\lim_{n\to\infty}\sum_{k=0}^{n}\left(\lambda\hat{K}\right)^k$ to $\hat{T}x$, one gets that

$$\begin{aligned}\left[\lim_{n\to\infty}\sum_{k=0}^{n}\left(\lambda\hat{K}\right)^k\right]\left[1+(\hat{T}-1)\right]x &= \left[\lim_{n\to\infty}\left(\sum_{k=0}^{n}\left(\lambda\hat{K}\right)^k\right)\left(1-\lambda\hat{K}\right)\right]x \\ &= \left[\lim_{n\to\infty}\left(1-\left(\lambda\hat{K}\right)^{n+1}\right)\right]x = \mathbb{1}x. \end{aligned} \tag{3.30}$$

Note that the last step follows from the "shrinkage" of $\left(\lambda\hat{K}\right)^{n+1}$ in the limit $n \to \infty$. Therefore, $\sum_{k=0}^{\infty}\left(\lambda\hat{K}\right)^k$ is the left inverse of $\hat{T}$.

A similar procedure with $\hat{T}$ applied to $\sum_{k=0}^{\infty}\left(\lambda\hat{K}\right)^k$ proves that the former is the left inverse of the latter.

Hence $\hat{T}^{-1} = \sum_{k=0}^{\infty}\lambda\hat{K}$ and the solution of (3.25) is

$$\phi(x) = T^{-1}f(x) = \left[\sum_{k=0}^{\infty}\left(\lambda\hat{K}\right)^k\right]f(x) \tag{3.31}$$

Let us define the *resolvent* operator $\hat{R}(\lambda)$ corresponding to $\hat{K}$ as

$$\hat{R}_\lambda \equiv \left(\sum_{k=1}^{\infty}\left(\lambda\hat{K}\right)^k\right), \tag{3.32}$$

so that one can reformulate the solution (3.31) in terms of $\hat{R}_\lambda$ as

$$\phi(x) = f(x) + \hat{R}_\lambda\left[f\right](x). \tag{3.33}$$

It is worth to briefly discuss the contribution of Fredholm (FREDHOLM, 1903). Although Neumann series were already known before Fredholm, the radius of convergence is determined by the restriction $\|\lambda\hat{K}\| < 1$. Fredholm not only lifted such restriction, but also proved seminal theorems that are summarized below.

1. If λ is not an eigenvalue of (3.25), i.e. for such value the homogeneous problem (3.25) ($f(x) = 0$) has no other solution than the trivial , then the non-homogeneous equation is soluble for any $f(x)$. The general solution is given by

$$\phi(x) = f(x) + \int_a^b \mathrm{d}x' \, R_\lambda(x, x') f(x'), \tag{3.34}$$

which corresponds to (3.33).

2. There is a bijection between the eigenvalues of the homogeneous equation (3.25) and the eigenvalues of the *adjoint* equation

$$\psi(x) = \lambda^* \int_a^b \mathrm{d}x' \, k^*(x', x)\psi(x'). \tag{3.35}$$

Moreover, both equations have the same (finite) number of eigenfunctions belonging to a given eigenvalue λ_k.

3. If $\lambda = \lambda_k$ is an eigenvalue of (3.25), i.e. the homogeneous problem is soluble, then the non-homogeneous problem is, in general, insoluble. It is soluble if and only if the free term $f(x)$ satisfies the condition

$$\int \mathrm{d}x' \, f(x')\psi_k(x') = 0 \tag{3.36}$$

for all k. The set $\{\psi_k\}$ is the set of eigenfunctions of the adjoint equation.

4. The spectrum of a Fredholm integral equation is a discrete set.

It is relevant to mention that if the kernel of (3.25) is *weakly singular*, for instance if it has the form $\frac{M(x,x')}{|x-x'|^\alpha}$ with $0 \le \alpha < 1$ and continuous function $M(x, x')$, then (3.25) exhibits all the properties of a Fredholm equation (GAKHOV, 1966). However, does it also apply to strongly singular kernels? Regardless of the answer, we will come across a kernel of that sort in Chapter 4.

3.3 INVERSION FORMULAE METHOD

As mentioned above, the main objective of this dissertation is to explore the application of the spectral relationship method to one dimensional random flights with heavy-tailed step distribution. In that regard, closed-form expressions for the mean number of steps and mean flight path length under a 1D Riesz potential were derived in (BULDYREV et al., 2001b) by means of Sonin inversion formula. Despite the reference to (POPOV, 1982), such formula is not widely documented in the literature. Hence, for the sake of completeness and curiosity, the purpose of this section is to make an attempt to prove it.

Let us consider the following integral equation of the first kind on the finite interval $(0, L)$

$$\fint_0^L \mathrm{d}x \, \frac{(c_1 \operatorname{sgn}(x_0 - x) + c_2)^\nu}{2|x - x_0|^\alpha} \phi(x) = h(x_0), \tag{3.37}$$

where the symbol $\fint$ denotes the Cauchy Principal Value of the integral.

According to (BULDYREV et al., 2001b), (3.37) has a solution of the form

$$\phi_\beta(y) = K(\alpha, \beta) \, y^{\beta-1} \frac{\mathrm{d}}{\mathrm{d}y} \int_y^L \mathrm{d}s \, s^{1-\alpha} (s - y)^{\alpha-\beta} \frac{\mathrm{d}}{\mathrm{d}s} \int_0^s \mathrm{d}t \, t^{\alpha-\beta} (s - t)^{\beta-1} h(t), \tag{3.38}$$

where

$$K(\alpha, \beta) = -\frac{2}{\pi} \sin \pi\beta \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(1 - \beta + \alpha)} (c_1 + c_2)^{-\nu} \tag{3.39}$$

and the admissible values of the parameter β fulfill the condition

$$\sin \pi(\beta - \alpha) = \left(\frac{c_2 - c_1}{c_2 + c_1}\right)^\nu \sin \pi\beta. \tag{3.40}$$

Eq. (3.38) is referred as *Sonin inversion formula* and an attempt to derive alternative inversion formulae is presented below.

3.3.1 Abel integral equation

Let us begin with Abel integral equation

$$\int_a^s \mathrm{d}\tau \, \frac{\phi(\tau)}{(s - \tau)^\alpha} \equiv \mathcal{I}_{a+}^{1-\alpha} \phi(s) = f(s) \tag{3.41}$$

with absolutely continuous $f(x)$ in $[a, b]$, $0 < \alpha < 1$ and $x > a$. Under those conditions (SAMKO; KILBAS; MARICHEV, 1993), (3.41) admits a unique solution in $L_1(a, b)$ of the form

$$\phi(s) = \frac{1}{\mathrm{B}(\alpha, 1 - \alpha)} \frac{\mathrm{d}}{\mathrm{d}s} \int_a^s \mathrm{d}t \, (s - t)^{\alpha-1} f(t). \tag{3.42}$$

Note the similarity between (3.42) and the innermost integral in (3.38). A proof of (3.42) involves substituting s by t in (3.41), multiply both sides with the function $(s - t)^{\alpha-1}$ and integrate with respect to t on $[a, s]$ (SAMKO; KILBAS; MARICHEV, 1993). After a suitable transformation of the domain of the double integral on the left hand side, one has that $\int_a^s \mathrm{d}\tau \, \phi(\tau) \int_\tau^s \mathrm{d}t \, (s - t)^{\alpha-1} (t - \tau)^{-\alpha} = \int_a^s \mathrm{d}t \, (s - t)^{\alpha-1} f(t)$. The integral with respect to

t in the left hand side can be cast into the form (3.8) by means of the change of variable $u = \dfrac{t-\tau}{s-\tau}$, leading to the factor $\mathrm{B}(\alpha, 1-\alpha)$. The last step is to take derivatives on both sides with respect to s.

With the definition

$$\mathcal{D}_{a+}^{1-\alpha}[f] \equiv \frac{1}{\mathrm{B}(\alpha,1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}s} \int_a^s \mathrm{d}t \, \frac{f(t)}{(s-t)^{1-\alpha}}, \tag{3.43}$$

one can summarise (3.41) and (3.42) into

$$\mathcal{D}_{a+}^{1-\alpha} \left[\mathcal{I}_{a+}^{1-\alpha} \phi \right] = \phi, \tag{3.44}$$

which means that $\mathcal{D}_{a+}^{1-\alpha}$ is the left inverse of $\mathcal{I}_{a+}^{1-\alpha}$. The operator $\mathcal{I}_{a+}^{1-\alpha}$ corresponds to the *left fractional integral* up to a factor $[\Gamma(1-\alpha)]^{-1}$), while $\mathcal{D}_{a+}^{1-\alpha}$ is the *left Riemann-Liouville fractional derivative*, up to a factor $\Gamma(1-\alpha)$. Both operators $\mathcal{I}_{a+}^{1-\alpha}$ and $\mathcal{D}_{a+}^{1-\alpha}$ are of order $1-\alpha$ with $0 < \alpha < 1$.

The upper limit of the integral in (3.41) can be held fixed instead of the lower limit. In such case, one has that $\mathcal{I}_{b-}^{1-\alpha}\phi(s) = \int_s^b \mathrm{d}\tau \, \frac{\phi(\tau)}{(\tau-s)^{\alpha}} = f(s)$, whose solution is given by $\phi(s) = -\frac{1}{\mathrm{B}(\alpha,1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}s} \int_s^b \mathrm{d}t \, (t-s)^{\alpha-1} f(t) \equiv \mathcal{D}_{b-}^{1-\alpha}$. The operators $\mathcal{I}_{b-}^{1-\alpha}$ and $\mathcal{D}_{b-}^{1-\alpha}$ are the *right fractional integral* and *right Riemann-Liouville fractional derivative* of order $1-\alpha$ with $0 < \alpha < 1$, respectively.

3.3.2 Singular integral equation

Eq (3.37) can be cast into the form

$$\mathcal{I}_{a+}^{1-\alpha}[u\phi] + \mathcal{I}_{b-}^{1-\alpha}[v\phi] = \int_a^{x_0} \mathrm{d}x \, \frac{u(x)\phi(x)}{(x_0-x)^{\alpha}} + \int_{x_0}^b \mathrm{d}x \, \frac{v(x)\phi(x)}{(x-x_0)^{\alpha}} = h(x_0) \tag{3.45}$$

with constant coefficients $u = \frac{1}{2}(c_2+c_1)^{\nu}$ and $v = \frac{1}{2}(c_2-c_1)^{\nu}$. Nevertheless, the more general case $u(x)$ and $v(x)$ will be kept for the sake of generality.

A remarkable fact regarding the operators $I_{a+}^{1-\alpha}$ and $I_{b-}^{1-\alpha}$ is that the following relation among them holds provided that $0 < \alpha < 1$ (SAMKO; KILBAS; MARICHEV, 1993)

$$\mathcal{I}_{b-}^{1-\alpha}\phi = \mathcal{I}_{a+}^{1-\alpha}\left[\cos\pi(1-\alpha)\,\phi + \sin\pi(1-\alpha)\, r_a^{\alpha-1}\hat{S}\left[r_a^{1-\alpha}\phi\right]\right], \tag{3.46}$$

where $\hat{S}$ stands for the singular operator with Cauchy kernel

$$\hat{S}\varphi = \frac{1}{\pi} \fint_a^b \mathrm{d}t \, \frac{\varphi(t)}{t-x}. \tag{3.47}$$

and the function $r_a(x)$ represents the position in Ω relative to the lower endpoint a, i.e. $r_a(x) = x - a$. As a consequence of (3.46), (3.45) can be transformed into Abel equation (3.41) with an embedded *singular* integral equation, namely

$$\mathcal{I}^{1-\alpha}_{a+}\left[r_a^{\alpha-1}\left((u - v\cos\pi\alpha)r_a^{1-\alpha}\phi + \hat{S}\left[v\sin\pi\alpha\, r_a^{1-\alpha}\phi\right]\right)\right] = h(x_0). \tag{3.48}$$

Indeed, by inverting $\mathcal{I}^{1-\alpha}_{a+}$ in (3.48) with the aid of (3.44) and defining $\psi(x) = r_a^{1-\alpha}(x)\phi(x)$, one arrives at the singular integral equation

$$a_1(s)\psi(s) + \frac{1}{\pi}\int_a^b \mathrm{d}t\,\frac{a_2(t)\,\psi(t)}{t-s} = r_a^{1-\alpha}(s)\mathcal{D}^{1-\alpha}_{a+}\left[h(t)\right], \tag{3.49}$$

where $a_1(x) = u(x) - v(x)\cos\pi\alpha$ and $a_2(x) = v(x)\sin\pi\alpha$.

Observe that eq (3.49) is already in the form of an *adjoint to the dominant* singular integral equation

$$K^{o\prime}\psi(s) = a(s)\psi(s) - \frac{1}{i\pi}\int_L \mathrm{d}t\,\frac{b(t)\psi(t)}{t-s} = f(s), \tag{3.50}$$

where L denotes a given arc on the complex plane, $a(s) = a_1(s)$ and $b(t) = -ia_2(t)$.

The solution of (3.50) is closely related to the Riemann boundary value problem in complex analysis: to find a sectionally holomorphic function $\Omega(z)$, i.e. $\Omega^+(z)$ and $\Omega^-(z)$ are analytic in the respective regions S^+ and S^- of the complex plane separated by the arc L that satisfy a non-trivial linear boundary condition on L, namely

$$\Omega^+(t) = \frac{1}{G(t)}\Omega^-(t) + \tilde{g}(t) \tag{3.51}$$

with $G(t) = \dfrac{a(t) - b(t)}{a(t) + b(t)}$ and $\tilde{g}(t) = \dfrac{b(t)f(t)}{a(t) - b(t)}$, among other conditions such as the order of the pole of $\Omega(z)$ at infinity (if any) (MUSKHELISHVILI, 2013).

Having solved the boundary value problem (3.51), one can find $\psi(t)$ on the boundary L by means of Plemelj-Sokhotski relation

$$b(t)\psi(t) = \Omega^+(t) - \Omega^-(t). \tag{3.52}$$

The details of the solution will not be discussed here. It suffices to say that the general solution of (3.50) is (GAKHOV, 1966)(MUSKHELISHVILI, 2013)

$$\psi(s) = a(s)f(s) + \frac{1}{\pi i Z(s)}\int_L \mathrm{d}t\,\frac{Z(t)b(t)f(t)}{t-s} + \frac{1}{Z(s)}P_{-\varkappa-1}(s), \tag{3.53}$$

where $Z(s) = (s-b)^{-\varkappa}\exp\Gamma(s)$ with $\Gamma(s)$ given by $\Gamma(s) = \dfrac{1}{2\pi i}\displaystyle\int_L \mathrm{d}t\,\frac{\log G(t)}{t-s}$ and $P_{-\varkappa-1}$ is a polynomial of degree $-\varkappa - 1$.

The index $\varkappa$ of the dominant equation and it is closely related to the behaviour of phase of $G(s)$ along the contour L and determines the solvability of the adjoint equation (3.50): when $-\varkappa \geq 0$, the latter is solvable. Conversely, when $-\varkappa < 0$, the problem is solvable on condition that $\int\limits_L \mathrm{d}t\, Z(t)b(t)f(t)t^{j-1} = 0$ for all integer $j \in \{1, 2, \cdots, \varkappa\}$.

In addition, $-\varkappa$ also represents the number of linearly independent solutions of the boundary value problem (3.51) with $\widetilde{g}(t) = 0$, i.e. the homogeneous problem, which in turn corresponds to the homogeneous version of (3.50). In fact, by setting $f(s) = 0$ in (3.53), one can see that the homogeneous solutions of (3.50) correspond to the term $\dfrac{1}{Z(s)}P_{-\varkappa-1}(s)$. If $-\varkappa \leq 0$, the homogeneous solutions of (3.50) are trivially zero ($P_{-\varkappa-1}(s) \equiv 0$).

Back to (3.49), one has that

$$G(t)^{-1} = \frac{a(t) + b(t)}{a(t) - b(t)} = \frac{u(t) - v(t)e^{i\pi\alpha}}{u(t) - v(t)e^{-i\pi\alpha}} = re^{i\theta}. \tag{3.54}$$

It is clear that the norm of the numerator and the denominator in (3.54) are the same, hence $r = 1$. In consequence, one has that $u(1 - e^{i\theta}) = v(e^{i\pi\alpha} - e^{i(\theta-\pi\alpha)})$, which is equivalent to

$$\frac{u}{v} = \frac{\left(e^{i\left(\frac{\theta}{2}-\pi\alpha\right)} - e^{-i\left(\frac{\theta}{2}-\pi\alpha\right)}\right)e^{i\frac{\theta}{2}}}{\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)e^{i\frac{\theta}{2}}} = \frac{\sin\pi\left(\frac{\theta}{2\pi} - \alpha\right)}{\sin\frac{\theta}{2}}. \tag{3.55}$$

Recalling that for the problem at hand $u = \frac{1}{2}(c_2 + c_1)^\nu$, $v = \frac{1}{2}(c_2 - c_1)^\nu$ and setting $\beta' \equiv \frac{\theta}{2\pi}$, one has that

$$\sin\pi\left(\beta' - \alpha\right) = \left(\frac{c_2 + c_1}{c_2 - c_1}\right)^\nu \sin\pi\beta', \tag{3.56}$$

which bears resemblance to the condition (3.40) on the admissible values of the parameter β, up to the sign of c_1. Also recall that the open contour L is the segment on the real line from a to b. Therefore, with regard to $\Gamma(s)$ on (a, b) one has that

$$\Gamma(s) = \frac{1}{2\pi i}\int\limits_L \mathrm{d}t\,\frac{-\log\widetilde{G}(t)}{t-s} = -\frac{i\theta}{2\pi i}\int_a^b \mathrm{d}t\,\frac{1}{t-s} = -\frac{\theta}{2\pi}\log\left(\frac{b-s}{s-a}\right), \tag{3.57}$$

which in turn leads to the *canonical function of the problem* $Z(s)$ given by

$$Z(s) = (s-b)^{-\varkappa}\exp\Gamma(s) = (-1)^\varkappa(b-s)^{-\varkappa}\left(\frac{s-a}{b-s}\right)^{\beta'} = (-1)^\varkappa\frac{r_a^{\beta'}(s)}{r_b^{\beta'+\varkappa}(s)}, \tag{3.58}$$

where $r_b(s) \equiv b - s$ and $r_a(s) \equiv s - a$. Therefore, by inserting (3.58) into (3.53), one has that

$$\psi(s) = (u - v\cos\pi\alpha)f(s) - \frac{v\sin\pi\alpha}{\pi}\frac{r_b^{\beta'+\varkappa}(s)}{r_a^{\beta'}(s)}\int_a^b \mathrm{d}t\,\frac{r_a^{\beta'}(t)}{r_b^{\beta'+\varkappa}(t)}\frac{f(t)}{t-s} + \frac{r_b^{\beta'+\varkappa}(s)}{r_a^{\beta'}(s)}P_{-\varkappa-1}(s). \tag{3.59}$$

Taking into account that $f(s) = r_a^{1-\alpha}(s)\mathcal{D}_{a+}^{1-\alpha}[h]$ and $\phi(s) = r_a^{\alpha-1}(s)\psi(s)$ that the solution $\phi(s)$ in (3.49), one has that

$$\phi(s) = (u - v\cos\pi\alpha)\mathcal{D}_{a+}^{1-\alpha}[h] - \frac{v\sin\pi\alpha}{\pi}\frac{r_b^{\beta'+\varkappa}(s)}{r_a^{\beta'-\alpha+1}(s)}\fint_a^b \mathrm{d}t\,\frac{r_a^{\beta'-\alpha+1}(t)}{r_b^{\beta'+\varkappa}(t)}\frac{\mathcal{D}_{a+}^{1-\alpha}[h]}{t-s} + \frac{r_b^{\beta'+\varkappa}(s)}{r_a^{\beta'-\alpha+1}(s)}P_{-\varkappa-1}(s). \quad (3.60)$$

An attempt to apply (3.60) to asymptotic Lévy flights is left to Chapter 4.

3.4 CLASSICAL JACOBI POLYNOMIALS

Although applications of Jacobi polynomials are scarce in Physics, standard orthogonal polynomials widely used in the field are special cases of Jacobi polynomials. For instance, Legendre polynomials, the azimuthally symmetric eigenfunctions of the Laplace operator on the 2-sphere correspond to Jacobi polynomials with parameters $(\gamma, \delta) = (0, 0)$. In fact, the Newton-Coulomb potential in $\mathbb{R}^n$ admit expansion in ultraspherical polynomials that are also particular cases of Jacobi polynomials.

A brief introduction of classical Jacobi polynomials and the most relevant properties is presented below.

3.4.1 Definition

Let us state the relations and properties concerning the Jacobi polynomials $P_m^{(\gamma,\delta)}$ that will be used throughout this report. Hereafter, we will be only interested in real $\gamma, \delta > -1$, i.e. the classical (orthogonal) Jacobi polynomials defined in the interval $[-1, 1]$. For abbreviation, $P_m^{(\gamma)}$ stands for $P_m^{(\gamma,\gamma)}$.

Let us define $P_n^{(\gamma,\delta)}(t)$ on $[-1, 1]$ as a particular solution to the following second-order linear homogeneous ordinary differential equation (SZEGŐ, 1939)

$$\left\{(1-t^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2} - [\gamma - \delta + (\gamma+\delta+2)t]\frac{\mathrm{d}}{\mathrm{d}t} + n(n+\gamma+\delta+1)\right\}P_n^{(\gamma,\delta)} = 0, \quad (3.61)$$

which is equivalent to the Sturm-Liouville form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t)^{\gamma+1}(1+t)^{\delta+1}\frac{\mathrm{d}}{\mathrm{d}t}\right]P_n^{(\gamma,\delta)} + n(n+\gamma+\delta+1)(1-t)^{\gamma}(1+t)^{\delta}P_n^{(\gamma,\delta)} = 0. \quad (3.62)$$

Note, by comparing (3.62) to (3.22), that its eigenvalues are $\lambda_n = n(n+\gamma+\delta+1)$ and the weight function ρ is given by $\rho^{(\gamma,\delta)} \equiv (1-t)^{\gamma}(1+t)^{\delta}$.

Let also define $\widetilde{P}_n^{(\gamma,\delta)} \equiv (1-t)^\gamma(1+t)^\delta P_n^{(\gamma,\delta)}$. Taking into account that

$$(1-t^2)\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{P}_n^{(\gamma,\delta)} = (1-t)^{\gamma+1}(1+t)^{\delta+1}\frac{\mathrm{d}}{\mathrm{d}t}P_n^{(\gamma,\delta)} - [\gamma-\delta+(\gamma+\delta)t]\,\widetilde{P}_n^{(\gamma,\delta)}, \tag{3.63}$$

one has from (3.61) that $\widetilde{P}_n^{(\gamma,\delta)}$ is a solution to the second-order linear differential equation

$$\left\{(1-t^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + [\gamma-\delta+(\gamma+\delta-2)t]\,\frac{\mathrm{d}}{\mathrm{d}t} + (n+1)(n+\gamma+\delta)\right\}\widetilde{P}_n^{(\gamma,\delta)} = 0, \tag{3.64}$$

equivalent to the SL form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(1-t^2)}{(1-t)^\gamma(1+t)^\delta}\frac{\mathrm{d}}{\mathrm{d}t}\right]\widetilde{P}_n^{(\gamma,\delta)} + (n+1)(n+\gamma+\delta)\frac{1}{(1-t)^\gamma(1+t)^\delta}\widetilde{P}_n^{(\gamma,\delta)} = 0. \tag{3.65}$$

The comparison of (3.65) and (3.22) indicates that the eigenvalues of the former are $\lambda_n = (n+1)(n+\gamma+\delta)$ and the weight function $\widetilde{\rho}$ is the reciprocal of ρ.

Figure 6 shows how the value of γ and δ modulate the behaviour of $\widetilde{P}_n^{(\gamma,\delta)}$ in the vicinity of the right an left boundary, respectively. Negative values produce a singularity at the boundary, whereas positive values produce a node.

The most important result follows from Theorem 3.1. It implies that $\{P_n^{(\gamma,\delta)}, n \in \mathbb{N}_0\}$ and $\{\widetilde{P}_n^{(\gamma,\delta)}, n \in \mathbb{N}_0\}$ are orthogonal bases that span $L^2_\rho[-1,1]$ and $L^2_{\widetilde{\rho}}[-1,1]$ respectively, provided that $\gamma,\delta > -1$. This restriction ensures that the weight function is integrable.

For $\{P_n^{(\gamma,\delta)}, n \in \mathbb{N}_0\}$, the orthogonality relation is given by (3.66) (HOCHSTRASSER, 1972)(SZEGŐ, 1939).

$$\left\langle P_m^{(\gamma,\delta)}\middle|P_n^{(\gamma,\delta)}\right\rangle_\rho = \int_{-1}^{1}\mathrm{d}t\,P_m^{(\gamma,\delta)}(t)\rho^{(\gamma,\delta)}(t)P_n^{(\gamma,\delta)}(t) = h_n^{(\gamma,\delta)}\delta_{m,n} \tag{3.66}$$

with the factor $h_n^{(\gamma,\delta)}$ defined as

$$h_n^{(\gamma,\delta)} \equiv \frac{2^{\gamma+\delta+1}}{2n+\gamma+\delta+1}\frac{\Gamma(n+\gamma+1)\Gamma(n+\delta+1)}{n!\,\Gamma(n+\gamma+\delta+1)}. \tag{3.67}$$

3.4.2 Recursion relations

It is known that orthogonal polynomials obey Rodrigues-type formulae provided certain conditions on the weight function of the related Sturm-Liouville problem, namely that the weight function ρ satisfies Pearson differential equation $[\sigma(x)\rho(x)]' = \tau(x)\rho(x)$ with polynomials $\sigma(x)$ and $\tau(x)$ of degree 2 at most and exactly 1 and boundary conditions $\sigma(x)\rho(x)|_\partial = 0$ (NODARSE, 2006), which are only satisfied by classical polynomials, i.e. Hermite, Laguerre, Bessel and Jacobi families. For the latter, one has that

$$(1-t)^\gamma(1+t)^\delta P_n^{(\gamma,\delta)}(t) = \frac{(-1)^n}{n!\,2^n}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[(1-t)^{n+\gamma}(1+t)^{n+\delta}\right], \tag{3.68}$$

which is valid for arbitrary γ and δ (SZEGŐ, 1939). From (3.68), one can prove the following property

$$P_n^{(\gamma,\delta)}(-t) = (-1)^n P_n^{(\delta,\gamma)}(t). \tag{3.69}$$

It also follows from (3.68) that the application of first-order differential operator on any weighted Jacobi polynomial $\widetilde{P}_n^{(\gamma,\delta)}(t)$ shifts its index up and its parameters down both by one, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{P}_n^{(\gamma,\delta)}(t) = -2(n+1)\widetilde{P}_{n+1}^{(\gamma-1,\delta-1)}(t). \tag{3.70}$$

At the same time, while acting on any Jacobi polynomial $\widetilde{P}_n^{(\gamma,\delta)}(t)$, it produces the opposite effect: the index is shifted down by one while the parameters are shifted up by one, explicitly

$$\frac{\mathrm{d}}{\mathrm{d}t}P_n^{(\gamma,\delta)}(t) = \frac{1}{2}\left(n+\gamma+\delta+1\right)P_{n-1}^{(\gamma+1,\delta+1)}(t). \tag{3.71}$$

Note that (3.71) follows from (3.68) by Leibniz product rule.

3.4.3 Notable identities

For $\mu > 0$ and $x > -1$, one has the following result from (ASKEY; FITCH, 1969)

$$\int_x^1 \mathrm{d}t\,(1-t)^\gamma\,P_n^{(\gamma,\delta)}(t)\,(t-x)^{\mu-1} = \frac{\Gamma(\gamma+1)\,\Gamma(\mu)}{\Gamma(\gamma+\mu+1)}\frac{P_n^{(\gamma,\delta)}(1)}{P_n^{(\gamma+\mu,\delta-\mu)}(1)}\,(1-x)^{\gamma+\mu}\,P_n^{(\gamma+\mu,\delta-\mu)}(x). \tag{3.72}$$

By taking $x \to -1$ and $\mu = \delta+\eta+1$, provided that $\eta > -\delta-1$, (3.72) leads to

$$\left\langle P_n^{(\gamma,\delta)}\middle|(1+t)^\eta\right\rangle = (-1)^n\frac{2^{\gamma+\delta+\eta+1}}{n!}\frac{\Gamma(\gamma+n+1)\,\Gamma(\delta+\eta+1)\,\Gamma(n-\eta)}{\Gamma(\gamma+\delta+\eta+n+2)\,\Gamma(-\eta)} \tag{3.73}$$

$$= \frac{2^{\gamma+\delta+\eta+1}}{n!}\frac{\Gamma(\gamma+n+1)\,\Gamma(\delta+\eta+1)\,\Gamma(\eta+1)}{\Gamma(\gamma+\delta+\eta+n+2)\,\Gamma(\eta-n+1)}, \tag{3.74}$$

where (3.73) follows from the Gamma function reflection formula (3.6).

