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ROBSON CARLOS DA SILVA REIS

UNBOUNDED HAMILTON-JACOBI-BELLMAN EQUATIONS WITH ONE CO-DIMENSIONAL DISCONTINUITIES

Recife

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Tese apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal de Pernambuco, como requisito parcial para obtenção do título de doutor em Matemática. Área de concentração: Análise.

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Orientadora (a): Silvia Sastre Gómez

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ABSTRACT

The aim of this thesis is to deal, of the point of view of viscosity solutions, with a discontinuous Hamilton-Jacobi equation in the whole euclidian N-dimensional space where the discontinuity is located on an hyperplane. The typical questions that arise this directions are concern the existence and uniqueness of solutions, and of course the definition itself of solution. Here we consider viscosity solutions in the sense of Ishii. Since we consider convex Hamiltonians, we can also associate the problem to a control problem with specific cost and dynamics given on each side of the hyperplane. We assume that those are Lipshichitz continuous but potentially unbounded, as well as the control spaces. Using Bellman's approach we construct two value functions which turn out to be the minimal and maximal solutions in the sense of Ishii. Moreover, we also build a whole family of value functions, which are still solutions in the sense of Ishii and connect continuously the minimal solution to the maximal one.

Keywords: optimal control; discontinuous dynamic; Hamilton-Jacobi-Bellman equation; viscosity solutions; Ishii problem.

RESUMO

O objetivo desta tese é lidar, do ponto de vista de soluções viscosas, com descontinuidades da equação de Hamilton-Jacobi no espaço euclidiano de dimensão N, onde a descontinuidade está localizada em um hiperplano. As típicas questões que surgem neste sentido estão relacionadas com a existência e unicidade de soluções, e naturalmente sobre a própria definição de solução. Nós consideramos soluções de viscosidade no sentido de Ishii. Desde que nós consideramos Hamiltonianos convexos, podemos associar o problema a um problema de controle com custo e dinâmica específicos dados em cada lado do hiperplano. Assumimos que esses são Lipschitz, mas potencialmente ilimitados, assim como os espaços de controle. Usando a abordagem de Bellman, construímos duas funções de valor que se tornam as soluções mínima e máxima no sentido de Ishii. Além disso, também construímos toda uma família de funções valores, que ainda são soluções no sentido de Ishii e conectam continuamente a solução mínima à máxima.

Palavras-chaves: controle ótimo; dinâmica descontínua; equação de Hamilton-Jacobi-Bellma; soluções viscosas; problema de Ishii.

CONTENTS

1 INTRODUCTION

This thesis focuses deeply on the concept of viscosity solution for Hamilton-Jacobi equations, investigating some non-uniqueness features that arise in presence of discontinuities.

This chapter is divided in three sections. In the first one, we give a quick overview of the theory of viscosity solutions, presenting the basic concepts and results. The second section presents the discontinuous specific we deal with and in the last section we present our results.

1.1 VISCOSITY SOLUTIONS

In 1981, M.G. Crandall and P.-L. Lions introduced the notion of a viscosity solution to address difficulties found in the study of Hamilton-Jacobi equations. The classical analysis, via the method of characteristics, only provides local results due to the crossing of characteristic curves. On the other hand, viscosity solutions allow to consider global solutions, and get uniqueness and stability results for many problems that arise in various applications ((CRANDALL; LIONS, 1983)). The initial notion of viscosity solution requires very little regularity of the function (only continuity) and was even extended later to discontinuous functions and Hamiltonians, as we explain below.

Due to its wide reach, this concept has aroused the interest of many important researchers such as G. Barles, L.C. Evans, M.G. Crandall, I. Ishii, P.-L. Lions, and viscosity analysis has become a very fertile field that has come a long way in the last 40 years (CRANDALL; EVANS; LIONS, 1984).

Initially the concept of viscosity solution was established for continuous Hamiltonians on $\Omega \times R \times R^N$, where Ω is open set of R^N. For example, we have the classical mechanics Hamiltonian

$$H(x,p) := \frac{1}{2}|p|^2 + V(x)$$

where $\frac{1}{2}|p|^2$ denote the kinetic energy and $V(x)$ the potential energy.

However later this concept was extended to Hamiltonians which are only locally bounded. We started with the definition of the continuous viscosity solution of the first-order stationary Hamilton-Jacobi equation for continuous Hamiltonians as presented in (CRANDALL; EVANS; LIONS, 1984).

Definition 1.1.1 *We say that $u \in C(\Omega)$ is a viscosity subsolution of H in Ω open set of R^N if given $\phi \in C^1(\Omega)$ and $x_0 \in \Omega$ local maximum point of $u - \phi$, then*

$$H(x_0, u(x_0), D\phi(x_0)) \leq 0. \tag{1.1.1}$$

We say that $u \in C(\Omega)$ is a viscosity supersolution of H in Ω open set of R^N if given $\phi \in C^1(\Omega)$ and $x_0 \in \Omega$ local minimum point of $u - \phi$, then

$$H(x_0, u(x_0), D\phi(x_0)) \geq 0. \tag{1.1.2}$$

We say that u is a viscosity solution of H if u is simultaneously subsolution and supersolution.

Functions ϕ above are called test functions. It is very convenient to use definition 1.1.1 to prove that the value function is a solution of the Hamilton-Jacobi-Bellman equation. This definition also plays an important role in technique of "doubling of variables".

The notion of viscosity solution is well-defined for any Ω open set of R^N. Yet to get specific results we can add hypotheses on the boundary of Ω. For example, Barles (BARLES, 1994, p. 111) works with boundary $W^{2,\infty}$. In (BLANC, 2001) it is required that the distance function to boundary be $C^{1,1}$.

An equivalent definition can be given, using sub and superdifferentials. While this approach may simplify some aspects of the theory, Barles (BARLES, 1994, p. 20) notes that it would be more technical to prove stability results using sub/super differentials.

Definition 1.1.2 *The **subdifferential** the v in point x is*

$$\partial^- v(x) := \{p \in R^N | \liminf_{y \to x} \frac{v(y) - v(x) - p \cdot (y - x)}{|y - x|} \geq 0\}$$

*The **Superdifferential** of v in point x is*

$$\partial^+ v(x) := \{p \in R^N | \limsup_{y \to x} \frac{v(y) - v(x) - p \cdot (y - x)}{|y - x|} \leq 0\}$$

With these elements, the equivalent definition is the following:

Definition 1.1.3 (Continuous Hamiltonians) *We say that $u \in C(\Omega)$ is a subsolution of H in Ω open set of R^N if*

$$H(x_0, u(x_0), p) \leq 0 \quad \forall\, p \in \partial^+ u(x_0). \tag{1.1.3}$$

We say that $u \in C(\Omega)$ is a supersolution of H in in Ω open set of R^N if

$$H(x_0, u(x_0), p) \geq 0, \quad \forall\, p \in \partial^- u(x_0). \tag{1.1.4}$$

We say that u is a viscosity solution of H if u is simultaneously subsolution and supersolution.

We see clearly by definition 1.1.3 that classical solutions are also viscosity solutions, since if v is differentiable at x, then $\partial^-(x) = Dv(x) = \partial^+(x)$. Moreover, this definition gives us regularity of the viscosity subsolutions of Coercive Hamiltonians, see Lemma 2.5 page 20 of (BARLES, 1994).

In many problems, classical solutions may not be unique or even exist. That is the case for the eikonal equation, $|\nabla u(x)| - 1 = 0$ in $(-1, 1)$ with boundary condition $u(-1) = u(1) = 0$. Thanks to Mean Value Theorem this problem does not have classical solution in $C^1(-1, 1)$. On the other hand, it has infinitely many weak solutions. For each n consider $u_n(-1) = 0$ and $\nabla u_n(x) = (-1)^j$ on $(-1 + \frac{j}{2n}, -1 + \frac{j+1}{2n})$..... for $j = 0, \cdots, 4n - 1$ and $u_n(-1) = u_n(1) = 0$. Hence, we do not have uniqueness for weak solution and that is a challenge in application problems, because of the need to identify the right solution.

Viscosity solution on the contrary are at the same time more flexible (they give existence easily) and allow to prove uniqueness results. For example, it can be proved that $u(x) = 1 - |x|$ is a unique viscosity solution of the eikonal problem above.

1.2 STABILITY

Another deficiency of the notion of weak solution is that uniform convergence of solutions does not necessarily yield a solution: in the eikonal problem above, (u_n) converges uniformly to 0 when n goes infinity, but 0 is not a weak solution of $|\nabla u(x)| - 1 = 0$ anywhere. However the uniform limit of viscosity solutions is still a viscosity solution as the following theorem shows.

Theorem 1.2.1 *Let $\varepsilon > 0$, and let $H^\varepsilon(y, r, p)$ be a family of continuous functions such that $H^\varepsilon(y, r, p)$ converges uniformly on compact subsets of $\Omega \times R \times R^N$ to some continuous function $H(y, r, p)$, as ε goes to 0. And let us assume that the u^ε is subsolution (supersolution) of H^ϵ converge uniformly on compact subsets of Ω to some $u \in C(\Omega)$. Then u is a viscosity subsolution (supersolution) of $H(y, r, p)$ in Ω.*

See Theorem 1.2 of (CRANDALL; EVANS; LIONS, 1984).

The name viscosity refers to the vanishing viscosity method, a kind of stability used in the existence results, and was chosen for want of a better idea', (CRANDALL; LIONS, 1983) page

2. Indeed, realize that if u is viscosity solution of $|\nabla u(x)| = 1$ then it is also viscosity solution of $|\nabla u(x)|^2 = 1$. Considering the approximate problem

$$-\varepsilon \Delta u^\varepsilon(x) + |\nabla u^\varepsilon(x)|^2 = 1 \text{ with } u^\varepsilon(-1) = u^\varepsilon(1) = 0, \tag{1.2.1}$$

we see that it has a unique classical solution u_ε. When passing to the limit, one can prove that $\lim_\varepsilon u_\varepsilon(x) = 1 - |x|$. More generally, the following result holds:

Theorem 1.2.2 (Vanishing viscosity Method) *Let $\varepsilon > 0$, and let $H^\varepsilon(y, r, p)$ be a family of continuous functions such that $H^\varepsilon(y, r, p)$ converges uniformly on compact subsets of $\Omega \times R \times R^N$ to some function $H(y, r, p)$, as ε goes to 0. Finally, suppose $u^\varepsilon \in C^2(\Omega)$ is a solution of*

$$-\varepsilon \Delta u^\varepsilon(x) + H^\varepsilon(x, u^\varepsilon(x) D u^\varepsilon(x)) = 0 \quad \textit{in } \Omega$$

and let us assume that the u^ε converge uniformly on compact subsets of Ω to some $u \in C(\Omega)$. Then u is a viscosity solution of $H(x, u, Du) = 0$ in Ω.

See Theorem 3.1 of(CRANDALL; EVANS; LIONS, 1984).

1.3 BOUNDARY CONDITIONS

The theory of viscosity solutions also allows to treat boundary conditions. Consider $\Omega \subset R^N$, an open bounded set and

$$\begin{cases} H(x, u(x), \nabla u(x)) = 0 \ \text{ in } \ \Omega \\ u(x) = g(x) \ \text{ on } \ \partial\Omega \end{cases}$$

This problem has no solution in general as we already saw in the eikonal case.

In particular,

$$\begin{cases} |\nabla u(x)| - 1 = 0 \ \text{ in } \ \Omega \\ u(x) = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

not have classical solution, yet has infinitely many Lipschitz continuous functions which satisfy the boundary condition and whose modulus of gradient is equal to 1 almost everywhere. However there is unique viscosity solution, $u(x) = d(x, \partial\Omega)$. In general, we can get viscosity solutions of boundary problems by requiring almost no regularity on g, see (BLANC, 1997), (BLANC, 2001).

But the boundary data has to be understood in a relaxed sense: for the subsolution condition, if ϕ is a test-function for u at $x \in \partial\Omega$, we require that

$$\min\Big\{H(x, u(x), \nabla\phi(x))\,,\, u(x) - g(x)\Big\} \le 0\,,$$
$$\max\Big\{H(x, u(x), \nabla\phi(x))\,,\, u(x) - g(x)\Big\} \ge 0\,.$$

In other words, either the equation holds, or the boundary condition. This relaxed boundary condition was introduced by I. Ishii (ISHII, 1985) in order to be compatible with the stability property of viscosity solutions. Indeed, when passing to the limit in $\overline{\Omega}$, the maximum (or minimum) points of $u - \phi$ can either come from the inside, or the boundary. So, when passing to the limit, on the boundary we get either the sub/supersolution condition, or the boundary condition, hence the formulation with min and max.

1.4 RELATION WITH OPTIMAL CONTROL

There exist a link between viscosity solutions and Optimal Deterministic Control: given the dynamical controlled system,

$$x'(t) = b(x(t), \alpha(t))$$

where $\alpha(\cdot)$ is the control variable with $\alpha(\cdot) \in \mathcal{A}$ admissible control space, we define a cost functions J, associated to the running cost function f and discount factor λ

$$J(x, \alpha(\cdot)) = \int_0^\infty l(x(t), \alpha(t))e^{-\lambda t}dt\,.$$

The purpose the Optimal control Theory is to optimize the functional cost for some optimal admissible control. This theory has several applications. For example, in economics or industry the cost functional can be the cost of production of some product that we want to minimize. In this context, the value function is defined as

$$u(x) = \inf_{\alpha \in \mathcal{A}} \int_0^\infty l(x(t), \alpha(t))e^{-\lambda t}dt.$$

The Classical Bellman approach consists in noticing that if u is differentiable, then this value function is a classical solution of the Bellman equation

$$\sup_{\alpha \in A}(-b(x, \alpha) \cdot \nabla u(x) - l(x, \alpha) + \lambda u(x)) = 0 \quad \in R^N.$$

That is consequence of the infinitesimal version of the Dynamic Programming Principle:

$$u(x) = \inf_{\alpha \in \mathcal{A}} \left(\int_0^T l(x(t), \alpha(t))e^{-\lambda t}dt + e^{-\lambda T}u(x(T)) \right), \quad \forall\, T > 0. \tag{1.4.1}$$

This equation is itself an integral version of the Principle of Optimality: "*An optimal policy has the property that what ever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*"where, "*a policy is any rule for making decisions which yields an allowable sequence of decisions; and an optimal policy is a policy which maximizes a preassigned function of the final state variables*", see page 82,83 of (BELLMAN, 2010).

Now we return to our heuristic consideration. Assume that in addition to the differentiability of u there is also an optimal control α^*, that is,for which infinity is reached

$$u(x) = \int_0^\infty l(x^*(t), \alpha^*(t))e^{-\lambda t}dt.$$

Using the Dynamic Programming Principle (DPP) we get

$$g(s) = \int_0^s l(x^*(t), \alpha^*(t))e^{-\lambda t}dt + u(x^*(s))e^{-\lambda s} = u(x) \quad \forall s > 0\,.$$

Note that g is constant if only if $x^*(t), \alpha^*(t)$ is optimal, if only if $g'(s) = 0$ which yields

$$\lambda u(x^*(s)) - b(x^*(s), \alpha^*(s)) \cdot \nabla u(x^*(s)) - l(x^*(s), \alpha^*(s)) = 0.$$

Since u is a classical solution of Bellman equation, then α^* is optimal if and only if

$$H(x^*(s), u(x^*(s)), \nabla u(x^*(s))) = \lambda u(x^*(s)) - b(x^*(s), \alpha^*(s)) \cdot \nabla u(x^*(s)) - l(x^*(s), \alpha^*(s)).$$

Now, suppose there is a map

$$S : R^N \to A$$

with $S(z) \in \arg\max(-b(z, \alpha) \cdot \nabla u(z) - l(z, \alpha) + \lambda u(z))$. We call such S a feedback map. Through this feedback, we may in turn find an optimal trajectory by solving:

$$x'(t) := \begin{cases} b(x(t), S(x(t))), & \text{if } t > 0 \\ x, & \text{if } t = 0. \end{cases}$$

The conclusion is that solving the Hamilton-Jacobi-Bellman equation $H(x, u, Du) = 0$ is closely related to finding optimal controls and trajectories.

1.5 DISCONTINUITIES IN HAMILTON-JACOBI EQUATIONS

Initially, viscosity solutions were defined in the context of continuous Hamiltonians and functions. However, as we saw above for boundary conditions, I. Ishii extended the concept of viscosity solutions to discontinuous Hamiltonians in 1985, (ISHII, 1985).

Now, as we saw above, the stability property of viscosity solutions requires some local uniform convergence of both the Hamiltonian and the solution. However, in many problems, such uniform convergence of the solutions is not easy to obtain.

1.6 DISCONTINUOUS VISCOSITY SOLUTIONS

In 1987, G. Barles and B. Perthame (BARLES; PERTHAME, 1987) introduced the concept of discontinuous viscosity solutions by using the **upper and lower semicontinuous envelopes**:

Definition 1.6.1 *Let g locally bounded, we define the* ***lower semicontinuous envelope*** *and* ***upper semicontinuous envelope*** *as $g_*(x) = \liminf_{y\to x} g(y)$, $g^*(x) = \limsup_{y\to x} g(y)$, respectively.*

This definition allows more flexibility to treat a priori discontinuous solutions, provided they are locally bounded (which is sufficient to define the semi-continuous envelopes). If the Hamiltonian is also discontinuous, we have to use its envelopes too and the definition becomes

Definition 1.6.2 (Discontinuous Hamiltonians) *We say that u locally bounded is a viscosity subsolution of H in Ω open set of R^N if given $\phi \in C^1(\Omega)$ and $x_0 \in \Omega$ maximum point of $u^* - \phi$, then*

$$H_*(x_0, u^*(x_0), D\phi\,(x_0)) \leq 0. \tag{1.6.1}$$

We say that u locally bounded is a viscosity supersolution of H in Ω open set of R^n if given $\phi \in C^1(\Omega)$ and $x_0 \in \Omega$ minimum point of $u_* - \phi$, then

$$H^*(x_0, u_*(x_0), D\phi\,(x_0)) \geq 0. \tag{1.6.2}$$

We say that u is a viscosity solution of H if u is simultaneously subsolution and supersolution.

If we consider a sequence of equations $H_n(x, u_n, Du_n) = 0$ where H_n converges to some Hamiltonian H, the advantage of the discontinuous approach is that we may consider the limits of the (u_n) with only local uniform bounds. Hence, introducing the half-relaxed limits

$$u^*(x) := \limsup_{n\to\infty, y\to x} u_n(y)\ ,\ u_*(x) := \liminf_{n\to\infty, y\to x} u_n(y)\ ,$$

it turns out that u^* is a subsolution of the limit equation $H(x,u,Du)=0$ while u_* is a supersolution of it. If the limit Hamiltonian enjoys a comparison property, we deduce that $u^* \leq u_*$, so that in the end $u^* = u_*$ and we get a solution. But of course, if H is discontinuous in x, the comparison principle may fail and recovering a stability result for discontinuous solutions requires a deeper analysis, part of which is done in this Thesis.

In the last two decades, many researchers have been interested in studying Hamilton-Jacobi equation with discontinuities, for example: Barles (BARLES, 1990), Blanc (BLANC, 1997), 2001. In (GIGA; HAMAMUKI, 2013) the following problem was studied

$$\begin{cases} \partial_t u + H(x, Du(x)) = 0, & \text{if } t > 0 \text{ in } R^N \times (0,T) \\ \quad u(0,x) = u_0(x), & \text{if } t = 0 \ \text{ in } R^N \end{cases}$$

where $u_0 \in BUC(R^N)$ and $H(x,p) = -|p| - cI(x)$

$$I(x) = \begin{cases} 1 & (x = 0) \\ 0 & (x \neq 0) \end{cases}$$

That is, H has discontinuous at $x = 0$. In (GIGA; GóRKA; RYBKA, 2011), Giga *et al.* consider the singular problem

$$d_t + H(t,x,d,d_x) = 0 \ \text{ in } \ (0,T) \times R$$

$$H(t,x,u,p) = \begin{cases} -\sigma(t, r^*(t), u)m(p) & |x| < r_0(t) \\ \quad \sigma(t,x,u)m(p) & |x| \geq r_0(t) \end{cases}$$

with $\sigma \in C^1 \ r_0, \ r^* \in C_0([0,T])$ and $r^*(t) > r_0(t)$ for all $t \in [0,T]$. Moreover the set $\{(t, r_0(t)) : t \in [0,T]\}$ is a Lipschitz curve.

1.7 THE ISHII PROBLEM

More recently, in (BARLES; BRIANI; CHASSEIGNE, 2013) Guy Barles, Ariela Briani and Emmanuel Chasseigne study an infinite horizon control problems with discontinuities. They consider the Hamilton-Jacobi-Bellmam problem in R^N with discontinuity on the hyperplane $\mathcal{H} = R^{N-1} \times \{0\}$.

In order to give a quick overview of their results and introduce some notations and concepts that we use throughout this thesis, let us decompose the space into three parts $R^N = \Omega_1 \cup \mathcal{H} \cup \Omega_2$, where $\Omega_1 = \{x \in R^N \mid x_N > 0\}$ and $\Omega_2 = \{x \in R^N \mid x_N < 0\}$. We will take $A_1, A_2 \subset R^d$ to be the sets of controls. Each domain Ω_i has associated a dynamic function $b_i : \overline{\Omega_i} \times A_i \to R^N$ and a control function $l_i : \overline{\Omega_i} \times A_i \to R$. Finally, let $\mathcal{H}$ be the intersection of $\overline{\Omega}_1$ and $\overline{\Omega}_2$, so that $\mathcal{H} = \{x \in R^N \mid x_N = 0\}$.

The Ishii problem consists in solving the set of (in)equations

$$\begin{cases} H_1(x, u(x), Du(x)) = 0 & \text{in } \Omega_1 \\ H_2(x, u(x), Du(x)) = 0, & \text{in } \Omega_2 \\ \min\{H_1(x, u(x), Du(x)), H_2(x, u(x), Du(x))\} \leq 0 & \text{in } \mathcal{H} \\ \max\{H_1(x, u(x), Du(x)), H_2(x, u(x), Du(x))\} \geq 0 & \text{in } \mathcal{H} \end{cases} \tag{1.7.1}$$

where

$$H_i(x, u, q) = \sup_{w \in A_i} (-b(x, w) \cdot q - l_i(x, w) + \lambda u) \ .$$

In order to describe the particularities on $\mathcal{H}$, let us introduce the extended control set as follows:

Definition 1.7.1 *Given $A_1,\ A_2$ as above, we set $A := A_1 \times A_2 \times [0, 1]$ and we consider the control set formed by bounded measurable functions $\mathcal{A} := L^\infty(0, \infty; A)$.*

Notice that if $x \notin \mathcal{H}$, only $\alpha_1 \in A_1$ or $\alpha_2 \in A_2$ is used. Then on $\mathcal{H}$ the authors introduce the dynamic $b_\mathcal{H} : \mathcal{H} \times A \to R^N$ given by a convex combination of b_1 and b_2, and a cost function $l_\mathcal{H} : \mathcal{H} \times A \to R$ given by a convex combination of l_1 and l_2. More precisely, they define $b_\mathcal{H}$ and $l_\mathcal{H}$ as

$$b_\mathcal{H}(x, \alpha) = \mu b_1(x, \alpha_1) + (1 - \mu) b_2(x, \alpha_2), \quad x \in \mathcal{H},\ \alpha = (\alpha_1, \alpha_2, \mu) \in A,$$

$$l_\mathcal{H}(x, \alpha) = \mu l_1(x, \alpha_1) + (1 - \mu) l_2(x, \alpha_2), \quad x \in \mathcal{H},\ \alpha = (\alpha_1, \alpha_2, \mu) \in A.$$

Given some control $(\alpha_1, \alpha_2, \mu) \in \mathcal{A} := L^\infty(0, \infty; A)$, the authors solve the ode

$$\begin{aligned} X_x'(t) = {} & b_1(X_x(t), \alpha_1(t)) 1_{\{X_x(t) \in \Omega_1\}} + b_2(X_x(t), \alpha_2(t)) 1_{\{X_x(t) \in \Omega_2\}} \\ & + b_\mathcal{H}(X_x(t), \alpha(t)) 1_{\{X_x(t) \in \Omega_\mathcal{H}\}} \quad \text{for a.e.} \quad t \in R^+ \end{aligned} \tag{1.7.2}$$

and denote by $\tau_A(x)$ the set of such trajectories starting from x.

The precise analysis of the situation shows that at least two specific value functions can be built, the "natural"one being defined by

$$U_A^-(x) = \inf_{\tau_A(x)} \left(\int_0^\infty l(X_x(t), \alpha(t)) e^{-\lambda t} dt \right) . \tag{1.7.3}$$

As one can expect, it turns out that U_A^- is an Ishii solution of the problem. But moreover, this special solution satisfies a complemented $(N-1)$-dimensional inequation on $\mathcal{H}$: $H_T(x, u, Du) \leq 0$, where the tangential Hamiltonian is defined by

$$H_T(x, \phi(x), D_{\mathcal{H}}\phi) = \sup_{(\alpha_1, \alpha_2, \mu) \in A_0(x)} -l_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) + \lambda \overline{\phi}(x) - b_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) \cdot (D_{\mathcal{H}}\phi(x), 0) . \tag{1.7.4}$$

Here, $A_0(x)$ is the set of controls that allow the trajectory to remain on $\mathcal{H}$, that is, the controls such that $b_{\mathcal{H}}(x, a) = 0$ and

$$D_{\mathcal{H}}\phi(x) = \left(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_{n-1}} \right)$$

But the authors also build a second solution in the sense of Ishii by introducing the **regular dynamics** on $\mathcal{H}$ as follows:

Definition 1.7.2 *We say that $b_{\mathcal{H}}(y, \alpha_1, \alpha_2, \mu)$ is regular if*

$$b_{\mathcal{H}}(y, \alpha_1, \alpha_2, \mu) \in \mathcal{H}, \ b_1(y, \alpha_1) \cdot e_N \leq 0 \ \text{ and } \ b_2(y, \alpha_2) \cdot e_N \geq 0 .$$

In some sense, these dynamics are maintaining the trajectory on $\mathcal{H}$ by "pushing"from either side of the hyperplane, contrary to singular dynamics which corresponds to an equilibrium obtained by "pulling"from both sides.

This allows to construct a second value function:

$$U_A^+(x) = \inf_{\tau_A^{reg}(x)} \left(\int_0^\infty l(X_x(t), \alpha(t)) e^{-\lambda t} dt \right) , \tag{1.7.5}$$

and it is shown in (BARLES; BRIANI; CHASSEIGNE, 2013) that while U_A^- is minimal solution of the Ishii problem, U_A^+ is the maximal one. However, only U_A^- enjoys the complementary property with H_T.

In (BARLES; BRIANI; CHASSEIGNE, 2014) the authors extend the results in three directions: they consider general domains; finite horizon control problems and weaken the controlability assumptions. An even more global approach is performed in (BARLES; CHASSEIGNE, 2018) but the authors always consider bounded control sets, bounded dynamics and costs functions.

1.8 OUR RESULTS

As we said, in (BARLES; BRIANI; CHASSEIGNE, 2013; BARLES; BRIANI; CHASSEIGNE, 2014; BARLES; CHASSEIGNE, 2018) the Ishii problem 1.7.1 was treated with compact control space and bounded data. In this thesis one of the main objectives is to verify which results of (BARLES; BRIANI; CHASSEIGNE, 2013), (BARLES; BRIANI; CHASSEIGNE, 2014) are still valid when dealing with non-compact controls space and unbounded data.

The motivation for considering non-compact controls space is that some Hamiltonians can only be approached in this context. For example,

$$H_i(x, u, p) = \lambda u + |p|^2 - f_i(x).$$

Moreover, several researchers have worked on the context of dynamics and costs unbounded with controls space non-compact, for example see (BARLES, 1990), (MISZTELA, 2020), (MOTTA, 2004) (SORAVIA, 1999).

1.8.1 One Central Idea

In this thesis we consider an unbounded set of controls such as R^N but we assume some growth conditions for the dynamic and cost that allows us to reduce, at least locally to the case of compact control sets. The central hypothesis that allows us to do that is the following:

$$\lim_{|\alpha_i|\to\infty} \frac{l_i(y,\alpha)}{1+|b_i(y,\alpha)|} = \infty, \quad \text{locally uniformly with respect to } \; y. \tag{1.8.1}$$

The key idea here is that the cost associated to large dynamics should be so big that such strategies are not optimal, which allows to recover some compactness of the trajectories and the associated controls.

We then approach the unbound control space by a sequence of compact sets; solve the problem for each compact in the sequence and pass the limit using the Half-Relaxed Limit Method **??** or some other argument.

1.8.2 Construction of U_A^- and U_A^+

In Chapter 1 we describe the Ishii problem, we fix the set of trajectories with which we work and build the value functions U_A^+ and U_A^- in the unbounded setting. We verify that they satisfy the Dynamic programming Principle and are locally Lipschitz. Furthermore, using 1.8.1 we show that the supremum defining H_i is locally achieved in a compact set of controls. Consequently our Hamiltonians H_i are well defined and are continuous functions. Moreover taking a compact sequence $(A_m)_m$ and considering $U_{A^m}^+$, $U_{A^m}^-$ we prove that both are solutions of an approximate Ishii problem. Thus, due to Half relaxed limit method sending m to infinity, we obtain that U_A^+ and U_A^- are viscosity solutions of the initial Ishii problem.

1.8.3 Comparison Results

In Chapter 2 we provide growth conditions that allow us to obtain comparison and uniqueness results. We first obtain a local comparison result assuming the complemented condition H_T for the subsolution, in the form:

$$\max_{B_R(x)}(u-v)^+ \leq \max_{\partial B_R(x)}(u-v)^+.$$

Notice that the H_T-subsolution condition is mandatory since there are counter-examples showing that in general $U^- \neq U^+$. Global uniqueness results, in general are obtained from global comparison results, that is,

$$u(x) \leq v(x),\ \forall\ x \in R^N\ .$$

Here also, u is a locally bounded Ishii subsolutions satisfying $H_T \leq 0$ while v is just a locally bounded supersolution in the sense of Ishii. However, our global comparison results are dependent on the growth of the dynamic and cost functions, as well as of the desired growth for the solutions. Therefore, we will present different versions of these results depending on the growth adopted. For instance, in order that global comparison result we add to hypothesis (1.8.1) the following growth condition

$$|b_i(y,\alpha)| \leq C|y|^\beta + \overline{C} \text{ and } l_i(y,\alpha) \leq C|y|^{\beta^*} + \overline{C},\ \forall\ y \in R^N \text{ and } \alpha \in A_i \tag{1.8.2}$$

in the set of solutions with growth $|x|^\gamma$, where β, β^*, γ are related.

One of the great difficulties of dealing with growth condition (1.8.2) is that we have some restriction not only in the control variable, but also in the spatial variable.

Another approach we take is to consider the hypothesis

$$\lim_{|\alpha_i|\to\infty} |b_i(y,\alpha)| = \infty \text{ and } \lim_{|\alpha_i|\to\infty} \frac{l_i(y,\alpha)}{1+|b_i(y,\alpha)|} = \infty, \text{ uniformly with respect to } y \in R^N. \tag{1.8.3}$$

combined with the following condition: let K be compact contained in A_i, then there exists $C_K(b)$ and $C_K(l)$ so that

$$|b_i(y,\alpha_i)| \leq C_K(b) \text{ and } l_i(y,\alpha_i) \leq C_K(l), \ \forall\, y \in R^N, \ \alpha_i \in K. \tag{1.8.4}$$

With hypotheses (1.8.3) and (1.8.4), the growth depends only on the control variable and this allows us to obtain desired results more easily.

Furthermore, under suitable growth conditions, we are able to prove that U_A^+, U_A^- are respectively the maximal subsolution and minimal supersolution of Ishii problem. Finally, we prove that under some suitable hypothesis, Filippov approximations converge to U^-.

1.8.4 The family of solutions U^η

In Chapter 3 we work with value functions U^η. We have already commented that problems with discontinuities, in general, do not have uniqueness. In fact, U^+ and U^- are viscosity solutions of the Ishii problem, but they are not the only ones. We build a family of solutions U^η, locally Lipschitz Ishii solutions. Under appropriate assumptions we obtain that the limit them when η goes to zero is U_A^+ and when η goes to infinite is U_A^-. Hence, this family can be seen as a continuous path between U_A^- and U_A^+.

Such solutions are built on the relaxed regular trajectories

$$\tau_A^\eta(x) := \{(X_x(\cdot),\alpha(\cdot)) \in \tau_A(x) \mid \text{for a.e. } t \in \mathcal{E}_\mathcal{H},\ b_2(X_x(t),\alpha(t))\cdot e_N \geq -\eta,\ b_1(X_x(t),\alpha(t))\cdot e_N \leq \eta\} \tag{1.8.5}$$

These $\eta-$trajectories may not be regular, but they are almost regular if η is close to 0. On the contrary, if η is close to ∞, we recover most of the trajectories. The η-value function is then defined as one can expect:

$$U_A^\eta(x) := \inf_{\tau_A^\eta(x)} \left(\int_0^\infty l(X_x(t),\alpha(t))e^{-\lambda t}dt \right). \tag{1.8.6}$$

1.8.5 Open problems and appendix

Finally, in Chapter 4, we have included some ideas that we intend to address in future work. An appendix collects some fundamental results used in this thesis. We discuss in particular the mechanism of the relaxed controls, the process of the Half relaxed limit method.

2 VALUE FUNCTIONS U^+ AND U^-

In this chapter we want to find solutions of the Hamilton-Jacobi-Bellmam problem in R^N with discontinuity on a hyperplane. Let us set the problem, defining first the

$$R^N = \Omega_1 \cup \mathcal{H} \cup \Omega_2,$$

where

$$\begin{aligned} \Omega_1 &= \{x \in R^N \mid x_N > 0\} \\ \Omega_2 &= \{x \in R^N \mid x_N < 0\} \\ \mathcal{H} &= \{x \in R^N \mid x_N = 0\}. \end{aligned}$$

We assume that each set Ω_i has an associated unbounded control set $A_i \subset R^d$. We define the control set as follows.

Definition 2.0.1 *Let $A_1,\ A_2$ be complete separable metric spaces,*

$$A := A_1 \times A_2 \times [0,1].$$

Then we consider the control set formed by bounded measurable functions $\mathcal{A} := L^\infty(0,\infty; A)$.

We consider that each domain Ω_i has a dynamic function associated $b_i : \overline{\Omega_i} \times A_i \to R^N$ and a cost function $l_i : \overline{\Omega_i} \times A_i \to R$. Let us state the main hypotheses on the dynamic and cost functions associated to Ω_i that will be used to prove our results.

(HA) $l_i : R^N \times A_i \to R$ and $b_i : R^N \times A_i \to R$ satisfy

$$\lim_{|\alpha|\to\infty} \frac{l_i(y,\alpha)}{1 + |b_i(y,\alpha)|} = \infty \quad \text{and} \quad \lim_{|\alpha|\to\infty} |b_i(y,\alpha)| = \infty$$

uniformly in compact sets of R^N with respect to $y \in R^N$.

(HB) l_i is continuous and b_i is Lipschitz continuous.

(HC) For any $x \in R^N$, the set $\{(b_i(x,\alpha_i), l_i(x,\alpha_i))|\ \alpha \in A_i^m\}$ is a closed and convex set. Moreover, there exists $\delta > 0$ such that

$$\overline{B_\delta(0)} \subset \mathcal{B}^{A_i}(x), \quad \text{where } \mathcal{B}^{A_i}(x) := \{b_i(x,\alpha)|\alpha \in A_i^m\}$$

where A_i^m is compact sequence which union equal a R^N.

Observe that $\mathcal{H}$ is the intersection of $\overline{\Omega}_1$ and $\overline{\Omega}_2$, then there exists a dynamic function $b_{\mathcal{H}} : \mathcal{H} \times A \to R^N$ associated to $\mathcal{H}$ given by a convex combination of b_1 and b_2, and a cost function $l_{\mathcal{H}} : \mathcal{H} \times A \to R$ given by a convex combination of l_1 and l_2. In particular, we define them as follows,

$$b_{\mathcal{H}}(x,\alpha) = \mu b_1(x,\alpha_1) + (1-\mu) b_2(x,\alpha_2), \quad x \in \mathcal{H},\ \alpha = (\alpha_1,\alpha_2,\mu) \in A,$$

$$l_{\mathcal{H}}(x,\alpha) = \mu l_1(x,\alpha_1) + (1-\mu) l_2(x,\alpha_2), \quad x \in \mathcal{H},\ \alpha = (\alpha_1,\alpha_2,\mu) \in A.$$

2.1 ISHII PROBLEM

We are interested in finding solutions of the Ishii problem given by

$$\begin{cases} H_1(x,u(x),Du(x)) = 0 & \text{in } \Omega_1, \\ H_2(x,u(x),Du(x)) = 0 & \text{in } \Omega_2, \\ \min\{H_1(x,u(x),Du(x)), H_2(x,u(x),Du(x))\} \leq 0 & \text{in } \mathcal{H}, \\ \max\{H_1(x,u(x),Du(x)), H_2(x,u(x),Du(x))\} \geq 0 & \text{in } \mathcal{H}. \end{cases} \tag{2.1.1}$$

where

$$H_i(x,u,q) = \sup_{\alpha_i \in A_i} (-b_i(x,\alpha_i)\cdot q - l_i(x,\alpha_i) + \lambda u).$$

with $(x,u,q) \in \overline{\Omega_i} \times R \times R^N$ and $\lambda \in R$ is a discount factor.

Since there is no uniqueness of solution of the Ishii problem (2.1.1), we define tangential Hamiltonians. To do this, we consider the control sets defined by

$$A_0(x) \quad = \{(\alpha_1,\alpha_2,\mu) \in A \mid b_{\mathcal{H}}(x,\alpha_1,\alpha_2)\cdot e_N = 0\}.$$

$$A_0^{reg}(x) \quad = \{(\alpha_1,\alpha_2,\mu) \in A_0(x) \mid b_{\mathcal{H}}(\alpha_1,\alpha_2,\mu) \text{ is regular}\}.$$

Then we can define the tangential Hamiltonians, for $(x,u,q) \in \mathcal{H} \times R \times R^{N-1}$

$$H_T(x,u,q) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0(x)} -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu) + \lambda u - b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot(q,0) \tag{2.1.2}$$

and

$$H_T^{reg}(x,u,q) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0^{reg}(x)} -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu) + \lambda u - b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot(q,0)\,. \tag{2.1.3}$$

Since there exists a link between the viscosity solutions of (2.1.1) and Optimal Deterministic Control, we need to define the set of trajectories with associated dynamic and cost functions. Let us introduce some definitions that will be needed.

Definition 2.1.1 *Let $x \in R^N$ and $\alpha \in A$, $\alpha_i \in A_i$ for $i = 1, 2$. We define the cost function as*

$$l(x,\alpha) := l_1(x,\alpha_1)1_{\Omega_1}(x) + l_2(x,\alpha_2)1_{\Omega_2}(x) + l_{\mathcal{H}}(x,\alpha)1_{\mathcal{H}}(x), \tag{2.1.4}$$

where 1_{Ω_i} and $1_{\mathcal{H}}$ are the characteristic functions of Ω_i and $\mathcal{H}$ respectively.

Definition 2.1.2 *Let $x \in R^N$ and $\alpha \in A$, $\alpha_i \in A_i$ for $i = 1, 2$. We define the dynamic function as*

$$b(x,\alpha) := b_1(x,\alpha)1_{\Omega_1}(x) + b_2(x,\alpha)1_{\Omega_2}(x) + b_{\mathcal{H}}(x,\alpha)1_{\mathcal{H}}(x). \tag{2.1.5}$$

Definition 2.1.3 *We say that $X_x(\cdot)$ is a trajectory if it is a Lipschitz continuous function that satisfies the following differential inclusion*

$$X_x'(t) \in \mathcal{B}(X_x(t)) \text{ for a.e. } t \in (0,\infty)\,; X_x\,(0) = x \tag{2.1.6}$$

where

$$\mathcal{B}(x) := \begin{cases} \mathcal{B}_1(x) & \text{if } x_N > 0 \\ \mathcal{B}_2(x) & \text{if } x_N < 0 \\ \overline{co}(\mathcal{B}_1(x) \cup \mathcal{B}_2(x)) & \text{if } x_N = 0. \end{cases} \tag{2.1.7}$$

with $\mathcal{B}_i(x) = \{b_i(x,\alpha) | \alpha \in A_i\}$ and $\overline{co}(\cdot)$ the convex hull.

A key hypothesis that we will consider throughout this work is hypothesis (HC), thanks to it, we can construct a solution of the differential inclusion (2.1.6), so we do not need to worry about the existence of a solution for differential inclusions. For example, for v such that $|v| \leq \delta$, we can always choose a control α such that $b_i\,(X(t),\alpha) = v$ and consider the trajectory $X(t) = tv$.

Let us introduce some measure results that will be used below to characterize the trajectories in Definition 2.1.3.

Theorem 2.1.1 (Measurable Selection Theorem) *Let X be a complete separable metric space, Z a measurable space and F a measurable set-valued map from Z to closed non empty subsets of X . Then there exist a measurable selection of F.*

See (AUBIN; FRANKOWSKA, 1990, Th 8.1.3).

Definition 2.1.4 *Let X be a topological space, Y a metrizable and separable space and Z a measurable space. A mapping $g : Z \times X \to Y$ will be called a Carathéodory mapping if $g(\cdot, x)$ is measurable with respect to the first variable for fixed $x \in X$ and $g(z, \cdot)$ is continuous with respect to the second variable for fixed $z \in Z$, (cf. (EKELAND; VALADIER, 1971)).*

The following theorem provides us a relation between a differential inclusion equation and an a differential equation, cf. (AUBIN; FRANKOWSKA, 1990).

Theorem 2.1.2 (Filippov) *Let (Z, Σ, ν) be a complete σ-finite measure space, X, Y be complete separable metric spaces, let $F : Z \to X$ be a measurable set-valued map with closed nonempty images, and let $g : Z \times X \to Y$ be a Carathéodory map, then for every measurable map $h : Z \to Y$ satisfying*

$$h(z) \in g(z, F(z)) \;\; a.e. \; z \in Z$$

there exists a measurable function $f : \Omega \to X$ with $f(z) \in F(z)$ such that

$$h(z) = g(z, f(z)) \;\; a.e. \; z \in Z.$$

See (AUBIN; FRANKOWSKA, 1990, p. 314).

In the result below we apply Filippov's Theorem 2.1.2 to provide a relation between the trajectories in Definition 2.1.3 and a control in $\mathcal{A}$, to obtain a differential equation almost everywhere.

Theorem 2.1.3 *For each solution $X_x(\cdot)$ of the differential inclusion* (2.1.6)*, there exists a control $\alpha(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot), \mu(\cdot)) \in \mathcal{A}$ such that*

$$\begin{aligned} X_x'(t) = {} & b_1 (X_x(t), \alpha_1(t)) 1_{\{X_x(t) \in \Omega_1\}} + b_2 (X_x(t), \alpha_2(t)) 1_{\{X_x(t) \in \Omega_2\}} \\ & + b_{\mathcal{H}} (X_x(t), \alpha(t)) 1_{\{X_x(t) \in \Omega_{\mathcal{H}}\}} \quad \text{for a.e.} \quad t \in R^+. \end{aligned} \tag{2.1.8}$$

Moreover, if $e_N = (0, \dots, 0, 1)$, then

$$b_{\mathcal{H}}(X_x(t)) \cdot e_N = 0 \;\; \text{on} \;\; \{t : (X_x(t))_N = 0\} \quad for \; a.e. \;\; t \in R^+. \tag{2.1.9}$$

Fix $x \in R^N$. We will prove the existence of the control α applying Filippov's Theorem 2.1.2. We consider $X := A$ which is complete and separable; $Z := (0, \infty)$ with associated Lebesgue measure. Let $h(\cdot) := X_x'(\cdot)$, let $F(t) := A$ for all $t \in (0, \infty)$. Thus, F has a non empty closed image. F is measurable since $F^{-1}(O) = Z$ for every open set O of X, because

$F(t) \cap O = A \cap O = O$. Let $Y = R^N$. We define the map $g : R^+ \times A \to R^N$ as follows

$$g(t,\alpha) = \begin{cases} b_1(X_x(t), \alpha_1) & \text{if } X_x(t)_N > 0 \\ b_2(X_x(t), \alpha_2) & \text{if } X_x(t)_N < 0 \\ b_{\mathcal{H}}(X_x(t), \alpha) & \text{if } X_x(t)_N = 0. \end{cases}$$

We claim that g is a Caratheodory map. Indeed, it is clear that, for fixed t, the function $g(t,\cdot)$ is continuous. To check that g is measurable with respect to t. We fix $\alpha \in A$, and an open set $O \subset R^N$, then

$$g_\alpha^{-1}(O) = \left\{g_\alpha^{-1}(O) \cap \{t : X_x(t)_N > 0\}\right\} \cup \left\{g_\alpha^{-1}(O) \cap \{t : X_x(t)_N = 0\}\right\} \cup \left\{g_\alpha^{-1}(O) \cap \{t : X_x(t)_N < 0\}\right\}$$

Since $g(\cdot,\alpha)$ is continuous the first and third components are opened. In the second component $\{t : X_x(t)_N = 0\}$ is closed and $g_\alpha^{-1}(O)$ is open, then the intersection is measurable. We also have that $h(\cdot) = X'_x(\cdot)$ is measurable since X_x is Lipschitz, and Z has Lebesgue measure. Finally the differential inclusion and definition of $g(\cdot)$ implies that

$$h(t) = X'_x(t) \in \mathcal{B}(X_x(t)) = g(t,A) = g(t,F(t)).$$

Thus, by Theorem 2.1.2 there exist $\alpha(t) = f(t)$ measurable function

$$\alpha : Z \to A$$

such that

$$\begin{aligned} X'_x(t) &= h(t) \\ &= g(t, f(t)) \\ &= g(t, \alpha(t)) \\ &= b_1(X_x(t), \alpha_1(t))\, 1_{\{X_x(t) \in t_1\}} + b_2(X_x(t), \alpha_2(t))\, 1_{\{X_x(t) \in t_2\}} + b_{\mathcal{H}}(X_x(t), \alpha(t))\, 1_{\{X_x(t) \in t_{\mathcal{H}}\}}. \end{aligned}$$

Let us prove now, the second part of the result. Let ν be a Lebesgue measure. Consider $E := X_x^{-1}(0)_N$, then we have E is closed, therefore $E_m := E \cap (0,m)$ is Lebesgue measurable. Thanks to Stampacchia's Theorem (cf. (BORWEIN; MOORS; WANG, 1997)) and since $X_x(\cdot)_N$ is Lipschitz, $\nu(X_x(E_m)_N) = \nu(\{0\}) = 0$, $X'_{x,N}(t) = 0$ for a.e. $t \in E_m$, then $X'_{x,N}(t) = 0$ for a.e. $t \in E$.

