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Some contributions to the study of incompressible micropolar fluids in two dimensional domains

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CLEYTON NATANAEL LOPES DE CARVALHO CUNHA

SOME CONTRIBUTIONS TO THE STUDY OF INCOMPRESSIBLE MICROPOLAR FLUIDS IN TWO DIMENSIONAL DOMAINS

Tese apresentada ao Programa de Pós-graduação em Matemática da Universidade Federal de Pernambuco como requisito parcial para obtenção do título de Doutor em Matemática

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To my family.

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ABSTRACT

This thesis deals with some theoretical aspects related to the two-dimensional incompressible micropolar fluids model. In particular, two problems were addressed. The first of them is known in the literature as the initialization problem. The fundamental idea of this type of problem is to recover information from the initial data based on observations of the state of the system. For this purpose, a bounded and smooth domain of $\mathbb{R}^2$ with Dirichlet boundary conditions was considered. Optimal control theory techniques ensure the existence of at least one and at most a finite number of solutions for the problem. We also provide sufficient conditions to guarantee uniqueness of the solution. The second problem was the study of well posedness, in Hadamard's sense, for the micropolar model with partial viscosity and singular initial data, including the possibility of measures as initial data in Morrey spaces. Through integral techniques and compactness arguments, the existence of a weak solution was established. Uniqueness and stability of these solutions were also analyzed, showing the model to be well posed.

Keywords: Micropolar fluids. Initialization problem. Singular initial data. Morrey spaces.

RESUMO

Esta tese trata de alguns aspectos teóricos relacionados ao modelo bidimensional para fluidos micropolares incompressíveis. Em particular, foram abordados dois problemas. O primeiro deles é conhecido na literatura como problema de inicialização. A ideia fundamental desse tipo de problema é partindo de observações do estado do sistema, buscar recuperar informações dos dados iniciais. Para tanto, foi considerado um domínio suave e limitado do $\mathbb{R}^2$ com condições de Dirichlet na fronteira. Através de técnicas da teoria de controle ótimo, assegura-se a existência de pelo menos uma e no máximo uma quantidade finita de soluções para o problema. Também foram estudadas condições suficientes para garantir a unicidade de solução. O segundo problema foi dedicado ao estudo da boa colocação, no sentido de Hadamard, para o modelo micropolar com viscosidade parcial e dados iniciais singulares, incluindo a possibilidade de medidas como dados iniciais em espaços de Morrey. Por meio de técnicas integrais e argumentos de compacidade foi estabelecida a existência de solução fraca. A unicidade e a estabilidade dessas soluções também foram analisadas, concluindo que o modelo é bem-posto.

Palavras-chaves: Fluidos micropolares. Problema de inicialização. Dados iniciais singulares. Espaços de morrey.

SUMÁRIO

1 INTRODUCTION

This thesis deals with the analysis of some problems on fluid mechanics. More specifically, we study some problems related to the system of nonlinear partial differential equations describing micropolar fluids in two spatial dimensions.

Initially, let us briefly recall some generalities related to the problems studied here. To describe the behavior of fluid particles over time, some assumptions and simplifications are necessary. Indeed, the initial idea is that of a fluid as a continuous medium, that is, the matter in the fluid is continuously distributed in order to fill the entire region of the space it occupies[1]. Then, based on the universal laws[2] and the constitutive laws, we obtain a set of partial differential equations (PDEs) for the variables of interest (velocity field, pressure, mass density, ...). This way, we have got a mathematical model.

The so called initial value problems or Cauchy problems arise naturally in the above context, being considered fundamental in continuum mechanics. The general idea is: *Assume that the mechanical state at time* $t = 0$ *(initial conditions) and the physical properties (PDEs and complementary conditions) for all* t *are known. Then, determine the mechanical state of this medium for all* t (see (CARA, 2005)). The analysis of this type of problem is based on the existence, uniqueness and stability of solution, that is, in the well-posedness of the system in the sense of Hadamard. By stability we mean the continuous dependence of the solution with respect to the data. In particular, the study of this type of problem for the Navier-Stokes equations has provided significant advances in several areas of the natural sciences and technology.

The Navier-Stokes equations form the fundamental mathematical model for describing the motion of Newtonian fluids as water and most gases. Since a wide variety of important fluids are non-Newtonian, new theories and mathematical models taking into account the complexity of these fluids are necessary. Over the time, the great advance in this direction is notorious. Micropolar fluids follow this line of advances, describing accurately the behavior of fluids with microstructures as, for example, polymeric suspensions, liquid crystals, blood or fluids containing certain additives (LUKASZEWICZ, 1999). From this point of view, in this work we aim to contribute to the study of micropolar fluids. Precisely, we address two problems, independent of each other and originally studied for the Navier-Stokes equations, and extend their results to the two-dimensional micropolar case, emphasizing the possible influences of the fluid structure on the obtained results.

The first problem is the so called initialization problem. It is strongly related to practical issues in the field of atmospheric sciences, in particular, in meteorology. In modern atmospheric science[3], the equations that govern atmospheric circulation (nonlinear PDEs) are analyzed

[1] More formally, a continuum media is a collection of elements (particle or material point) which can be put into one-to-one correspondence with some region (open connected set) of Euclidean space.

[2] Laws of conservation of mass, momentum and energy, see (CHORIN; MARSDEN, 1990).

[3] This is a field that combines, in general, meteorology, physics, mathematics, chemistry and computer sciences (JACOBSON, 2005).

through numerical and computational approaches. Thus, it is very important the initial conditions of this system to be completely defined in order to allow the execution of the algorithms. This is considered central to numerical weather prediction (GAL-CHEN, 1978; LEDIMET; TALAGRAND, 1986). Based on some technological devices (weather balloons, reconnaissance aircraft and satellites), it is possible to obtain observational data for the system, while the complete definition of the initial conditions is not ensured *a priori*. The initialization problem consists in to use such observed data from the system to recover the initial data. In the Navier-Stokes case, this problem has been studied in the works (NOTTE-CUELLO; ROJAS-MEDAR; SANTOS, 2002; WHITE, 1993; ZHOU, 1995). More specifically, in (NOTTE-CUELLO; ROJAS-MEDAR; SANTOS, 2002), the authors was investigated a initialization problem for the Magnetohydrodinamic type equations (MHD), where the Navier-Stokes model is a particular case. In (WHITE, 1993), the author studies initialization problem for the Burgers equation with viscosity, which is a conservation law related to the Navier-Stokes model in one-dimensional case. In (ZHOU, 1995), the two-dimensional Navier-Stokes case is in fact studied.

The second problem is the study of the well posedness for the micropolar model with singular initial data. The case in which the initial data are measures (in convenient Morrey spaces) is included in the analysis. For this, partial viscosity and the evolution of vorticity in a micropolar fluid are considered, as well as their relationship with the angular velocity of the particles. In the Navier-Stokes case, there are several works with singular initial data, of which we mention, for instance (BEN-ARTZI, 1994; BEN-ARTZI M.; CROISILLE; FISHELOV, 2013; BENFATTO; ESPOSITO; PULVIRENTI, 1985; COTTET, 1986; GALLAGHER; GALLAY, 2005; GIGA; MIYAKAWA; OSADA, 1986), and the references therein. The study of the evolution of vorticity for the case of singular initial data goes beyond the classical theory of Leray. Indeed, in this theory the initial data has "finite energy", while measures have, in general, infinite energy (see (BEN-ARTZI M.; CROISILLE; FISHELOV, 2013)).

Below, we outline the contents of this work. In chapter 2, we describe the model of micropolar fluids and summarize some information and related results that will be used throughout this work (for more details see (GIGA; GIGA; SAAL, 2010; LUKASZEWICZ, 1999; LUKASZEWICZ, 2001; MAJDA; BERTOZZI; OGAWA, 2002; ROJAS-MEDAR, 1997; ROJAS-MEDAR, 1998)). In chapter 3, we investigate the initialization problem for the micropolar fluids model. The problem is formulated as an optimal control problem. We prove that this problem has at least one and at most finitely many solutions. Finally, we determine sufficient conditions to assure uniqueness of the solution. The results of this chapter were accepted for publication in *Applied Mathematics & Optimization* (SILVA; CUNHA; ROJAS-MEDAR, 2020). In chapter 4, we deal with the Cauchy problem for the partial viscous micropolar fluids model in the whole space. We assume null angular viscosity and singular initial data, which includes the possibility of vortex sheets or measures as initial data in Morrey spaces. For this purpose, we will consider the velocity-vorticity formulation for the micropolar fluid model presented in chapter 2. The results of this chapter will soon be submitted for publication in an indexed journal (BÉJAR-LÓPEZ A.; CUNHA; SOLER,

). In chapter 5, we briefly present some current projects and future works that arise naturally from what was discussed in previous chapters.

2 MICROPOLAR FLUIDS MODEL

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $n = 2$ or 3, with smooth boundary $\partial\Omega$. For a $T > 0$, we write $Q_T := \Omega \times (0, T)$ and $\Sigma_T := \partial\Omega \times (0, T)$. The behavior of an incompressible fluid, with constant density $\rho = 1$, confined to Ω is described by the following system of equations,

$$\begin{aligned} \partial_t \mathbf{u} - \operatorname{div} \tau + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \mathbf{f} \ \text{ in } Q_T, \\ \operatorname{div} \mathbf{u} = 0 \ \text{ in } Q_T, \\ \mathbf{u} = 0 \ \text{ in } \Sigma_T, \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 \ \text{ in } \Omega, \end{aligned} \tag{2.1}$$

where we use boldface letters to denote vector fields in $\mathbb{R}^n$. In system (2.1), the unknown $\mathbf{u} = \mathbf{u}(x, t) = (u_1(x, t), \cdots, u_n(x, t))$ represents the linear velocity field. The functions $\mathbf{u}_0 = \mathbf{u}_0(x)$ and $\mathbf{f} = \mathbf{f}(x, t) = (f_1(x, t), \cdots, f_n(x, t))$ are given and denote, respectively, the initial linear velocity and external force. The tensor $\tau : \mathbb{R}^{n^2}_{sym} \to \mathbb{R}^{n^2}_{sym}$ is called stress tensor. We denote by $\mathbb{R}^{n^2}_{sym}$ the set of all symmetric matrices $n \times n$. The first equation above it is frequently called the *e*quation of motion of a fluid.