3.4.4 Explicit formulae

Let us perform the change of variable $\frac{1-t}{2} \equiv w$ in eq. (3.61). Thus one has that

$$\left\{w(1-w)\frac{\mathrm{d}^2}{\mathrm{d}w^2} + [\gamma+1-(\gamma+\delta+2)w]\frac{\mathrm{d}}{\mathrm{d}w} + n(n+\gamma+\delta+1)\right\}P_n^{(\gamma,\delta)}(1-2w) = 0, \tag{3.75}$$

which is a particular case of the Gaussian hypergeometric equation (3.17) with parameters $a=-n$, $b=n+\gamma+\delta+1$ and $c=\gamma+1$. Therefore, the representation

$$P_n^{(\gamma,\delta)}(t) = \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)\Gamma(\gamma+1)}\,{}_2F_1\left(-n, n+\gamma+\delta+1; \gamma+1; \frac{1-t}{2}\right) \tag{3.76}$$

is valid. The identity ${}_2F_1\left(a, b; c; z\right) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$ (OBERHETTINGER, 1972) leads to

$$P_n^{(\gamma,\delta)}(t) = \left(\frac{1+t}{2}\right)^n \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)\Gamma(\gamma+1)}\, {}_2F_1\left(-n, -n-\delta; \gamma+1; \frac{t-1}{t+1}\right), \tag{3.77}$$

which takes the explicit form

$$P_n^{(\gamma,\delta)}(t) = \frac{1}{2^n}\sum_{\nu=0}^{n}\binom{n+\gamma}{n-\nu}\binom{n+\delta}{\nu}(t-1)^\nu(t+1)^{n-\nu} \tag{3.78}$$

$$= \frac{1}{2^n}\sum_{\nu=0}^{n}\frac{\mathrm{b}(n-\nu+1,\nu+\gamma+1)}{n+\gamma+1}\frac{\mathrm{b}(\nu+1,n-\nu+\delta+1)}{n+\delta+1}(t-1)^\nu(t+1)^{n-\nu}, \tag{3.79}$$

where the latter follows from (3.7). Also observe that

$$P_n^{(\gamma,\delta)}(1) = \frac{\mathrm{b}(n+1,\gamma+1)}{n+\gamma+1} = \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)\Gamma(\gamma+1)} = \frac{1}{n!}\left(\gamma+1\right)_n. \tag{3.80}$$

For illustration purposes, the explicit formulae for the Jacobi polynomials of the lowest order are presented below and the corresponding plots of $P_n^{(\gamma)}$ and $\widetilde{P}_n^{(\gamma)}$ are presented in Figure 6 for selected values of the parameter γ. Further details, representations and interesting properties, albeit beyond the scope of this dissertation, can be found in (SZEGŐ, 1939).

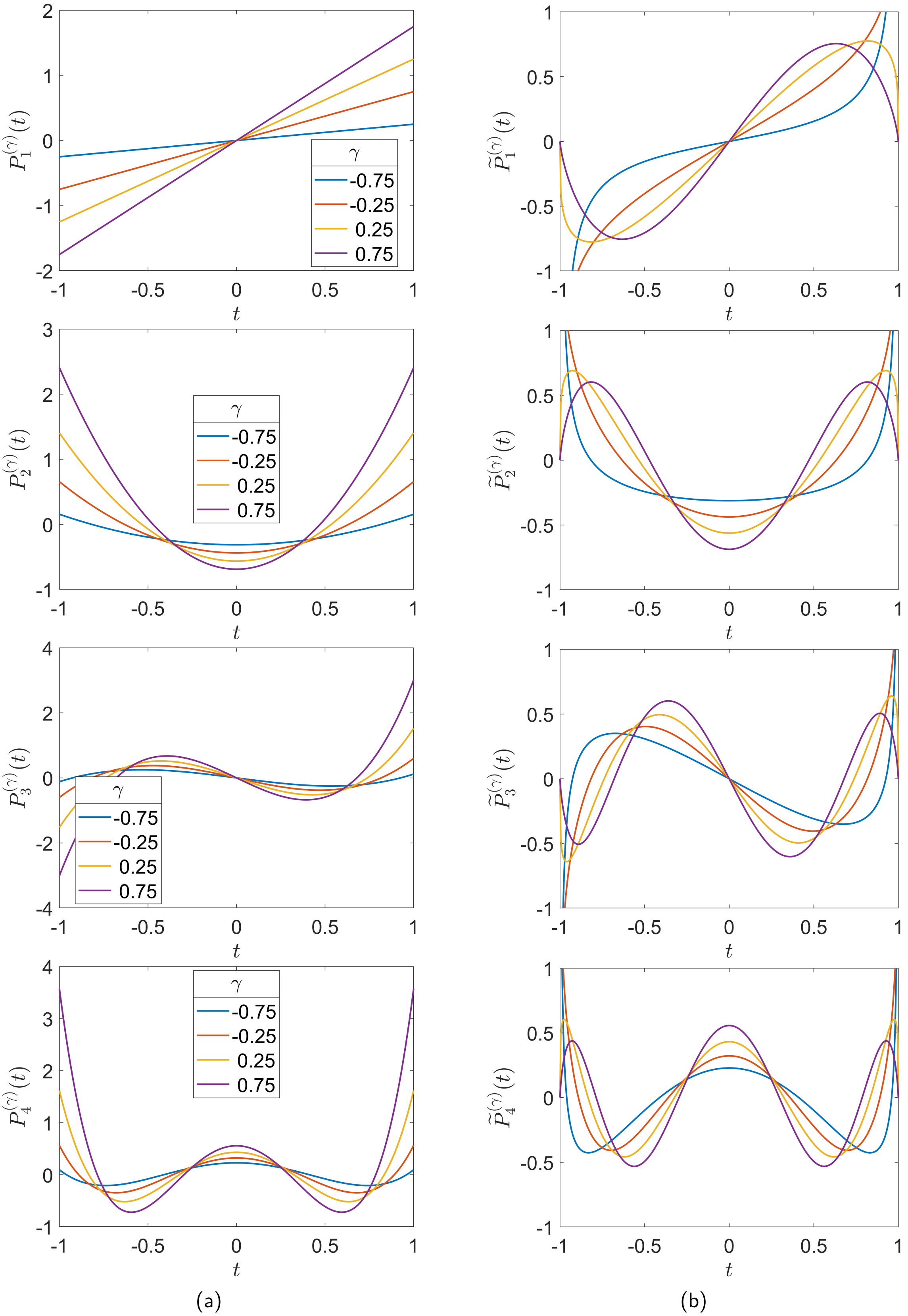
$$P_0^{(\gamma,\delta)}(t) = 1 \tag{3.81}$$

$$P_1^{(\gamma,\delta)}(t) = \frac{1}{2^1}\left[(\gamma-\delta)+(\gamma+\delta+2)t\right] \tag{3.82}$$

$$P_2^{(\gamma,\delta)}(t) = \frac{1}{2^2}\left[\frac{1}{2!}\Big((\gamma-\delta)^2-(\gamma+\delta+2)-2\Big) + (\gamma-\delta)(\gamma+\delta+3)\,t + \frac{1}{2!}(\gamma+\delta+3)_2\,t^2\right] \tag{3.83}$$

$$P_3^{(\gamma,\delta)}(t) = \frac{1}{2^3}\left[\frac{1}{3!}(\gamma-\delta)\Big((\gamma-\delta)^2-3(\gamma+\delta+4)-4\Big) + \frac{1}{2!}\Big((\gamma-\delta)^2-(\gamma+\delta+3)-3\Big)(\gamma+\delta+4)\,t + \frac{1}{2!}(\gamma-\delta)(\gamma+\delta+4)_2\,t^2 + \frac{1}{3!}(\gamma+\delta+4)_3\,t^3\right] \tag{3.84}$$

Figure 6 – Plots of Jacobi polynomials and weighted Jacobi polynomials. (a) Jacobi polynomials (JPs) of order 1, 2, 3 and 4 for selected values of $\gamma \in \{-0.75, -0.25, 0.25, 0.75\}$ with $\gamma = \delta$ and (b) corresponding weighted Jacobi polynomials (wJPs). For wJPs, the parameters γ and δ modulate the weight assigned to the right and left boundary respectively with negative values resulting in large amplitude close to the boundaries. Due to the restriction on the parameters, JP and wJP of odd (even) order exhibit odd (even) parity. The number of zeroes, excluding the boundaries, equals the order of the function.



Source: The author (2023)

3.5 SPECTRAL RELATIONS

The method of *orthogonal polynomials* is an approach developed in (POPOV, 1963) to solve integral equations of the first kind $\hat{K}\,[G]\,(t) \equiv \int_{[a,b]} \mathrm{d}\tau\, K(t,\tau)G(\tau) = g(t)$ i.e., given the function $g(t)$ on $[a,b]$ and the kernel $K(t,\tau) = K_0(t,\tau) + K_1(t,\tau)$, provided that a suitable *spectral relationship* exists for, let us say, $\hat{K}_0$ and $\hat{K}_1$ is regular (PODLUBNY, 1999). A *spectral relationship* exists for $K_0(t,\tau)$ if there are two families of *orthogonal* polynomials, $\{p_m(t)\}$ and $\{\pi_m(t)\}$, and scalars $\{\lambda_m\}$ such that

$$\hat{K}_0[\phi\, p_m] = \lambda_m \psi\, \pi_m, \tag{3.85}$$

with functions ϕ and ψ such that the weight function ρ under which any pair of distinct elements in $\{p_m(t)\}$ are orthogonal, is given by $\rho = \phi\widetilde{\psi}$. Likewise, one has functions $\widetilde{\phi}$ and ψ such that $\widetilde{\rho} = \widetilde{\phi}\psi$, where $\widetilde{\rho}$ is the weight function under which any pair of distinct elements in $\{\pi_m(t)\}$ are orthogonal, that is

$$\langle p_n | p_m \rangle_{\phi\widetilde{\psi}} = \int\limits_{[a,b]} \mathrm{d}t\; p_n(t)\widetilde{\psi}(t)\phi(t)\, p_m(t) = \delta_{n\,m} \qquad \text{and} \tag{3.86}$$

$$\langle \pi_n | \pi_m \rangle_{\widetilde{\phi}\psi} = \int\limits_{[a,b]} \mathrm{d}t\; \pi_n(t)\widetilde{\phi}(t)\psi(t)\, \pi_m(t) = \delta_{n\,m}. \tag{3.87}$$

Let $G(t) = \phi(t)\sum_{m>0} \bar{G}_m\, p_m(t)$ be the solution ansatz. Plugging it into the integral equation along with the spectral relation (3.85) and taking the inner product with $\pi_n\,\widetilde{\phi}$ on both sides leads to a system of linear algebraic equations (3.88) to be solved for $\bar{G}_m$

$$\begin{aligned}
\hat{K}[G](t) &= g(t) \\
\sum_{m>0} \bar{G}_m \left\{ \left\langle \pi_n\,\widetilde{\phi} \middle| \hat{K}_0[\phi\, p_m] \right\rangle + \left\langle \pi_n\,\widetilde{\phi} \middle| \hat{K}_1[\phi\, p_m] \right\rangle \right\} &= \left\langle \pi_n\,\widetilde{\phi} \middle| g \right\rangle \\
\sum_{m>0} \bar{G}_m \left\{ \lambda_m \langle \pi_n | \pi_m \rangle_{\widetilde{\phi}\psi} + \left\langle \pi_n\,\widetilde{\phi} \middle| \hat{K}_1[\phi\, p_m] \right\rangle \right\} &= \left\langle \pi_n\,\widetilde{\phi} \middle| g \right\rangle \\
\sum_{m>0} (\lambda_m \delta_{n\,m} + \kappa_{n\,m})\, \bar{G}_m &= \left\langle \pi_n\,\widetilde{\phi} \middle| g \right\rangle \\
K_{n\,m}\bar{G}_m &= \widetilde{g}_n.
\end{aligned} \tag{3.88}$$

If $\hat{K} = \hat{K}_0$, then $K_{n\,m}$ would have no off-diagonal terms and the solution to (3.88) would simply be $\bar{G}_n = \lambda_n^{-1}\widetilde{g}_n$.

3.5.1 Riesz potential

For the present work, the key spectral relation involving the Riesz potential and the classical Jacobi polynomials is proven below using an approach similar to the one used in (ARUTIUNIAN, 1959) to find the solution of the plane contact problem in nonlinear creep theory, i.e. the distribution of the pressure intensity along the contact between compressed bodies as a function of time. Nevertheless, the result in (ARUTIUNIAN, 1959) is restricted to the integral

$$\fint_{-a}^{a} \mathrm{d}s \sin\tfrac{\pi\mu}{2} \frac{(a^2-s^2)^{-\frac{\mu}{2}}}{\pi|s-x|^{1-\mu}} = 1 \tag{3.89}$$

with $0<\mu<1$. The proof presented below is original, as far as the author is aware.

Theorem 3.3. *If* $0<\eta<1$ *and* $k\in\mathbb{N}$*, then for* $\tau\in(-1,1)$ *it holds that*

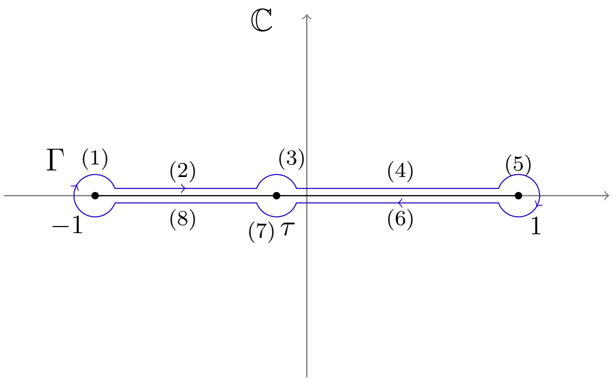
$$\fint_{-1}^{1} \mathrm{d}t\, \frac{\operatorname{sgn}(\tau-t)}{|\tau-t|^{\eta}} \widetilde{P}_k^{(\frac{\eta}{2}-1)}(t) = -\frac{\pi}{2\,k!\,\sin\frac{\eta\pi}{2}} \frac{\Gamma(k+\eta)}{\Gamma(\eta)} P_{k-1}^{(\frac{\eta}{2})}(\tau). \tag{3.90}$$

Proof. Let us consider the function $f_{k,\frac{\eta}{2}-1}(z)$ defined on the extended complex plane as

$$f_{k,\frac{\eta}{2}-1}(z) = \frac{(z-1)^{\frac{\eta}{2}-1}(z+1)^{\frac{\eta}{2}-1}}{(z-\tau)^{\eta}} P_k^{(\frac{\eta}{2}-1)}(z) \tag{3.91}$$

with branch cut along $[-1,1]$ and the contour Γ depicted in Figure 7 around the branch cut.

Figure 7 – Contour Γ around the branch cut $[-1,1]$ of $f_{k,\frac{\eta}{2}-1}(z)$.



Source: The author (2023)

Let us consider the integral of $f_{k,\frac{\eta}{2}-1}(z)$ along Γ, $\int\limits_{\Gamma} f_{k,\frac{\eta}{2}-1}(z)\,\mathrm{d}z$. The arcs (1) and (5). For instance, let us take (1): by having $z+1=\epsilon e^{i\theta}$, $\int_{(1)} \mathrm{d}z\, f_{k,\frac{\eta}{2}-1}(z) \sim \int_{(1)} \mathrm{d}z\,(z+1)^{\frac{\eta}{2}-1} = -i\epsilon^{\frac{\eta}{2}} \int_0^{2\pi} \mathrm{d}\theta\, e^{i\theta(\frac{\eta}{2}-1)} \xrightarrow{\epsilon\to 0} 0$. The arcs (3) and (7) do not contribute to the contour integral either. Take (3), for example, with $z-\tau=\epsilon e^{i\theta}$: $\int_{(3)} \mathrm{d}z\, f_{k,\frac{\eta}{2}-1}(z) \sim \int_{(3)} \mathrm{d}z\,(z-\tau)^{-\eta} = -i\epsilon^{1-\eta}\int_0^{\pi} \mathrm{d}\theta\, e^{-i\theta\eta} \xrightarrow{\epsilon\to 0} 0$, provided that $0<\eta<1$.

As for the remaining paths, note that winding counterclockwise around the branch cut produces a phase shift in $f_{k,\frac{\eta}{2}-1}(z)$ equal to $\pi\eta$ for $-1 < \mathrm{Re}\{z\} < \tau$, i.e. sweeping from segment (2) towards (8) in Figure 7, and equal to $-\pi\eta$ for $\tau < \mathrm{Re}\{z\} < 1$, i.e. from segment (4) towards (6). Hence

$$\int\limits_{\Gamma} f_{k,\frac{\eta}{2}-1}(z)\,\mathrm{d}z = \lim_{\epsilon\to 0^+} \Big[\int\limits_{(2)} \mathrm{d}t\,(1-e^{i\pi\eta}) f(t+i\epsilon) + \int\limits_{(4)} \mathrm{d}t\,(1-e^{-i\pi\eta}) f(t+i\epsilon) \Big]\,, \tag{3.92}$$

which in turn leads to

$$\int\limits_{\Gamma} f_{k,\frac{\eta}{2}-1}(z)\,\mathrm{d}z = 2i\sin\left(\pi\tfrac{\eta}{2}\right) ⨍_{-1}^{1} \mathrm{d}t\, \frac{\mathrm{sgn}(\tau-t)}{|\tau-t|^{\eta}} \widetilde{P}_k^{(\frac{\eta}{2}-1)}(t) \tag{3.93}$$

On the other hand, note that Γ encloses the singularity of $f(z)$ at the point at infinity. Therefore, by Cauchy's residue theorem one has that

$$\int\limits_{\Gamma} f_{k,\frac{\eta}{2}-1}(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}\Big[f_{k,\frac{\eta}{2}-1}(z);\infty\Big] = -2\pi i\,\mathrm{Res}\Big[\frac{1}{z^2} f_{k,\frac{\eta}{2}-1}\left(\tfrac{1}{z}\right);0\Big]. \tag{3.94}$$

In virtue of (3.93) and (3.94), the value of the left hand side of (3.90) is given by

$$⨍_{-1}^{1} \mathrm{d}t\, \frac{\mathrm{sgn}(\tau-t)}{|\tau-t|^{\eta}} \widetilde{P}_k^{(\frac{\eta}{2}-1)}(t) = -\frac{\pi}{\sin\frac{\eta\pi}{2}}\,\mathrm{Res}\Big[\frac{1}{z^2} f_{k,\frac{\eta}{2}-1}\left(\tfrac{1}{z}\right);0\Big], \tag{3.95}$$

where the function whose residue at 0 is sought has the form

$$\frac{1}{z^2} f_{k,\frac{\eta}{2}-1}\left(\tfrac{1}{z}\right) = \frac{(1-z^2)^{\frac{\eta}{2}-1}}{(1-z\tau)^{\eta}} P_k^{(\frac{\eta}{2}-1)}\left(\tfrac{1}{z}\right). \tag{3.96}$$

At this point, the proof of the theorem relies on proving that

$$\mathrm{Res}\left[\frac{(1-z^2)^{\frac{\eta}{2}-1}}{(1-z\tau)^{\eta}} P_k^{(\frac{\eta}{2}-1)}\left(\tfrac{1}{z}\right);0\right] = \frac{(\eta)_k}{2\,k!} P_{k-1}^{(\frac{\eta}{2})}(\tau). \tag{3.97}$$

For that purpose, let us expand every factor in (3.96) into power series around $z=0$ as

$$(1-z^2)^{\frac{\eta}{2}-1} = \sum_{j=0} a_{2j} z^{2j} \quad \text{with} \quad a_{2j} \equiv \frac{(1-\frac{\eta}{2})_j}{j!} \tag{3.98}$$

and

$$(1-z\tau)^{-\eta} = \sum_{j=0} b_j (\tau z)^j \quad \text{with} \quad b_j \equiv \frac{(\eta)_j}{j!}, \tag{3.99}$$

respectively. Recall the definition of the Pochhammer symbol $(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)}$.

The product of (3.98) and (3.99) yields

$$\begin{aligned}(1-z^2)^{\frac{\eta}{2}-1}(1-z\tau)^{-\eta} &= a_0b_0 + a_0b_1\tau z + (a_2b_1\tau + a_0b_3\tau^3)z^3 + (a_4b_1\tau + a_2b_3\tau^3 + a_0b_5\tau^5)z^5 + \cdots \\ &+ (a_2 + a_0b_2\tau^2)z^2 + (a_4 + a_2b_2\tau^2 + a_0b_4\tau^4)z^4 + (a_6 + a_4b_2\tau^2 + a_2b_4\tau^4 + a_0b_6\tau^6)z^6 + \cdots\,.\end{aligned} \tag{3.100}$$

Note that the coefficient of the term proportional to z^j with odd (even) j is a polynomial involving odd (even) powers of τ up to j. Moreover, the coefficients of those polynomials in τ are combinations of the coefficients a_r and b_s such that $r+s=j$.

With the definition of the polynomials $\sigma^{(e)}_{2h}(\tau)=\sum_{h'=0}^{h}a_{2(h-h')}b_{2h'}\tau^{2h'}$ and $\sigma^{(o)}_{2h+1}(\tau)=\sum_{h'=0}^{h}a_{2(h-h')}b_{2h'+1}\tau^{2h'+1}$, one has that (3.100) corresponds to

$$(1-z^2)^{\frac{\eta}{2}-1}(1-z\tau)^{-\eta}=\sum_{h=0}^{\infty}\left(\sigma^{(e)}_{2h}(\tau)z^{2h}+\sigma^{(o)}_{2h+1}(\tau)z^{2h+1}\right). \tag{3.101}$$

As for the Jacobi polynomial in (3.96), let us begin with the j-th order derivative of the Jacobi polynomial of order k given by

$$\frac{\mathrm{d}^j}{\mathrm{d}w^j}P_k^{\left(\frac{\eta}{2}-1\right)}(w)=\frac{1}{2^j}\left(\eta+k-1\right)_j P_{k-j}^{\left(\frac{\eta}{2}-1+j\right)}(w), \tag{3.102}$$

which is a direct result of the iterative application of (3.71) j times.

With that in mind, $P_k^{\left(\frac{\eta}{2}-1\right)}\left(\frac{1}{z}\right)$ admits the expansion given by

$$\begin{aligned}P_k^{\left(\frac{\eta}{2}-1\right)}\left(\tfrac{1}{z}\right)&=\sum_{j=0}^{k}\left(\frac{1}{z}\right)^j\left[\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}w^j}P_k^{\left(\frac{\eta}{2}-1\right)}(w)\Big|_{w=0}\right]\\&=\sum_{j=0}^{k}\left(\frac{1}{z}\right)^j\left[\frac{1}{j!\,2^j}\left(\eta+k-1\right)_j P_{k-j}^{\left(\frac{\eta}{2}-1+j\right)}(0)\right].\end{aligned} \tag{3.103}$$

Observe that if k is even, then only terms with even j contribute. To see this, recall that the parity property (3.69) for $\gamma=\delta$ implies that $P_{k-j}^{\left(\frac{\eta}{2}-1+j\right)}(0)=0$ for odd $k-j$, i.e. odd j. Conversely, if k is odd, then only terms with odd j contribute.

The same expansion procedure applied to $P_{k-1}^{\left(\frac{\eta}{2}\right)}(\tau)$ leads to

$$P_{k-1}^{\left(\frac{\eta}{2}\right)}(\tau)=\sum_{j=0}^{k-1}(\tau)^j\left[\frac{1}{j!\,2^j}\left(\eta+k\right)_j P_{k-1-j}^{\left(\frac{\eta}{2}+j\right)}(0)\right]. \tag{3.104}$$

Let us consider (3.102) with **odd** index k, i.e. $k=2l+1$ with $l\in\mathbb{N}_0$. By considering odd values of j equal to $2h+1$ with $h\in\mathbb{N}_0$ in (3.103), one has that

$$P_{2l+1}^{\left(\frac{\eta}{2}-1\right)}\left(\tfrac{1}{z}\right)=\sum_{h'=0}^{l}\left(\frac{1}{z}\right)^{2h'+1}\left[\frac{(\eta+2l)_{2h'+1}}{(2h'+1)!\,2^{2h'+1}}P_{2(l-h')}^{\left(\frac{\eta}{2}+2h'\right)}(0)\right]. \tag{3.105}$$

It follows from (3.105) and (3.101) that the statement in (3.97), namely

$$\mathrm{Res}\Big[\frac{1}{z^2}f_{2l+1,\frac{\eta}{2}-1}\left(\tfrac{1}{z}\right);0\Big]=\frac{(\eta)_{2l+1}}{2\,(2l+1)!}P_{2l}^{\left(\frac{\eta}{2}\right)}(\tau), \tag{3.106}$$

is equivalent to

$$\sum_{h'=0}^{l}\frac{(\eta+2l)_{2h'+1}}{(2h'+1)!\,2^{2h'+1}}P_{2(l-h')}^{\left(\frac{\eta}{2}+2h'\right)}(0)\,\sigma^{(e)}_{2h'}(\tau)-\frac{(\eta)_{2l+1}}{2\,(2l+1)!}P_{2l}^{\left(\frac{\eta}{2}\right)}(\tau)=0. \tag{3.107}$$

To prove (3.107), let us consider the explicit polynomial forms of $\sigma_{2h'}^{(e)}(\tau)$ and $P_{2l}^{\left(\frac{\eta}{2}\right)}(\tau)$ and group terms with common power in τ. Then, one needs to show that the coefficient of each term equals zero. In general, the coefficient corresponding to the power $\tau^{2(l-l')}$, with $0 \leq l' \leq l$, is given by

$$c_{2(l-l')} = \overbrace{\frac{(\eta)_{2(l-l')}}{[2(l-l')]!}}^{b_{2(l-l')}} \sum_{h=0}^{l'} \overbrace{\frac{\left(1-\frac{\eta}{2}\right)_h}{h!}}^{a_{2h}} \frac{(\eta+2l)_{2(l-l')+2h+1}}{(2(l-l')+2h+1)!\; 2^{2(l-l')+2h+1}} P_{2(l'-h)}^{\left(\frac{\eta}{2}+2(l-l')+2h\right)}(0) - \frac{(\eta)_{2l+1}}{2\,(2l+1)!} \frac{(\eta+2l+1)_{2(l-l')}}{[2(l-l')]!\; 2^{2(l-l')}} P_{2l'}^{\left(\frac{\eta}{2}+2(l-l')\right)}(0). \tag{3.108}$$

In order to simplify (3.108), one has that $(\eta)_{2l+1} = (\eta)_{2(l-l')}\,(\eta+2(l-l'))_{2l'}\,(\eta+2l)$ and also that $(\eta+2l)_{2(l-l')+2h+1} = (\eta+2l)\,(\eta+2l+1)_{2(l-l')}\,(\eta+2l+1+2(l-l'))_{2h}$. Therefore, it follows that

$$c_{2(l-l')} = \frac{(\eta)_{2(l-l')}(\eta+2l+1)_{2(l-l')}(\eta+2l)}{[2(l-l')]!\; 2^{2(l-l')+1}} \cdot \left(\sum_{h=0}^{l'} \frac{\left(1-\frac{\eta}{2}\right)_h}{h!} \frac{(\eta+2l+1+2(l-l'))_{2h}}{(2(l-l')+2h+1)!\; 2^{2h}} P_{2(l'-h)}^{\left(\frac{\eta}{2}+2(l-l')+2h\right)}(0) - \frac{(\eta+2(l-l'))_{2l'}}{(2l+1)!} P_{2l'}^{\left(\frac{\eta}{2}+2(l-l')\right)}(0)\right). \tag{3.109}$$

One can also relate every $P_{2(l'-h)}^{\left(\frac{\eta}{2}+2(l-l')+2h\right)}(0)$ to $P_{2l'}^{\left(\frac{\eta}{2}+2(l-l')\right)}(0)$ by means of

$$P_{2(l'-h)}^{\left(\frac{\eta}{2}+2(l-l')+2h\right)}(0) = \frac{(-1)^h\, 2^{2h}\, (l'-h+1)_h}{\left(\frac{\eta}{2}+2l-l'+1\right)_h} P_{2l'}^{\left(\frac{\eta}{2}+2(l-l')\right)}(0). \tag{3.110}$$

A proof of (3.110) can be achieved from the following identities involving ultraspherical polynomials $C_{2m}^{(\gamma')}$ (HOCHSTRASSER, 1972)

$$C_{2m}^{(\gamma')}(0) = (-1)^m \frac{(\gamma')_m}{m!}, \qquad \gamma' > -\frac{1}{2}, \gamma' \neq 0 \tag{3.111}$$

$$P_{2m}^{\left(\gamma'-\frac{1}{2}\right)}(0) = \frac{\left(\gamma'+\frac{1}{2}\right)_{2m}}{(2\gamma')_{2m}} C_{2m}^{(\gamma')}(0), \tag{3.112}$$

both of which lead to

$$P_{2m}^{(\gamma)}(0) = \frac{(-1)^m}{m!\; 2^{2m}} \frac{(\gamma+1)_{2m}}{(\gamma+1)_m} = \frac{(-1)^m}{m!\; 2^{2m}} (\gamma+1+m)_m \tag{3.113}$$

for $\gamma > -1$. Also note that in order to have (3.113) one can make use of the relation

$$(2\gamma')_{2m} = 2^{2m}\,(\gamma')_m \left(\gamma'+\frac{1}{2}\right)_m, \tag{3.114}$$

which can be easily verified from the ratio ${}^{(2\gamma')_{2m}}/_{(\gamma')_m}$ and the definition of the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1)$.

By having (3.113) for $P_{2m}^{(\gamma)}(0)$ and $P_{2m-2h}^{(\gamma+2h)}(0)$ separately and then calculating the ratio, one gets to

$$\begin{aligned} P_{2(m-h)}^{(\gamma+2h)}(0) &= \frac{(-1)^h\, m!\, 2^{2h}}{(m-h)!} \frac{(\gamma+m+h+1)_{m-h}}{(\gamma+m+1)_m} P_{2m}^{(\gamma)}(0) \\ &= \frac{(-1)^h\, 2^{2h}\, (m-h+1)_h}{(\gamma+m+1)_h} P_{2m}^{(\gamma)}(0), \end{aligned} \tag{3.115}$$

from which (3.110) follows.