Below we define the the set of trajectories that will be considered throughout this work.

$$\tau_A(x) := \{(X_x(\cdot), \alpha(\cdot)) \in Lip(R^+; R^n) \times \mathcal{A} \ \text{ such that(2.1.8) is fulfilled and } X_x(0) = x\} \tag{2.1.10}$$

Let us introduce the following notation for the sets of times where the trajectories are in Ω_i or $\mathcal{H}$,

$$\mathcal{E}_1 := \{t : \ X_x(t) \in \Omega_1\}, \ \mathcal{E}_2 := \{t : \ X_x(t) \in \Omega_2\}, \ \mathcal{E}_{\mathcal{H}} := \{t : \ X_x(t) \in \mathcal{H}\}.$$

Definition 2.1.5 *Let $(y, \alpha) = (y, \alpha_1, \alpha_2, \mu) \in Lip(R^+; R^n) \times \mathcal{A}$, we say that the dynamic $b_{\mathcal{H}}(y, \alpha_1, \alpha_2, \mu)$ is regular if*

$$b_1(y, \alpha_1) \cdot e_N \leq 0 \quad \textit{and} \quad b_2(y, \alpha_2) \cdot e_N \geq 0 \quad \textit{and} \quad b_{\mathcal{H}}(y, \alpha_1, \alpha_2, \mu) \in \mathcal{H}.$$

We define the set of regular trajectories as

$$\tau_A^{reg}(x) := \{(X_x(\cdot), \alpha(\cdot)) \in \tau_A(x), \quad \textit{for a.e. } t \in \mathcal{E}_{\mathcal{H}} \quad \textit{and} \ \ b_{\mathcal{H}}(X_x(t), \alpha(t)) \textit{ is regular}\}. \tag{2.1.11}$$

Finally we define the value functions

$$U_A^-(x) = \inf_{\tau_A(x)} \left(\int_0^\infty l(X_x(t), \alpha(t)) e^{-\lambda t} dt \right). \tag{2.1.12}$$

$$U_A^+(x) = \inf_{\tau_A^{reg}(x)} \left(\int_0^\infty l(X_x(t), \alpha(t)) e^{-\lambda t} dt \right). \tag{2.1.13}$$

In this chapter our aim is to prove that the value functions are solutions of the Ishii problem and to prove their regularity. In particular, we will prove that the value functions are locally bounded and locally Lipschitz. Since the control set A in unbounded in this work, to be able to prove the results for the Value Functions, we need to consider compact control sets for which we will prove the results, and we will take limits to recover our initial value functions defined in an unbounded control set.

Let us introduce the sequence of compact sets such that $A_i^1 \subset A_i^2 \subset \cdots \subset A_i^m$ and

$$A_i = \bigcup_{m \in N} A_i^m$$

and denote

$$A^m := A_1^m \times A_2^m \times [0, 1] \subset A,$$

then $A^1 \subset A^2 \subset \cdots \subset A^m$ and

$$A = \bigcup_{m \in N} A^m.$$

Then we can consider the Ishii problem associated to the compact control sets A^m, defined as

$$\begin{cases} H_1^m(x,u(x),Du(x)) = 0 & \text{in } \Omega_1 \\ H_2^m(x,u(x),Du(x)) = 0 & \text{in } \Omega_2 \\ \min\{H_1^m(x,u(x),Du(x)), H_2^m(x,u(x),Du(x))\} \le 0 & \text{on } \mathcal{H} \\ \max\{H_1^m(x,u(x),Du(x)), H_2^m(x,u(x),Du(x))\} \le 0 & \text{on } \mathcal{H} \end{cases} \tag{2.1.14}$$

where

$$H_i^m(x,u,q) = \sup_{\alpha\in A_i^m}\{-b_i(x,\alpha)\cdot q - l_i(x,\alpha)\} + \lambda u.$$

We consider the control sets defined by

$$A_0^m(x) \quad = \{(\alpha_1,\alpha_2,\mu)\in A^m \mid b_{\mathcal{H}}(x,\alpha_1,\alpha_2)\cdot e_N = 0\}.$$

$$A_0^{m,reg}(x) \quad = \{(\alpha_1,\alpha_2,\mu)\in A_0^m \mid b_{\mathcal{H}}(\alpha_1,\alpha_2,\mu) \text{ is regular}\}.$$

Then we can define the tangential Hamiltonians

$$H_T^m(x,u(x),D_{\mathcal{H}}u) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0^m(x)} -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\underline{u}(x)-b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot(D_{\mathcal{H}}u(x),0) \tag{2.1.15}$$

and

$$H_T^{m,reg}(x,u(x),D_{\mathcal{H}}u) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0^{m,reg}(x)} -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\underline{u}(x)-b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot(D_{\mathcal{H}}u(x),0) \tag{2.1.16}$$

To describe the value functions, which we will prove that are solutions of the Ishii problem (2.1.14), we need to introduce the trajectories that will be considered associated to the compact control sets A^m,

$$\tau_{A^m}(x) := \{(X_x(\cdot),\alpha(\cdot))\in Lip(R^+;R^n)\times\mathcal{A}^m \text{ such that(2.1.8) is fulfilled and } X_x(0)=x\} \tag{2.1.17}$$

and

$$\tau_{A^m}^{reg}(x) := \{(X_x(\cdot),\alpha(\cdot))\in\tau_{A^m}(x), \quad \text{for a.e. } t\in\mathcal{E}_{\mathcal{H}} \quad \text{and} \quad b_{\mathcal{H}}(X_x(t),\alpha(t)) \text{ is regular}\}. \tag{2.1.18}$$

Then the value functions associated to A^m are defined as

$$U_{A^m}^-(x) = \inf_{\tau_{A^m}(x)}\left(\int_0^\infty l(X_x(t),\alpha(t))e^{-\lambda t}dt\right). \tag{2.1.19}$$

$$U_{A^m}^+(x) = \inf_{\tau_{A^m}^{reg}(x)}\left(\int_0^\infty l(X_x(t),\alpha(t))e^{-\lambda t}dt\right). \tag{2.1.20}$$

2.2 DYNAMIC PROGRAMMING PRINCIPLE

Let us prove the following theorem where we prove the Dynamic Programming Principle which will be very useful to prove that the value functions are subsolutions and supersolutions of the Ishii problem (2.1.1).

Theorem 2.2.1 (Dynamic Programming Principle)

$$U_A^-(x) = \inf_{\tau_A(x)} \left(\int_0^T l(X(t), \alpha(t)) e^{-\lambda t} dt + U_A^- \left(X\left(T \right) \right) e^{-\lambda T} \right). \tag{2.2.1}$$

$$U_A^+(x) = \inf_{\tau_A^{reg}(x)} \left(\int_0^T l(X(t), \alpha(t)) e^{-\lambda t} dt + U_A^+ \left(X\left(T \right) \right) e^{-\lambda T} \right). \tag{2.2.2}$$

Let us prove this result for U_A^-. We denote the right hand side of (2.2.1) as $\overline{U}$. We prove first that $U_A^- \leq \overline{U}$. Consider $(X_x(\cdot), \nu) \in \tau_A(x)$ and and let $z = X_x\left(T\right)$. By definition of U_A^-, there exists a sequence $(X_z^\epsilon(\cdot), \nu^\epsilon(\cdot)) \in \tau_A(z)$ with $X_z^\epsilon(0) = z$ such that

$$U_A^-(z) + \epsilon \geq \int_0^\infty l\left(X_z^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt.$$

Let us consider

$$\overline{\nu}(t) := \begin{cases} \nu(t) & t \leq T, \\ \nu^\epsilon(t-T) & t > T. \end{cases}$$

Analogously,

$$\overline{X}(t) := \begin{cases} X_x(t) & t \leq T, \\ X_z^\epsilon(t-T) & t > T. \end{cases}$$

Since $X_x(t)$ and X_z^ϵ are Lipschitz then we have $\left(\overline{X}(t), \overline{\nu}(t)\right) \in \tau_A(x)$. Thus, by definition of U_A^- and by the change of variables $s = t - T$, we obtain

$$\begin{aligned} U_A^-\left(x\right) &\leq \int_0^\infty l\left(\overline{X}(t), \overline{\nu}(t)\right) e^{-\lambda t} dt \\ &= \int_0^T l\left(X_x(t), \nu(t)\right) e^{-\lambda t} dt + \int_T^\infty l\left(X_z^\epsilon(t-T), \nu^\epsilon(t-T)\right) e^{-\lambda t} dt \\ &= \int_0^T l\left(X_x(t), \nu(t)\right) e^{-\lambda t} dt + e^{-\lambda T} \int_0^\infty l\left(X_z^\epsilon(s), \nu^\epsilon(s)\right) e^{-\lambda s} ds \\ &\leq \int_0^T l\left(X_x(t), \nu(t)\right) e^{-\lambda t} dt + e^{-\lambda T} \left(U_A^-\left(z\right) + \epsilon\right). \end{aligned}$$

Taking limits as ϵ goes to 0, we obtain that for any trajectory $(X_x(t), \nu(t)) \in \tau_A(x)$ we have

$$U_A^-(x) \leq \int_0^T l\left(X_x(t), \nu(t)\right) e^{-\lambda t} dt + e^{-\lambda T}\left(U_A^-\left(X_x(T)\right)\right). \tag{2.2.3}$$

Therefore we have $U_A^- \leq \overline{U}$.

Now, let us prove that $\overline{U} \leq U_A^-$. Consider a sequence $(X_x^\epsilon, \nu^\epsilon) \in \tau_A(x)$ such that

$$\begin{aligned} U_A^-(x) + \epsilon &\geq \int_0^\infty l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt \\ &= \int_0^T l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt + \int_T^\infty l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt \\ &= \int_0^T l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt + e^{-\lambda T} \int_0^\infty l\left(X_x^\epsilon(s+T), \nu^\epsilon(s+T)\right) e^{-\lambda s} ds. \end{aligned} \tag{2.2.4}$$

Define $X_{X_x^\epsilon(T)}(s) := X_x^\epsilon(s+T)$ and $\nu(s) := \nu^\epsilon(s+T)$, then second integral on the right hand side can be rewritten as

$$\int_0^\infty l\left(X_x^\epsilon(s+T), \nu^\epsilon(s+T)\right) e^{-\lambda s} ds = \int_0^\infty l\left(X_{X_x^\epsilon(T)}(s), \nu(s)\right) e^{-\lambda s} ds. \tag{2.2.5}$$

we have $\left(X_{X_x^\epsilon(T)}(\cdot), \nu(\cdot)\right) \in \tau_A(X_x^\epsilon(T))$ and then $U_A^-\left(X_x^\epsilon(T)\right) \leq \int_0^\infty l\left(X_{X_x^\epsilon(T)}(s), \nu(s)\right) e^{-\lambda s} ds.$ Thanks to (2.2.4) and (2.2.5), we have

$$U_A^-(x) + \epsilon \geq \int_0^T l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt + U_A^-\left(X_x^\epsilon(T)\right) e^{-\lambda T} \geq \overline{U}(x). \tag{2.2.6}$$

Taking limits as ϵ goes to 0, we obtain

$$U_A^-(x) \geq \overline{U}(x).$$

The proof for U_A^+ is analogous.

Remark 2.2.2 *Consider a sequence* $(X_x^\epsilon, \nu^\epsilon) \in \tau_{A^m}(x)$ *such that*

$$U_{A^m}^-(x) + \epsilon \geq \int_0^\infty l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt.$$

Since $U_A^-(x) = \overline{U}(x)$ *due to the inequality* (2.2.6) *we obtain*

$$U_A^-(x) = \lim_{\epsilon \to 0}\left(\int_0^T l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt + U_A^-\left(X_x^\epsilon(T)\right) e^{-\lambda T}\right) = \overline{U}(x).$$

Analogously, if a sequence $(X_x^\epsilon, \nu^\epsilon) \in \tau_{A^m}^{reg}(x)$ *satisfies that*

$$U_{A^m}^+(x) + \epsilon \geq \int_0^\infty l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt,$$

then

$$U_A^+(x) = \lim_{\epsilon \to 0}\left(\int_0^T l\left(X_x^\epsilon(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt + U_A^+\left(X_x^\epsilon(T)\right) e^{-\lambda T}\right). \tag{2.2.7}$$

The lemma below, gives a bound of the trajectories, that will be very useful to consider that the trajectories are in bounded sets.

Lemma 2.2.3 *Assume that there exists C such that*

$$|b_i(x,\alpha_i)| < C(1+|\alpha_i|+|x|) \text{ for all } x \in R^N \text{ and } \alpha_i \in A_i^m.$$

Then, given $x \in R^N$ and $T > 0$ there exist $R_T^m > 0$ such that for any $(X_x^m, \alpha^m) \in \tau_{A^m}(x)$, we have

$$X_x^m(t) \in B_{R_T^m}(x) \text{ for all } \ t \in [0,T].$$

Since $(X_x^m, \alpha^m) \in \tau_{A^m}(x)$, we have

$$\begin{aligned} |X_x^m(t)| &\le |x| + \int_0^t |b(X_x^m(s), \alpha^m(s))| ds \\ &\le |x| + C(1+\sup_{\alpha \in A_i^m} |\alpha|)t + \int_0^t C|X_x^m(s)| ds \end{aligned}$$

Thanks to Gronwall inequality

$$|X_x^m(t)| \le C_T^m e^{CT} \ \forall \ t \in [0,T]$$

where $C_T^m := |x| + C(1+\sup_{\alpha \in A_i^m} |\alpha|)T$

Then,

$$|X_x^m(t) - x| \le |x| + C_T^m e^{CT} \ \forall \ t \in [0,T].$$

Thus, we take $R_T^m := |x| + C_T^m e^{CT}$. We need to introduce the lemma below to prove the following result.

Lemma 2.2.4 *Assume that*

1. *$f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(0,T;R^N)$.*
2. *There exists a measurable set map $s \mapsto K(s) \subset R^N$ defined on $[0,T]$ such that for any $n \in N$, and $s \in (0,T)$, $f_n(s) \in K(s)$ a.e..*

Then for any $s \in (0,T)$, the weak limit also satisfies $f(s) \in K(s)$ a.e..*

Let $E = L^\infty(0,T;R^N)$ and consider $f \in E$. We define the functional G by

$$G(f) := \int_0^T dist(f(s), K(s)) ds \ \in R_+ \cup \{\infty\}. \tag{2.2.8}$$

Observe that G has the following properties,

(a) $G : E \to R$ is convex. Indeed, the distance function is convex (this follows from the triangle inequality and positive homogeneity).

(b) $G : E \to R$ is continuous for the strong topology. Indeed, if $f_n \to f$ strongly in E, then $|dist(f_n, K) - dist(f, K)| \leq |f_n - f| \to 0$ in $L^\infty(0, T)$. Then, $G(f_n) \to G(f)$.

(c) Since G is convex and continuous it is lower semi-continuous with respect to the weak* topology.

Since $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(0, T; R^N)$, thanks to Proposition **??**

$$G(f) \leq \liminf_{n\to\infty} G(f_n).$$

But since $f_n(s) \in K(s)$ a.e., then $G(f_n) = 0$ for any $n \in N$. We deduce that $G(f) = 0$, which implies that $f(s) \in K(s)$ a.e. on $(0, T)$. The following result will be used to prove comparison results related to $U^+_{A^m}$

Lemma 2.2.5 *Under the hypotheses of Lemma 2.2.3 and hypothesis (HC). Consider a sequence* $(X^\epsilon_x, \nu^\epsilon) \in \tau^{reg}_{A^m}(x)$ *such that*

$$U^+_{A^m}(x) + \epsilon \geq \int_0^\infty l\left(X^\epsilon_x(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt.$$

Then given $T > 0$ *there exists* $(X^T_x, \alpha^T) \in \tau^{reg}_{A^m}(x)$ *satisfying*

$$\int_0^s l(X^T_x(t), \alpha^T(t)) e^{-\lambda t} dt + U^+_{A^m}\left(X^T_x(s)\right) e^{-\lambda s} = \lim_{\epsilon\to 0}\left(\int_0^s l(X^\epsilon_x(t), \alpha^\epsilon(t)) e^{-\lambda t} dt + U^+_{A^m}\left(X^\epsilon_x(s)\right) e^{-\lambda s}\right)$$

for any $s \leq T$. *Moreover*

$$\int_0^s l(X^T_x(t), \alpha^T(t)) e^{-\lambda t} dt + U^+_{A^m}\left(X^T_x(s)\right) e^{-\lambda s} = U^+_{A^m}(x) \quad \forall s \leq T.$$

We define

$$Y^\epsilon(s) := \int_0^s l(X^\epsilon_x(t), \alpha^\epsilon(t)) dt,$$

and consider the trajectories

$$(X^\epsilon_x, Y^\epsilon) : (0, T) \longrightarrow R^{N+1}$$

Due to Lemma 2.2.3 there exists R^m_T such that $|X^{T,\epsilon}_x(t) - x| \leq R^m_T$ for any $t \in (0, T]$, then we have

$$|b(X^\epsilon_x(t), \alpha^\epsilon(t))| \leq ||b||_{L^\infty(B_{R_T}(x)\times A^m)}, \; |l(X^\epsilon_x(t), \quad \alpha^\epsilon(t))| \leq ||l||_{L^\infty(B_{R_T}(x)\times A^m)}.$$

Thus, the curves $(X^\epsilon_x, Y^\epsilon)$ are equicontinuous and uniformly bounded in $[0,T]$. Thus, thanks to Ascoli Arzela's Theorem, we can extract a subsequence $(X^\epsilon_x, Y^\epsilon)$ which converges uniformly in $[0,T]$ to (X^T_x, Y^T). Moreover, thanks to Banach–Alaoglu Theorem we can extract a subsequence of $(\dot{X}^\epsilon_x, \dot{Y}^\epsilon)$ that converges weakly* in $L^\infty(0,T,R^{N+1})$ to $(\dot{X}^T_x, \dot{Y}^T)$. Indeed, given $\phi \in C^\infty_0(R^N)$, by definition of weak derivative

$$\int_0^T (\dot{X}^\epsilon_x, \dot{Y}^\epsilon)(s)\phi(s)ds = \int_0^T (X^\epsilon_x, Y^\epsilon)(s)\nabla\phi(s)ds. \tag{2.2.9}$$

Let g be a weak* limit of $(\dot{X}^\epsilon_x, \dot{Y}^\epsilon)(s)$ then taking limits on the left hand side of (2.2.9) as ϵ goes to 0, we obtain

$$\int_0^T (\dot{X}^\epsilon_x, \dot{Y}^\epsilon)(s)\phi(s)ds \to \int_0^T g(s)\phi(s)ds$$

Thanks to uniform limit of $(X^\epsilon_x, Y^\epsilon)$, then taking limits on the right hand side of (2.2.9) as ϵ goes to 0, we obtain

$$\int_0^T (X^\epsilon_x, Y^\epsilon)(s)\nabla\phi(s)ds \to \int_0^T (X_x, Y)(s)\nabla\phi(s)ds.$$

Therefore,

$$\int_0^T g(s)\phi(s)ds = \int_0^T (X_x, Y)(s)\nabla\phi(s)ds \quad \forall\phi \in C^\infty_0(R^N). \tag{2.2.10}$$

Thus, g is the weak derivative of (X^T_x, Y^T).

We define the set of dynamic and cost functions,

$$BL_i(Z^T) := \{(b_i(X^T_x, \alpha_i), l^T_i(X^T_x, \alpha_i)) | \alpha_i \in A_i\}$$

and

$$\mathcal{BL}(Z) = \begin{cases} BL_1(Z) & \text{if } Z_N > 0 \\ BL_2(Z) & \text{if } Z_N < 0 \\ \overline{co}(BL_1(Z) \cup BL_2(Z)) & \text{if } Z_N = 0, \end{cases} \tag{2.2.11}$$

where Z_N is the N-th coordinate of Z. Thanks to the uniform convergence of Z_ϵ and since l and b are Lipschitz. Given $\gamma > 0$, then for ϵ small enough $Z_\epsilon := (X^\epsilon_x, Y^\epsilon)$ is solution of the differential inclusion

$$\dot{Z}_\epsilon(t) = (b(Z_\epsilon(t), \alpha_\epsilon(t)), l(Z_\epsilon(t), \alpha_\epsilon(t)) \in \mathcal{BL}(Z_\epsilon(t)) \subset \mathcal{BL}(Z^T(t)) + \gamma B_1(0)$$

Let $Z^T := (X^T_x, Y^T)$. Applying Lemma 2.2.4 to $K(s) = \mathcal{BL}(Z^T(s)) + \gamma B_1(0)$, we have

$$\dot{Z}^T(s) \in \mathcal{BL}(Z^T(s)) + \gamma B_1(0) \quad a.e.$$

Taking limits as γ goes to 0, we obtain

$$\dot{Z}^T(s) \in \overline{\mathcal{BL}(Z^T(s))} \quad a.e.$$

Since $\overline{\mathcal{BL}(Z^T(s))}$ is closed, then

$$\dot{Z}^T(s) \in \mathcal{BL}(Z^T(s)) \quad a.e.$$

We use Filippov's Theorem 2.1.2 with

$$F(t) = A, \quad h(t) = \dot{Z}(t), \quad \Omega = [0, T], \qquad g(t, \alpha) = (b(X_x^T(t), \alpha), l(X_x^T(t), \alpha)),$$

we obtain that there exists a measurable control $\alpha^T(\cdot)$ such that

$$(\dot{X}_x^T(t), \dot{Y}^T(t)) = (b(X_x^T(t), \alpha^T(t)), l(X_x^T(t), \alpha^T(t))).$$

Therefore for $s \leq T$, we have $l(X_x^\epsilon(\cdot), \alpha^\epsilon(\cdot))$ converges weakly* in $L^\infty([0, s]; R)$ to $l(X_x^T(t), \alpha^T(t))$. Moreover, since $e^{-\lambda t} \in L^1([0, s]; R)$ and thanks to (2.2.7), we obtain that

$$\begin{aligned} U^+_{A^m}(x) &= \lim_{\epsilon \to \infty} \left(\int_0^s l(X_x^\epsilon(t), \alpha^\epsilon(t)) e^{-\lambda t} dt + U^+_{A^m}\left(X_x^\epsilon(T)\right) e^{-\lambda s} \right) \\ &= \int_0^s l(X_x^T(t), \alpha^T(t)) e^{-\lambda t} dt + U^+_{A^m}\left(X_x^T(s)\right) e^{-\lambda s}. \end{aligned}$$

Moreover, thanks to (BARLES; CHASSEIGNE, 2018, Lemma 6.3.1, p. 115), we have $(X_x^T(\cdot), \alpha^T(\cdot)))$ is regular in $[0, T]$, that is, the limit of regular trajectory is also a regular trajectory. Thanks to hypothesis (HC) there exists a control $\alpha_i^* \in A_i$ with $b_i(X_x^T(T), \alpha_i^*) = 0$ then we can define $X_x^T(t) := X_x^T(T)$ and $\alpha_i^T(t) := \alpha_i^*$ for any $t \geq T$. Thus, $(X_x^T, \alpha^T) \in \tau^{reg}_{A^m}(x)$. We have a slightly weaker version of the previous Lemma 2.2.5 without assuming the growing hypothesis on the dynamic function. To do this we consider the following result, where we state a bound of the trajectories in $\tau_{A^m}(x)$.

Lemma 2.2.6 *Let $m \in N$ and let $b_i : \overline{\Omega_i} \times A_i^m \to R^N$ be continuous. Given $r > 0$ and $x \in R^N$, there exist $\bar{t} > 0$ such that for any trajectory $(X_x, \alpha) \in \tau_{A^m}(x)$, we obtain*

$$|X_x(s) - x| \leq r, \ \forall \ s \leq \bar{t}.$$

Since

$$\begin{aligned} X_x'(t) = {} & b_1\left(X_x(t), \alpha_1(t)\right) 1_{\{X_x(t) \in \Omega_1\}} + b_2\left(X_x(t), \alpha_2(t)\right) 1_{\{X_x(t) \in \Omega_2\}} \\ & + b_{\mathcal{H}}\left(X_x(t), \alpha(t)\right) 1_{\{X_x(t) \in \Omega_{\mathcal{H}}\}} \quad \text{for a.e.} \quad t \in R^+. \end{aligned}$$

Integrating the expression above, we can consider $\bar{t} \leq \frac{r}{||b||_{B_r(x)\times A^m}}$ to obtain the result. Below we state the weaker version of Lemma 2.2.5 without assuming the growing hypothesis on the dynamic function. This result will be useful to prove comparison results related to the value function $U^+_{A^m}$

Lemma 2.2.7 *Under hypotheses in Lemma 2.2.6 and hypothesis (HC). Consider a sequence* $(X^\epsilon_x, \nu^\epsilon) \in \tau^{reg}_{A^m}(x)$ *such that*

$$U^+_{A^m}(x) + \epsilon \geq \int_0^\infty l\left(X^\epsilon_x(t), \nu^\epsilon(t)\right) e^{-\lambda t} dt.$$

Then for $T > 0$ *there exists a trajectory* $(X^T_x, \alpha^T) \in \tau^{reg}_{A^m}(x)$ *such that*

$$\int_0^s l(X^T_x(t), \alpha^T(t)) e^{-\lambda t} dt + U^+_{A^m}\left(X^T_x(s)\right) e^{-\lambda s} = \lim_{\epsilon\to 0}\left(\int_0^s l(X^\epsilon_x(t), \alpha^\epsilon(t)) e^{-\lambda t} dt + U^+_{A^m}\left(X^\epsilon_x(s)\right) e^{-\lambda s}\right)$$

for any $s \leq T$. *Moreover,*

$$\int_0^s l(X^T_x(t), \alpha^T(t)) e^{-\lambda t} dt + U^+_{A^m}\left(X^T_x(s)\right) e^{-\lambda s} = U^+_{A^m}(x), \quad \forall s \in [0, T].$$

We just replace Lemma 2.2.3 by Lemma 2.2.6 in the proof of lemma 2.2.5.

2.3 REGULARITY OF VALUE FUNCTIONS

Let us recall the main hypotheses on the dynamic and cost functions associated to Ω_i that will be used to prove the results in this section.

(HA) $l_i : R^N \times A_i \to R$ and $b_i : R^N \times A_i \to R$ satisfy

$$\lim_{|\alpha|\to\infty} \frac{l_i(y,\alpha)}{1 + |b_i(y,\alpha)|} = \infty \quad \text{and} \quad \lim_{|\alpha|\to\infty} |b_i(y,\alpha)| = \infty$$

uniformly in compact sets of R^N with respect to $y \in R^N$.

(HB) l_i is continuous and b_i is Lipschitz continuous.

(HC) For any $x \in R^N$, the set $\{(b_i(x,\alpha_i), l_i(x,\alpha_i))|\ \alpha \in A^m_i\}$ is a closed and convex set. Moreover, there exists $\delta > 0$ such that

$$\overline{B_\delta(0)} \subset \mathcal{B}^{A_i}(x), \quad \text{where } \mathcal{B}^{A_i}(x) := \{b_i(x,\alpha)|\alpha \in A^m_i\}$$

where A^m_i is compact sequence which union equal a R^N.

In the following theorem we prove the regularity of the value functions.

Proposition 2.3.1 *Under hypotheses (HA), (HB) and (HC), then the value functions U_A^- and U_A^+ are locally bounded and locally Lipschitz continuous functions.*

We prove the result for U_A^+, the proof for U_A^- is analogous. Initially we prove that U_A^+ is locally bounded. Consider a convex compact set $V \subset R^N$. Thanks to hypothesis (HA) there exists $R > 0$ such that if $|b_i(x, \alpha_i)| \leq \delta$, for some $x \in V$, then $\alpha_i \in B_R^{A_i}(0)$, where $B_R^{A_i}(0) := \{\gamma_i \in A_i| \ \|\gamma_i\| \leq R\}$.

If $x \in \Omega_i \cap V$, thanks to (HC) we can choose $\alpha_i^* \in B_R^{A_i}(0)$ such that $b_i(x, \alpha_i^*) = 0$. Thus, we can associate the trajectory $X_x(t) = x$ with the control, $\alpha_i(t) = \alpha_i^*$, for all t, since we have

$$X_x'(t) = b_i(X_x(t), \alpha_i(t)) = b_i(x, \alpha_i^*) = 0.$$

In this particular case $(X_x(\cdot), \alpha(\cdot)) \in \tau_A^{reg}(x)$. Using the definition of U_A^+, hypotheses (HA) and (HC), we have

$$U_A^+(x) \leq \int_0^\infty l_i(X_x(t), \alpha_i(t))e^{-\lambda t}dt = \int_0^\infty l_i(x, \alpha_i^*)e^{-\lambda t}dt = \frac{l_i(x, \alpha_i^*)}{\lambda} \leq \frac{||l_i||_{V\times B_R^{A_i}(0)}}{\lambda}$$

for all $x \in V$. Then we have proved that U_A^+ is bounded for $x \in \Omega_i \cup V$.

Now, let $x \in \mathcal{H} \cap V$, thanks to hypothesis (HC), we can choose $\alpha_1^* \in A_1$ and $\alpha_2^* \in A_2$ such that $b_1(x, \alpha_1^*) = 0 = b_2(x, \alpha_2^*)$. Considering $X_x(t) = x$ and $\alpha(t) = (\alpha_1^*, \alpha_2^*, \mu)$, with $\mu \in [0, 1]$, then

$$X_x'(t) = b_{\mathcal{H}}(X_x(t), \alpha(t)) = b_{\mathcal{H}}(x, (\alpha_1^*, \alpha_2^*, \mu)) = 0.$$

Since $(X_x(\cdot), \alpha(\cdot)) \in \tau_A^{reg}(x)$,

$$\begin{aligned} U_A^+(x) &\leq \int_0^\infty l_{\mathcal{H}}(X_x(t), \alpha(t))e^{-\lambda t}dt \\ &= \frac{l_{\mathcal{H}}(x, (\alpha_1^*, \alpha_2^*, \mu))}{\lambda} \\ &= \frac{(1-\mu)}{\lambda} l_1(x, \alpha_1^*) + \frac{\mu}{\lambda} l_2(x, \alpha_2^*) \\ &\leq \frac{1-\mu}{\lambda} ||l_1||_{V\times B_R^{A_i}(0)} + \frac{\mu}{\lambda} ||l_2||_{V\times B_R^{A_i}(0)} \\ &\leq \frac{||l_1||_{V\times B_R^{A_i}(0)} + ||l_2||_{V\times B_R^{A_i}(0)}}{\lambda}. \end{aligned}$$

Therefore, we have proved that U_A^+ is locally bounded.

Now will prove that U_A^+ is locally Lipschitz. Let $x, z \in V$ and consider the unitary vector $w = \frac{x-z}{|x-z|}$. Let us define the trajectory

$$X_z(t) := \begin{cases} z + t\delta w & \text{if } 0 \leq t < \frac{|x-z|}{\delta} \\ x & \text{if } t \geq \frac{|x-z|}{\delta}. \end{cases} \tag{2.3.1}$$

which satisfies the differential inclusion $X_x'(t) \in \mathcal{B}(X_x(t))$. Below we build regular controls for the trajectory X_z.

We divide this part of the proof in two cases: $\{z,x\} \subset \mathcal{H}$ and $\{z,x\} \not\subset \mathcal{H}$.

Case 1: $\{z,x\} \subset \mathcal{H}$. In this case w and $X_z(t)$ are in $\mathcal{H}$ for all t. By hypothesis (HC) there exists $\alpha_i(t) \in A_i$ such that

$$b_1(X_z(t), \alpha_1(t)) = \delta w \quad \text{and} \quad b_2(X_z(t), \alpha_2(t)) = 0$$

Moreover $b_1(X_z(t), \alpha_1(t)) \cdot e_N = \delta w \cdot e_N = 0$ and $b_2(X_z(t), \alpha_2(t)) = 0$. Furthermore, thanks to Filippov's Theorem 2.1.2 we can choose the control $\alpha(t) := (\alpha_1(t), \alpha_2(t), 1)$ measurable. Hence $(X_z(\cdot), \alpha(\cdot)) \in \tau_A^{reg}(z)$.

Thanks to the construction of the trajectory $X_z(t)$, (2.3.1), we have $X_z(t) \in V$ and $|b_i(X_z(t), \alpha_i(t))| \leq \delta$ a.e then by hypothesis (HA), there exists R such that $\alpha_i(t) \in B_R^{A_i}(0)$ for a.e. t. Consequently $|l(X_z(t), \alpha_i(t))| \leq C$ for a.e. t.

Case 2: $\{y,z\}\mathcal{H}$. Thanks to Theorem 2.1.3 there exist a measurable control, $\alpha(\cdot)$, for which we can express $X_z'(\cdot)$ as in (2.1.8). Since $\{y,z\}\mathcal{H}$ then there exist at most one point of the trajectory $X_z(\cdot)$ that belongs to $\mathcal{H}$. Due hypothesis (HA), we have $\alpha_i(t) \in B_R^{A_i}(0)$ for a.e. $t \leq \frac{|x-z|}{\delta}$ and thanks to hypothesis (HC), we can keep the trajectory at $x = X_z(\frac{|x-z|}{\delta})$ for all $t \geq \frac{|x-z|}{\delta}$ and therefore we can assume that $(X_z(\cdot), \alpha(\cdot)) \in \tau_A^{reg}(z)$.

Since $X_z(t) \in V$ for all t and a $\alpha_i(t) \in B_R^{A_i}(0)$ for a.e. t, we have $|l(X_z(t), \alpha_i(t))| \leq C$ for a.e. t.

Now, we us prove that $U_A^+(\cdot)$ is locally Lipschitz. Let $0 < M < \infty$ be a constant such that

$$\max\{\|l_i\|_{L^\infty(V \times B_R^{A_i}(0))},\ \|U_A^+\|_{L^\infty(V)}\} < M.$$

Let $T := \frac{|x-z|}{\delta}$. we have $X_z(0) = z$ and $X_z(T) = x$. Since $(X_z(\cdot), \alpha(\cdot)) \in \tau_A^{reg}(z)$, and thanks to the Dynamic Programming Principle Theorem 2.2.1, we have

$$U_A^+(z) \leq \int_0^T l(X_z(t), \alpha(t)) e^{-\lambda t} dt + e^{-\lambda T} U_A^+(x).$$

Thus, since $\alpha_i(t) \in B_R^{A_i}(0)$, for a.e t, and $T = \frac{|x-z|}{\delta}$ then by Mean Value Theorem, we obtain

$$\begin{aligned} U_A^+(z) - U_A^+(x) &\leq \int_0^T l\left(X_z(t), \alpha(t)\right) e^{-\lambda t} dt + (e^{-\lambda T} - 1) U_A^+(x) \\ &\leq MT + \left|1 - e^{-\lambda T}\right| U_A^+(x) \\ &\leq MT + M\lambda T \\ &= \left(\frac{M}{\delta} + \frac{M\lambda}{\delta}\right) |x-z| = C|x-z|. \end{aligned}$$

Therefore, we have proved that U_A^+ is a locally Lipschitz continuous function.

Remark 2.3.2 *In the previous Proposition 2.3.1, if we add the hypothesis*

(i) Let K be a compact subset of A_i, then there exists $C_K(b)$ and $C_K(l)$ such that

$$|b_i(y,\alpha)| \leq C_K(b) \;\; \text{and} \;\; l_i(y,\alpha) \leq C_K(l), \; \forall \; y \in R^N, \; \alpha \in K. \qquad (2.3.2)$$

we obtain that U^+ and U^- are globally bounded and Lipschitz continuous functions.

In the following result we prove that for $(x,u,q) \in \overline{\Omega_i} \times R \times R^N$, the Hamiltonian

$$H_i(x,u,q) = \sup_{\alpha \in A_i} \{-b_i(x,\alpha) \cdot q - l_i(x,\alpha) + \lambda u\}$$

is well defined, continuous and locally coercive with respect to the third variable, that is, it grows to infinity on the extremes where it is defined.

Lemma 2.3.3 *Under hypotheses (HA), (HB) and (HC). Given $(x,u,q) \in \overline{\Omega_i} \times R \times R^N$ then*

(a) $H_i(x,u,q)$ is well-define and is continuous with respect to all the variables. H_i attains its supremum locally in a compact set of A_i.

(b) $H_i(x,u,q)$ is locally coercive.

Consider $g_i : A_i \times \overline{\Omega_i} \times R \times R^N \longrightarrow R$ defined as

$$g_i(\alpha,x,u,q) := -b_i(x,\alpha) \cdot q - l_i(x,\alpha) + \lambda u.$$

Note that g_i is a continuous function. Let $V_i \times U \times D \subset \overline{\Omega_i} \times R \times R^N$ be compacts sets. Thanks to hypothesis (HA), $\{\alpha \in A_i \mid b_i(x,\alpha) = 0\} \subset B_R^{A_i}(0) \;\; \forall \, x \in V_i$. Let $x \in V_i$, $\alpha_i^x \in A_i$ with $b_i(x,\alpha_i^x) = 0$ then

$$\sup_{\alpha \in A_i} \{-b_i(x,\alpha) \cdot q - l_i(x,\alpha)\} \geq -b_i(x,\alpha_i^x) \cdot q - l_i(x,\alpha_i^x) \geq -\|l_i\|_{L^\infty(V_i \times B_R^{A_i}(0))}.$$

We consider M_i such that,

$$M_i > \max\left\{\max_{q\in D}|q|, \|l_i\|_{L^\infty(V_i\times B_R^{A_i}(0))}\right\}.$$

Thanks to hypothesis (HA), there exists $\Gamma_{M_i} > 0$ such that if $|\alpha| > \Gamma_{M_i}$ then

$$l_i(x,\alpha) > M_i(1+|b_i(x,\alpha)|) \quad \forall x \in V_i.$$

That is,

$$-l_i(x,\alpha) + M_i|b_i(x,\alpha)| < -M_i.$$

Thus for $|\alpha| \geq \Gamma_{M_i}$, we have

$$-l_i(x,\alpha)-b_i(x,\alpha)\cdot q \leq -l_i(x,\alpha)+|b_i(x,\alpha)||q| \leq -l_i(x,\alpha)+M_i|b_i(x,\alpha)| < -M_i \leq -\|l_i\|_{L^\infty(V_i\times B_R^{A_i}(0))}.$$

Hence,

$$\begin{aligned}
-l_i(x,\alpha) - b_i(x,\alpha)\cdot q + \lambda u &\leq -l_i(x,\alpha) + M_i|b_i(x,\alpha)| + \lambda u \\
&< -M_i + \lambda u \\
&< -\|l_i\|_{L^\infty(V_i\times B_R^{A_i}(0))} + \lambda u \\
&\leq H_i(x,u,q).
\end{aligned}$$

Therefore, if $(x,u,q) \in V_i \times U \times D$

$$\begin{aligned}
H_i(x,u,q) &= \sup_{\alpha\in A_i}\{-b_i(x,\alpha)\cdot q - l_i(x,\alpha)\} + \lambda u \\
&= \sup_{\alpha\in \overline{B^{A_i}_{\Gamma_{M_i}}(0)}}\{-b_i(x,\alpha)\cdot q - l_i(x,\alpha)\} + \lambda u \\
&= \sup_{\alpha\in \overline{B^{A_i}_{\Gamma_{M_i}}(0)}} g_i(\alpha,x,u,q).
\end{aligned}$$

Therefore, H_i is a continuous function because is the supremum of a continuous function over a compact set.

Now let us prove that H_i is locally coercive. Let V be a compact set of $\overline{\Omega_i}$. Given $x \in V$ and $q \in R^N$, there exists $r > 0$ and $\alpha_i \in B_r^{A_i}(0)$ such that

$$|b_i(x,\alpha_i)| = \delta, \quad \text{and} \quad -b_i(x,\alpha_i)\cdot q = \delta|q|.$$

Moreover, we consider u in a compact set such that $|u| < R$. Thus,

$$H_i(x,u,q) \geq -b_i(x,\alpha_i)\cdot q - l_i(x,\alpha_i) + \lambda u \geq \delta|q| - ||l_i||_{V\times B_r^{A_i}(0)} - \lambda R$$

Hence,

$$\lim_{|q|\to\infty} H_i(x,u,q) = \infty.$$

Thus, H_i is locally coercive.

Remark 2.3.4 *If the previous Lemma 2.3.3, if we add the hypothesis*

(i) Let K be compact contained in A_i, then there exists $C_K(b)$ and $C_K(l)$ such that

$$|b_i(y,\alpha)| \le C_K(b) \ \text{ and } \ l_i(y,\alpha) \le C_K(l),\ \forall\ y \in R^N,\ \alpha \in K. \tag{2.3.3}$$

then for $(x,v,q) \in \overline{\Omega_i} \times U \times D$, where $U \times D \subset R \times R^N$ are compact, we obtain

$$H_i(x,v,q) = \sup_{\alpha\in A_i}\{-b_i(x,\alpha)\cdot q - l_i(x,\alpha)\} + \lambda v = \sup_{\alpha\in \overline{B^{A_i}_{\Gamma_{M_i}}(0)}}\{-b_i(x,\alpha)\cdot q - l_i(x,\alpha)\} + \lambda v$$

That is, the compact where the supremum is reached depends only on the modulus of the third variable.

2.4 VALUE FUNCTIONS ARE SOLUTIONS OF THE ISHII PROBLEM

The aim o this section is to prove the that the value functions U^+ and U^- are solutions of the Ishii problem (2.1.1). To do this we will prove first that the value functions in compact control set are solutions of the auxiliar Ishii problem associated to compact control sets, and finally we pass to the limit.

The following theorem is an auxiliar result which proves that the value functions associated to compact control sets are locally bounded and locally Lipschitz continuous functions, and they are solutions of the Ishii problem associated to compact control sets.

Theorem 2.4.1 *Under hypotheses (HA), (HB) and (HC). Then $U^+_{A^m}$ and $U^-_{A^m}$ are viscosity solutions of the Ishii problem* (2.1.14) *associated to compact control sets.*
Furthermore,

(i) $U^-_{A^m}$ is subsolution of of the tangential Hamiltonian H^m_T, (2.1.15)*, that is, for $\bar{x} \in R^{N-1}$, then $\bar{x} \to U^-_{A^m}(\bar{x},0)$ satisfies*

$$H^m_T(x,u,D_{\mathcal{H}u}) \le 0.$$

(ii) $U^+_{A^m}$ *is subsolution of the tangential Hamiltonian* $H_T^{m,reg}$*,* (2.1.16)*. That is, for* $\bar{x} \in R^{N-1}$*, then* $\bar{x} \to U^+_{A^m}(\bar{x}, 0)$ *satisfies*

$$H_T^m(x, u, D_{\mathcal{H}u}) \leq 0.$$

Thanks to Proposition 2.3.1 we have $U^+_{A^m}$ and $U^-_{A^m}$ are locally bounded and locally Lipschitz functions. Since the control sets A^m are compact, we can apply Theorem 2.5 in (BARLES; BRIANI; CHASSEIGNE, 2013), to obtain that $U^+_{A^m}$ and $U^-_{A^m}$ are subsolutions of (2.1.14). Let us prove that $U^+_{A^m}$ and $U^-_{A^m}$ are supersolutions. We will denote $U^+_{A^m}$ and $U^-_{A^m}$ by U for the simplicity of the notation. Let ϕ be a test function, consider x a local minimum of $U - \phi$ and assume $U(x) = \phi(x)$ without lost of generality. Then there exists $r \geq 0$ such that $U \geq \phi$ in $B_r(x)$. Thanks to Lemma 2.2.6, there exist $\bar{t} > 0$ such that for any trajectory $(X_x, \alpha) \in \tau_{A^m}(x)$, we have $|X_x(s) - x| \leq r, \quad s \leq \bar{t}$. Then the trajectories are in a bounded subset, and we can apply Theorem 2.5 in (BARLES; BRIANI; CHASSEIGNE, 2013).

The theorem below is the principal result of this section. It states that the value functions are solution of the Ishii problem (2.1.1).

Theorem 2.4.2 *Under hypotheses (HA), (HB) and (HC). Then the value functions* U^-_A*,* U^+_A *are viscosity solutions of the Hamilton-Jacobi-Bellman problem* (2.1.1)*. Moreover* U^-_A *is a subsolution of the tangential Hamiltonian* H_T*, and* U^+_A *is a subsolution of the tangential Hamiltonian* H_T^{reg}*.*

We will follow the ideas in (BARLES, 1990) to prove that U^+_A and U^-_A are supersolutions. Consider a sequence of compacts $A^m_i \subset A_i$. Recall that if $m \leq m'$, then $A^m_i \subset A^{m'}_i$, and thanks to the definition of $U^+_{A^m}$ and $U^-_{A^m}$, we have

$$U^+_{A^{m'}}(x) \leq U^+_{A^m}(x), \qquad U^-_{A^{m'}}(x) \leq U^-_{A^m}(x).$$

This implies together with Proposition 2.3.1, that $U^-_{A^m}$ and $U^+_{A^m}$ are uniformly locally bounded. Moreover, since the sequences $U^-_{A^m}$ and $U^+_{A^m}$ are decreasing we have

$$\inf_m U^+_{A^m}(x) = U^+_A(x), \qquad \inf_m U^-_{A^m}(x) = U^-_A(x).$$

Thanks to Proposition 2.3.1 and Theorem (2.4.1), $U^-_{A_m}, U^-_A$, $U^+_{A_m}, U^+_A$ are continuous, then due Lemma **??**

$$\begin{aligned} \liminf_{(m,y)\to(\infty,x)} U^+_{A^m}(y) &= \left(\inf_m U^+_{A_m}\right)_*(x) = U^+_A(x) \\ \liminf_{(m,y)\to(\infty,x)} U^-_{A^m}(y) &= \left(\inf_m U^-_{A^m}\right)_*(x) = U^-_A(x), \end{aligned} \tag{2.4.1}$$

where $(\cdot)_*$ is the envelope of l.s.c functions.