Fluids for which the stress tensor can be written as

$$\tau(e(\mathbf{u})) = -\Pi\, \mathbf{I}\mathrm{d} + 2\nu e(\mathbf{u}) - \frac{2}{3}\nu(\operatorname{div} \mathbf{u})\mathbf{I}\mathrm{d}, \tag{2.2}$$

are called *N*ewtonian fluids and they have been extensively studied in the last decades. In identity (2.2), we have that the unknown $\Pi = \Pi(x, t)$ represents the pressure, the viscosity ν is a positive constant given, $\mathbf{I}$d is the identity matrix in $\mathbb{R}^{n^2}$ and

$$e(u) = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right]$$

is the shear-stress tensor due to friction forces. A large class of real fluids are Newtonian fluids, for example water and most gases. In this case, the previous system becomes the well-known Navier-Stokes system

$$\begin{aligned} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla\Pi = \mathbf{f} \ \text{ in } Q_T, \\ \operatorname{div} \mathbf{u} = 0 \ \text{ in } Q_T, \\ \mathbf{u} = 0 \text{ in } \Sigma_T, \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0, \ \text{ in } \Omega. \end{aligned}$$

This model does not consider certain fluids where the stress tensor is not symmetric, for example, *p*olar fluids, and the same goes for fluids that can support stress moments and body moments, for example, *m*icrofluids. In this context, we can find fluids with microstructures as polymeric fluids, liquid crystals and biological fluids as blood or fluids containing certain additives, which in general are of great interest for applications in the field of industry, engineering and biology. These fluids are called micropolar fluids and constitute a subclass of *s*imple microfluids (ERINGEN, 1964), which exhibit only micro-rotational effects and micro-rotational inertia of its elements. From a physical point of view (LUKASZEWICZ, 1999), micropolar fluids describe the motion of fluids consisting of rigid, randomly oriented (or spherical) particles suspended in a viscous medium, where the deformation of fluid particles is ignored. In (ERINGEN, 1966), A. C. Eringen developed the mathematical model of micropolar fluids as a simplification of the theory of simple microfluids. According to this theory, one considers the interaction between the fluid motion and rotational motion of micro–particles. Moreover, they are generalization of the Navier-Stokes model in the sense that the micropolar fluids model allows considering some physical phenomena that cannot be treated by the classical Navier-Stokes approach, given that the particles of the fluid are subject to both translation and rotation motion.

The basic micropolar model, with constant density $\rho = 1$, is

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \operatorname{div} \tau + \mathbf{f},$$
$$\partial_t \mathbf{l} + (\mathbf{u} \cdot \nabla)\mathbf{l} = \operatorname{div} \Gamma + \tau_\times + \mathbf{g},$$

where

- $\tau = -(\Pi + \frac{2}{3}\nu \operatorname{div} \mathbf{u})\mathbf{Id} + \nu\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right] + \kappa\left[\nabla \mathbf{u} - (\nabla \mathbf{u})^t\right] - 2\kappa \boldsymbol{w}_\times,$
- $\mathbf{l} = \mathbb{I}\boldsymbol{w}$, where $\mathbb{I} \in \mathbb{R}^{3\times3}$ is the called microinertia density tensor,
- $\Gamma = \gamma_0(\operatorname{div} \boldsymbol{w})\mathbf{Id} + \gamma_d\left[\nabla \boldsymbol{w} + (\nabla \boldsymbol{w})^t\right] + \gamma_a\left[\nabla \boldsymbol{w} - (\nabla \boldsymbol{w})^t\right],$
- $(\boldsymbol{w}_\times)_{ij} = \epsilon_{kij} w_k$ and
- $(\tau_\times)_i = \epsilon_{ijk}\tau_{jk}$[1], where ϵ_{ijk} is the Levi-Civita symbol.

As above, the unknowns $\mathbf{u} = \mathbf{u}(x, t)$, $\boldsymbol{w} = \boldsymbol{w}(x, t)$ and $\Pi = \Pi(x, t)$ represent, respectively, the linear velocity field, the microrotation field (interpreted as the angular velocity field of rotation of particles), and the pressure. The functions $\mathbf{f}$ and $\mathbf{g}$ are given and denote, respectively, the density of external body forces per unit mass and a body source of moments. The terms τ, $\mathbf{l}$ and Γ represent, respectively, stress tensor, angular momentum per unit mass and the moment stress tensor. The positive constants ν, κ, γ_0, γ_d, γ_a are viscosity coefficients given, satisfying

$$\gamma_0 + \gamma_d > \gamma_a.$$

If $\mathbf{l} = 0$, $\Gamma = 0$ and $\mathbf{g} = 0$, then the stress tensor τ is symmetric ($\tau_\times = 0$), which is the situation generally considered in the literature. In this case, the system above reduces to the Navier-Stokes equations presented above.

[1] Here we use the Einstein notation for tensors.

Assuming, in addition, the fluid to be incompressible, we have the micropolar fluids model in Ω

$$
\begin{aligned}
\partial_t \mathbf{u} - (\nu+\kappa)\Delta \mathbf{u} + \nabla \Pi + (\mathbf{u}\cdot\nabla)\mathbf{u} &= 2\kappa \,\mathbf{curl}\, \boldsymbol{w} + \mathbf{f} \quad &&\text{in} \quad Q_T,\\
\partial_t \boldsymbol{w} - \gamma\Delta\, \boldsymbol{w} - (\alpha+\beta)\nabla \operatorname{div} \boldsymbol{w} + 4\kappa \boldsymbol{w} + (\mathbf{u}\cdot\nabla)\boldsymbol{w} &= 2\kappa \,\mathbf{curl}\, \mathbf{u} + \mathbf{g} \quad &&\text{in} \quad Q_T, \qquad (2.3)\\
\operatorname{div} \mathbf{u} &= 0 \quad &&\text{in} \quad Q_T,
\end{aligned}
$$

together with boundary and initial conditions

$$\mathbf{u} = 0,\ \boldsymbol{w} = 0 \text{ on } \Sigma_T, \qquad (2.4)$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ \boldsymbol{w}(\mathbf{x},0) = \boldsymbol{w}_0(\mathbf{x}),\ \ \mathbf{x}\in\Omega, \qquad (2.5)$$

with $\gamma = \gamma_a + \gamma_d > 0$ and $\alpha + \beta = \gamma_0 + \gamma_d - \gamma_a > 0$. For more details see (LUKASZEWICZ, 1999).

In the present work, we restrict ourselves to the case $n = 2$. Such a case can be understood as a special case of the three-dimensional model (LUKASZEWICZ, 2001). To this end, one considers that the flow itself and the external fields do not depend on the x_3-coordinate. Moreover, assume the velocity component u_3 in the x_3-direction to be zero and the axes of rotation of the particles to be parallel to the x_3-axis, that is, $\mathbf{u} = (u_1(\mathbf{x},t),\ u_2(\mathbf{x},t),\ 0)$ and $\boldsymbol{w} = (0,\ 0,\ w_3(\mathbf{x},t))$ with $\mathbf{x} = (x_1,\ x_2) \in \Omega \subset \mathbb{R}^2$. The external forces are $\mathbf{f} = (f_1(\mathbf{x},t),\ f_2(\mathbf{x},t),\ 0)$ and $\mathbf{g} = (0,\ 0,\ g_3(\mathbf{x},t))$. Then, using the notation $\mathbf{u} = (u_1(\mathbf{x},t),\ u_2(\mathbf{x},t))$, $w = w_3(\mathbf{x},t)$, $\Pi = \Pi(\mathbf{x})$, $\mathbf{f} = (f_1(\mathbf{x},t),\ f_2(\mathbf{x},t))$, $g = g_3(\mathbf{x},t)$ and

$$\operatorname{curl}\mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},\ \operatorname{div}\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2},\ \mathbf{curl}\, w = \left(\frac{\partial w_3}{\partial x_2}, -\frac{\partial w_3}{\partial x_1}\right),$$

we can replace $\mathbf{u}$, w, $\mathbf{f}$ and g as above into system (2.3) to obtain the system

$$
\begin{aligned}
\partial_t \mathbf{u} - (\nu+\kappa)\Delta \mathbf{u} + \nabla \Pi + (\mathbf{u}\cdot\nabla)\mathbf{u} &= 2\kappa \,\mathbf{curl}\, w + \mathbf{f} \quad &&\text{in} \quad Q_T,\\
\partial_t w - \gamma\Delta w + 4\kappa w + (\mathbf{u}\cdot\nabla) w &= 2\kappa \operatorname{curl} \mathbf{u} + g \quad &&\text{in} \quad Q_T,\\
\operatorname{div} \mathbf{u} &= 0 \quad &&\text{in} \quad Q_T, \qquad (2.6)\\
\mathbf{u} = 0,\ w &= 0 \quad &&\text{in} \quad \Sigma_T,\\
\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ w(\mathbf{x},0) &= w_0(\mathbf{x}) \quad && \mathbf{x}\in\Omega.
\end{aligned}
$$