Inserting (3.110) into (3.109) leads to

$$\begin{aligned} c_{2(l-l')} = {} & \frac{(\eta)_{2(l-l')}(\eta+2l+1)_{2(l-l')}(\eta+2l)}{[2(l-l')]!\, 2^{2(l-l')+1}} P_{2l'}^{\left(\frac{\eta}{2}+2(l-l')\right)}(0)\cdot \\ & \cdot\left(\sum_{h=0}^{l'} \frac{\left(1-\frac{\eta}{2}\right)_h}{h!} \frac{(\eta+4l-2l'+1)_{2h}}{(2(l-l')+2h+1)!} \frac{(-1)^h\, (l'-h+1)_h}{\left(\frac{\eta}{2}+2l-l'+1\right)_h} \right. \\ & \left. - \frac{(\eta+2(l-l'))_{2l'}}{(2l+1)!}\right). \end{aligned} \tag{3.116}$$

By means of (3.114) and the fact that $(2l+1)! = (2(l-l')+2h+1)!\,(2(l-l'+h+1))_{2(l'-h)}$, one can simplify (3.116) into the form

$$\begin{aligned} c_{2(l-l')} = {} & \frac{(\eta)_{2(l-l')}(\eta+2l+1)_{2(l-l')}(\eta+2l)}{[2(l-l')]!\, 2^{2(l-l')+1}\, (2l+1)!} P_{2l'}^{\left(\frac{\eta}{2}+2(l-l')\right)}(0) \left(\sum_{h=0}^{l'} \frac{(-1)^h\, (l'-h+1)_h}{h!} \cdot\right. \\ & \left. \cdot 2^{2h}\, (2(l-l'+h+1))_{2(l'-h)} \left(\tfrac{\eta}{2}+2l-l'+\tfrac{1}{2}\right)_h \left(1-\tfrac{\eta}{2}\right)_h - (\eta+2(l-l'))_{2l'}\right). \end{aligned} \tag{3.117}$$

Eq. (3.117) admits further simplification by applying (3.114) in the last two terms, leading to

$$\begin{aligned} c_{2(l-l')} = {} & \frac{(\eta)_{2(l-l')}(\eta+2l+1)_{2(l-l')}(\eta+2l)}{[2(l-l')]!\, 2^{2(l-l')+1}\, (2l+1)!} P_{2l'}^{\left(\frac{\eta}{2}+2(l-l')\right)}(0)\, 2^{l'}\cdot \\ & \cdot\left(\sum_{h=0}^{l'} \frac{(-1)^h\, (l'-h+1)_h}{h!} (l-l'+h+1)_{l'-h} \left(l-l'+h+\tfrac{3}{2}\right)_{l'-h} \cdot \right. \\ & \left. \cdot \left(\tfrac{\eta}{2}+2l-l'+\tfrac{1}{2}\right)_h \left(1-\tfrac{\eta}{2}\right)_h - \left(\tfrac{\eta}{2}+l-l'\right)_{l'} \left(\tfrac{\eta}{2}+l-l'+\tfrac{1}{2}\right)_{l'}\right). \end{aligned} \tag{3.118}$$

Therefore, in order to prove that the coefficients $c_{2(l-l')}$ equal zero for any l and l', provided that $0 \le l' \le l$, one needs to show that

$$\mathring{c}(l,l') = \left(\tfrac{\eta}{2}+l-l'\right)_{l'} \left(\tfrac{\eta}{2}+l-l'+\tfrac{1}{2}\right)_{l'}, \tag{3.119}$$

where $\mathring{c}(l,l')$ is defined as

$$\mathring{c}(l,l') = \sum_{h=0}^{l'} \frac{(l'-h+1)_h}{h!} (l-(l'-1)+h)_{l'-h} \left(l-(l'-1)+h+\tfrac{1}{2}\right)_{l'-h} \cdot \\ \cdot \left(\tfrac{\eta}{2}+l-(l'-1)+l-\tfrac{1}{2}\right)_h \left(\tfrac{\eta}{2}-h\right)_h . \quad (3.120)$$

The statement (3.119) was verified with the aid of symbolic computations carried out in the Mathematica (WOLFRAM RESEARCH INC., 2022) and the code is included in Appendix A. Nevertheless, a sketch of a possible proof is presented below.

The case $l'=0$ is trivial as $\mathring{c}(l,0)=1$ in (3.119) and so is the only term in the right hand side of (3.120). For $l'=1$, one has that

$$\begin{aligned} \mathring{c}(l,1) &= (l)_1 \left(l+\tfrac{1}{2}\right)_1 - \left(\tfrac{\eta}{2}+l-1+l+\tfrac{1}{2}\right)\left(1-\tfrac{\eta}{2}\right) \\ &= \left(l+\tfrac{1}{2}\right)\left(l+\tfrac{\eta}{2}-1\right) + \left(\tfrac{\eta}{2}+l-1\right)\left(\tfrac{\eta}{2}-1\right) \\ &= \left(\tfrac{\eta}{2}+l-1\right)\left(\tfrac{\eta}{2}+l-1+\tfrac{1}{2}\right). \end{aligned}$$

Having proved (3.119) for $l'=1$, let us assume as induction hypothesis that

$$\mathring{c}(l,l') = \left(\tfrac{\eta}{2}+l-l'\right)_{l'} \left(\tfrac{\eta}{2}+l-l'+\tfrac{1}{2}\right)_{l'} \quad (3.121)$$

holds for any $1<l'<l$. Multiplying both sides of (3.121) by $\frac{\eta}{2}+l-l'-1$ and $\frac{\eta}{2}+l-l'-1+\frac{1}{2}$ leads to

$$\left(\tfrac{\eta}{2}+l-l'-1\right)\left(\tfrac{\eta}{2}+l-l'-1+\tfrac{1}{2}\right)\mathring{c}(l,l') \\ = \left(\tfrac{\eta}{2}+l-(l'+1)\right)_{l'+1}\left(\tfrac{\eta}{2}+l-(l'+1)+\tfrac{1}{2}\right)_{l'+1} . \quad (3.122)$$

The right-hand side of (3.122) is already in the form of the right-hand side of the induction hypothesis (3.122) for $l'+1$. With regard to the left-hand side of (3.122), one can start with the $h=0$ term

$$\begin{aligned} &\left(l-l'+\tfrac{\eta}{2}-1\right)\left(l-l'+\tfrac{1}{2}+\tfrac{\eta}{2}-1\right)(l-(l'-1))_{l'}\left(l-(l'-1)+\tfrac{1}{2}\right)_{l'} \\ &= (l-l')_{l'+1}\left(l-l'+\tfrac{1}{2}\right)_{l'+1} + \\ &\quad \left(\tfrac{\eta}{2}-1\right)\left(\tfrac{\eta}{2}-1+2l-2l'+\tfrac{1}{2}\right)(l-(l'-1))_{l'}\left(l-(l'-1)+\tfrac{1}{2}\right)_{l'} . \end{aligned} \quad (3.123)$$

The first term of (3.123) corresponds to the term of $\mathring{c}(l,l'+1)$ with $h=0$. The second term of (3.123) is recombined with the term of $\mathring{c}(l,l')$ with $h=1$. This process of forward propagation is carried out $l'+1$ times and eventually the $h=l'+1$ term of $\mathring{c}(l,l'+1)$ comes up.

Finally, the proof of (3.102) with **even** index k follows an analogous, if not identical, approach to the one just presented for odd k. For that reason, it will not be included here.

□

The spectral relation (3.90) is a particular case of the following general result reported in (PODLUBNY, 1999) concerning the Riesz potential:

Theorem 3.4. *[I. Podlubny] If $\gamma, \delta > -1$, $0 < \eta < 1$ and θ is an arbitrary real number, then for $-1 < \tau < 1$ the following holds*

$$\fint_{-1}^{1} \mathrm{d}t \left(\mathrm{sgn}(\tau - t) + \frac{\tan \pi\theta}{\tan \frac{\eta\pi}{2}} \right) \frac{\widetilde{P}_k^{(\gamma,\delta)}(t)}{|\tau - t|^{\eta}} = \frac{\Phi_1(\tau) \sin \pi \left(\theta - \frac{\eta}{2}\right) + \Phi_2(\tau) \sin \pi \left(\theta + \frac{\eta}{2} - \delta\right)}{\Gamma(\eta) \sin \frac{\eta\pi}{2} \cos \theta\pi}, \tag{3.124}$$

where

$$\Phi_1(\tau) = \frac{(-1)^k \Gamma(k + \gamma + 1)\Gamma(k + \eta)\Gamma(\delta - \eta + 1)}{2^{-\gamma-\delta+\eta-1}\Gamma(k + \gamma + \delta - \eta + 2)\, k!} {}_2F_1(k+\eta, \eta-k-\gamma-\delta-1; -\delta+\eta; \frac{1+\tau}{2}) \tag{3.125}$$

and

$$\Phi_2(\tau) = \frac{(-1)^{k+1}\Gamma(k + \delta + 1)\Gamma(\eta - \delta - 1)}{2^{-\gamma}(1 + \tau)^{\eta-\delta-1}} {}_2F_1(k+\delta+1, -k-\gamma; \delta-\eta+2; \frac{1+\tau}{2}). \tag{3.126}$$

Proof. Here follows the proof sketched in (PODLUBNY, 1999) thoroughly "decompressed". Consider the function $r(\omega)$ given by

$$r(s) = \exp(i\pi\theta) \int_0^{\infty} \mathrm{d}\omega\, \omega^{\eta-1} \exp(is\omega) = \exp(i\pi\theta)\mathcal{F}[\omega^{\eta-1}\mathbb{1}_{(0,\infty)}(\omega)], \tag{3.127}$$

where $\mathcal{F}$ is the Fourier transform operator.

Taking into account that the application of the Fourier operator twice produces a reflection of the original function, i.e. $\mathcal{F}^2[f(t)] = f(-t)$ and that

$$\mathcal{F}\left[(it)^{-\eta}\right] = \frac{1}{\Gamma(\eta)} \omega^{\eta-1}\mathbb{1}_{(0,\infty)}(\omega) \tag{3.128}$$

holds for $0 < \eta < 1$, one has that $r(s) = \exp(i\pi\theta)\Gamma(\eta)\,[-is]^{-\eta}$ which is equivalent to $r(s) = \exp(i\pi\theta)\Gamma(\eta)\,[-i\,\mathrm{sgn}(s)|s|]^{-\eta}$. The phase factor $(-i\,\mathrm{sgn}(s))^{-\eta}$ can be reformulate as $\exp\left(-i\pi\frac{\eta}{2}\,\mathrm{sgn}(s)\right)$, leading in the end to

$$r(s) = \exp(i\pi\theta)\Gamma(\eta)\,[-is]^{-\eta} = \Gamma(\eta)|s|^{-\eta} \exp\left[i\pi\left(\theta + \frac{\eta}{2}\,\mathrm{sgn}(s)\right)\right]. \tag{3.129}$$

Let $J(\tau)$ be the function defined as

$$J(\tau) = \int_{-1}^{1} \mathrm{d}t\, r(\tau - t)\widetilde{P}_k^{(\gamma,\delta)}(t). \tag{3.130}$$

On the one hand, the insertion of (3.129) into (3.130) leads to

$$J(\tau) = \Gamma(\eta)\int_{-1}^{1} \mathrm{d}t\,|\tau - t|^{-\eta}\exp\left[i\pi\left(\theta + \frac{\eta}{2}\,\mathrm{sgn}(\tau - t)\right)\right]\widetilde{P}_k^{(\gamma,\delta)}(t). \tag{3.131}$$

On the other hand, let us plug (3.127) into (3.131) and swap the integration order

$$\begin{aligned} J(\tau) &= \exp(i\pi\theta)\int_{-1}^{1}\mathrm{d}t\int_{0}^{\infty}\mathrm{d}\omega\,\omega^{\eta-1}\exp\left[i(\tau - t)\omega\right]\widetilde{P}_k^{(\gamma,\delta)}(t) \\ &= \exp(i\pi\theta)\int_{0}^{\infty}\mathrm{d}\omega\,\omega^{\eta-1}\exp\left(i\tau\omega\right)\int_{-1}^{1}\mathrm{d}t\exp\left(-it\omega\right)\widetilde{P}_k^{(\gamma,\delta)}(t). \end{aligned} \tag{3.132}$$

It follows from Rodrigues' formula (3.68) that $\widetilde{P}_k^{(\gamma,\delta)}(t) = \frac{(-1)^k}{k!\,2^k}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[(1-t)^{k+\gamma}(1+t)^{k+\delta}\right]$, hence

$$\begin{aligned} J(\tau) = \frac{(-1)^k}{k!\,2^k}\exp(i\pi\theta)\int_{0}^{\infty}\mathrm{d}\omega\,\omega^{\eta-1}\exp\left(i\tau\omega\right)\cdot \\ \cdot\int_{-1}^{1}\mathrm{d}t\exp\left(-it\omega\right)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[(1-t)^{k+\gamma}(1+t)^{k+\delta}\right]. \end{aligned} \tag{3.133}$$

Regarding the inner integral, let us change variables $s \equiv \frac{1}{2}(1+t)$ and perform iterated partial integration k times starting with $v = \exp(-i2\omega s)$, hence $\mathrm{d}v = -i2\omega v\,\mathrm{d}s$, and $\mathrm{d}u = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\left[(1-s)^{k+\gamma}s^{k+\delta}\right]$. As for the boundary terms, note that they vanish in virtue of (3.68), at least for positive values for the parameters γ and δ. With that in hand, one has that

$$\begin{aligned} &\int_{-1}^{1}\mathrm{d}t\exp\left(-it\omega\right)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[(1-t)^{k+\gamma}(1+t)^{k+\delta}\right] \\ &\quad= \exp\left(i\omega\right)\int_{0}^{1}2\,\mathrm{d}s\exp\left(-i2\omega s\right)2^{-k}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left[2^{2k+\gamma+\delta}(1-s)^{k+\gamma}s^{k+\delta}\right] \\ &\quad= \exp\left(i\omega\right)2^{k+\gamma+\delta+1}\int_{0}^{1}\mathrm{d}s\exp\left(-i2\omega s\right)\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left[(1-s)^{k+\gamma}s^{k+\delta}\right] \\ &\quad= \exp\left(i\omega\right)2^{k+\gamma+\delta+1}(-1)^k(-i2\omega)^k\int_{0}^{1}\mathrm{d}s\exp\left(-2i\omega s\right)s^{k+\delta}(1-s)^{k+\gamma}. \end{aligned} \tag{3.134}$$

The confluent hypergeometric function ${}_1F_1(a;b;z)$ or Kummer function, also denoted by $M(a,b,z)$, admits the integral representation given by (SLATER, 1972)

$$\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}{}_1F_1(a;b;z) = \int_{0}^{1}\exp(zt)t^{a-1}(1-t)^{b-a-1}. \tag{3.135}$$

By considering that $\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)} = \mathrm{B}(a, b-a)$ and by taking $a = k+\delta+1$ and $b = 2k+\gamma+\delta+2$, (3.134) becomes

$$\begin{aligned} \int_{-1}^{1}\mathrm{d}t\exp\left(-it\omega\right)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[(1-t)^{k+\gamma}(1+t)^{k+\delta}\right] = \exp\left(i\omega\right)2^{k+\gamma+\delta+1}(-i2\omega)^k\cdot \\ \cdot\,\mathrm{B}(k+\delta+1, k+\gamma+1){}_1F_1(k+\delta+1; 2k+\gamma+\delta+2; -2i\omega), \end{aligned} \tag{3.136}$$

which in turn leads from (3.133) to

$$J(\tau) = \frac{(-i)^k}{k!} \exp(i\pi\theta) 2^{k+\gamma+\delta+1} \mathrm{B}(k+\delta+1, k+\gamma+1) \\ \int_0^\infty \mathrm{d}\omega \exp\left[i(\tau+1)\omega\right] \omega^{k+\eta-1} {}_1F_1\left(k+\delta+1; 2k+\gamma+\delta+2; -2i\omega\right). \quad (3.137)$$

The Laplace transform $\mathcal{L}\{f(x)\}(z) \equiv \int_0^\infty \mathrm{d}x\, e^{-zx} f(x)$ of the function $f(x) = x^{s-1} {}_1F_1(a; b; \lambda x)$ is given by

$$\mathcal{L}\{x^{s-1} {}_1F_1(a; b; \lambda x)\}(z) = \Gamma(s) z^{-s} {}_2F_1(a, s; b; \lambda z^{-1}), \quad (3.138)$$

provided that $\mathrm{Re}\, s > 0$ and $\mathrm{Re}\, z > \max(0, \mathrm{Re}\,\lambda)$ (ERDÉLYI et al., 1953). To see this, one can expand the Kummer function in the power series ${}_1F_1(a; b; \lambda x) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n} \frac{1}{n!} (\lambda x)^n$ and perform the Laplace transform term by term $\mathcal{L}\{x^q \mathbb{1}_{(0,+\infty)}(x)\} = \Gamma(q+1) z^{-(q+1)}$ with $\mathrm{Re}\, q > -1$ and $\mathrm{Re}\, s > 0$. Thus one arrives at $\Gamma(s) z^{-s} \sum_{n=0}^\infty \frac{(a)_n (s)_n}{(b)_n} \frac{1}{n!} \left(\lambda z^{-1}\right)^n$, which is equivalent to the right hand side of (3.138) in virtue of (3.16).

By comparing (3.138) to (3.137), one can identify $s = k + \eta > 0$, $a = k + \delta + 1$ and $b = 2k + \gamma + \delta + 2$. Even though $z = -i(\tau + 1)$ and $\lambda = -2i$ are purely imaginary, one can take $z + \epsilon$ and check that the Laplace integral converges in the limit $\epsilon \to 0$.

Inserting (3.138) into (3.137) and performing some simplifications to the phase factor arising from the powers of i involved lead to

$$J(\tau) = \frac{1}{k!} \exp(i\pi\theta)\, 2^{k+\gamma+\delta+1}\, \Gamma(k+\eta)(-1)^{-\eta} \exp\!\left(-i\tfrac{\pi\eta}{2}\right)(\tau+1)^{-k-\eta} \\ \mathrm{B}(k+\delta+1, k+\gamma+1)\, {}_2F_1\left(k+\delta+1, k+\eta; 2k+\gamma+\delta+2; \tfrac{2}{1+\tau}\right). \quad (3.139)$$

The linear transformation formula (3.18) with $a = k+\delta+1$, $b = k+\eta$ and $c = 2k+\gamma+\delta+2$, allows to invert the argument of ${}_2F_1$ in (3.139). In so doing, two terms A_1 and A_2 arise that will be treated separately, namely

$$\mathrm{B}(k+\delta+1, k+\gamma+1) {}_2F_1(k+\delta+1, k+\eta; 2k+\gamma+\delta+2; \tfrac{2}{1+\tau}) = A_1 + A_2 \quad (3.140)$$

with

$$A_1 = \frac{\Gamma(k+\gamma+1)\Gamma(-\eta+\delta+1)}{\Gamma(-\eta+k+\gamma+\delta+2)} \left(-\tfrac{1+\tau}{2}\right)^{k+\eta} {}_2F_1\left(k+\eta, \eta-k-\gamma-\delta-1; \eta-\delta; \tfrac{1+\tau}{2}\right) \quad (3.141)$$

and

$$A_2 = \frac{\Gamma(k+\delta+1)\Gamma(\eta-\delta-1)}{\Gamma(k+\eta)} \left(-\tfrac{1+\tau}{2}\right)^{k+\delta+1} {}_2F_1\left(k+\delta+1, -k-\gamma; -\eta+\delta+2; \tfrac{1+\tau}{2}\right). \quad (3.142)$$

Let us split $J(\tau)$ accordingly. For the term $J_1(\tau)$ corresponding to A_1 one has that

$$\begin{aligned}
J_1(\tau) = {} & \frac{1}{k!}\exp(i\pi\theta)\,2^{k+\gamma+\delta+1}\,\Gamma(k+\eta)(-1)^{-\eta}\exp\!\left(-i\tfrac{\pi\eta}{2}\right)(\tau+1)^{-k-\eta}\cdot \\
& \cdot\frac{\Gamma(k+\gamma+1)\Gamma(-\eta+\delta+1)}{\Gamma(-\eta+k+\gamma+\delta+2)}\left(-\tfrac{1+\tau}{2}\right)^{k+\eta}{}_2F_1\left(k+\eta,\eta-k-\gamma-\delta-1;\eta-\delta;\tfrac{1+\tau}{2}\right) \\
= {} & \exp(i\pi\theta)\exp\left(-i\tfrac{\pi\eta}{2}\right)\frac{(-1)^k}{2^{-\gamma-\delta+\eta-1}}\,\frac{\Gamma(k+\gamma+1)\Gamma(k+\eta)\Gamma(\delta-\eta+1)}{k!\,\Gamma(k+\gamma+\delta-\eta+2)}\cdot \\
& \cdot{}_2F_1\left(k+\eta,\eta-k-\gamma-\delta-1;\eta-\delta;\tfrac{1+\tau}{2}\right) \\
= {} & \exp\left(i\pi\left(\theta-\tfrac{\eta}{2}\right)\right)\Phi_1(\tau),
\end{aligned} \tag{3.143}$$

while A_2 leads to the term $J_2(\tau)$ given by

$$\begin{aligned}
J_2(\tau) = {} & \frac{1}{k!}\exp(i\pi\theta)\,2^{k+\gamma+\delta+1}\,\Gamma(k+\eta)(-1)^{-\eta}\exp\!\left(-i\tfrac{\pi\eta}{2}\right)(\tau+1)^{-k-\eta}\cdot \\
& \cdot\frac{\Gamma(k+\delta+1)\Gamma(\eta-\delta-1)}{\Gamma(k+\eta)}\left(-\tfrac{1+\tau}{2}\right)^{k+\delta+1}{}_2F_1\left(k+\delta+1,-k-\gamma;-\eta+\delta+2;\tfrac{1+\tau}{2}\right). \\
= {} & \exp(i\pi\theta)\exp(i\pi\eta)\exp\left(-i\tfrac{\pi\eta}{2}\right)\exp(-i\pi\delta)\frac{(-1)^{k+1}}{2^{-\gamma}}\,\frac{\Gamma(k+\delta+1)\Gamma(\eta-\delta-1)}{k!(\tau+1)^{\eta-\delta-1}}\cdot \\
& \cdot{}_2F_1\left(k+\delta+1,-k-\gamma;-\eta+\delta+2;\tfrac{1+\tau}{2}\right) \\
= {} & \exp\left(i\pi\left(\theta+\tfrac{\eta}{2}-\delta\right)\right)\Phi_2(\tau).
\end{aligned} \tag{3.144}$$

It follows from (3.143) and (3.144) that

$$J(\tau) = \Phi_1(\tau)\exp\left[i\pi\left(\theta-\tfrac{\eta}{2}\right)\right] + \Phi_2(\tau)\exp\left[i\pi\left(\theta+\tfrac{\eta}{2}-\delta\right)\right]. \tag{3.145}$$

Let us calculate the imaginary part of $J(\tau)$. On the one hand, (3.131) leads to

$$\begin{aligned}
\mathrm{Im}\{J(\tau)\} &= \Gamma(\eta)\int_{-1}^{1}\mathrm{d}t\sin\pi\left(\theta+\tfrac{\eta}{2}\operatorname{sgn}(\tau-t)\right)\frac{\widetilde{P}_k^{(\gamma,\delta)}(t)}{|\tau-t|^\eta} \\
&= \Gamma(\eta)\int_{-1}^{1}\mathrm{d}t\left[\sin\pi\theta\cos\tfrac{\pi\eta}{2}+\operatorname{sgn}(\tau-t)\cos\pi\theta\sin\tfrac{\pi\eta}{2}\right]\frac{\widetilde{P}_k^{(\gamma,\delta)}(t)}{|\tau-t|^\eta} \\
&= \Gamma(\eta)\sin\tfrac{\pi\eta}{2}\cos\pi\theta\int_{-1}^{1}\mathrm{d}t\left[\frac{\tan\pi\theta}{\tan\frac{\pi\eta}{2}}+\operatorname{sgn}(\tau-t)\right]\frac{\widetilde{P}_k^{(\gamma,\delta)}(t)}{|\tau-t|^\eta}.
\end{aligned} \tag{3.146}$$

On the other hand, one has from (3.145) that

$$\mathrm{Im}\{J(\tau)\} = \Phi_1(\tau)\sin\pi\left(\theta-\tfrac{\eta}{2}\right) + \Phi_2(\tau)\sin\pi\left(\theta+\tfrac{\eta}{2}-\delta\right). \tag{3.147}$$

The statement of the theorem (3.124) follows from the equivalence of (3.146) and (3.147).

□

Let us find the restrictions on the parameters that produce a polynomial in the right-hand side of (3.124). First of all, if $\delta-\theta-\frac{\eta}{2}=r$ with $r\in\mathbb{Z}$, the term proportional to Φ_2 vanishes due to the phase of the sine function. Note that Φ_2 has a factor $(1+\tau)^{-(\eta-\delta-1)}$ that forbids the goal of getting a polynomial. On the other hand, the power series of the hypergeometric function in Φ_1 becomes a finite sum if either $k+\eta$ or $\eta-k-\gamma-\delta-1$ is a negative integer. Considering the latter, one has that $\eta-k-\gamma-\delta-1=-l$ with $l\in\mathbb{Z}^+$.

The two restrictions above imply that $\delta=\frac{\eta}{2}+\theta+r$ and $\gamma=\frac{\eta}{2}-\theta+(l-k-r-1)=\frac{\eta}{2}-\theta+s$ with $s\in\mathbb{Z}$ such that $l=k+r+s+1$. So far, one has that (3.124) turns into

$$\begin{aligned}\fint_{-1}^{1}\mathrm{d}t\left(\operatorname{sgn}(\tau-t)\cos\pi\theta\sin\tfrac{\eta\pi}{2}+\sin\pi\theta\cos\tfrac{\eta\pi}{2}\right)\frac{\widetilde{P}_k^{\left(\frac{\eta}{2}-\theta+s,\frac{\eta}{2}+\theta+r\right)}(t)}{|\tau-t|^{\eta}}\\ =\frac{\sin\pi\left(\theta-\frac{\eta}{2}\right)}{\Gamma(\eta)}\frac{(-1)^k\Gamma(k+\gamma+1)\Gamma(k+\eta)\Gamma(\delta-\eta+1)}{2^{-\gamma-\delta+\eta-1}\Gamma(k+\gamma+\delta-\eta+2)\,k!}\cdot\\ \cdot\underbrace{{}_2F_1(k+\eta,\eta-k-\gamma-\delta-1;-\delta+\eta;\frac{1+\tau}{2})}_{(F)}.\end{aligned}\tag{3.148}$$

It follows from (3.76) and (3.69) that

$${}_2F_1\left(-n',n'+\gamma'+\delta'+1;\gamma'+1;\frac{1+t}{2}\right)=\frac{\Gamma(n'+1)\Gamma(\gamma'+1)}{\Gamma(n'+\gamma'+1)}(-1)^{n'}P_{n'}^{(\delta',\gamma')}(t).\tag{3.149}$$

By comparing the left-hand side of (3.149) with (F), one is able to identify $n'=k+r+s+1$, $\gamma'=\frac{\eta}{2}-\theta-r-1$ and $\delta'=\frac{\eta}{2}+\theta-s-1$, leading to

$$(F)=\frac{\Gamma(k+r+s+2)\Gamma(\frac{\eta}{2}-\theta-r)}{\Gamma(k+s+\frac{\eta}{2}-\theta+1)}(-1)^{k+r+s+1}P_{k+r+s+1}^{(\frac{\eta}{2}+\theta-s-1,\frac{\eta}{2}-\theta-r-1)}(\tau).\tag{3.150}$$

Taking into account that $k+\gamma+\delta-\eta+2=k+r+s+2$, $\delta-\eta+1=1-(\frac{\eta}{2}-\theta-r)$ and $k+\gamma+1=k+s+\frac{\eta}{2}-\theta+1$ as well the reflection formula for the Gamma function (3.4), one has that (3.148) becomes

$$\begin{aligned}\fint_{-1}^{1}\mathrm{d}t\left(\operatorname{sgn}(\tau-t)\cos\pi\theta\sin\tfrac{\eta\pi}{2}+\sin\pi\theta\cos\tfrac{\eta\pi}{2}\right)\frac{\widetilde{P}_k^{\left(\frac{\eta}{2}-\theta+s,\frac{\eta}{2}+\theta+r\right)}(t)}{|\tau-t|^{\eta}}\\ =\frac{\sin\pi\left(\theta-\frac{\eta}{2}\right)}{k!\,\Gamma(\eta)}\frac{(-1)^{2k+r+s+1}\Gamma(k+\eta)}{2^{-\gamma-\delta+\eta-1}}\frac{\pi}{\sin\pi\left(\frac{\eta}{2}-\theta-r\right)}P_{k+r+s+1}^{(\frac{\eta}{2}+\theta-s-1,\frac{\eta}{2}-\theta-r-1)}(\tau)\\ =\frac{(-1)^{r+s+1}2^{r+s+1}\,\pi\,\Gamma(k+\eta)\sin\pi\left(\theta-\frac{\eta}{2}\right)}{k!\,\Gamma(\eta)\,\sin\pi\left(\frac{\eta}{2}-\theta-r\right)}P_{k+r+s+1}^{(\frac{\eta}{2}+\theta-s-1,\frac{\eta}{2}-\theta-r-1)}(\tau).\end{aligned}\tag{3.151}$$

Observe that, in order to restrict the spectral relations to classical Jacobi polynomials, the following conditions $k+r+s+1\geq 0$, $s>\theta-\frac{\eta}{2}-1$ and $r>-\theta-\frac{\eta}{2}-1$ must be fulfilled.

Taking $\theta = 0$ in (3.151) yields

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{\operatorname{sgn}(\tau - t)}{|\tau - t|^{\eta}} \widetilde{P}_k^{\left(\frac{\eta}{2}+s, \frac{\eta}{2}+r\right)}(t)$$

$$= \frac{(-1)^{r+s+1} 2^{r+s+1} \, \pi \, \Gamma(k+\eta)}{k! \, \sin\frac{\eta\pi}{2} \Gamma(\eta)} \frac{\sin\pi\left(-\frac{\eta}{2}\right)}{\sin\pi\left(\frac{\eta}{2} - r\right)} P_{k+r+s+1}^{\left(\frac{\eta}{2}-s-1, \frac{\eta}{2}-r-1\right)}(\tau). \quad (3.152)$$

The Jacobi polynomials involved in the right-hand side of (3.152) have parameters greater than -1 if $s, r > -\frac{\eta}{2} - 1$, i.e. $s, r \geq -1$ for all $\eta \in (0,1)$. Meanwhile, the left-hand side polynomial requires that $s, r < \frac{\eta}{2}$, i.e. $s, r \leq 0$ for $\eta \in (0,1)$. These restrictions produce the four combinations shown below.