Consider the discontinuous Hamiltonians H^m and H given by

$$H^m(x,u,p) := \begin{cases} H_1^m(x,u,p) & \text{in } \Omega_1 \\ H_2^m(x,u,p) & \text{in } \Omega_2 \\ \max\{H_1^m(x,u,p),\ H_2^m(x,u,p)\} & \text{on } \mathcal{H} \end{cases} \tag{2.4.2}$$

and

$$H(x,u,p) := \begin{cases} H_1(x,u,p) & \text{in } \Omega_1 \\ H_2(x,u,p) & \text{in } \Omega_2 \\ \max\{H_1(x,u,p),\ H_2(x,u,p)\} & \text{on } \mathcal{H}. \end{cases} \tag{2.4.3}$$

Let (y,v,q) and (x,u,p) in $R^N \times R \times R^N$, then

$$\begin{aligned} \limsup_{(m,y,v,q)\to(\infty,x,u,p)}(H^m(y,v,q)) &= \lim_{m\to\infty} \sup\{H^m(y,v,q) \mid (y,v,q) \in B_{1/m}(x,u,p) \setminus \{(x,u,p)\}\} \\ &= \limsup_{(y,v,q)\to(x,u,p)} H(y,v,q) \\ &= H^*(x,u,p) \\ &= H(x,u,p), \end{aligned} \tag{2.4.4}$$

where $(\cdot)^*$ is the envelope of u.s.c functions.

Thanks to Theorem 2.4.1 we know that U_m^+ and U_m^- are supersolutions of H^m, and thanks to Proposition 2.3.1 and Lemma 2.3.3 we have U_m^+, U_m^- and H^m are locally bounded. From (2.4.1) and (2.4.4), and thanks to the Half-Relaxed Limit Method Theorem **??**, we obtain that U_A^- and U_A^+ are supersolutions of Ishii problem (2.1.1).

The proof that U_A^+ and U_A^- are subsolutions of H_i in Ω_i is classical, cf. (BARLES, 1994). We prove below that U_A^+, U_A^- are subsolutions of $\min\{H_1, H_2\}$ and that U_A^+, U_A^- are subsolutions of the tangential Hamiltonians H_T^{reg} and H_T respectively. We prove the result for U_A^+, the proof for U_A^- is analogous. Let $x \in \mathcal{H}$ be a local maximum of $U_A^+ - \phi$, with $\phi \in C^1(R^N)$. We assume that this maximum is zero for simplicity. Thus, there exist $r > 0$ with $U_A^+(y) - \phi(y) \leq 0$ for all $y \in B_r(x) \subset R^N$ and $U_A^+(x) = \phi(x)$.

Initially we will show that U_A^+ is subsolution of $\min\{H_1, H_2\}$. Thanks to Lemma 2.3.3, since H_i attains its supremum in a bounded control set, then there exist $(\alpha_1, \alpha_2) \in A_1 \times A_2$

such that

$$H_1\left(x,\phi\left(x\right),\nabla\phi\left(x\right)\right)=-l_1(x,\alpha_1)+\lambda\phi\left(x\right)-b_1\left(x,\alpha_1\right)\cdot\nabla\phi\left(x\right)$$

$$H_2\left(x,\phi\left(x\right),\nabla\phi\left(x\right)\right)=-l_2(x,\alpha_2)+\lambda\phi\left(x\right)-b_2\left(x,\alpha_2\right)\cdot\nabla\phi\left(x\right).$$

In order to prove that U_A^+ is subsolution of $\min\{H_1,H_2\}$, we use specific trajectories that we build using the constant controls α_1 and α_2. To do this we divide the proof in cases for the different combinations of signs of $b_i(x,\alpha_i)\cdot e_N$.

Case 1

Let (α_1,α_2) such that

$$(b_1(x,\alpha_1)-b_2(x,\alpha_2))\cdot e_N<0.$$

We construct regular trajectories, $(X_x(\cdot),\alpha(\cdot))\in\tau_A^{reg}(x)$ which stay on $\mathcal{H}$, at least for a while. We emphasize that it is essential to consider regular trajectories in order to be able to use the Dynamic Programming Principle for U_A^+. We divide Case 1 in three sub-cases:

1.A $b_1\left(x,\alpha_1\right)\cdot e_N<0$ and $b_2\left(x,\alpha_2\right)\cdot e_N>0$.

1.B $b_1\left(x,\alpha_1\right)\cdot e_N=0$ and $b_2\left(x,\alpha_2\right)\cdot e_N>0$.

1.C $b_1\left(x,\alpha_1\right)\cdot e_N<0$ and $b_2\left(x,\alpha_2\right)\cdot e_N=0$.

Case 1.A. Let $\bar{\delta}>0$, $y\in\mathcal{H}\cap B_{\bar{\delta}}(x)$. We define

$$\mu(y):=\frac{-b_2(y,\alpha_2)\cdot e_N}{(b_1(y,\alpha_1)-b_2(y,\alpha_2))\cdot e_N} \tag{2.4.5}$$

Note that since $(b_1(\cdot,\alpha_1)-b_2(\cdot,\alpha_2))\cdot e_N$ is continuous then $(b_1(\cdot,\alpha_1)-b_2(\cdot,\alpha_2))\cdot e_N<0$ on a neighbourhood of x, therefore μ is well defined in this neighbourhood and $b_1(\cdot,\alpha_1)\cdot e_N<0$, $-b_2(\cdot,\alpha_2)\cdot e_N<0$. So,

$$0<\mu(y)=\frac{1}{1-\frac{(b_1(y,\alpha_1))\cdot e_N}{b_2(y,\alpha_2)\cdot e_N}}<1$$

in such neighbourhood. Now, consider the local trajectory that satisfies

$$X'\left(t\right)=\mu(X(t))b_1\left(X\left(t\right),\alpha_1\right)+(1-\mu(X(t)))b_2\left(X\left(t\right),\alpha_2\right),\quad X\left(0\right)=x. \tag{2.4.6}$$

Since b_1, b_2 and μ are continuous functions, then (2.4.6) has a local solution. Substituting in (2.4.6), $\mu(X(t))$ by its expression (2.4.5), we get

$$X'\left(t\right)=\frac{(-b_2(X(t),\alpha_2)\cdot e_N)b_1(X(t),\alpha_1)+(b_1(X(t),\alpha_1)\cdot e_N)b_2(X(t),\alpha_2)}{(b_1(X(t),\alpha_1)-b_2(X(t),\alpha_2))\cdot e_N}.$$

Thus, the trajectory satisfies locally $X'(t) \cdot e_N = 0$. Thus, since $x \in \mathcal{H}$ then $X(t)$ stays on $\mathcal{H}$ for a while.

Since $0 < \mu(X(0)) = \mu(x) < 1$, thanks to the continuity of $\mu(X(\cdot))$, there exist $T > 0$ such that

$$0 < \mu(X(t)) < 1, \quad b_1(X(t), \alpha_2) \cdot e_N < 0, \quad b_2(X(t), \alpha_2) \cdot e_N > 0 \quad \text{for} \quad 0 \leq t \leq T.$$

Let us define the following trajectory

$$X_x(t) := \begin{cases} X(t), & \text{if } 0 \leq t < T \\ X(T), & \text{if } t \geq T. \end{cases} \tag{2.4.7}$$

Furthermore, thanks to the continuity of b_i and the trajectory, there exists a constant C_T such that

$$X'(t) = \mu(X(t))b_1(X(t), \alpha_1) + (1 - \mu(X(t)))b_2(X(t), \alpha_2) \leq C_T, \quad \forall t \leq T.$$

Hence $X_x(\cdot)$ is Lipschitz. Thanks to hypothesis (HC), there exists $\alpha_i^* \in A_i$ such that $b_i(X_x(T), \alpha_i^*) = 0$, then we consider the control

$$\alpha(t) := \begin{cases} (\alpha_1, \alpha_2, \mu(X(t))) & \text{if } 0 \leq t < T, \\ (\alpha_1^*, \alpha_2^*, \mu) & \text{if } t \geq T. \end{cases} \tag{2.4.8}$$

Hence $(X_x(\cdot), \alpha(\cdot)) \in \tau_A^{reg}$.

Recall $U_A(y) - \phi(y) \leq 0$ for all $y \in B_r(x)$ and $U_A(x) = \phi(x)$. Since X_x is Lipschitz, then there exists $T' > 0$ such that $X_x(t) \in B_r(x)$ for all $t < T' < T$. By the Dynamic Programming Principle we have

$$\phi(x) \leq \int_0^{T'} l_{\mathcal{H}}(X_x(t), (\alpha_1, \alpha_2, \mu(X(t)))e^{-\lambda t}dt + \phi(X_x(T'))e^{-\lambda T'}$$

By the Fundamental Calculus Theorem, we obtain

$$0 \geq \int_0^{T'} \left[-l_{\mathcal{H}}(X_x(t), (\alpha_1, \alpha_2, \mu(X(t))) + \lambda\phi(X_x(t)) - b_{\mathcal{H}}(X_x(t), (\alpha_1, \alpha_2, \mu(X_x(t)))) \cdot \nabla\phi(X_x(t))\right] e^{-\lambda t} d$$

Dividing by T' and taking limits as T' goes to 0 yields

$$\begin{aligned} 0 &\geq -l_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu(x)) + \lambda\phi(x) - b_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu(x)) \cdot \nabla\phi(x) \\ &= \mu(x)(-l_1(x, \alpha_1) + \lambda\phi(x) - b_1(x, \alpha_1) \cdot \nabla\phi(x)) + (1 - \mu(x))(-l_2(x, \alpha_2) + \lambda\phi(x) - b_2(x, \alpha_2) \cdot \nabla\phi(\end{aligned}$$

Thus,

$$\begin{aligned} 0 &\geq \min\{-l_1(x,\alpha_1)+\lambda\phi(x)-b_1(x,\alpha_1)\cdot\nabla\phi(x),-l_2(x,\alpha_2)+\lambda\phi(x)-b_2(x,\alpha_2)\cdot\nabla\phi(x)\} \\ &=\min\{H_1(x,\phi(x),\nabla\phi(x)),H_2(x,\phi(x),\nabla\phi(x))\}. \end{aligned}$$

Case 1.B Let $(\alpha_1,\alpha_2,\mu)\in A_0(x)$ such that

$$b_1(x,\alpha_1)\cdot e_N=0 \quad \text{and} \quad b_2(x,\alpha_2)\cdot e_N>0.$$

Thanks to hypothesis (HC), there exist $\alpha_1^+\in A_1$ such that $b_1\left(x,\alpha_1^+\right)=-\delta e_N$, and for $0<\nu<1$, by convexity, there exists α_1^ν satisfying

$$\begin{aligned} b_1(x,\alpha_1^\nu)&=\nu b_1\left(x,\alpha_1^+\right)+(1-\nu)b_1(x,\alpha_1) \\ l_1(x,\alpha_1^\nu)&=\nu l_1\left(x,\alpha_1^+\right)+(1-\nu)l_1(x,\alpha_1). \end{aligned}$$

Therefore, $b_1(x,\alpha_1^\nu)\cdot e_N=-\nu\delta<0$ and we argue as in case 1.A with control $(\alpha_1^\nu,\alpha_2,\mu^\nu(x))$ in $A_0^{reg}(x)$ where

$$\mu^\nu(x)=\frac{-b_2(x,\alpha_2)\cdot e_N}{(b_1(x,\alpha_1^\nu)-b_2(x,\alpha_2))\cdot e_N}.$$

We obtain that

$$0\geq -l_{\mathcal{H}}(x,\alpha_1^\nu,\alpha_2,\mu^\nu(x))+\lambda\phi(x)-b_{\mathcal{H}}(x,\alpha_1^\nu,\alpha_2,\mu^\nu(x))\cdot\nabla\phi(x)$$

Hence,

$$0\geq\mu^\nu(x)(-l_1(x,\alpha_1^\nu)+\lambda\phi(x)-b_1(x,\alpha_1^\nu)\cdot\nabla\phi(x))+(1-\mu^\nu(x))(-l_2(x,\alpha_2)+\lambda\phi(x)-b_2(x,\alpha_2)\cdot\nabla\phi(x)). \tag{2.4.9}$$

By construction, $(b_1(x,\alpha_1^\nu),l_1(x,\alpha_1^\nu),\mu^\nu(x))$ converges to $(b_1(x,\alpha_1),l_1(x,\alpha_1),1)$ as ν goes to 0. Therefore taking limits as ν go to 0 in (2.4.9), we have

$$\begin{aligned} 0 &\geq -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,1)+\lambda\phi(x)-b_{\mathcal{H}}(x,\alpha_1,\alpha_2,1)\cdot\nabla\phi(x) \\ &=(-l_1(x,\alpha_1)+\lambda\phi(x)-b_1(x,\alpha_1)\cdot\nabla\phi(x)) \\ &=H_1(x,\phi(x)),\nabla\phi(x)). \end{aligned}$$

Case 1.C We proceed analogously to case 1.B. There exists $\alpha_2^+\in A_2$ such that $b_2\left(x,\alpha_2^+\right)=\delta e_N$. Let $0<\nu<1$. We can find $\alpha_2^\nu\in A_2$ satisfying

$$\begin{aligned} b_2(x,\alpha_2^\nu)&=\nu b_2\left(x,\alpha_2^+\right)+(1-\nu)b_2(x,\alpha_2) \\ l_2(x,\alpha_2^\nu)&=\nu l_2\left(x,\alpha_2^+\right)+(1-\nu)l_2(x,\alpha_2). \end{aligned}$$

By construction, $(b_2(x,\alpha_2^\nu), l_2(x,\alpha_2^\nu), \mu^\nu(x))$ goes to $(b_2(x,\alpha_2), l_2(x,\alpha_2), 1)$ as ν goes to 0. Thus,

$$0 \geq (-l_2(x,\alpha_2) + \lambda\phi(x) - b_2(x,\alpha_2)\cdot\nabla\phi(x)) = H_2(x,\phi(x),\nabla\phi(x)).$$

Case 2 We construct regular trajectories, $(X_x(\cdot),\alpha(\cdot)) \in \tau_A^{reg}(x)$, which stay in Ω_i at least for a while. We divide this case in the following subcases.

2.A $b_1(x,\alpha_1)\cdot e_N > 0$ and $b_2(x,\alpha_2)\cdot e_N < 0$

2.B $b_1(x,\alpha_1)\cdot e_N = 0$ and $b_2(x,\alpha_2)\cdot e_N < 0$

2.C $b_1(x,\alpha_1)\cdot e_N > 0$ and $b_2(x,\alpha_2)\cdot e_N = 0$.

2.D $b_1(x,\alpha_1)\cdot e_N > 0$ and $b_2(x,\alpha_2)\cdot e_N > 0$.

2.E $b_1(x,\alpha_1)\cdot e_N < 0$ and $b_2(x,\alpha_2)\cdot e_N < 0$.

In all theses cases we can build trajectories that stay in Ω_i for all $s \in]0,T]$ for some $T > 0$. So,

$$0 \geq (-l_i(x,\alpha_i) + \lambda\phi(x) - b_i(x,\alpha_i)\cdot\nabla\phi(x)) = H_i(x,\phi(x),\nabla\phi(x)).$$

In the cases 2.A, 2.B and 2.C we can also build trajectories that stay on $\mathcal{H}$ for a while and such that

$$0 \geq -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu) + \lambda\phi(x) - b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot\nabla\phi(x).$$

Case 3 Let $(\alpha_1,\alpha_2) \in A_1 \times A_2$ such that

$$b_1(x,\alpha_1)\cdot e_N = 0 \quad \text{and} \quad b_2(x,\alpha_2)\cdot e_N = 0.$$

From hypothesis (HC), there exist controls $\alpha_1^- \in A_1$ and $\alpha_2^+ \in A_2$ such that $b_1\left(x,\alpha_1^-\right) = -\delta e_N$ and $b_2\left(x,\alpha_2^+\right) = \delta e_N$. For $0 < \nu,\nu' < 1$, thanks to the convexity, we can find $\alpha_1^\nu \in A_1$ and $\alpha_2^{\nu'} \in A_2$ such that

$$b_1(x,\alpha_1^\nu) := \nu b_1\left(x,\alpha_1^-\right) + (1-\nu)b_1(x,\alpha_1)$$
$$l_1(x,\alpha_1^\nu) := \nu l_1\left(x,\alpha_1^-\right) + (1-\nu)l_1(x,\alpha_1)$$

and

$$b_2(x,\alpha_2^{\nu'}) := \nu' b_2\left(x,\alpha_2^+\right) + (1-\nu')b_2(x,\alpha_2)$$
$$l_2(x,\alpha_2^{\nu'}) := \nu' l_2\left(x,\alpha_2^+\right) + (1-\nu')l_2(x,\alpha_2).$$

Therefore, $b_1(x,\alpha_1^\nu)\cdot e_N = -\nu\delta < 0$ and $b_2\left(x,\alpha_2^{\nu'}\right)\cdot e_N = \nu'\delta > 0$. Arguing as in case 1.A, we obtain $(X_x(\cdot),(\alpha_1^\nu,\alpha_2^{\nu'},\mu(X_x(\cdot))) \in \tau_A^{reg}(x)$ satisfying

$$0 \geq \mu(\nu,\nu')(-l_1(x,\alpha_1^\nu)+\lambda\phi(x)-b_1(x,\alpha_1^\nu)\cdot\nabla\phi(x))+(1-\mu(\nu,\nu'))(-l_2(x,\alpha_2^{\nu'})+\lambda\phi(x)-b_2\left(x,\alpha_2^{\eta'}\right)\cdot\nabla\phi(\tag{2.4.10}$$

where $\mu(\nu,\nu')=\mu(X_x(0))=\frac{\nu'}{\nu'+\nu}$ and $a=(\alpha_1^{\nu},\alpha_1^{\nu'},\mu(\nu,\nu'))\in A_0^{reg}(x)$. By construction, we have the following convergences

$$b_1(x,\alpha_1^{\nu})\longrightarrow b_1(x,\alpha_1),\quad l_1(x,\alpha_1^{\nu})\longrightarrow l_1(x,\alpha_1)\quad \text{as}\quad \nu\to 0$$
$$b_2(x,\alpha_2^{\nu'})\longrightarrow b_2(x,\alpha_2),\quad l_2\left(x,\alpha_2^{\nu'}\right)\longrightarrow l_2(x,\alpha_2)\quad \text{as}\quad \nu'\to 0$$

Given $0\le\mu\le 1$, we can consider the sequences $(\nu_n)_n$, $(\nu'_n)_n$ that converge to 0 and such that $\frac{\nu_n}{\nu'_n}=\frac{1-\mu}{\mu}$, then $\mu(\nu_n,\nu'_n)$ converges to μ. Taking limits in (2.4.10) as ν and ν' go to 0 we have

$$\begin{aligned}0\ &\ge\mu(-l_1(x,\alpha_1)+\lambda\phi(x)-b_1(x,\alpha_1)\cdot\nabla\phi(x))+(1-\mu)(-l_2(x,\alpha_2)+\lambda\phi(x)-b_2(x,\alpha_2)\cdot\nabla\phi(x))\\ &\ge\min\{H_1(x,\phi(x),\nabla\phi(x)),H_2(x,\phi(x),\nabla\phi(x))\}.\end{aligned}$$

Therefore, we have proved that in all the cases, U_A^+ is subsolution of $\min\{H_1,H_2\}$ on $\mathcal{H}$.

Now, let us show that U_A^- is subsolution of H_T. We consider $\phi\in C^1(R^{N-1})$ and $x=(x',0)\in\mathcal{H}$ such that $x'\longmapsto U_A^-(x',0)-\phi(x')$ has a local maximum at x'_0. We assume that this maximum is equal to 0 for simplicity. We extend the test function defining a new function as follows $\overline{\phi}$

$$\overline{\phi}(x',x_N)=\phi(x'),$$

then $\overline{\phi}\in C^1(R^N)$ and $\overline{\phi}$ is independent of the N-th coordinate. Notice that

$$D\overline{\phi}(x)=(D_{\mathcal{H}}\phi(x'),0).$$

Consider $(\alpha_1,\alpha_2,\mu)\in A_0(x)$, then (α_1,α_2) is in some of the following cases: case 1, case 2.A, case 2.B, case 2.C or case 3. Then have that

$$-l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\overline{\phi}(x)-b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot\nabla\overline{\phi}(x)\le 0.$$

Thus, taking supremum in $A_0(x)$ we obtain that

$$H_T(x,\phi(x),D_{\mathcal{H}})=\sup_{(\alpha_1,\alpha_2,\mu)\in A_0(x)}-l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\overline{\phi}(x)-b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot(D_{\mathcal{H}}\phi(x),0)\le 0.$$

Analogously we prove that U_A^+ is subsolution of H_T^{reg}. Given a control $(\alpha_1,\alpha_2,\mu)\in A_0^{reg}(x)$, we have (α_1,α_2) is in case 1 or case 3. Then

$$-l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\overline{\phi}(x)-b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot\nabla\overline{\phi}(x)\le 0.$$

Thus, taking supremum in $A_0^{reg}(x)$, we have

$$H_T^{reg}(x,\phi(x),D_{\mathcal{H}}\phi)=\sup_{(\alpha_1,\alpha_2,\mu)\in A_0^{reg}(x)}-l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\overline{\phi}(x)-b_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)\cdot(D_{\mathcal{H}}\phi(x),0)\le 0.$$

3 COMPARISON AND UNIQUENESS RESULTS

In this chapter, we prove comparison results under suitable growing conditions on the dynamic and cost functions. This way, we are able to prove uniqueness of solution of the Ishii problem. Initially we prove the results locally and use the fact that under suitable conditions we have a relation between the local and global comparison results, cf. (BARLES; CHASSEIGNE, 2018). We also present specific examples of unbounded dynamic and cost functions with non-compact controls space and consider Filippov approximations.

Let us recall the definition of the Ishii problem

$$\begin{cases} H_1(x, u(x), Du(x)) = 0 & \text{in } \Omega_1, \\ H_2(x, u(x), Du(x)) = 0 & \text{in } \Omega_2, \\ \min\{H_1(x, u(x), Du(x)), H_2(x, u(x), Du(x))\} \leq 0 & \text{in } \mathcal{H}, \\ \max\{H_1(x, u(x), Du(x)), H_2(x, u(x), Du(x))\} \leq 0 & \text{in } \mathcal{H}. \end{cases} \tag{3.0.1}$$

where

$$H_i(x, u, q) = \sup_{\alpha_i \in A_i} (-b_i(x, \alpha_i) \cdot q - l_i(x, \alpha_i) + \lambda u).$$

with $(x, u, q) \in \overline{\Omega_i} \times R \times R^N$ and $\lambda \in R$ is a discount factor.

Let us recall the definition of the control sets

$$A_0(x) \quad = \{(\alpha_1, \alpha_2, \mu) \in A \mid b_{\mathcal{H}}(x, \alpha_1, \alpha_2) \cdot e_N = 0\}.$$

$$A_0^{reg}(x) \quad = \{(\alpha_1, \alpha_2, \mu) \in A_0 \mid b_{\mathcal{H}}(\alpha_1, \alpha_2, \mu) \text{ is regular}\}.$$

Then we define the tangential Hamiltonians, for $(x, u, q) \in \mathcal{H} \times R \times R^{N-1}$

$$H_T(x, u, q) = \sup_{(\alpha_1, \alpha_2, \mu) \in A_0(x)} -l_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) + \lambda u - b_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) \cdot (q, 0) \tag{3.0.2}$$

and

$$H_T^{reg}(x, u, q) = \sup_{(\alpha_1, \alpha_2, \mu) \in A_0^{reg}(x)} -l_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) + \lambda u - b_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) \cdot (q, 0). \tag{3.0.3}$$

We consider also the Ishii problem associated to the compact control sets A^m,

$$\begin{cases} H_1^m(x, u(x), Du(x)) = 0 & \text{in } \Omega_1 \\ H_2^m(x, u(x), Du(x)) = 0 & \text{in } \Omega_2 \\ \min\{H_1^m(x, u(x), Du(x)), H_2^m(x, u(x), Du(x))\} \leq 0 & \text{on } \mathcal{H} \\ \max\{H_1^m(x, u(x), Du(x)), H_2^m(x, u(x), Du(x))\} \leq 0 & \text{on } \mathcal{H} \end{cases} \tag{3.0.4}$$

where

$$H_i^m(x,u,q) = \sup_{\alpha \in A_i^m} \{-b_i(x,\alpha)\cdot q - l_i(x,\alpha)\} + \lambda u.$$

We consider the associated control sets defined by

$$A_0^m(x) \quad = \{(\alpha_1,\alpha_2,\mu) \in A^m \mid b_{\mathcal{H}}(x,\alpha_1,\alpha_2)\cdot e_N = 0\}.$$

$$A_0^{m,reg}(x) \quad = \{(\alpha_1,\alpha_2,\mu) \in A_0^m \mid b_{\mathcal{H}}(\alpha_1,\alpha_2,\mu) \text{ is regular}\}.$$

Then we can define the tangential Hamiltonians for $x \in \mathcal{H}$

$$H_T^m(x,u(x),D_{\mathcal{H}}u) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0^m(x)} -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\overline{u}\left(x\right)-b_{\mathcal{H}}\left(x,\alpha_1,\alpha_2,\mu\right)\cdot\left(D_{\mathcal{H}}u\left(x\right),0\right) \tag{3.0.5}$$

where $\overline{u}(x',x_N) = u(x')$, with $x' \in R^{N-1}$ and

$$H_T^{m,reg}(x,u(x),D_{\mathcal{H}}u) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0^{m,reg}(x)} -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\overline{u}\left(x\right)-b_{\mathcal{H}}\left(x,\alpha_1,\alpha_2,\mu\right)\cdot\left(D_{\mathcal{H}}u\left(x\right),0\right) \tag{3.0.6}$$

3.1 COMPARISON RESULTS WITH COMPACT CONTROL SETS

In the following result we prove that if we have a supersolution of the Ishii problem, then it is also a supersolution of the Ishii problem associated to compact control sets on compact subsets of R^N.

Proposition 3.1.1 *Assume hypotheses (HA), (HB) and (HC). Let $V \subset R^N$ be compact. If v is a locally bounded lsc supersolution of* (3.0.1), *then there exists $m \in N$ such that v is supersolution of* (3.0.4) *on V.*

Due to hypotheses (HA) and (HC) there exists a control set $A_i^\delta \subset A_i$ such that given $\omega \in B_\delta(0)$ and $x \in V$ there exists $\alpha_i \in A_i^\delta$ with $b_i(x,\alpha_i) = \omega$. Thanks to hypothesis (HA) we have A_i^δ is bounded, then there exists $\bar{m}$ big enough such that $A_i^\delta \subset A_i^{\bar{m}}$.

Let $\phi \in C_1(V)$ and $x \in V$ be a local minimum point of $v - \phi$. We divide the proof in two cases for ϕ such that $|\nabla\phi(x)| \geq \frac{||l||_{L^\infty(V\times A_i^\delta)}+\lambda||v||_{L^\infty(V)}}{\delta}$ and ϕ such that $|\nabla\phi(x)| < \frac{||l||_{L^\infty(V\times A_i^\delta)}+\lambda||v||_{L^\infty(V)}}{\delta}$.

Case 1: $|\nabla\phi(x)| \geq \frac{||l||_{L^\infty(V\times A_i^\delta)}+\lambda||v||_{L^\infty(V)}}{\delta}$.

Since v is supersolutionSince exist $\alpha_i \in A_i$ such that $-b_i(x,\alpha_i)\cdot\nabla\phi(x) = \delta|\nabla\phi(x)|$, and then $\alpha_i \in A_i^\delta$, and since $A_i^\delta \subset A_i^{\bar{m}}$. we have for $x \in V$

$$H_i^{\overline{m}}(x,v(x),\nabla\phi(x)) \geq -b_i(x,\alpha_i)\cdot\nabla\phi(x)-l_i(x,\alpha_i)+\lambda v \geq \delta|\nabla\phi(x)|-||l_i||_{L^\infty(V\times A_i^\delta)}-\lambda||v||_{L^\infty(V)} \geq 0.$$

Case 2: $|\nabla\phi(x)| < \frac{||l||_{L^\infty(V\times A_i^\delta)}+\lambda||v||_{L^\infty(V)}}{\delta}$.

Since $|\nabla\phi(x)|$ is bounded, v is supersolution of (2.1.1), and thanks to Lemma 2.3.3, there exists $m_v > 0$ such that

$$H_i^{m_v}(x, v(x), \nabla\phi(x)) = H_i(x, v(x), \nabla\phi(x)) \geq 0.$$

Choosing $m > \max\{\overline{m}, m_v\}$,

$$H_i^m(x, v(x), \nabla\phi(x)) \geq 0.$$

Therefore, v is supersolution of H_i^m.

In the particular case in which $x \in V \cap \mathcal{H}$ then

$$\max\{H_1(x, v(x), \nabla\phi(x)), H_2(x, v(x), \nabla\phi(x))\} \geq 0.$$

Consequently there exists $i \in \{1, 2\}$ with $H_i(x, v(x), \nabla\phi(x)) \geq 0$. From the prove above, we know that there exists $m_i \in N$ big enough such that $H_i^{m_i}(x, v(x), \nabla\phi(x)) \geq 0$. So, taking $m > m_i$, we have

$$\max\{H_1^m(x, v(x), \nabla\phi(x)), H_2^m(x, v(x), \nabla\phi(x))\} \geq 0.$$

In the result below we add and extra hypothesis to obtain a better version of the previous result. In particular it states that if $v(x)$ is a supersolution of (3.0.1) then $v(x)$ is a supersolution of (3.0.4) for m big enough and for any $x \in R^N$.

Proposition 3.1.2 *Under hypotheses (HA), (HB) and (HC). Let $A_i^m \subset A_i$ be compact, then there exists $C_m(b)$ and $C_m(l)$ such that*

$$|b_i(y, \alpha)| \leq C_m(b) \ \ \textit{and} \ \ l_i(y, \alpha) \leq C_m(l), \ \forall\, y \in R^N, \ \ \textit{and} \ \ \forall \alpha \in A_i^m. \qquad (3.1.1)$$

Let v be a bounded lsc supersolutions of (3.0.1)*, then there exists $m \in N$ such that v is supersolution of* (3.0.4)*.*

The proof is analogous to the proof in Proposition 3.1.1, where we just need to substitute the norm $L^\infty(V)$ by the norm $L^\infty(R^N)$.

Remark 3.1.3 *Thanks to hypothesis* (3.1.1) *in the previous Proposition 3.1.2, we have U_A^+ and U_A^- are bounded and hence they are supersolutions of 3.0.4 for some $m \in N$.*

Below we state some results for which under hypotheses in Proposition 3.1.2 we obtain that the supersolutions are supersolutions of the Ishii problem associated to a compact control set and then we can apply the results in (BARLES; BRIANI; CHASSEIGNE, 2013), obtaining some comparison results.

The following results gives a comparison result under hypothesis (3.1.1). If we have a bounded subsolution and a bounded supersolution, then they are ordered.

Corollary 3.1.3.1 *Under hypotheses of Proposition 3.1.2. Given u bounded subsolution of Ishii problem of* (3.0.1), (3.0.2) *and and v a bounded supersolution of* (3.0.1) *then*

$$u \leq v.$$

Due to Proposition 3.1.2, we have v is supersolution of (2.1.14) for some m. Applying Corollary 4.4 in (BARLES; BRIANI; CHASSEIGNE, 2013), we obtain that $u \leq v$.

The following result we prove that under assumption (3.1.1), we have U_A^- is a minimal supersolution of (3.0.1), U_A^+ is a maximal subsolution of the Ishii problem and the value function U_A^- is actually equal to the value function $U_{A^m}^-$ associated to a compact control set A^m.

Corollary 3.1.3.2 *Under hypotheses in Proposition 3.1.2.*

(a) The value function U_A^- is a minimal supersolution of the Ishii problem (3.0.1).

(b) The value function U_A^+ is a bounded maximal subsolution of (3.0.1).

(c) There exists $m \in N$ such that $U_A^- = U_{A^m}^-$.

(a) Thanks to Theorem 2.4.2, we have U_A^- is subsolution and supersolution of (3.0.1). Given v a bounded supersolution of (3.0.1), thanks to Corollary 3.1.3.1, we obtain that $U_A^- \leq v$.

(b) Thanks to the definition of A^m. Given u subsolution of (3.0.1) then u is subsolution of (3.0.4) for all m. Applying Corollary 4.4 in (BARLES; BRIANI; CHASSEIGNE, 2013), we obtain that $u \leq U_{A^m}^+$ for all $m \in N$. Then,

$$u \leq \inf_m U_{A^m}^+ = U_A^+.$$

(c) Thanks to Proposition 3.1.2, we have U_A^- is a supersolution of (2.1.14) for some $m \in N$, then applying Corollary 4.4 in (BARLES; BRIANI; CHASSEIGNE, 2013), we have $U_{A^m}^- \leq U_A^-$, but by definition $U_A^- \leq U_{A^m}^-$. Thus, the result.

3.2 LOCAL COMPARISON RESULTS

The Proposition 3.1.1, allows us to obtain a Local Comparison Result. To do this we need to define below the local version of the Ishii problem (3.0.1), (3.0.2) and (3.0.3) as well as of (3.0.4), (3.0.5) and (3.0.6).

Let $x \in R^N$ and $R > 0$. We denote

$$\Omega_i^R := B_R(x) \cap \Omega_i \qquad \text{and} \qquad \mathcal{H}^R := B_R(x) \cap \mathcal{H}.$$

We consider the local version of the Ishii problem (3.0.1), given by

$$\begin{cases} H_1\left(x, u(x), Du(x)\right) = 0 & \text{in } \Omega_1^R \\ H_2\left(x, u(x), Du(x)\right) = 0 & \text{in } \Omega_2^R \\ \max\{H_1\left(x, u(x), Du(x)\right),\ H_2\left(x, u(x), Du(x)\right)\} \geq 0 & \text{on } \mathcal{H}^R \\ \min\{H_1\left(x, u(x), Du(x)\right),\ H_2\left(x, u(x), Du(x)\right)\} \leq 0 & \text{on } \mathcal{H}^R. \end{cases} \tag{3.2.1}$$

We consider also the local version of the tangential Hamiltonians

$$H_T(x, \phi(x), D_{\mathcal{H}}\phi) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0(x)} -l_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) + \lambda\overline{\phi}\,(x) - b_{\mathcal{H}}\,(x, \alpha_1, \alpha_2, \mu)\cdot(D_{\mathcal{H}}\phi\,(x), 0) \text{ on } \mathcal{H}^R. \tag{3.2.2}$$

$$H_T^{reg}(x, \phi(x), D_{\mathcal{H}}\phi) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0^{reg}(x)} -l_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) + \lambda\overline{\phi}\,(x) - b_{\mathcal{H}}\,(x, \alpha_1, \alpha_2, \mu)\cdot(D_{\mathcal{H}}\phi\,(x), 0) \text{ on } \mathcal{H}^R. \tag{3.2.3}$$

Let us describe below the local version of the Ishii problem associated to compact control sets, (3.0.4), given by

$$\begin{cases} H_1^m\left(x, u(x), Du(x)\right) = 0 & \text{in } \Omega_1^R \\ H_2^m\left(x, u(x), Du(x)\right) = 0 & \text{in } \Omega_2^R \\ \max\{H_1^m\left(x, u(x), Du(x)\right),\ H_2^m\left(x, u(x), Du(x)\right)\} \geq 0 & \text{on } \mathcal{H}^R \\ \min\{H_1^m\left(x, u(x), Du(x)\right),\ H_2^m\left(x, u(x), Du(x)\right)\} \leq 0 & \text{on } \mathcal{H}^R. \end{cases} \tag{3.2.4}$$

We consider the operator H_T^m locally defined on $\mathcal{H}^R$ as follows

$$H_T^m(x,\phi(x),D_{\mathcal{H}}\phi) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0^m(x)} -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\overline{\phi}\,(x)-b_{\mathcal{H}}\,(x,\alpha_1,\alpha_2,\mu)\cdot(D_{\mathcal{H}}\phi\,(x),0) \text{ on } \mathcal{H}^R. \tag{3.2.5}$$

$$H_T^{reg,m}(x,\phi(x),D_{\mathcal{H}}\phi) = \sup_{(\alpha_1,\alpha_2,\mu)\in A_0^{reg,m}(x)} -l_{\mathcal{H}}(x,\alpha_1,\alpha_2,\mu)+\lambda\overline{\phi}\,(x)-b_{\mathcal{H}}\,(x,\alpha_1,\alpha_2,\mu)\cdot(D_{\mathcal{H}}\phi\,(x),0) \text{ on } \mathcal{H}^R. \tag{3.2.6}$$

The following result is a result from (BLANC, 2001) which will be used in this section to prove some comparison results.

In what follows, we assume that Y_x^i is a trajectory that satisfies

$$\dot{Y}_x^i(s) = b_i(Y_x^i(s),\alpha_i(s)) \quad \text{for} \quad s>0, \quad \text{and} \quad Y_x^i(0)=x.$$

Theorem 3.2.1 *Let $D_i \subset \Omega_i$ be a bounded open domain with C^1-boundary, let $x \in D_i$. If v is a locally bounded lsc supersolutions of* (3.2.4), *then*

$$v(x) \geq \inf_{\alpha(\cdot)\in\mathcal{A}_i^m,\theta_i}\left(\int_0^{t\wedge\theta_i} l_i(Y_x^i(s),\alpha(s))e^{-\lambda s}ds + v(Y_x^i(t\wedge\theta))e^{-\lambda(t\wedge\theta)}\right) \tag{3.2.7}$$

On the other the hand, if u is a locally bounded subsolutions of (3.2.4) *and* (3.2.5)

$$u(x) \leq \inf_{\alpha(\cdot)\in\mathcal{A}_i^m,\theta_i}\int_0^{t\wedge\theta_i} l_i(Y_x^i(s),\alpha(s))e^{-\lambda s}ds + u(Y_x^i(t\wedge\theta_i))e^{-\lambda(t\wedge\theta_i)} \tag{3.2.8}$$

where $t\wedge\theta_i = \min\{t,\ \theta_i\}$, θ_i is an stopping time such that $Y_x^i(\theta_i)\in\partial D_i$ and $\tau_i \leq \theta_i \leq \overline{\tau}_i$, where τ_i is the exit time of the trajectory Y_x^i from D_i and $\overline{\tau}_i$ is the exit time from $\overline{D}_i$. Moreover, the infimum is reached for some control $\alpha \in A_i^m$ and $\theta_i \in [\tau_i,\overline{\tau}_i]$.

Observe that if v satisfies (3.2.7), then it is a supersolution of

$$\begin{cases} w_t(t,x) + H_i^m(x,w(t,x),Dw(t,x)) = 0 & \text{in } D_i\times(0,\infty), \\ w(x,0) = v(x) & \text{on } \overline{D}_i, \\ w(x,t) = v(x) & \text{on } \partial D_i\times(0,\infty). \end{cases} \tag{3.2.9}$$

Besides that, the right hand side of (3.2.7) is the minimal supersolution of (3.2.9).

Observe also that if u satisfies (3.2.8), then it is a supersolution of

$$\begin{cases} w_t(t,x) + H_i^m(x,w(t,x),Dw(t,x)) = 0 & \text{in } D_i\times(0,\infty), \\ w(x,0) = u(x) & \text{on } \overline{D}_i, \\ w(x,t) = u(x) & \text{on } \partial D_i\times(0,\infty). \end{cases} \tag{3.2.10}$$

Moreover, the right hand side of (3.2.8) is the maximal subsolution of (3.2.10).

For the local problem in a ball of radius R centered in x, we have a Local Comparison Result. Indeed, fixed $m \in N$, we have the following local comparison result for the local Ishii problem associated to the compact control set A^m.

Theorem 3.2.2 *Assume hypotheses (HB) and (HC). Let $x \in R^N$, $R > 0$, let u be a bounded subsolutions of* (3.2.4) *and* (3.2.5)*, v be a bounded lsc supersolution of* (3.2.4) *in $B_R(x)$, we obtain*

$$\max_{B_R(x)}(u-v)^+ \leq \max_{\partial B_R(x)}(u-v)^+.$$

Before presenting the proof of Theorem 3.2.2, let us see its fundamental consequence, which is the local comparison result for the local version of the Ishii problem associated to the unbounded control set A.

Theorem 3.2.3 *Assume (HA), (HB) and (HC). Let $x \in R^N$, $R > 0$, let u be a locally bounded subsolution of* (3.2.1) *and* (3.2.2)*, let v be a locally bounded lsc supersolutions of* (3.2.1)*. Then*

$$\max_{B_R(x)}(u-v)^+ \leq \max_{\partial B_R(x)}(u-v)^+.$$

Thanks to the definition of A^m, since u is a locally bounded subsolution of (3.2.1) and (3.2.2), then u is a subsolution of (3.2.4) and (3.2.5). Furthermore, thanks to Proposition 3.1.1 we have v is a supersolution of (3.2.4) for some $m \in N$. Thus, u and v satisfy the hypotheses of Theorem 3.2.2 and the proof is complete.

To prove Theorem 3.2.2 we need some previous results. The next Theorem plays a fundamental role. It gives an alternative for the supersolutions, or they consider trajectories that stay on the hyperplane for a while or the supersolutions satisfy that the tangential Hamiltonian is greater or equal to zero.

Theorem 3.2.4 *Assume hypotheses (HB) and (HC). Let v be locally bounded lsc supersolution of* (3.2.4)*. Let $x_0 = (x'_0, 0) \in \mathcal{H}$. Fix $R > \bar{R} > 0$. Let $D_i \subset \Omega_i$ be a bounded open domain with C^1-boundary such that $B_{\bar{R}}(x_0) \cap \Omega_i \subset D_i \subset B_R(x_0) \cap \Omega_i$. Consider $\phi \in C^1(\overline{D}_i \cap \mathcal{H})$ such that $x' \longmapsto v(x', 0) - \phi(x')$ has a local minimum at x'_0. Then, either*

A) there exists $\eta > 0$, and a sequence $x_k \in D_i$ that converges to $x_0 = (x'_0, 0)$ as k goes to ∞, such that $v(x_k)$ converges to $v(x_0)$ as k goes to ∞, and, for each k, there exists a *control* $\alpha_i^k(\cdot) \in \mathcal{A}_i^m$ *such that the corresponding trajectory* $Y_{x_k}^i(s) \in \overline{D_i}$ *for all* $s \in [0, \eta]$ *and*

$$v(x_k) \geq \int_0^\eta l_i \left(Y_{x_k}^i(s), \alpha_i^k(s)\right) e^{-\lambda s} ds + v(Y_{x_k}^i(\eta))e^{-\lambda \eta}$$

or

B) $H_T^m\left(x_0, v\left(x_0\right), D_{\mathcal{H}}\phi(x_0)\right) \geq 0.$

We are going to prove that, if A does not hold, then we have B. To do so, we first claim that $(x_0', 0)$ is an strict local minimum of $v(x', 0) - \phi(x') + |x' - x_0'|^2$ in $B_r(x_0') \subset R^{N-1}$ for $0 < r < \bar{R}$. Then we define the function $\overline{\phi}(x', x_N) = \phi(x')$ and we have $D\overline{\phi}(x) = (D_{\mathcal{H}}\phi(x'), 0)$. For $\epsilon > 0$ and fixed $\delta \in R$, consider

$$\Gamma_\epsilon(x', x_N) := v(x', x_N) - \overline{\phi}(x', x_N) + |x - x_0|^2 - \delta x_N + \frac{x_N^2}{\epsilon^2}. \tag{3.2.11}$$

Since $\Gamma_\epsilon(x', x_N)$ is lsc and $\overline{B_r(x_0)} \subset R^N$ is compact, then Γ_ϵ achieves its minimum in $\overline{B_r(x_0)}$ in x_ϵ, and

$$\Gamma_\epsilon(x_\epsilon) \leq \Gamma_\epsilon(x_0) = v(x_0', 0) - \phi({x_0}') = C.$$

We are going show that $x_\epsilon \to x_0$ when $\epsilon \to 0$. Since $\Gamma_\epsilon(x_\epsilon) \leq C$, $(v - \overline{\phi})$ is bounded in $\overline{B_r(x_0)}$ and by (3.2.11), there exists a constant C' satisfying

$$0 \leq \frac{(x_\epsilon)_N^2}{\epsilon^2} \leq C' + \delta(x_\epsilon)_N - |x_\epsilon - x_0|^2 \leq C' + \delta(x_\epsilon)_N.$$

Thus

$$(x_\epsilon)_N^2 \leq C'\epsilon^2 + \epsilon^2\delta(x_\epsilon)_N.$$

Since $x_\epsilon \in \overline{B_r(x_0)}$ and δ is fixed, then $\lim_{\epsilon \to 0}(x_\epsilon)_N = 0$ and hence $x_\epsilon \to (y, 0)$ for some $y \in R^{N-1}$. Then, either $v(x_0', 0) - \phi({x_0}') \leq v(x_\epsilon) - \phi(x_\epsilon') + |x_\epsilon - x_0|^2$ for ϵ small enough or either there exists a subsequence satisfying $v(x_\epsilon) - \phi(x_\epsilon') + |x_\epsilon - x_0|^2 \leq v(x_0', 0) - \phi({x_0}')$.

Case a) if $v(x_0', 0) - \phi({x_0}') \leq v(x_\epsilon) - \phi(x_\epsilon') + |x_\epsilon - x_0|^2$ for ϵ small enough, then

$$v(x_0', 0) - \phi({x_0}') - \delta(x_\epsilon)_N + \frac{(x_\epsilon)_N^2}{\epsilon^2} \leq \Gamma_\epsilon(x_\epsilon) \leq \Gamma_\epsilon(x_0) = v(x_0', 0) - \phi({x_0}'). \tag{3.2.12}$$

Thus,

$$\frac{(x_\epsilon)_N^2}{\epsilon^2} \leq \delta(x_\epsilon)_N \implies \lim_{\epsilon \to 0} \frac{(x_\epsilon)_N^2}{\epsilon^2} = 0.$$

Therefore, $-\frac{(x_\epsilon)_N^2}{\epsilon^2} + \delta(x_\epsilon)_N$ converges to 0 and for ϵ small enough and there exists a constant C_Γ such that $|\Gamma_\epsilon(x_\epsilon)| < C_\Gamma$.