There are several articles dealing with the mathematical analysis of the micropolar fluids model and important results have been obtained, among them we have well-posedness, asymptotic behavior and stability results. We briefly mention some results: In (ROJAS-MEDAR, 1998), the authors proved the existence of weak solutions in bounded regular domains Ω. Existence of local and global in time strong solutions was obtained in (ORTEGA-TORRES; ROJAS-MEDAR, 1999) and (ROJAS-MEDAR, 1997). In (ORTEGA-TORRES; ROJAS-MEDAR; CABRALES, 2012) and (ROJAS-MEDAR; BOLDRINI, 1997), the authors studied local and uniform in time convergence rates for Galerkin approximations of the solutions. Regarding the several aspects of long time behavior of micropolar fluid flows, let us highlight the following results. In (SILVA et al., 2019), the long time behavior of weak solutions, in L^2 sense, for the micropolar fluids equations in the whole space $\mathbb{R}^3$ was studied through Fourier splitting. In (SILVA; FRIZ; ROJAS-MEDAR, 2016), the authors studied the convergence of nonstationary solutions to stationary solutions of system (2.3)-(2.5) when $t \to \infty$ in the L^2 and L^α case, with $\alpha > 3$ (see also (LUKASZEWICZ, 2001)). In (SILVA; FERNANDEZ-CARA; ROJAS-MEDAR, ; SILVA; CRUZ; ROJAS-MEDAR, 2014) the authors proved the existence of a small time interval where the fluid variables converge uniformly as the viscosities tend to zero. The case of unbounded domains is less studied. For the case of an exterior domain, we highlight (BOLDRINI; DURAN; ROJAS-MEDAR, 2010), where the authors proved the existence and uniqueness of a strong solution. For two-dimensional unbounded domains, one can look at the work (LUKASZEWICZ, 2003) (see also (LUKASZEWICZ; SADOWSKI, 2004)). In the general case of a connected open set of $\mathbb{R}^n$, with $n = 2, 3$, in (GALDI; RIONERO, 1977) the authors proved theorems of existence and uniqueness of weak solutions. For applications, for example in theory of lubrification, theory of porous media and natural sciences, see (FERRARI, 1983; LUKASZEWICZ, 1999). Other related works are (ELLAHI et al., 2014; LUKASZEWICZ, 1990; SZOPA, 2007; YAMAGUCHI, 2005).

2.1 FUNCTION SPACES

As stated above, we use boldface letters to denote vector fields in $\mathbb{R}^n$, as well as to indicate spaces whose elements are of this nature. We also use the standard notation for the usual Lebesgue and Sobolev spaces. The inner product in $L^2(\Omega)$ and $H^1_0(\Omega)$ are represented by $(\cdot,\cdot)$ and $((\cdot,\cdot))$, respectively. We denote

$$\mathbf{C}^\infty_{0,\sigma}(\Omega) = \{\boldsymbol{v} \in (C^\infty_0(\Omega))^2 \ ; \ \operatorname{div} \boldsymbol{v} = 0\},$$

$$\mathbf{H} = \text{the clousure of } \mathbf{C}^\infty_{0,\sigma}(\Omega) \text{ in } \mathbf{L}^2(\Omega),$$

$$\mathbf{V} = \text{the clousure of } \mathbf{C}^\infty_{0,\sigma}(\Omega) \text{ in } \mathbf{H}^1_0(\Omega).$$

The spaces $\mathbf{H}$ and $\mathbf{V}$ are Hilbert spaces with respect to the inner products induced by $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1_0(\Omega)$, respectively. Moreover,

$$\mathbf{V} = \{\boldsymbol{v} \in \mathbf{H}^1_0(\Omega) \ ; \ \operatorname{div} \boldsymbol{v} = 0\}$$

and

$$\mathbf{V} \hookrightarrow \mathbf{H} \equiv \mathbf{H}^* \hookrightarrow \mathbf{V}^*,$$

with each space dense in the following one and with compact injections (see e.g. (TEMAM; FORSTE, 1979)). The norm in L^2 and $\mathbf{H}$ will be denoted by $|\cdot|$, while the norm in H^1_0 and $\mathbf{V}$ will be denoted by $\|\cdot\|$. We indicate by P the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$. The well known Stokes operator is $A : D(A) \hookrightarrow \mathbf{H}$ given by $A = -P\Delta$ with domain

$D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}$. It is characterized by

$$(A\mathbf{u}, \mathbf{v}) = ((\mathbf{u}, \mathbf{v})),$$

for all $\mathbf{u} \in D(A)$ and $\mathbf{v} \in \mathbf{V}$. When $A\mathbf{u} \in \mathbf{L}^2(\Omega)$ one has $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and the norms $\|\mathbf{u}\|_{\mathbf{H}^2}$ and $|A\mathbf{u}|$ are equivalent if Ω is regular. Similar properties hold for the Laplacian operator $B \equiv -\Delta : D(B) \to L^2(\Omega)$ with Dirichlet boundary conditions and domain $D(B) = H^2(\Omega) \cap H_0^1(\Omega)$. If X is a Banach space, T is either a positive real number or $T = +\infty$ and $1 \leqslant p \leqslant \infty$, we denote by $L^p(0,T;X)$ the Banach space of all measurable functions $v : (0,T) \longrightarrow X$, such that $t \mapsto \|v(t)\|_X$ is in $L^p(0,T)$ with norm

$$\|v\|_p = \Big(\int_0^T \|v(t)\|_X^p dt\Big)^{1/p}, \quad \text{if} \quad 1 \leqslant p < \infty,$$

and

$$\|v\|_\infty = \operatorname*{ess\,sup}_{t\in[0,T]} \|v(t)\|_X,$$

if $p = \infty$. The space $C([0,T], X)$ is understood in a similar manner. We define the trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i d\mathbf{x}$$

for vector-valued functions $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$, and

$$b_1(\mathbf{u}, v, w) = \sum_{j=1}^{2} \int_\Omega u_j \frac{\partial v}{\partial x_j} w d\mathbf{x},$$

for $\mathbf{u} \in \mathbf{V}$ and scalar functions $v, w \in H_0^1(\Omega)$. These forms satisfy the following properties (see (TEMAM; FORSTE, 1979; TEMAM, 1983)):

Lemma 2.1. *(i)* $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$

1. *is a continuous form on* $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$;
2. $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$, $\forall\, \mathbf{u} \in \mathbf{V}$ *and* $\forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$;
3. $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v})$, $\forall\, \mathbf{u} \in \mathbf{V}$ *and* $\forall\, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)$;
4. $|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leqslant \sqrt{2}|\mathbf{u}|^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\||\mathbf{w}|^{1/2}\|\mathbf{w}\|^{1/2}$, $\forall\, \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbf{H}_0^1(\Omega)$;
5. $|b(\mathbf{u}, \mathbf{u}, \mathbf{v})| \leqslant \sqrt{2}|\mathbf{u}|\|\mathbf{u}\|\|\mathbf{v}\|$, $\forall\, \mathbf{u},\ \mathbf{v} \in \mathbf{V}$;
6. $|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leqslant C|\mathbf{u}|^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\|^{1/2}|A\mathbf{v}|^{1/2}|\mathbf{w}|$, $\forall\, \mathbf{u} \in \mathbf{V}$, $\mathbf{v} \in D(A)$, $\mathbf{w} \in \mathbf{H}$;
7. $|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leqslant C|\mathbf{u}|^{1/2}|A\mathbf{u}|^{1/2}\|\mathbf{v}\||\mathbf{w}|$, $\forall\, \mathbf{u} \in D(A)$, $\mathbf{v} \in \mathbf{V}$, $\mathbf{w} \in \mathbf{H}$.

(ii) $b_1(\mathbf{u}, v, w)$

1. *is a continuous form on* $\mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$;
2. $b_1(\mathbf{u}, v, v) = 0$, $\forall\, \mathbf{u} \in \mathbf{V}$ *and* $\forall\, v \in H_0^1(\Omega)$;
3. $b_1(\mathbf{u}, v, w) = -b_1(\mathbf{u}, w, v)$, $\forall\, \mathbf{u} \in \mathbf{V}$ *and* $\forall\, v, w \in H_0^1(\Omega)$;
4. $|b_1(\mathbf{u}, v, w)| \leqslant C|\mathbf{u}|^{1/2}\|\mathbf{u}\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}$, $\forall\, \mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *and* $v,\ w \in H_0^1(\Omega)$.

2.2 WEAK AND STRONG SOLUTIONS IN L^2 SENSE

The weak formulation of problem (2.6) is as follows.

Definition 2.2. *A pair of functions* $(\mathbf{u}, w)$ *defined on* $\Omega \times (0,T)$, $T > 0$, *is called a* ***w****eak solution of system* (2.6) *if*

$$\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H}), \;\; \partial_t \mathbf{u} \in L^2(0,T;\mathbf{V}^*), \tag{2.7}$$

$$w \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \;\; \partial_t w \in L^2(0,T;H^{-1}(\Omega)), \tag{2.8}$$

and they satisfy

$$(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + (\nu+\kappa)((\mathbf{u}, \boldsymbol{\varphi})) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\varphi}) = 2\kappa(\textbf{\textit{curl}}\, w, \boldsymbol{\varphi}), \tag{2.9}$$

$$(\partial_t w, \phi) + \gamma((w, \phi)) + 4\kappa(w, \phi) + b_1(\mathbf{u}, w, \phi) = 2\kappa(\textit{curl}\, \mathbf{u}, \phi), \tag{2.10}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \tag{2.11}$$

$$w(x,0) = w_0(x), \tag{2.12}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}$, $\phi \in H_0^1(\Omega)$.

Remark 2.3. *The regularity conditions* (2.7) *and* (2.8) *imply* $\mathbf{u} \in C([0,T];\mathbf{H})$ *and* $w \in C([0,T];L^2(\Omega))$. *Therefore, the initial conditions* (2.11) *and* (2.12) *make sense with* $\mathbf{u}_0 \in \mathbf{H}$ *and* $w_0 \in L^2$.

Strong solutions are defined as follows.