1. $r, s = -1$ with $k \in \mathbb{N}$. This case corresponds to (3.90):

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{\operatorname{sgn}(\tau - t)}{|\tau - t|^{\eta}} \widetilde{P}_k^{\left(\frac{\eta}{2}-1\right)}(t) = -\frac{\pi}{2\,k! \, \sin\frac{\eta\pi}{2}} \frac{\Gamma(k+\eta)}{\Gamma(\eta)} P_{k-1}^{\left(\frac{\eta}{2}\right)}(\tau); \quad (3.153)$$

2. $r = -1$ and $s = 0$ with $k \in \mathbb{N}_0$:

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{\operatorname{sgn}(\tau - t)}{|\tau - t|^{\eta}} \widetilde{P}_k^{\left(\frac{\eta}{2}-1, \frac{\eta}{2}\right)}(t) = -\frac{\pi}{k! \, \sin\frac{\eta\pi}{2}} \frac{\Gamma(k+\eta)}{\Gamma(\eta)} P_{k}^{\left(\frac{\eta}{2}, \frac{\eta}{2}-1\right)}(\tau); \quad (3.154)$$

3. $r = 0$ and $s = -1$. This case corresponds to (3.154) after swapping the parameters of the Jacobi polynomials $\frac{\eta}{2} \longleftrightarrow \frac{\eta}{2} - 1$;

4. $r = 0$ and $s = 0$ with $k \in \mathbb{N}_0$:

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{\operatorname{sgn}(\tau - t)}{|\tau - t|^{\eta}} \widetilde{P}_k^{\left(\frac{\eta}{2}, \frac{\eta}{2}\right)}(t) = \frac{2\pi \, \Gamma(k+\eta)}{k! \, \sin\frac{\eta\pi}{2} \Gamma(\eta)} P_{k+1}^{\left(\frac{\eta}{2}-1, \frac{\eta}{2}-1\right)}(\tau). \quad (3.155)$$

Now consider (3.151) in the case $\theta = \frac{1}{2}$,

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{1}{|\tau - t|^{\eta}} \widetilde{P}_k^{\left(\frac{\eta-1}{2}+s, \frac{\eta+1}{2}+r\right)}(t)$$

$$= \frac{(-1)^{r+s+1} 2^{r+s+1} \, \pi \, \Gamma(k+\eta) \sin\pi\left(-\frac{\eta-1}{2}\right)}{k! \, \Gamma(\eta) \cos\frac{\eta\pi}{2} \, \sin\pi\left(\frac{\eta-1}{2} - r\right)} P_{k+r+s+1}^{\left(\frac{\eta+1}{2}-s-1, \frac{\eta-1}{2}-r-1\right)}(\tau), \quad (3.156)$$

for which $s > -\frac{\eta+1}{2}$ and $r > -\frac{\eta+3}{2}$, i.e. $s > -\frac{1}{2}$ and $r > -\frac{3}{2}$ for all $\eta \in (0,1)$. Nevertheless, the parameters of the polynomial in the right-hand side of (3.156) impose the restriction $s < \frac{\eta+1}{2}$ and $r < \frac{\eta-1}{2}$, i.e. $s < \frac{1}{2}$ and $r < 0$ for all $\eta \in (0,1)$. Hence $s = 0$ and $r = -1$, which lead to (3.157)

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{1}{|\tau - t|^{\eta}} \widetilde{P}_k^{\left(\frac{\eta-1}{2}\right)}(t) = \frac{\pi}{k! \, \cos\frac{\eta\pi}{2}} \frac{\Gamma(k+\eta)}{\Gamma(\eta)} P_{k}^{\left(\frac{\eta-1}{2}\right)}(\tau) \quad (3.157)$$

with $k \in \mathbb{N}_0$. One can also observe that (3.89) in (ARUTIUNIAN, 1959) is a particular case of (3.157) with $k = 0$.

3.5.2 Extension to $\alpha \in [0, 2)$

Let us consider the equation

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{1}{|\tau - t|^{\eta}} f(t) = \lim_{\epsilon \to 0^+} \int_{-1}^{\tau-\epsilon} \mathrm{d}t \, \frac{1}{(\tau - t)^{\eta}} f(t) + \int_{\tau+\epsilon}^{1} \mathrm{d}t \, \frac{1}{(t - \tau)^{\eta}} f(t) = h(\tau) \quad (3.158)$$

with functions $f(\tau)$ and $h(\tau)$ belonging to $C^1[-1, 1]$ and $0 < \eta < 1$. Due to the (weak) singularity, the integral is understood in the sense of Cauchy principal value.

Taking the derivative with respect to τ on both sides and applying the rule for differentiation under the integral operator (so called Leibniz integral rule) yields

$$h'(\tau) = \lim_{\epsilon \to 0^+} \left[\frac{f(\tau - \epsilon)}{(\tau - (\tau - \epsilon))^{\eta}} - \eta \int_{-1}^{\tau-\epsilon} \mathrm{d}t \, \frac{1}{(\tau - t)^{\eta+1}} f(t) \right. \quad (3.159)$$

$$\left. - \frac{f(\tau + \epsilon)}{((\tau + \epsilon) - \tau)^{\eta}} + \eta \int_{\tau+\epsilon}^{1} \mathrm{d}t \, \frac{1}{(t - \tau)^{\eta+1}} f(t) \right]$$

$$= \lim_{\epsilon \to 0^+} \left[-\eta \left(\int_{-1}^{\tau-\epsilon} \mathrm{d}t \, \frac{1}{(\tau - t)^{\eta+1}} f(t) + \int_{\tau+\epsilon}^{1} \mathrm{d}t \, \frac{-1}{(t - \tau)^{\eta+1}} f(t) \right) \right. \quad (3.160)$$

$$\left. - \frac{f(\tau + \epsilon) - f(\tau - \epsilon)}{\epsilon^{\eta}} \right]. \quad (3.161)$$

As any $f(\tau)$ in $C^1[-1, 1]$ is also Lipschitz continuous in the vicinity of any interior τ, i.e. $|f(\tau + \epsilon) - f(\tau - \epsilon)| \leq M\epsilon$, the third term in the left-hand side of (3.161) is proportional to $\epsilon^{1-\eta}$, namely $-2\epsilon^{1-\eta} f'(\tau)$, which vanishes in the limit $\epsilon \to 0^+$. The non-vanishing terms correspond to

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{\mathrm{sgn}\,(\tau - t)}{|\tau - t|^{\eta+1}} f(t) = -\frac{1}{\eta} h'(\tau). \quad (3.162)$$

Let us define $\alpha \equiv \eta + 1$ and take $f(t)$ and $h(\tau)$ from (3.157), namely

$$f(t) = \widetilde{P}_k^{\left(\frac{\eta-1}{2}\right)}(t) = \widetilde{P}_k^{\left(\frac{\alpha}{2}-1\right)}(t) \quad (3.163)$$

$$h(\tau) = \frac{\pi}{k! \, \cos \frac{\eta\pi}{2}} \frac{\Gamma(k + \eta)}{\Gamma(\eta)} P_k^{\left(\frac{\eta-1}{2}\right)}(\tau) = \frac{\pi}{k! \, \cos \frac{(\alpha-1)\pi}{2}} \frac{\Gamma(k + \alpha - 1)}{\Gamma(\alpha - 1)} P_k^{\left(\frac{\alpha}{2}-1\right)}(\tau). \quad (3.164)$$

Eqs. (3.71), (3.164) and the Gamma function recursion relation lead to

$$h'(\tau) = \frac{\pi}{k! \, \sin \frac{\alpha\pi}{2}} \frac{\Gamma(k + \alpha - 1)}{\Gamma(\alpha - 1)} \frac{\mathrm{d}}{\mathrm{d}t} P_k^{\left(\frac{\alpha}{2}-1\right)}(\tau)$$

$$= \frac{\pi}{2\,k! \, \sin \frac{\alpha\pi}{2}} \frac{(\alpha - 1)(k + \alpha - 1)\Gamma(k + \alpha - 1)}{(\alpha - 1)\Gamma(\alpha - 1)} P_{k-1}^{\left(\frac{\alpha}{2}\right)}(\tau)$$

$$= \frac{(\alpha - 1)\pi}{2\,k! \, \sin \frac{\alpha\pi}{2}} \frac{\Gamma(k + \alpha)}{\Gamma(\alpha)} P_{k-1}^{\left(\frac{\alpha}{2}\right)}(\tau). \quad (3.165)$$

Eqs. (3.165) and (3.162) lead to (3.166) which is an extension of (3.90) to $1 < \alpha < 2$.

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{\mathrm{sgn}(\tau - t)}{|\tau - t|^{\alpha}} \widetilde{P}_k^{\left(\frac{\alpha}{2}-1\right)}(t) = -\frac{\pi}{2\,k! \, \sin \frac{\alpha\pi}{2}} \frac{\Gamma(k + \alpha)}{\Gamma(\alpha)} P_{k-1}^{\left(\frac{\alpha}{2}\right)}(\tau), \quad k \in \mathbb{N}. \quad (3.166)$$

4 APPLICATION TO RANDOM SEARCHES IN 1D

4.1 OVERVIEW

This chapter presents a proof that the expectation value of flight observables in the presence of absorbing boundaries satisfy a Fredholm integral equation of the second kind under certain assumptions. Furthermore, an approximate solution of this integral equation is obtained with the aid of the properties of Jacobi polynomials for observables such as the absorption probability at one boundary, the mean number of steps and the mean flight path length before absorption, under a power law step length distribution. The rationale behind this choice of distribution is that, even though the scale parameter imposes a threshold that forbids any step shorter and the central part is by no means similar to the α-stable Levy distribution, the Pareto Type I provides a suitable proxy for mimicking the heavy-tailed behaviour of the latter, which is tractable by analytic methods at the same time.

Hereafter, the random flyer is restricted to a bounded interval of $\mathbb{R}$ denoted by Ω. For convenience and without loss of generality, Ω is centered at 0 and its length is denoted by L, hence $\Omega = \left(-\frac{L}{2}, \frac{L}{2}\right)$. Conversely, Ω^c denotes the complement of Ω in $\mathbb{R}$. Whenever a landing in Ω^c occurs, the target is hit, the flight ends and the flyer is said to be absorbed.

4.2 EXPECTATION VALUE OF ADDITIVE OBSERVABLES

The purpose of this section is to prove that the expectation value of any observable is the solution to an integral equation, provided certain conditions. Therefore, closed-form expressions for the expectation value of certain observables can be obtained from integral equations with known solutions, such as the spectral relations introduced in section 3.5.

Let $\mathcal{Q}(x_{j-1}, x_j)$ represent any observable measured during the j-th step departing from position x_{j-1} and landing in any point x_j. If x_j lies in Ω^c, the flyer is absorbed at $\partial\Omega$. Otherwise, the flyer remains unabsorbed and can take the $(j+1)$-th step. Let also state that $\mathcal{Q}$ is *additive* in the sense that the value assigned to any flight path with vertex sequence $(x_0, x_1, \cdots, x_{n-1}, x_n)$ is the sum of $\mathcal{Q}(x_{j-1}, x_j)$ of each *independent* step j, i.e. $\mathcal{Q}(x_0, x_1, \cdots, x_{n-1}, x_n) = \sum_{i=0}^{n-1} \mathcal{Q}(x_i, x_{i+1})$.

Let $\hat{K}$ denote the operator that maps a function $f(x)$ defined on Ω into $\hat{K}[f](x')$, also defined on Ω, by means of the step distribution $p(x', x)$, i.e.

$$\hat{K}[f](x') \equiv \int_{\Omega} \mathrm{d}x\, p(x', x) f(x). \tag{4.1}$$

Considering that the kernel of $\hat{K}$ is a transition probability between states x and x', $\hat{K}[f](x')$ can be interpreted as the single step expectation value of f given the initial state x' and that the flyer lands inside the search interval or the expectation value of f given any initial state x in the search space and landing in x'.

Let us consider the k-th step of a flight $(x_0, x_1, \cdots, x_{n-1}, x_n)$ and define the *single-step* expectation value of $\mathcal{Q}$, regardless of whether absorption occurs or not as $q(x_{k-1})$, namely

$$q(x_{k-1}) = \int_{\mathbb{R}} \mathrm{d}x_k\, p(x_{k-1}, x_k) \mathcal{Q}(x_{k-1}, x_k) \overset{sh.}{=} \int_{\bar{k}} {}_{k-1}p_k \cdot {}_{k-1}\mathcal{Q}_k, \tag{4.2}$$

where $\overset{sh.}{=}$ stands for *shorthand*.

Let $q^*(x_{k-1})$ represent the *single-step* expectation value of $\mathcal{Q}$ given absorption during the k-th step, that is

$$q^*(x_{k-1}) = \int_{\Omega^c} \mathrm{d}x_k\, p(x_{k-1}, x_k) \mathcal{Q}(x_{k-1}, x_k) \overset{sh.}{=} \int_{k^*} {}_{k-1}p_k \cdot {}_{k-1}\mathcal{Q}_k. \tag{4.3}$$

Conversely, let $q^o(x_{k-1})$ in (4.3) represents the *single-step* expectation value of $\mathcal{Q}$ given that the flyer is not absorbed during the step k, that is

$$q^o(x_{k-1}) = \int_{\Omega} \mathrm{d}x_k\, p(x_{k-1}, x_k) \mathcal{Q}(x_{k-1}, x_k) \overset{sh.}{=} \int_{k} {}_{k-1}p_k \cdot {}_{k-1}\mathcal{Q}_k. \tag{4.4}$$

It is clear from these definitions that $q(x) = q^o(x) + q^*(x)$ for any $x \in \Omega$.

For the sake of generality, let $P_0(x_0)$ be the pdf of the departure site location of the flyer. If $P_0(x_0) = \delta(x_0 - x)$, i.e. a flyer localized in x at $t = 0$.

Let $\langle Q|P_0\rangle$ denote the expectation value of $\mathcal{Q}$ before absorption. Considering the partition of flights according to the index k of the step in which the flyer is absorbed and defining $\langle Q_k|P_0\rangle = \int_{\Omega} \mathrm{d}x_0\, P_0(x_0) Q_k(x_0)$ as the conditional expectation value of $\mathcal{Q}$ given absorption during the k-th step, it holds that

$$\langle Q|P_0\rangle = \lim_{n\to\infty} \sum_{k=1}^{n} \langle Q_k|P_0\rangle \,. \tag{4.5}$$

For illustration, let us consider the smallest values of k. For $k = 1$, it is clear that $\langle Q_1|P_0\rangle = \langle q^*|P_0\rangle$.

For $k = 2$, one has that

$$\langle Q_2|P_0\rangle = \int_{\Omega^2} \mathrm{d}x_0\,\mathrm{d}x_1 \int_{\Omega^c} \mathrm{d}x_2\, P_0(x_0)p(x_0,x_1)p(x_1,x_2)\left[Q(x_0,x_1)+Q(x_1,x_2)\right]$$
$$\overset{sh.}{=} \int_{0,1,2^*} P_0 \cdot {}_0p_1 \cdot {}_1p_2 \cdot ({}_0\mathcal{Q}_1 + {}_1\mathcal{Q}_2)$$
$$\overset{sh.}{=} \int_{0,1,\bar{2}} P_0 \cdot {}_0p_1 \cdot {}_0\mathcal{Q}_1 \cdot {}_1p_2 - \int_{0,1,2} P_0 \cdot {}_0p_1 \cdot {}_0\mathcal{Q}_1 \cdot {}_1p_2 + \int_{0,1,2^*} P_0 \cdot {}_0p_1 \cdot {}_1p_2 \cdot {}_1\mathcal{Q}_2$$
$$\overset{sh.}{=} \int_{0,1} P_0 \cdot {}_0p_1 \cdot {}_0\mathcal{Q}_1 + \int_{0,1,2^*} P_0 \cdot {}_0p_1 \cdot {}_1p_2 \cdot {}_1\mathcal{Q}_2 - \int_{0,1,2} P_0 \cdot {}_0p_1 \cdot {}_0\mathcal{Q}_1 \cdot {}_1p_2$$
$$= \langle q^o|P_0\rangle + \left\langle \hat{K}q^* \middle| P_0 \right\rangle - \int_{0,1,2} P_0 \cdot {}_0p_1 \cdot {}_0\mathcal{Q}_1 \cdot {}_1p_2. \tag{4.6}$$

For $k = 3$, setting aside the term corresponding to the absorption step $({}_2\mathcal{Q}_3)$ and performing the integration over the final position in Ω^c (3^*) for the remaining terms as the difference between the integrals over $\mathbb{R}$ $(\bar{3})$ and Ω (3) lead to

$$\langle Q_3|P_0\rangle \overset{sh.}{=} \int_{0,1,2,3^*} P_0 \cdot {}_0p_1 \cdot {}_1p_2 \cdot {}_2p_3 \cdot ({}_0\mathcal{Q}_1 + {}_1\mathcal{Q}_2 + {}_2\mathcal{Q}_3)$$
$$\overset{sh.}{=} \int_{0,1,2} P_0 \cdot {}_0p_1 \cdot {}_1p_2 \cdot {}_1\mathcal{Q}_2 + \int_{0,1,2,3^*} P_0 \cdot {}_0p_1 \cdot {}_1p_2 \cdot {}_2p_3 \cdot {}_2\mathcal{Q}_3$$
$$+ \int_{0,1,2} P_0 \cdot {}_0p_1 \cdot {}_0\mathcal{Q}_1 \cdot {}_1p_2 - \int_{0,1,2,3} P_0 \cdot {}_0p_1 \cdot \left[{}_0\mathcal{Q}_1 \cdot {}_1p_2 \cdot {}_2p_3 + {}_1p_2 \cdot {}_1\mathcal{Q}_2 \cdot {}_2p_3\right]$$
$$= \left\langle \hat{K}q^o \middle| P_0 \right\rangle + \left\langle \hat{K}^2 q^* \middle| P_0 \right\rangle + \int_{0,1,2} P_0 \cdot {}_0p_1 \cdot {}_0\mathcal{Q}_1 \cdot {}_1p_2$$
$$- \int_{0,1,2,3} P_0 \cdot {}_0p_1 \cdot \left[{}_0\mathcal{Q}_1 \cdot {}_1p_2 \cdot {}_2p_3 + {}_1p_2 \cdot {}_1\mathcal{Q}_2 \cdot {}_2p_3\right]. \tag{4.7}$$

The same procedure that led to (4.6) and (4.7) can be generalized to any k as follows

$$\langle Q_k|P_0\rangle = \left\langle \hat{K}^{k-1} q^o \middle| P_0 \right\rangle + \left\langle \hat{K}^k q^* \middle| P_0 \right\rangle + R_{k-1} - R_k. \tag{4.8}$$

The "remainder" R_k is defined as $R_k \equiv \int_{0,1,\cdots,k} P_0 \cdot \prod_{i=0}^{k-1} {}_ip_{i+1} \cdot \sum_{j=0}^{k-2} {}_j\mathcal{Q}_{j+1}$ with the particular case $R_1 = 0$.

Inserting (4.8) back into (4.5) leads to

$$\langle Q|P_0\rangle = \langle q^*|P_0\rangle + \lim_{n\to\infty} \left[\sum_{k=1}^{n} \left(\left\langle \hat{K}^{k-1} q^o \middle| P_0 \right\rangle + \left\langle \hat{K}^k q^* \middle| P_0 \right\rangle \right) - R_n \right]. \tag{4.9}$$

The remainder R_n goes to zero as $n \to \infty$. The underlying intuition is that $\int_\Omega \mathrm{d}x\, p(x,x') < 1$ for every $x' \in \Omega$ due to the probability leaked into Ω^c at the (absorbing) boundary. Therefore, with the aid of the identity $q(x) = q^o(x) + q^*(x)$, (4.9) becomes

$$\langle Q|P_0\rangle = \sum_{k=0}^{\infty} \left\langle \hat{K}^k q \middle| P_0 \right\rangle . \tag{4.10}$$

Let us consider hereafter the case of an initially localized flyer $P_0 = \delta(x_0 - x)$ in (4.10), namely

$$Q(x) = \sum_{k=0}^{\infty} \hat{K}^k q(x). \tag{4.11}$$

As the premises of Theorem 3.2 are fulfilled, it is safe to say that $\hat{T} \equiv 1 - \hat{K}$ is the inverse of the operator $\hat{T}^{-1}$ formally defined as $\hat{T}^{-1} \equiv \sum_{k=0}^{\infty} \hat{K}^k$. Applying $\hat{T}$ on both sides of (4.11) leads to the Fredholm equation of the second kind $(1 - \hat{K})Q(x) = q(x)$, which is equivalent to (4.12).

$$\hat{D}Q(x) \equiv (\hat{K} - 1)Q(x) = -q(x) \tag{4.12}$$

4.3 POWER LAW SINGLE-STEP LENGTH DISTRIBUTION

In virtue of the asymptotic power law behaviour of the Lévy α-stable distribution in the form of (2.34), a way to circumvent the predicament of dealing with an α-stable distribution in Euclidean space is to consider a Pareto type I step length distribution, which corresponds to an inverse power pdf with a lower cutoff length l_0 in order to ensure it is an integrable function.

Based on that reasoning, let us consider the power law kernel given by (4.13), with $0 < \alpha < 2$ and normalisation constant $\frac{\alpha l_0^\alpha}{2}$, ensuring that $\int_\Omega \mathrm{d}x' \, \delta(x' - x_0) \int_{\mathbb{R}} \mathrm{d}x \, p(x', x) = 1$ for any starting point x_0.

$$p(x', x) = \frac{\alpha l_0^\alpha}{2} \frac{\mathbb{1}_{(0,\infty)}(|x - x'| - l_0)}{|x - x'|^{\alpha+1}}, \tag{4.13}$$

where $\mathbb{1}_{(0,\infty)}(s)$ stands for the Heaviside step function in the notation of an indicator function, i.e. $\mathbb{1}_{(0,\infty)}(s) = 1$ whenever $s \in (0, \infty)$; otherwise it takes null value.

The integral operator $\hat{K}$ with the kernel (4.13) acting on $Q(x)$ is given by

$$\hat{K}Q(x_0) = \frac{\alpha l_0^\alpha}{2} \int_\Omega \mathrm{d}x' \, \delta(x' - x_0) \int_\Omega \underbrace{\mathrm{d}x \, |x - x'|^{-(\alpha+1)}}_{\mathrm{d}v} \underbrace{\mathbb{1}_{(0,\infty)}(|x - x'| - l_0) Q(x)}_{u} . \tag{4.14}$$

The equation above can be cast into the form

$$\hat{K}Q(x_0) = \frac{l_0^\alpha}{2}\int_\Omega \mathrm{d}x'\,\delta(x'-x_0)\Bigg[\int_\Omega \mathrm{d}x\,\frac{\mathrm{sgn}(x-x')\mathbb{1}_{(0,\infty)}(|x-x'|-l_0)}{|x-x'|^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}Q(x) - \frac{Q\left(-\frac{L}{2}\right)}{\left(\frac{L}{2}+x'\right)^\alpha}\mathbb{1}_{(0,\infty)}(x'-(-\tfrac{L}{2}+l_0)) - \frac{Q\left(\frac{L}{2}\right)}{\left(\frac{L}{2}-x'\right)^\alpha}\mathbb{1}_{(0,\infty)}(\tfrac{L}{2}-l_0-x') + l_0^{-\alpha}\left[Q(x'+l_0)+Q(x'-l_0)\right]\Bigg] \quad (4.15)$$

by performing the following partial integration in x

$$\hat{K}Q(x_0) = \frac{\alpha l_0^\alpha}{2}\int_\Omega \mathrm{d}x'\,\delta(x'-x_0)\left[-\int_\Omega \mathrm{d}x\,\frac{\mathrm{d}u}{\mathrm{d}x}v(x) + uv\Big|_{\partial\Omega}\right] \quad (4.16)$$

with $u'(x)$ and $v(x)$ given by

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \left[\delta(x-(x'+l_0)) - \delta(x'-l_0-x)\right]Q(x) + \mathbb{1}_{(0,\infty)}(|x-x'|-l_0)\frac{\mathrm{d}}{\mathrm{d}x}Q(x) \quad (4.17)$$

$$v = -\frac{1}{\alpha}\,\mathrm{sgn}(x-x')|x-x'|^{-\alpha}, \quad (4.18)$$

respectively. For the former, recall that $\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{1}_{(0,\infty)}(s) = \delta(s)$ and that $\mathbb{1}_{(0,\infty)}(|x-x'|-l_0)$ can be split into two terms $\mathbb{1}_{(0,\infty)}(x-(x'+l_0)) + \mathbb{1}_{(0,\infty)}(x'-l_0-x)$.

On the other hand, it follows from (4.12) that the expectation value $Q(x_0)$ is the solution of the equation

$$\hat{D}_\alpha Q(x_0) = -l_0^{-\alpha}q(x_0), \quad (4.19)$$

where the operator $\hat{D}_\alpha$ is defined as $\hat{D}_\alpha Q(x_0) \equiv l_0^{-\alpha}(\hat{K}-1)Q(x_0)$.

After carrying out the integration with respect to x' in (4.15), the operator $\hat{D}_\alpha$ takes the form

$$\hat{D}_\alpha Q(x_0) = \frac{1}{2}\Bigg[\,⨍_\Omega \mathrm{d}x\,\frac{\mathrm{sgn}(x-x_0)}{|x-x_0|^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}Q(x) - \underbrace{⨍_{B(x_0,l_0)} \mathrm{d}x\,\frac{\mathrm{sgn}(x-x_0)}{|x-x_0|^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}Q(x)}_{(A)} - \frac{Q\left(-\frac{L}{2}\right)}{\left(\frac{L}{2}+x_0\right)^\alpha}\mathbb{1}_{(0,\infty)}(\tfrac{L}{2}-l_0+x_0) - \frac{Q\left(\frac{L}{2}\right)}{\left(\frac{L}{2}-x_0\right)^\alpha}\mathbb{1}_{(0,\infty)}(\tfrac{L}{2}-l_0-x_0) + \underbrace{l_0^{-\alpha}\left[Q(x_0+l_0)-2Q(x_0)+Q(x_0-l_0)\right]}_{(B)}\Bigg]. \quad (4.20)$$

Note that the integral term on $\Omega - B(x_0,l_0)$ is split in two integrals defined on Ω and $B(x_0,l_0)$ respectively. Due to the singular kernel, these two integrals are understood in the principal value sense.

The term (A) in (4.20) removes the contribution of the jumps within the interval $(x_0 - l_0, x_0 + l_0)$ which do not take place due to the cutoff l_0 implicit in the power law step length distribution. Provided l_0 small enough, one has the lowest-order approximation

$$Q'(x) \approx Q'(x_0) + Q''(x_0)(x - x_0) \tag{4.21}$$

in $(x_0 - l_0, x_0 + l_0)$. The constant term does not contribute to the result due to the odd-parity integrand around x_0, whereas the first term equals $l_0^{2-\alpha}\frac{2}{2-\alpha}Q''(x_0)$. As for (B), the leading term of the series around x_0 with small l_0 equals $l_0^{2-\alpha}Q''(x_0)$. Hence, the aggregate contribution of the leading terms of (A) and (B) equals $-l_0^{2-\alpha}\frac{\alpha}{2-\alpha}Q''(x_0)$ and $\hat{D}_\alpha Q(x_0)$ takes the *approximate* form

$$\hat{D}_\alpha Q(x_0) \approx \frac{1}{2}\Bigg[\underbrace{\fint_\Omega \mathrm{d}x \, \frac{\operatorname{sgn}(x-x_0)}{|x-x_0|^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}Q(x) - \frac{Q\left(-\frac{L}{2}\right)}{\left(\frac{L}{2}+x_0\right)^\alpha} - \frac{Q\left(\frac{L}{2}\right)}{\left(\frac{L}{2}-x_0\right)^\alpha}}_{\hat{D}_\alpha^{(0)}} \\ - l_0^{2-\alpha}\frac{\alpha}{2-\alpha}\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}Q(x)\bigg|_{x=x_0}\Bigg]. \tag{4.22}$$

The remainder of this chapter will be restricted to the solution of the equation

$$\hat{D}_\alpha^{(0)}Q(x_0) = -l_0^{-\alpha}q(x_0) \tag{4.23}$$

by means of the "benchmark to beat", i.e. Sonin inversion formula, for some relevant quantities such as the average number of steps before absorption and the absorption probability at a specific boundary site. The Jacobi Polynomials spectral relations approach will be the matter of Chapter 5.

4.4 INVERSION FORMULA APPROACH

Eq (4.23) corresponds to the case $c_1 = -1$, $c_2 = 0$ and $\nu = 1$ in (3.37), provided that the boundary terms implicit in $\hat{D}_\alpha^{(0)}$ vanish. With those parameters, (3.56) becomes

$$\tan \pi\beta' = \frac{\sin \pi\alpha}{1 + \cos \pi\alpha} = \tan \tfrac{\pi\alpha}{2}. \tag{4.24}$$

Hence, $\beta' = \frac{\alpha}{2} + k'$ with $k' \in \mathbb{Z}$.

Let us take $k' = 0$, for which $0 < \beta' < \alpha$, which leads to unbounded solutions $\psi(s)$ of (3.49) at a. In addition, taking $-\varkappa = 1$ leads to unbounded $\psi(s)$ at b. This choice allows the widest class of solutions and, since the function we seek in (4.22) is not $Q(x)$, but $\frac{\mathrm{d}Q}{\mathrm{d}x}$, for which there is no *a priori* restriction on whether it should be bounded at the endpoints or not. With those choices of β' and $\varkappa$, one has that (3.60) becomes

$$\phi(s) = -\frac{(1+\cos\pi\alpha)}{2}\mathcal{D}_{a+}^{1-\alpha}[h] - \frac{\sin\pi\alpha}{2\pi}\frac{r_b^{\frac{\alpha}{2}-1}(s)}{r_a^{\frac{\alpha}{2}-\alpha+1}(s)}\fint_a^b \mathrm{d}t\,\frac{r_a^{\frac{\alpha}{2}-\alpha+1}(t)}{r_b^{\frac{\alpha}{2}-1}(t)}\frac{\mathcal{D}_{a+}^{1-\alpha}[h]}{t-s} + \frac{r_b^{\frac{\alpha}{2}-1}(s)}{r_a^{\frac{\alpha}{2}-\alpha+1}(s)}P_0(s). \quad (4.25)$$

Now we proceed to find the solution $Q(x)$ to (4.23) for the absorption probability at a given absorbing endpoint and the mean number of steps, by obtaining $\frac{\mathrm{d}Q}{\mathrm{d}x}$ from (3.60) and then integrating to obtain a solution that satisfies the boundary conditions.