So there exists a convergent subsequence $(\Gamma_\epsilon(x_\epsilon))_\epsilon$ and we have the convergence as ϵ goes to zero of

$$v(x_\epsilon) - \phi(x'_\epsilon) = \Gamma_\epsilon(x_\epsilon) - \frac{(x_\epsilon)_N^2}{\epsilon^2} + \delta(x_\epsilon)_N.$$

Since v is lsc, thanks to (3.2.12) and $\lim_{\epsilon\to 0} x_\epsilon = (y,0)$, we have

$$\begin{aligned} v(x'_0,0) - \phi(x_0') &\geq \lim_{\epsilon\to 0} \Gamma_\epsilon(x_\epsilon) \\ &= \lim_{\epsilon\to 0} \left(v(x_\epsilon) - \phi(x'_\epsilon) + |x_\epsilon - x_0|^2 - \delta(x_\epsilon)_N + \tfrac{(x_\epsilon)_N^2}{\epsilon^2} \right) \\ &= \lim_{\epsilon\to 0} \left(v(x_\epsilon) - \overline{\phi}(x'_\epsilon,(x_\epsilon)_N) + |x_\epsilon - x_0|^2 \right) \\ &= \lim_{\epsilon\to 0} (v(x_\epsilon)) - \overline{\phi}(y,0) + |(y,0) - x_0|^2 \\ &\geq v(y,0) - \overline{\phi}(y,0) + |(y,0) - x_0|^2. \end{aligned} \tag{3.2.13}$$

Since x'_0 is strict local minimum of $v(x',0) - \overline{\phi}(x',0) + |x' - x'_0|^2$, then

$$v(x'_0,0) - \overline{\phi}(x'_0,0) = v(y,0) - \overline{\phi}(y,0), \text{ and } (y,0) = x_0. \tag{3.2.14}$$

Thhanks to (3.2.13) and (3.2.14), we have the result

$$\lim_{\epsilon\to 0} v(x_\epsilon) = v(x_0) \qquad \lim_{\epsilon\to 0} x_\epsilon = x_0.$$

Case b) If $v(x'_\epsilon,(x_\epsilon)_N) - \overline{\phi}(x'_\epsilon,(x_\epsilon)_N) + |x_\epsilon - x_0|^2 \leq v(x'_0,0) - \phi(x_0')$. Since v is lsc, taking limits in the inequality, we have

$$\begin{aligned} v(y,0) - \overline{\phi}(y,0) + |(y,0) - (x'_0,0)|^2 &\leq \lim\nolimits_{\epsilon\to 0}(v(x'_\epsilon,(x_\epsilon)_N) - \overline{\phi}(x'_\epsilon,(x_\epsilon)_N) + |x_\epsilon - x_0|^2) \\ &\leq v(x'_0,0) - \phi(x_0'). \end{aligned}$$

Since x'_0 is strict local minimum of $v(x',0) - \overline{\phi}(x',0) + |x' - x'_0|^2$, then

$$v(x'_0,0) - \overline{\phi}(x'_0,0) = v(y,0) - \overline{\phi}(y,0),$$

and $(y,0) = x_0$. Moreover

$$\lim_{\epsilon\to 0} v(x_\epsilon) = v(x_0).$$

Thus, we have proved that $\lim_{\epsilon\to 0} x_\epsilon = x_0$.

Let us consider the test function

$$\psi_\epsilon(x',x_N) := \overline{\phi}(x',x_N) + \delta x_N - \frac{x_N^2}{\epsilon^2} - |x - x_0|^2.$$

Case 1: Assume that the sequence $(x_\epsilon)_\epsilon$ that converges to x_0, satisfies that $x_\epsilon \in \mathcal{H}$, $\forall\, \delta \in R$. In this case, we will prove that B) occurs.

Thus, the sequence is given by $x_\epsilon = (x'_\epsilon, 0)$. By definition, x'_ϵ is the local minimum point of $v(x',0) - \phi(x') + |x - x_0|^2$ at $B_r(x'_0)$. Since x'_0 is a strict minimum point, we have $x'_\epsilon = x'_0$. Since v is a supersolution of (3.2.4), considering the minimum point $x'_\epsilon = x'_0$ of $v - \psi_\epsilon = \Gamma_\epsilon$, then

$$f(\delta) = \max\{H_1(x_0, v(x_0), (D_{\mathcal{H}}\phi(x_0), \delta)), H_2(x_0, v(x_0), (D_{\mathcal{H}}\phi(x_0), \delta))\} \geq 0$$

we have f is coercive and convex, since it is the maximum of H_1 and H_2 which are convex and coercive functions. In fact, $f(s) := H_i(x_0, v(x_0), (D_{\mathcal{H}}\phi(x_0), s))$ for $i = 1$ or $i = 2$, and there exists $\alpha_i \in A_i^m$ such that

$$\begin{aligned} f((1-\mu)\delta_1 + \mu\delta_2) \quad &= -b_i(x_0, \alpha_i) \cdot (D_{\mathcal{H}}\phi(x'_0), (1-\mu)\delta_1 + \mu\delta_2) - l_i(x_0, \alpha_i) + \lambda v(x_0) \\ &= (1-\mu)\left(-b_i(x_0, \alpha_i) \cdot (D_{\mathcal{H}}\phi(x'_0), \delta_1) - l_i(x_0, \alpha_i) + \lambda v(x_0)\right) \\ &+ \mu\left(-b_i(x_0, \alpha_i) \cdot (D_{\mathcal{H}}\phi(x'_0), \delta_2) - l_i(x_0, \alpha_i) + \lambda v(x_0)\right) \\ &\leq (1-\mu) f(\delta_1) + \mu f(\delta_2). \end{aligned}$$

Thus, f has a global minimum point at $\overline{\delta}$, and we have

$$0 \in \partial^- f(\overline{\delta}).$$

Let $\alpha \in A_1^m \cup A_2^m$ and consider a new function

$$f_\alpha(\delta) := -b(x_0, \alpha) \cdot (D_{\mathcal{H}}\phi(x'_0), \delta) - l(x_0, \alpha) + \lambda v(x_0)$$

where $(b, l) = (b_i, l_i)$, when $\alpha \in A_i^m$. Then we have

$$f(\delta) = \sup_{\alpha \in A_1^m \cup A_2^m} f_\alpha(\delta)$$

Since f_α is differentiable and $\partial^- f_\alpha(\delta) = b(x_0, \alpha) \cdot e_N$, by classical result of Convex Analysis

$$\partial^- f(\delta) = co\Big(\bigcup_{\alpha \in A_1^m \cup A_2^m,\ f_\alpha(\delta) = f(\delta)} \partial^- f_\alpha(\delta)\Big) = co\Big(\bigcup_{\alpha \in K,\ f_\alpha(\delta) = f(\delta)} b(x_0, \alpha) \cdot e_N\Big). \qquad (3.2.15)$$

Since $0 \in \partial^- f(\overline{\delta})$, by (3.2.15) there exist $r_1, \cdots, r_k \geq 0$ with $\sum_{i=1}^k r_i = 1$ such that

$$0 = \left(\sum_{i=1}^k r_i b(x_0, \alpha_{r_i})\right) \cdot e_N = \left(\sum_{i=1}^{k_1} r_{1i} b_1(x_0, \alpha_{r_{1i}}) + \sum_{j=1}^{k_2} r_{2j} b_2(x_0, \alpha_{r_{2j}})\right) \cdot e_N,$$

with $k_1 + k_2 = k$. Let $\mu := \sum_{i=1}^{k_1} r_{1i}$ and $1 - \mu = \sum_{j=1}^{k_2} r_{2j}$. By convexity of $\mathcal{B}_i(x_0)$, there exist $\alpha_1^* \in A_1^m,\ \alpha_2^* \in A_2^m$ with $(\alpha_1^*, \alpha_2^*, \mu) \in A_0(x_0)$, such that

$$0 = \left(\mu \sum_{i=1}^{k_1} \frac{r_{1i}}{\mu} b_1(x_0, \alpha_{r_{1i}}) + (1-\mu) \sum_{j=1}^{k_2} \frac{r_{2j}}{1-\mu} b_2(x_0, \alpha_{r_{2j}})\right) \cdot e_N = (\mu b_1(x_0, \alpha_1^*) + (1-\mu) b_2(x_0, \alpha_2^*)) \cdot e_N.$$

Moreover, thanks to (3.2.15), we have for all $i = 1, \dots, k_1$ and $j = 1, \dots, k_2$

$$f(\overline{\delta}) = -b_1(x_0, \alpha_{r_{1i}}) \cdot (D_{\mathcal{H}}\phi(x_0'), \overline{\delta}) - l_1(x_0, \alpha_{r_{1i}}) + \lambda v(x_0)$$

$$f(\overline{\delta}) = -b_2(x_0, \alpha_{r_{2j}}) \cdot (D_{\mathcal{H}}\phi(x_0'), \overline{\delta}) - l_2(x_0, \alpha_{r_{2j}}) + \lambda v(x_0)$$

Since $(\alpha_1^*, \alpha_2^*, \mu) \in A_0(x)$, then $b_{\mathcal{H}}(x_0, \alpha_1^*, \alpha_2^*, \mu) \cdot (\overline{\delta} e_N) = 0$. Thus

$$\begin{aligned}
0 \ &\leq f(\overline{\delta}) = \textstyle\sum_{i=1}^{k} f(\overline{\delta}) r_i \\
&= \Big(-\textstyle\sum_{i=1}^{k_1} r_{1i} b_1(x_0, \alpha_{r_{1i}})\Big) \cdot (D_{\mathcal{H}}\phi(x_0'), \overline{\delta}) - \Big(\textstyle\sum_{i=1}^{k_1} r_{1i} l_1(x_0, \alpha_{r_{1i}})\Big) + \Big((\textstyle\sum_{i=1}^{k_1} r_{1i} \lambda v(x_0)\Big) \\
&+ \Big(-\textstyle\sum_{i=1}^{k_2} r_{2j} b_2(x_0, \alpha_{r_{2j}})\Big) \cdot (D_{\mathcal{H}}\phi(x_0'), \overline{\delta}) - \Big(\textstyle\sum_{i=1}^{k_2} r_{2j} l_1(x_0, \alpha_{r_{2j}})\Big) + \Big((\textstyle\sum_{i=1}^{k_2} r_{2j} \lambda v(x_0)\Big) \\
&= -\mu b_1(x_0, \alpha_1^*) \cdot (D_{\mathcal{H}}\phi(x_0'), \overline{\delta}) - \mu l_1(x_0, \alpha_1^*) + \mu \lambda v(x_0) \\
&-(1-\mu) b_2(x_0, \alpha_2^*) \cdot (D_{\mathcal{H}}\phi(x_0'), \overline{\delta}) - (1-\mu) l_2(x_0, \alpha_2^*) + (1-\mu)\lambda v(x_0) \\
&= -b_{\mathcal{H}}(x_0, \alpha_1^*, \alpha_2^*, \mu) \cdot (D_{\mathcal{H}}\phi(x_0'), 0) - l_{\mathcal{H}}(x_0, \alpha_1^*, \alpha_2^*, \mu) + \lambda v(x_0) \\
&\leq H_T(x_0, v(x_0), D_{\mathcal{H}}\phi(x_0'))
\end{aligned}$$

Thus, we are on case B).

The arguments in Case 1 work whenever $f(\overline{\delta}) \geq 0$ even if x_ϵ is not in $\mathcal{H}$.

Case 2: Let $f(\overline{\delta}) < 0$ and $x_\epsilon \in B_{\bar{R}}(x_0) \setminus \mathcal{H}$ for ϵ small enough, then $x_\epsilon \in D_i$. Thanks to Theorem 3.2.1 in the domain D_i we have

$$v(x_\epsilon) \geq \inf_{\alpha^\epsilon(\cdot), \theta_\epsilon} \int_0^{t \wedge \theta_\epsilon} l_i(Y^i_{x_\epsilon}(s), \alpha^\epsilon(s)) e^{-\lambda s} ds + v(Y^i_{x_\epsilon}(t \wedge \theta_\epsilon)) e^{-\lambda(t \wedge \theta_\epsilon)}. \quad (3.2.16)$$

Assume that the infimum in (3.2.16) is reached for $(\alpha^\epsilon(\cdot), \theta_\epsilon)$ which exists due to the convex hypothesis (HC), then

$$v(x_\epsilon) \geq \int_0^{t \wedge \theta_\epsilon} l_i(Y^i_{x_\epsilon}(s), \alpha^\epsilon(s)) e^{-\lambda s} ds + v(Y^i_{x_\epsilon}(t \wedge \theta_\epsilon)) e^{-\lambda(t \wedge \theta_\epsilon)}.$$

Since we are assuming that A) does not hold, we have $\lim_{\epsilon \to 0} \theta_\epsilon = 0$. Then for ϵ small enough

$$v(x_\epsilon) \geq \int_0^{\theta_\epsilon} l_i(Y^i_{x_\epsilon}(s), \alpha^\epsilon(s)) e^{-\lambda s} ds + v(Y^i_{x_\epsilon}(\theta_\epsilon)) e^{-\lambda \theta_\epsilon}. \quad (3.2.17)$$

We claim that for ϵ small enough $Y^i_{x_\epsilon}(\theta_\epsilon) \notin \partial D_i \setminus \mathcal{H}$. This follows from the fact that there exists $M > 0$ such that $||b_i||_{L^\infty(\overline{D_i} \times A_i^m)} \leq M$, $\lim_{\epsilon \to 0} x_\epsilon = x_0$, $\lim_{\epsilon \to 0} \theta_\epsilon = 0$ and $B_{\bar{R}}(x_0) \cap \Omega_i \subset D_i$. Whence $Y^i_{x_\epsilon}(\theta_\epsilon) \in \mathcal{H} \cap \partial D_i$.

Now we subtract $\psi_\epsilon(x_\epsilon)$ in (3.2.17) and we assume that $v(x_\epsilon) - \psi_\epsilon(x_\epsilon) = 0$ then

$$0 = v(x_\epsilon) - \psi_\epsilon(x_\epsilon) \geq \int_0^{\theta_\epsilon} l_i(Y^i_{x_\epsilon}(s), \alpha^\epsilon(s)) e^{-\lambda s} ds + v(Y^i_{x_\epsilon}(\theta_\epsilon)) e^{-\lambda(\theta_\epsilon)} - \psi_\epsilon(x_\epsilon). \quad (3.2.18)$$

Since x_ϵ is the where the minimum of $v-\psi_\epsilon$ is attained, then

$$0=-v(x_\epsilon)+\psi_\epsilon(x_\epsilon)\geq -v(Y^i_{x_\epsilon}(\theta_\epsilon))+\psi_\epsilon(Y^i_{x_\epsilon}(\theta_\epsilon)).$$

Thus adding $-v(Y^i_{x_\epsilon}(\theta_\epsilon))+\psi_\epsilon(Y^i_{x_\epsilon}(\theta_\epsilon))$ to (3.2.18), and by the Fundamental Calculus Theorem, we obtain

$$\begin{aligned}&0\geq \int_0^{\theta_\epsilon} l_i(Y^i_{x_\epsilon}(s),\alpha^\epsilon(s))e^{-\lambda s}ds+v(Y^i_{x_\epsilon}(\theta_\epsilon))(e^{-\lambda(\theta_\epsilon)}-1)+(\psi_\epsilon(Y^i_{x_\epsilon}(\theta_\epsilon))-\psi_\epsilon(x_\epsilon))\\&=\int_0^{\theta_\epsilon} l_i(Y^i_{x_\epsilon}(s),\alpha^\epsilon(s))e^{-\lambda s}ds+v(Y^i_{x_\epsilon}(t_\epsilon))(e^{-\lambda\theta_\epsilon}-1)+\int_0^{\theta_\epsilon}\Big[D_{\mathcal{H}}\phi(Y^i_{x_\epsilon}(s))+\delta e_N\Big]\cdot b_i(Y^i_{x_\epsilon}(s),\alpha^\epsilon(s))ds\\&-\frac{(Y^i_{x_\epsilon}(\theta_\epsilon))^2_N}{\epsilon^2}+\frac{(x_\epsilon)^2_N}{\epsilon^2}\end{aligned}$$

Since $-\frac{(Y^i_{x_\epsilon}(\theta_\epsilon))^2_N}{\epsilon^2}+\frac{(x_\epsilon)^2_N}{\epsilon^2}\geq 0$, because $(Y^i_{x_\epsilon}(\theta_\epsilon))_N=0$. Then,

$$\begin{aligned}0\ &\leq \int_0^{\theta_\epsilon} -l_i(Y^i_{x_\epsilon}(s),\alpha^\epsilon(s))e^{-\lambda s}-b_i(Y^i_{x_\epsilon}(s),\alpha^\epsilon(s))\cdot\Big[D_{\mathcal{H}}\phi(Y^i_{x_\epsilon}(s))+\delta)e_N\Big]ds-v(Y^i_{x_\epsilon}(\theta_\epsilon))(e^{-\lambda\theta_\epsilon}-1)\\&\leq\int_0^{\theta_\epsilon}\sup_{\alpha^\epsilon\in\mathcal{A}^m_i}\Big(-l_i(Y^i_{x_\epsilon}(s),\alpha^\epsilon)e^{-\lambda s}-b_i(Y^i_{x_\epsilon}(s),\alpha^\epsilon)\cdot\Big[D_{\mathcal{H}}\phi(Y^i_{x_\epsilon}(s))+\delta)e_N\Big]\Big)ds\\&-v(Y^i_{x_\epsilon}(\theta_\epsilon))(e^{-\lambda\theta_\epsilon}-1).\end{aligned}\tag{3.2.19}$$

For $s\leq\theta_\epsilon$, there exists $M>0$ such that $||b||_{L^\infty(\overline{D_i}\times A^m_i)}<M$. Thus,

$$|Y^i_{x_\epsilon}(s)-x_0|\leq|x_0-x_\epsilon|+|Y^i_{x_\epsilon}(s)-x_\epsilon|\leq|x_0-x_\epsilon|+M\theta_\epsilon.$$

Denote $r_\epsilon=|x_0-x_\epsilon|+M\theta_\epsilon$. Since A^m_i is compact, then the function

$$\zeta(y,t):=\sup_{\alpha_i\in A^m_i}\Big[-l_i(y,\alpha_i)e^{-\lambda t}-b_i(y,\alpha_i)\cdot[D_{\mathcal{H}}\phi(y)+\delta e_N]\Big]$$

is continuous. Moreover, since $s\leq\theta_\epsilon$ then $(Y^i_{x_\epsilon}(s),s)\to(x_0,0)$ when ϵ goes to 0, ans thanks to the continuity of the function ζ, there exists $R_\epsilon>0$, such that

$$|\zeta(Y^i_{x_\epsilon}(s),s)-\zeta(x_0,0)|<R_\epsilon,\quad \zeta(Y^i_{x_\epsilon}(s),s)\leq\zeta(x_0,0)+R_\epsilon$$

and

$$\lim_{\epsilon\to0}R_\epsilon=0.$$

Rewriting (3.2.19), we have

$$\begin{aligned}0\ &\leq\int_0^{\theta_\epsilon}\zeta(Y^i_{x_\epsilon}(s),s)ds-v(Y^i_{x_\epsilon}(\theta_\epsilon))(e^{-\lambda\theta_\epsilon}-1)\\&\leq\int_0^{\theta_\epsilon}(\zeta(x_0,0)+R_\epsilon)ds-v(Y^i_{x_\epsilon}(\theta_\epsilon))(e^{-\lambda\theta_\epsilon}-1)\\&=(\zeta(x_0,0)+R_\epsilon)\,\theta_\epsilon-v(Y^i_{x_\epsilon}(\theta_\epsilon))(e^{-\lambda\theta_\epsilon}-1)\end{aligned}\tag{3.2.20}$$

We recall that $\lim_{\epsilon\to 0} v(x_\epsilon) = v(x_0)$ and $\lim_{\epsilon\to 0}\theta_\epsilon = 0$. Taking limits as ϵ goes to 0 in (3.2.17), we have $v(x_0) \geq \lim_{\epsilon\to 0} v(Y^i_{x_\epsilon}(\theta_\epsilon))$. Moreover since v is lsc then $\lim_{\epsilon\to 0} v(Y^i_{x_\epsilon}(\theta_\epsilon)) \geq v(x_0)$. Thus $\lim_{\epsilon\to 0} v(Y^i_{x_\epsilon}(\theta_\epsilon)) = v(x_0)$. Dividing (3.2.20) by θ_ϵ and taking limits as ϵ goes to 0, we obtain

$$0 \leq \zeta(x_0, 0) + \lambda v(x_0) = H_i\left(x_0, v(x_0), D_{\mathcal{H}}\phi(x_0') + \delta e_N\right).$$

Therefore, $f(\overline{\delta}) \geq 0$ which is a contradiction. Thus, the proof is complete.

To prove the Theorem 3.2.2 we will also need to regularize the subsolution, we will do this via convolution. Let

$$V = B_R(0) \quad \text{ and } \quad V_\epsilon := \{x \in V |\ dist(x, \partial V) > 2\epsilon\}.$$

In the following result we will consider the convolution of u with a $C^\infty_C(R^{N-1})$ function. We will denote this convolution by u_ϵ. In particular, we can define it as follows. Let $\rho \in C^\infty_C(R^{N-1})$, with $\int_{R^{N-1}} \rho(y)dy = 1$ and $supp(\rho) \subset B_1(0)$. We define

$$\rho_\epsilon(y) := \frac{1}{\epsilon^{N-1}}\rho(\frac{y}{\epsilon}).$$

For any $x \in V_\epsilon$, we define

$$u_\epsilon(x) := \int_{R^{N-1}} u(x' - y, x_N)\rho_\epsilon(y)dy.$$

We will need the following lemma which states that any convex combination of subsolutions is a subsolution, under suitable convex hypotheses on the operator that defines the equation.

Lemma 3.2.5 *Let $\mathcal{V}$ be an open set of $R^N \times R \times R^N$. Assume that $G : \mathcal{V} \to R$ is l.s.c. and G is convex in the last two variables, that is, for $\mu_1 + \mu_2 = 1$*

$$G(x,\, \mu_1 r_1 + \mu_2 r_2,\, \mu_1 p_1 + \mu_2 p_2) \leq \mu_1 G(x,\, r_1,\, p_1) + \mu_2 G(x,\, r_2,\, p_2).$$

Then any convex combination of Lipschitz continuous subsolutions of $G = 0$ in $\mathcal{V}$ is a subsolution of $G = 0$ in $\mathcal{V}$.

We only have to prove the result for a convex combination of two subsolutions ω_1 and ω_2 of $G = 0$, given by

$$\omega := \mu_1\omega_1 + \mu_2\omega_2,$$

with $\mu_1 + \mu_2 = 1$. The general case involving n subsolutions for $n > 2$ derives immediately by iteration of the result. We assume without loss of generality that μ_1 and μ_2 are strictly positive.

Let ϕ be a smooth test-function and $\tilde{x} \in B_r(\overline{x})$ be a local strict maximum point of $w - \phi$ in $\overline{B_{\tilde{r}}(\tilde{x})} \subset B_r(\overline{x})$. We use a tripling of variables by considering in $\overline{B_{\tilde{r}}(\tilde{x})} \times \overline{B_{\tilde{r}}(\tilde{x})} \times \overline{B_{\tilde{r}}(\tilde{x})}$ the function

$$\Psi(x_1, x_2, x) := \mu_1\omega_1(x_1) + \mu_2\omega_2(x_2) - \phi(x) - \mu_1\frac{|x_1 - x|^2}{\epsilon} - \mu_2\frac{|x_2 - x|^2}{\epsilon}.$$

Let $(x_1^\epsilon, x_2^\epsilon, x^\epsilon)$ be a local maximum point of Ψ. we have $(x_1^\epsilon, x_2^\epsilon, x^\epsilon)$ converges to $(\tilde{x}, \tilde{x}, \tilde{x})$ as ϵ goes to 0. Thus, x_1^ϵ is local maximum point of

$$\mu_1\omega_1(x_1) + \mu_2\omega_2(x_2^\epsilon) - \phi(x^\epsilon) - \mu_1\frac{|x_1 - x^\epsilon|^2}{\epsilon} - \mu_2\frac{|x_2^\epsilon - x^\epsilon|^2}{\epsilon}.$$

Hence, x_1^ϵ is local maximum point of

$$\omega_1(x_1) + \frac{\mu_2}{\mu_1}\omega_2(x_2^\epsilon) - \frac{\phi(x^\epsilon)}{\mu_1} - \frac{|x_1 - x^\epsilon|^2}{\epsilon} - \frac{\mu_2}{\mu_1}\frac{|x_2^\epsilon - x^\epsilon|^2}{\epsilon}.$$

Then $\nabla\omega_1$ is given by

$$\nabla\left(-\frac{\mu_2}{\mu_1}\omega_2(x_2^\epsilon) + \frac{\phi(x^\epsilon)}{\mu_1} + \frac{|x_1 - x^\epsilon|^2}{\epsilon} + \frac{\mu_2}{\mu_1}\frac{|x_2^\epsilon - x^\epsilon|^2}{\epsilon}\right) = \frac{2(x_1 - x^\epsilon)}{\epsilon}. \tag{3.2.21}$$

Since ω_1 is a subsolution of $G = 0$, and thanks to (3.2.21), we have

$$G\left(x_1^\epsilon, \omega_1(x_1^\epsilon), \frac{2(x_1 - x^\epsilon)}{\epsilon}\right) \leq 0. \tag{3.2.22}$$

Analogously, we obtain that

$$G\left(x_2^\epsilon, \omega_2(x_2^\epsilon), \frac{2(x_2 - x^\epsilon)}{\epsilon}\right) \leq 0. \tag{3.2.23}$$

Finally x^ϵ is a local maximum point of

$$\mu_1\omega_1(x_1^\epsilon) + \mu_2\omega_2(x_2^\epsilon) - \phi(x) - \mu_1\frac{|x_1^\epsilon - x|^2}{\epsilon} - \mu_2\frac{|x_2^\epsilon - x|^2}{\epsilon}.$$

Whence,

$$\nabla\phi(x^\epsilon) = \mu_1\frac{2(x_1^\epsilon - x^\epsilon)}{\epsilon} + \mu_2\frac{2(x_2^\epsilon - x^\epsilon)}{\epsilon}. \tag{3.2.24}$$

Since w_i is Lipschitz continuous due to (BARLES, 1994, Th. 2.2 p. 17), $|\frac{2(x_i^\epsilon - x^\epsilon)}{\epsilon}| \leq C_i$, where C_i is the Lipschitz constant of w_i. Extracting if necessary a subsequence, we can assume that the gradient $|\frac{2(x_i^\epsilon - x^\epsilon)}{\epsilon}|$ converges to a function that we denote by P_i as ϵ goes to 0.

Taking limits as ϵ goes to 0 in (3.2.22) and (3.2.23), and since G is lower semi-continuous, we obtain that

$$G(\tilde{x}, \omega_1(\tilde{x}), P_1) \leq 0 \text{ and } G(\tilde{x}, \omega_2(\tilde{x}), P_2) \leq 0. \tag{3.2.25}$$

Because of the continuity of $\nabla\phi$, taking limits as ϵ goes to 0 in (3.2.24), we obtain

$$\nabla\phi(\tilde{x}) = \mu_1 P_1 + \mu_2 P_2.$$

So, thanks to (3.2.25) and the convexity of G, we finally get

$$\begin{aligned} G(\tilde{x}, \omega(\tilde{x}), \nabla\phi(\tilde{x})) &= G(\tilde{x}, \mu_1\omega_1(\tilde{x}) + \mu_2\omega_2(\tilde{x}), \mu_1 P_1 + \mu_2 P_2) \\ &\le \mu_1 G(\tilde{x}, \omega_1(\tilde{x}), P_1) + \mu_2 G(\tilde{x}, \omega_2(\tilde{x}), P_2) \\ &\le 0, \end{aligned}$$

which proves that ω is a viscosity subsolution of $G = 0$.

In the following result we proof that the Hamiltonians associated to compact control sets, H_i^m and H_T^m, are convex, continuous and we give some bounds of these Hamiltonians.

Lemma 3.2.6 *Under hypotheses (HB) and (HC). H_i^m is continuous, convex in the last variable and satisfies that there exists $M > 0$ such that*

$$|H_i^m(x, u, p) - H_i^m(x_1, u, p)| \le M|x - x_1||p| \quad \text{for } x, x_1 \in \Omega_i \text{ and } (u, p) \in R \times R^N. \tag{3.2.26}$$

Moreover, H_T^m is continuous, convex in the last variable and satisfies

$$|H_T^m(z, u, q) - H_T^m(\overline{z}, u, q)| \le M|z - \overline{z}||q| \quad \text{for } z, \overline{z} \in \mathcal{H} \text{ and } (u, q) \in R \times R^{N-1}. \tag{3.2.27}$$

Let $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2, \bar{\mu}) \in A_0^m(\bar{z})$ such that

$$\tilde{H}_T^m(\bar{z}, q) = -b_{\mathcal{H}}(\bar{z}, \bar{\alpha}) \cdot q - l_{\mathcal{H}}(\bar{z}, \bar{\alpha}).$$

We will show that there exist $\tilde{\alpha} \in A_0(z)$ such that

$$\begin{cases} b_{\mathcal{H}}(\bar{z}, \bar{\alpha}) = b_{\mathcal{H}}(z, \tilde{\alpha}) + \sigma(|\bar{z} - z|) \\ l_{\mathcal{H}}(\bar{z}, \bar{\alpha}) = l_{\mathcal{H}}(z, \tilde{\alpha}) + \gamma(|\bar{z} - z|) \end{cases} \tag{3.2.28}$$

with σ and γ going to 0 when $|\bar{z} - z|$ goes to 0. We will prove this approach for $b_{\mathcal{H}}$, the proof for $l_{\mathcal{H}}$ is analogous. First, let us define

$$\tilde{\sigma}_i(|z - \bar{z}|) := b_i(\bar{z}, \bar{\alpha}_i) - b_i(z, \bar{\alpha}_i).$$

Since b_i is Lipschitz, then $|\tilde{\sigma}_i(|z - \bar{z}|)| \le C_{b_i}|z - \bar{z}|$, where C_{b_i} is the Lipschitz constant of b_i. Consider $\beta := \min\{\frac{\delta - |\tilde{\sigma}_1(z - \bar{z})|}{\delta}, \frac{\delta - |\tilde{\sigma}_2(z - \bar{z})|}{\delta}\}$ and define

$$\sigma_i(|z - \bar{z}|) := (1 - \beta)(b_i(z, \bar{\alpha}_i) + \tilde{\sigma}_i(|z - \bar{z}|)). \tag{3.2.29}$$

Note that

$$|\sigma_i(|z-\bar{z}|)| \leq (1-\beta)(\|b_i\|_{L^\infty(V\times A_i^m)} + |\tilde{\sigma}_i(|z-\bar{z}|)|).$$

Thus, $\sigma_i(|z-\bar{z}|)$ is going to 0 when $|z-\bar{z}|$ goes to 0, because $1-\beta = \max\{\frac{|\tilde{\sigma}_1(z-\bar{z})|}{\delta}, \frac{|\tilde{\sigma}_2(z-\bar{z})|}{\delta}\}$.

Now, we proof that there exists $\tilde{\alpha}_i \in A_i^m$ satisfying

$$b_i(\bar{z},\bar{\alpha}_i) = b_i(z,\tilde{\alpha}_i) + \sigma_i(|z-\bar{z}|).$$

Indeed, by definition of $\tilde{\sigma}_i(|z-\bar{z}|)$, we have $|\frac{\beta\tilde{\sigma}_i(|z-\bar{z}|)}{1-\beta}| \leq \delta$, and there exists $\tilde{\alpha}_i \in A_i^m$ such that

$$\begin{aligned} \beta b_i(\bar{z},\bar{\alpha}_i) &= \beta(b_i(z,\bar{\alpha}_i) + \tilde{\sigma}_i(|z-\bar{z}|)) \\ &= \beta b_i(z,\bar{\alpha}_i) + (1-\beta)\tfrac{\beta\tilde{\sigma}_i(|z-\bar{z}|)}{1-\beta} \\ &= b_i(z,\tilde{\alpha}_i). \end{aligned} \tag{3.2.30}$$

Thanks to (3.2.29) and (3.2.30), we obtain

$$\begin{aligned} b_i(\bar{z},\bar{\alpha}_i) &= \beta b_i(\bar{z},\bar{\alpha}_i) + (1-\beta) b_i(\bar{z},\bar{\alpha}_i) \\ &= b_i(z,\tilde{\alpha}_i) + (1-\beta)(b_i(z,\bar{\alpha}_i) + \tilde{\sigma}_i(|z-\bar{z}|)) \\ &= b_i(z,\tilde{\alpha}_i) + \sigma_i(|\bar{z}-z|). \end{aligned}$$

Furthermore, we have

$$[\bar{\mu} b_1(z,\tilde{\alpha}_1) + (1-\bar{\mu}) b_2(z,\tilde{\alpha}_2)] \cdot e_N = \beta\,[\bar{\mu} b_1(\bar{z},\bar{\alpha}_1) + (1-\bar{\mu}) b_2(\bar{z},\bar{\alpha}_2)] \cdot e_N = 0.$$

Hence, $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \bar{\mu}) \in A_0^m(z)$ and

$$b_{\mathcal{H}}(\bar{z},\bar{\alpha}) = b_{\mathcal{H}}(z,\tilde{\alpha}) + \sigma(|\bar{z}-z|)$$

where $\sigma = \bar{\mu}\sigma_1 + (1-\bar{\mu})\sigma_2$. Therefore,

$$\begin{aligned} \tilde{H}_T^m(\bar{z},q) &= -b_{\mathcal{H}}(\bar{z},\bar{\alpha}) \cdot q - l_{\mathcal{H}}(\bar{z},\bar{\alpha}) \\ &= -b_{\mathcal{H}}(z,\tilde{\alpha}) \cdot q - \sigma(|\bar{z}-z|) \cdot q - l_{\mathcal{H}}(z,\tilde{\alpha}) - \gamma(|\bar{z}-z|) \\ &\leq \tilde{H}_T^m(z,q) - \sigma(|\bar{z}-z|) \cdot q - \gamma(|\bar{z}-z|). \end{aligned}$$

Symmetrically,

$$\tilde{H}_T^m(z,q) \leq \tilde{H}_T^m(\bar{z},q) - \bar{\sigma}(|\bar{z}-z|) \cdot q - \bar{\gamma}(|\bar{z}-z|)$$

with $\bar{\sigma}$ and $\bar{\gamma}$ going to 0 when $|z-\bar{z}|$ goes to 0.

$$|\tilde{H}_T^m(z,q) - \tilde{H}_T^m(\bar{z},q)| \leq \max\{(|\bar{\sigma}(|\bar{z}-z|)||q| + |\bar{\gamma}(|\bar{z}-z|)|), (\sigma(|\bar{z}-z|)||q| + |\gamma(|\bar{z}-z|)|)\}. \tag{3.2.31}$$

We denote by $m(|\bar{z}-z|)$ the right hand side of the inequality (3.2.31). we have m goes to 0 when $|\bar{z}-z|$ goes to 0. Thus,

$$\tilde{H}^m_T(z,q) \le \tilde{H}^m_T(\bar{z},q) + m(|\bar{z}-z|), \quad \tilde{H}^m_T(\bar{z},q) \le \tilde{H}^m_T(z,q) + m(|\bar{z}-z|).$$

In the following result we prove that if u is a subsolution of the Ishii problem associated to a compact control set locally, in a ball of radius R, then the convolution u_ϵ minus an term that goes to zero as ϵ goes to zero, is also a subsolution of the Ishii problem in a ball with a slightly smaller radius.

Lemma 3.2.7 *We assume hypotheses (HB) and (HC). Let u be a locally Lipschitz subsolution of* (3.2.4), (3.2.5) *in the ball of radius R. Let $\epsilon > 0$. There exists a function $\zeta : R^+ \to R^+$ with $\lim_{s\to 0^+} \zeta(s) = 0$ such that the function $u_\epsilon - \zeta(\epsilon)$ is a viscosity subsolution of* (3.2.4), (3.2.5) *in the ball of radius $\overline{R}$ with $\overline{R} = R - 2\epsilon$.*

In this proof, we denote H^m_T and H^m_i by G. we have $u_\epsilon(x) := \int_{R^{N-1}} u(x'-y, x_N)\rho_\epsilon(y)dy$ is smooth in the first $N-1$ variables. Let us approximate $u_\epsilon(x)$ by Riemann sums

$$u^n_\epsilon(x) := \sum_{j=1}^n V_j u(x'-e_j, x_N)\rho_\epsilon(e_j) = \sum_{j=1}^n \mu_j u(x'-e_j, x_N)$$

where $e_j \in B_\epsilon(0)$, and $\mu_j := V_j\rho_\epsilon(e_j)$ and V_j is the volume of the j-th block of the partition. Thus,

$$\lim_{n\to\infty} \sum_{j=1}^n \mu_j = \lim_{n\to\infty} \sum_{j=1}^n V_j\rho_\epsilon(e_j) = \int_{B_\epsilon(0)} \rho_\epsilon(y)dy = 1.$$

We exchange μ_n by $\overline{\mu}_n$ such that $\overline{\mu}_n + \sum_{i=1}^{n-1}\mu_i = 1$. Clearly $\overline{\mu}_n - \mu_n$ goes to 0 when n goes to ∞. The function

$$\tilde{u}_j(x', x_N) := u(x'-e_j, x_N)$$

approximates well a subsolution. Indeed, consider the test function ϕ and (x'_0, x^0_N) a local maximum point of

$$\tilde{u}_j(x', x_N) - \phi(x', x_N).$$

Consider $\tilde{\phi}(x', x_N) := \phi(x'+e_j, x_N)$, then $\nabla\tilde{\phi}(x', x_N) = \nabla\phi(x'+e_j, x_N)$. Thus, (x'_0, x^0_N) is a local maximum point of

$$\tilde{u}_j(x', x_N) - \tilde{\phi}(x'-e_j, x_N) = u(x'-e_j, x_N) - \tilde{\phi}(x'-e_j, x_N).$$

Hence, $(x'_0 - e_j, x^0_N)$ is local maximum point of $u - \tilde{\phi}$. Since u is a subsolution of G then

$$G((x'_0-e_j, x^0_N), u(x'_0-e_j, x^0_N), \nabla\tilde{\phi}(x'_0-e_j, x^0_N)) \le 0$$

where $\nabla\tilde{\phi}((x_0' - e_j, x_N^0)) = \nabla\phi(x_0', x_N^0)$. Thus,

$$G((x_0' - e_j, x_N^0), \tilde{u}_j(x_0', x_N^0), \nabla\phi(x_0', x_N^0)) \leq 0. \tag{3.2.32}$$

Since $e_j \in B_\epsilon(0)$ and thanks to Lemma 3.2.6, we have

$$|G((x_0', x_N^0), \tilde{u}_j(x_0', x_N^0), \nabla\phi(x_0', x_N^0)) - G((x_0' - e_j, x_N^0), \tilde{u}(x_0', x_N^0), \nabla\phi(x_0', x_N^0))| \leq \eta(\epsilon). \tag{3.2.33}$$

where $\eta(\epsilon)$ goes to 0 when ϵ goes to 0. From (3.2.32) and (3.2.33), we obtain,

$$G((x_0', x_N^0), \tilde{u}_j(x_0', x_N^0), \nabla\phi(x_0', x_N^0)) - \eta(\epsilon) \leq 0.$$

That is, $\tilde{u}_j$ is subsolution of $G - \eta(\epsilon)$. Thanks to Lemma 3.2.5, the convex combination defined by $\overline{u}_\eta^n := \sum_{j=1}^{n-1} \mu_j \tilde{u}_j + \overline{\mu}_n \tilde{u}_n$ is subsolution of $G - \eta(\epsilon)$. Taking limits as n goes to infinity, we have

$$\lim_{n\to\infty} \overline{u}_\eta^n(x', x_N) = u_\epsilon(x', x_N) = \lim_{n\to\infty} u_\epsilon^n(x', x_N).$$

Moreover note that u_η^n and $\overline{u}_\eta^n$ are equi-Lipschitz equi-bounded and consequently the convergence is uniform. By stability, (BARLES, 1994, Th 2.3 p.21), u_ϵ is a subsolution of $G - \eta(\epsilon)$. So, we can take $\zeta(\epsilon) := \frac{\eta(\epsilon)}{\lambda}$.

Let us prove below the main local comparison result of this section.

Proof of Theorem 3.2.2. Let $V_\epsilon := \{x \in B_R(x) |\ dist(x, \partial B_R(x)) > 2\epsilon\}$, and define

$$\bar{u}_\epsilon := u_\epsilon - \zeta(\epsilon),$$

where ζ is the function in Lemma 3.2.7. Let

$$M_\epsilon := \sup_{\overline{V}_\epsilon} (\bar{u}_\epsilon - v).$$

Since $\bar{u}_\epsilon - v$ is usc then the supreme is achieved on the compact $\overline{V}_\epsilon$. Thus, there exists $\overline{x}_\epsilon \in \overline{V}_\epsilon$ such that $M_\epsilon = \bar{u}_\epsilon(\overline{x}_\epsilon) - v(\overline{x}_\epsilon)$. If $M_\epsilon \leq 0$ we have the result. So let us assume that $M_\epsilon > 0$.

Case 1: $\overline{x}_\epsilon \in \Omega_i \cap V_\epsilon$.

There exists $r > 0$ such that $B_r(\overline{x}_\epsilon) \subset \Omega_i \cap V_\epsilon$. Consider the usc function

$$\xi_\beta(x, y) := \bar{u}_\epsilon(x) - v(y) - \frac{|x - y|^2}{\beta^2} - |\overline{x}_\epsilon - x|^2.$$

Let (x_β, y_β) be the maximum point of ξ_β in $\overline{B_r(\overline{x}_\epsilon)} \times \overline{B_r(\overline{x}_\epsilon)}$ and denote

$$M_\epsilon^\beta = \xi_\beta(x_\beta, y_\beta) = \bar{u}_\epsilon(x_\beta) - v(y_\beta) - \frac{|x_\beta - y_\beta|^2}{\beta^2} - |\overline{x}_\epsilon - x_\beta|^2 \tag{3.2.34}$$

Thus

$$\begin{aligned} 0 \quad &< M_\epsilon = \bar{u}_\epsilon(\overline{x}_\epsilon) - v(\overline{x}_\epsilon) - \tfrac{|\overline{x}_\epsilon - \overline{x}_\epsilon|^2}{\beta^2} - |\overline{x}_\epsilon - \overline{x}_\epsilon|^2 \\ &\leq \bar{u}_\epsilon(x_\beta) - v(y_\beta) - \tfrac{|x_\beta - y_\beta|^2}{\beta^2} - |\overline{x}_\epsilon - x_\beta|^2 \\ &< 2C - \tfrac{|x_\beta - y_\beta|^2}{\beta^2} - |\overline{x}_\epsilon - x_\beta|^2 \end{aligned}$$

where $C > \max\{||\bar{u}_\epsilon||_{L^\infty(V_\epsilon)}, ||v||_{L^\infty(V_\epsilon)}\}$. Consequently,

$$\frac{|x_\beta - y_\beta|^2}{\beta^2} + |\overline{x}_\epsilon - x_\beta|^2 < 2C \implies |x_\beta - y_\beta| \leq \sqrt{2C}\beta \tag{3.2.35}$$

That is why, $\lim_{\beta\to 0} x_\beta = z = \lim_{\beta \to 0} y_\beta$. Moreover from (3.2.34), we have

$$\begin{aligned} M_\epsilon \quad &\leq M_\epsilon^\beta \\ &\leq \bar{u}_\epsilon(x_\beta) - v(y_\beta) - |\overline{x}_\epsilon - x_\beta|^2 \end{aligned}.$$

Taking limits as β goes to 0 and since $-v$ is usc and $\bar{u}_\epsilon$ is differentiable, we obtain

$$M_\epsilon \leq \lim_{\beta\to 0} M_\epsilon^\beta = \bar{u}_\epsilon(z) - v(z) - |\overline{x}_\epsilon - z|^2 \leq \bar{u}_\epsilon(z) - v(z).$$

Since $\overline{x}_\epsilon$ is an strict maximum of $\bar{u}_\epsilon(x) - v(x) - |\bar{x}_\epsilon - x|^2$ in $\overline{V}_\epsilon$, then $z = \overline{x}_\epsilon$ and

$$\lim_{\beta\to 0} M_\epsilon^\beta = M_\epsilon. \tag{3.2.36}$$

Thus, taking limits as β goes to 0 in (3.2.34), we have

$$\lim_{\beta\to 0} \frac{|x_\beta - y_\beta|^2}{\beta^2} = 0. \tag{3.2.37}$$

We prove below that x_β is the maximum point of $\bar{u}_\epsilon(x) - \left(v(y_\beta) + \frac{|x-y_\beta|^2}{\beta^2} + |\overline{x}_\epsilon - x_\beta|^2\right)$ in $B_r(\bar{x})$ and y_β is a minimum point of $v(y) - \left(\bar{u}_\epsilon(x_\beta) - \frac{|y-x_\beta|^2}{\beta^2} - |\overline{x}_\epsilon - x_\beta|^2\right)$ in $B_r(\bar{x})$. By definition, $\frac{|x_\beta - y_\beta|^2}{\beta^2} + |\overline{x}_\epsilon - x_\beta| + M_\epsilon^\beta = \bar{u}_\epsilon(x_\beta) - v(y_\beta)$. Since $\lim_{\beta\to 0} M_\epsilon^\beta = M_\epsilon > 0$, then $M_\epsilon^\beta > 0$ for β small enough then $\bar{u}_\epsilon(x_\beta) - v(y_\beta) > 0$.