Definition 2.4. *If* $\mathbf{u}$ *and* w *are such that*

$$\mathbf{u} \in L^2(0,T;D(A)) \cap L^\infty(0,T;\mathbf{V}),$$

$$w \in L^2(0,T;D(B)) \cap L^\infty(0,T;H_0^1(\Omega)),$$

and satisfy (2.9) - (2.12), *the pair* $(\mathbf{u}, w)$ *is said to be* ***s****trong solutions of system* (2.6).

Note that in this case the regularity of the solutions imply the initial conditions to be satisfied pointwisely. We state known results about the existence of such solutions. For their proof, see (ROJAS-MEDAR, 1997; ROJAS-MEDAR, 1998).

Theorem 2.5. *Given* $T > 0$, *if* $\mathbf{u}_0 \in \mathbf{H}$ *and* $w_0 \in L^2(\Omega)$, *then there exists a unique weak solution* $(\mathbf{u}, w)$ *of system* (3.1). *Moreover, this solution satisfy the energy inequality*

$$|\mathbf{u}(t)|^2 + |w(t)|^2 + (\nu+\kappa)\int_0^T \|\mathbf{u}(t)\|^2 dt + \gamma \int_0^T \|w(t)\|^2 dt \leqslant \Big(|\mathbf{u}_0|^2 + |w_0|^2\Big) e^{C(\nu,\kappa,\gamma)T}, \tag{2.13}$$

for all $t \in [0,T]$.

Theorem 2.6. *Let $\bar{T} > 0$, if $\mathbf{u}_0 \in \mathbf{V}$ and $w_0 \in H_0^1(\Omega)$ then, in a (possibly small) time interval* $[0, T]$*, with* $0 < T \leqslant \bar{T}$*, the problem* (3.1) *has a unique strong solution. This solution belongs to* $C([0, T]; \mathbf{V}) \times C([0, T]; H_0^1(\Omega))$ *and satisfies*

$$\|\mathbf{u}(t)\|^2 + \|w(t)\|^2 + C\int_0^T \left\{|A\mathbf{u}(t)|^2 + |Bw(t)|^2\right\}dt \leqslant K_0 = K_0(\nu, \kappa, \gamma, \|\mathbf{u}_0\|, \|w_0\|, T), \tag{2.14}$$

for all $t \in [0, T]$*, with* $K_0 \to \|\mathbf{u}_0\|^2 + \|w_0\|^2$ *for* $T \to 0$*. In addition, if* $\mathbf{u}_0 \in D(A)$ *and* $w_0 \in D(B)$*, then the solution* $(\mathbf{u}, w)$ *satisfies, in particular,*

$$|\partial_t \mathbf{u}(t)|^2 + |\partial_t w(t)|^2 + C_0\int_0^T \left\{\|\partial_t \mathbf{u}(t)\|^2 + \|\partial_t w(t)\|^2\right\}dt \leqslant Ce^{CK_0 + CT}, \tag{2.15}$$

for every $t \in [0, T]$*. Also,*

$$\mathbf{u} \in C^1([0, T]; \mathbf{H}) \cap C([0, T]; D(A)),$$
$$w \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; D(B)).$$

The results above will be widely used in chapter 3.

2.3 VORTICITY EQUATION

In this section we present some notations and basic results that will be used in chapter 4, where the micropolar model will be studied through the velocity-vorticity formulation.

The previously mentioned works have in common the use of velocity-pressure formulation and the solution in L^2 sense. However, in the context of fluid dynamics it is interesting to describe the events in terms of the evolution of the vorticity field. The vorticity of the vector field $\mathbf{v}$ in $\mathbb{R}^n$ is denoted by **curl** $\mathbf{v}$ and represents the tendency of the fluid to rotate around $\mathbf{x} \in \mathbb{R}^n$. For $n = 2$, the vorticity is a scalar real-valued function and, in the present work, we write as $\omega(\mathbf{x}, t) = \text{curl } \mathbf{u}(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^2$ and $t \geqslant 0$. Under the perspective of numerical and computational methods, such approach is of great importance since it simplifies significantly some numerical methods as the vortex method, which is widely used in the context of the Navier-Stokes equations (COTTET, 1988). Moreover, the vorticity contains all the necessary information for velocity field reconstruction from the Biot-Savart Law:

$$\mathbf{u}(\mathbf{x}, t) = (\mathbf{K} * \omega)(\mathbf{x}, t) = \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{x} - \mathbf{y})\omega(\mathbf{y}, t)d\mathbf{y}, \tag{2.16}$$

where $\mathbf{K}$ is the Biot-Savart kernel, that is,

$$\mathbf{K}(\mathbf{x}) = \frac{1}{2\pi}|\mathbf{x}|^{-2}\mathbf{x}^\perp, \ \mathbf{x} \in \mathbb{R}^2, \tag{2.17}$$

where $\mathbf{x}^\perp = (-x_2, x_1)$. The velocity fields given by (2.16) may include, in particular, the case of vortex sheet.

Remark 2.7. *Here we use the same notation for the convolution* $A * \mathbf{B} = (A * B_1, A * B_2)$ *of a scalar function* A *and a vector function* $\mathbf{B} = (B_1, B_2)$ *that are defined on* $\mathbb{R}^2$*. It is also convenient to fix the same notation for the convolution of a function* f *and a measure* μ*, that is, we consider*

$$(f * \mu)(x) = \int_{\mathbb{R}^n} f(x - y) d\mu(y).$$

The use of the velocity-vorticity formulation to describe fluid dynamics, especially for the Navier-Stokes system, has been studied since the 1980's. In particular, the works (CHORIN, 1982; MAJDA, 1986; MAJDA; BERTOZZI; OGAWA, 2002) bring more details about such a study and the equivalence between the cited formulations. The main advantage of this approach, with regard to this thesis work, is the parabolic character of the vorticity equation as well as the fact that the pressure term is eliminated from the system.

The formulation of system (2.6) in terms of velocity-vorticity is

$$\begin{aligned}
\partial_t \omega - (\nu + \kappa)\Delta\omega + (\mathbf{u} \cdot \nabla)\omega &= -2\kappa\Delta w + \text{curl}\, \mathbf{f}, \\
\partial_t w - \gamma\Delta w + 4\kappa w + (\mathbf{u} \cdot \nabla) w &= 2\kappa\omega + g, \\
\mathbf{u} &= \mathbf{K} * \omega, \\
\omega(\cdot, 0) &= \omega_0, \\
w(\cdot, 0) &= w_0,
\end{aligned}$$

where we take the curl operator of the first equation in (2.6), and we use that curl $((\mathbf{u} \cdot \nabla)\mathbf{u}) = (\mathbf{u} \cdot \nabla)\omega$, curl $(\Delta\mathbf{u}) = \Delta\omega$ and curl $(\nabla\Pi) = 0$. Moreover, in order to apply the Biot-Savart law, the boundary condition is replaced by a condition of decay at infinity. Then, we can use the equivalence between the identity (2.16) and the system

$$\begin{aligned}
\text{curl}\, \mathbf{u} &= \omega, \\
\text{div}\, \mathbf{u} &= 0, \\
|\mathbf{u}(x, t)| \to 0 &\text{ as } |x| \to +\infty.
\end{aligned}$$

Note that the pressure field can also be recovered considering the incompressibility of the fluid and the velocity fields $(\mathbf{u}, w)$:

$$\Delta\Pi = \text{div}\, \big(\mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u} + (\nu + \kappa)\Delta\mathbf{u} + 2\kappa\, \mathbf{curl}\, w\big),$$

when the drivatives make sense.

To finish this section, we emphasize that in chapter 4 we will use for simplicity $w := b$ to represent the angular velocity, in order to avoid possible confusion between the notations for angular velocity and the vorticity.

3 THE INITIALIZATION PROBLEM

The fundamental idea of initialization problem is to seek to recover information from the initial data, based on observations of the state of the system. Due to its importance in a great variety of problems, as for example in numerical weather prediction, and more generally in numerical simulations for prediction models, initialization problems have been studied by several authors, for instance, in (GAL-CHEN, 1978; LEDIMET; TALAGRAND, 1986; LORENC, 1986; NOTTE-CUELLO; ROJAS-MEDAR; SANTOS, 2002; WHITE, 1993; ZHOU, 1995). In (LEDIMET; TALAGRAND, 1986), the use of variational methods is proposed, including techniques of optimal control. In this method, the initial condition is understood as a control variable. Then, a control functional is introduced comparing the observed data with the solution of the mathematical model, which depends on the control variable. In this way, the problem is therefore viewed as a minimization problem. The control formulation of initialization problems is discussed in (LIONS, 1971). Through this approach, the initialization problems for Navier-Stokes and MHD equations have been treated in (NOTTE-CUELLO; ROJAS-MEDAR; SANTOS, 2002; WHITE, 1993; ZHOU, 1995). In these works, it has been proven that if the time interval on which the observations are taken is short enough, then there exists at most a finite number of solutions to the initialization problem. In (ZHOU, 1995), it is proven that such a solution is indeed unique. Our main goal in this chapter is to extend these results for the micropolar fluids case.

Thus, let us consider the following system in $\mathbb{R}^2$

$$\begin{aligned}
\partial_t \mathbf{u} - (\nu+\kappa)\Delta \mathbf{u} + \nabla \Pi + (\mathbf{u}\cdot\nabla)\mathbf{u} &= 2\kappa\,\mathbf{curl}\, w + \mathbf{f} && \text{in} \quad Q_T,\\
\partial_t w - \gamma \Delta w + 4\kappa w + (\mathbf{u}\cdot\nabla) w &= 2\kappa\,\mathrm{curl}\,\mathbf{u} + g && \text{in} \quad Q_T,\\
\operatorname{div}\mathbf{u} &= 0 && \text{in} \quad Q_T,\\
\mathbf{u} = 0,\ w &= 0 && \text{in} \quad \Sigma_T,\\
\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ w(\mathbf{x},0) &= w_0(\mathbf{x}) && \mathbf{x}\in\Omega,
\end{aligned} \tag{3.1}$$

where we assume, without loss of generality to our ends, $\mathbf{f} = 0$ and $g = 0$.