4.4.1 Absorption probability

Let us consider the absorption probability $P_+(x_0)$ at the boundary $x=b$ for a flyer initially located at x_0. . The single step expectation value function $q(x_0)$ denoted in this case by $p_+(x_0)$ is given by

$$\begin{aligned} p_+(x_0) &= \int_\Omega \mathrm{d}x\, p(x_0,x)\mathbb{1}_{\left[\frac{L}{2},\infty\right)}(x) \\ &= \frac{\alpha l_0^\alpha}{2}\int_{[b,\infty)} \mathrm{d}x\,\frac{\mathbb{1}_{(0,\infty)}(x-x_0-l_0)}{(x-x_0)^{\alpha+1}} \\ &= \frac{l_0^\alpha}{2}\frac{1}{(b-x_0)^\alpha}\mathbb{1}_{(a,b-l_0]}(x_0) + \frac{1}{2}\mathbb{1}_{(b-l_0,b)}(x_0). \end{aligned} \quad (4.26)$$

The boundary conditions for the current problem are $P_+(a)=0$ and $P_+(b)=1$. As long as $x_0\in(a,b-l_0]$, the boundary term implicit in $\hat{D}_\alpha^{(0)}$ corresponding to b in (4.20) cancels out the first term in (4.26), leading to the homogeneous equation

$$\hat{D}_\alpha^{(0)}P_+(x_0) = \fint_\Omega \mathrm{d}x\,\frac{\operatorname{sgn}(x-x_0)}{|x-x_0|^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}P_+(x) = 0. \quad (4.27)$$

In such case, the inversion formula (3.60) leads to a solution of the form

$$\phi_H(s) = \frac{r_b^{\frac{\alpha}{2}-1}(s)}{r_a^{1-\frac{\alpha}{2}}(s)}P_0(s) = C^{-1}(s-a)^{\frac{\alpha}{2}-1}(b-s)^{\frac{\alpha}{2}-1}. \quad (4.28)$$

Note the correspondence between (4.28) and the spectral relation (3.90) with $k=0$.

With the result (4.28) in hand, one has that the solution to (4.27) is given by

$$P_+(x_0) = \int_a^{x_0}\mathrm{d}s\,\phi_H(s), \quad (4.29)$$

where the constant C must be chosen consistently with the boundary condition $P_+(b) = 1$. Note that the boundary condition $P_+(a) = 0$ is already satisfied by (4.29). Therefore, one has that

$$\int_a^b \mathrm{d}s\, \phi_H(s) = C^{-1} L^{\alpha-1} \mathrm{B}\left(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}\right) = 1. \tag{4.30}$$

On the other hand, it follows from (3.8), after a suitable change of variables, that

$$\int_a^{x_0} \mathrm{d}s\, (s-a)^{\frac{\alpha}{2}-1} (b-s)^{\frac{\alpha}{2}-1} = (b-a)^{\alpha-1}\, \mathrm{B}\left(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}; \tfrac{x_0-a}{b-a}\right). \tag{4.31}$$

Also recall that the incomplete Beta function is related to the Gaussian hypergeometric function ${}_2F_1$ through (3.9). Therefore, one has that the absorption probability at the upper boundary for a flyer starting at x_0 is given by

$$P_+(x_0) = \left(\frac{x_0-a}{b-a}\right)^{\frac{\alpha}{2}} \frac{{}_2F_1\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \frac{\alpha}{2}+1, \frac{x_0-a}{b-a}\right)}{\frac{\alpha}{2}\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)}, \tag{4.32}$$

in agreement with the results reported in (BULDYREV et al., 2001a) and (BULDYREV et al., 2001b).

4.4.2 Mean number of steps

Let $N(x_0)$ denote the mean number of steps before absorption taken by a random flyer with state transitions given by (4.13) whose initial state is x_0. It is clear that the absorbing boundary conditions imply that $N|_{\partial\Omega} = 0$ as an absorbed flyer cannot take further steps and that the single-step expectation value $n(x_0)$ is given by $n(x_0) = \mathbb{1}_{(a,b)}(x_0)$, i.e. $n(x_0)$ takes the value 1 whenever x_0 is in the interior of the search interval $[a, b]$.

Back to (4.25), one has that $h(\tau) = 1$. It follows from the definition of $\mathcal{D}_{a+}$ (3.43) that

$$\mathcal{D}_{a+}^{1-\alpha}[h] = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}(t-a)^{\alpha-1} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} r_a^{\alpha-1}(t), \tag{4.33}$$

which leads to

$$\begin{aligned} \phi(s) = &-\frac{(1+\cos\pi\alpha)}{2\Gamma(\alpha)\Gamma(1-\alpha)} r_a^{\alpha-1}(s) \\ &- \frac{\sin\pi\alpha}{2\pi\Gamma(\alpha)\Gamma(1-\alpha)} \frac{1}{r_a^{1-\frac{\alpha}{2}}(s)\, r_b^{1-\frac{\alpha}{2}}(s)} \fint_a^b \mathrm{d}t\, \frac{r_a^{\frac{\alpha}{2}}(t)\, r_b^{1-\frac{\alpha}{2}}(t)}{t-s} \\ &+ \frac{1}{r_a^{1-\frac{\alpha}{2}}(s)\, r_b^{1-\frac{\alpha}{2}}(s)} P_0(s). \end{aligned} \tag{4.34}$$

The second term in (4.34) seems daunting, as it involves an integral in the principal value sense of a singular integrand. However, it is known that (SAMKO; KILBAS; MARICHEV, 1993)

$$\fint_a^b \mathrm{d}t\, \frac{r_a^{\frac{\alpha}{2}}(t)\, r_b^{1-\frac{\alpha}{2}}(t)}{t-s} = -\pi \left(r_a^{\frac{\alpha}{2}}(s) r_b^{1-\frac{\alpha}{2}}(s) \cot \tfrac{\pi\alpha}{2} - r_b(s) \csc \tfrac{\pi\alpha}{2} + L\tfrac{\alpha}{2} \csc \tfrac{\pi\alpha}{2} \right), \quad (4.35)$$

hence (4.34) becomes

$$\phi(s) = -\frac{(1+\cos\pi\alpha)}{2\Gamma(\alpha)\Gamma(1-\alpha)} r_a^{\alpha-1}(s) + \frac{\sin\pi\alpha}{2\Gamma(\alpha)\Gamma(1-\alpha)} \Big[r_a^{\alpha-1}(s) \cot \tfrac{\pi\alpha}{2} - r_a^{\frac{\alpha}{2}-1}(s)\, r_b^{\frac{\alpha}{2}}(s) \csc \tfrac{\pi\alpha}{2} + \tfrac{L\alpha}{2} r_a^{\frac{\alpha}{2}-1}(s)\, r_b^{\frac{\alpha}{2}-1}(s) \csc \tfrac{\pi\alpha}{2} \Big] + r_a^{\frac{\alpha}{2}-1}(s)\, r_b^{\frac{\alpha}{2}-1}(s) P_0(s). \quad (4.36)$$

Since $\sin\pi\alpha\, \cot\frac{\pi\alpha}{2} = 1+\cos\pi\alpha$, the first two terms in (4.36) cancel out. In addition, the fourth and fifth terms correspond to the homogeneous solution (4.28) and can be grouped together, leading to

$$\phi(s) = -\frac{\cos\frac{\pi\alpha}{2}}{\Gamma(\alpha)\Gamma(1-\alpha)} \left[r_a^{\frac{\alpha}{2}-1}(s)\, r_b^{\frac{\alpha}{2}}(s) - C r_a^{\frac{\alpha}{2}-1}(s)\, r_b^{\frac{\alpha}{2}-1}(s) \right], \quad (4.37)$$

where the constant C stands for the factor $\frac{L\alpha}{2} + P_0(s)$ of the homogeneous solution term that needs to be calibrated according to the boundary condition $N(b) = 0$. Some simplifications were also carried out by means of the identity $\sin\pi\alpha \csc\frac{\pi\alpha}{2} = 2\cos\frac{\pi\alpha}{2}$.

The constant C is such that

$$\int_a^b \mathrm{d}s\, \phi(s) \sim (b-a)^\alpha \left[\mathrm{B}\left(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}+1\right) - C\,(b-a)^{-1} \mathrm{B}\left(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}\right) \right] = 0. \quad (4.38)$$

Having in mind (3.12), one has that

$$C = \frac{b-a}{\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)} \mathrm{B}\left(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}+1\right) = \frac{b-a}{\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)} \mathrm{B}\left(\tfrac{\alpha}{2}, \tfrac{\alpha}{2}\right) \frac{\frac{\alpha}{2}}{\alpha} = \frac{b-a}{2}, \quad (4.39)$$

i.e. C is half the length of the search interval.

Finally, one has that

$$\begin{aligned} N(x_0) &\sim \int_a^{x_0} \mathrm{d}s\, r_a^{\frac{\alpha}{2}-1}(s)\, r_b^{\frac{\alpha}{2}-1}(s) \left[r_b(s) - \tfrac{b-a}{2} \right] \\ &\sim \frac{1}{2} \int_a^{x_0} \mathrm{d}s\, [(s-a)(b-s)]^{\frac{\alpha}{2}-1} \left[-2s + (b+a) \right] \\ &\sim \frac{1}{\alpha} \left[(x_0-a)(b-x_0) \right]^{\frac{\alpha}{2}}. \end{aligned} \quad (4.40)$$

Taking into account the factor $-l_0^\alpha$ in (4.23) and the factor in (4.37), one has that

$$N_\alpha(x) = \frac{\sin\pi\frac{\alpha}{2} \cos^2\pi\frac{\alpha}{2}}{\pi\frac{\alpha}{2}} \left[\left(\frac{x_0-a}{l_0} \right) \left(\frac{b-x}{l_0} \right) \right]^{\frac{\alpha}{2}}. \quad (4.41)$$

Observe that (4.41) differs from (1.1) in (BULDYREV et al., 2001a) and (BULDYREV et al., 2001b) by a factor $\cos^2 \pi \frac{\alpha}{2}$, a situation that clearly requires double-checking. Nevertheless, it is worth recalling that the main result of this work concerns the solution of the problem by means of Jacobi polynomials spectral relations, which is the subject of Chapter 5.

5 JACOBI POLYNOMIALS APPROACH

5.1 OVERVIEW

In this chapter, the solution of (4.23) with homogeneous Dirichlet (fixed) boundary for an asymptotic Lévy flyer is obtained by means of spectral relations valid for Jacobi polynomials. Besides reproducing the results from the previous chapter, this approach allows the inclusion of correction terms that were neglected in order to permit the use of closed-form inversion formulae. In so doing, the fit of the simulated date is significantly improved.

We will show that, by means of spectral relations of the Jacobi polynomials, the solution to the absorption probability is straightforward compared to the inversion formulae. Next, a general method for solving (4.23) with homogeneous Dirichlet (fixed) boundary conditions is presented in subsection 5.2.2 along with the applications to the mean number of steps and the mean flight path length.

Finally, the neglected term proportional to $Q''(x)$ resulting from the finite l_0 correction will be treated numerically in section 5.3.

5.2 FIRST-ORDER APPROXIMATION REVISITED

5.2.1 Absorption probability

Hereafter, without loss of generality, we will take the convenient choice of a search interval of length L centered at 0, that is the lower endpoint previously denoted as a corresponds to $-\frac{L}{2}$, while the upper endpoint b corresponds to $\frac{L}{2}$.

In Chapter 4 we concluded that the absorption probability at the right boundary $P_+(x_0)$ is the solution of (4.27), namely

$$\hat{D}_\alpha^{(0)} P_+(x_0) = \fint_\Omega \mathrm{d}x \, \frac{\mathrm{sgn}(x - x_0)}{|x - x_0|^\alpha} \frac{\mathrm{d}}{\mathrm{d}x} P_+(x) = 0. \tag{4.27'}$$

Due to the symmetry of the system centered at 0 under $x \leftrightarrow -x$, the absorption probability P_- at $x = -\frac{L}{2}$ is related to P_+ through $P_-(x_0) = P_+(-x_0)$.

Note that (4.27) is identical to the JP spectral relation (3.166) with $n = 0$. Recalling (3.68) and also that $P_{n-1}^{(\frac{\alpha}{2})}$ is proportional to $\frac{\mathrm{d}}{\mathrm{d}t} P_n^{(\frac{\alpha}{2}-1)}$ in (3.71), the right hand side of (3.166) vanishes for $n = 0$, that is

$$\fint_{-1}^{1} \mathrm{d}t \, \frac{\mathrm{sgn}\,(t-\tau)}{|t-\tau|^{\alpha}} \; \widetilde{P}_0^{\left(\frac{\alpha}{2}-1\right)}(t) = 0. \tag{5.1}$$

For the time being, assume that $L = 2$ in order to spare us the scale transformation from Ω into the JP domain $[-1, 1]$. The solution for (5.1) is given by

$$P_{+}(\tau) = C \int_{-1}^{\tau} \mathrm{d}t \; \widetilde{P}_0^{\left(\frac{\alpha}{2}-1\right)}(t), \tag{5.2}$$

which already satisfies the boundary condition $P_{+}(-1) = 0$

The value of C is determined by the boundary condition $P_{+}(1) = 1$. By means of the variable change $s = \frac{1}{2}(1+t)$, it follows from the definition of the incomplete Beta function (3.8) that $C = \left[\kappa \mathrm{B}\left(1; \frac{\alpha}{2}, \frac{\alpha}{2}\right)\right]^{-1}$. The constant factor κ that arises from the aforementioned variable change turns out to be irrelevant in this case as it also comes out of the integral in (5.2) after performing the same variable change from t to s. Therefore, for $\Omega = [0, 1]$, the solution to P_{+} and P_{-} are given by

$$P_{+}(\widetilde{x}_0) = \frac{\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}; \widetilde{x}_0\right)}{\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}; 1\right)} = \widetilde{x}_0^{\frac{\alpha}{2}} \, \frac{{}_2F_1\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \frac{\alpha}{2}+1; \widetilde{x}_0\right)}{{}_2F_1\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \frac{\alpha}{2}+1; 1\right)} \tag{5.3}$$

and

$$P_{-}(\widetilde{x}_0) = \frac{\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}; 1-\widetilde{x}_0\right)}{\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}; 1\right)} = (1-\widetilde{x}_0)^{\frac{\alpha}{2}} \, \frac{{}_2F_1\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \frac{\alpha}{2}+1; 1-\widetilde{x}_0\right)}{{}_2F_1\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \frac{\alpha}{2}+1; 1\right)}, \tag{5.4}$$

respectively.

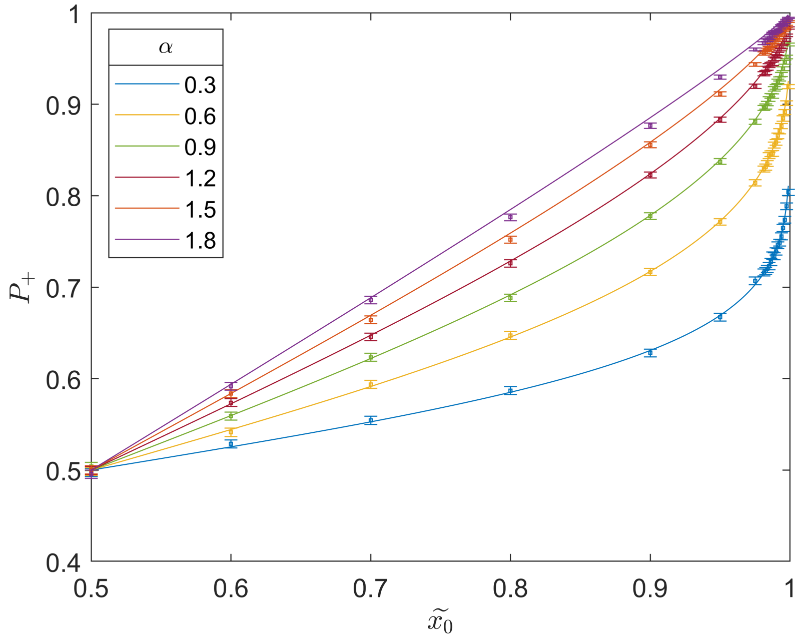
The solution for any L can be obtained from (5.3) and (5.4) by means of the substitution $\widetilde{x}_0 = \frac{1}{2} + \frac{x_0}{L}$, leading to

$$P_{\pm}(x_0) = \frac{\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}; \frac{1}{2} \pm \frac{x_0}{L}\right)}{\mathrm{B}\left(\frac{\alpha}{2}, \frac{\alpha}{2}; 1\right)} = \left(\frac{1}{2} \pm \frac{x_0}{L}\right)^{\frac{\alpha}{2}} \frac{{}_2F_1\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \frac{\alpha}{2}+1; \frac{1}{2} \pm \frac{x_0}{L}\right)}{{}_2F_1\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \frac{\alpha}{2}+1; 1\right)}. \tag{5.5}$$

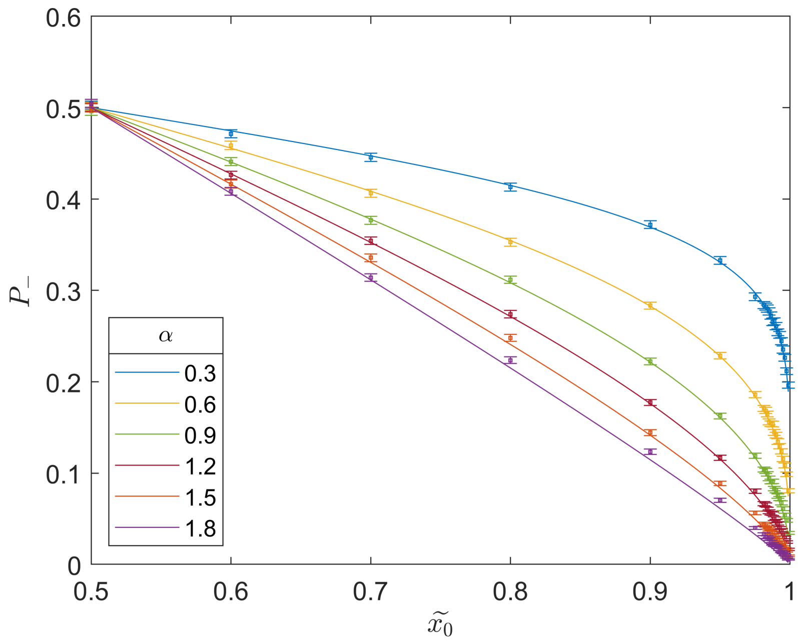
Observe that (5.5) agrees with (4.32) and with the expressions reported in (BULDYREV et al., 2001a) and (BULDYREV et al., 2001b) obtained from Sonine inversion formula.

The comparison between P_{+} and P_{-} given by (5.5) and the results from Monte Carlo simulations are presented in Figure 8 for $L = 1$ and $M = 800$, so that the single-step distribution scale parameter $l_0 = \frac{1}{800}$. As for the simulation results, the average was calculated over $50\,000$ MC flight samples and the 95% confidence intervals were obtained by means of bootstrap with 999 replicates. Unless otherwise stated, these are the same parameters used to produce the results presented hereafter.

Figure 8 – Absorption probability P_+ at boundary $\frac{L}{2}$ (a) and P_- at $-\frac{L}{2}$ (b) as functions of the dimensionless initial position $\widetilde{x}_0 = \frac{x_0}{L} + \frac{1}{2}$ for several values of α in $(0, 2)$. There is no need to plot the range $0 \leq \widetilde{x}_0 \leq 0.5$ due to $P_-(x_0) = P_+(-x_0)$. The continuous approximation deviates from the results of the simulations for $\alpha \geq 1.5$. The parameters of the system are $L = 1$ and $M = 800$ so the step length distribution scale parameter l_0 equals $\frac{1}{800}$. Dots depict the average over $50\,000$ MC flight samples with error bars representing the 95% confidence intervals from bootstrap with 999 replicates.



(a)

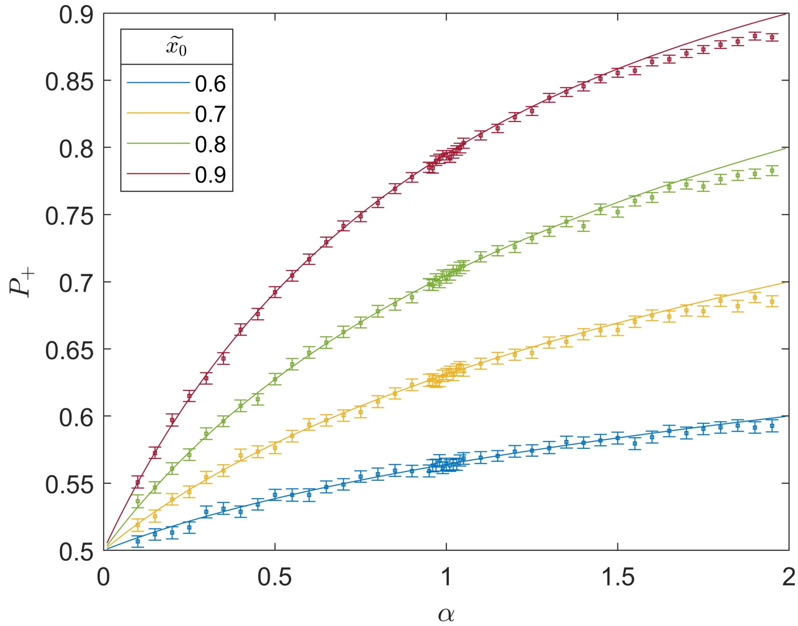


(b)

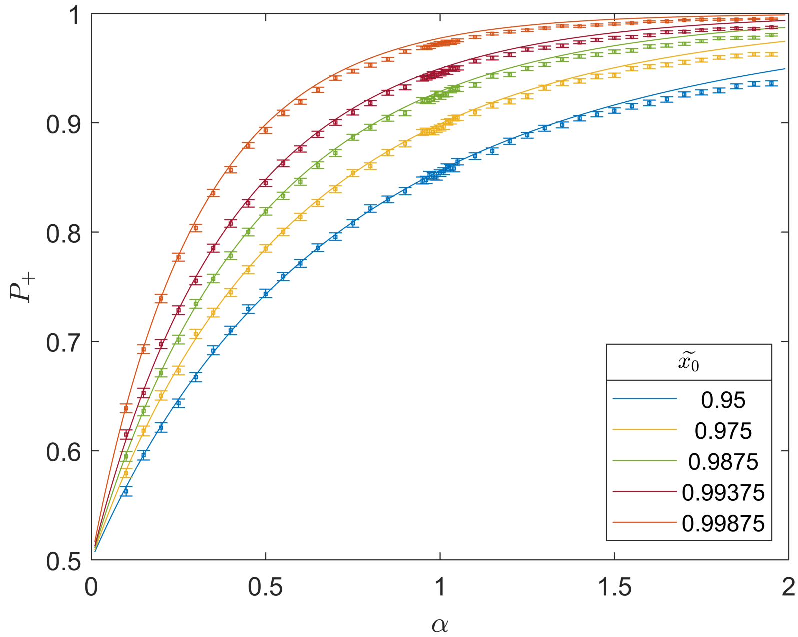
Source: The author (2023)

Observe in Figure 8 that as one approaches the Gaussian $\alpha = 2$ regime, P_+ and P_- become straight lines connecting 0.5 to 1 or 0, respectively. However, as α becomes smaller, both curves approach the constant line $P_* = 0.5$ for $\widetilde{x}_0$ in the middle of the interval. This is consistent with the fact that events where the flyer reaches the initially *distal* target site in a single jump or very few jumps, albeit rare, become more frequent with smaller α.

Figure 9 – Absorption probability P_+ as a function of α for selected values of $\widetilde{x}_0$ located "far" from the absorbing boundary (a) and in its vicinity (b). Overall, good fit is attained for $\alpha \lessapprox 1.5$, provided that $0.05 \lessapprox \widetilde{x}_0 \lessapprox 0.95$. Nevertheless, as one approaches the boundary, the continuous solution overestimates the absorption probability from the simulations to such an extent that, for $\widetilde{x}_0 = 1-\frac{1}{8}$, lack of fit is evident from $\alpha \approx 0.5$ on.



(a)



(b)

Source: The author (2023)

Figure 9 better depicts the deviations of the continuous solution from the simulations that are already noticeable in Figure 8. In general, the closed-form continuous solution fits the simulations well whenever $\alpha \lessapprox 1.5$ and $0.05 \lessapprox \widetilde{x}_0 \lessapprox 0.95$. Nevertheless, as one approaches the absorbing sites, the continuous solution overestimates the absorption probability to such an extent that for $\widetilde{x}_0 = 1 - \frac{1}{800}$, extremely good fit is restricted to $\alpha \lessapprox 0.5$.

5.2.2 General solution to $Q(x_0)$ with homogeneous fixed boundary conditions

Let us consider any observable $Q(x_0)$ for which the assumptions that led to (4.12) and the boundary conditions $Q\left(-\frac{L}{2}\right) = 0$ and $Q\left(\frac{L}{2}\right) = 0$ hold. As a consequence of Theorem 3.1, a suitable solution ansatz to (4.23) in terms of the basis conformed by $\{\widetilde{P}_k^{\left(\frac{\alpha}{2}\right)}, k \in \mathbb{N}_0\}$ is given by

$$Q(x) = \sum_{m\geq 0} \bar{Q}_m \underbrace{\left[\left(\frac{L}{2} - x\right)\left(\frac{L}{2} + x\right)\right]^{\frac{\alpha}{2}}}_{\rho_\Omega^{\left(\frac{\alpha}{2}\right)}} P_{2m}^{\left(\frac{\alpha}{2}\right)}\left(\frac{2x}{L}\right) = \left(\frac{L}{2}\right)^{\alpha} \sum_{m\geq 0} \bar{Q}_m \, \widetilde{P}_{2m}^{\left(\frac{\alpha}{2}\right)}\left(\frac{2x}{L}\right). \quad (5.6)$$

Both the omission of the functions of odd order $2m+1$ and the restriction $\gamma = \delta$ imposed on the parameters of $\widetilde{P}_k^{(\gamma,\delta)}$ arise from (3.69) and the $x \leftrightarrow -x$ symmetry expected from $Q(x)$.

As for the parameter $\frac{\alpha}{2}$, it is a convenient choice due to the fact that the operator $\hat{D}_\alpha^{(0)}$ is a one -to-one correspondence between the bases $\{\widetilde{P}_k^{\left(\frac{\alpha}{2}\right)}, k \in \mathbb{N}_0\}$ and $\{P_k^{\left(\frac{\alpha}{2}\right)}, k \in \mathbb{N}_0\}$. Indeed, (3.70) means that differentiation maps $\widetilde{P}_k^{\left(\frac{\alpha}{2}\right)}$ into $\widetilde{P}_{k+1}^{\left(\frac{\alpha}{2}-1\right)}$, while the operator $⨍_{-1}^{1} \mathrm{d}t \,\mathrm{sgn}\,(t-\tau)|t-\tau|^{-\alpha}\psi(t)$ maps $\widetilde{P}_{k+1}^{\left(\frac{\alpha}{2}-1\right)}$ into $P_k^{\left(\frac{\alpha}{2}\right)}$, as implied by the spectral relation (3.166). Therefore, by assuming (5.6) and taking the inner product with $P_{2n}^{\left(\frac{\alpha}{2}\right)}$ on both sides of (4.23), the solution of the Fredholm equation reduces to a set of uncoupled algebraic linear equations

$$\begin{gathered} \left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| \hat{D}_\alpha^{(0)} Q \right\rangle_{\rho\Omega} = -l_0^{-\alpha} \left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| q \right\rangle_{\rho\Omega} \\ \Updownarrow \\ D_{nm}^{(0)} \bar{Q}_m = -l_0^{-\alpha} \bar{q}_n, \end{gathered} \quad (5.7)$$

where $D_{mn}^{(0)}$ is given by

$$D_{nm}^{(0)} = -\frac{1}{2}\left(\frac{L}{2}\right)^{\alpha+1} \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma(\alpha)} \frac{2^{\alpha+1}}{4n+\alpha+1} \left[\frac{\Gamma\left(2n+\frac{\alpha}{2}+1\right)}{(2n)!}\right]^2 \delta_{nm} \quad (5.8)$$

which is clearly a diagonal matrix.

The proof of (5.7) and (5.8) relies on the spectral relation (3.166) and the orthogonality relation (3.66) satisfied by the classical Jacobi polynomials as shown below.

Let us apply $\hat{D}_\alpha^{(0)}$ onto $Q(x)$. First of all, (5.9) follows from (3.70) applied to (5.6)

$$\frac{\mathrm{d}}{\mathrm{d}x}Q(x) = -\left(\frac{L}{2}\right)^{\alpha-1} \sum_{m\geq 0} 2(2m+1)\widetilde{P}_{2m+1}^{\left(\frac{\alpha}{2}-1\right)}\left(\tfrac{2x}{L}\right)\bar{Q}_m, \tag{5.9}$$

hence

$$\hat{D}_\alpha^{(0)}Q = -\left(\frac{L}{2}\right)^{\alpha-1} \sum_{m\geq 0}(2m+1) \fint_\Omega \mathrm{d}x\, \frac{\operatorname{sgn}(x_0-x)}{|x_0-x|^\alpha}\widetilde{P}_{2m+1}^{\left(\frac{\alpha}{2}-1\right)}\left(\tfrac{2x}{L}\right)\bar{Q}_m. \tag{5.10}$$

Let us rescale from Ω to $[-1,1]$ by means of the substitutions $x \leftrightarrow \frac{L}{2}t$ and $x_0 \leftrightarrow \frac{L}{2}\tau$. The spectral relation (3.166) leads to

$$\begin{aligned}
\hat{D}_\alpha^{(0)}Q &= \left(\frac{L}{2}\right)^{\alpha-1} \sum_{m\geq 0}(2m+1)\left(\frac{L}{2}\right)^{1-\alpha} \fint_{-1}^{1} \mathrm{d}t\, \frac{\operatorname{sgn}(t-\tau)}{|\tau-t|^\alpha}\widetilde{P}_{2m+1}^{\left(\frac{\alpha}{2}-1\right)}(t)\bar{Q}_m \\
&= -\frac{1}{2}\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma(\alpha)} \sum_{m\geq 0}(2m+1)\frac{\Gamma(2m+\alpha+1)}{(2m+1)!}P_{2m}^{\left(\frac{\alpha}{2}\right)}(\tau)\bar{Q}_m.
\end{aligned} \tag{5.11}$$

Note that the substitutions under the integral render $\hat{D}_\alpha^{(0)}Q$ independent of the size of Ω.