Since $\bar{u}_\epsilon,\ v$ are respectively subsolution and supersolution of H_i, and b_i and l_i are Lipchitz, we obtain

$$
\begin{aligned}
0 &< \lambda\left(\tfrac{|x_\beta-y_\beta|^2}{\beta^2}+|\overline{x}_\epsilon-x_\beta|+M_\epsilon^\beta\right)\\
&= \lambda(\bar{u}_\epsilon(x_\beta)-v(y_\beta))\\
&= H_i\left(x_\beta,\bar{u}_\epsilon(x_\beta),\tfrac{2(x_\beta-y_\beta)}{\beta^2}\right)-H_i\left(x_\beta,v(y_\beta),\tfrac{2(x_\beta-y_\beta)}{\beta^2}\right)\\
&\leq H_i\left(y_\beta,v(y_\beta),\tfrac{2(x_\beta-y_\beta)}{\beta^2}\right)-H_i\left(x_\beta,v(y_\beta),\tfrac{2(x_\beta-y_\beta)}{\beta^2}\right)\\
&\leq -b_i(y_\beta,\alpha_y)\cdot\tfrac{2(x_\beta-y_\beta)}{\beta^2}+l_i(y_\beta,\alpha_y)-(-b_i(x_\beta,\alpha_y)\cdot\tfrac{2(x_\beta-y_\beta)}{\beta^2}+l_i(x_\beta,\alpha_y))\\
&\leq C|x_\beta-y_\beta|\left(\left|\tfrac{2(x_\beta-y_\beta)}{\beta^2}\right|+1\right)\\
&= C\left(\tfrac{2|x_\beta-y_\beta|^2}{\beta^2}+|x_\beta-y_\beta|\right),
\end{aligned}
$$

where $\alpha_y \in A_i^m$ is the control for which the supremum of $H_i\left(y_\beta, v(y_\beta), \frac{2(x_\beta-y_\beta)}{\beta^2}\right)$ is attained. Since $\lim_{\beta\to 0} M_\epsilon^\beta = M_\epsilon$, $\lim_{\beta\to 0}\frac{|x_\beta-y_\beta|^2}{\beta^2}=0$, $\lim_{\beta\to 0}|x_\beta-y_\beta|=0$ and $\lim_{\beta\to 0}|x_\beta-\bar{x}|=0$, taking the limit as β goes to 0, we obtain a contradiction.

Case 2: $\overline{x}_\epsilon \in \mathcal{H}\cap V_\epsilon$

Recall that for r small enough, $B_r(x_\epsilon)\cap\Omega_i \subset V_\epsilon$. Since $\bar{u}_\epsilon$ is a subsolution and it is differentiable with respect to the first $N-1$ variables, we have

$$H_T(\overline{x}_\epsilon, \bar{u}_\epsilon(\overline{x}_\epsilon), D_{\mathcal{H}}\bar{u}_\epsilon(\overline{x}_\epsilon)) \leq 0.$$

Now we consider a set D_i as in Theorem 3.2.4. We apply this Theorem for $x_0=\overline{x}_\epsilon$, $0<\bar{r}<r$ such that $B_{\bar{r}}(\overline{x}_\epsilon)\cap\Omega_i\subset D_i\subset B_r(\overline{x}_\epsilon)\cap\Omega_i$.

Either we are in case **B)** $H_T(\overline{x}_\epsilon, v(\overline{x}_\epsilon), D_{\mathcal{H}}(\overline{x}_\epsilon))\geq 0$. Therefore,

$$\lambda(\bar{u}_\epsilon(\overline{x}_\epsilon)-v(\overline{x}_\epsilon)) = H_T(\overline{x}_\epsilon,\bar{u}_\epsilon(\overline{x}_\epsilon),D_{\mathcal{H}}\bar{u}_\epsilon(\overline{x}_\epsilon)) - H_T(\overline{x}_\epsilon, v(\overline{x}_\epsilon), D_{\mathcal{H}}\bar{u}_\epsilon(\overline{x}_\epsilon)) \leq 0.$$

Thus, $M_\epsilon \leq 0$, what leads to contradiction.

Or either we are in case **A)**. Then there exists a sequence $x_k\to\overline{x}_\epsilon$ and $v(x_k)\to v(\overline{x}_\epsilon)$ such that

$$v(x_k) \geq \int_0^\eta l_i\left(Y_{x_k}^i(s),\alpha_i^k(t)\right)+v(Y_{x_k}^i(\eta))e^{-\lambda\eta} \tag{3.2.38}$$

with $Y_{x_k}^i(s)\in\overline{D}_i$ for $s\in[0,\eta]$. Consequently the exit time from $\overline{D}_i$ of this trajectory is greater than η. Since $\bar{u}_\epsilon$ is subsolution of (3.2.4), and thanks to Theorem 3.2.1, we know that

$$\bar{u}_\epsilon(x_k) \leq \int_0^\eta l_i\left(Y_{x_k}^i(s),\alpha_i^k(t)\right)+\bar{u}_\epsilon(Y_{x_k}^i(\eta))e^{-\lambda\eta}. \tag{3.2.39}$$

Subtracting inequalities (3.2.38) and (3.2.39), we obtain

$$\bar{u}_\epsilon(x_k) - v(x_k) \leq (\bar{u}_\epsilon(Y^i_{x_k}(\eta)) - v(Y^i_{x_k}(\eta)))e^{-\lambda\eta} \leq M_\epsilon e^{-\lambda\eta}. \tag{3.2.40}$$

Taking limits in (3.2.40) as k goes to infinity, we have

$$M_\epsilon \leq M_\epsilon e^{-\lambda\eta}.$$

which is a contradiction. Thus, if $M_\epsilon = \bar{u}_\epsilon(\overline{x}_\epsilon) - v(\overline{x}_\epsilon) > 0$ then $\overline{x}_\epsilon \in \partial V_\epsilon$, that is

$$\max_{V_\epsilon}(\bar{u}_\epsilon - v)^+ < \max_{\partial V_\epsilon}(\bar{u}_\epsilon - v)^+$$

Since $\bar{u}_\epsilon$ is continuous then letting ϵ go to 0 , we obtain

$$\max_{V}(\bar{u} - v)^+ \leq \max_{\partial V}(\bar{u} - v)^+.$$

.

3.3 GLOBAL COMPARISON RESULTS

To obtain global comparison result in R^N, we need two extra hypotheses that we denote by LOC 1 and LOC 2. Let $\mathcal{C}_*$ be a set of subsolutions with certain growing hypotheses and let $\mathcal{C}^*$ a set of supersolutions with certain growing hypotheses, which will be specified in the examples in section 3.5.

LOC 1: Given a subsolution $u \in \mathcal{C}_*$ and supersolution $v \in \mathcal{C}^*$ then there exists a sequence of subsolutions $(u_\beta)_\beta$ such that for all $x \in R^N$,

$$\lim_{|x|\to\infty}(u_\beta - v)(x) = -\infty \quad \text{and} \quad \lim_{\beta\to 0} u_\beta(x) = u(x).$$

LOC 2: Let $x \in R^N$, $r > 0$, and consider u_β of LOC 1. There exists a sequence $(u_{\beta\gamma})_\gamma$ of subsolutions such that

$$u_{\beta\gamma}(x) - u_\beta(x) \geq u_{\beta\gamma}(y) - u_\beta(y) + d(\gamma) \quad \text{for all} \quad y \in \partial B_r(x) \quad \text{with} \quad d(\gamma) > 0.$$

Moreover, $u_{\beta\gamma}(z) \to u_\beta(z)$ as $\gamma \to 0$ for any $z \in B_r(x)$.

The following result states that if we have local comparison results as in Theorem 3.2.3, then we have also global comparison results for the problem (3.0.1).

Theorem 3.3.1 *Under the hypotheses LOC 1 and LOC 2, Local Comparison Result implies Global Comparison Result, that is*

$$u(x) \leq v(x) \quad \forall x \in R^N.$$

Thanks to LOC 1 there exists $x_\beta \in R^N$ such that

$$M_\beta := \max_{R^N}(u_\beta - v) = u_\beta(x_\beta) - v(x_\beta).$$

Assume by contradiction that $M_\beta > 0$. Thanks to LOC2, there exists a sequence of subsolutions $(u_{\beta\gamma})_\gamma$ that is an approximation of u_β. We will use Local Comparison Results with $u_{\beta\gamma}$ and v on $B_R(x_\beta)$, that is

$$\max_{B_R(x_\beta)}(u_{\beta\gamma} - v)^+ \leq \max_{\partial B_R(x_\beta)}(u_{\beta\gamma} - v)^+.$$

We divide the proof in two cases:

Case 1 $\max_{\overline{B_R(x_\beta)}}(u_{\beta\gamma} - v) \leq 0$.

Since $M_\beta > 0$, and thanks to LOC 2, we have $\lim_{\gamma\to} u_{\beta\gamma}(x_\beta) = u_\beta(x_\beta)$, then for γ small enough we have a contradiction with $\max_{\overline{B_R(x_\beta)}}(u_{\beta\gamma} - v) \leq 0$.

Case 2 $\max_{\overline{B_R(x_\beta)}}(u_{\beta\gamma} - v) > 0$. Thanks to the Local Comparison Result and hypothesis LOC2, we have

$$\begin{aligned} u_{\beta\gamma}(x_\beta) - v(x_\beta) &\leq \max_{\partial B_R(x_\beta)}(u_{\beta\gamma} - v) \\ &\leq \max_{\partial B_R(x_\beta)}(u_\beta - v) + \max_{\partial B_R(x_\beta)}(u_{\beta\gamma} - u_\beta) \\ &\leq M_\beta + (u_{\beta\gamma}(x_\beta) - u_\beta(x_\beta)) - d(\gamma). \end{aligned}$$

Consequently,

$$M_\beta = u_\beta(x_\beta) - v(x_\beta) \leq M_\beta - d(\gamma),$$

which is a contradiction. Therefore, $M_\beta \leq 0$ and $u_\beta(x) \leq v(x)$ for any $x \in R^N$. Taking limits as β goes to 0, we obtain that $u(x) \leq v(x)$ for any $x \in R^N$.

3.4 U_A^- MINIMAL SUPERSOLUTION AND U_A^+ MAXIMAL SUBSOLUTION OF ISHII PROBLEM

In the result below we prove that U_A^- is a minimal supersolution of the Ishii problem.

Proposition 3.4.1 *Under the hypotheses (HA), (HB), (HC), LOC 1 and LOC 2, U_A^- is a minimal locally bounded lsc supersolution of the Ishii problem* (3.0.1)*.*

U_A^- is subsolution of (3.0.1) and (3.0.2), then given a l.s.c supersolution v of (3.0.1). Due to Theorem 3.2.3, we obtain Local Comparison and Theorem 3.3.1 provide us the Global Comparison Result, that is $U_A^-(x) \leq v(x)$ for all $x \in R^N$.

Now, we will introduce several results that will be used to prove that U_A^+ is a maximal subsolution of the Ishii problem.

The result below can be found in (BARLES; BRIANI; CHASSEIGNE, 2013). It states that any subsolution of the Ishii problem is a subsolution of the regular tangential Hamiltonian.

Theorem 3.4.2 *Let u be a subsolution locally bounded of* (3.0.1) *then u is subsolution of* (3.0.3)*.*

See (BARLES; BRIANI; CHASSEIGNE, 2013, Th. 3.1).

The following theorem is similar to Theorem 3.2.4. It gives an alternative for $U_{A^m}^+$, or it considers trajectories that stay on the hyperplane for a while or it satisfies that the regular tangential Hamiltonian is greater or equal to zero. The difference here is that it considers the regular tangential Hamiltonian, and the result is only valid for the supersolution $U_{A^m}^+$.

Theorem 3.4.3 *Assume hypothesis (HC). Let $x_0 = (x_0', 0) \in \mathcal{H}$. Fix $R > \bar{R} > 0$. Let $D_i \subset \Omega_i$ be an open bounded domain with C^1-boundary and such that $D_i \cap \mathcal{H} \neq \emptyset$, and $B_{\bar{R}}(x_0) \cap \Omega_i \subset D_i \subset B_R(x_0) \cap \Omega_i$. Consider $\phi \in C^1(D_i \cap \mathcal{H})$ such that $x' \longmapsto U_{A^m}^+(x', 0) - \phi(x')$ has a local minimum at x_0'. Then, either*

A) there exists $\eta > 0$, and a control $\alpha_i(\cdot) \in \mathcal{A}_i^m$ such that the corresponding trajectory $Y_{x_0}^i(s) \in D_i$ for all $s \in (0, \eta]$ and

$$U_{A^m}^+(x_0) \geq \int_0^\eta l_i\left(Y_{x_0}^i(s), \alpha_i(s)\right) e^{-\lambda s} ds + U_{A^m}^+(Y_{x_0}^i(\eta)) e^{-\lambda \eta}$$

or

B) $(H_T^m)^{reg}\left(x_0, U_{A^m}^+(x_0), D_{\mathcal{H}}\phi(x_0)\right) \geq 0.$

By Lemma 2.2.7 there exists T and $(X_{x_0}^T, \alpha^T) \in \tau_{A^m}^{reg}(x_0)$ such that for $s \leq T$

$$U_{A^m}^+(x_0) = \int_0^s l(X_{x_0}^T(t), \alpha^T(t)) e^{-\lambda t} dt + U_{A^m}^+\left(X_{x_0}^T(s)\right) e^{-\lambda s}.$$

If case A) does not hold then there exists sequence t_k that goes to 0 as k goes to ∞ such that $X_{x_0}^T(t_k) \in \partial D_i$. However, for k large enough $X_{x_0}^T(t_k) \notin \partial D_i \setminus \mathcal{H}$. This follows from the fact that there exists $M > 0$ such that $||b_i||_{L^\infty(\overline{D_i}\times A_i^m)} \leq M$, $d(x_0, \partial D_i \setminus \mathcal{H}) > \bar{R}$ and $(B_{\bar{R}}(x_0) \cap \Omega_i) \subset D_i$. Whence $X_{x_0}^T(t_k) \in \mathcal{H} \cap \partial D_i$. Consequently,

$$U_{A^m}^+(x_0) = \int_0^{t_k} l(X_{x_0}^T(t), \alpha^T(t))e^{-\lambda t}dt + U_A^+\left(X_{x_0}^T(t_k)\right)e^{-\lambda t_k}.$$

We assume without lost of generality that the minimum $U_{A^m}^+(x_0', 0) - \phi(x_0')$ is equal to 0, then

$$\phi(x_0') \geq \int_0^{t_k} l(X_{x_0}^T(t), \alpha^T(t))e^{-\lambda t}dt + \phi\left(X_{x_0}^T(t_k)\right)e^{-\lambda t_k}.$$

Let us define

$$A[\phi](s) := \dot{X}_{x_0}^T(s) \cdot \nabla\phi(X_{x_0}^T(s)) + \lambda\phi(X_{x_0}^T(s)) - l(X_{x_0}^T(s), \alpha^T(s)).$$

Then

$$\int_0^{t_k} A[\phi](s)ds \geq 0. \tag{3.4.1}$$

Now, assume by contradiction that

$$H_T^{m,reg}\left(x_0, U_{A^m}^+(x_0), D_{\mathcal{H}}\phi(x_0)\right) < 0. \tag{3.4.2}$$

Define $\varepsilon_i^k := \{s \in (0, t_k) : X_{x_0}^T(s) \in (B_{\bar{R}}(x_0) \cap \Omega_i)\}$ and $\varepsilon_{\mathcal{H}}^k := \{s \in (0, t_k) : X_{x_0}^T(s) \in (B_{\bar{R}}(x_0) \cap \mathcal{H})\}$. Since $X_{x_0}^T$ is regular and thanks to Lemma 3.2.6, $H_T^{m,reg}$ is continuous, then by (3.4.2), for k large enough, we obtain

$$\int_0^{t_k} A[\phi](s)1_{\varepsilon_{\mathcal{H}}^k}(s)ds \leq \int_0^{t_k} H_T^{m,reg}\left(X_{x_0}^T(s), U_{A^m}^+\left(X_{x_0}^T(s)\right), D_{\mathcal{H}}\phi(X_{x_0}^T(s))\right)ds < 0. \tag{3.4.3}$$

On the other the hand, since ε_i^k is an open set then

$$\varepsilon_i^k = \bigcup_{j=1}^{n_k^i}(c_{i,j}^k, d_{i,j}^k)$$

with $X_{x_0}^T(c_{i,j}^k), X_{x_0}^T(d_{i,j}^k) \in \mathcal{H}$ and $\dot{X}_{x_0}^T = b_i(X_{x_0}^T, \alpha)$ on $(c_{i,j}^k, d_{i,j}^k)$. Consider $d(x) := x_N$, then

$$0 = d(X_{x_0}^T(d_{i,j}^k)) - d(X_{x_0}^T(c_{i,j}^k)) = \int_{c_{i,j}^k}^{d_{i,j}^k} b_i(X_{x_0}^T(s), \alpha(s)) \cdot e_N ds. \tag{3.4.4}$$

Furthermore,

$$\int_{c_{i,j}^k}^{d_{i,j}^k} b_i(X_{x_0}^T(s), \alpha(s))ds = \int_{c_{i,j}^k}^{d_{i,j}^k} b_i(X_{x_0}^T(s), \alpha(s)) - b_i(x_0, \alpha(s))ds + \int_{c_{i,j}^k}^{d_{i,j}^k} b_i(x_0, \alpha(s))ds.$$

Since

$$\left|\int_{c_{i,j}^k}^{d_{i,j}^k} b_i(X_{x_0}^T(s), \alpha(s)) - b_i(x_0, \alpha(s))ds\right| \leq C_b||X_{x_0}^T(\cdot) - x_0||(d_{i,j}^k - c_{i,j}^k).$$

Therefore,

$$\int_{c_{i,j}^k}^{d_{i,j}^k} b_i(X_{x_0}^T(s),\alpha(s))ds = \int_{c_{i,j}^k}^{d_{i,j}^k} b_i(x_0,\alpha(s))ds + O(k) \tag{3.4.5}$$

with $\frac{O(k)}{(d_{i,j}^k - c_{i,j}^k)} \to 0$ when $k \to \infty$. Now we calculate the Riemann integral $\int_{c_{i,j}^k}^{d_{i,j}^k} b_i(x,\alpha(s))ds$. For each k. We partition $(d_{i,j}^k - c_{i,j}^k)$ in n equal parts, $I_1, \cdots, I_n$ of length $\frac{d_{i,j}^k - c_{i,j}^k}{n}$. For each $z \le n$, we choose $s^z \in I_z$. By convexity of $\mathcal{B}_i(x_0)$ there exists $\alpha_i^{k,n*} \in A_i^m$ such that

$$\sum_{z=1}^n \frac{d_{i,j}^k - c_{i,j}^k}{n} b_i(x_0,\alpha(s^z)) = (d_{i,j}^k - c_{i,j}^k)\sum_{z=1}^n \frac{1}{n} b_i(x_0,\alpha(s^z)) = (d_{i,j}^k - c_{i,j}^k) b_i(x_0,\alpha_{i,j}^{k,n*})$$

Since $\mathcal{B}_i(x_0)$ is a closed set, taking limits as n goes to infinity, we find $\alpha_{i,j}^{k*} \in A_i^m$ such that

$$\int_{c_{i,j}^k}^{d_{i,j}^k} b_i(x_0,\alpha(s))ds = (d_{i,j}^k - c_{i,j}^k) b_i(x_0,\alpha_{i,j}^{k*}) \tag{3.4.6}$$

By (3.4.6), (3.4.4) and taking the inner product in (3.4.5) with e_N, we obtain

$$b_{i,j,N}^{k*} := -b_i(x_0,\alpha_{i,j}^{k*})\cdot e_N = -\frac{O(k)\cdot e_N}{(d_{i,j}^k - c_{i,j}^k)} \to 0. \tag{3.4.7}$$

The above definition, plays a key role in approximating $\alpha_{i,j}^{k*}$ by regular controls. In fact, let $\delta > 0$ and define

$$\beta_j^k := \min\left\{\frac{\delta - 2|b_{1,j,N}^{k*}|}{\delta}, \frac{\delta - 2|b_{2,j,N}^{k*}|}{\delta}\right\}.$$

Since $b_{i,j,N}^{k*}$ goes to 0 when $k \to \infty$ then $\beta_j^k \to 1$ when $k \to \infty$.

If $\beta_j^k = 1$, then $|b_{1,j,N}^{k*}| = 0 = |b_{2,j,N}^{k*}|$. Thus, we can take

$$\alpha_{i,j}^{k**} := \alpha_{i,j}^{k*}, \quad \text{such that} \quad b_i(x_0,\alpha_{i,j}^{k**})\cdot e_N = 0.$$

On the other hand, if $\beta_j^k \neq 1$. Since for k large enough $|\beta_j^k| \le 1$ and $\frac{|b_{i,j,N}^{k*}|}{1-\beta_j^k} \le \frac{\delta}{2}$, then by convexity of $\mathcal{B}_i(x_0)$, there exists $\alpha_{i,j}^{k**} \in A_i^m$ satisfying

$$\begin{aligned} b_i(x_0,\alpha_{i,j}^{k**}) &:= \beta_j^k b_i(x_0,\alpha_{i,j}^{k*}(s)) + (1-\beta_j^k)\left(\tfrac{\beta_j^k b_{i,j,N}^{k*}\ e_N}{1-\beta_j^k}\right) \\ &= \beta_j^k\left(b_i(x_0,\alpha_{i,j}^{k*}) + b_{i,j,N}^{*k}\ e_N\right). \end{aligned}$$

Therefore, due to (3.4.7), $b_i(x_0,\alpha_{i,j}^{k**})\cdot e_N = 0$, then $\alpha_{i,j}^{k**} \in A_0^{reg}(x_0)$.

$$\begin{aligned} \int_{c_{i,j}^k}^{d_{i,j}^k} A[\phi](s)ds &= (d_{i,j}^k - c_{i,j}^k)\left(b_i(x_0,\alpha_{i,j}^{k**}(s))\cdot\nabla\phi(x_0) + \lambda\phi(x_0) - l_i(x_0,\alpha_{i,j}^{k**}(s))\right) + O(k) \\ &\le (d_{i,j}^k - c_{i,j}^k) H_T^{m,reg}\left(x_0, U_{A^m}^+(x_0), D_{\mathcal{H}}\phi(x_0)\right) + O(k). \end{aligned}$$

Due to (3.4.2), for k large enough

$$\int_{c_{i,j}^k}^{d_{i,j}^k} A[\phi](s)ds < 0. \tag{3.4.8}$$

From (3.4.3) and (3.4.8) we obtain that,

$$\int_0^{t_k} A[\phi](s)ds < 0$$

which is a contradiction with (3.4.1).

In the following result we obtain a local comparison result for $U_{A^m}^+$ in the ball of radius R, where we compare $U_{A^m}^+$ with any subsolution of the Ishii problem, and not with subsolutions of the Ishii problem and the tangential Hamiltonian at the same time, as we did in Theorem 3.2.2.

Theorem 3.4.4 *Assume hypotheses (HB) and (HC). Let u be a locally bounded subsolution of* (3.0.4). *Let $x \in R^N$, $R > 0$, we obtain*

$$\max_{B_R(x)} (u - U_{A^m}^+)^+ \leq \max_{\partial B_R(x)} (u - U_{A^m}^+)^+.$$

The proof is analogous to the one in Theorem 3.2.2.

As before, we regularize u via convolution, obtaining u_ϵ in $V_\epsilon := \{x \in B_R(x) | \, dist(x, \partial B_R(x)) > 2\epsilon\}$. Define $\bar{u}_\epsilon := u_\epsilon - \zeta(\epsilon)$, where ζ is as in Lemma 3.2.7, and $M_\epsilon := \sup_{\overline{V}_\epsilon}(\bar{u}_\epsilon - U_{A^m}^+)$. Since $\bar{u}_\epsilon - U_{A^m}^+$ is continuous functions then the supreme is achieved on the compact $\overline{V}_\epsilon$. Thus, there exists $x_\epsilon \in \overline{V}_\epsilon$ such that $M_\epsilon = \bar{u}_\epsilon(x_\epsilon) - U_{A^m}^+(x_\epsilon)$. Let us assume that $M_\epsilon > 0$. We divide the proof in two cases:

Case 1: $x_\epsilon \in \Omega_i \cap V_\epsilon$, and case 2: $x_\epsilon \in \mathcal{H} \cap V_\epsilon$.

The proof is the same as in Theorem 3.2.2 for case 1. For case 2, we change H_T^m by $H_T^{m,reg}$. Thanks to Theorem 3.4.2, since $\bar{u}_\epsilon$ is a subsolution of the Ishii problem (3.0.4), then we have

$$H_T^{m,reg}(x_\epsilon, \bar{u}_\epsilon(x_\epsilon), D_{\mathcal{H}}\bar{u}_\epsilon(x_\epsilon)) \leq 0.$$

Then we apply Theorem 3.4.3, and the rest of the proof is analogous.

The following results states that U_A^+ is the maximal subsolution of the Ishii problem.

Corollary 3.4.4.1 *Assume hypotheses (HA), (HB), (HC), LOC 1 and LOC 2. Let u be a locally bounded subsolution of the Ishii problem* (3.0.1). *Then U_A^+ is maximal subsolution of* (3.0.1), *that is*

$$u(x) \leq U_A^+(x) \quad \text{for all} \quad x \in R^N.$$

Let u be a locally bounded subsolution of 3.0.1 then u is a subsolution of the Ishii problem associated to the compact control set A^m, 3.0.4, for all $m \in N$. Thanks to Theorem 3.4.4 and Theorem 3.3.1, we have $u(x) \leq U^+_{A^m}(x)$ for all $x \in R^N$. Taking limits as m goes to infinity we obtain $u(x) \leq U^+_A(x)$ for all $x \in R^N$.

3.5 EXAMPLE OF NONLINEAR HAMILTONIAN WITH UNBOUNDED ASSOCIATED CONTROL SET

Let us consider the following nonlinear Hamiltonian which has associated unbounded control sets

$$H_i(x,u,p) = \lambda u + d_i(x)^\xi |p|^\xi - f_i(x).$$

For this example, we consider the dynamic function

$$b_i(x,\alpha_i) := c_\xi d(x)|\alpha_i|^{\xi-2}\alpha_i, \text{ where } 0 < d_i(x) \leq |x|^\beta \text{ is locally Lipschitz and } c_\xi = \frac{\xi}{\sqrt[\xi]{(\xi-1)^{\xi-1}}} \tag{3.5.1}$$

and the cost function

$$l_i(x,\alpha_i) := f_i(x) + |\alpha_i|^\xi \text{ with } 0 \leq f_i(x) \leq C_f|x|^{\zeta-\varepsilon} + \overline{C}_f, \text{ with } f_i \text{ continuous in } \overline{\Omega}_i.$$

We consider the constant λ satisfying

$$\lambda \geq \zeta^\xi.$$

Observe that b_i and l_i satisfy hypothesis (HA), (HB) and (HC). Indeed $b_i(x,0) = 0$, and given v in R^N different from zero and $x \in R^N$, there exists a control α_i such that $b_i(x,\alpha_i) = v$, where α_i is given by

$$\alpha_i := \left(\frac{|v|}{c_\xi d(x)}\right)^{\frac{1}{\xi-1}} \frac{v}{|v|}$$

This implies that, $R^N \subset \mathcal{B}_i(x)$ for any $x \in R^N$, and hypothesis (HC) is satisfied. Let us verify that hypothesis (HA) is satisfied,

$$\lim_{|\alpha_i|\to\infty} \frac{l_i(x,\alpha_i)}{1+|b_i(x,\alpha_i)|} = \lim_{|\alpha_i|\to\infty} \frac{f_i(x)+|\alpha_i|^\xi}{c_\xi d(x)|\alpha_i|^{\xi-1}} = \infty.$$

Let $m \in (0,\infty)$, we consider the compact control sets

$$A^m := B_m(0) \subset R^N,$$

then we denote

$$B_i^m(x) := \{b_i(x,\alpha_i) \ | \alpha_i \in B_m(0)\}$$

and

$$L_i^m(x) := \{l_i(x,\alpha_i)\ |\alpha_i \in B_m(0)\}.$$

The sets $B_i^m(x)$ and $L_i^m(x)$ are convex, but $BL_i^m(x) := \{(b_i(x,\alpha_i), l_i(x,\alpha_i))|\alpha_i \in B_m(0)\}$ may not be convex, what is necessary in hypothesis (HC). Indeed, let $\alpha_i, \overline{\alpha}_i \in B_m(0)$

$$(1-\mu)b_i(x,\alpha_i) + \mu b_i(x,\overline{\alpha}_i) = b_i(x,\alpha_i^{b*})$$

$$(1-\mu)l_i(x,\alpha_i) + \mu l_i(x,\overline{\alpha}_i) = l_i(x,\alpha_i^{l*})$$

where

$$|\alpha_i^{b*}| = (|(1-\mu)|\alpha_i|^{\xi-2}\alpha_i + \mu|\overline{\alpha}_i|^{\xi-2}\overline{\alpha}_i|)^{\frac{1}{\xi-1}}$$

$$|\alpha_i^{*l}| = ((1-\mu)|\alpha_i|^{\xi} + \mu|\overline{\alpha}_i|^{\xi})^{\frac{1}{\xi}}.$$

Consequently,

$$|\alpha_i^{b*}|, |\alpha_i^{l*}| \leq \max\{|\alpha_i|, |\overline{\alpha}_i|\}.$$

Thus, $\alpha_i^{b*}, \alpha_i^{l*} \in B_m(0)$ and $B_i^m(x)$ and $L_i^m(x)$ are convex . To address this problem we consider the convex hull of $BL_i^m(x)$ and we work with $\mathcal{BL}^m(x)$ defined as

$$\mathcal{BL}^m(x) := \begin{cases} \overline{co}\ (B_1^m(x) \times L_1^m(x)) & \text{if } x_N > 0 \\ \overline{co}\ (B_2^m(x) \times L_2^m(x)) & \text{if } x_N < 0 \\ \overline{co}((B_1^m(x) \times L_1^m(x)) \cup (B_2^m(x) \times L_2^m(x))) & \text{if } x_N = 0. \end{cases} \tag{3.5.2}$$

To define the value functions and the Hamiltonians associated to $\mathcal{BL}^m(x)$. We consider first, $\overline{\tau}_x(A^m)$, the set of all trajectories, denoted by (X,Y), which are Lipschitz functions, solutions of the differential inclusion $(\dot{X},\dot{L})(s) \in \mathcal{BL}^m(X(s))$ with $X(0)=x$. Now, we define the value function

$$\overline{U}^-_{A^m}(x) := \inf_{\overline{\tau}_x(A^m)} \int_0^\infty \dot{L}(s)e^{-\lambda s}ds$$

We define the Hamiltonian associated to compact control set as follow

$$\overline{H}_i^m(x,u,p) := \sup_{\mathcal{BL}^m(x)} -b\cdot p - l + \lambda u$$

Let us introduce the dynamic and cost functions such that will be considered to define the tangential Hamiltonians. We recall that $b_{\mathcal{H}}(x,\alpha) = \mu b_1(x,\alpha_1) + (1-\mu)b_2(x,\alpha_2)$, where $(\alpha_1,\alpha_2,\mu) \in A_1^m \times A_2^m \times [0,1]$,then we consider

$$\mathcal{BL}_T^m(x) := \{(b_{\mathcal{H}}, l_{\mathcal{H}}), with (b,l) \in \mathcal{BL}^m(x)\ |\ b_{\mathcal{H}}\cdot e_N = 0\},$$

then we define the tangential Hamiltonian as

$$\overline{H}_T^m(x,u,p) := \sup_{\mathcal{BL}_T^m(x)} -b\cdot p - l + \lambda u.$$

Now, we consider the regular trajectories. We say that $b_{\mathcal{H}}$ is regular if $b_1 \cdot e_N \leq 0$ and $b_2 \cdot e_N \geq 0$, then we define

$$\mathcal{BL}_T^{m,reg}(x) := \{(b_{\mathcal{H}}, l_{\mathcal{H}}) \in \mathcal{BL}_T^m(x) \mid b_{\mathcal{H}} \text{ is regular}\},$$

then the regular tangential Hamiltonian is defined as

$$\overline{H}_T^{m,reg}(x,u,p) := \sup_{\mathcal{BL}_T^{m,reg}(x)} -b\cdot p - l + \lambda u.$$

Finally we define

$$\overline{U}^+_{A^m}(x) := \inf_{\overline{\tau}_x^{reg}(A^m)} \int_0^\infty \dot{L}(s) e^{-\lambda s} ds.$$

where

$$\overline{\tau}_x^{reg}(A^m) := \{(X,Y) \in \overline{\tau}_x(A^m) \mid \dot{X}(s) \text{ is regular for } X(s) \in \mathcal{H}\}$$

Below we give some results related to the convex hull, that will be useful to prove that the Hamiltonians, $\overline{H_i}$, that we have defined, with the convex hull of the dynamic and cost functions, are equal to the Hamiltonians, H_i, that we consider without the convex hull.

Let us consider first the definition of a convex hull of a compact set $K \subset R^N$,

$$\overline{co}(K) = \left\{ \int_K \Lambda d\mu(\Lambda) \ : \ \mu \text{ probability measure on } K \right\}.$$

The result below states that the supremum in the convex hull of a compact set is equal to the supremum in the compact set.

Lemma 3.5.1 *Let $F : R^p \to R$ be a convex function and $K \subset R^p$ compact. Then*

$$\sup_{\Lambda \in \overline{co}(K)} F(\Lambda) = \sup_{\Lambda \in K} F(\Lambda).$$

For each $\epsilon > 0$, we consider $F_\epsilon(\Lambda) := F(\Lambda) + \epsilon|\Lambda|^2$ which is strictly convex. Notice that $\overline{co}(K)$ is compact: it is clearly closed by definition and since K is bounded, any convex combination of elements of K is also bounded by some fixed constant.

The supremum of F_ϵ over $\overline{co}(K)$ is attained at a point $\Lambda_\epsilon \in \overline{co}(K)$, associated to a measure μ_ϵ. If μ is not trivial (i.e. not a Dirac Delta), then we reach a contradiction. From Jensen's inequality (for a strictly convex function) we have

$$\sup_{\Lambda \in \overline{co}(K)} F_\epsilon(\Lambda) = F_\epsilon(\int_K \Lambda d\mu_\epsilon(\Lambda)) < \int_K F_\epsilon(\Lambda) d\mu_\epsilon(\Lambda)) \leq \sup_{\Lambda \in K} F_\epsilon(\Lambda).$$

So, necessarily the convex combination for Λ_ϵ is trivial, in other words $\mu_\epsilon = \delta_{\Lambda_\epsilon}$ and $\Lambda_\epsilon \in K$. Now,

$$\sup_{\overline{co}(K)} F \leq \sup_{\overline{co}(K)} F_\epsilon = F(\Lambda_\epsilon) + \epsilon|\Lambda_\epsilon|^2 \leq \sup_K F + \epsilon|\Lambda_\epsilon|^2$$

and since K is compact, $\Lambda_\epsilon \to \Lambda_* \in K$ at least along a subsequence. So, passing to the limit as ϵ goes to 0, yields that

$$\sup_{\overline{co}(K)} F \leq \sup_K F.$$

which gives the result. The other inequality is trivial.

We apply now this lemma to the Hamiltonians associated to compact control sets.

Corollary 3.5.1.1

$$H_i^m = \overline{H}_i^m, \quad H_T^m = \overline{H}_T^m, \quad \textit{and} \quad H_T^{m,reg} = \overline{H}_T^{m,reg}.$$

Thanks to Lemma 3.5.1, considering $F(b,l) = -b \cdot p - l$ and $K = BL_i(x)$, $K = BL_T(x)$ and $K = BL_T^{reg}(x)$, respectively, we obtain the results.

Since we have proved that hypotheses (HA), (HB) and (HC) are satisfied for the convex hull of b and l in the example, then we can obtain the main results in Chapter 1 and Chapter 2. In particular we have

Proposition 3.5.2 *$\overline{U}^-_{A^m}(x)$ and $\overline{U}^+_{A^m}(x)$ are solutions of the Ishii problem. Moreover, $\overline{U}^-_{A^m}(x)$ is subsolution of H_T^m and $\overline{U}^+_{A^m}(x)$ is subsolution of $H_T^{m,reg}$*

We denote the value functions associated to the unbounded control set, R^N, by $\overline{U}^-_A(x)$, and $\overline{U}^+_A(x)$. We define them analogously to $\overline{U}^-_{A^m}(x)$, and $\overline{U}^+_{A^m}(x)$ just changing the control set A^m by R^N.

Proposition 3.5.3 *$\overline{U}^-_A(x)$, $\overline{U}^+_A(x)$ are solutions of Ishii problem* (3.0.1)*. Moreover, $\overline{U}^-_A(x)$ is subsolution of H_T and $\overline{U}^+_A(x)$ is subsolution of H_T^{reg}.*

Proof of Propositions 3.5.2 and 3.5.3: is analogous to the proof of Theorems 2.4.1 and 2.4.2. We do not use the convexity of $BL_i(x)$ to prove that the value functions are supersolutions of the respective Hamiltonians. But we use the convexity to proof that they are subsolutions of $H = \min\{H_1, H_2\}$, H_T and H_T^{reg}. We use convexity to approximate a dynamic $b_i(x, \alpha_i)$ such that $b_i(x, \alpha_i) \cdot e_N = 0$ and

$$H_i(x, \phi(x), \nabla\phi(x)) = (-l_i(x, \alpha_i) + \lambda\phi(x) - b_i(x, \alpha_i) \cdot \nabla\phi(x)).$$

For regular trajectories, we approximate by dynamics $b_i(x,\alpha_i^{\nu b*})$ such that $b_i(x,\alpha_i^{\nu b*})\cdot e_N$ is strictly positive or negative with $0<\nu<1$. We choose controls $\alpha_i^{\nu b*}$ and $\alpha_i^{\nu l*}$ that satisfy

$$b_i\left(x,\alpha_i^{\nu b*}\right)=\nu b_i\left(x,\tilde{\alpha}_i\right)+(1-\nu)b_i\left(x,\alpha_i\right)$$
$$l_i\left(x,\alpha_i^{\nu l*}\right)=\nu l_i\left(x,\tilde{\alpha}_i\right)+(1-\nu)l_i\left(x,\alpha_i\right).$$

where $b_i(x,\tilde{\alpha}_i)=(-1)^i\delta e_N$. Moreover, we have $b_i\left(x,\alpha_i^{\nu b*}\right),l_i\left(x,\alpha_i^{\nu l*}\right)$ converges to $\left(b_i\left(x,\alpha_2\right),l_i\left(x,\alpha_i\right)\right)$ as ν goes to 0. Arguing as in Theorem 2.4.2 and thanks to Lemma 3.5.1, we obtain

$$\overline{H}(x,\phi(x),\nabla\phi(x))=H(x,\phi(x),\nabla\phi(x))\leq 0.$$

We obtain also a local comparison result for this example.

Theorem 3.5.4 *Let u be a locally bounded subsolution of* (3.0.1). *Let $x\in R^N$, $R>0$, then*

$$\max_{B_R(x)}(u-\overline{U}^+_{A^m})^+\leq\max_{\partial B_R(x)}(u-\overline{U}^+_{A^m})^+$$

The proof is similar to the one in Theorem 3.4.4 We just modify the part that uses the convexity of $BL_i(x)$. In fact, we only need the convexity of $B_i(x)$ and $L_i(x)$ this is enough so we do not need to use relaxed controls in Theorem 3.2.1. Moreover, let μ_j with $0\leq\mu_j\leq 1$ and $\sum_{j=1}^n\mu_j=1$ then there exists $\alpha_i^{b*},\alpha_i^{l*}\in A_i^m$ such that

$$\sum_{j=1}^n\mu_j(b_i(x,\alpha_j),l_i(x,\alpha_j))=(b_i(x,\alpha_i^{b*}),l_i(x,\alpha_i^{l*})).$$

Thus, if $(\alpha_1^{b*},\alpha_2^{b*})\in\mathcal{BL}_T^{m,reg}(x)$ then by definition of $\overline{H}_T^{m,reg}(x,u,p)$ and Lemma 3.5.1, we obtain

$$-b_{\mathcal{H}}(x,\alpha_1^{b*},\alpha_2^{b*})\cdot p-l_{\mathcal{H}}(x,\alpha_1^{l*},\alpha_2^{l*})+\lambda u\quad\leq\overline{H}_T^{m,reg}(x,u,p)=H_T^{m,reg}(x,u,p).$$

To obtain global comparison results we need to verify that for our example, hypotheses LOC 1 and LOC 2 are satisfied. To do this, we assume some growing conditions on the subsolutions and supersolutions.

Recall that in the example, we consider the dynamic function

$$b_i(x,\alpha_i):=c_\xi d(x)|\alpha_i|^{\xi-2}\alpha_i,\ \text{ where }\ 0<d_i(x)\leq|x|^\beta\ \text{ is locally Lipschitz and }\ c_\xi=\frac{\xi}{\sqrt[\xi]{(\xi-1)^{\xi-1}}},$$

with $\xi > 1$, and cost function

$$l_i(x,\alpha_i) := f_i(x) + |\alpha_i|^\xi \quad \text{with} \quad 0 \le f_i(x) \le C_f|x|^{\zeta-\varepsilon} + \overline{C}_f, \quad \text{with} \quad f_i \quad \text{continuous in } \overline{\Omega}_i.$$

We consider the constant λ satisfying

$$\lambda \ge \zeta^\xi.$$

Let $\mathcal{C}$ be the set of subsolutions of (3.0.1),(3.0.2) and supersolutions of (3.0.1) with growth $|x|^{\zeta-\epsilon}$, where ζ is related with β and ξ, constants that define the dynamic and cost functions. Then any $\omega \in \mathcal{C}$ satisfies

$$|\omega(x)| \le C_\omega|x|^{\zeta-\epsilon} + C_\omega \quad \text{where} \quad \zeta \ge (\zeta-1)\xi + \beta\xi. \tag{3.5.3}$$

We will prove that any function in $\mathcal{C}$ satisfies hypotheses LOC 1, LOC 2 and then use Theorem 3.3.1, to obtain the global comparison result. Moreover since

$$\overline{U}_A^-(x) \le \overline{U}_{A^m}^-(x), \ \overline{U}_A^+(x) \le \overline{U}_{A^m}^+(x) \ \text{ and } \ \overline{U}_{A^m}^+(x), \ \overline{U}_{A^m}^-(x) \le \frac{C_{f_i}}{\lambda}|x|^{\zeta-\epsilon} + C_m$$

the value functions $\overline{U}_A^+(x), \ \overline{U}_A^-(x)$ are in the set of solutions $\mathcal{C}$ for which we will prove global comparison results.

Let us start computing the Hamiltonian

$$H_i(x,u,p) \ = \lambda u - f_i(x) + \sup_{A_i}\left(c_\xi d_i(x)|\alpha_i|^{\xi-2}\alpha_i\cdot p - |\alpha_i|^\xi\right).$$

Let $y = |\alpha_i|$,

$$c_\xi|\alpha_i^{\xi-2}|\alpha_i\cdot p - |\alpha_i|^\xi \le -y^\xi + c_\xi d_i(x)|p|y^{\xi-1}$$

Thus differentiating, we obtain that $-y^\xi + c_\xi d_i(x)|p|y^{\xi-1}$ reaches its maximum in $y = 0$ or in $y = c_\xi d_i(x)|p|\frac{\xi-1}{\xi}$. Due to the choice of c_ξ the maximum is $d_i(x)^\xi|p|^\xi$. So that,

$$H_i(x,u,p) = \lambda u + d_i(x)^\xi|p|^\xi - f_i(x).$$

Fixed $\zeta > 1$, we want to find a differentiable subsolution with growth $\sim |x|^\zeta$. A good candidate is

$$-(1+|x|^{2\zeta})^{1/2} \sim -|x^\zeta|$$

which is differentiable for $2\zeta > 1$.

Let us compute for which values of ξ and β we have $-(1+|x|^{2\zeta})^{1/2}$ is a subsolution.

we have

$$|\nabla(1+|x|^{2\zeta})^{1/2}| = \zeta\frac{|x|^{2\zeta-1}}{\sqrt{|x|^{2\zeta}+1}} < \zeta\frac{|x|^{2\zeta-1}}{\sqrt{|x|^{2\zeta}}} = \zeta|x|^{\zeta-1}.$$

Therefore,

$$|\nabla(1+|x|^{2\zeta})^{1/2}|^{\xi} < \zeta^{\xi}|x|^{(\zeta-1)\xi}.$$

Since $d_i(x)$ has growth $|x|^{\beta}$ and $\lambda \geq \zeta^{\xi}$ then for (ζ, ξ, β) satisfying

$$\zeta \geq (\zeta-1)\xi + \beta\xi, \tag{3.5.4}$$

we obtain that for $|x| \geq 1$,

$$\begin{aligned} H_i(x, -(1+|x|^{2\zeta})^{1/2}, -\nabla(1+|x|^{2\zeta})^{1/2}) &\leq -\lambda(1+|x|^{2\zeta})^{1/2} + \zeta^{\xi}|x|^{\xi(\zeta-1+\beta)} - f_i(x) \\ &\leq -\lambda|x|^{\zeta} + \zeta^{\xi}|x|^{\xi(\zeta-1+\beta)} - f_i(x) \\ &\leq 0. \end{aligned}$$

For $|x| < 1$

$$\begin{aligned} H_i(x, -(1+|x|^{2\zeta})^{1/2}, -\nabla(1+|x|^{2\zeta})^{1/2}) &\leq -\lambda(1+|x|^{2\zeta})^{1/2} + \zeta^{\xi}|x|^{\xi(\zeta-1+\beta)} - f_i(x) \\ &\leq -\lambda + \zeta^{\xi} - f_i(x) \\ &\leq 0. \end{aligned}$$

By definition of c_{ξ}, we have $\xi > 1$.

Fix $\zeta > 1$. Now, let us focus on finding values for ξ and β such that for which $-(1+|x|^{2\zeta})^{1/2}$ is subsolution of (3.0.1) and 3.0.3, and

$$\zeta \geq (\zeta-1)\xi + \beta\xi.$$

we have $1 < \xi \leq \frac{\zeta}{\zeta-1+\beta}$ and consequently $\beta < 1$. Thus, fixed $\beta \in [0,1)$, we have for any $\xi \in (1, \frac{\zeta}{\zeta-1+\beta}]$.