This chapter is organized according to its published version (SILVA; CUNHA; ROJAS-MEDAR, 2020). In section 3.1, we show the continuity of the flux of (3.1) as well as its Gâteaux differentiability up to second order. In sections 3.2 and 3.3 we present our main results.

3.1 CONTINUITY AND GÂTEAUX DIFFERENTIABILITY OF THE FLUX

Consider the map $S : \mathbf{V}\times H_0^1(\Omega) \longrightarrow L^2(0,T;\mathbf{V})\times L^2(0,T;H_0^1(\Omega))$, which takes the initial data $(\mathbf{u}_0, w_0)\in \mathbf{V}\times H_0^1(\Omega)$ to the corresponding solution of problem (3.1) in a time interval $[0,T]$. We denote

$$S(\mathbf{u}_0, w_0) = (\mathbf{u}(\mathbf{u}_0, w_0), w(\mathbf{u}_0, w_0))$$

and

$$S(\mathbf{u}_0, w_0)(t) = (\mathbf{u}(t; \mathbf{u}_0, w_0), w(t; \mathbf{u}_0, w_0)),$$

for each $t \in [0,T]$. In view of the Definition 2.2, we can also see S as a map from $\mathbf{V} \times H_0^1(\Omega)$ into $L^\infty(0,T;\mathbf{H}) \times L^\infty(0,T;L^2(\Omega))$. We show that these maps are continuous.

Proposition 3.1. *Given* $(\mathbf{u}_0, w_0) \in \mathbf{V} \times H_0^1(\Omega)$, *there exists a constant* $C = C(\nu, \kappa, \gamma, T, |\mathbf{u}_0|, |w_0|)$ *such that*

$$\begin{aligned}\|S(\mathbf{u}_0, w_0) - S(z, s)\|^2_{L^\infty(0,T;\mathbf{H})\times L^\infty(0,T;L^2)} + \|S(\mathbf{u}_0, w_0) - S(z, s)\|^2_{L^2(0,T;\mathbf{V})\times L^2(0,T;H_0^1)} \\ \leqslant C\|(\mathbf{u}_0, w_0) - (z, s)\|^2_{\mathbf{V}\times H_0^1},\end{aligned} \tag{3.2}$$

for all $(z, s) \in \mathbf{V} \times H_0^1(\Omega)$.

Proof. Let

$$\boldsymbol{\delta}(t) = S(\mathbf{u}_0, w_0)(t) - S(z, s)(t) = (\boldsymbol{\delta}_1(t), \delta_2(t))$$

and

$$\boldsymbol{\chi} = (\mathbf{u}_0 - z, w_0 - s) = (\boldsymbol{\chi}_1, \chi_2),$$

where $(z, s) \in \mathbf{V} \times H_0^1(\Omega)$. In view of the embedding $\mathbf{V} \hookrightarrow \mathbf{H}$ and $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, it is sufficient to prove that

$$\|\boldsymbol{\delta}\|^2_{L^\infty(0,T;\mathbf{H})\times L^\infty(0,T;L^2)} + \|\boldsymbol{\delta}\|^2_{L^2(0,T;\mathbf{V})\times L^2(0,T;H_0^1)} \leqslant C\|\boldsymbol{\chi}\|^2_{\mathbf{H}\times L^2}. \tag{3.3}$$

Note that $\boldsymbol{\delta}_1$ and δ_2 satisfy the system

$$\begin{aligned}
(\partial_t\boldsymbol{\delta}_1, \boldsymbol{\varphi}) + (\nu+\kappa)((\boldsymbol{\delta}_1, \boldsymbol{\varphi})) + b(\boldsymbol{\delta}_1, \mathbf{u}(\mathbf{u}_0, w_0), \boldsymbol{\varphi}) \\
+ b(\mathbf{u}(z, s), \boldsymbol{\delta}_1, \boldsymbol{\varphi}) &= 2\kappa(\mathbf{curl}\,\delta_2, \boldsymbol{\varphi}), \\
(\partial_t\delta_2, \phi) + \gamma((\delta_2, \phi)) + 4\kappa(\delta_2, \phi) + b_1(\boldsymbol{\delta}_1, w(\mathbf{u}_0, w_0), \phi) \\
+ b_1(\mathbf{u}(z, s), \delta_2, \phi) &= 2\kappa(\mathrm{curl}\,\boldsymbol{\delta}_1, \phi), \\
\boldsymbol{\delta}_1(0) &= \boldsymbol{\chi}_1, \\
\delta_2(0) &= \chi_2,
\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}$, $\phi \in H_0^1(\Omega)$. Taking $\boldsymbol{\varphi} = \boldsymbol{\delta}_1$, $\phi = \delta_2$ and adding these equations we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\{|\boldsymbol{\delta}_1(t)|^2 + |\delta_2(t)|^2\right\} + (\nu+\kappa)\|\boldsymbol{\delta}_1(t)\|^2 + \gamma\|\delta_2(t)\|^2 + 4\kappa|\delta_2(t)|^2 \\
+ b(\boldsymbol{\delta}_1(t), \mathbf{u}(t; \mathbf{u}_0, w_0), \boldsymbol{\delta}_1(t)) + b_1(\boldsymbol{\delta}_1(t), w(t; \mathbf{u}_0, w_0), \delta_2(t)) \\
= 2\kappa\left\{(\mathbf{curl}\,\delta_2(t), \boldsymbol{\delta}_1(t)) + (\mathrm{curl}\,\boldsymbol{\delta}_1(t), \delta_2(t))\right\}.
\end{aligned}$$

By Lemma 2.1 and Young's Inequality,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\right\}+(\nu+\kappa)\|\boldsymbol{\delta}_1(t)\|^2+\gamma\|\delta_2(t)\|^2+4\kappa|\delta_2(t)|^2\\
&\leqslant \frac{\nu+\kappa}{2}\|\delta_1(t)\|^2+\frac{\gamma}{2}\|\delta_2(t)\|^2+\frac{4\kappa^2}{\nu+\kappa}|\delta_2(t)|^2+\frac{4\kappa^2}{\gamma}|\boldsymbol{\delta}_1(t)|^2\\
&\quad+\frac{4}{\nu+\kappa}\left\{\|\mathbf{u}(t;\mathbf{u}_0,w_0)\|^2+\|w(t;\mathbf{u}_0,w_0)\|^2\right\}|\boldsymbol{\delta}_1(t)|^2+\frac{1}{8\gamma}\|w(t;\mathbf{u}_0,w_0)\|^2|\delta_2(t)|^2\\
&\leqslant \frac{\nu+\kappa}{2}\|\boldsymbol{\delta}_1(t)\|^2+\frac{\gamma}{2}\|\delta_2(t)\|^2+\frac{4\kappa^2}{\nu+\kappa}|\delta_2(t)|^2+\frac{4\kappa^2}{\gamma}\left\{|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\right\}\\
&\quad+C\left\{(\nu+\kappa)\|\mathbf{u}(t;\mathbf{u}_0,w_0)\|^2+\gamma\|w(t;\mathbf{u}_0,w_0)\|^2\right\}\left\{|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\right\},
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{d}{dt}\left\{|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\right\}+(\nu+\kappa)\|\boldsymbol{\delta}_1(t)\|^2+\gamma\|\delta_2(t)\|^2+\frac{8\kappa\nu}{\nu+\kappa}|\delta_2(t)|^2\\
&\leqslant C\left\{1+(\nu+\kappa)\|\mathbf{u}(t;\mathbf{u}_0,w_0)\|^2+\gamma\|w(t;\mathbf{u}_0,w_0)\|^2\right\}\left\{|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\right\}.
\end{aligned}$$

Let $F(t;\mathbf{u}_0,w_0)=1+(\nu+\kappa)\|\mathbf{u}(t;\mathbf{u}_0,w_0)\|^2+\gamma\|w(t;\mathbf{u}_0,w_0)\|^2$. Then,

$$\frac{d}{dt}\left\{|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\right\}+(\nu+\kappa)\|\boldsymbol{\delta}_1(t)\|^2+\gamma\|\delta_2(t)\|^2\leqslant CF(t;\mathbf{u}_0,w_0)\left\{|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\right\}.$$

Integrating from 0 to t, we have

$$\begin{aligned}
&|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2+(\nu+\kappa)\int_0^t\|\boldsymbol{\delta}_1(\xi)\|^2d\xi+\gamma\int_0^t\|\delta_2(\xi)\|^2d\xi\\
&\leqslant C_0\|\chi\|^2_{H\times L^2}+C\int_0^t F(\xi;\mathbf{u}_0,w_0)\left\{|\boldsymbol{\delta}_1(\xi)|^2+|\delta_2(\xi)|^2\right\}d\xi. \qquad (3.4)
\end{aligned}$$

In particular, by Gronwall's Inequality

$$|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\leqslant C_0\|\chi\|^2_{H\times L^2}\exp\left(C\int_0^t F(\xi;\mathbf{u}_0,w_0)d\xi\right),\ \forall\ t\in[0,T].$$

Note that due to the energy inequality (2.13),

$$C\int_0^t F(\xi;\mathbf{u}_0,w_0)d\xi\leqslant C(\nu,\kappa,\gamma,T,|\mathbf{u}_0|,|w_0|),\ \forall\ t\in[0,T].$$

Then,

$$|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2\leqslant C\|\boldsymbol{\chi}\|^2_{H\times L^2},\ \forall\ t\in[0,T]. \qquad (3.5)$$

Using inequalities (3.5) and (3.4), we obtain

$$|\boldsymbol{\delta}_1(t)|^2+|\delta_2(t)|^2+\theta\left\{\int_0^t\|\boldsymbol{\delta}_1(\xi)\|^2d\xi+\int_0^t\|\delta_2(\xi)\|^2d\xi\right\}\leqslant C\|\boldsymbol{\chi}\|^2_{H\times L^2}, \qquad (3.6)$$

for all $t\in[0,T]$ and $\theta=\min\{\nu+\kappa,\gamma\}$. This implies the map S to be continuous from $\mathbf{V}\times H^1_0(\Omega)$ into $L^\infty(0,T;\mathbf{H})\times L^\infty(0,T;L^2(\Omega))$ and from $\mathbf{V}\times H^1_0(\Omega)$ into $L^2(0,T;\mathbf{V})\times L^2(0,T;H^1_0(\Omega))$. □

Next we show that the map $S : \mathbf{V} \times H_0^1(\Omega) \longrightarrow L^2(0,T;\mathbf{V}) \times L^2(0,T;H_0^1(\Omega))$ is Gâteaux differentiability. The Gâteaux derivative of S at $(z,s) \in \mathbf{V} \times H_0^1(\Omega)$ along the direction $(\boldsymbol{\pi},\tau)$ is denoted by

$$\eta = DS(z,s)\cdot(\boldsymbol{\pi},\tau) = (\boldsymbol{\eta}_1,\eta_2),$$

which, if it exists, is a continuous linear map from $\mathbf{V}\times H_0^1(\Omega)$ into $L^2(0,T;\mathbf{V})\times L^2(0,T;H_0^1(\Omega))$.