Let us find the inner product between $\hat{D}_\alpha^{(0)}Q$ and $P_{2n}^{\left(\frac{\alpha}{2}\right)}(x_0)$ under the weight $\rho_\Omega^{\left(\frac{\alpha}{2}\right)}$ on Ω, by means of the application of the orthogonality relation (3.66), namely

$$\begin{aligned}
\left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| \hat{D}_\alpha^{(0)}Q \right\rangle_{\rho_\Omega} &= -\frac{1}{2}\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma(\alpha)} \sum_{m\geq 0}\frac{\Gamma(2m+\alpha+1)}{(2m)!}\left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| P_{2m}^{\left(\frac{\alpha}{2}\right)} \right\rangle_{\rho_\Omega}\bar{Q}_m \\
&= -\frac{1}{2}\left(\frac{L}{2}\right)^{\alpha+1}\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma(\alpha)}\frac{\Gamma(2n+\alpha+1)}{(2n)!}h_{2n}^{\left(\frac{\alpha}{2}\right)}\delta_{nm}\,\bar{Q}_m \\
&= -\frac{1}{2}\left(\frac{L}{2}\right)^{\alpha+1}\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma(\alpha)}\frac{2^{\alpha+1}}{4n+\alpha+1}\left[\frac{\Gamma\left(2n+\frac{\alpha}{2}+1\right)}{(2n)!}\right]^2\delta_{nm}\,\bar{Q}_m.
\end{aligned} \tag{5.12}$$

Observe that factor $\left(\frac{L}{2}\right)^{\alpha+1}$ in (5.12) emerges while scaling from Ω to $[-1,1]$.

The next subsections are devoted to the expansion coefficient $\bar{q}_n$ of the single step function $q(x_0)$ corresponding to $\widetilde{P}_{2n}^{\left(\frac{\alpha}{2}\right)}$ and, in so doing, to finding expressions for $\bar{Q}_n$ regarding the mean number of steps $N(x_0)$ and the mean flight path length $S(x_0)$.

5.2.3 Mean number of steps

In this particular case, it is clear that the single-step expectation value $n(x_0)$ is given by $n(x_0) = \mathbb{1}_{\left(-\frac{L}{2},\frac{L}{2}\right)}(x_0)$. Therefore, it follows directly from the orthogonality relation (3.66) applied to the Jacobi polynomial of degree zero (3.81) that

$$-l_0^{-\alpha}\bar{n}_n = -l_0^{-\alpha}\left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)}\middle|P_0^{\left(\frac{\alpha}{2}\right)}\right\rangle = -l_0^{-\alpha}\left(\frac{L}{2}\right)^{\alpha+1} h_{2n}^{\left(\frac{\alpha}{2}\right)}\delta_{0,n}. \tag{5.13}$$

Thus, (5.8) and (5.13) imply that the only non-vanishing coefficient is $\bar{N}_0$ given by (5.14)

$$\bar{N}_0 = \frac{l_0^{-\alpha}}{\frac{\alpha}{2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}. \tag{5.14}$$

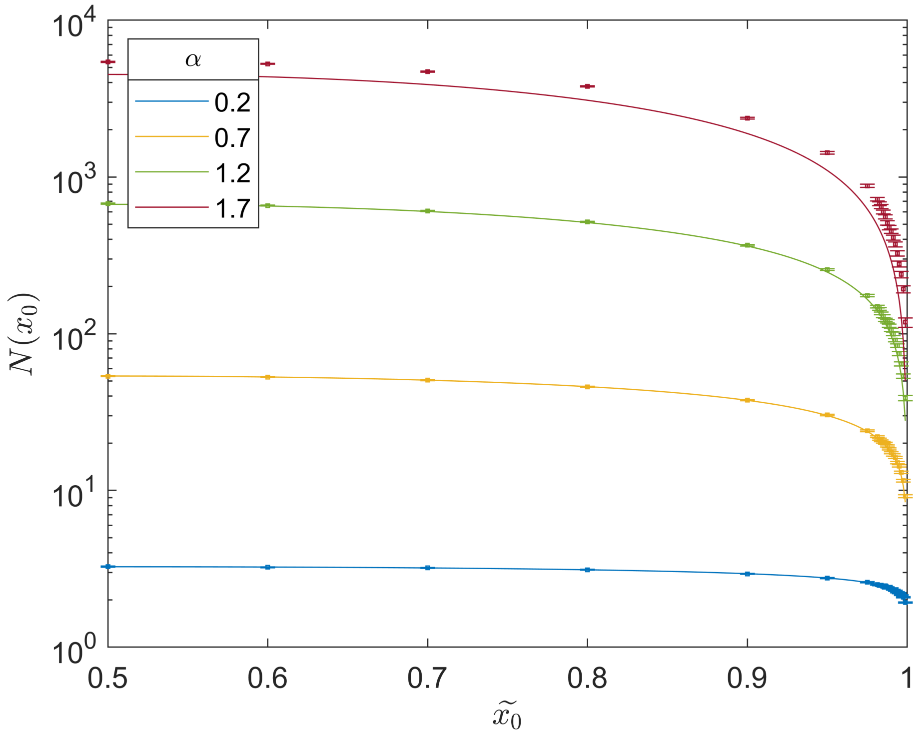
In conclusion, the continuous approximate solution to the average number of steps before absorption is given by

$$\begin{aligned} N(x_0) &= \frac{l_0^{-\alpha}}{\frac{\alpha}{2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}\left[\left(\frac{L}{2}+x_0\right)\left(\frac{L}{2}-x_0\right)\right]^{\frac{\alpha}{2}} \\ &= \frac{M^\alpha}{\frac{\alpha}{2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}\left[\widetilde{x}_0(1-\widetilde{x}_0)\right]^{\frac{\alpha}{2}}, \end{aligned} \tag{5.15}$$

where $M \equiv \frac{L}{l_0}$ and $\widetilde{x}_0 \equiv \frac{x_0}{L} + \frac{1}{2}$. Bearing in mind (3.4), one can see that (5.15) is equivalent to (1.1) from (BULDYREV et al., 2001b) and (BULDYREV et al., 2001a).

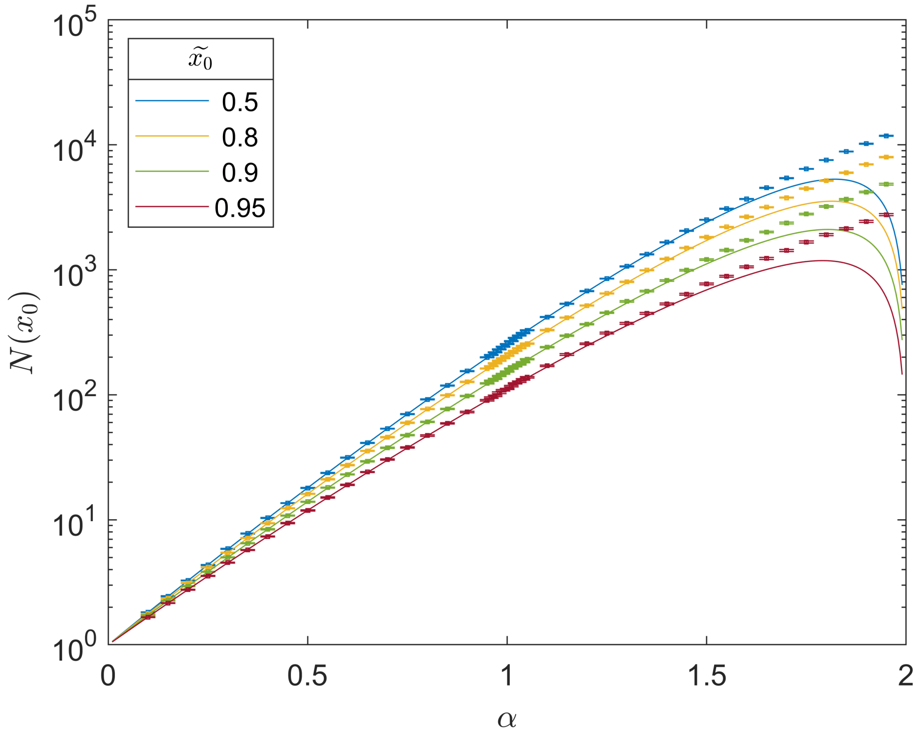
Plots of $N(x_0)$ are presented in Figure 10 as a function of $\widetilde{x}_0$ for several values of α in the interval of interest and as a function of α for departure sites located from the center of the interval ($\widetilde{x}_0 = 0.5$) up to the upper boundary ($\widetilde{x}_0 = 0.95$). Similar to the absorption probability, the continuous approximate solution fits well in that interval of departure sites provided that $\alpha \lessapprox 1.5$. Nevertheless, as seen in Figure 11, the approximation strongly deviates from the simulations in the vicinity of the absorbing site.

Figure 10 – Slice plots of the mean number of steps $N(x_0)$ as a function of the dimensionless initial position $\widetilde{x}_0$ for selected values of α. The continuous solution fits well except for $\alpha = 1.7$ (crimson solid line). Deviations close to the upper boundary are noticeable for $\alpha = 1.2$ The plot range is restricted to $0.5 \leq \widetilde{x}_0 \leq 1$ from which the other half can be inferred due to the $x_0 \leftrightarrow -x_0$ symmetry.

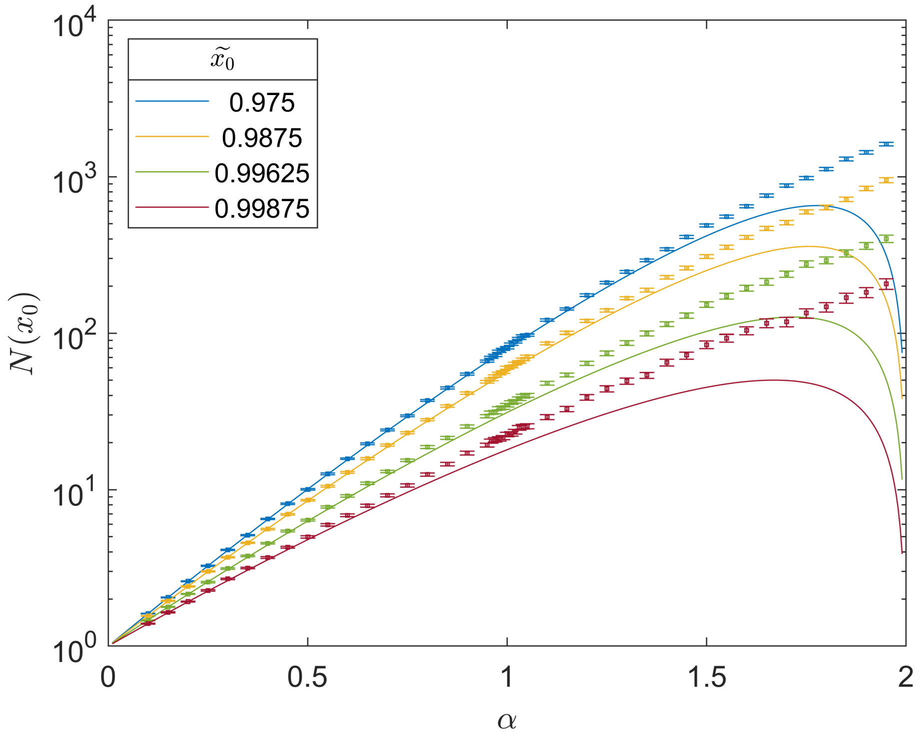


Source: The author (2023)

Figure 11 – Mean number of steps $N(x_0)$ versus α for selected values of $\widetilde{x}_0$ in semi logarithmic scale. (a) The departure site is "far" from the absorption sites, from the center of the interval $\widetilde{x}_0 = 0.5$ (blue) to $\widetilde{x}_0 = 0.95$ (crimson). Note the lack of fit for $\alpha \gtrapprox 1.5$. (b).The departure site is close to the boundary ranging from $\widetilde{x}_0 = 1 - \frac{20}{800}$ (blue) to $\widetilde{x}_0 = 1 - \frac{1}{800}$ (crimson). For the latter, no fit is observed beyond $\alpha \approx 0.5$. Both plots are in semilogarithmic scale.



(a)



(b)

Source: The author (2023)

5.2.4 Mean flight path length

Provided that $x_0 \in (-\frac{L}{2} + l_0, \frac{L}{2} - l_0)$ and $\alpha \neq 1$, the expectavion value of the single-step length $s(x_0)$ is given by

$$\begin{aligned} s(x_0) &= \int_{\Omega} \mathrm{d}x\, p(x_0,x)|x-x_0| \\ &\quad + \int_{\Omega^c} \mathrm{d}x\, p(x_0,x) \left[\left(\tfrac{L}{2} - x_0\right) \mathbb{1}_{\left(\frac{L}{2},\infty\right)}(x) + \left(\tfrac{L}{2} + x_0\right) \mathbb{1}_{\left(-\infty,-\frac{L}{2}\right)}(x) \right] \\ &= \frac{\alpha l_0^\alpha}{2(1-\alpha)} \left[\left(\frac{L}{2} - x_0\right)^{1-\alpha} + \left(\frac{L}{2} + x_0\right)^{1-\alpha} - 2l_0^{1-\alpha} \right. \\ &\qquad \left. + \frac{1-\alpha}{\alpha} \left(\left(\frac{L}{2} - x_0\right)^{1-\alpha} + \left(\frac{L}{2} + x_0\right)^{1-\alpha} \right) \right] \\ &= \frac{l_0^\alpha}{2(1-\alpha)} \left[\left(\frac{L}{2} - x_0\right)^{1-\alpha} + \left(\frac{L}{2} + x_0\right)^{1-\alpha} - 2\alpha l_0^{1-\alpha} \right]. \end{aligned} \tag{5.16}$$

In this case, one has that the right-hand side of (5.7) takes the form

$$\begin{aligned} -l_0^{-\alpha} \bar{s}_n &= -l_0^{-\alpha} \left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| s \right\rangle \\ &= \frac{1}{2(1-\alpha)} \left[2\alpha l_0^{1-\alpha} \left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| P_0^{\left(\frac{\alpha}{2}\right)} \right\rangle \right. \\ &\quad \left. - \left(\frac{L}{2}\right)^2 \underbrace{\left(\left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| (1-\tau)^{1-\alpha} \right\rangle + \left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| (1+\tau)^{1-\alpha} \right\rangle \right)}_{(*)} \right]. \end{aligned} \tag{5.17}$$

The first term of the right hand side of (5.17) is a particular case of the orthogonality relation (3.66). The second term in $(*)$ is a straightforward application of (3.74). The first term in $(*)$ is also obtained from (3.74) by means of a substitution $t \leftrightarrow -t$ that produces a $(-1)^{2n}$ factor. These considerations lead to

$$\begin{aligned} -l_0^{-\alpha} \left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| s \right\rangle = \frac{1}{2(1-\alpha)} &\left[2\alpha l_0^{1-\alpha} \left(\frac{L}{2}\right)^{\alpha+1} h_{2n}^{\left(\frac{\alpha}{2}\right)} \delta_{2n,0} \right. \\ &\left. - (2-\alpha) L^2 \frac{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma(2-\alpha)\Gamma\left(\frac{\alpha}{2}+2n+1\right)}{(2n)!\,(2n+2)!\Gamma(2-\alpha-2n)} \right]. \end{aligned} \tag{5.18}$$

By matching (5.12) and (5.18) term by term, one has that the expansion coefficients $\bar{S}_n$ of $S(x_0)$ are given by

$$\begin{aligned} \bar{S}_n = &\frac{2}{(\alpha-1)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)} l_0^{1-\alpha} \delta_{0\,2n} + \\ &\frac{(2-\alpha)(4n+\alpha+1)}{(1-\alpha)(2n+2)\left(2n+\frac{\alpha}{2}\right)(2n+1)} \frac{\Gamma(\alpha)\Gamma(2-\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}+2n\right)\Gamma(2-\alpha-2n)} L^{1-\alpha}. \end{aligned} \tag{5.19}$$

For the sake of completeness, let us consider the case $\alpha = 1$. For that purpose, one has that $(\alpha-1)\bar{S}_0$ admits the Taylor expansion around $\alpha = 1$ given by

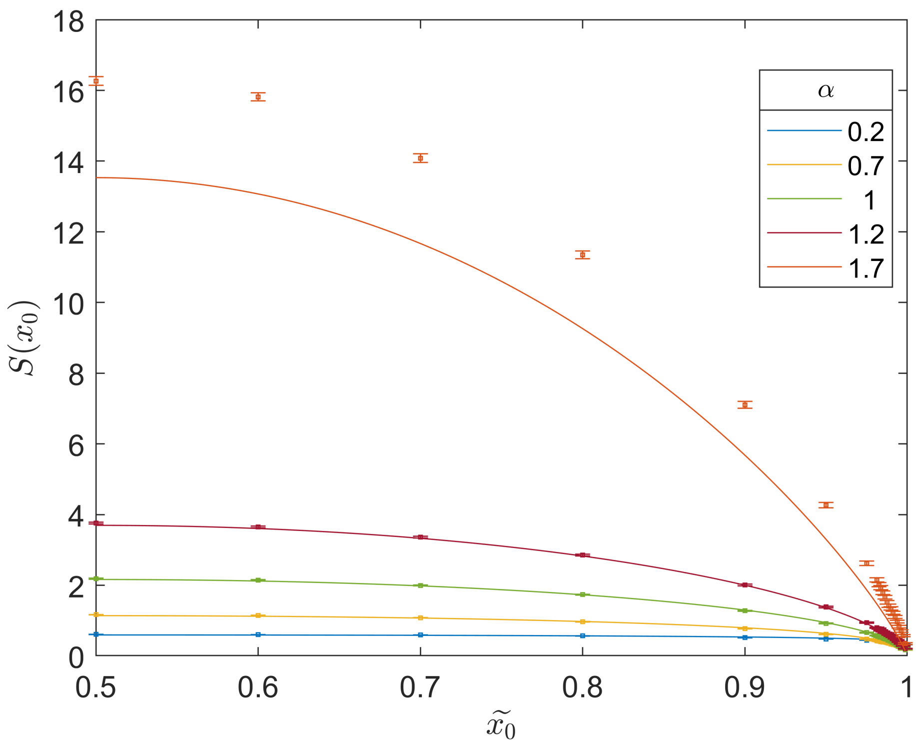
$$(\alpha-1)\bar{S}_0 = (\alpha-1)\left(\frac{2}{\pi}\right)\left(\ln\left(\frac{L}{l_0}\right)+\frac{3}{2}-2\ln(2)\right)+O\left((\alpha-1)^2\right), \tag{5.20}$$

which leads to the following approximate expression

$$S(x_0) \approx \left(\frac{2L}{\pi}\right)\left[\ln\left(\frac{L}{l_0}\right)+\frac{3}{2}-2\ln(2)\right]\left[\widetilde{x}_0(1-\widetilde{x}_0)\right]^{\frac{1}{2}}. \tag{5.21}$$

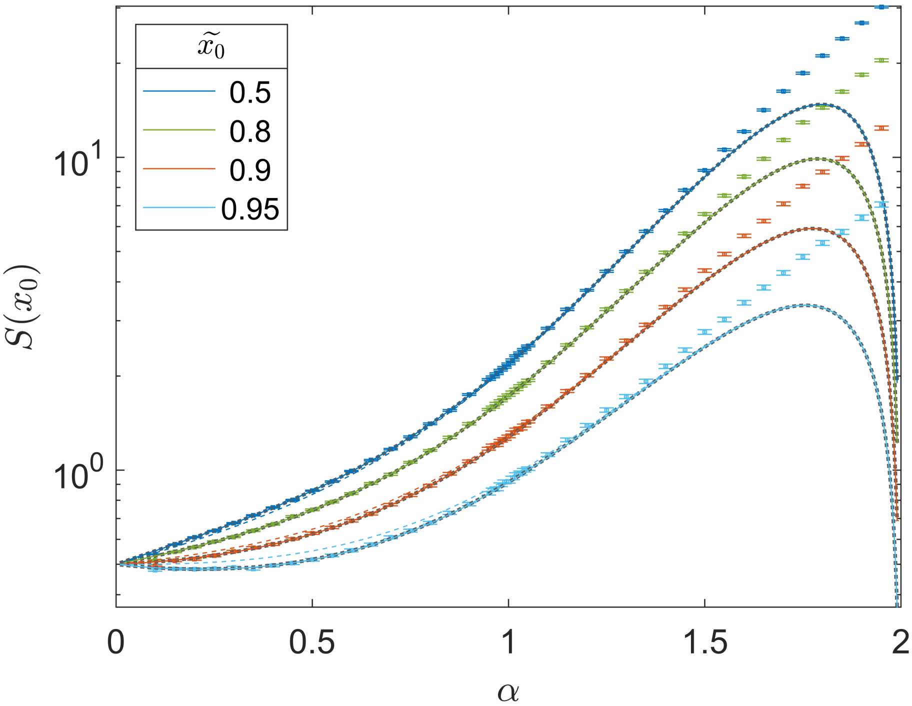
Back to the general case $0 < \alpha < 2$, plots of $S(x_0)$ are presented in Figure 12 in a fashion analogous to the mean number of steps: as a function of $\widetilde{x}_0$ for selected values of α and as a function of α for departure sites located from $\widetilde{x}_0 = 0.5$ up to $\widetilde{x}_0 = 0.95$. This plots reveal that, identically to the mean number of steps, the continuous approximate solution fits well whenever the departure site is in the aforementioned range and $\alpha \lessapprox 1.5$, but notoriously fails otherwise as depicted in Figure 13. For the sake of comparison, the solution (1.2) from (BULDYREV et al., 2001b) is also plotted (dotted grey line) and the difference between the approximate JP solution with four modes and (1.2) is found to be negligible.

Figure 12 – Slice plots of the mean flight path length $S(x_0)$ versus $\widetilde{x}_0$ and α. Similar to $N(x_0)$, the continuous solution fits well except for $\alpha = 1.7$ (crimson solid line). The cause of the deviation will become clearer in Section 5.3.

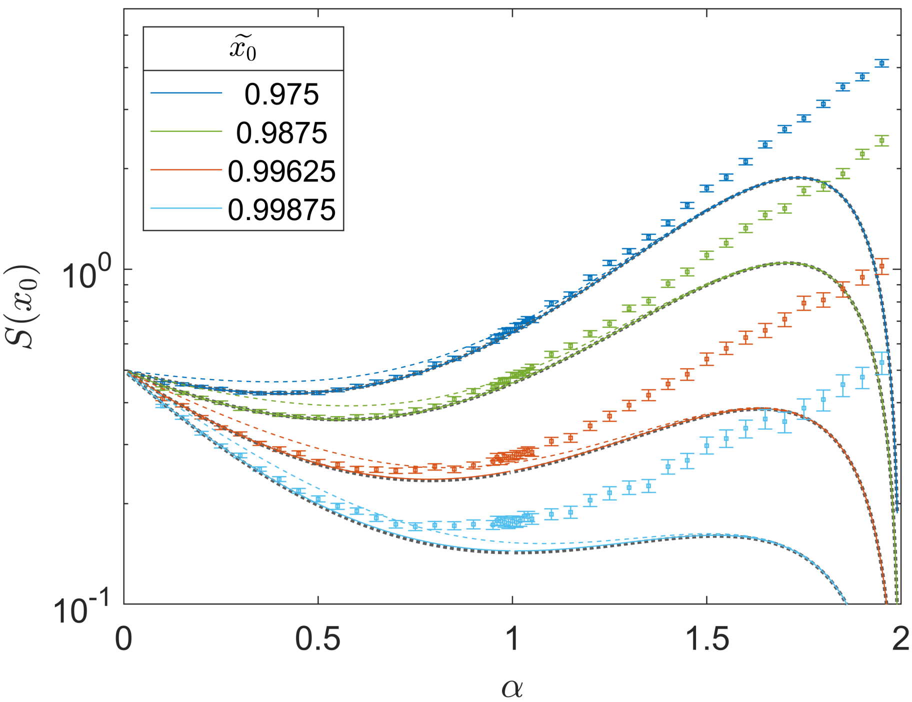


Source: The author (2023)

Figure 13 – Effect of additional modes upon the accuracy of $S(x_0)$ for (a) departure sites "far" from the boundary and (b) in the vicinity of the upper absorbing site with $\widetilde{x}_0$ ranging from $1-\frac{20}{800}$ to $1-\frac{1}{800}$. For comparison, the solution (1.2) from (BULDYREV et al., 2001b) is plotted in gray dotted lines hereafter. As for the JP solutions, the approximation to $S(x_0)$ with the single mode $m=0$ (dashed line) also exhibits noticeable deviations from the simulations for $\alpha \leq 1$. For small values of α and closer to the boundary, the zero mode tends to overestimate the simulated results. The lack of fit in the region $\alpha \leq 0.5$ significantly improves with the inclusion of additional modes as conveyed by the approximation to $S(x_0)$ with four even modes with $m \in \{0,1,2,3\}$ (solid line). Observe also that the differences between the JP approximation with four modes and (1.2) are negligible.



(a)



(b)

Source: The author (2023)

With regard to Figure 13, where the single (zero) mode approximation $m = 0$ (dashed line) is compared to the approximation with the first four non-vanishing modes $m \in \{0, 1, 2, 3\}$ (solid line), it is worth noting that the contribution of the first term in (5.19), which is proportional to $l_0^{1-\alpha}$, is negligible when $\alpha < 1$. Nevertheless, it becomes the leading term for $\alpha > 1$. Conversely, the remaining terms are proportional to $L^{1-\alpha}$ and thus become meaningful only when $\alpha < 1$. This is the reason why increasing the number of modes in the expansion in (5.6) only improves the accuracy of the approximation for small values of $\alpha \lessapprox 0.5$.

5.3 FINITE SIZE CORRECTION: NUMERICAL RESULTS

It was concluded in the previous section that the approximate continuous solution obtained by neglecting the second-order derivative in (4.22) proved to work very well provided that $\alpha \lessapprox 1.5$ and the departure site is "far" from the absorbing boundary, a conclusion that applies for the three quantities considered: absorption probability at a given site, mean number of steps and mean flight-path length. Nevertheless, this section deals with the inclusion of the neglected term using a numerical approach. For that purpose, the evaluation of the expansion coefficients d_{nm} of the second-order derivative in terms of Jacobi polynomials was carried out. Afterwards, the corrected expansion coefficients $\bar{Q}_m$ of $Q(x)$ are obtained by solving the system of linear (algebraic) equations

$$\left[D_{nm}^0 + d_{nm}\right] \bar{Q}_m = -l_0^{-\alpha} \bar{q}_n \tag{5.22}$$

where the diagonal "unperturbed" D_{nm}^0 is given by (5.8) and the amplitude $\bar{q}_n$ of the $2n$-th mode of the single-step observable is given by (5.13) and (5.17) for the mean number of steps and the mean flight path length respectively. The off-diagonal elements $d_n m$ are the subject of the upcoming subsection.

Perhaps it is worth to clarify at this point that n, m are indices over the *even* modes, so $n, m \in \{0, 1, 2, 3, ..., N-1\}$ with N the size of the linear equation system (5.22). In that sense, the n-th mode corresponds to the Jacobi polynomial of degree $2n$.