On the other hand, let us fix ξ, β and seek values of ζ for which $-(1+|x|^{2\zeta})^{1/2}$ is subsolution of (3.0.1) and 3.0.3. Since $\xi > 1$, then $0 < \zeta \leq \frac{\xi}{\xi-1}(1-\beta)$. In order to obtain that $-(1+|x|^{2\zeta})^{1/2}$ is differentiable, we need that $\frac{\xi}{\xi-1}(1-\beta) > \frac{1}{2}$, that is,

$$\beta < \frac{\xi+1}{2\xi}.$$

Chosen $\beta \in (0, \frac{\xi+1}{2\xi})$, we can take any $\zeta \in (\frac{1}{2}, \frac{\xi}{\xi-1}(1-\beta))$.

Now since (ζ, β, ξ) satisfies (3.5.4) and $\lambda \geq \zeta^{\xi}$. In particular if $y \in \mathcal{H}$ then

$$H_1(y, -(|y|^{2\zeta}+1)^{\frac{1}{2}}, -\nabla(|y|^{2\zeta}+1)^{\frac{1}{2}}) \leq 0, \quad \text{and} \quad H_2(y, -(|y|^{2\zeta}+1)^{\frac{1}{2}}, -\nabla(|y|^{2\zeta}+1)^{\frac{1}{2}}) \leq 0. \tag{3.5.5}$$

Consequently,

$$\min\{H_1(y,-(|y|^{2\zeta}+1)^{\frac{1}{2}},-\nabla(|y|^{2\zeta}+1)^{\frac{1}{2}}),\ H_2(y,-(|y|^{2\zeta}+1)^{\frac{1}{2}},-\nabla(|y|^{2\zeta}+1)^{\frac{1}{2}})\}\leq 0.$$

Moreover defining

$$S(y_1,\cdots,y_N):=(y_1,\cdots,y_{N-1},0)\in R^N$$

then $|S(y)|\leq |y|$ and from (3.5.5), we obtain

$$\begin{aligned}&H_T(y,-(|y|^{2\zeta}+1)^{\frac{1}{2}},S(-\nabla(|y|^{2\zeta}+1)^{\frac{1}{2}}))\\ &\leq H_1(y,-(|y|^{2\zeta}+1)^{\frac{1}{2\zeta}},S(-\nabla(|y|^{2\zeta}+1)^{\frac{1}{2}}))+H_2(y,-(|y|^{2\zeta}+1)^{\frac{1}{2}},S(-\nabla(|y|^{2\zeta}+1)^{\frac{1}{2}}))\\ &\leq 0.\end{aligned}$$

Let u be a subsolution of (3.0.1) in $\mathcal{C}$ and let us define,

$$u_\beta(y):=(1-\beta)u(y)-\beta(|y|^{2\zeta}+1)^{\frac{1}{2}}$$

Note that u_β is a subsolution of 3.0.1. Let $\phi\in C^1(R^N)$ be a test function and y_0 be a local maximum point of $u_\beta-\phi=(1-\beta)u(y)-(\beta(|y|^{2\zeta}+1)^{\frac{1}{2}}+\phi(y))$. Then y_0 is local maximum point of $u(y)-\frac{\beta(|y|^{2\zeta}+1)^{\frac{1}{2}}+\phi(y)}{1-\beta}$ and thanks to the the convexity of H_i in the last two variables, we obtain

$$\begin{aligned}&H_i(y_0,u_\beta(y_0),\nabla\phi(y_0))\\ &=H_i\left(y_0,u_\beta(y_0),\nabla\phi(y_0)+\tfrac{\beta\zeta|y_0|^{2\zeta-2}y_0}{(|y_0|^{2\zeta}+1)^{\frac{1}{2}}}-\tfrac{\beta\zeta|y_0|^{2\zeta-2}y_0}{(|y_0|^{2\zeta}+1)^{\frac{1}{2}}}\right)\\ &\leq(1-\beta)H_i\left(y_0,u(y_0),\tfrac{\nabla\phi(y_0)}{1-\beta}+\tfrac{\beta\zeta|y_0|^{2\zeta-2}y_0}{(1-\beta)(|y_0|^{2\zeta}+1)^{\frac{1}{2}}}\right)+\beta H_i\left(y_0,-(|y_0|^{2\zeta}+1)^{\frac{1}{2}},-\tfrac{\zeta|y_0|^{2\zeta-2}y_0}{(|y_0|^{2\zeta}+1)^{\frac{1}{2}}}\right)\\ &\leq 0.\end{aligned}$$

Now assume that $y_0\in\mathcal{H}$, arguing as above, we have

$$\min\{H_1(y_0,u_\beta(y_0),\nabla\phi(y_0)),H_2(y_0,u_\beta(y_0),\nabla\phi(y_0))\}\leq 0.$$

Take $\phi\in C^1(R^{N-1})$ and y_0 a local maximum point of $u_\beta(y)-\phi(y)=(1-\beta)u(y)-(\beta(|y|^{2\zeta}+1)^{\frac{1}{2}}+\phi(y))$. Then y_0 is a local maximum point of $u(y)-\frac{\beta(|y|^{2\zeta}+1)^{\frac{1}{2}}+\phi(y)}{1-\beta}$ and from the convexity of H_T in the last two variables, we have

$$\begin{aligned}&H_T(y_0,u_\beta(y_0),\nabla\phi(y_0))\\ &=H_T\left(y_0,u_\beta(y_0),\nabla\phi(y_0)+D_{\mathcal{H}}(\tfrac{\beta(|y|^{2\zeta}+1)^{\frac{1}{2}}}{1-\beta})-D_{\mathcal{H}}(\tfrac{\beta(|y|^{2\zeta}+1)^{\frac{1}{2}}}{1-\beta})\right)\\ &\leq(1-\beta)H_T\left(y_0,u(y_0),\tfrac{\nabla\phi(y_0)}{1-\beta}+D_{\mathcal{H}}(\tfrac{\beta(|y|^{2\zeta}+1)^{\frac{1}{2}}}{1-\beta})\right)+\beta H_T\left(y_0,-(|y_0|^{2\zeta}+1)^{\frac{1}{2}},D_{\mathcal{H}}(\tfrac{-(|y|^{2\zeta}+1)^{\frac{1}{2}}}{1-\beta})\right)\\ &\leq 0.\end{aligned}$$

Let v be a supersolution of (3.0.1) in $\mathcal{C}$, then

$$\begin{aligned} u_\beta(y) - v(y) &= (1-\beta)u(y) - \beta(|y|^{2\zeta}+1)^{\frac{1}{2}} - v(y) \\ &\leq |u(y)| + |v(y)| - \beta\lambda(|y|^{2\zeta}+1)^{\frac{1}{2}} \\ &\leq (C_u + C_v)|y|^{\zeta-\epsilon} + (\overline{C}_u + \overline{C}_v) - \beta(|y|^{2\zeta}+1)^{\frac{1}{2}} \\ &\leq |y|^{\zeta-\epsilon}(C_u + C_v + \tfrac{\overline{C}_u+\overline{C}_v}{|y|^{\zeta-\epsilon}} - \beta|y|^{\epsilon}). \end{aligned}$$

Taking limits as $|y|$ goes to infinity, we obtain that $u_\beta(y) - v(y)$ goes to $-\infty$. Thus, the sequence u_β satisfies LOC 1. Therefore, there exist x_β such that $M_\beta := \max_{R^N}(u_\beta - v) = u_\beta(x_\beta) - v(x_\beta)$.

Now we build a subsequence, $u_{\beta\gamma}$, of the sequence u_β, satisfying LOC 2. Consider

$$u_{\beta\gamma}(x) := u_\beta(x) - \gamma(|x-x_\beta|^2+1)^{\frac{1}{2}}.$$

We will show that there exists a neighbourhood of x_β where $u_{\beta\gamma}$ is subsolution of H_i. Let x be a local maximum point of $u_{\beta\gamma} - \phi$ then x is local maximum point of $u_\beta(x) - ((\gamma(|x-x_\beta|^2+1)^{\frac{1}{2}} + \phi(x))$. Due to Lemma 2.3.3, there exists A_i^m such that the supremum $H_i(x, u_{\beta\gamma}(x), \nabla\phi(x))$ is reached in A_i^m for any $x \in B_r(x_\beta)$, for $r > 0$. Thus, given $x \in B_r(x_\beta)$, there exists $\alpha_i^x \in A_i^m$ such that

$$\begin{aligned} &H_i(x, u_{\beta\gamma}(x), \nabla\phi(x)) = H_i\left(x, u_{\beta\gamma}(x), \nabla\phi(x) + \tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} - \tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}\right) \\ &= -l_i(x,\alpha_i^x) + \lambda u_{\beta\gamma}(x) - b_i(x,\alpha_i^x)\cdot\left(\nabla\phi(x) + \tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} - \tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}\right) \\ &= -l_i(x,\alpha_i^x) + \lambda u_\beta(x) - b_i(x,\alpha_i^x)\cdot\left(\nabla\phi(x) + \tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}\right) - \lambda\gamma(|x-x_\beta|^2+1)^{\frac{1}{2}} - b_i(x,\alpha_i^x)\cdot\tfrac{-\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} \\ &\leq H_i\left(x, u_\beta(x), \nabla\phi(x) + \tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}\right) - \lambda\gamma(|x-x_\beta|^2+1)^{\frac{1}{2}} + b_i(x,\alpha_i^x)\cdot\tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} \\ &\leq -\lambda\gamma(|x-x_\beta|^2+1)^{\frac{1}{2}} + b_i(x,\alpha_i^x)\cdot\tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} \end{aligned}$$

Let $||b_i||_{L^\infty(B_r(x_\beta)\times A_i^m)} < B$, and let as choose r such that $r < \frac{\lambda}{B}$, then $|x - x_\beta| < r < \frac{\lambda}{B}$, and

$$-\lambda\gamma(|x-x_\beta|^2+1)^{\frac{1}{2}} + b_i(x,\alpha_i^x)\cdot\frac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} \leq -\lambda\gamma(|x-x_\beta|^2+1)^{\frac{1}{2}} + \lambda\gamma \leq 0.$$

Thus, we obtain that

$$H_i(x, u_{\beta\gamma}(x), \nabla\phi(x)) \leq 0, \quad \forall x \in B_r(x_\beta).$$

Assume that $x_\beta \in \mathcal{H}$. Take $x \in B_r(x_\beta)$. If $x \in \Omega_i$ there is nothing to prove. Hence we can assume that $x \in \mathcal{H}$. Arguing as for $x \in \Omega_i$, we obtain

$$\min\{H_1(x, u_{\beta\gamma}(x), \nabla\phi(x)), H_2(x, u_{\beta\gamma}(x), \nabla\phi(x))\} \leq 0.$$

Therefore $u_{\beta\gamma}$ is subsolution of the Ishii problem. Now let us prove that $u_{\beta\gamma}$ is subsolution of H_T. Consider the test function $\phi \in R^{N-1}$ and $x = (x', 0)$ a local maximum point of $u_{\beta\gamma} - \phi$. Given $y \in B_r(x_\beta)$, we denote by $\alpha_{n^*}(y) := (\mu_{n^*}, \alpha^1_{n^*}(y), \alpha^2_{n^*}(y))$ a control that satisfies

$$\tilde{H}_T(y, \nabla\phi(y)) - \frac{1}{n^*} \le -l_{\mathcal{H}}(y, \alpha_{n^*}(y)) - S(b_{\mathcal{H}}(y, \alpha_{n^*}(y))) \cdot \nabla\phi(y). \tag{3.5.6}$$

This way,

$$\begin{aligned}
&\lambda u_{\beta\gamma}(x) + \tilde{H}_T(x, \nabla\phi(x)) - \tfrac{1}{n} \\
&= H_T\left(x, u_{\beta\gamma}(x), \nabla\phi(x) + \tfrac{\gamma S(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} - \tfrac{\gamma S(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}\right) - \tfrac{1}{n} \\
&\le -l_{\mathcal{H}}(x, \alpha_n(x)) + \lambda u_{\beta\gamma}(x) - S(b_{\mathcal{H}}(x, \alpha_n(x))) \cdot \left(\nabla\phi(x) + \tfrac{\gamma S(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} - \tfrac{\gamma S(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}\right) \\
&= -l_{\mathcal{H}}(x, \alpha_n(x)) + \lambda u_\beta(x) - S(b_{\mathcal{H}}(x, \alpha_n(x))) \cdot \left(\nabla\phi(x) + \tfrac{\gamma S(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}\right) \\
&-\lambda\gamma(|x-x_\beta|^2+1)^{\frac{1}{2}} + S(b_{\mathcal{H}}(x, \alpha_n(x))) \cdot \tfrac{\gamma S(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}} \\
&\le H_T\left(x, u_\beta(x), \nabla\phi(x) + \tfrac{\gamma(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}\right) \\
&-\lambda\gamma(|x-x_\beta|^2+1)^{\frac{1}{2}} + S(b_{\mathcal{H}}(x, \alpha_n(x))) \cdot \tfrac{\gamma S(x-x_\beta)}{(|x-x_\beta|^2+1)^{\frac{1}{2}}}
\end{aligned}$$

where α_n is a control in A. Let us prove that $S(b_{\mathcal{H}}(x, \alpha_n(x)))$ is uniformly bounded $B_r(x_\beta)$. Initially note that there exists C_r such that

$$|\tilde{H}_T(x, \nabla\phi(x))| < C_r, \ \forall\ x \in B_r(x_\beta). \tag{3.5.7}$$

Indeed,

$$-||l_1||_{L^\infty(B_r(x_\beta)\times A_1^*)} - ||l_2||_{L^\infty(B_r(x_\beta)\times A_2^*)} \le \tilde{H}_T(x, \nabla\phi(x)) \le \tilde{H}_1(x, \nabla\phi(x)) + \tilde{H}_2(x, \nabla\phi(x)).$$

where A_i^* is a compact set such that for each $x \in R^N$ there exists $\alpha_i \in A_i^*$ with $b(x, \alpha_i) = 0$.

Now assume by contradiction that $D := \{S(b_{\mathcal{H}}(x, \alpha_n(x)))|x \in B_r(x_\beta),\ n \in N\}$ is unbounded. Then there exists sequence $(x_n, \alpha_{n^*}(x_n))$ satisfying

$$\lim_{n\to\infty} S(b_{\mathcal{H}}(x_n, \alpha_{n^*}(x_n))) = \infty. \tag{3.5.8}$$

Since $b_{\mathcal{H}}(x_n, \alpha_{n^*}(x_n)) = \mu^i_{n^*} b_1(x_n, \alpha_{n^*}(x_n)) + \mu^2_{n^*} b_2((x_n, \alpha_{n^*}(x_n)))$, where $\mu^1_{n^*} + \mu^2_{n^*} = 1$. Let $\{i, j\} = \{1, 2\}$, then we have $\lim_{n\to\infty} \mu^i_{n^*} S(b_i(x_n, \alpha^i_{n^*}(x_n))) = \infty$. Thanks hypothesis (HA) we have $|\alpha^i_{n^*}(x)| \to \infty$, and

$$\lim_{n\to\infty} \frac{l_i(x_n, \alpha^i_{n^*}(x_n))}{1 + |b_i(x_n, \alpha^i_{n^*}(x_n)|} \to \infty.$$

Given $M > 0$, there exist C_M so that for $n > C_M$:

$$\frac{l_i(x_n, \alpha^i_{n^*}(x_n))}{1 + |b_i(x_n, \alpha^i_{n^*}(x_n)|} \geq 2M \quad \text{and} \quad |\mu^i_{n^*} S(b_i(x_n, \alpha^i_{n^*}(x_n)))| > 1,$$

then

$$\begin{aligned} l_i(x_n, \alpha^i_{n^*}(x_n)) &\geq 2M + 2M|b_i(x_n, \alpha^i_{n^*}(x))| \\ &\geq 2M|b_i(x_n, \alpha^i_{n^*}(x_n))| \\ &\geq 2M|S(b_i(x_n, \alpha^i_{n^*}(x_n)))| \end{aligned}$$

which implies that for $n > C_M$

$$\begin{aligned} \mu^i_{n^*} l_i(x_n, \alpha^i_{n^*}(x_n)) &\geq 2M\mu^i_{n^*}|S(b_i(x_n, \alpha^i_{n^*}(x_n)))| \\ &\geq M + M\mu^i_{n^*}|S(b_i(x_n, \alpha^i_{n^*}(x_n)))| \end{aligned}$$

Consequently, for $M > ||\nabla\phi||_{L^\infty(B_r(x_\beta))}$, we have

$$\begin{aligned} &-\mu^i_{n^*} l_i(x_n, \alpha^i_{n^*}(x_n)) + \mu^i_{n^*} \nabla\phi(x_n) \cdot S(b_i(x_n, \alpha^i_{n^*}(x_n))) \\ &\leq -\mu^i_{n^*} l_i(x_n, \alpha^i_{n^*}(x_n)) + M\mu^i_{n^*}|S(b_i(x_n, \alpha^i_{n^*}(x_n)))| \\ &\leq -M. \end{aligned}$$

That is,

$$-\mu^i_{n^*} l_i(x_n, \alpha^i_{n^*}(x_n)) + \mu^i_{n^*} \nabla\phi(x_n) \cdot S(b_i(x_n, \alpha^i_{n^*}(x))) \leq -M.$$

Since $\{i, j\} = \{1, 2\}$, from (3.5.8), taking subsequence if necessary, for j we have two possibilities, either $\mu^j_{n^*} S(b_j(x_n, \alpha^j_{n^*}(x))) = \infty$ or $\mu^j_{n^*} S(b_j(x_n, \alpha^j_{n^*}(x))) < \infty$. In the first case, arguing as above we have

$$-\mu^j_{n^*} l_j(x_n, \alpha^j_{n^*}(x_n)) + \mu^j_{n^*} \nabla\phi(x_n) \cdot S(b_i(x_n, \alpha^j_{n^*}(x_n))) \leq -M.$$

Thus, for $M > \max\{||\nabla\phi||_{L^\infty(B_r(x_\beta))}, C_r + 2\}$, from (3.5.7), we obtain

$$\begin{aligned} -l_{\mathcal{H}}(x_n, \alpha_{n^*}(x_n)) - S(b_{\mathcal{H}}(x_n, \alpha_{n^*}(x_n))) \cdot \nabla\phi(x_n) &\leq -2M \\ &\leq -2|\tilde{H}_T(x_n, \nabla\phi(x_n))| - 4 \\ &< \tilde{H}_T(x_n, \nabla\phi(x_n)) - \tfrac{1}{n^*} \end{aligned}$$

which is contradiction with (3.5.6).

Now, let us assume that $\mu^j_{n^*} S(b_j(x_n, \alpha^j_{n^*}(x))) < \infty$, then there exists C_j such that

$$-\mu^j_n l_j(x_n, \alpha^j_{n^*}(x_n)) + \mu^j_n \nabla\Phi(x_n) \cdot S(b_j(x_n, \alpha^j_{n^*}(x_n))) \leq C_j$$

Hence, taking $M > \max\{|\nabla\phi(x)|_{B_r(x_\beta)}, C_r + C_j + 4\}$, we obtain

$$\begin{aligned} -l_{\mathcal{H}}(x_n, \alpha_{n^*}(x_n)) - S(b_{\mathcal{H}}(x_n, \alpha_{n^*}(x_n))) \cdot \nabla\phi(x_n) &\leq -M + C_j \\ &\leq -|H_T(x_n, \nabla\phi(x_n))| - 4 \\ &< H_T(x_n, \nabla\phi(x_n)) - \tfrac{1}{n^*} \end{aligned}$$

which is again a contradiction with (3.5.6). Therefore, $D = \{S(b_{\mathcal{H}}(x, \alpha_n(x)))| x \in B_r(x_\beta),\ n \in N\}$ is bounded. Then, there exists $B > 0$ such that $\|b\| < B$ for all $b \in D$. Consider r such that $r < \frac{\lambda}{B}$, and $x \in B_r(x_\beta)$, then

$$-\lambda\gamma(|x - x_\beta|^2 + 1)^{\frac{1}{2}} + S(b_{\mathcal{H}}(x, \alpha_n(x))) \cdot \frac{\gamma(x - x_\beta)}{(|x - x_\beta|^2 + 1)^{\frac{1}{2}}} \leq -\lambda\gamma(|x - x_\beta|^2 + 1)^{\frac{1}{2}} + \lambda\gamma \leq 0.$$

Thus, $u_{\beta\gamma}$ is subsolution of the Ishii problem. Let us prove that $u_{\beta\gamma}$ satisfies LOC 2. Indeed, let $x \in \partial B_r(x_\beta)$

$$u_{\beta\gamma}(x_\beta) - u_\beta(x_\beta) = -\gamma > -\gamma(r^2 + 1)^{\frac{1}{2}} = -\gamma(|x - x_\beta|^2 + 1)^{\frac{1}{2}} = u_{\beta\gamma}(x) - u_\beta(x).$$

We can choose $d(\gamma) := \frac{\gamma(r^2+1)^{\frac{1}{2}} - \gamma}{2} > 0$. Moreover, $u_{\beta\gamma}(x) \to u_\beta(x)$ as $\gamma \to 0$ for any $x \in B_r(x_\beta)$.

Therefore, the set $\mathcal{C}$ of subsolutions u of (3.0.1),(3.0.2) and supersolutions v of (3.0.1), satisfying the growth condition (3.5.3), satisfy the hypotheses LOC 1 and LOC 2. Hence, thanks to Theorem (3.3.1), we obtain a global comparison result,

$$u(x) \leq v(x) \ \ \forall\ x \in R^N.$$

Due to Proposition 3.5.3, we have also that $\overline{U}_A^-$ is a minimal locally bounded l.s.c supersolution of the Ishii problem. Moreover, thanks to Theorem 3.5.4, we have $\overline{U}_A^+$ is a maximal subsolution of the Ishii problem.

3.6 FILIPPOV APROXIMATIONS

In this section we consider consider Filippov approximations. Let $\varphi : R \to [0, 1]$ be a continuous functions satisfying $\lim_{y\to\infty} \varphi(y) = 1$ and $\lim_{y\to-\infty} \varphi(y) = 0$. Let

$$\varphi_\varepsilon(y) := \varphi(\frac{y}{\varepsilon})$$

We extend H_1 and H_2 for any $x \in R^N$ as follows

$$H_\varepsilon(x,u,p) := \varphi_\varepsilon(x_N)H_1(x,u,p) + (1-\varphi_\varepsilon(x_N))H_2(x,u,p), \quad \forall x \in R^N \tag{3.6.1}$$

We assume the hypotheses

(HA) $l_i : R^N \times A_i \to R$ and $b_i : R^N \times A_i \to R$ satisfy

$$\lim_{|\alpha|\to\infty} \frac{l_i(y,\alpha)}{1+|b_i(y,\alpha)|} = \infty \text{ and } \lim_{|\alpha|\to\infty} |b_i(y,\alpha)| = \infty$$

uniformly in compact sets of R^N with respect to $y \in R^N$.

(HB) b_i, l_i are continuous functions and b_i, l_i are locally Lipschitz in the spacial variable.

(HC) For any $x \in R^N$, the set $\{(b_i(x,\alpha_i), l_i(x,\alpha_i))|\ \alpha \in A_i^m\}$ is a closed and convex set. Moreover, there exists $\delta > 0$ such that

$$\overline{B_\delta(0)} \subset \mathcal{B}^{A_i^m}(x), \quad \text{where } \mathcal{B}^{A_i^m}(x) := \{b_i(x,\alpha)|\alpha \in A_i^m\}.$$

(HCC) Let $A_i^m \subset A_i$ be compact, then there exists $C_m(b)$ and $C_m(l)$ such that

$$|b_i(y,\alpha)| \le C_m(b) \text{ and } l_i(y,\alpha) \le C_m(l),\ \forall\ y \in R^N,\ \alpha \in A_i^m. \tag{3.6.2}$$

where A_i^m is a chain of compact set which union is equal to R^N. Assuming also hypotheses (HA), (HB), (HC) and (HCC) we are able to prove uniqueness of solution of H_ε, and the solution converges as ε goes to zero to U_A^-. To do this is very important to have hypotheses (HCC), which assumes that the dynamic and cost functions are bounded when restricted to compact set.

Theorem 3.6.1 *Under Hypotheses (HA), (HB), (HC) and (HCC). There exists a unique bounded Lipschitz continuous solution u_ε of* (3.6.1)*. Moreover, u_ε converges to U_A^- as ε goes to 0 locally uniformly in R^N.*

Fix $m_0 \in N$. Note that $H_\varepsilon^{m_0}$ is coercive, because H_1 and H_2 are coercive. Thus, there exists $C_{m_0} > 0$ such that

$$H_\varepsilon^{m_0}(x,u,p) > 0, \text{ for } |u| < ||U^-_{A^{m_0}}||_{L^\infty(R^N)} + ||U_A^-||_{L^\infty(R^N)} + 1,\ x \in R^N \text{ and } |p| > C_{m_0}. \tag{3.6.3}$$

If $m_0' > m_0$, by definition, we have

$$H_\varepsilon^{m_0'}(x,u,p) \geq H_\varepsilon^{m_0}(x,u,p).$$

By Lemma (2.3.3) there exists $m > m_0'$ such that

$$H_\varepsilon(x,u,p) = H_\varepsilon^{m}(x,u,p), \text{ if } |p| \leq C_{m_0}. \tag{3.6.4}$$

Thanks to Corollary 3.1.3.2, let m' be large enough, with $m' > m$ such that $U_A^- = U_{A^{m'}}^-$. Thanks to (BARLES; BRIANI; CHASSEIGNE, 2013, Th 6.1) there exists a unique bounded Lipschitz continuous solution $u_\varepsilon^{m'}$ of $H_\varepsilon^{m'}$ such that $u_\varepsilon^{m'}$ converges to $U_{A^{m'}}^-$ as ε goes to 0, locally uniformly in R^N. Therefore $\lim_{\epsilon\to 0} u_\epsilon^{m'} = U_A^-$ locally uniformly in R^N. We know that $u_\varepsilon^{m'}$ is supersolution of H_ε. To conclude we just have to prove that $u_\varepsilon^{m'}$ is subsolution of H_ε.

Since $u_\epsilon^{m'}$ is Lipschitz continuous subsolution of $H_\varepsilon^{m'}$, then

$$H_\varepsilon^{m'}(x, u_\varepsilon^{m'}(x), \nabla u_\varepsilon^{m'}(x)) \leq 0, \ a.e. \tag{3.6.5}$$

Since $u_\varepsilon^{m'}$ converges to U_A^- locally uniformly, for ε small enough we have $|u_\varepsilon^{m'}(y)| \leq |U_A^-(y)|+1$ for all $y \in B_r(x)$. If $u_\varepsilon^{m'}$ is differentiable in x then by (3.6.3), we have $|\nabla u_\varepsilon^{m'}(x)| \leq C_{m_0}$. Consequently, thanks to (BREZIS; BRÉZIS, 2011, Remark 7, p. 269), we have

$$|u_\varepsilon^{m'}(x) - u_\varepsilon^{m'}(y)| \leq ||\nabla u_\varepsilon^{m'}||_{L\infty(B_r(x))}|x-y| \leq C_{m_0}|x-y|.$$

Thus, $u_\varepsilon^{m'}$ is uniformly locally Lipschitz with Lipschitz constant C_{m_0}, for ε small enough. Thus, if $p_\varepsilon \in \partial^+ u_\varepsilon^{m'}(x)$, then $|p_\varepsilon| < C_{m_0}$, cf. (BARLES, 1994, Lemma 2.4 p. 31). Thus, by (3.6.4) and (3.6.5), $u_\varepsilon^{m'}$ is subsolution of H_ε.

Assume that H_ε has another bounded Lipschitz continuous solution $v_\varepsilon \neq u_\varepsilon^{m'}$, then since $||\nabla v_\varepsilon|| \leq C_v$. Thanks to Lemma (2.3.3), there exist $m^* > m'$ such that v_ε is solution of $H_\varepsilon^{m^*}$. Thanks to (BARLES; BRIANI; CHASSEIGNE, 2013, Th 6.1), $H_\varepsilon^{m^*}$ has a unique solution, and then $v_\varepsilon = u_\epsilon^{m^*}$. Since $u_\varepsilon^{m'}$ is solution of H_ε then it is also a solution of $H_\varepsilon^{m^*}$. Thus, $u_\varepsilon^{m'} = u_\varepsilon^{m^*} = v_\varepsilon$.

Let us focus now on the example of nonlinear Hamiltonian which satisfies only hypotheses (HA), (HB) and (HC). We will obtain this result of uniqueness of solution for the Filippov approximated Hamiltonian H_ε, and the convergence to $\overline{U}_A^-$. Recall that the Hamiltonian considered in the previous section is given by

$$H_i(x,u,p) = \lambda u + d(x)^\xi |p|^\xi - f_i(x)$$

with associated dynamic function

$$b_i(x,\alpha_i) := c_\xi d(x)|\alpha_i|^{\xi-2}\alpha_i, \text{ where } 0 < d_i(x) \le |x|^\beta \text{ is locally Lipschitz and } c_\xi = \frac{\xi}{\sqrt[\xi]{(\xi-1)^{\xi-1}}} \quad (3.6.6)$$

and the cost function

$$l_i(x,\alpha_i) := f_i(x) + |\alpha_i|^\xi \text{ with } 0 \le f_i(x) \le C_f|x|^{\zeta-\varepsilon} + \overline{C}_f, \text{ with } f_i \text{ continuous in } \overline{\Omega}_i.$$

We consider the constant λ satisfying

$$\lambda \ge \zeta^\xi,$$

where $\zeta > 1$, ξ and β satisfy

$$\zeta \ge (\zeta-1)\xi + \beta\xi.$$

We proved in the previous section that hypotheses (HA), (HB) and (HC) are satisfied. We also have that

$$\overline{U}_A^-(x) \le \overline{U}_{A^m}^-(x),\ \overline{U}_A^+(x) \le \overline{U}_{A^m}^+(x) \text{ and } \overline{U}_{A^m}^+(x),\ \overline{U}_{A^m}^-(x) \le \frac{C_{f_i}}{\lambda}|x|^{\zeta-\epsilon} + C_m.$$

These value functions $\overline{U}_A^+(x)$, $\overline{U}_A^-(x)$ are in the set $\mathcal{C}$ of solutions for which we have comparison results.

In the following result we prove that there exists a unique solution of the Filippov aproximation Hamiltonian, and this solution converges to the value function $\overline{U}_A^-$.

Theorem 3.6.2 *There exists a unique locally Lipschitz continuous solution u_ε of* (3.6.1) *satisfying*

$$-C(1+|x|^{2\zeta})^{\frac{1}{2}-\epsilon} - C \le u_\varepsilon(x) \le C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C$$

where $C > \max\left\{C_m, \frac{C_f}{\lambda}, \frac{\overline{C}_f}{\lambda}\right\}$. Moreover, u_ε converges to $\overline{U}_A^-$ as ε goes to 0, locally uniformly in R^N.

First we prove the existence of a solution for (3.6.1) using the Perron Method, see (BARLES, 1994, p. 52). In order to proof uniqueness, we first need to fix the set where we will prove comparison result

$$\mathcal{S}_\varepsilon := \{u \text{ solution of (3.6.1)}| - C(1+|x|^{2\zeta})^{\frac{1}{2}-\epsilon} - C \le u(x) \le C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C \text{ for all } x \in R^N\}.$$

Since H_ε is continuous, $\lambda(u-v) = H_\varepsilon(x,v,p) - H_\varepsilon(x,u,p)$ and H_ε are locally coercive then thanks to (BARLES, 1994, Cor. 2.2, p. 27), we obtain Local Comparison Result.

Moreover we have $-(1+|x|^{2\zeta})^{\frac{1}{2}}$ is subsolution of (3.6.1), therefore hypotheses LOC 1 and LOC 2 are satisfied. We have LOC 1 with

$$u_\beta(y) := \beta u(y) - (1-\beta)(1+|y|^{2\zeta})^{\frac{1}{2}},$$

and LOC 2 with

$$u_{\beta\gamma}(y) := u_\beta(y) - \gamma(|y-x_\beta|^2+1)^{\frac{1}{2}}.$$

Finally, thanks to Theorem 3.3.1 we obtain global comparison result for H_ε in $\mathcal{S}_\varepsilon$, what proves the uniqueness.

Now let us prove the existence via Perron's Method. It is important to note that $C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C$ is a differentiable supersolution of (3.6.1). In fact,

$$(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} > (|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} = |x|^{\zeta-\epsilon}$$

and by choosing C big enough we have

$$C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C > f_i(x).$$

Consider

$$\underline{\mathcal{S}_\varepsilon} := \{\underline{u} \text{ subsolution of (3.6.1)}| - C(1+|x|^{2\zeta})^{\frac{1}{2}-\epsilon} - C \le \underline{u}(x) \le C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C\ \forall x \in R^N\}$$

Note that $\underline{\mathcal{S}_\varepsilon}$ is non-empty because $0,\ \overline{U}_A^+$ and $\overline{U}_A^- \in \underline{\mathcal{S}_\varepsilon}$. Indeed, by choosing C

$$\overline{U}_A^+(x), \overline{U}_A^-(x) \le \frac{C_{f_i}}{\lambda}|x|^{\zeta-\epsilon} + C_m \le C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C$$

Thus for any $x \in R^N$, we define

$$u_\varepsilon(x) := \sup_{\underline{u}\in\underline{\mathcal{S}_\varepsilon}} \underline{u}(x). \tag{3.6.7}$$

Thanks to (BARLES, 1994, Lemma 2.5, p.33), we have any subsolution $\underline{u}$ in $\underline{\mathcal{S}_\varepsilon}$ is a Lipschitz function on compact sets $K \subset R^N$ with Lipschitz constant C_K.

Now the aim is to prove that $u_\varepsilon \in \underline{\mathcal{S}_\varepsilon}$. we have

$$-C(1+|x|^{2\zeta})^{\frac{1}{2}-\epsilon} - C \le u_\varepsilon(x) \le C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C.$$

Let $x, y \in K$

$$\begin{aligned} u_\varepsilon(x) - u_\varepsilon(y) &= \sup_{\underline{\mathcal{S}}_\varepsilon} \underline{u}(x) - \sup_{\underline{\mathcal{S}}_\varepsilon} \underline{u}(y) \\ &\leq \sup_{\underline{\mathcal{S}}_\varepsilon} (\underline{u}(x) - \underline{u}(y)) \end{aligned}$$

Let $\underline{u}_n \in \underline{\mathcal{S}}_\varepsilon$ be a sequence such that $\lim_{n\to\infty} \underline{u}_n(x) = u_\varepsilon(x)$. Then by definition of u_ε, we have $u_\varepsilon(x) - u_\varepsilon(y) \leq \underline{u}_n(x) - \underline{u}_n(y)$. Taking limits as n goes to infinity,

$$\begin{aligned} u_\varepsilon(x) - u_\varepsilon(y) &\leq \lim_{n\to\infty} (\underline{u}_n(x) - \underline{u}_n(y)) \\ &\leq \sup_{\underline{u} \in \underline{\mathcal{S}}_\varepsilon} (\underline{u}(x) - \underline{u}(y)) \\ &\leq C_{\underline{\mathcal{S}}_\varepsilon, K} |x - y|. \end{aligned}$$

Now to prove that $u_\varepsilon \in \underline{\mathcal{S}}_\varepsilon$, we just need to show that u_ε is subsolution of (3.6.1). To do this we use the following lemmas that state that the supremum of two subsolutions is also a subsolution; the supremum of a sequence of subsolutions is also a subsolutions, and the supremum of a family of subsolutions is also a subsolution.

Lemma 3.6.3 *Let $\underline{u}_1,\ \underline{u}_2$ be subsolutions then $\underline{u} := \sup\{\underline{u}_1, \underline{u}_2\}$ also is subsolution.*

Let $x \in R^N$. There exist three possibilities :

Case 1. $\underline{u}(x) > \underline{u}_1(x)$

Case 2. $\underline{u}(x) > \underline{u}_2(x)$

Case 3. $\underline{u}(x) = \underline{u}_1(x) = \underline{u}_2(x)$

We use the definition of subsolution via superdifferential, see (BARLES, 1994, Th. 2.2 p. 17). In cases 1 and 2, we have $\underline{u} = \underline{u}_i$ in a neighborhood of x, hence $\partial^+ \underline{u}_i(x) = \partial^+ \underline{u}(x)$. In Case 3, let $p \in \partial^+ \underline{u}(x)$. Since $\underline{u}_i(y) \leq \underline{u}(y)$

$$\frac{\underline{u}_i(y) - \underline{u}_i(x) - p \cdot (y - x)}{|y - x|} \leq \frac{\underline{u}(y) - \underline{u}(x) - p \cdot (y - x)}{|y - x|}.$$

Taking $\limsup$

$$\limsup_{y\to x} \frac{\underline{u}_i(y) - \underline{u}_i(x) - p \cdot (y - x)}{|y - x|} \leq \limsup_{y\to x} \frac{\underline{u}(y) - \underline{u}(x) - p \cdot (y - x)}{|y - x|} \leq 0.$$

Hence $p \in \partial^+ \underline{u}_i(x)$.

Lemma 3.6.4 *Let $\underline{u}_n$ be a sequence of elements of $\underline{\mathcal{S}}_\varepsilon$ then $\underline{u} := \sup_n \underline{u}_n$ also is subsolution.*

Let us define $\vartheta_n = \sup_{p\leq n} \underline{u}_p$. Thanks to Lemma 3.6.3, ϑ_n is a subsolution. Since $\underline{u} = \sup_n \vartheta_n$ by Dini's Theorem, ϑ_n converges locally uniformly to $\underline{u}$. Therefore, by (BARLES, 1994, Th 2.3 p. 21) $\underline{u}$ is subsolution of H_ε.

Lemma 3.6.5 *Let $(\underline{u}_\kappa)_{\kappa\in\mathcal{F}}$ be a family of elements of $\underline{\mathcal{S}}_\varepsilon$ then $\underline{u} := \sup_{\kappa\in\mathcal{F}} \underline{u}_\kappa$ also is a subsolution.*

Given $x \in R^N$ there exists sequence κ_n such that $\lim_{n\to\infty} \underline{u}_{\kappa_n}(x) = \underline{u}(x)$. Let $y \in R^N$, and consider $\underline{u}^*(y) := \lim \underline{u}_{\kappa_n}(y)$. Then $\underline{u}^*(y)$ is subsolution by Lemma 3.6.4 with $\underline{u}(y) \geq \underline{u}^*(y)$ and $\underline{u}(x) = \underline{u}^*(x)$. Using the same arguments of lemma 3.6.3, we obtain $\partial^+\underline{u}^*(x) \subset \partial^+\underline{u}(x)$.

As a consequence of Lemma 3.6.5, considering $\mathcal{F} = \underline{\mathcal{S}}_\varepsilon$ we obtain that u_ε is a subsolution of (3.6.1).

Now let us prove that u_ε is a supersolution of (3.6.1).

Assume by contradiction that u_ε is not supersolution, then there exists a test function ϕ and x a local minimum point of $u_\varepsilon - \phi$ on $B_r(x)$, with $r > 0$, such that

$$H_\varepsilon(x, \phi(x), D\phi(x)) < 0. \tag{3.6.8}$$

We assume without loss of generality that $\phi(x) = u_\varepsilon(x)$. We assume also that x is a global minimum point of $u_\varepsilon - \phi$. To do this, we consider a bump function, Γ, with $\Gamma(y) = 1$ for any $y \in B_{\frac{r}{2}}(x)$ and $\Gamma(y) = 0$ for any y outside $B_r(x)$. We just need to exchange $\phi(y)$ by

$$\Gamma(y)\phi(y) + (1 - \Gamma(y))\left(-C(1 + |y|^{2\zeta})^{\frac{1}{2}-\epsilon} - C\right) \leq u_\varepsilon(y) \text{ for any } y \in R^N.$$

Let us define

$$u_\nu(y) := \max\{u_\varepsilon(y),\ \phi(y) + \nu - |y - x|^2\}.$$

We will prove that for ν small enough $u_\nu \in \underline{\mathcal{S}}_\varepsilon$ which is a contradiction with (3.6.7) since $u_\nu(x) = \phi(x) + \nu = u_\varepsilon(x) + \nu > u_\varepsilon(x)$. Indeed, by definition, $u_\nu(y) \geq u_\varepsilon(y) \geq -C(1+|y|^{2\zeta})^{\frac{1}{2}-\epsilon} - C$. Moreover, $u_\nu(y) \leq C(1+|y|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C$ for ν small enough. Otherwise, there exists a sequence (ν_n, y_n) such that $|y_n - x| \leq \nu_n$, ν_n goes to zero as n goes to infinity, and such that

$$C(1 + |y_n|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C < u_\nu(y_n) = \phi(y_n) + \nu_n - |y_n - x|^2.$$

Taking limits as n goes to infinity, we obtain $C(1 + |x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C \leq \phi(x) = u_\varepsilon(x)$. Hence, since $u_\varepsilon \in \underline{\mathcal{S}}_\varepsilon$, we have

$$C(1 + |y|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C = \phi(x) = u_\varepsilon(x).$$

Besides that, since x is global minimum point of $u_\varepsilon - \phi$ we have

$$\phi(y) \le u_\varepsilon(y) \le C(1+|y|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C \quad \text{for all } y \in R^N.$$

Therefore x is global minimum point of $C(1+|y|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C - \phi(y)$ which has null gradient at x, that is,

$$\nabla\phi(x) = \nabla(C(1+|y|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C).$$

Since $C(1+|y|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C$ is a differentiable supersolution and from (3.6.8),

$$0 > H_\varepsilon(x, \phi(x), \nabla\phi(x)) = H_\varepsilon(x, C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C, \nabla(C(1+|x|^{2\zeta})^{\frac{1}{2}-\frac{\epsilon}{2\zeta}} + C)) \ge 0$$

which is a contradiction.

Now just need to prove that u_ν is a subsolution of (3.6.1). Thanks to (3.6.8) and the continuity of H_ε we obtain that $\phi(y) + \nu - |y-x|^2$ is a subsolution of H_ε for y such that $|y-x|^2 \le \nu$ for ν small enough. Therefore, due to Lemma 3.6.3, u_ν is subsolution of H_ε for y such that $|y-x|^2 \le \nu$. Moreover, if $|y-x|^2 > \nu$, $u_\nu = u_\varepsilon$, because $u_\nu(y) \ge u_\varepsilon(y) \ge \phi(y)$ for all $y \in R^N$. Hence, u_ν is a subsolution.

Let us take limit of u_ε when ε goes to 0. We will prove that this limit is a viscosity solution of the Ishii Problem 3.0.1 and H_T. Therefore, by uniqueness we obtain that the limit of u_ε is U_A^-.

By definition, $(u_\varepsilon)_\varepsilon$ is locally uniformly bounded. Moreover, by coercivity of H_1 and H_2, we get that $(u_\varepsilon)_\varepsilon$ is locally uniformly Lipschitz. Thanks to Arzela-Ascoli's Theorem, we can extract a convergent subsequence which converges to a continuous function u. Therefore,

$$(\liminf)_* u_\varepsilon(x) = u(x) = (\limsup)^* u_\varepsilon(x).$$

Moreover H_ε also converges uniformly to H_i on compact sets $K \times W \times V \subset \Omega_i \times R \times R^N$

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon(x_N) H_1(x,t,p) + (1-\varphi_\varepsilon(x_N)) H_2(x,t,p) = H_i(x,t,p).$$

Since H_i is continuous on $\Omega_i \times R \times R^N$, then

$$(\liminf)_* H_\varepsilon(x,t,p) = H_i(x,t,p) = (\limsup)^* H_\varepsilon(x,t,p).$$

For $x \in \mathcal{H}$

$$\begin{aligned} (\liminf)_* H_\varepsilon(x,t,p) &= \min\{H_1(x,t,p), H_2(x,t,p)\}, \\ (\limsup)^* H_\varepsilon(x,t,p) &= \max\{H_1(x,t,p), H_2(x,t,p)\}. \end{aligned}$$

Thanks to Half-Relaxed limit Method, Theorem **??** we obtain that u is solution of Ishii Problem 1.7.1.

Now let us prove that u is a subsolution of H_T and consequently by Global Comparison Result Theorem 3.3.1, we obtain $u = U_A^-$.