Proposition 3.2. *The map* S *is Gâteaux differentiable and* η *is the solution of the linear system*

$$\begin{aligned}(\partial_t\boldsymbol{\eta}_1,\boldsymbol{\varphi}) + (\nu+\kappa)((\boldsymbol{\eta}_1,\boldsymbol{\varphi})) + b(\boldsymbol{\eta}_1,\mathbf{u},\boldsymbol{\varphi}) + b(\mathbf{u},\boldsymbol{\eta}_1,\boldsymbol{\varphi}) &= 2\kappa(\textbf{\textit{curl}}\,\eta_2,\boldsymbol{\varphi})\\ (\partial_t\eta_2,\phi) + \gamma((\eta_2,\phi)) + 4\kappa(\eta_2,\phi) + b_1(\boldsymbol{\eta}_1,w,\phi) + b_1(\mathbf{u},\eta_2,\phi) &= 2\kappa(\textit{curl}\,\boldsymbol{\eta}_1,\phi),\\ \boldsymbol{\eta}_1(0) &= \boldsymbol{\pi},\\ \eta_2(0) &= \tau,\end{aligned} \tag{3.7}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}, \phi \in H_0^1(\Omega)$, *where* $\mathbf{u} = \mathbf{u}(t;z,s)$ *and* $w = w(t;z,s)$. *The function* η *is estimated by*

$$|\boldsymbol{\eta}_1(t)|^2 + |\eta_2(t)|^2 + \theta\left\{\int_0^t \|\boldsymbol{\eta}_1(\xi)\|^2 d\xi + \int_0^t \|\eta_2(\xi)\|^2 d\xi\right\} \leqslant C\|(\boldsymbol{\pi},\tau)\|^2_{\mathbf{V}\times H_0^1}, \tag{3.8}$$

for all $t \in [0,T]$ *and* $\theta = \min\{\nu+\kappa,\gamma\}$.

Proof. Let $(\boldsymbol{\eta}_1,\eta_2)$ be the solution of system (3.7). We need to show

$$\lim_{h\to 0}\left\{\|\boldsymbol{\eta}_{1,h} - \boldsymbol{\eta}_1\|_{L^2(0,T;\mathbf{V})} + \|\eta_{2,h} - \eta_2\|_{L^2(0,T;H_0^1)}\right\} = 0, \tag{3.9}$$

where

$$\boldsymbol{\eta}_{1,h} = \frac{\mathbf{u}(z+h\pi,s+h\tau) - \mathbf{u}(z,s)}{h}$$

and

$$\eta_{2,h} = \frac{w(z+h\pi,s+h\tau) - w(z,s)}{h},$$

with $h \in \mathbb{R}$. The pair of functions $(\boldsymbol{\eta}_{1,h},\eta_{2,h})$ satisfy

$$\begin{aligned}(\partial_t\boldsymbol{\eta}_{1,h},\boldsymbol{\varphi}) + (\nu+\kappa)((\boldsymbol{\eta}_{1,h},\boldsymbol{\varphi})) &+ b(\boldsymbol{\eta}_{1,h},\mathbf{u}(z+h\pi,s+h\tau),\boldsymbol{\varphi})\\ &+ b(\mathbf{u}(z,s),\boldsymbol{\eta}_{1,h},\boldsymbol{\varphi}) = 2\kappa(\mathbf{curl}\,\eta_{2,h},\boldsymbol{\varphi}),\\ (\partial_t\eta_{2,h},\phi) + \gamma((\eta_{2,h},\phi)) &+ 4\kappa(\eta_{2,h},\phi) + b_1(\boldsymbol{\eta}_{1,h},w(z+h\pi,s+h\tau),\phi)\\ &+ b_1(\mathbf{u}(z,s),\eta_{2,h},\phi) = 2\kappa(\mathrm{curl}\,\boldsymbol{\eta}_{1,h},\phi),\\ \boldsymbol{\eta}_{1,h}(0) &= \pi,\\ \eta_{2,h}(0) &= \tau,\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}$, $\phi \in H_0^1(\Omega)$. Therefore, $\mathbf{r} = \boldsymbol{\eta}_{1,h} - \boldsymbol{\eta}_1$ and $l = \eta_{2,h} - \eta_2$ satisfy the system

$$\begin{aligned}
(\partial_t \mathbf{r}, \boldsymbol{\varphi}) + (\nu + \kappa)((\mathbf{r}, \boldsymbol{\varphi})) + b(\mathbf{u}(z, s), \mathbf{r}, \boldsymbol{\varphi}) \\
+ b(\mathbf{r}, \mathbf{u}(z, s), \boldsymbol{\varphi}) = 2\kappa(\mathbf{curl}\, l, \boldsymbol{\varphi}) - b(\boldsymbol{\eta}_{1,h}, h\boldsymbol{\eta}_{1,h}, \boldsymbol{\varphi}), \\
(\partial_t l, \phi) + \gamma((l, \phi)) + 4\kappa(l, \phi) + b_1(\mathbf{u}(z, s), l, \phi) \\
+ b_1(\boldsymbol{\eta}_{1,h}, h\eta_{2,h}, \phi) = 2\kappa(\mathrm{curl}\, \mathbf{r}, \phi) + b_1(\mathbf{r}, \phi, w(z, s)), \\
\mathbf{r}(0) = 0, \\
l(0) = 0,
\end{aligned} \tag{3.10}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}$, $\phi \in H_0^1(\Omega)$. Taking $\boldsymbol{\varphi} = \mathbf{r}$, $\phi = l$ in system (3.10) and adding these equations we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{|\mathbf{r}(t)|^2 + |l(t)|^2\right\} + (\nu + \kappa)\|\mathbf{r}(t)\|^2 + \gamma\|l(t)\|^2 + 4\kappa|l(t)|^2 \\
&= 4\kappa(l(t), \mathrm{curl}\, \mathbf{r}(t)) + b(\mathbf{r}(t), \mathbf{u}(t; z, s), \mathbf{r}(t)) + b(\boldsymbol{\eta}_{1,h}(t), \mathbf{r}(t), h\boldsymbol{\eta}_{1,h}(t)) \\
&\quad + b_1(\mathbf{r}(t), l(t), w(t; z, s)) + b_1(\boldsymbol{\eta}_{1,h}(t), l(t), h\eta_{2,h}(t)).
\end{aligned}$$

Using Lemma 2.1, Young's inequality and the fact that $(\mathbf{u}, w) \in C([0, T]; \mathbf{V}) \times C([0, T]; H_0^1(\Omega))$, we obtain the following estimates

$$\begin{aligned}
4\kappa|(l(t), \mathrm{curl}\, \mathbf{r}(t))| &\leqslant C_{\epsilon_1}|l(t)|^2 + \epsilon_1\|\mathbf{r}(t)\|^2, \\
|b(\mathbf{r}(t), \mathbf{u}(t; z, s), \mathbf{r}(t))| &\leqslant C_{\epsilon_2}|\mathbf{r}(t)|^2 + \epsilon_2\|\mathbf{r}(t)\|^2, \\
|b(\boldsymbol{\eta}_{1,h}(t), \mathbf{r}(t), h\boldsymbol{\eta}_{1,h}(t))| &\leqslant C_{\epsilon_3}|\boldsymbol{\eta}_{1,h}(t)|^2\|h\boldsymbol{\eta}_{1,h}(t)\|^2 + \epsilon_3\|\mathbf{r}(t)\|^2, \\
|b_1(\mathbf{r}(t), l(t), w(t; z, s))| &\leqslant C_{\epsilon_4}|\mathbf{r}(t)|^2 + \epsilon_4\|\mathbf{r}(t)\|^2 + C_{\epsilon_5}|l(t)|^2 \\
&\quad + \epsilon_5\|l(t)\|^2, \\
|b_1(\boldsymbol{\eta}_{1,h}(t), l(t), h\eta_{2,h}(t))| &\leqslant C_1|\boldsymbol{\eta}_{1,h}(t)|^2\|h\boldsymbol{\eta}_{1,h}(t)\|^2 + C_2|\eta_{2,h}(t)|^2\|h\eta_{2,h}(t)\|^2 \\
&\quad + \epsilon_6\|l(t)\|^2,
\end{aligned}$$

with $\epsilon_i > 0$, $i = 1, 2, 3, 4, 5, 6$. In particular, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\{|\mathbf{r}(t)|^2 + |l(t)|^2\right\} + (\nu + \kappa)\|\mathbf{r}(t)\|^2 + \gamma\|l(t)\|^2 \leqslant\ & (C_{\epsilon_2} + C_{\epsilon_4})|\mathbf{r}(t)|^2 \\
& + (C_{\epsilon_1} + C_{\epsilon_5})|l(t)|^2 \\
& + (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)\|\mathbf{r}(t)\|^2 \\
& + (\epsilon_5 + \epsilon_6)\|l(t)\|^2 \\
& + C_1|\boldsymbol{\eta}_{1,h}(t)|^2\|h\boldsymbol{\eta}_{1,h}(t)\|^2 \\
& + C_2|\eta_{2,h}(t)|^2\|h\eta_{2,h}(t)\|^2.
\end{aligned}$$