5.3.1 Second-order derivative in terms of Jacobi polynomials

Let us find the representation of the second-order derivative operator in the basis of functions $\widetilde{P}_n^{\left(\frac{\alpha}{2}\right)}$. The expansion coefficients are obtained through

$$\begin{aligned}\left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| Q'' \right\rangle_\rho &= \left(\frac{L}{2}\right)^\alpha \int_\Omega \mathrm{d}x\, \widetilde{P}_{2n}^{\left(\frac{\alpha}{2}\right)}\left(\tfrac{2x}{L}\right) \frac{\mathrm{d}^2 Q(x)}{\mathrm{d}x^2} \\ &= \left(\frac{L}{2}\right)^\alpha \left[\widetilde{P}_{2n}^{\left(\frac{\alpha}{2}\right)} \frac{\mathrm{d}Q}{\mathrm{d}x}\bigg|_{\partial\Omega} + 2(2n+1)\left(\frac{2}{L}\right) \int_\Omega \mathrm{d}x\, \widetilde{P}_{2n+1}^{\left(\frac{\alpha}{2}-1\right)} \frac{\mathrm{d}Q}{\mathrm{d}x} \right],\end{aligned} \tag{5.23}$$

where the last steps arises from partial integration with $u \equiv \widetilde{P}_{2n}^{\left(\frac{\alpha}{2}\right)}$ and $v \equiv \frac{\mathrm{d}Q}{\mathrm{d}x}$ along with the application of the identity (3.70). Inserting the ansatz (5.6) into (5.23) and recalling (5.9) yield

$$\begin{aligned}\left\langle P_{2n}^{\left(\frac{\alpha}{2}\right)} \middle| Q'' \right\rangle_\rho = -\left(\frac{L}{2}\right)^{2\alpha-1} \sum_{m\geq 0} \Bigg[& 2(2m+1) \widetilde{P}_{2n}^{\left(\frac{\alpha}{2}\right)}\left(\tfrac{2x}{L}\right) \widetilde{P}_{2m+1}^{\left(\frac{\alpha}{2}-1\right)}\left(\tfrac{2x}{L}\right)\bigg|_{\partial\Omega} \\ & + 4(2n+1)(2m+1) \underbrace{\int_{-1}^{1} \mathrm{d}t\, \widetilde{P}_{2n+1}^{\left(\frac{\alpha}{2}-1\right)}(t) \widetilde{P}_{2m+1}^{\left(\frac{\alpha}{2}-1\right)}(t)}_{I_{m,n}^{(1)}} \Bigg] \bar{Q}_m,\end{aligned} \tag{5.24}$$

Let us consider the case when $1 < \alpha < 2$. In such case, the boundary term in (5.24) vanishes due to the weight function $\rho^{(\alpha-1)}|_{\partial\Omega} = 0$. Concerning the integral term in (5.24), the weight function $\rho^{(\alpha-2)}(t)$ is even under $t \leftrightarrow -t$ as well as the product of any two Jacobi polynomials of odd order, hence $I_{m,n}^{(1)}$ is different from zero for any pair $(m,n) \in \mathbb{N}_0^2$. Moreover, due to (3.79), $I_{m,n}^{(1)}$ can be expressed as a finite sum of integrals of the form (5.25) , where $J \equiv 2m+2n+2 \geq 2$, $0 \leq \mu \leq 2m+1$, $0 \leq \nu \leq 2n+1$, and $j \equiv \mu+\nu$ such that $0 \leq j \leq J$.

$$\int_{-1}^{1} \mathrm{d}t\, \frac{(t-1)^j (t+1)^{J-j}}{[(1-t)(1+t)]^{2-\alpha}} = (-1)^j\, 2^{2\alpha+J-3} \mathrm{B}(\alpha+j-1, \alpha+J-j-1) \tag{5.25}$$

Note that the integral (5.25) is convergent provided that $\alpha + j > 1$ and $\alpha + J > j+1$, both of which are true for the case at hand, i.e. $1 < \alpha < 2$.

In general,

$$\begin{aligned} I_{m,n}^{(1)} = 2^{2\alpha-3} \sum_{\mu=0}^{2m+1} \sum_{\nu=0}^{2n+1} (-1)^{\mu+\nu} W(2m+1, \tfrac{\alpha}{2}-1, \mu)\, W(2n+1, \tfrac{\alpha}{2}-1, \nu) \cdot \\ \cdot \mathrm{B}(\alpha+\mu+\nu-1, \alpha+2m+1-\mu+2n+1-\nu-1) \end{aligned} \tag{5.26}$$

where

$$\begin{aligned} W(k,\gamma,\kappa) = \frac{\mathrm{b}(k-\kappa+1, \kappa+\gamma+1)}{k+\gamma+1} \frac{\mathrm{b}(\kappa+1, k-\kappa+\gamma+1)}{k+\gamma+1} \mathbb{1}_{\{0<\kappa<k,\, \kappa\in\mathbb{N}\}}(\kappa) \\ + \frac{\mathrm{b}\,(k+1, \gamma+1)}{k+\gamma+1} \mathbb{1}_{\{0,k\}}(\kappa) \end{aligned} \tag{5.27}$$

For illustrative purposes, the explicit value of $I^{(1)}_{m,n}$ is presented below for the smallest n and m.

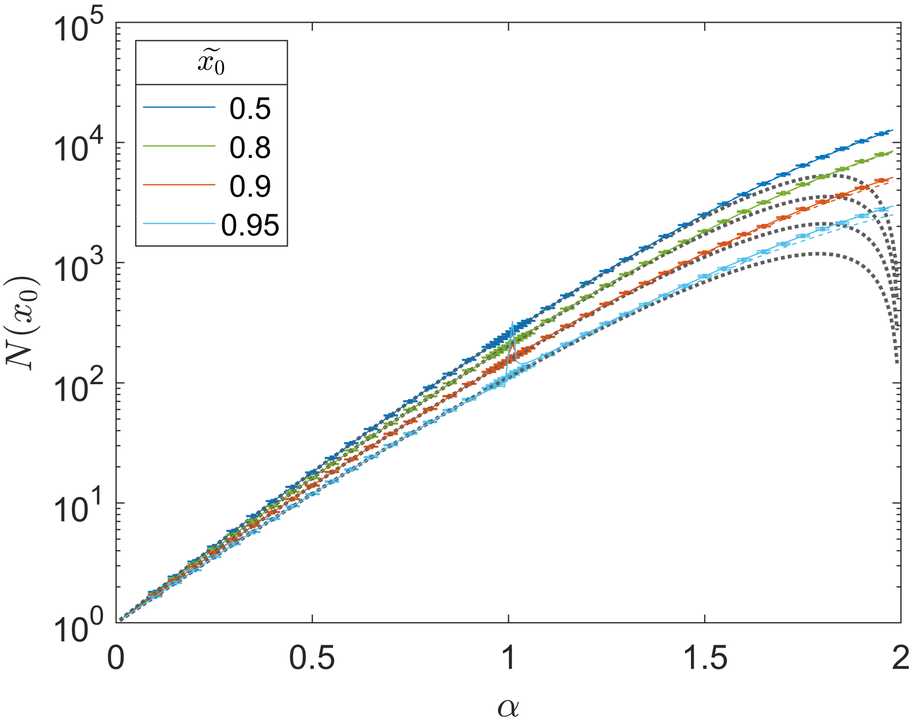
$$\begin{aligned} I^{(1)}_{0,0} &= 2^{2\alpha-3}\left[W\left(1,\tfrac{\alpha}{2}-1,0\right)\right]^2\left[\mathrm{B}(\alpha-1,\alpha+1)-2\mathrm{B}(\alpha,\alpha)+\mathrm{B}(\alpha+1,\alpha-1)\right] \\ &= 2^{2\alpha-2}\left(\frac{\alpha}{2}\right)^2\left[\mathrm{B}(\alpha-1,\alpha+1)-\mathrm{B}(\alpha,\alpha)\right] \end{aligned} \tag{5.28}$$

$$\begin{aligned} I^{(1)}_{0,1} &= 2^{2\alpha-3}W\left(1,\tfrac{\alpha}{2}-1,0\right)\Big(2W\left(3,\tfrac{\alpha}{2}-1,0\right)\left[\mathrm{B}(\alpha-1,\alpha+3)-\mathrm{B}(\alpha,\alpha+2)\right] \\ &\qquad +2W\left(3,\tfrac{\alpha}{2}-1,1\right)\left[\mathrm{B}(\alpha+1,\alpha+1)-\mathrm{B}(\alpha+2,\alpha)\right]\Big) \\ &= 2^{2\alpha-2}\left(\frac{\alpha}{2}\right)\left[\frac{\left(2+\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}\right)}{3!}\left[\mathrm{B}(\alpha-1,\alpha+3)-\mathrm{B}(\alpha,\alpha+2)\right]\right. \\ &\qquad \left.+\frac{\left(2+\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2}\right)\left(2+\frac{\alpha}{2}\right)}{2}\left[\mathrm{B}(\alpha+1,\alpha+1)-\mathrm{B}(\alpha+2,\alpha)\right]\right] \end{aligned} \tag{5.29}$$

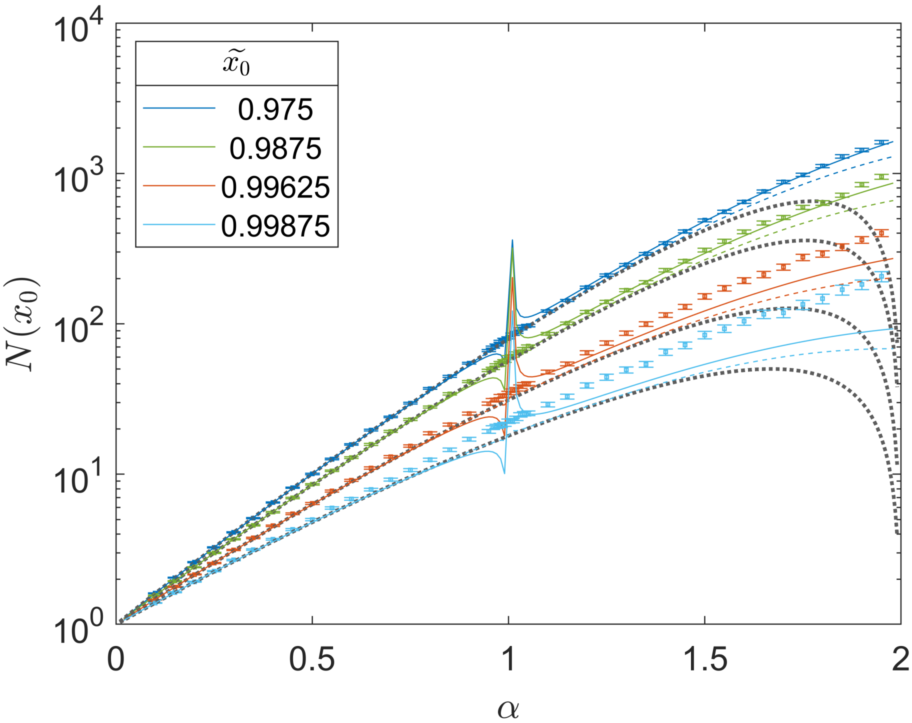
$$\begin{aligned} I^{(1)}_{1,1} &= 2^{2\alpha-3}\left[\left(W\left(3,\tfrac{\alpha}{2}-1,0\right)\right)^2\left[\mathrm{B}(\alpha-1,\alpha+5)-\mathrm{B}(\alpha+2,\alpha+2)\right]\right. \\ &\qquad -2W\left(3,\tfrac{\alpha}{2}-1,0\right)W\left(3,\tfrac{\alpha}{2}-1,1\right)\left[\mathrm{B}(\alpha,\alpha+4)-\mathrm{B}(\alpha+1,\alpha+3)\right] \\ &\qquad \left.+\left(W\left(3,\tfrac{\alpha}{2}-1,1\right)\right)^2\left[\mathrm{B}(\alpha+1,\alpha+3)-\mathrm{B}(\alpha+2,\alpha+2)\right]\right] \\ &= 2^{2\alpha-2}\left[\left(\frac{\left(2+\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}\right)}{3!}\right)^2\left[\mathrm{B}(\alpha-1,\alpha+5)-\mathrm{B}(\alpha+2,\alpha+2)\right]\right. \\ &\qquad -2\frac{\left(2+\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}\right)}{3!}\frac{\left(2+\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2}\right)\left(2+\frac{\alpha}{2}\right)}{2}\left[\mathrm{B}(\alpha,\alpha+4)-\mathrm{B}(\alpha+1,\alpha+3)\right] \\ &\qquad \left.+\left(\frac{\left(2+\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2}\right)\left(2+\frac{\alpha}{2}\right)}{2}\right)^2\left[\mathrm{B}(\alpha+1,\alpha+3)-\mathrm{B}(\alpha+2,\alpha+2)\right]\right] \end{aligned} \tag{5.30}$$

The approximate solutions to the mean number of steps $N(x_0)$ and the mean flight path length $S(x_0)$, including the aforementioned corrections, are presented in Figure 14 and 15 respectively. The depicted solutions were obtained by solving systems of linear equations (5.22) of sizes $N = 1$ and $N = 4$. The mere inclusion of the correction term proportional to $I^{(1)}_{0,0}$ represents a significant improvement in the fit to the results of the simulations for $\widetilde{x}_0$ up to 0.95. Although retaining additional modes improves the fit of the corrected approximate solution for values of α closer to 2, regardless of the departure site, a "jittery" behaviour around $\alpha = 1$ also becomes apparent. Note also that every $I^{(1)}_{m,n}$ includes a term proportional to $\mathrm{B}(\alpha-1, \alpha+2m+2n+1)$ which is troublesome for α close to 1, approaching from both sides. Indeed, it follows from (3.12) that $\mathrm{B}(\alpha-1, \alpha+2m+2n+1) = \frac{2(\alpha+m+n)}{\alpha-1}\mathrm{B}(\alpha, \alpha+2m+2n+1)$.

Figure 14 – Mean number of steps $N(x_0)$ with first-order corrections. Approximate forms of $N(x_0)$ obtained from the solutions to the system (5.22) $N = 4$ (solid line). For comparison, the *uncorrected* approximate solution with $N = 1$ (dashed line) and the graph of(1.1) (dotted line) are also included.. Recall that $0 \leq n, m < N$. (a) When the departure site $\widetilde{x}_0$ is between 0.5 and 0.95, good fit is achieved with just four modes of the lowest even order. Note that the approximation with the zero mode is also fair enough. (b) For departure sites close to the boundary, from $\widetilde{x}_0 = 1 - \frac{20}{800}$ (dark blue) to $\widetilde{x}_0 = 1 - \frac{1}{800}$ (light blue), the numerical solution still exhibits increasingly strong deviations from the random flight simulations, although the fit improves in the vicinity of $\alpha \approx 2$ compared to the uncorrected solution. It is also noticeable the discontinuity at $\alpha = 1$ and the sign flip of the correction across that value.



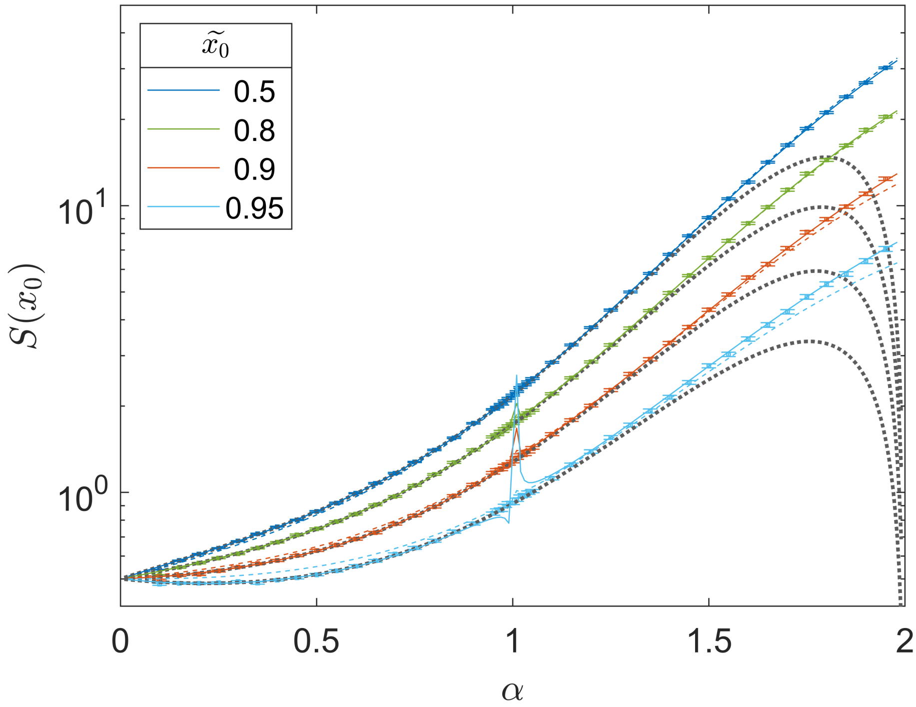
(a)



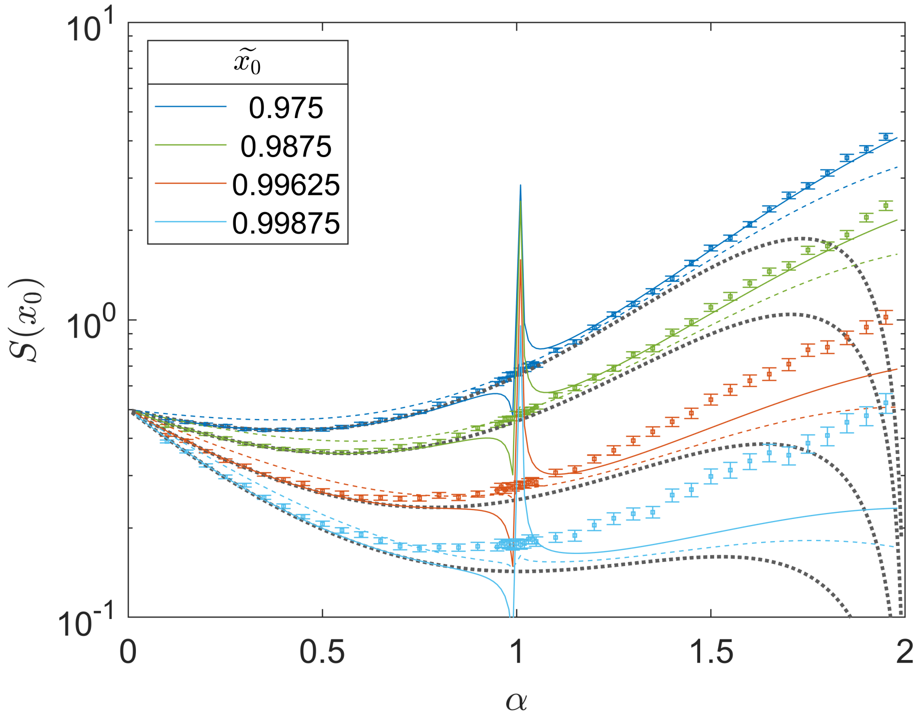
(b)

Source: The author (2023)

Figure 15 – Mean flight path length $S(x_0)$ with first-order corrections. Approximate forms of $S(x_0)$ were obtained from the solutions to the system (5.22) of size $N = 4$ (solid line). For comparison, the *uncorrected* approximate solution with $N = 1$ (dashed line) and the solution (1.2) (dotted line) are also included. Recall that $0 \leq n, m < N$. (a) When the departure site $\widetilde{x}_0$ is between 0.5 and 0.95, good fit is achieved with just four modes of the lowest even order. Note that the approximation with the zero mode is also fair enough. (b) For departure sites close to the boundary, from $\widetilde{x}_0 = 1 - \frac{20}{800}$ (dark blue) to $\widetilde{x}_0 = 1 - \frac{1}{800}$ (light blue), the numerical solution still exhibits increasingly strong deviations from the random flight simulations, although the fit improves in the vicinity of $\alpha \approx 2$ compared to the uncorrected solution. It is also noticeable the discontinuity at $\alpha = 1$ and the sign flip of the correction across that value.



(a)



(b)

Source: The author (2023)

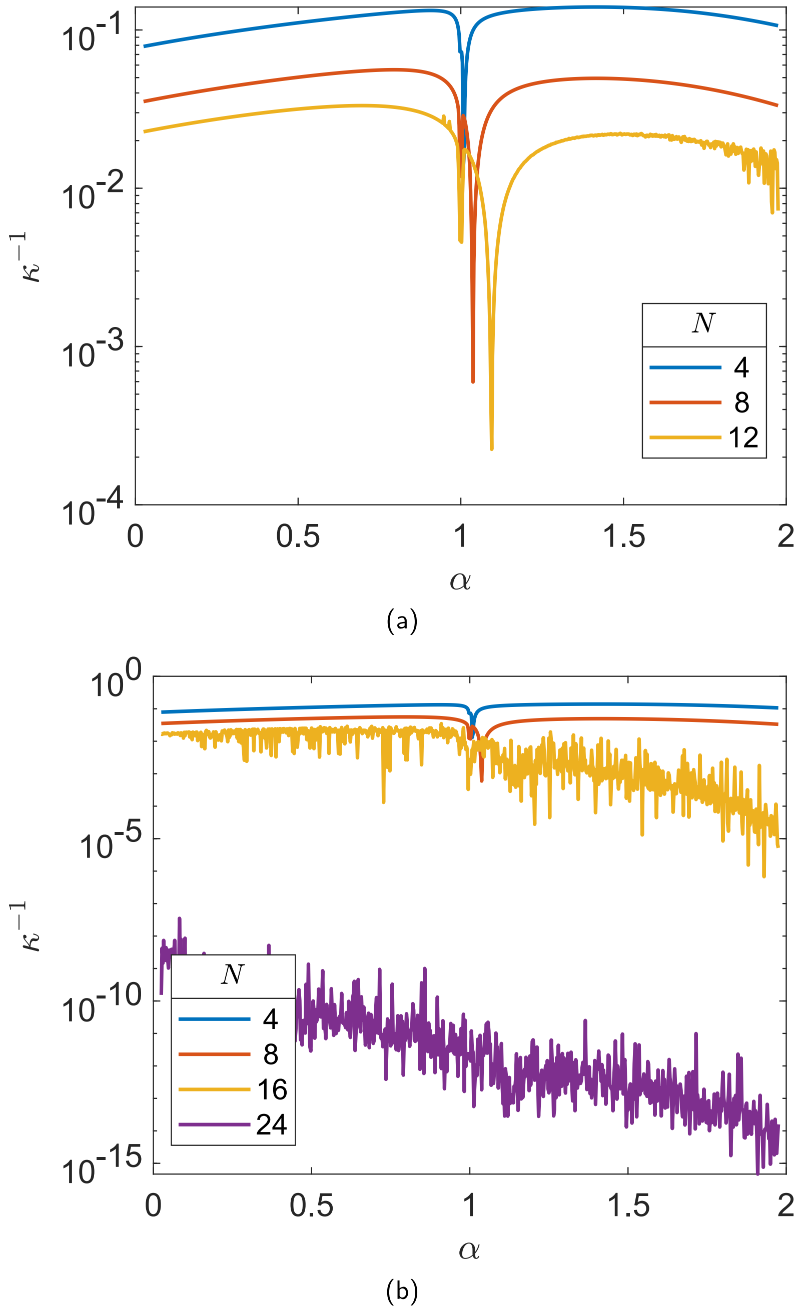
Regarding the regime $0 < \alpha < 1$, the right-hand side of (5.24) has no divergent terms for $1 < \alpha < 2$, as mentioned earlier, but is highly problematic elsewhere. Nevertheless, an attempt was made to assign a finite value to the integral (5.25) for $\alpha < 1$ by extending the beta function to negative arguments through (3.10). Also, the boundary term in (5.24), which diverges due to $\rho^{(\alpha-2)}(t \to 1)$, was discarded. The plots presented in Figure 14 and 15 suggest that such an arbitrary attempt was not successful.

It is worth bearing in mind that, for the mean number of steps, the expansion of the single-step expectation value $n(x_0)$ in terms of the modes $\widetilde{P}_{2m}^{(\frac{\alpha}{2})}$ given by (5.13) is exact, which is not the case for the mean flight path length expansion in (5.16) due to the particular form that $s(x_0)$ take in $(-\frac{L}{2}, -\frac{L}{2} + l_0)$ and $(\frac{L}{2} - l_0, \frac{L}{2})$, hence the emphasis on the former.

5.3.2 Numerical stability

As shown in Figure 16, the linear equation system (5.22) is ill-conditioned particularly in the vicinity of $\alpha = 1$. The reciprocal condition number κ^{-1}, defined as the quotient between the minimum and the maximum singular values of $D_{nm}^{(\alpha)}$, approaches zero as the system size increases, which indicates that the equation system is highly unstable and considering modes of higher orders does not improve overall convergence, but adds noise.

Figure 16 – Reciprocal condition number of $D_{nm}^{(\alpha)}$ versus α for several number of modes. A reciprocal condition number κ^{-1} approaching zero indicates an ill-conditioned linear equation system for which small variations in the input produce large changes in the solution. N denotes the number of modes considered, i.e. the size of (5.22). (a) For small N, it is interesting to note a fix tip at $\alpha = 1$ and a tip moving to the right with increasing number of modes. (b) For $N = 24$, κ^{-1} reaches an order of magnitude of -10 in the entire region $0 < \alpha < 2$, for which regularization methods are required.

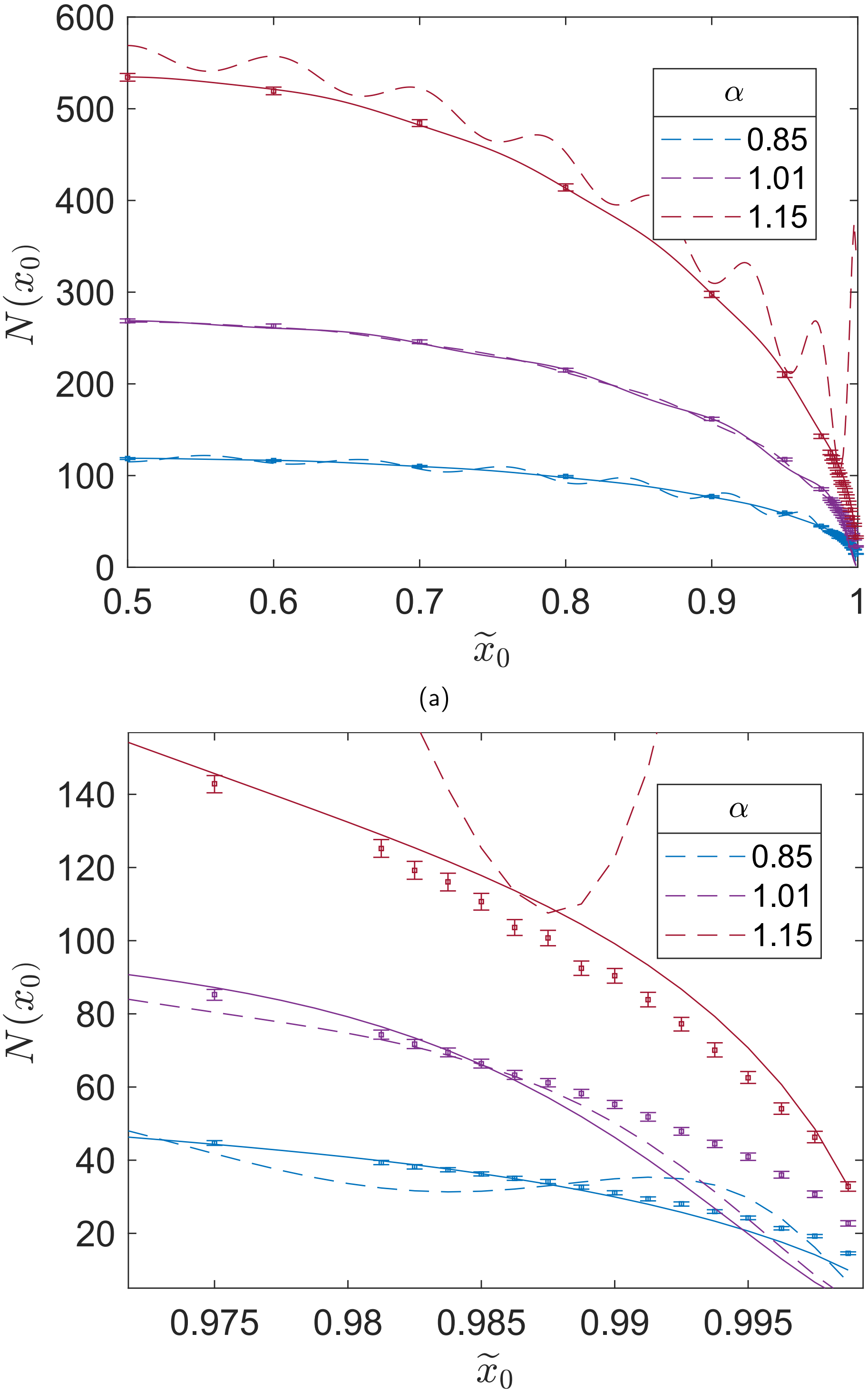


Source: The author (2023)

In order to get an approximate noise-free solution to (5.22), the truncated Singular Value Decomposition (tSVD) regularization technique was applied (NEUMAIER, 1998). The key idea is to rid D_{mn} of the high-frequency noise by suppressing the singular values σ_k of rank $k > \bar{K}$

without compromising the accuracy of the solution itself.

Figure 17 – Non-regularised and regularised numerical solutions to $N(x_0)$ as a function of $\widetilde{x}_0$ for the upper half of the search interval (a) and for a small neighbourhood around the absorbing site (b). The non-regularised approximation to $N(x_0)$ with $N = 32$ (dashed lines) exhibits some high-frequency artifacts ("ripples") that become larger near the boundary. Those artifacts are successfully removed by tSVD with cutoff rank $\bar{K} = 26$, albeit the regularised solution (solid lines) still deviates from the the simulations in the vicinity of the absorbing site for $N(x_0)$.