Let x' be strict local maximum point of $u(y',0) - \phi(y')$. We need to proof

$$H_T(x, u(x), D_{\mathcal{H}}\phi(x')) \leq 0.$$

That is,

$$-l_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) + \lambda\phi\left(x\right) - b_{\mathcal{H}}\left(x, \alpha_1, \alpha_2, \mu\right) \cdot \left(D_{\mathcal{H}}\phi\left(x\right), 0\right) \leq 0$$

for all $(\alpha_1, \alpha_2, \mu) \in A_0(x)$. Take $(\alpha_1, \alpha_2, \mu) \in A_0(x)$, there exists $s \in R$ such that $\varphi(s) = \mu$, where φ is the function we use to define the Filippov approximation. We define

$$u_\varepsilon(y) - \phi(y') - \frac{1}{\varepsilon}\left|\frac{y_N}{\varepsilon} - s\right|^2 \text{ as } \varepsilon \text{goes to } 0$$

Let x_ε be the local maximum point of of $u_\varepsilon(y) - \phi(y') - \frac{1}{\varepsilon}\left|\frac{y_N}{\varepsilon} - s\right|^2$, then

$$\frac{1}{\varepsilon}\left|\frac{x_{\varepsilon N}}{\varepsilon} - s\right|^2 \to 0$$

Since u_ε is locally Lipschitz functions then $\partial^+ u_\varepsilon(x)$ is bounded, ϕ is fixed and $x_\varepsilon \to x$ as $\varepsilon \to 0$, then for ε small enough $\nabla\phi(x_\varepsilon)$ is bounded. Then $g_{\varepsilon,N} := \frac{2}{\varepsilon^2}(\frac{x_{\varepsilon N}}{\varepsilon} - s) < C$. Thus, since $(\alpha_1, \alpha_2, \mu) \in A_0(x)$,

$$\lim_{\varepsilon \to 0}(b_1(x_\varepsilon, \alpha_1) \cdot e_N + b_2(x_\varepsilon, \alpha_2) \cdot e_N) g_{\varepsilon,N} = 0 \tag{3.6.9}$$

As $|\frac{x_{\varepsilon N}}{\varepsilon} - s| \to 0$ and φ is continuous function

$$\varphi_\varepsilon(x_{\varepsilon N}) = \varphi(s) + o(1). \tag{3.6.10}$$

Since u_ε is subsolution of H_ε

$$\varphi_\varepsilon(x_{\varepsilon N}) H_1(x_\varepsilon, u_\epsilon(x_\varepsilon), D_{\mathcal{H}}\phi(x') + g_{\varepsilon,N}) + (1 - \varphi_\varepsilon(x_{\varepsilon N})) H_2(x_\varepsilon, u_\varepsilon(x_\varepsilon), D_{\mathcal{H}}\phi(x') + g_{\varepsilon,N}) \leq 0.$$

Thanks to (3.6.10)

$$\varphi(s) H_1(x_\varepsilon, u_\varepsilon(x_\varepsilon), D_{\mathcal{H}}\phi(x'_\varepsilon) + g_{\varepsilon,N}) + (1 - \varphi(s)) H_2(x_\varepsilon, u_\varepsilon(x_\varepsilon), D_{\mathcal{H}}\phi(x'_\varepsilon) + g_{\varepsilon,N}) \leq o(1)$$

Using the expression of H_i, we obtain

$$\begin{aligned}
&\varphi(s)(-l_1(x_\varepsilon, \alpha_1) + \lambda u_\varepsilon\left(x\right) - b_1\left(x_\varepsilon, \alpha_1\right) \cdot \left(D_{\mathcal{H}}\phi\left(x_\varepsilon\right), g_{\varepsilon,N}\right)) \\
&+(1 - \varphi(s))((-l_2(x_\varepsilon, \alpha_1) + \lambda u_\varepsilon\left(x_\varepsilon\right) - b_2\left(x_\varepsilon, \alpha_1\right) \cdot \left(D_{\mathcal{H}}\phi\left(x_\varepsilon\right), g_{\varepsilon,N}\right))) \leq o(1).
\end{aligned}$$

By (3.6.9), taking limits as ε goes to 0, we obtain

$$-l_{\mathcal{H}}(x, \alpha) - b_{\mathcal{H}}(x, \alpha) \cdot D_{\mathcal{H}}\phi\left(x\right) + \lambda u(x) \leq 0.$$

4 U^η FUNCTIONS

In this chapter we build a whole family of value functions, U^η, which are locally Lipschitz Ishii solutions. Under appropriate assumptions we obtain that the limit when η goes to zero is U_A^+ and when η goes to infinite is U_A^-. Hence, this family can be seen as a continuous path between U_A^- and U_A^+.

Let $\eta > 0$. We define the η-trajectories as

$$\tau_A^\eta(x) := \{(X_x(\cdot),\alpha(\cdot)) \in \tau_A(x) \mid \text{for a.e. } t \in \mathcal{E}_\mathcal{H},\ \ b_2(X_x(t),\alpha(t))\cdot e_N \geq -\eta,\ b_1(X_x(t),\alpha(t))\cdot e_N \leq \eta\} \tag{4.0.1}$$

These $\eta-$trajectories may not be regular for small η, but they are almost regular. Moreover $\tau_A^{reg}(x) \subset \tau_A^\eta(x)$ and if $\eta' \leq \eta$ we have $\tau_{\eta'}^A(x) \subset \tau_\eta^A(x)$.

Now, we define the associated value function.

$$U_A^\eta(x) = \inf_{\tau_A^\eta(x)} \left(\int_0^\infty l(X_x(t),\alpha(t))e^{-\lambda t}dt \right). \tag{4.0.2}$$

Let $\eta \leq \eta'$, as a consequence of the definitions we have the following relations between them

$$U_A^- \leq U_A^{\eta'} \leq U_A^\eta \leq \lim_{\eta\to 0} U_A^\eta \leq U_A^+, \tag{4.0.3}$$

4.1 DYNAMIC PROGRAMMING PRINCIPLE AND REGULARITY OF U^η FUNCTIONS

We have also the Dynamic Programming Principle for the value functions.

Theorem 4.1.1 (Dynamic Programming principle) *Let $K \subset A$ be a compact controls set, then*

$$U_K^-(x) = \inf_{\tau_K(x)} \left(\int_0^T l(X(t),\alpha(t))e^{-\lambda t}dt + U_K^-(x)\left(X\left(T\right)\right)e^{-\lambda T} \right). \tag{4.1.1}$$

$$U_A^\eta(x) = \inf_{\tau_A^\eta(x)} \left(\int_0^T l(X(t),\alpha(t))e^{-\lambda t}dt + U_A^\eta\left(X\left(T\right)\right)e^{-\lambda T} \right). \tag{4.1.2}$$

The proof is analogous to Theorem 2.2.1 in Chapter 1. We prove that U_A^η has the same regularity of U_A^+ and U_A^-, under the hypotheses in Proposition 2.3.1. Then, U_A^η is locally bounded and locally Lipschitz.

Theorem 4.1.2 *We assume that (HA), (HB) and (HC) then, for η small enough, the value functions U_A^η, are locally bounded and locally Lipschitz continuous functions defined from R^N to R.*

Since in Proposition 2.3.1 we proved the result with trajectories in $\tau_A^{reg}(x)$, and we have $\tau_A^{reg}(x) \subset \tau_A^\eta(x)$, then we have the proof is analogous for U_A^η. In the above Theorem prove regularity of U^η only for η small enough, but after proving that U^η is subsolution the Ishii Problem, we obtain regularity for all η.

4.2 U_A^η SOLUTIONS OF THE ISHII PROBLEM

In this section we prove that the value functions U_A^η are solutions of the Hamilton-Jacobi-Bellman problem,

$$\begin{cases} H_1(x, u(x), Du(x)) = 0 & \text{in } \Omega_1 \\ H_2(x, u(x), Du(x)) = 0, & \text{in } \Omega_2 \\ \min\{H_1(x, u(x), Du(x)), H_2(x, u(x), Du(x))\} \leq 0 & \text{in } \mathcal{H} \\ \max\{H_1(x, u(x), Du(x)), H_2(x, u(x), Du(x))\} \leq 0 & \text{in } \mathcal{H} \end{cases} \tag{4.2.1}$$

where

$$H_i(x, u, q) = \sup_{w \in A_i} (-b(x, w) \cdot q - l_i(x, w) + \lambda u)$$

We define the set of controls

$$A_0^\eta(x) = \{(\alpha_1, \alpha_2, \mu) \in A_0 \mid b_1(x, \alpha_1, \alpha_2) \cdot e_N \leq \eta,\ b_2(x, \alpha_1, \alpha_2) \cdot e_N \geq -\eta\}.$$

and the tangential Hamiltonian

$$H_T^\eta(x, \phi(x), D_{\mathcal{H}}\phi) = \sup_{(\alpha_1, \alpha_2, \mu) \in A_0^\eta(x)} -l_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) + \lambda\phi(x) - b_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) \cdot (D_{\mathcal{H}}\phi(x), 0)$$

Theorem 4.2.1 *Under hypotheses (HA), (HB) e (HC) the values functions U_A^η are viscosity solutions of the Hamilton-Jacobi-Bellman problem* (4.2.1). *Moreover, U_A^η is subsolution of H_T^η.*

Since we have the Dynamic Programming Principle 4.1.1, the proof of U_A^η being sub and supersolutions of (4.2.1) is analogous to the proof in Theorem 2.4.2.

Let us prove the second part of the theorem that states that the η-value functions are subsolutions of the tangential Hamiltonian H_T^η. Let ϕ be a test function, and let x be the point where the local maximum of $U_A^\eta - \phi$ is attained, and consider without lost of generality that the maximum is equal to zero, then $U_A^\eta(x) = \phi(x)$. Thanks to the proof of Lemma 2.3.3 we know that there exist $(\alpha_1, \alpha_2) \in A_1 \times A_2$ such that

$$H_1\left(x, \phi\left(x\right), \nabla\phi\left(x\right)\right) = -l_1(x, \alpha_1) + \lambda\phi\left(x\right) - b_1\left(x, \alpha_1\right) \cdot \nabla\phi\left(x\right)$$

$$H_2\left(x, \phi\left(x\right), \nabla\phi\left(x\right)\right) = -l_2(x, \alpha_2) + \lambda\phi\left(x\right) - b_2\left(x, \alpha_2\right) \cdot \nabla\phi\left(x\right).$$

We construct specific trajectories for these constant controls α_i that will help to prove that U_A^η is subsolution of H_T^η. Let $(\alpha_1, \alpha_2, \mu) \in A_0^\eta(x)$. Let us consider three cases:

Case 1: Let α_1, α_2 be such that

$$b_1(x, \alpha_1) \cdot e_N < \eta \quad \text{and} \quad b_2(x, \alpha_2) \cdot e_N > -\eta.$$

Let us consider a trajectory that satisfies the following differential equation locally

$$X_x'\left(t\right) = \frac{(-b_2(X_x(t), \alpha_2) \cdot e_N) b_1(X_x(t), \alpha_1) + (b_1(X_x(t), \alpha_1) \cdot e_N) b_2(X_x(t), \alpha_2)}{(b_1(X_x(t), \alpha_1) - b_2(X_x(t), \alpha_2)) \cdot e_N}, \;\; X_x(0) = x.$$

We build an η-trajectory $X_x(\cdot)$ that stays in $\mathcal{H}$ for all $t \in [0, T]$ and such that

$$b_1(X_x(t), \alpha_1) \cdot e_N < \eta \text{ and } b_2(X_x(t), \alpha_2) \cdot e_N > -\eta \text{ for } t \leq T.$$

we have $U_A^\eta(y) - \phi(y) \leq 0$ for all $y \in B_r(x)$ for r small enough, and $U_A^\eta(x) = \phi(x)$. Thanks to the continuity of the trajectory, there exists $T' > 0$ such that $X_x\left(t\right) \in B_r(x)$ for $t < T' < T$. By the Dynamic Programming Principle we have

$$\phi\left(x\right) \leq \int_0^{T'} l_{\mathcal{H}}(X_x(t), (\alpha_1, \alpha_2, \mu(X_x(t))) e^{-\lambda t} dt + \phi\left(X_x\left(T'\right)\right) e^{-\lambda T'}.$$

By the Fundamental Calculus Theorem, we obtain

$$0 \geq \int_0^{T'} \left[-l_{\mathcal{H}}(X_x(t), (\alpha_1, \alpha_2, \mu(X(t))) + \lambda\phi\left(X_x(t)\right) - b_{\mathcal{H}}\left(X_x(t), (\alpha_1, \alpha_2, \mu(X_x(t)))\right) \cdot \nabla\phi\left(X_x(t)\right)\right] e^{-\lambda t} d$$

Dividing by T' and taking limits as T' goes to 0 yields,

$$-b_{\mathcal{H}}(x, \alpha_1, \alpha_2, \mu) + \lambda\overline{\phi}\left(x\right) - l_{\mathcal{H}}\left(x, \alpha_1, \alpha_2, \mu\right) \cdot \left(D_{\mathcal{H}}\phi\left(x\right), 0\right) \leq 0.$$

Thus we have proved that U_A^η is a subsolution of H_T^η.

Case 2: Let α_1, α_2 be such that

$$b_1(x,\alpha_1)\cdot e_N = \eta \quad \text{and} \quad b_2(x,\alpha_2)\cdot e_N = -\eta.$$

Let $\varepsilon > 0$, thanks to the controllability hypothesis (HC), there exist controls α_1^- and α_2^+ such that for $\varepsilon < \eta < \delta$

$$b_1\left(x,\alpha_1^-\right) = (\eta-\varepsilon)e_N \;\text{ and }\; b_2\left(x,\alpha_2^+\right) = (-\eta+\varepsilon)e_N.$$

Thanks to the convexity hypothesis on the control variable, we have for $0<\nu,\nu'<1$, there exist controls α_1^ν and $\alpha_2^{\nu'}$ such that

$$\begin{aligned} b_1\left(x,\alpha_1^\nu\right) &= \nu b_1\left(x,\alpha_1^-\right) + (1-\nu) b_1\left(x,\alpha_1\right) \\ l_1\left(x,\alpha_1^\nu\right) &= \nu l_1\left(x,\alpha_1^-\right) + (1-\nu) l_1\left(x,\alpha_1\right) \end{aligned}$$

and

$$\begin{aligned} b_2(x,\alpha_2^{\nu'}) &= \nu' b_2\left(x,\alpha_2^+\right) + (1-\nu') b_2\left(x,\alpha_2\right) \\ l_2(x,\alpha_2^{\nu'}) &= \nu' l_2\left(x,\alpha_2^+\right) + (1-\nu') l_2\left(x,\alpha_2\right). \end{aligned}$$

Then we have

$$\begin{aligned} b_1\left(x,\alpha_1^\nu\right)\cdot e_N &= \nu(\eta-\varepsilon) + (1-\nu)\eta &&= \eta - \nu\epsilon &&< \eta \\ b_2\left(x,\alpha_2^{\nu'}\right)\cdot e_N &= \nu'(-\eta+\varepsilon) + (1-\nu')(-\eta) &&= -\eta + \nu'\epsilon &&> -\eta \end{aligned}$$

and

$$0 \geq \mu(\nu,\nu')(-l_1(x,\alpha_1^\nu)+\lambda\phi(x)-b_1(x,\alpha_1^\nu)\cdot\nabla\phi(x))+(1-\mu(\nu,\nu'))(-l_2(x,\alpha_2^{\nu'})+\lambda\phi(x)-b_2\left(x,\alpha_2^{\nu'}\right)\cdot\nabla\phi($$

where $\bar\mu(\nu,\nu') = \bar\mu(X_x(0)) = \frac{\nu'}{\nu'+\nu}$ and $(\alpha_1^\nu, \alpha_2^{\nu'}, \mu(\nu,\nu')) \in A_0^{reg}(x)$.

We consider sequences ν_n and ν_n' that go to 0 as n goes to ∞, we obtain by construction the following convergences as n goes to ∞

$$b_1\left(x,\alpha_1^{\nu_n}\right) \longrightarrow b_1\left(x,\alpha_1\right), \quad l_1\left(x,\alpha_1^{\nu_n}\right) \longrightarrow l_1\left(x,\alpha_1\right)$$

and

$$b_2(x,\alpha_2^{\nu_n'}) \longrightarrow b_2\left(x,\alpha_2\right), \quad l_2\left(x,\alpha_2^{\nu_n'}\right) \longrightarrow l_2\left(x,\alpha_2\right).$$

Given $0\leq\mu\leq 1$, we consider sequences ν_n, ν_n' that converge to 0 and $\frac{\nu_n}{\nu_n'} = \frac{1-\mu}{\mu}$. Then

$$\bar\mu(\nu_n,\nu_n') \longrightarrow \mu.$$

Since $(\alpha_1^{\nu_n}, \alpha_2^{\nu'_n}, \bar{\mu}(\nu_n, \nu'_n))$ is in case 1. Taking limits as n goes to ∞ we obtain

$$0 \geq \mu(-l_1(x,\alpha_1)+\lambda\phi(x)-b_1(x,\alpha_1)\cdot\nabla\phi(x))+(1-\mu)(-l_2(x,\alpha_2)+\lambda\phi(x)-b_2(x,\alpha_2)\cdot\nabla\phi(x)).$$

Arguing as in case 1 we obtain that U_A^η is a subsolution of H_T^η.

Case 3: We divide this case in two cases:

Case 3.1 Let α_1, α_2 be such that

$$b_1(x,\alpha_1)\cdot e_N = \eta \quad \text{and} \quad b_2(x,\alpha_2)\cdot e_N > -\eta.$$

Arguing as in case 2, let $\varepsilon > 0$, thanks to hypothesis (HC), there exists a control α_1^- such that such that

$$b_1\left(x,\alpha_1^-\right) = (\eta-\varepsilon)e_N.$$

For $0 < \nu < 1$, we can find a control α_1^ν such that

$$\begin{aligned} b_1(x,\alpha_1^\nu) &= \nu b_1\left(x,\alpha_1^-\right) + (1-\nu)b_1(x,\alpha_1) \\ l_1(x,\alpha_1^\nu) &= \nu l_1\left(x,\alpha_1^-\right) + (1-\nu)l_1(x,\alpha_1) \end{aligned}$$

Thus,

$$b_1(x,\alpha_1^\nu)\cdot e_N = \nu(\eta-\varepsilon) + (1-\nu)\eta < \eta - \nu\epsilon < \eta.$$

Taking limits as ν goes to 0, we obtain

$$0 \geq \mu(-l_1(x,\alpha_1)+\lambda\phi(x)-b_1(x,\alpha_1)\cdot\nabla\phi(x))+(1-\mu)(-l_2(x,\alpha_2)+\lambda\phi(x)-b_2(x,\alpha_2)\cdot\nabla\phi(x)).$$

Arguing as in case 1, we obtain that U_A^η is a subsolution of H_T^η, since $H_T^\eta(x,\phi,D_{\mathcal{H}}\phi) \leq 0$.

Case 3.2 Let α_1, α_2 be such that

$$b_1(x,\alpha_1)\cdot e_N < \eta \quad \text{and} \quad b_2(x,\alpha_2)\cdot e_N = -\eta.$$

Let $\varepsilon > 0$, thanks to hypothesis (HC), there exists a control α_2^+ such that for $\varepsilon < \eta < \delta$

$$\text{and} \quad b_2\left(x,\alpha_2^+\right) = (-\eta+\varepsilon)e_N.$$

For $0 < \nu' < 1$, we can find a control $\alpha_2^{\nu'}$ such that

$$\begin{aligned} b_2(x,\alpha_2^{\nu'}) &= \nu' b_2\left(x,\alpha_2^+\right) + (1-\nu')b_2(x,\alpha_2) \\ l_2(x,\alpha_2^{\nu'}) &= \nu' l_2\left(x,\alpha_2^+\right) + (1-\nu')l_2(x,\alpha_2) \end{aligned}$$

Then

$$b_2\left(x, \alpha_2^{\nu'}\right) \cdot e_N = \nu'(-\eta+\varepsilon) + (1-\nu')(-\eta) = -\eta + \nu'\epsilon > -\eta.$$

Taking limits as ν' goes to 0, we obtain

$$0 \geq \mu(-l_1(x,\alpha_1)+\lambda\phi(x)-b_1(x,\alpha_1)\cdot\nabla\phi(x))+(1-\mu)(-l_2(x,\alpha_2)+\lambda\phi(x)-b_2(x,\alpha_2)\cdot\nabla\phi(x)).$$

Therefore, U_A^η is a subsolution of H_T^η, since $H_T^\eta(x,\phi,D_{\mathcal{H}}\phi) \leq 0$.

Remark 4.2.2 *Observe that in the proof above, in all the three cases, we have that $H_T^\eta \geq H_T^{reg}$.*

Convergence of U_A^η Our η trajectories approximate regular trajectories when η goes to zero. This suggests that the limit of U^η when η goes to zero is U^+ . Moreover, when we increase η the set of trajectories grow, which indicates that the limit of U^η when η goes to ∞ is U^-. In this section we prove the two convergences.

In the two lemmas below we prove some relations between U_A^+, U_A^- and U_A^η.

Lemma 4.2.3 *Let $\eta \leq \eta'$, then*

$$U_A^- \leq U_A^{\eta'} \leq U_A^\eta \leq \lim_{\eta\to 0} U_A^\eta \leq U_A^+. \tag{4.2.2}$$

By definition, $\tau_A^{reg}(x) \subset \tau_A^\eta(x)$ and $U_A^{\eta'}(x) \leq U_A^\eta(x)$ for $\eta \leq \eta'$. Thus the result.

Lemma 4.2.4 *Let $\delta > 0$,*

$$\overline{B_\delta(0)} \subset \mathcal{B}_i^{A_i}(x), \tag{4.2.3}$$

where $\mathcal{B}_i^{A_i}(x) := \{b_i(x,w) | w \in A_i\}$. Assume that there exist λ, C and $\overline{C}$ such that

(HD)

$$|b_i(x,\alpha_i^m)| \leq \lambda(1+|\alpha_i^m|+|x|),$$

$$l_i(x,\alpha_i^m) \leq C|x|^{1-\epsilon} + \overline{C}|\alpha_i|.$$

for all $x \in R^n$ and $\alpha_i^m \in A_i^m$ for $i = 1,2$. Let $(X_x, \alpha^m) \in \tau_x(A_i^m)$. Then,

$$\lim_{T\to\infty} U_{A^m}^\eta(X_x(T))e^{-\lambda T} = \lim_{T\to\infty} U_{A^m}^-(X_x(T))e^{-\lambda T} = \lim_{T\to\infty} U_{A^m}^+(X_x(T))e^{-\lambda T} = 0$$

Denote $U^+_{A^m}$, $U^-_{A^m}$ and $U^\eta_{A^m}$ by U_{A^m}. Thanks to hypothesis (4.2.3), there exists a control $\alpha^m_{X_x(T)}$ such that $b_i(X_x(T),\alpha^m_{X_x(T)})=0$. Thus , we can consider the fixed trajectory at $X_x(T)$, then by definition of U_{A^m}, hypothesis (HD) and Lemma 2.2.3 we have

$$\begin{aligned} U^+_{A^m}(X_x(T))e^{-\lambda T} &\le \frac{l(X_x(T),\alpha^m_{X_x(T)})}{\lambda}e^{-\lambda T}\\ &\le \frac{C|X_x(T)|^{1-\epsilon}+\overline{C}}{\lambda}e^{-\lambda T}\\ &\le \frac{C(|x|+C^m_T e^{\lambda T})^{1-\epsilon}}{\lambda}e^{-\lambda T}+\frac{\overline{C}}{\lambda}e^{-\lambda T}\\ &= \frac{Ce^{\lambda T(1-\epsilon)}(|x|e^{-\lambda T}+C^m_T)^{1-\epsilon}}{\lambda}e^{-\lambda T}+\frac{\overline{C}}{\lambda}e^{-\lambda T}\\ &= \frac{C(|x|e^{-\lambda T}+C^m_T)^{1-\epsilon}}{\lambda}e^{-\lambda T\epsilon}+\frac{\overline{C}}{\lambda}e^{-\lambda T}. \end{aligned}$$

Hence, $\lim_{T\to\infty} U^+_{A^m}(X_x(T))e^{-\lambda T}=0$.

The following result proves the convergence of $U^\eta_{A^m}$ to $U^-_{A^m}$ as η goes to infinity in compact control set A^m.

Theorem 4.2.5 *Under the hypotheses (HA), (HB) (HC) and (HD), we have*

$$\lim_{\eta\to\infty} U^\eta_{A^m}(x)=U^-_{A^m}(x),$$

where $A^m\subset A$ is a compact control set.

Fix $T>0$. Thanks to the Dynamic Programming Theorem 4.1.1, there exists a sequence $(X^n_x,\alpha^n)\in\tau_{A^m}(x)$ such that

$$\lim_{n\to\infty}\left(\int_0^T l(X^n_x(t),\alpha^n(t))e^{-\lambda t}dt+U^-_{A^m}\left(X^n_x\left(T\right)\right)e^{-\lambda T}\right)=U^-_{A^m}(x).$$

We define

$$Y^n(s)=\int_0^s l(X^n_x(t),\alpha^n(t))dt,$$

and consider the trajectories

$$(X^n_x,Y^n):(0,T)\longrightarrow R^{N+1}.$$

Due to the Lemma 2.2.3 there exist R^m_T such that $|X^n_x(t)-x|\le R^m_T$ for any $t\in(0,T]$ then

$$|b(X^n_x(t),\alpha^n(t))|\le ||b||_{B_{R_T}(x)\times A^m},\quad \text{and}\quad |l(X^n_x(t),\alpha^n(t))|\le ||l||_{B_{R_T}(x)\times A^m}\quad \forall t\in(0,T].$$

Arguing as in Lemma 2.2.5, we obtain trajectories $(X^T_x,Y^T)(\cdot)$ in $[0,T]$ which satisfy that

$$\dot{X}^T_x(s)=b(X^T(s),\alpha^T(s))\quad \text{for a.e. } s\in[0,T]$$

and

$$U^-_{A^m}(x) = \lim_{n\to\infty}\left(\int_0^T l(X^n_x(t),\alpha^n(t))e^{-\lambda t}dt + U^-_{A^m}\left(X^n_x(T)\right)e^{-\lambda T}\right)$$
$$= \int_0^T l(X^T_x(t),\alpha^T(t))e^{-\lambda t}dt + U^-_{A^m}\left(X^T_x(T)\right)e^{-\lambda T}.$$

Due to Lemma 4.2.4, we know that $\lim_{T\to\infty} U^-_{A^m}\left(X^T_x(T)\right)e^{-\lambda T} = 0$. Thus,

$$U^-_{A^m}(x) = \lim_{T\to\infty}\int_0^T l(X^T_x(t),\alpha^T(t))e^{-\lambda t}dt. \tag{4.2.4}$$

Thanks to hypothesis (HC) there exists a control α_i^* such that $b_i(X^T_x(T),\alpha_i^*) = 0$. Then we can define a trajectory such that

$$X^T_x(t) := X^T_x(T) \quad \text{and} \quad \alpha^T_i(t) := \alpha_i^* \quad \text{for any} \quad t \geq T.$$

Taking $\eta^T = \max\{||b_1||_{R^m_T\times A^m_1}, ||b_2||_{R^m_T\times A^m_i}, T\}$ then $(X^T_x,\alpha^T) \in \tau^{\eta^T}_{A^m}(x)$. Hence, by definition of $U^{\eta^T}_{A^m}$

$$U^-_{A^m}(x) \leq U^{\eta^T}_{A^m}(x) \leq \int_0^T l(X^T_x(t),\alpha^T(t))e^{-\lambda t}dt + U^{\eta^T}_{A^m}\left(X^T_x(T)\right)e^{-\lambda T}. \tag{4.2.5}$$

Moreover, by definition

$$U^-_{A^m}\left(X^T_x(T)\right)e^{-\lambda T} \leq U^{\eta^T}_{A^m}\left(X^T_x(T)\right)e^{-\lambda T} \leq U^+_{A^m}\left(X^T_x(T)\right)e^{-\lambda T}.$$

Thanks to Lemma 4.2.4, we have

$$\lim_{T\to\infty} U^{\eta^T}_{A^m}\left(X^T_x(T)\right)e^{-\lambda T} = 0. \tag{4.2.6}$$

Taking limits as T goes to ∞ in (4.2.5) and thanks to (4.2.4) and (4.2.6), we obtain

$$U^-_{A^m}(x) \leq \lim_{T\to\infty} U^{\eta^T}_{A^m}(x) \leq \lim_{T\to\infty}\int_0^T l(X^T_x(t),\alpha^T(t))e^{-\lambda t}dt = U^-_{A^m}(x).$$

Therefore,

$$\lim_{T\to\infty} U^{\eta^T}_{A^m}(x) = U^-_{A^m}(x).$$

Observe that $\eta^T > T$ then $\lim_{T\to\infty}\eta^T = \infty$.

The following result is a consequence of Theorem 4.2.5 that states the convergence of U^η_A to U^-_A as η goes to infinity in a general unbounded control set A.

Corollary 4.2.5.1 *Under the hypotheses (HA), (HB) (HC) and (HD), we have*

$$\lim_{\eta\to\infty} U^\eta_A(x) = U^-_A(x).$$

Thanks to the definition of $U^-_{A^m}$, we have given $j \in N$ there exists m_j such that for $m \geq m_j$ then

$$|U^-_A(x) - U^-_{A^m}| < \frac{1}{4j}. \tag{4.2.7}$$

Then there exists $\eta(j)$ such that

$$|U^-_{A^{m_j}} - U^{\eta(j)}_{A^{m_j}}| \leq \frac{1}{4j}. \tag{4.2.8}$$

From (4.2.7) and (4.2.8), we obtain

$$|U^-_A(x) - U^{\eta(j)}_{A^{m_j}}| \leq |U^-_A(x) - U^-_{A^{m_j}}| + |U^-_{A^{m_j}} - U^{\eta(j)}_{A^{m_j}}| < \frac{1}{2j}.$$

Moreover, there exists $\overline{m} > m_j$ such that $|U^{\eta(j)}_{A^{\overline{m}}}(x) - U^{\eta(j)}_A(x)| < \frac{1}{2j}$. Furthermore,

$$U^{\eta(j)}_{A^{m_j}}(x) \geq U^{\eta(j)}_{A^{\overline{m}}}(x) \geq U^-_{A^{\overline{m}}}(x) \geq U^-_A(x).$$

Hence,

$$|U^-_A(x) - U^{\eta(j)}_{A^{\overline{m}}}(x)| \leq |U^-_A(x) - U^{\eta(j)}_{A^{m_j}}(x)| \leq \frac{1}{2j}.$$

So that,

$$|U^-_A(x) - U^{\eta(j)}_A(x)| \leq |U^-_A(x) - U^{\eta(j)}_{A^{\overline{m}}}| + |U^{\eta(j)}_{A^{\overline{m}}} - U^{\eta(j)}_A(x)| \leq \frac{1}{j}.$$

The following theorem states the convergence of $U^\eta_{A^m}$ to $U^+_{A^m}$ as η goes to zero where A^m is a compact control set.

Theorem 4.2.6 *Under the hypotheses (HA), (HB) (HC) and (HD), we have*

$$\lim_{\eta \to 0} U^\eta_{A^m}(x) = U^+_{A^m}(x),$$

where $A^m \subset A$ is a compact control set.

Consider a strictly positive decreasing sequence $\{\eta_n\}_{n \geq 0}$ such that $\lim_{n\to\infty} \eta_n = 0$. Since

$$U^\epsilon_{A^m}(x) \geq U^{\epsilon'}_{A^m}(x) \quad \text{for} \quad \epsilon \leq \epsilon'.$$

We assume that

$$U^{\eta_n}_{A^m}(x) < U^+_{A^m}(x), \tag{4.2.9}$$

otherwise, there exists $N \in N$ such that $U^{\eta_n}_{A^m}(x) = U^+_{A^m}(x)$ for all $n \geq N$ and we have the result. Given $T > 0$, thanks to (4.2.9), and due to the Dynamic Programming Principle Theorem 4.1.1, there exists $(X^{\eta_n}_x, \alpha^{\eta_n}) \in \tau^{\eta_n}_{A^m}(x)$ satisfying

$$\int_0^T l(X^{\eta_n}_x(t), \alpha^{\eta_n}(t))e^{-\lambda t}dt + U^{\eta_n}_{A^m}(X^{\eta_n}_x(T))e^{-\lambda T} \leq U^{\eta_n}_{A^m}(x) + \frac{U^+_{A^m}(x) - U^{\eta_n}_{A^m}(x)}{2} < U^+_{A^m}(x). \tag{4.2.10}$$

Let us define

$$Y^{\eta_n}(s) := \int_0^s l(X_x^{\eta_n}(t), \alpha^{\eta_n}(t))dt.$$

Since $b_i(X_x^{\eta_n}, \alpha^{\eta_n})$ and $l_i(X_x^{\eta_n}, \alpha^{\eta_n})$ are bounded in $[0,T]$ and thanks to Lemma 2.2.3, we have the curves $(X_x^{\eta_n}, Y^{\eta_n})(\cdot)$ are equicontinuous and uniformly bounded in $[0,T]$. Therefore, by Ascoli Azela's Theorem, we can extract a subsequence $(X_x^{\eta_n}, Y^{\eta_n})(\cdot)$ which converges uniformly to $Z^T := (X_x^T, Y^T)$ in $[0,T]$. Proceeding as in Lemma 2.2.5, we find a measurable control $\alpha^T(\cdot)$ such that

$$(\dot{X}_x^T(t), \dot{Y}^T(t)) = (b(X_x^T(t), \alpha^T(t)), l(X_x^T(t), \alpha^T(t))) \quad \forall t \in [0,T]. \tag{4.2.11}$$

Thus, $((b(X^{\eta_n}(t), \alpha^{\eta_n}(t)), l(X^{\eta_n}(t), \alpha^{\eta_n}(t))))$ converges weakly* to $\Big(b(X^T(t), \alpha^T(t)), l(X^T(t), \alpha^T(t))\Big)$ in $L^\infty([0,T];R)^2$.

Below we want to prove that the limit control α^T is regular. To do this we need to define for $z \in \mathcal{H}$,

$$K(z) := \{b_{\mathcal{H}}(z, \alpha') : \alpha' \in A_0^{reg}(z)\}.$$

$$E_{sing}^{\Upsilon}(T) := \{s \in [0,T] : X_x^T(s) \in \mathcal{H} \text{ and } dist(b_{\mathcal{H}}(X_x^T(s), \alpha^T(s)), K(X_x^T(s))) \geq \Upsilon\}.$$

$$E_{sing}(T) := \{s \in [0,T] : X_x^T(s) \in \mathcal{H} \text{ and } dist(b_{\mathcal{H}}(X_x^T(s), \alpha^T(s)), K(X_x^T(s))) > 0\}.$$

where $dist$ is the euclidian distance in R^N.

Note that

$$E_{sing}(T) = \bigcup_{j \in N} E_{sing}^{\frac{1}{j}}(T).$$

We claim that $|E_{sing}^{\Upsilon}(T)| = 0$ for any $\Upsilon > 0$. Then,

$$|E_{sing}(T)| \leq |\bigcup_{j \in N} E_{sing}^{\frac{1}{j}}(T)| = 0.$$

Since $l(X_x^{\eta_n}(t), \alpha^{\eta_n}(t))$ converges weakly* to $l(X_x^T(t), \alpha^T(t))$ in $L^\infty([0,T];R)$ and $e^{-\lambda t}$ is in $L^1([0,T];R)$, we obtain

$$\lim_{n\to\infty} \int_0^T l(X_x^{\eta_n}(t), \alpha^{\eta_n}(t))e^{-\lambda t}dt = \int_0^T l(X_x^T(t), \alpha^T(t))e^{-\lambda t}dt.$$

Thanks to hypothesis (HC), there exists a control α_i^* such that $b_i(X_x^T(T), \alpha_i^*) = 0$ then we can define $X_x^T(t) := X_x^T(T)$ and $\alpha_i^T(t) := \alpha_i^*$ for any $t \geq T$. Thus, $(X_x^T(\cdot), \alpha^T(\cdot)) \in \tau_{A^m}^{reg}(x)$.

Thanks to Dynamic Programming Theorem 4.1.1, (4.2.10) and (4.2.9), we have

$$\begin{aligned} U^+_{A^m}(x) - U^+_{A^m}(X^T_x(T))e^{-\lambda T} &\le \int_0^T l(X^T_x(t), \alpha^T(t))e^{-\lambda t}dt \\ &= \lim_{n\to\infty} \int_0^T l(X^{\eta_n}_x(t), \alpha^{\eta_n}(t))e^{-\lambda t}dt \\ &\le \lim_{n\to\infty} \left(\int_0^T l(X^{\eta_n}_x(t), \alpha^{\eta_n}(t))e^{-\lambda t}dt + U^{\eta_n}_{A^m}(X^{\eta_n}_x(T))e^{-\lambda T} \right) \\ &\le \lim_{n\to\infty} U^{\eta_n}_{A^m}(x) + \lim_{n\to\infty} \frac{U^+_{A^m}(x) - U^{\eta_n}_{A^m}(x)}{2} \\ &\le U^+_{A^m}(x). \end{aligned} \tag{4.2.12}$$

That is,

$$U^+_{A^m}(x) - U^+_{A^m}(X^T_x(T))e^{-\lambda T} \le \lim_{n\to\infty} U^{\eta_n}_{A^m}(x) + \lim_{n\to\infty} \frac{U^+_{A^m}(x) - U^{\eta_n}_{A^m}(x)}{2} \le U^+_{A^m}(x)$$

By Lemma 4.2.4 we know that

$$\lim_{T\to\infty} U^+_{A^m}(X^T_x(T))e^{-\lambda T} = 0.$$

Taking limits in (4.2.12) as T goes to infinity, we obtain that

$$U^+_{A^m}(x) \le \lim_{n\to\infty} U^{\eta_n}_{A^m}(x) + \lim_{n\to\infty} \frac{U^+_{A^m}(x) - U^{\eta_n}_{A^m}(x)}{2} \le U^+_{A^m}(x)$$

Thus, $U^+_{A^m}(x) = \lim_{n\to\infty} U^{\eta_n}_{A^m}(x) + \lim_{n\to\infty} \frac{U^+_{A^m}(x) - U^{\eta_n}_{A^m}(x)}{2}$. Therefore,

$$\lim_{n\to\infty} U^{\eta_n}_{A^m}(x) = U^+_{A^m}(x).$$

Since $U^{\epsilon}_{A^m}(x) \ge U^{\epsilon'}_{A^m}(x)$ for $\epsilon \le \epsilon'$, we have the convergence as η goes to zero, i.e.,

$$\lim_{\eta\to 0} U^{\eta}_{A^m}(x) = U^+_{A^m}(x).$$

To conclude the proof, we need to show that $|E^{\Upsilon}_{sing}(T)| = 0$ for any $\Upsilon > 0$. Assume by contradiction that there exist $\Upsilon > 0$ such that $|E^{\Upsilon}_{sing}(T)| > 0$. To prove this we need the following result.

Lemma 4.2.7 *Let $E \subset R^N$ and $C(R^N)$ be the set of all compact, convex subsets of R^N. Let $F : E \to R^N$ be a measurable function and $\mathcal{K} : E \to C(R^N)$ be a continuous set-valued map. Moreover we assume that there exists $\Upsilon > 0$ such that for almost everywhere $s \in E$, $dist(F(s), \mathcal{K}(s)) \ge \Upsilon$. Then there exists an affine function*

$$\Psi_s(z) : c(s) \cdot z + d(s)$$

with measurable coefficients c, d such that for almost any $s \in E$, $\Psi_s(F(s)) \le -1$ and $\Psi_s(z) \ge 1$ for z on $\mathcal{K}(s)$.

For fixed $s \in E$, $\mathcal{K}(s)$ is compact and convex, then the projection $\pi_{\mathcal{K}(s)} : R^N \to \mathcal{K}(s)$ is well-defined. Moreover, $s \longmapsto \mathcal{K}(s)$ is continuous set-valued map. Now, since $s \longmapsto F(s)$ is measurable, $s \longmapsto \pi_s(F(s))$ is also measurable and we denote by $P(s) := \pi_{\mathcal{K}(s)}(F(s))$ the projection of $F(s)$ in $\mathcal{K}(s)$. Thanks to (AUBIN; FRANKOWSKA, 1990, Cor. 8.2.13 p. 317) P is measurable.

Let us define by $M(s) := (F(s) + P(s))/2$ the middle point of segment $[F(s), P(s)]$, and set

$$e_s := \frac{P(s) - F(s)}{||P(s) - F(s)||}$$

so that e_s is the unit vector pointing in the direction from $F(s)$ towards $P(s)$. Then we define the affine function

$$\Psi_s(z) := \frac{2e_s}{\Upsilon} \cdot (z - M(s)) = c(s) \cdot z + d(s)$$

where $c(s) := \frac{2e_s}{\Upsilon}$ and $d(s) := -\frac{2e_s}{\Upsilon} \cdot M(s)$. Notice that since $P(s)$ is measurable, then so is $M(s)$ and e_s. So, the coefficients $c(s)$ and $d(s)$ are indeed measurable.

Let us recall that by assumption, $||F(s) - P(s)|| \geq \Upsilon$, then

$$(F(s) - M(s)) \cdot e_s \leq -\frac{\Upsilon}{2}, \quad (P(s) - M(s)) \cdot e_s \geq \frac{\Upsilon}{2}$$

This first inequality implies immediately that $\Psi_s(F(s)) = \frac{2}{\Upsilon} e_s \cdot (F(s) - M(s)) \leq -1$ and $\Psi_s(P(s)) = \frac{2}{\Upsilon} e_s \cdot (P(s) - M(s)) \geq 1$. For the second one, notice that since $K(s)$ is convex, for any $w \in K(s)$ there exists $v \in R^N$ such that $w = P(s) + v$ with $e_s \cdot v \geq 0$, which implies by our choice of Ψ_s that for any $w \in K(s)$

$$\Psi_s(w) \geq \Psi_s(P(s)) \geq 1$$

which gives the second property and completes the proof. Now, we apply Lemma 4.2.7 with $E = E^{\Upsilon}_{sing}(T)$, $F(s) = b_{\mathcal{H}}(X^T_x(s), \alpha^T(s))$ and $\mathcal{K}(s) = K(X^T_x(s))$. Then, there exists ψ_s defined as

$$\psi_s(z) = c(s) \cdot z + d(s) \tag{4.2.13}$$

with $(c, d)(\cdot) \in L^\infty([0, T], R^{N+1})$ such that for a.e. $s \in E^{\Upsilon}_{sing}(T)$, that is, we have $b_{\mathcal{H}}(X^T_x(s), \alpha^T(s))$ is not in $K(X^T_x(s))$ and

$$\psi_s(b_{\mathcal{H}}(X^T_x(s), \alpha^T(s))) \leq -1, \quad \psi_s(z) \geq +1 \text{ on } K(X^T_x(s)). \tag{4.2.14}$$

Observe that the affine function ψ_s is less or equal than -1 on the times where the trajectory is singular and ψ_s is greater or equal than 1 where the trajectory is regular.

Recall that we are assuming by contradiction that there exists $\Upsilon > 0$ such that $|E^{\Upsilon}_{sing}(T)| > 0$.

Note that $\dot{X}^{\eta_n}_x(\cdot)$ converges weakly* to $\dot{X}^T_x(\cdot)$ in $L^\infty([0,T];R^N)$. We define

$$\begin{aligned} I^{\eta_n} &:= \int_0^T \psi_s(\dot{X}_{X^{\eta_n}_x(s)})\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds \\ &= \int_0^T c(s)\cdot(\dot{X}_{X^{\eta_n}_x(s)})\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds + \int_0^T d(s)\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds. \end{aligned}$$

We will arrive to contradiction proving that I^{η_n} is strictly negative and positive.

Since $\dot{X}^{\eta_n}(\cdot)$ converges weakly* to $\dot{X}^T_x(\cdot) = b_{\mathcal{H}}(X^T_x(\cdot),\alpha^T(\cdot))$, $c \in L^\infty([0,T],R^N) \subset L^1([0,T],R^N)$, and thanks to (4.2.13) and (4.2.14), we obtain

$$\begin{aligned} \lim_{n\to\infty} I^{\eta_n} &= \lim_{\eta_n\to 0}\int_0^T c(s)\cdot(\dot{X^{\eta_n}_x}(s))\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds + \int_0^T d(s)\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds \\ &= \int_0^T c(s)\cdot(\dot{X}^T_x(s))\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds + \int_0^T d(s)\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds \\ &= \int_0^T \psi_s(\dot{X}^T_x(s))\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds \\ &= \int_0^T \psi_s(b_{\mathcal{H}}(X^T_x(s),\alpha^T(s)))\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds \\ &\le -|E^{\Upsilon}_{sing}(T)| < 0. \end{aligned}$$

Below we prove that $\lim\limits_{n\to\infty} I^{\eta_n} \ge 0$ what leads to contradiction.

Note that

$$\begin{aligned} \dot{X}^{\eta_n}_x(s) &= b(X^{\eta_n}_x(s),\alpha^{\eta_n}(s)) \\ &= b_1(X^{\eta_n}_x(s),\alpha^{\eta_n}(s))1_{\{X^{\eta_n}_x(s)\in\Omega_1\}} + b_2(X^{\eta_n}_x(s),\alpha^{\eta_n}(s))1_{\{X^{\eta_n}_x(s)\in\Omega_2\}}(x) \\ &+ b_{\mathcal{H}}(X^{\eta_n}_x(s),\alpha^{\eta_n}(s))1_{\{X^{\eta_n}_x(s)\in\mathcal{H}\}}(x). \end{aligned}$$

Then

$$\begin{aligned} &I^{\eta_n} = \\ &\int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)\left(b_1(X^{\eta_n}_x(s),\alpha^{\eta_n}(s))1_{\Omega_1}(X^{\eta_n}_x(s))ds + b_2(X^{\eta_n}_x(s),\alpha^{\eta_n}(s))1_{\Omega_2}(X^{\eta_n}_x(s))\right)ds \\ &+ \int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)b_{\mathcal{H}}(X^{\eta_n}_x(s),\alpha^{\eta_n}(s))1_{\mathcal{H}}(X^{\eta_n}_x(s))ds + \int_0^T d(s)\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds \end{aligned} \tag{4.2.15}$$

Let us divide I^{η_n} in two parts, we denote

$$I^{\eta_n}_1 = \int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)\left(b_1(X^{\eta_n}_x(s),\alpha^{\eta_n}(s))1_{\Omega_1}(X^{\eta_n}_x(s)) + b_2(X^{\eta_n}_x(s),\alpha^{\eta_n}(s))1_{\Omega_2}(X^{\eta_n}_x(s))\right)ds \tag{4.2.16}$$

$$I_2^{\eta_n} = \int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s) b_{\mathcal{H}}(X_x^{\eta_n}(s), \alpha^{\eta_n}(s))1_{\mathcal{H}}(X_x^{\eta_n}(s))ds + \int_0^T d(s)\mathbf{1}_{E^{\Upsilon}_{sing}(T)}(s)ds \tag{4.2.17}$$

Observe that

$$b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n}(s))1_{\{X_x^{\eta_n}\in\Omega_i\}}(s) = \int_{A_i} b_i(X_x^{\eta_n}(s), \alpha_i) d\vartheta_i^{\eta_n}(s)$$

where $\vartheta_i^{\eta_n}(s)$ is a Borel measure defined on A_i^m as $\vartheta_i^{\eta_n}(s)(V) := \delta_{\alpha_i^{\eta_n}(s)}(V)1_{\{X_x^{\eta_n}\in\Omega_i\}}(s)$. Thus,

$$\int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s) b_1(X_x^{\eta_n}(s), \alpha^{\eta_n}(s))1_{\Omega_1}(X_x^{\eta_n}(s))ds = \int_0^T \int_{A_i^m} c(s)1_{E^{\Upsilon}_{sing}(T)}(s) b_i(X_x^{\eta_n}(s), \alpha_i) d\vartheta_i^{\eta_n}(s)ds.$$

Consequently,

$$\begin{aligned}&\int_0^T \int_{A_i^m} b_i(X_x^{\eta_n}(s), \alpha_i) d\vartheta_i^{\eta_n}(s) = \\ &\int_0^T \int_{A_i^m} b_i(X_x^{\eta_n}(s), \alpha_i) - b_i(X_x^T(s), \alpha_i) d\vartheta_i^{\eta_n}(s)ds + \int_0^T \int_{A_i^m} b_i(X_x^T(s), \alpha_i) d\vartheta_i^{\eta_n}(s)ds\end{aligned}$$

Since $X_x^{\eta_n}$ converges uniformly to X_x^T and b_i is Lipschitz, then

$$\begin{aligned}&\lim_{n\to\infty} \int_0^T \int_{A_i^m} c(s)1_{E^{\Upsilon}_{sing}(T)}(s) b_i(X_x^{\eta_n}(s), \alpha_i) d\vartheta_i^{\eta_n}(s)ds \quad = \\ &\lim_{n\to\infty} \int_0^T \int_{A_i^m} c(s)1_{E^{\Upsilon}_{sing}(T)}(s) b_i(X_x^T(s), \alpha_i) d\vartheta_i^{\eta_n}(s)ds.\end{aligned}$$

Let $\mathcal{R}(A_i^m)$ be the Radon measures in A_i^m, and assume that $\vartheta_i^{\eta_n} : [0,T] \longrightarrow \mathcal{R}(A_i^m)$ is convergent, taking subsequence if necessary, with limit ϑ_i. Then thanks to Theorem **??**, we have

$$\lim_{n\to\infty} \int_0^T \int_{A_i^m} c(s)1_{E^{\Upsilon}_{sing}(T)}(s) b_i(X_x^T(s), \alpha_i) d\vartheta_i^{\eta_n}(s)ds = \int_0^T \int_{A_i^m} c(s)1_{E^{\Upsilon}_{sing}(T)}(s) b_i(X_x^T(s), \alpha_i) d\vartheta_i(s)ds.$$

Let us compute $\int_{A_i^m} b_i(X_x^T(s), \alpha_i) d\vartheta_i(s)$. For each $s \in [0,T]$, we have either $\vartheta_i(s)(A_i^m) = 0$, or $\vartheta_i(s)(A_i^m) \neq 0$ and $\frac{\vartheta_i(s)}{\vartheta_i(s)(A_i^m)}$ is a Borel probability measure on A_i^m.