Choosing the ϵ_i's so that $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4=\kappa$ and $\epsilon_5+\epsilon_6=\dfrac{\gamma}{2}$, we obtain

$$\begin{aligned}\frac{d}{dt}\Big\{|\mathbf{r}(t)|^2+|l(t)|^2\Big\}+2\nu\|\mathbf{r}(t)\|^2+\gamma\|l(t)\|^2 \leqslant C_0\Big\{|\mathbf{r}(t)|^2+|l(t)|^2\Big\}&\\ +C_1|\boldsymbol{\eta}_{1,h}(t)|^2\|h\boldsymbol{\eta}_{1,h}(t)\|^2&\\ +C_2|\eta_{2,h}(t)|^2\|h\eta_{2,h}(t)\|^2.&\end{aligned}\tag{3.11}$$

By Gronwall's Inequality and $\mathbf{r}(0)=0$, $l(0)=0$, we have

$$|\mathbf{r}(t)|^2+|l(t)|^2\leqslant C\int_0^T\Big\{C_1|\boldsymbol{\eta}_{1,h}(t)|^2\|h\boldsymbol{\eta}_{1,h}(t)\|^2+C_2|\eta_{2,h}(t)|^2\|h\eta_{2,h}(t)\|^2\Big\}dt.$$

Since

$$\begin{aligned}&\int_0^T\Big\{C_1|\boldsymbol{\eta}_{1,h}(t)|^2\|h\boldsymbol{\eta}_{1,h}(t)\|^2+C_2|\eta_{2,h}(t)|^2\|h\eta_{2,h}(t)\|^2\Big\}dt\\ &=\frac{C}{h^2}\int_0^T|\mathbf{u}(t;\boldsymbol{z}+h\pi,s+h\tau)-\mathbf{u}(t;\boldsymbol{z},s)|^2\|\mathbf{u}(t;\boldsymbol{z}+h\pi,s+h\tau)-\mathbf{u}(t;\boldsymbol{z},s)\|^2dt\\ &\quad+\frac{C}{h^2}\int_0^T|w(t;\boldsymbol{z}+h\pi,s+h\tau)-w(t;\boldsymbol{z},s)|^2\|w(t;\boldsymbol{z}+h\pi,s+h\tau)-w(t;\boldsymbol{z},s)\|^2dt\\ &\leqslant\frac{C}{h^2}\|\mathbf{u}(\boldsymbol{z}+h\pi,s+h\tau)-\mathbf{u}(\boldsymbol{z},s)\|^2_{L^\infty(0,T;\mathbf{H})}\|\mathbf{u}(\boldsymbol{z}+h\pi,s+h\tau)-\mathbf{u}(\boldsymbol{z},s)\|^2_{L^2(0,T;\mathbf{V})}\\ &\quad+\frac{C}{h^2}\|w(\boldsymbol{z}+h\pi,s+h\tau)-w(\boldsymbol{z},s)\|^2_{L^\infty(0,T;L^2)}\|w(\boldsymbol{z}+h\pi,s+h\tau)-w(\boldsymbol{z},s)\|^2_{L^2(0,T;H^1_0)},\end{aligned}$$

we have by Proposition 3.1

$$|\mathbf{r}(t)|^2+|l(t)|^2\leqslant Ch^2.\tag{3.12}$$

By inequalities (3.11) and (3.12)

$$\begin{aligned}\int_0^T\Big\{2\nu\|\mathbf{r}(t)\|^2+\gamma\|l(t)\|^2\Big\}dt&\leqslant C_0\int_0^T\Big\{|\mathbf{r}(t)|^2+|l(t)|^2\Big\}dt\\ &\quad+C_1\int_0^T|\boldsymbol{\eta}_{1,h}(t)|^2\|h\boldsymbol{\eta}_{1,h}(t)\|^2dt\\ &\quad+C_2\int_0^T|\eta_{2,h}(t)|^2\|h\eta_{2,h}(t)\|^2dt\\ &\leqslant Ch^2,\end{aligned}$$

that is,

$$\int_0^T\Big\{2\nu\|\mathbf{r}(t)\|^2+\gamma\|l(t)\|^2\Big\}dt\leqslant Ch^2,\tag{3.13}$$

proving (3.9). To prove the estimate (3.8), just proceed as in Proposition 3.1. Indeed, taking $\boldsymbol{\varphi}=\boldsymbol{\eta}_1$, $\phi=\eta_2$ in system (3.7). Then, adding these equations and considering Lemma 2.1, Young's and Gronwall's inequalities, we obtain the estimate (3.8). □

Similar arguments can be applied to the second order Gâteaux derivative of S. For $\boldsymbol{\psi}=(\boldsymbol{z},s)\in\mathbf{V}\times H^1_0(\Omega)$, we denote the second order Gâteaux derivative of S at $\boldsymbol{\psi}$ along the direction $(\boldsymbol{\pi},\boldsymbol{\tau})=((\pi_1,\pi_2),(\tau_1,\ \tau_2))$ by $D^2S(\boldsymbol{\psi})\cdot\boldsymbol{\pi}\cdot\boldsymbol{\tau}$. We are interested only in

$$\boldsymbol{\zeta}=D^2S(\boldsymbol{\psi})\cdot\boldsymbol{\pi}\cdot\boldsymbol{\pi}=(\zeta_1,\zeta_2).$$

Proposition 3.3. *The second order Gâteaus derivative ζ exists and is the solution of the linear system*

$$
\begin{aligned}
(\partial_t \boldsymbol{\zeta}_1, \boldsymbol{\varphi}) + (\nu+\kappa)((\boldsymbol{\zeta}_1, \boldsymbol{\varphi})) + b(\boldsymbol{\zeta}_1, \mathbf{u}, \boldsymbol{\varphi})& \\
+b(\mathbf{u}, \boldsymbol{\zeta}_1, \boldsymbol{\varphi}) &= 2\kappa(\textbf{\textit{curl}}\, \zeta_2, \boldsymbol{\varphi}) - 2b(\boldsymbol{\eta}_1, \boldsymbol{\eta}_1, \boldsymbol{\varphi}), \\
(\partial_t \zeta_2, \phi) + \gamma((\zeta_2, \phi)) + 4\kappa(\zeta_2, \phi) + b_1(\boldsymbol{\zeta}_1, w, \phi)& \\
+b_1(\mathbf{u}, \zeta_2, \phi) &= 2\kappa(\textit{curl}\, \boldsymbol{\zeta}_1, \phi) - 2b_1(\boldsymbol{\eta}_1, \eta_2, \phi), \\
\boldsymbol{\zeta}_1(0) &= 0, \\
\zeta_2(0) &= 0,
\end{aligned} \tag{3.14}
$$

for all $\boldsymbol{\varphi} \in \mathbf{V}$, $\phi \in H_0^1(\Omega)$, where $\mathbf{u} = \mathbf{u}(t; \boldsymbol{z}, s)$, $w = w(t; \boldsymbol{z}, s)$ and $\boldsymbol{\eta}(t) = (DS(\boldsymbol{\psi})\cdot\boldsymbol{\pi})(t) = (\boldsymbol{\eta}_1(t), \eta_2(t))$. The function ζ is estimated by

$$
\|\boldsymbol{\zeta}\|_{L^2(0,T;\mathbf{V})\times L^2(0,T;H_0^1)} \leqslant M\|\boldsymbol{\pi}\|^2_{\mathbf{V}\times H_0^1}, \tag{3.15}
$$

with $M = M(\nu, \kappa, \gamma, T, \mathbf{u}, w) > 0$.

Proof. We consider

$$
\boldsymbol{\zeta}_{1,h} = \frac{\boldsymbol{\eta}_1(\boldsymbol{z}+h\pi_1, s+h\pi_2) - \boldsymbol{\eta}_1(\boldsymbol{z}, s)}{h}
$$

and

$$
\zeta_{2,h} = \frac{\eta_2(\boldsymbol{z}+h\pi_1, s+h\pi_2) - \eta_2(\boldsymbol{z}, s)}{h},
$$

with $h \in \mathbb{R}$, and the associated system with the pair $(\boldsymbol{\zeta}_{1,h}, \zeta_{2,h})$. Then, we proceed as in the proof of Proposition 3.2. □

3.2 FORMULATION OF THE INITIALIZATION PROBLEM

Here, we present the problem of initialization as an optimal control problem whose control variables are the initial data. In this way, we look for solving a minimization problem, as well as to characterize its solutions through the "adjoint problem" (optimality conditions). Uniqueness of the solution is investigated in the next section.