(a)

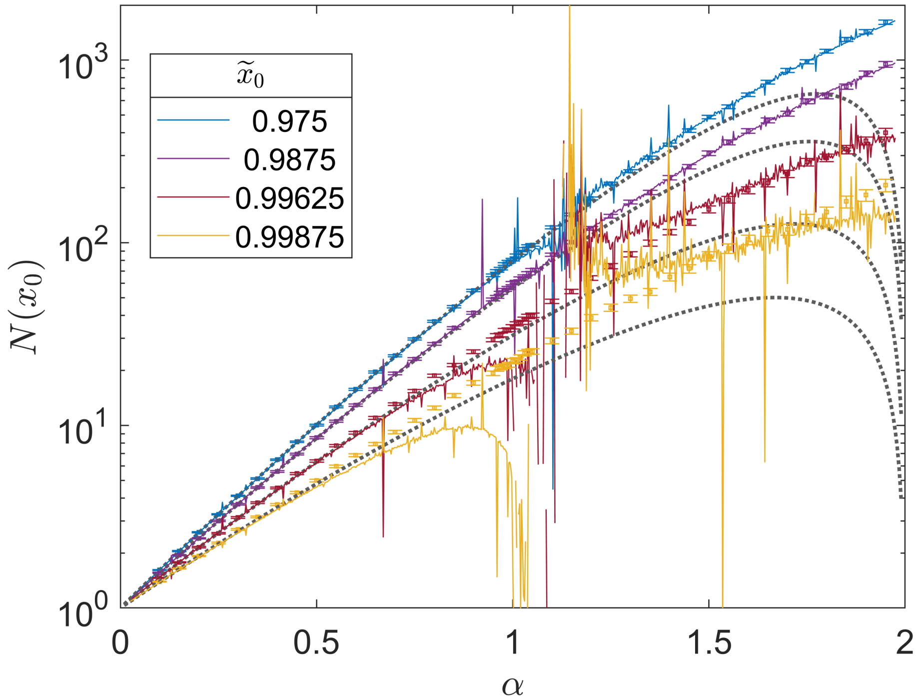
(b)

Source: The author (2023)

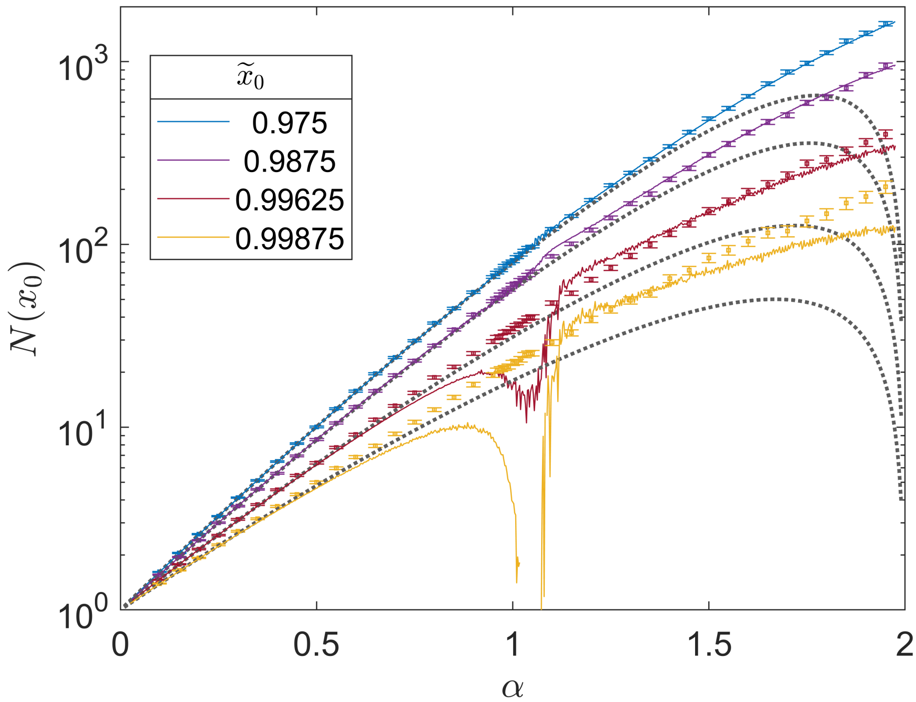
The non-regularized and regularized solutions for $N(x)$ are depicted in Figure 17 for $\alpha \in \{0.85, 1.01, 1.15\}$ for $N = 32$ and cutoff rank $\bar{K} = 26$ (solid line) compared with the oscillating non-regularized solution (dashed line). Even though the tSVD irons out the "ripples", the regularised solution still misses the behaviour of the simulations in the vicinity of the boundary. As mentioned earlier, the reason for presenting $N(x_0)$ is to leave aside the approximations made while computing the coefficients $\bar{s}_n$ required for solving $S(x_0)$.

Figure 18 and 19 compare the unregularised solutions of $N(x_0)$ and $S(x_0)$ considering 32 JP modes and the tSVD regularisation obtained by retaining 26 modes, with $\widetilde{x}_0$ in the vicinity of the target site. Despite the evident noise reduction and overall good fit for $\widetilde{x}_0 \in \{1-\frac{20}{800}, 1-\frac{10}{800}\}$, even for $\alpha \approx 1$ and the improvement in the near Gaussian regime compared to (1.1) and (1.2), the regularised solution deviates from the data for $\widetilde{x}_0 \in \{1 - \frac{3}{800}, 1 - \frac{1}{800}\}$ and $\alpha \approx 1$. In that regime, expressions (1.1) and (1.2) outperform the regularised solutions of (5.22) with correction terms.

Figure 18 – Non-regularised and regularised numerical solutions $N(x_0)$ as a function of α in the vicinity of the upper absorbing site. (a) Non-regularised $N(x_0)$ with $N = 32$. (b) tSVD-regularised $N(x_0)$ with cutoff rank $\bar{K} = 26$. Despite the lack of fit around $\alpha = 1$ increases as one approaches the boundary, both solutions gets closer to the simulation results as α approaches 2.



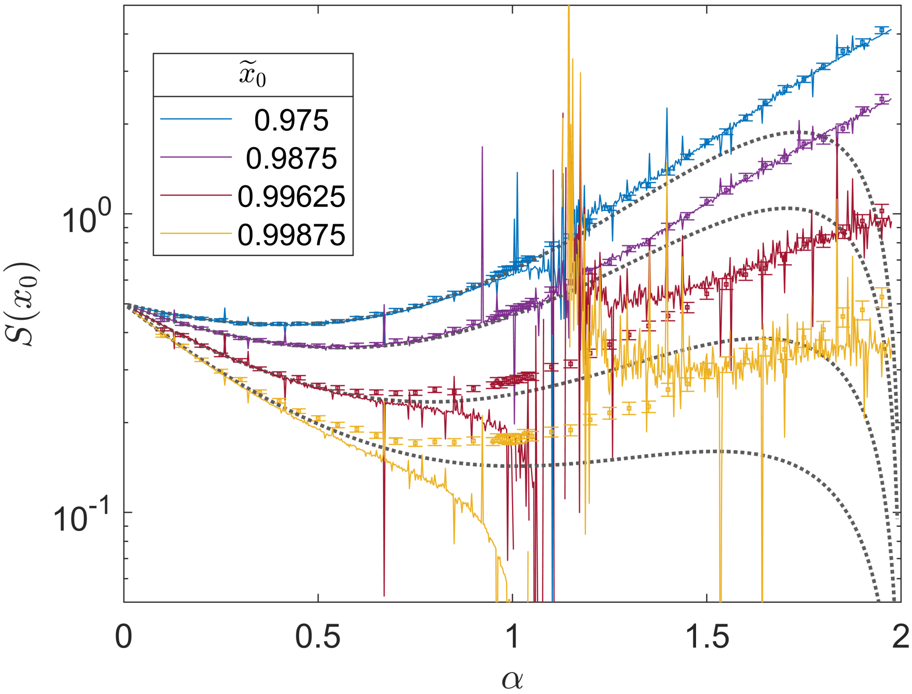
(a)



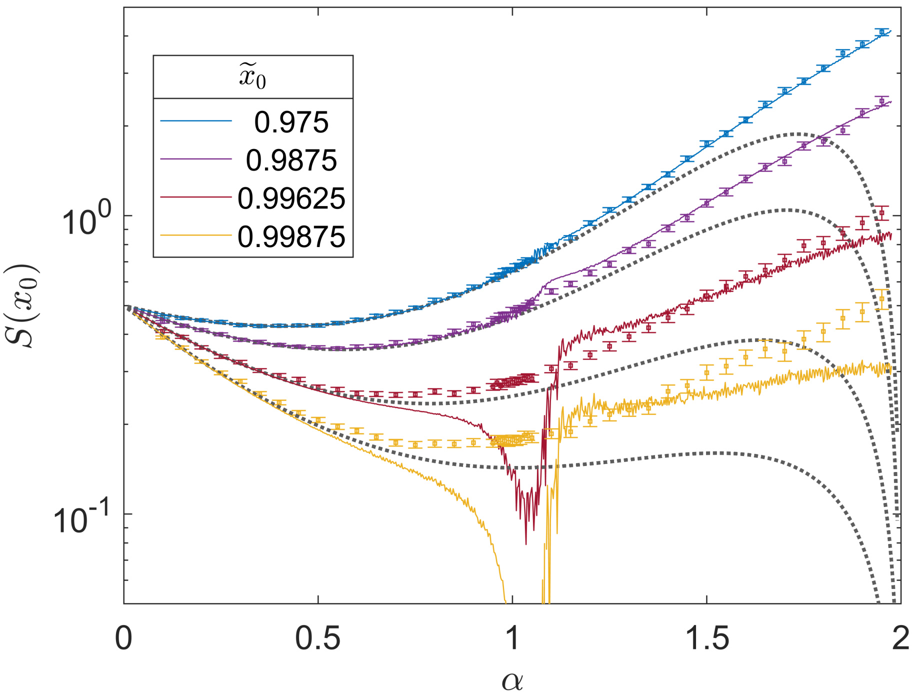
(b)

Source: The author (2023)

Figure 19 – Non-regularised and regularised numerical solutions $S(x_0)$ as a function of α in the vicinity of the upper absorbing site. (a) Non-regularised approximate $S(x_0)$ with $N = 32$. (b) tSVD-regularised $S(x_0)$ with cutoff rank $\bar{K} = 26$. Despite the lack of fit around $\alpha = 1$ increases as one approaches the boundary, both solutions gets closer to the simulation results as α approaches 2.



(a)



(b)

Source: The author (2023)

6 CLOSING REMARKS AND OUTLOOK

The initial motivation of this project was to delve into the solution of the integral equation (4.22) and find strategies to fix the lack of fit of the known closed-form approximate solutions close to the boundary and approaching the diffusive regime (BULDYREV et al., 2001a)(BULDYREV et al., 2001b).

In order to solve the singular integral equation, alternative approaches to the Sonine inversion formula were explored. The first approach involves the solution to Abel integral equation, Riemann-Liouville fractional integrals and derivatives and the application of boundary value problem theory. This *tour de force* has some drawbacks. First of all, it is restricted to weakly singular kernels such as the Riesz potential with $0 < \alpha < 1$, which leaves out half of the range of interest. Second, the decision of inverting Abel equation with the aid of (3.46) before solving the Cauchy-type singular integral equation prevents the inclusion of additional terms. One could also solve the singular integral equation before inverting Abel equation, which is the approach for which explicit formulae are reported in the reviewed literature (SAMKO; KILBAS; MARICHEV, 1993) (GAKHOV, 1966). However, the attempts made indicate that the application of the latter is no less challenging than the former. In either case, one has to deal with the operator $\mathcal{D}_{a+}^{1-\alpha}[h]$ and with integrals with a kernel $(\tau - t)^{-1}$ for which the principal value needs to be sought.

The second approach explored in this project is the spectral relationship method with Jacobi polynomials. It turns out that Jacobi polynomials have some special bond with the Riesz potential stated in 3.3 and 3.4. Although the former is a particular case of the latter, there are some interesting intermediate results such as (3.97), which nicely depicts the link between (weighted) Jacobi polynomials and Riesz potential in the form of a residue formula. It is worth recalling that the key JP spectral relation (3.90), valid for $0 < \alpha < 1$ at first, was extended to $1 < \alpha < 2$.

The foundations underlying the spectral relationship method and the JPs spectral relations are by no means trivial, yet they are not as inscrutable as inversion formulae. Furthermore, the former allows for the inclusion of certain correction terms that greatly improve the accuracy of the solution, which is a game changer. Although no explicit closed-form expressions were obtained for the corrected mean number of steps and mean flight path length, the relatively fast convergence with the number of JP modes observed in Figure 14 and 15 could be exploited to obtain approximate closed form expressions. In any case, converting an integral equation of the first kind such as (4.19) into a linear equation system for the amplitudes of the JPs modes is advantageous from the computational (numerical) point of view, even tough the ill-conditioned nature of the problem comes from (4.19) itself.

Regarding plausible future steps, it is mandatory to tackle the instability problem of the numerical solution in the vicinity of $\alpha \approx 1$ as one approaches the target sites, which might entail taking into account higher order corrections, such as considering non-vanishing contributions from additional terms in (4.21) (e.g. fourth-order term). At first, it seems that computing the matrix elements arising from those terms is by no means a trivial task.

With regard to the absorption probability and the mean flight path length, the particular behaviour of p_+ (4.26) and $s(x)$ (5.16) in the l_0- neighbourhood around the endpoint was neglected. Some preliminary calculations indicate those adjustment do not improve overall fitness with respect to the simulations for the parameters l_0 and L of the system considered. The key might be in the expansion of the regularising term (A) in (4.20).

It would also be interesting to refine the calculation of the absorption probability by including correction terms. This calculation differs from the mean number of steps and the mean path length as the boundary conditions are not fixed to zero in both ends, hence the ansatz (5.6) no longer works.

There is also an open problem, as far as the author is aware, regarding the solution of the fractional diffusion equation (2.47). The solution in free space is a Lévy α-stable distribution. Nevertheless, attempts to find the solution in a finite interval with absorbing endpoints have not been successful (CHECHKIN et al., 2003). Perhaps Jacobi polynomials may shed some light on that problem.

REFERENCES

AL-GWAIZ, M. *Sturm-Liouville Theory and its Applications*. [S.l.]: Springer-Verlag London Limited, 2008. (Springer Undergraduate Mathematics Series). ISBN 9781846289729.

ARFKEN, G.; WEBER, H.; HARRIS, F. *Mathematical Methods for Physicists: A Comprehensive Guide*. [S.l.]: Academic Press, 2013. ISBN 9780123846549.

ARUTIUNIAN, N. The plane contact problem of the theory of creep. *Journal of Applied Mathematics and Mechanics*, v. 23, n. 5, p. 1283–1313, 1959. ISSN 0021-8928. Available at: <https://www.sciencedirect.com/science/article/pii/0021892859901315>.

ASKEY, R.; FITCH, J. Integral representations for jacobi polynomials and some applications. *Journal of Mathematical Analysis and Applications*, v. 26, n. 2, p. 411–437, 1969. ISSN 0022-247X. Available at: <https://www.sciencedirect.com/science/article/pii/0022247X69901656>.

BARABÁSI, A.-L.; VICSEK, T. Multifractality of self-affine fractals. *Phys. Rev. A*, American Physical Society, v. 44, p. 2730–2733, Aug 1991. Available at: <https://link.aps.org/doi/10.1103/PhysRevA.44.2730>.

BARTUMEUS, F.; PETERS, F.; PUEYO, S.; MARRASé, C.; CATALAN, J. Helical lévy walks: adjusting searching statistics to resource availability in microzooplankton. *Proceedings of the National Academy of Sciences of the United States of America*, v. 100, n. 5, p. 12771–12775, 2003.

BULDYREV, S.; GITTERMAN, M.; HAVLIN, S.; KAZAKOV, A.; da Luz, M.; RAPOSO, E.; STANLEY, H.; VISWANATHAN, G. Properties of lévy flights on an interval with absorbing boundaries. *Physica A: Statistical Mechanics and its Applications*, v. 302, n. 1, p. 148–161, 2001. ISSN 0378-4371. Proc. Int. Workshop on Frontiers in the Physics of Complex Systems. Available at: <https://www.sciencedirect.com/science/article/pii/S0378437101004617>.

BULDYREV, S. V.; HAVLIN, S.; KAZAKOV, A. Y.; LUZ, M. G. E. da; RAPOSO, E. P.; STANLEY, H. E.; VISWANATHAN, G. W. Average time spent by lévy flights and walks on an interval with absorbing boundaries. *Physical Review E*, v. 64, n. 041108, 2001.

CALVO, I.; CUCHI, J. C.; ESTEVE, J. G.; FALCETO, F. Generalized central limit theorem and renormalization group. *Journal of Statistical Physics*, v. 141, n. 3, p. 409–421, 2010. ISSN 1572-9613. Available at: <https://doi.org/10.1007/s10955-010-0065-y>.

CHECHKIN, A. V.; METZLER, R.; GONCHAR, V. Y.; KLAFTER, J.; TANATAROV, L. V. First passage and arrival time densities for lévy flights and the failure of the method of images. *Journal of Physics A: Mathematical and General*, IOP Publishing Ltd., v. 36, n. 41, p. L537, oct 2003. Available at: <https://dx.doi.org/10.1088/0305-4470/36/41/L01>.

CHECHKIN, A. V.; METZLER, R.; KLAFTER, J.; GONCHAR, V. Y. Introduction to the theory of Lévy Flights. In: REINER, K.; GUNTHER, R.; IGOR, S. (Ed.). *Anomalous Transport*. John Wiley & Sons, Ltd, 2008. chap. 5, p. 129–162. ISBN 9783527622979. Available at: <https://onlinelibrary.wiley.com/doi/abs/10.1002/9783527622979.ch5>.

COLE, B. J. Fractal time in animal behaviour: the movement activity of drosophila. *Animal Behaviour*, v. 50, n. 22, p. 1317–1324, 1995. ISSN 0003-3472. Available at: <https://www.sciencedirect.com/science/article/pii/0003347295800476>.

COSTIN, R. An introduction to hilbert spaces. 2011. Available at: <https://people.math.osu.edu/costin.10/5101/Hilbert%20Spaces.pdf>.

DAVIS, P. J. Gamma function and related functions. In: Abramowitz, M.; Stegun, I. A. (Ed.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. [S.l.]: National Bureau of Standards, US Government Printing Office. Tenth printing with corrections, 1972, (Applied Mathematics, v. 55).

EIDELMAN, Y.; MILMAN, V.; TSOLOMITIS, A. *Functional Analysis: An Introduction*. American Mathematical Society, 2004. (Graduate studies in mathematics). ISBN 9780821836460. Available at: <https://books.google.com.co/books?id=eejpBwAAQBAJ>.

EINSTEIN, A. Über die von der molekularkinetischen theorie der wärme geforderte bewegung von in ruhenden flüssigkeiten suspendierten teilchen. *Annalen der Physik*, v. 322, n. 8, p. 549–560, 1905. Available at: <https://onlinelibrary.wiley.com/doi/abs/10.1002/andp.19053220806>.

ERDÉLYI, A.; MAGNUS, W.; OBERHETTINGER, F.; TRICOMI, F. G. *Higher Transcendental Functions. Vol. I*. New York-Toronto-London: McGraw-Hill Book Company, Inc., 1953. xxvi+302 p. Reprinted by Robert E. Krieger Publishing Co. Inc., 1981. Table errata: Math. Comp. v. 65 (1996), no. 215, p. 1385, v. 41 (1983), no. 164, p. 778, v. 30 (1976), no. 135, p. 675, v. 25 (1971), no. 115, p. 635, v. 25 (1971), no. 113, p. 199, v. 24 (1970), no. 112, p. 999, v. 24 (1970), no. 110, p. 504, v. 17 (1963), no. 84, p. 485.

ERDÉLYI, A.; MAGNUS, W.; OBERHETTINGER, F.; TRICOMI, F. G. *Tables of Integral Transforms. Vol. I*. New York-Toronto-London: McGraw-Hill Book Company, Inc., 1954. xx+391 p. Table errata: Math. Comp. v. 66 (1997), no. 220, p. 1766–1767, v. 65 (1996), no. 215, p. 1384, v. 50 (1988), no. 182, p. 653, v. 41 (1983), no. 164, p. 778–779, v. 27 (1973), no. 122, p. 451, v. 26 (1972), no. 118, p. 599, v. 25 (1971), no. 113, p. 199, v. 24 (1970), no. 109, p. 239-240.

FORMAGLIO, P.; WOSNIACK, M.; TROMER, R.; POLLI, J.; MATOS, Y.; ZHONG, H.; RAPOSO, E.; da Luz, M.; AMINO, R. Plasmodium sporozoite search strategy to locate hotspots of blood vessel invasion. *Nature Communications*, v. 14, n. 1, p. 2965, 2023. ISSN 2041-1723.

FREDHOLM, I. Sur une classe d'équations fonctionnelles. *Acta Mathematica*, Institut Mittag-Leffler, v. 27, n. none, p. 365 – 390, 1903. Available at: <https://doi.org/10.1007/BF02421317>.

FRISTEDT, B.; LAWRENCE, G. *A Modern Approach to Probability Theory*. [S.l.]: Birkhäuser Boston, 1996. (Probability and Its Applications). ISBN 9780817638078.

GAKHOV, F. D. *Boundary Value Problems*. [S.l.]: Pergamon, 1966. (International Series of Monographs on Pure and Applied Mathematics, v. 85).

GOMES, A.; RAPOSO, E.; MOURA, A.; FAWO, S.; PINCHEIRA, P.; JEREZ, V.; MAIA, L.; ARAúJO, C. d. Observation of lévy distribution and replica symmetry breaking in random lasers from a single set of measurements. *Scientific Reports*, v. 6, n. 27987, 2016.

HOCHSTRASSER, U. W. Orthogonal polynomials. In: Abramowitz, M.; Stegun, I. A. (Ed.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*.

[S.l.]: National Bureau of Standards, US Government Printing Office. Tenth printing with corrections, 1972, (Applied Mathematics, v. 55).

HUGHES, B. D.; SHLESINGER, M. F.; MONTROLL, E. W. Random walks with self-similar clusters. *Proceedings of the National Academy of Sciences of the United States of America*, National Academy of Sciences, v. 78, n. 6, p. 3287–3291, 1981. ISSN 00278424. Available at: <http://www.jstor.org/stable/10528>.

HUMPHRIES, N.; WEIMERSKIRCH, H.; QUEIROZ, N.; SOUTHALL, E. J.; SIMS, D. W. Foraging success of biological lévy flights recorded in situ. *Proceedings of the National Academy of Sciences of the United States of America*, v. 109, n. 19, p. 7169–7174, 2012.

JAFFARD, S. The multifractal nature of lévy processes. *Probability Theory and Related Fields*, v. 114, n. 2, p. 207–227, 1999. ISSN 1432-2064.

KANTELHARDT, J. W.; ZSCHIEGNER, S. A.; KOSCIELNY-BUNDE, E.; HAVLIN, S.; BUNDE, A.; STANLEY, H. Multifractal detrended fluctuation analysis of nonstationary time series. *Physica A: Statistical Mechanics and its Applications*, v. 316, n. 1, p. 87–114, 2002. ISSN 0378-4371. Available at: <https://www.sciencedirect.com/science/article/pii/S0378437102013833>.

KANWAL, R. *Linear Integral Equations*. [S.l.]: Birkhäuser Boston, 2013. ISBN 9781461207658.

LANG, S. *Complex Analysis*. Fourth edition. [S.l.]: Springer Science+Business Media New York, 1999. (Graduate Texts in Mathematics). ISBN 9781441931351.

MANTEGNA, R. N.; STANLEY, H. E. Stochastic process with ultraslow convergence to a gaussian: The truncated lévy flight. *Phys. Rev. Lett.*, American Physical Society, v. 73, p. 2946–2949, Nov 1994. Available at: <https://link.aps.org/doi/10.1103/PhysRevLett.73.2946>.

MANTEGNA, R. N.; STANLEY, H. E. Scaling behaviour in the dynamics of an economic index. *Nature*, v. 376, n. 1, p. 46–49, 1995. ISSN 1476-4687. Available at: <https://doi.org/10.1038/376046a0>.

METZLER, R.; KLAFTER, J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, v. 339, n. 1, p. 1–77, 2000. ISSN 0370-1573. Available at: <https://www.sciencedirect.com/science/article/pii/S0370157300000703>.

MONTROLL, E. W.; WEST, B. J. On an enriched collection of stochastic processes. In: MONTROLL, E.; LEBOWITZ, J. (Ed.). *Fluctuation Phenomena*. Elsevier, 1979. chap. 2, p. 61–175. ISBN 978-0-444-85248-9. Available at: <https://www.sciencedirect.com/science/article/pii/B9780444852489500054>.

MUSKHELISHVILI, N. *Singular Integral Equations: Boundary Problems of Function Theory and Their Application to Mathematical Physics*. [S.l.]: Dover Publications, 2013. (Dover Books on Mathematics). ISBN 9780486145068.

NAKAO, H. Multi-scaling properties of truncated lévy flights. *Physics Letters A*, v. 266, n. 4, p. 282–289, 2000. ISSN 0375-9601. Available at: <https://www.sciencedirect.com/science/article/pii/S0375960100000591>.

NEUMAIER, A. Solving ill-conditioned and singular linear systems: A tutorial on regularization. *SIAM Review*, Society for Industrial and Applied Mathematics, v. 40, n. 3, p. 636–666, 1998. ISSN 00361445. Available at: <http://www.jstor.org/stable/2653234>.

NODARSE, R. Álvarez. On characterizations of classical polynomials. *Journal of Computational and Applied Mathematics*, v. 196, n. 1, p. 320–337, 2006. ISSN 0377-0427. Available at: <https://www.sciencedirect.com/science/article/pii/S0377042705005662>.

OBERHETTINGER, F. Hypergeometric functions. In: Abramowitz, M.; Stegun, I. A. (Ed.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. [S.l.]: National Bureau of Standards, US Government Printing Office. Tenth printing with corrections, 1972, (Applied Mathematics, v. 55).

PFEILSTICKER, K. First geometrical path length probability density function derivation of the skylight from high-resolution oxygen a-band spectroscopy: 2. derivation of the lévy index for the skylight transmitted by midlatitude clouds. *Journal of Geophysical Research: Atmospheres*, v. 104, n. D4, p. 4101–4116, 1999. Available at: <https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/1998JD200081>.

PODLUBNY, I. *Fractional Differential Equations*. [S.l.]: Academic Press, 1999. (Mathematics in Science and Engineering, v. 198).

POPOV, G. Y. Some properties of classical polynomials and their application to contact problems. *Prikl. Math. Mekh.*, v. 27, p. 821–832, 1963. (In Russian).

POPOV, G. Y. Concentration of elastic stresses near stamps, cuts, thin inclusions, and reinforcements. *Nauka, Moscow*, 1982. (In Russian).

PóLYA, G. Über eine aufgabe der wahrscheinlichkeitsrechnung betreffend die irrfahrt im straßennetz. *Mathematische Annalen*, v. 84, p. 149–160, 1921. Available at: <http://eudml.org/doc/158886>.

RATHIE, P.; OZELIM, L. S.; OTINIANO, C. Exact distribution of the product and the quotient of two stable lévy random variables. *Communications in Nonlinear Science and Numerical Simulation*, v. 36, p. 204–218, 2016. ISSN 1007-5704. Available at: <https://www.sciencedirect.com/science/article/pii/S1007570415003895>.

REICHL, L. *A Modern Course in Statistical Physics*. 2nd edition. ed. [S.l.]: John Wiley I& Sons, 1998. ISBN 0471595209.

SAMKO, S. G.; KILBAS, A. A.; MARICHEV, O. I. Fractional integrals and derivatives: Theory and applications. In: . [S.l.]: Gordon and Breach Science, Switzerland, 1993. Originally published in Russian by Nauka i Tekhnika, Minsk in 1987.

SCHNEIDER, W. R. Stable distributions: Fox function representation and generalization. In: ALBEVERIO, S.; CASATI, G.; MERLINI, D. (Ed.). *Stochastic Processes in Classical and Quantum Systems*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1986. p. 497–511. ISBN 978-3-540-47222-3.

SHLESINGER, M. F.; KLAFTER, J.; J. West, B. Levy walks with applications to turbulence and chaos. *Physica A: Statistical Mechanics and its Applications*, v. 140, n. 1, p. 212–218, 1986. ISSN 0378-4371. Available at: <https://www.sciencedirect.com/science/article/pii/0378437186902244>.

SHLESINGER, M. F.; ZASLAVSKY, G. M.; KLAFTER, J. Strange kinetics. *Nature*, v. 363, n. 1, p. 31–37, 1993. ISSN 1476-4687.

SIMS, D. W.; REYNOLDS, A. M.; HUMPHRIES, N. E.; SOUTHALL, E. J.; WEARMOUTH, V. J.; METCALFE, B.; TWITCHETT, R. J. Hierarchical random walks in trace fossils and the origin of optimal search behavior. *Proceedings of the National Academy of Sciences of the United States of America*, v. 111, n. 30, p. 11073–11078, 2014.

SLATER, L. J. Confluent hypergeometric functions. In: Abramowitz, M.; Stegun, I. A. (Ed.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. [S.l.]: National Bureau of Standards, US Government Printing Office. Tenth printing with corrections, 1972, (Applied Mathematics, v. 55).

SZEGŐ, G. *Orthogonal Polynomials*. [S.l.]: American Mathematical Society, 1939. (American Mathematical Society Colloquium Publications, XXIII). (Fourth Edition, 1975). ISBN 0821810235.

VISWANATHAN, G.; RAPOSO, E.; LUZ, M. d. Lévy flights and superdiffusion in the context of biological encounters and random searches. *Physics of Life Reviews*, v. 5, n. 3, p. 133–150, 2008. ISSN 1571-0645. Available at: <https://www.sciencedirect.com/science/article/pii/S1571064508000146>.

VLADIMIROV, V. *Equations of Mathematical Physics*. [S.l.]: Mir, 1984. ISBN 9780828528771.

WOLFRAM RESEARCH INC. *Mathematica, Version 13.2*. 2022. Champaign, IL. Available at: <https://www.wolfram.com/mathematica>.

YOSHIDA, K. *Functional Analysis*. [S.l.]: Springer Verlag Berlin Heidelberg, 1980. (Classics in mathematics). ISBN 9780387102108.

ZASLAVSKY, G. *Hamiltonian Chaos and Fractional Dynamics*. [S.l.]: Oxford University Press, 2005. ISBN 9780198526049.

ZUMOFEN, G.; KLAFTER, J. Spectral random walk of a single molecule. *Chemical Physics Letters*, v. 219, n. 3, p. 303–309, 1994. ISSN 0009-2614. Available at: <https://www.sciencedirect.com/science/article/pii/0009261494870624>.

APPENDIX A – SYMBOLIC CHECK

In[62]:=
```
f[η_, L_, l_, h_] := (Pochhammer[l - h + 1, h] / Factorial[h]) *
   Pochhammer[L - (l - 1) + h, l - h] * Pochhammer[L - (l - 1) + h + 1 / 2, l - h] *
   Pochhammer[η / 2 + L - (l - 1) + L - 1 / 2, h] * Pochhammer[η / 2 - h, h];
f1[η_, L_, l_] := Sum[ f[η, L, l, h] , {h, 0, l}]

Expand[f1[η, L, l ]]

FullSimplify[ % == Pochhammer[η / 2 + L - l, l] * Pochhammer[η / 2 + L - l + 1 / 2, l],
 Assumptions → Element[η, Reals] && 0 < η < 2 &&
   Element[L, Integers] && Element[l, Integers] && 0 ≤ l && l ≤ L]
```

Out[64]=

$$\frac{2^{-2\,l}\,(2\,l-2\,L-\eta)\,!}{(-2\,L-\eta)\,!}$$

Out[65]=

True