If $\vartheta_i(s)(A_i^m) \neq 0$, since A_i^m is a separable metric space, then there exists an enumerable set $\{\alpha_i^1, \cdots, \alpha_i^\zeta, \cdots\}$ dense in A_i^m, i.e., there exist coefficients $\mu_{i,\zeta}^1(s), \cdots, \mu_{i,\zeta}^m(s) \in [0,1]$ with $\sum\limits_{j=1}^{\zeta} \mu_{i,\zeta}^j(s) = 1$, such that

$$\lim_{\zeta\to\infty} \mu_{i,\zeta}^1(s)\delta_{\alpha_i^1} + \mu_{i,\zeta}^2(s)\delta_{\alpha_i^2} + \cdots + \mu_{i,\zeta}^\zeta(s)\delta_{\alpha_i^\zeta} = \frac{\vartheta_i(s)}{\vartheta_i(s)(A_i^m)}.$$

By convexity of $\mathcal{B}_i(X_x^T(s))$ there exists $\overline{\alpha}_i^\zeta \in A_i^m$ such that

$$\begin{aligned} b_i(X_x^T(s),\overline{\alpha}_i^\zeta) &= \mu_{i,\zeta}^1(s)b_i(X_x^T(s),\alpha_i^1)+\mu_{i,\zeta}^2(s)b_i(X_x^T(s),\alpha_i^2)+\cdots+\mu_{i,\zeta}^\zeta(s)b_i(X_x^T(s),\alpha_i^\zeta) \\ &= \int_{A_i^m} b_i(X_x^T(s),\alpha_i)\ d\left(\mu_{i,\zeta}^1(s)\delta_{\alpha_i^1}+\mu_{i,\zeta}^2(s)\delta_{\alpha_i^2}+\cdots+\mu_{i,\zeta}^\zeta(s)\delta_{\alpha_i^\zeta}\right). \end{aligned} \tag{4.2.18}$$

Thus, by definition of weak* convergence of Radon measures, we have

$$\lim_{\zeta\to\infty}\int_{A_i^m} b_i(X_x^T(s),\alpha_i)d(\mu_{i,\zeta}^1(s)\delta_{\alpha_i^1}+\mu_{i,\zeta}^2(s)\delta_{\alpha_i^2}+\cdots+\mu_{i,\zeta}^\zeta(s)\delta_{\alpha_i^\zeta}) = \int_{A_i^m} b_i(X_x^T(s),\alpha_i)d\frac{\vartheta_i(s)}{\vartheta_i(s)(A_i^m)}. \tag{4.2.19}$$

From (4.2.18) and (4.2.19), we have

$$\lim_{\zeta\to\infty} b_i(X_x^T(s),\overline{\alpha}_i^\zeta) = \int_{A_i^m} b_i(X_x^T(s),\alpha_i)d\frac{\vartheta_i(s)}{\vartheta_i(s)(A_i^m)}. \tag{4.2.20}$$

From (4.2.20), and since $\mathcal{B}_i(X_x^T(s))$ is closed there exists a control $\overline{\alpha}_i(s)\in A_i^m$ satisfying

$$\vartheta_i(s)(A_i^m)b_i(X_x^T(s),\overline{\alpha}_i(s)) = \int_{A_i^m} b_i(X_x^T(s),\alpha_i)d\vartheta_i(s). \tag{4.2.21}$$

On the other hand, if $\vartheta_i(s)(A_i^m)=0$ then for any $\overline{\alpha}_i(s)\in A_i^m$

$$\vartheta_i(s)(A_i^m)b_i(X_x^T(s),\overline{\alpha}_i(s)) = 0 = \int_{A_i^m} b_i(X_x^T(s),\alpha_i)d\vartheta_i(s).$$

Thus, we arrive to the same expression for the integral of b_i in A_i^m.

Thanks to the Measurable Selection Theorem 2.1.1 we we can choose the control $\overline{\alpha}_i(\cdot)\in L^\infty([0,T];R^N)$ and thanks to (4.2.21), we have

$$\begin{aligned} &\lim_{n\to\infty}\int_0^T c(s)1_{E^\Upsilon_{sing}(T)}(s)b_i(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))1_{\Omega_i}(X_x^{\eta_n}(s))ds \\ &\qquad = \int_0^T c(s)1_{E^\Upsilon_{sing}(T)}(s)\vartheta_i(s)(A_i^m)b_i(X_x^T(s),\overline{\alpha}_i(s))ds. \end{aligned} \tag{4.2.22}$$

Analogously we consider measures $\vartheta_{\mathcal{H}}^{\eta_n}:[0,T]\longrightarrow\mathcal{R}(A^m)$ such that

$$\int_{A^m} b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha)d\vartheta_{\mathcal{H}}^{\eta_n}(s) := b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))1_{\{X^{\eta_n}\in\mathcal{H}\}}(s).$$

Similarly we can assume that $\vartheta_{\mathcal{H}}^{\eta_n}$ converges to some $\vartheta_{\mathcal{H}}$ and there exists $\overline{\alpha}_{\mathcal{H}}\in A^m$ such that

$$\lim_{n\to\infty} b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))1_{\{X_x^{\eta_n}\in\mathcal{H}\}}(s) = \vartheta_{\mathcal{H}}(s)(A^m)b_{\mathcal{H}}(X_x^T(s),\overline{\alpha}_{\mathcal{H}}(s)). \tag{4.2.23}$$

Consequently,

$$\begin{aligned} &\lim_{n\to\infty}\int_0^T c(s)1_{E^\Upsilon_{sing}(T)}(s)b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))1_{\mathcal{H}}(X_x^{\eta_n}(s))ds \\ &\qquad = \int_0^T c(s)1_{E^\Upsilon_{sing}(T)}(s)\vartheta_{\mathcal{H}}(s)(A)b_{\mathcal{H}}(X_x^T(s),\overline{\alpha}_{\mathcal{H}}(s))ds. \end{aligned} \tag{4.2.24}$$

For $s \in E^{\Upsilon}_{sing}(T)$ and the trajectories on the hyperplane, $X^T_x(s) \in \mathcal{H}$, we claim that

$$b_i(X^T_x(s), \overline{\alpha}_i(s)) \cdot e_N = 0.$$

Indeed, consider the sequence of Lipschitz continuous functions, $(X^{\eta_n}_{x,N})_+ := \max\{0, X^{\eta_n}_{x,N}\}$, which converges uniformly to $(X^T_{x,N})_+ := \max\{0, X^T_{x,N}\}$ on $[0,T]$, up to an additional extraction of subsequence. Thus, we have weak* convergence of the derivative

$$(\dot{X}^{\eta_n}_N)_+ \overset{*}{\rightharpoonup} (\dot{X}_N)_+. \tag{4.2.25}$$

Furthermore,

$$(\dot{X}^{\eta_n}_{x,N})_+(s) = (\dot{X}^{\eta_n}_{x,N})(s)1_{\{\dot{X}^{\eta_n}_{x,N} \in \Omega_i\}}(s) \quad \text{and} \quad (\dot{X}^T_{x,N})_+ = (\dot{X}^T_{x,N})(s)1_{\{X^T_{x,N} \in \Omega_i\}}(s). \tag{4.2.26}$$

The above derivative is in the sense of distributions. Therefore, we have the weak* convergence

$$(\dot{X}^{\eta_n}_{x,N})(s)1_{\{X^{\eta_n}_{x,N} \in \Omega_i\}}(s)1_{\{X^T_{x,N} \in \mathcal{H}\}}(s) \overset{*}{\rightharpoonup} (\dot{X}^T_{x,N})(s)1_{\{X^T_x \in \Omega_i\}}(s)1_{\{X^T_x \in \mathcal{H}\}}(s).$$

Since $1_{\{X^T_x \in \Omega_i\}}(s)1_{\{X^T_x \in \mathcal{H}\}}(s) = 0$, we have

$$(\dot{X}^{\eta_n}_{x,N})(s)1_{\{X^{\eta_n}_{x,N} \in \Omega_i\}}(s)1_{\{X^T_{x,N} \in \mathcal{H}\}}(s) \overset{*}{\rightharpoonup} 0.$$

Since

$$(\dot{X}^{\eta_n}_{x,N})(s)1_{\{X^{\eta_n}_{x,N} \in \Omega_i\}}(s)1_{\{X^T_{x,N} \in \mathcal{H}\}}(s) = b_i(X^{\eta_n}_x(s), \alpha^{\eta_n}_i(s)) \cdot e_N 1_{\{X^T_x(s) \in \mathcal{H}\}}$$

Since the left hand side converges weakly* to zero and the right hand side converges to $\vartheta_i(s)(A_i)b_i(X^T_x(s), \overline{\alpha}_i)(s) \cdot e_N 1_{\{X^T_x(s) \in \mathcal{H}\}}$, we obtain that

$$\vartheta_i(s)(A_i)b_i(X^T_x(s), \overline{\alpha}_i)(s) \cdot e_N 1_{\{X^T_x(s) \in \mathcal{H}\}}(s) = 0.$$

Then

$$b_i(X^T_x(s), \overline{\alpha}_i)(s) \cdot e_N = 0 \quad \text{on } \{X^T_x(s) \in \mathcal{H}\},$$

Now, we can choose a control in A^{reg}_0 such that $b_{\mathcal{H}}(X^T_x(s), \cdot)$ is in $K(X^T_x(s))$. Thanks to hypothesis (HC), there exists $\alpha^{**}_j(s)$ such that $b_j(X^T_x(s), \alpha^{**}_j(s)) = 0$, then $(\overline{\alpha}_i(s), \alpha^{**}_j(s), 1) \in A^{reg}_0(X^T_x(s))$ and

$$b_i(X^T_x(s), \overline{\alpha}_i(s)) = b_{\mathcal{H}}(X^T_x(s)\overline{\alpha}_i(s), \alpha^{**}_j(s), 1) \in K(X^T_x(s)).$$

Let us focus now on the term in $I_2^{\eta_n}$ given by

$$\int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))1_{\mathcal{H}}(X_x^{\eta_n}(s))ds$$

We proceed as in (BARLES; BRIANI; CHASSEIGNE, 2014, Lemma 5.3). What changes in our case is that we are not taking the limit of a sequence of regular trajectories but of η_n-trajectories. We prove that for s in

$$\Delta_n := (X_x^{\eta_n})^{-1}(\mathcal{H}) \cap E^{\Upsilon}_{sing}(T),$$

there exists a regular control $\tilde{\alpha}(s) \in A_0^{reg}(X_x^T(s))$ satisfying

$$b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s)) = b_{\mathcal{H}}(X_x^T(s),\tilde{\alpha}(s)) + \varsigma^n(s) \tag{4.2.27}$$

with $\varsigma^n(\cdot)$ measurable going uniformly to 0 when n goes to infinity and $\tilde{\alpha}(\cdot) \in L^\infty(\Delta_n; A^m)$.

We will focus on proving (4.2.27). To do this, first we will find a regular control $\alpha^n(\cdot)$ for the trajectory $X_x^{\eta_n}(\cdot)$ such that $b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^n(s))$ approaches well $b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))$. After that, we find a regular control $\tilde{\alpha}(\cdot)$ for trajectory $X_x^T(\cdot)$ such that $b_{\mathcal{H}}(X_x^T(s),\tilde{\alpha}(\cdot))$ approaches well $b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^n(s))$.

Let us find the regular controls α^n. For the first step, we define below the functions

$$\begin{aligned} \gamma_1^n(s) &:= -\max\{0, b_1(X_x^{\eta_n}(s),\alpha_1^{\eta_n}(s))\cdot e_N\} \\ \gamma_2^n(s) &:= -\min\{0, b_2(X_x^{\eta_n}(s),\alpha_2^{\eta_n}(s))\cdot e_N\}. \end{aligned} \tag{4.2.28}$$

On one hand, if $(\alpha_1^{\eta_n}(s),\alpha_2^{\eta_n}(s),\mu^{\eta_n}(s)) \in A_0^{reg}(X_x^{\eta_n}(s))$ then $\gamma_1^n(s) = 0 = \gamma_2^n(s)$.

Since $X_x^{\eta_n}(s) \in \mathcal{H}$, we have $b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))\cdot e_N = 0$ almost everywhere on Δ_n, then if $(\alpha_1^{\eta_n}(s),\alpha_2^{\eta_n}(s),\mu^{\eta_n}(s)) \notin A_0^{reg}(X_x^{\eta_n}(s))$, we have

$$\mu^{\eta_n} b_1(X_x^{\eta_n}(s),\alpha_1^{\eta_n}(s))\cdot e_N + (1-\mu^{\eta_n})b_2(X_x^{\eta_n}(s),\alpha_2^{\eta_n}(s))\cdot e_N = 0. \tag{4.2.29}$$

If $\mu^{\eta_n}(s) \in (0,1)$, we obtain $\gamma_i^n(s) := -b_i(X_x^{\eta_n}(s),\alpha_i^{\eta_n}(s))\cdot e_N$ and

$$\mu^{\eta_n}\gamma_1^n(s)+(1-\mu^{\eta_n})\gamma_2^n(s) = -(\mu^{\eta_n}b_1(X_x^{\eta_n}(s),\alpha_1^{\eta_n}(s))\cdot e_N+(1-\mu^{\eta_n})b_2(X_x^{\eta_n}(s),\alpha_2^{\eta_n}(s))\cdot e_N) = 0.$$

If $\mu^{\eta_n}(s) = 1$, $\gamma_1^n(s) = 0$ and $1-\mu^{\eta^n}(s) = 0$.
If $\mu^{\eta_n}(s) = 0$, we have $\gamma_2^n(s) = 0$. Then

$$\mu^{\eta_n}\gamma_1^n(s) + (1-\mu^{\eta_n})\gamma_2^n(s) = 0, \text{ for a.e. } s \in \Delta_n. \tag{4.2.30}$$

Moreover since b_i and $X_x^{\eta_n}$ are continuous and $\alpha_i^{\eta_n}(\cdot)$ is measurable, then γ_i^n is measurable. Since $|\gamma_i^n(s)| \leq \eta_n$ almost everywhere on Δ_n, then

$$\gamma_i^n \to 0, \quad \text{uniformly as } \; n \to \infty. \tag{4.2.31}$$

Moreover,

$$(b_1(X_x^{\eta_n}(s), \alpha_1^{\eta_n}(s)) + \gamma_1^n(s) e_N) \cdot e_N \leq 0, \;\; (b_2(X_x^{\eta_n}(s), \alpha_2^{\eta_n}(s)) + \gamma_2^n(s) e_N) \cdot e_N \geq 0. \tag{4.2.32}$$

Now we are going to use γ_i^n to build the regular control $\alpha^n(\cdot)$. To do so, we define

$$\beta^n(s) := \min \left\{ \frac{\delta - 2|\gamma_1^n(s)|}{\delta}, \frac{\delta - 2|\gamma_2^n(s)|}{\delta} \right\}.$$

Thanks to (4.2.31), we have $\beta^n(s) \to 1$ when $n \to \infty$.

We claim that there exists $\alpha_i^{\eta_n *}(s) \in A_i$ such that

$$b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n}(s)) = b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n *}(s)) + p_i^n(s), \tag{4.2.33}$$

where

$$p_i^n(s) := (1 - \beta^n(s)) \left(b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n}(s)) + \gamma_i^n(s) \right) - \gamma_i^n(s).$$

Note that p_i^n is measurable and goes uniformly to 0 when n goes to infinity.

Indeed, if $\beta^n(s) = 1$, then $|\gamma_1^n(s)| = 0 = |\gamma_2^n(s)|$. Thus, we can take

$$\alpha_i^{\eta_n *}(s) := \alpha_i^{\eta_n}(s) \text{ and } p_i^n(s) = (1-1)(b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n}(s)) + 0) - 0 = 0.$$

Otherwise, if $\beta^n(s) \neq 1$. For n large enough $0 \leq \beta^n(s) \leq 1$ and $\frac{|\gamma_i^n(s)|}{1-\beta^n(s)} \leq \frac{\delta}{2}$. Thanks to hypothesis (HC), there exist $\alpha_i^{\eta_n *}(s) \in A_i^m$ such that

$$\begin{aligned} \beta^n(s) \left(b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n}(s)) + \gamma_i^n(s) e_N \right) &= \beta^n(s) b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n}(s)) + (1 - \beta^n(s)) \left(\tfrac{\beta^n(s) \gamma_i^n(s) e_N)}{1-\beta^n(s)} \right. \\ &= b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n *}(s)). \end{aligned}$$

Furthermore, thanks to (4.2.32)

$$b_1(X_x^{\eta_n}(s), \alpha_1^{\eta_n *}(s)) \cdot e_N = \beta^n(s) (b_1(X_x^{\eta_n}(s), \alpha_1^{\eta_n}(s)) + \gamma_1^n(s)) \cdot e_N \leq 0$$

$$b_2(X_x^{\eta_n}(s), \alpha_2^{\eta_n *}(s)) \cdot e_N = \beta^n(s) (b_2(X_x^{\eta_n}(s), \alpha_2^{\eta_n}(s)) + \gamma_2^n(s)) \cdot e_N \geq 0$$

Thanks to (4.2.29) and (4.2.32)

$$\mu^{\eta_n}(s) b_1(X_x^{\eta_n}(s), \alpha_1^{\eta_n *}(s)) \cdot e_N + (1 - \mu^{\eta_n}(s)) b_2(X_x^{\eta_n}(s), \alpha_2^{\eta_n *}(s)) \cdot e_N$$

$$= \mu^{\eta_n}(s)\beta^n(s)b_1(X_x^{\eta_n}(s),\alpha_1^{\eta_n}(s))\cdot e_N + (1-\mu^{\eta_n}(s))\beta^n(s)(b_2(X_x^{\eta_n}(s),\alpha_2^{\eta_n}(s))$$

$$+\beta^n(s)\mu^{\eta_n}\gamma_1^n(s) + \beta^n(s)(1-\mu^{\eta_n})\gamma_2^n(s) = 0 \;\; a.e \text{ on } \mathcal{H}.$$

Then, we have $(\alpha_1^{\eta_n *}(s)\alpha_2^{\eta_n *}(s), \mu^{\eta_n}(s)) \in A_0^{reg}(X_x^{\eta_n}(s))$.

However $(\alpha_1^{\eta_n *}(\cdot)\alpha_2^{\eta_n *}(\cdot), \mu^{\eta_n}(\cdot))$ is not necessarily measurable. But we can use Filippov Theorem 2.1.2 to obtain measurable controls. We use Filippov in the following way: $h(s) = b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n}(s)) - p_i^n(s)$ on $[0,T]$, $F(s) = A_i$ on $[0,T]$, $Z = \Delta_n$ and $g(s,\alpha_i) = b_i(X_x^{\eta_n}(s), \alpha_i)$. Indeed,

$$b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n *}(s)) \in \mathcal{B}_i(X_x^{\eta_n}(s)).$$

So, we can find $\alpha_i^n(\cdot)$ measurable such that

$$b_i(X_x^{\eta_n}(s), \alpha_i^{\eta_n}(s)) - p_i^n(s) = b_i(X_x^{\eta_n}(s), \alpha_i^n(s)).$$

Thus, $\alpha^n(\cdot) := (\alpha_1^n(\cdot), \alpha_2^n(\cdot), \mu^n(\cdot)) \in A_0^{reg}(X_x^{\eta_n}(s))$ is measurable.

Consequently,

$$b_{\mathcal{H}}(X_x^{\eta_n}(s), \alpha^{\eta_n}(s)) = b_{\mathcal{H}}(X_x^{\eta_n}(s), \alpha^n(s)) + p^n(s)$$

where $p^n(s) = \mu^{\eta_n}(s)p_1^n(s) + (1-\mu^{\eta_n}(s))p_2^n(s)$. Note that p^n is measurable and goes uniformly to 0 when n goes to infinity since p_i^n goes to zero as n goes to infinity.

Since $(\alpha_1^n(s), \alpha_2^n(s), \mu^{\eta_n}(s)) \in A_0^{reg}(X_x^{\eta_n}(s))$ and thanks to (BARLES; BRIANI; CHASSEIGNE, 2014, Lemma 5.3) there is a measurable control $\tilde{\alpha} \in A_0^{reg}(X_x^T(s))$ satisfying

$$b_{\mathcal{H}}(X_x^{\eta_n}(s), \alpha^n(s)) = b_{\mathcal{H}}(X_x^T(s), \tilde{\alpha}(s)) + \sigma^n(s). \tag{4.2.34}$$

with $\sigma^n(\cdot)$ measurable that goes uniformly to 0 when n goes to infinity. Therefore,

$$b_{\mathcal{H}}(X_x^{\eta_n}(s), \alpha^{\eta_n}(s)) = b_{\mathcal{H}}(X_x^T(s), \tilde{\alpha}(s)) + \sigma^n(s) + p^n(s)$$

and we can choose $\varsigma^n(s) := \sigma^n(s) + p^n(s)$ in (4.2.27).

We prove (4.2.34) as follows. Define,

$$\tilde{\sigma}_i^n(s) := b_i(X_x^{\eta_n}(s), \alpha_i^n(s)) - b_i(X_x^T(s), \alpha_i^n(s)).$$

Since b_i, X^{η_n} and X_x^T are continuous and $\alpha_i^n(\cdot)$ is measurable, then $\tilde{\sigma}_i^n(\cdot)$ is measurable. Furthermore, $\tilde{\sigma}_i^n(\cdot)$ converges uniformly to 0 as n goes to infinity, because

$$|\tilde{\sigma}_i^n(s)| = |b_i(X_x^{\eta_n}(s), \alpha_i^n(s)) - b_i(X_x^T(s), \alpha_i^n(s))| \leq C_b|X_x^{\eta_n}(s) - X_x^T(s)|$$

and $X_x^{\eta_n}(\cdot)$ converges uniformly to $X_x^T(\cdot)$. Thus,

$$b_i(X_x^{\eta_n}(s),\alpha_i^n(s)) = b_i(X_x^T(s),\alpha_i^n(s)) + \tilde{\sigma}_i^n(s).$$

There exist $\tilde{\alpha}_i^*(s) \in A_i$ and $\sigma_i^n(s)$ such that

$$b_i(X_x^T(s),\alpha_i^n(s)) = b_i(X_x^T(s),\tilde{\alpha}_i^*(s)) + \sigma_i^n(s).$$

with $\sigma_i^n(s) = (1-\beta^n(s))(b_i(X_x^T(s),\alpha_i^n(s))+\tilde{\sigma}_i^n(s))-\tilde{\sigma}_i^n(s)$, where $\beta^n(s) = \min\{\frac{\delta-2|\tilde{\sigma}_1^n(s)|}{\delta}, \frac{\delta-2|\tilde{\sigma}_2^n(s)|}{\delta}\}$. Thus, σ_i^n is measurable and goes uniformly to 0 when n goes to infinity.

If $\beta^n(s) = 1$, then $|\tilde{\sigma}_1^n(s)| = 0 = |\tilde{\sigma}_2^n(s)|$. Thus, we can take $\tilde{\alpha}_i^*(s) := \alpha_i^n(s)$ and $\sigma_i^n(s) = (1-1)(b_i(X_x^{\eta_n}(s),\alpha_i^n(s)) + 0) - 0 = 0$.

On the other hand, if $\beta^n(s) \neq 1$, we proceed as follows,

$$\begin{aligned}\beta^n(s)b_i(X_x^{\eta_n}(s),\alpha_i^n(s)) &= \beta^n(s)\left(b_i(X_x^T(s),\alpha_i^n(s)) + \tilde{\sigma}_i^n(s)\right)\\ &= \beta^n(s)b_i(X_x^T(s),\alpha_i^n(s)) + (1-\beta^n(s))\left(\tfrac{\beta^n(s)\tilde{\sigma}_i^n(s)}{1-\beta^n(s)}\right)\\ &= b_i(X_x^T(s),\tilde{\alpha}_i^*(s)).\end{aligned}$$

Thus,

$$b_i(X_x^T(s),\tilde{\alpha}_i^*(s))\cdot e_N = \beta^n(s)b_i(X_x^{\eta_n}(s),\alpha_i^n(s))\cdot e_N,$$

and

$$\begin{aligned}&\mu^{\eta_n}(s)b_1(X_x^T(s),\tilde{\alpha}_1^*(s))\cdot e_N + (1-\mu^{\eta_n}(s)))b_2(X_x^T(s),\tilde{\alpha}_2^*(s)(s))\cdot e_N\\ &= \mu^{\eta_n}(s)\beta^n(s)b_1(X_x^{\eta_n}(s),\alpha_1^n(s))\cdot e_N + (1-\mu^{\eta_n}(s))\beta^n(s)(b_2(X_x^{\eta_n}(s),\alpha_2^n(s))\cdot e_N\\ &= \beta^n(s)b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^n(s))\cdot e_N = 0.\end{aligned}$$

Hence $(\tilde{\alpha}_1^*(s),\tilde{\alpha}_2^*(s),\mu^{\eta_n}(s)) \in A_0^{reg}(X_x^T(s))$, however it is not necessarily measurable. Arguing as above, thanks to Filippov's Theorem 2.1.2, we obtain measurable controls $\tilde{\alpha}_1(\cdot), \tilde{\alpha}_2(\cdot)$ such that

$$b_i(X_x^T(s),\tilde{\alpha}_i(s)) = b_i(X_x^{\eta_n}(s),\alpha_i^n(s)) - \sigma_i^n(s) = b_i(X_x^T(s),\tilde{\alpha}_i^*(s)).$$

Thus, $\tilde{\alpha}(s) = (\tilde{\alpha}_1(s),\tilde{\alpha}_2(s),\mu^{\eta_n}(s)) \in A_0^{reg}(X_x^T(s))$ is measurable and

$$b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^n(s)) = b_{\mathcal{H}}(X_x^T(s),\tilde{\alpha}(s)) + \sigma^n(s),$$

with $\sigma^n(s) = \mu^{\eta_n}(s)\sigma_1^n(s) + (1-\mu^{\eta_n}(s))\sigma_2^n(s)$. Note that σ^n is measurable and go uniformly to 0 when n goes to infinity.

Now we consider α_* satisfying

$$c(s)b_{\mathcal{H}}(X_x^T(s),\alpha_*(s)) = \min_{\left\{a\in A_0^{reg}\ :\ b_{\mathcal{H}}(X_x^T,a)\in K(X_x^T)\right\}} c(s)b_{\mathcal{H}}(X_x^T(s),a). \tag{4.2.35}$$

The minimum on the left hand side exists due to the compactness of $K(X_x^T(s))$. We can assume that α_* is measurable by the Measurable Selection Theorem 2.1.1. Also, note that α_* does not depend on η_n. Thanks to (4.2.35), we have,

$$\begin{aligned}
&\int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))1_{\mathcal{H}}(X_x^{\eta_n}(s))ds\\
&= \int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)b_{\mathcal{H}}(X_x^T(s),\tilde{\alpha}(s))1_{\mathcal{H}}(X_x^{\eta_n}(s))ds + o_{\eta_n}\\
&\geq \int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)b_{\mathcal{H}}(X_x^T(s),\alpha_*(s))1_{\mathcal{H}}(X_x^{\eta_n}(s))ds + o_{\eta_n}.
\end{aligned}$$

Taking limits as n goes to infinity, we obtain thanks to (4.2.24), that

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)b_{\mathcal{H}}(X_x^{\eta_n}(s),\alpha^{\eta_n}(s))1_{\mathcal{H}}(X_x^{\eta_n}(s))ds\\
&\geq \int_0^T c(s)1_{E^{\Upsilon}_{sing}(T)}(s)\vartheta_{\mathcal{H}}(s)(A^m)b_{\mathcal{H}}(X_x^T(s),\alpha_*(s))ds.
\end{aligned} \tag{4.2.36}$$

5 OTHER PROBLEMS

5.1 STRATIFICATIONS

In this thesis we work with discontinuities in the hyperplane $\mathcal{H}$, but our set of discontinuities can be much more general. Emmanuel and Guy have done this via Whitney Stratification in (BARLES; CHASSEIGNE, 2018). Now, let us present the definition of stratification. Let $M^0, M^1, \cdots, M^N$ be disjoint submanifolds of R^N such that $R^N = M^0 \cup M^1 \cup \cdots \cup M^N$ where each M^k decomposes as $M^k = \bigcup_j M_j^k$, for $k \in \{0, 1, \ldots, N\}$.

Definition 5.1.1 *We say that $M = (M^k)_{k=0}^N$ is an Admissible Flat Stratification of R^N if the following hypotheses are satisfied*

If $x \in M^k$ for $k = 0, 1 \cdots, N$ there exists $r = r_x$ such that

- *$B(x,r) \cap M^k = B(x,r) \cap (x + V_k)$, where V_k is a k-dimensional linear subspace of R^N.*
- *if $l < k$, $B(x,r) \cap M^l = \emptyset$*
- *For any $l > k$, then $B(x,r) \cap M^l$ is either empty or has at most a finite number of connected components;*
- *For any $l > k$, $B(x,r) \cap M^l \neq \emptyset$ if and only if $x \in \partial M_j^l$.*

If $M_i^k \cap \overline{M}_j^l \neq \emptyset$ with $l > k$ then $M_i^k \subset \overline{M}_j^l$;

Moreover, we have $\overline{M}^k \subset M^0 \cup M^1 \cup \cdots \cup M^k$.

With this definition we can present a more general notion of stratification.

Definition 5.1.2 *We say that $M = (M^k)_{k=0}^N$ is a Regular General Stratification of R^N if the following hypotheses are satisfied*

- *$R^N = M^0 \cup M^1 \cup \cdots \cup M^N$;*
- *for any $x \in R^N$, there exists $r = r(x) > 0$ and a $C^{1,1}$ change of coordinates $\psi^x : B(x,r) \to R^N$ such that $\psi^x(x) = x$ and $\{\psi^x(M \cap B(x,r))\}_{k=0}^N$ is the restriction to $\psi^x(B(x,r))$ of an admissible flat stratification.*

When the set of discontinuities form an stratification, then the results that we have throughout this Thesis, may be reached with these new discontinuities.

In (BARLES; CHASSEIGNE, 2018) Guy and Emmanuel handle the problem with bounded data. Our aim is to prove the results under the same settings in this Thesis, with unbounded cost and dynamic functions, and unbounded control set.

5.2 KPP EQUATION

Reaction-diffusion equation of type KPP appears in different models in Physics, combustion for example, and Biology. In simple format, the KPP equation is expressed as

$$u_t - \frac{1}{2}\nabla u = cu(1-u) \text{ in } R^N \times (0,\infty).$$

The important thing in applications is to understand the behavior of u for long times and large space. Therefore, we consider the rescaling $u^\epsilon(x,t) = u(\frac{x}{\epsilon}, \frac{t}{\epsilon})$ which solves

$$u_t^\epsilon - \epsilon\nabla u^\epsilon = \epsilon^{-1}cu(1-u) \text{ in } R^N \times (0,\infty).$$

In (BARLES; CHASSEIGNE, 2018), Emanuel and Guy have studied in the viscosity sense, the KPP equation with discontinuities. We would like to apply the unbounded framework to deal with these type of problems.

5.3 OTHER VALUES FUNCTIONS

There exist many family of functions that may be considered, as we did in Chapter 3. Another family of value functions that we can consider is the following defined below.

$$\tau^\nu_{A(\mathcal{S})}(x) := \{(X_x(\cdot),\alpha(\cdot)) \in \tau_A(x) | \text{for a.e. } t \in \mathcal{E}_\mathcal{H},\ b_2(X_x(t),\alpha(t))\cdot e_N \leq \nu,\ b_1(X_x(t),\alpha(t))\cdot e_N \geq -\nu\} \tag{5.3.1}$$

We define

$$U^\nu_{A(\mathcal{S})}(x) := \inf_{\tau^\nu_A(x)} \left(\int_0^T l(X(t),\alpha(t))e^{-\lambda t}dt + U^\nu_{A(\mathcal{S})}\left(X\left(T\right)\right)e^{-\lambda T} \right). \tag{5.3.2}$$

Of course two questions arise: what is the limit of $U^\nu_A(x)$ when ν goes to zero? And when ν goes to infinity?

A good candidate as a limit of $U^\nu_{A(\mathcal{S})}$ when ν goes to zero is

$$U^\mathcal{S}_A(x) := \inf_{\tau^\mathcal{S}_A(x)} \int_0^\infty l(X(t),\alpha(t))e^{-\lambda t}dt \tag{5.3.3}$$

where,

$$\tau_A^{\mathcal{S}}(x) := \{(X_x(\cdot),\alpha(\cdot)) \in \tau_A(x) \,|\, \text{for a.e. } t \in \mathcal{E}_{\mathcal{H}},\ \ b_2(X_x(t),\alpha_2(t))\cdot e_N \le 0,\ b_1(X_x(t),\alpha_1(t))\cdot e_N \ge 0\}. \tag{5.3.4}$$

And when ν goes to infinity we suspect that the limit of $U^{\nu}_{A(\mathcal{S})}$ is 0.

5.4 CONTROLS IN L_p

We work with controls in $L^{\infty}(A)$. But what happens if we work with controls in $L^{p}(A)$? The main difficulty of this approach is to obtain comparison and limit results.

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APPENDIX A – DISCONTINUOUS APPROACH

In the discontinuous approach, we work with supersolutions that are lower semicontinuous

Definition 6.3. *Let X be a topological space and $f : X \to (-\infty, \infty]$. We say that f is lower semicontinuous when* $f(x_0) \leq \liminf_{x \to x_0} f(x)$.

Equivalently we have the definitions in the proposition below.

Proposition 6.1. *Let X a topological space and $f : X \to (-\infty, \infty]$. Then f is lower semicontinuous iff*

(i) $f^{-1}((-\infty, \lambda])$ *is closed set for all* $\lambda \in \mathbb{R}$.

(ii) $f(x_0) \leq \liminf_{x \to x_0} f(x)$.

Thanks to characterization (i) we obtain the following result.

Theorem 6.2. *Let X be a normed space and f strong lower continuous and convex function then f is lower semicontinuous with respect a weak topology.*

Proof. Since f is lower semicontinuous and convex function then the set $f^{-1}(-\infty, \lambda]$ is convex and strong closed set. Thus, thanks to Mazur Theorem (see [13] page 61), $f^{-1}(-\infty, \lambda]$ is convex and weak closed set. Therefore, due Proposition 6.1 f is weak lower semicontinuous. □

The advantage of the discontinuous approach is that we can cross the limit without requiring almost anything, just uniform local bound. To do this, we need the following notion of limit

Given g_ε a sequence of uniformly locally bounded functions, the **half-relaxed limits** of $(g_\varepsilon)_\varepsilon$ is defined by

$$(\liminf)_* g_\varepsilon(x) = \liminf_{(\varepsilon,\ y) \to (0,\ x)} g_\varepsilon(y) := \sup_{\varepsilon \to 0} \inf\{g_\gamma(y) : 0 < \gamma < \varepsilon, |x - y| < \varepsilon\}$$

$$(\limsup)^* g_\varepsilon(x) = \limsup_{(\varepsilon,\ y) \to (0,\ x)} g_\varepsilon(y) := \inf_{\varepsilon \to 0} \sup\{g_\gamma(y) : 0 < \gamma < \varepsilon, |x - y| < \varepsilon\}$$

This notion of limit allows us to obtain the following result

Theorem 6.3 (Half-Relaxed Limit Method). *Given sequences $(u_\varepsilon)_\varepsilon$ and $(H^\varepsilon)_\varepsilon$ uniformly locally bounded such that u_ε is supersolution of*

$$H^\varepsilon(x, v(x), Dv(x)) = 0 \quad \text{in} \ \ \mathbb{R}^N.$$

Let

$$u(x) := (\liminf)_* u_\varepsilon(x) \quad \text{and} \quad H(x, u, p) = (\limsup)^* H_\varepsilon(x, u, q).$$

Then u is a viscosity supersolution of

$$H(x, v(x), Dv(x)) = 0 \quad \text{in} \ \ \mathbb{R}^N.$$

Analogously, given sequences $(u_\varepsilon)_\varepsilon$ *and* $(H^\varepsilon)_\varepsilon$ *uniformly locally bounded such that* u_ε *is subsolution of*

$$H^\varepsilon(x, v(x), Dv(x)) = 0 \quad \text{in} \ \ \mathbb{R}^N.$$

Let

$$u(x) := (\limsup)^* u_\varepsilon(x) \quad \text{and} \quad H(x, u, p) = (\liminf)_* H_\varepsilon(x, u, q).$$

Then u *is a viscosity subsolution of*

$$H(x, v(x), Dv(x)) = 0 \quad \text{in} \ \ \mathbb{R}^N.$$

Proof. See [20, Th A.2, p. 577], [4, Th 4.1, p. 85] and [7, Th 3.1.3, p. 29]. □

In particular we have the following lemma

Lemma 6.4. *Consider* u_ε *a sequence of uniformly locally bounded continuous functions. If* u_ε *is non-increasing, then*

$$\limsup_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y) = \inf_\varepsilon u_\varepsilon(x) \qquad \text{and} \qquad \liminf_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y) = \left(\inf_\varepsilon u_\varepsilon\right)_* (x).$$

If u_ε *is non-decreasing, then*

$$\liminf_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y) = \sup_\varepsilon u_\varepsilon(x) \qquad \text{and} \qquad \limsup_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y) = \left(\sup_\varepsilon u_\varepsilon\right)^* (x).$$

where $(\cdot)^*$ *and* $(\cdot)_*$ *are the envelope of u.s.c and l.s.c functions respectively.*

Proof. See [4, p. 91]. □

APPENDIX B – RELAXED CONTROL

The relaxed control method plays an important role in the theory of viscosity solutions of the Bellman-Hamilton-Jacobi equation. In this section we present this method and its main result.

We denote a Radon measure in a compact set A by $\mathcal{R}(A)$ and A^r set of Radon probability measure, $\mathcal{A}^r$ the set of measurable functions $(0,\infty) \to A^r$ and $\mathcal{A}^r_T$ set of measurable functions $(0,T) \to A^r$.

Let $C(A)$ be the set of continuous functions of $A \to \mathbb{R}$ and $(C(A))^*$ dual space of $C(A)$. we have

$$(C(A))^* = \mathcal{R}(A).$$

Since $C(A)$ is separable, due to Alaoglu Theorem the unit ball in the dual space $(C(A))^*$ is compact in the weak* topology, (metrizable compact). Thus,

$$A^r \hookrightarrow (C(A))^*.$$

Consequently, A^r is compact metrizable space, with the weak* topology. Beyond that

$$A \hookrightarrow A^r$$

with $a \to \delta_a$ where δ_a is Dirac measure concentrated in a.

So relaxing the controls means extending the control set, A, to a larger compact, A^r. Since we changed the set of controls then we need to change too the cost function and dynamic function:

$$b^r(x,m) = \int_A b(x,a)d\mu \quad \mu \in A^r$$

$$l^r(x,m) = \int_A l(x,a)d\mu \quad \mu \in A^r$$

Then b^r, l^r satisfy

$$co(l(x,A) \times b(x,A)) = l^r(x,A^r) \times b^r(x,A^r)$$

We define:

- **relaxed control:** $\mathcal{A}^r$, the set of measurable functions $(0,\infty) \to A^r$.
- **Relaxed trajectories:**
$$y(t) - x = \int_0^t b^r(y(s),\mu(s))ds.$$
- **Relaxed cost function:**
$$J^r(x,\mu) = \int_0^\infty l^r(y(s),\mu(s))ds.$$
- **Relaxed value function:**
$$v^r(x) := \inf_{\mu \in \mathcal{A}^r} \int_0^\infty l^r(y(s),\mu(s))ds. \tag{6.5}$$

The advantage of considering relaxed problem is that it always has an **optimal trajectory** that is trajectories which reach the infimum (6.5), cf. [2, Cor 1.4 p. 368].

The result below is the main Theorem in this section, this theorem allows us to get the relaxed controls limit.

Theorem 6.5. *[Bunford-Pettis] Consider*

$$\mathcal{V} := \{\nu : [0,T] \to \mathcal{R}(A) |\ \nu \text{ is measurable and } \sup_{t\in[0,T]} \text{ess}\, |\nu(t)| \leq \infty\}$$

and

$$\mathcal{B} := \{\phi : [0,T] \times A \to \mathbb{R} | \phi(\cdot,\alpha) \text{ is measurable for all } \alpha \in A,\ \phi(s,\cdot) \text{ is continuous } \sup_A \leq |\phi(s,\alpha)| \leq \psi(s) \forall s \in [0,T] \text{ for some } \psi \in L^1([0,T]).\}$$

Then there exists an isomorphism $\varphi : \mathcal{V} \to \mathcal{B}'$. Given $\nu \in \mathcal{V}$, we define Λ_ν as

$$\Lambda_\nu(\phi) := \int_0^T \int_A \phi(s,a) d\mu(s) ds, \quad \text{for} \quad \phi \in \mathcal{B} \quad \text{and} \quad |\Lambda_\nu|_{\mathcal{B}} = \sup_{t\in[0,T]} \text{ess}\, |\nu(t)|$$

where $\{\mathcal{B}, |\cdot|_{\mathcal{B}}\}$ is isometrically isomorphic to the space $L^1([0,T]; C(A))$. Moreover, if $(\nu_n(t)(A))_n$ is uniformly bounded for any $t \in [0,T], n \in \mathbb{N}$ then there exists subsequence such that Λ_{ν_n} converges weakly to Λ_ν, that is,*

$$\lim_{n\to\infty} \int_0^T \int_A \phi(s,a) d\nu_n(s) ds = \int_0^T \int_A \phi(s,a) d\nu(s) ds.$$

Proof. See [24, p. 268] and [2, p. 166]. □

We apply Theorem 6.5 with the sequence of Dirac measures $\mu_n(t) = \delta_{\alpha_n(t)}$. We can extract convergent subsequence of μ_n which converges weakly* to $\mu \in \mathcal{A}^r$.

Theorem 6.6 (Relaxed Control)**.** *Given a sequence of classical controls, $\alpha_n \in L^\infty(0,\infty; A)$. Let y^n be the trajectory associated with α_n. Then, extract a subsequence , y^n uniformly convergent to y and there exists $\mu \in \mathcal{A}^r$ such that*

$$\lim_{n\to\infty} \int_0^t l(y^n(s), \alpha^n(s)) e^{-\lambda s} ds = \int_0^t \int_A l(y(s), \alpha) e^{-\lambda s} d\mu(s) ds.$$

Proof. Since b is Lipschitz

$$y^n(t) - x = \int_0^t b(y^n(s), \alpha_n(s)) ds = \int_0^t b(y^n(s), \alpha_n(s)) - b(x, \alpha_n(s)) ds + \int_0^t b(x, \alpha_n(s)) ds.$$

Then

$$|y^n(t) - x| \leq C \int_0^t |y^n(s) - x| ds + t \sup_{\alpha\in A} |b(x,\alpha)|.$$

Taking $t \leq T$, thanks to Gronwall inequality y^n is uniformly bounded in $[0,T]$ and since the dynamics are bounded, y^n is a sequence of equicontinuous functions. Due to Arzela-Ascoli Theorem, We can extract a subsequence y^n uniformly convergent to y. Consequently,

$$\begin{aligned} y(t) - x &= \lim_{n\to\infty} (y^n(t) - x) = \lim_{n\to\infty} \int_0^t \int_A b(y^n(s), \alpha) d\mu_n ds \\ &= \lim_{n\to\infty} \int_0^t \int_A b(y(s), \alpha) d\mu_n ds \\ &= \int_0^t \int_A b(y(s), \alpha) d\mu ds. \end{aligned}$$

Therefore, y is a relaxed trajectory associated the measure μ. Now, take $\phi(s,\alpha) = l(y(s),\alpha)e^{-\lambda s}$ and use the sequence of Dirac measures $\mu_n(t) = \delta_{\alpha_n(t)}$ then

$$\lim_{n\to\infty} \int_0^t l(y^n(s), \alpha^n(s)) e^{-\lambda s} ds = \lim_{n\to\infty} \int_0^t \int_A l(y(s), \alpha) e^{-\lambda s} d\mu_n(s) ds = \int_0^t \int_A l(y(s), \alpha) e^{-\lambda s} d\mu(s) ds. \tag{6.6}$$

□