Let $T > 0$, $\beta > 0$ and Z_1, Z_2 be Hilbert spaces serving as the observation space $z(t) = (z_1(t), z_2(t))$. Let $C : \mathbf{V} \times H_0^1(\Omega) \longrightarrow Z_1 \times Z_2$ be a bounded linear mapping. We assume $z \in L^2(0,T;Z_1) \times L^2(0,T;Z_2)$. The control-to-state mapping S from $\mathbf{V} \times H_0^1(\Omega)$ into $L^2(0,T;\mathbf{V}) \times L^2(0,T;H_0^1(\Omega))$ is well defined and continuous (see Proposition 3.1). It follows that the map $\boldsymbol{\psi} \mapsto CS(\boldsymbol{\psi})$ is a continuous mapping from $\mathbf{V} \times H_0^1(\Omega)$ into $L^2(0,T;Z_1) \times L^2(0,T;Z_2)$. The constant β serves as a regularization coefficient. The state is compared with the data by the control functional

$$
J(\boldsymbol{\psi}) = \int_0^T \|CS(\boldsymbol{\psi})(t) - z(t)\|^2_{Z_1\times Z_2} dt + \beta\|\boldsymbol{\psi}\|^2_{\mathbf{V}\times H_0^1}. \tag{3.16}
$$

The optimal control problem is to find $\psi_0 \in \mathbf{V} \times H_0^1(\Omega)$ such that

$$J(\psi_0) = \min\{J(\psi) \; ; \; \psi \in \mathbf{V} \times H_0^1(\Omega)\}. \tag{3.17}$$

We consider the following lemmas.

Lemma 3.4. *Let $\{\psi_k\}$ be a sequence in $\mathbf{V} \times H_0^1(\Omega)$ such that $\psi_k \rightharpoonup \psi_0$ in $\mathbf{V} \times H_0^1(\Omega)$ and let $S(\psi_k)$, $S(\psi_0)$ be the corresponding solutions of* (3.1)*. Then, $S(\psi_k) \rightharpoonup S(\psi_0)$ in $L^2(0, T; \mathbf{V}) \times L^2(0, T; H_0^1(\Omega))$.*

Proof. Since $\mathbf{V} \times H_0^1(\Omega) \hookrightarrow \mathbf{H} \times L^2(\Omega)$ with compact injection, $\psi_k \to \psi_0$ strongly in $\mathbf{H} \times L^2(\Omega)$. By Proposition 3.1, $S(\psi_k) \to S(\psi_0)$ in $L^\infty(0, T; \mathbf{H}) \times L^\infty(0, T; L^2(\Omega))$, hence in $L^2(0, T; \mathbf{H}) \times L^2(0, T; L^2(\Omega))$. By estimate (2.13), $\{S(\psi_k)\}_k$ is bounded in $L^2(0, T; \mathbf{V}) \times L^2(0, T; H_0^1(\Omega))$. Therefore, there exists a subsequence $\{S(\psi_{k_j})\}_j$ that converges weakly to some $\hat{\psi}$ in $L^2(0, T; \mathbf{V}) \times L^2(0, T; H_0^1(\Omega))$. Since $L^2(0, T; \mathbf{V}) \times L^2(0, T; H_0^1(\Omega)) \hookrightarrow L^2(0, T; \mathbf{H}) \times L^2(0, T; L^2(\Omega))$, we have $\hat{\psi} = S(\psi_0)$. Then, by a subsequence argument, we conclude that $S(\psi_k) \rightharpoonup S(\psi_0)$ in $L^2(0, T; \mathbf{V}) \times L^2(0, T; H_0^1(\Omega))$. □

Lemma 3.5. *Let $\{\psi_k\}$ be a sequence in $\mathbf{V} \times H_0^1(\Omega)$ such that $\psi_k \rightharpoonup \psi_0$ in $\mathbf{V} \times H_0^1(\Omega)$. Then, $CS(\psi_k) \rightharpoonup CS(\psi_0)$ in $L^2(0, T; Z_1) \times L^2(0, T; Z_2)$.*

Proof. Let $\chi \in L^2(0, T; Z_1) \times L^2(0, T; Z_2)$. Then,

$$\int_0^T (CS(\psi_k), \chi)_{Z_1 \times Z_2} dt = \int_0^T (S(\psi_k), C^*\chi)_{\mathbf{V} \times H_0^1(\Omega)} dt,$$

where $(\cdot, \cdot)_{Z_1 \times Z_2}$ is the inner product in $Z_1 \times Z_2$ and C^* denotes the adjoint of C. By Lemma 3.4,

$$\begin{aligned}\int_0^T (S(\psi_k), C^*\chi)_{\mathbf{V} \times H_0^1(\Omega)} dt \to & \int_0^T (S(\psi_0), C^*\chi)_{\mathbf{V} \times H_0^1(\Omega)} dt \\ = & \int_0^T (CS(\psi_0), \chi)_{Z_1 \times Z_2} dt.\end{aligned}$$

By the definition of weak convergence, we obtain $CS(\psi_k) \rightharpoonup CS(\psi_0)$ in $L^2(0, T; Z_1) \times L^2(0, T; Z_2)$. □

The main result of this section is the following.

Theorem 3.6 (Optimality Criterion)**.** *There exists a solution ψ_0 of the minimization problem* (3.17)*.*

Proof. Initially note that

$$J(\psi) = \|CS(\psi) - z\|^2_{L^2(0,T;Z_1) \times L^2(0,T;Z_2)} + \beta \|\psi\|^2_{\mathbf{V} \times H_0^1} \geqslant 0,$$

for all $\psi \in \mathbf{V} \times H_0^1(\Omega)$. Then, let $d = \inf\{J(\psi)\ ;\ \psi \in \mathbf{V} \times H_0^1(\Omega)\}$ and let $\{\psi_k\}$ be a minimizing sequence for J, that is, $J(\psi_k) \to d$. Since $\beta > 0$, we have

$$\|\psi_k\|^2_{\mathbf{V}\times H_0^1} \leqslant \frac{1}{\beta} J(\psi_k) \leqslant C,\ \forall\ k \in \mathbb{N}.$$

This implies that there exists a subsequence, which we still denote by $\{\psi_k\}$, such that $\psi_k \rightharpoonup \psi_0$ in $\mathbf{V} \times H_0^1(\Omega)$. By Lemma 3.5, $CS(\psi_k) \rightharpoonup CS(\psi_0)$ in $L^2(0,T;Z_2) \times L^2(0,T;Z_2)$. Since in a Hilbert space the norm is weakly lower semicontinuous, it follows that

$$\begin{aligned} d &= \liminf_{k\to\infty} J(\psi_k) \\ &= \liminf_{k\to\infty} \left\{ \|CS(\psi_k) - z\|^2_{L^2(0,T;Z_1)\times L^2(0,T;Z_2)} + \beta\|\psi_k\|^2_{\mathbf{V}\times H_0^1} \right\} \\ &\geqslant J(\psi_0). \end{aligned}$$

Hence, $J(\psi_0) \leqslant d$ and we conclude that $J(\psi_0) = d$, that is, ψ_0 is a solution of the minimization problem (3.17). This finishes the proof □

Under the above conditions, it is not always possible to guarantee the uniqueness of the solution, since the state is a nonlinear function of the control (initial data). Hence, we define the set

$$U_0 = \{\psi_0\ ;\ \psi_0 \text{ is a solution of problem (3.17)}\}. \tag{3.18}$$

From the continuity of J, it follows that U_0 is a closed set in $\mathbf{V} \times H_0^1(\Omega)$. We will obtain more information about the set U_0 in the next section. In this direction, the following theorem is of great importance.

Theorem 3.7. *Suppose $\psi_0 = (\mathbf{u}_0, w_0) \in U_0$. Then, there exist $\mathbf{u}, \mathbf{p} \in L^2(0,T;\mathbf{V})$ and $w, q \in L^2(0,T;H_0^1(\Omega))$ such that*

$$\begin{aligned} (\partial_t \mathbf{u}, \boldsymbol{\varphi}) + (\nu+\kappa)((\mathbf{u}, \boldsymbol{\varphi})) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\varphi}) &= 2\kappa(\mathbf{curl}\, w, \boldsymbol{\varphi}), \\ (\partial_t w, \phi) + \gamma((w, \phi)) + 4\kappa(w, \phi) + b_1(\mathbf{u}, w, \phi) &= 2\kappa(\mathit{curl}\, \mathbf{u}, \phi), \\ (\mathbf{u}(0), w(0)) &= \psi_0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} -(\partial_t \mathbf{p}, \boldsymbol{\varphi}) &+ (\nu+\kappa)((\mathbf{p}, \boldsymbol{\varphi})) + b(\boldsymbol{\varphi}, \mathbf{u}, \mathbf{p}) \\ &+ b(\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p}) = 2\kappa(\mathbf{curl}\, q, \boldsymbol{\varphi}) + (C_1 S(\psi_0) - z_1, C_1\xi)_{Z_1}, \\ -(\partial_t q, \phi) &+ \gamma((q, \phi)) + 4\kappa(q, \phi) + b_1(\mathbf{p}, w, \phi) \\ &+ b_1(\mathbf{u}, \phi, q) = 2\kappa(\mathit{curl}\, \mathbf{p}, \phi) + (C_2 S(\psi_0) - z_2, C_2\xi)_{Z_2}, \\ &(p(T), q(T)) = 0, \end{aligned} \tag{3.20}$$

$$\begin{aligned} 2\Big\{(\mathbf{p}(0), \boldsymbol{\varphi}) + (q(0), \phi)\Big\} &+ \beta((\psi_0, \xi)) \\ &+ \int_0^T \Big\{ b_1(\mathbf{p}(t), w(t), \eta_2(t)) - b_1(\boldsymbol{\eta}_1(t), w(t), q(t)) \Big\} dt = 0, \end{aligned} \tag{3.21}$$

for all $\xi = (\boldsymbol{\varphi}, \phi) \in \mathbf{V} \times H_0^1(\Omega)$ and $C = (C_1, C_2)$.

Proof. We begin by computing the derivative of the functional J. For a fixed $\boldsymbol{\psi_0}$, we have

$$\frac{1}{2}DJ(\boldsymbol{\psi_0})\cdot\boldsymbol{\xi} = \int_0^T (CS(\boldsymbol{\psi_0})(t) - z(t), C\boldsymbol{\eta}(t))_{Z_1\times Z_2}dt + \beta((\boldsymbol{\psi_0},\boldsymbol{\xi})), \tag{3.22}$$

for all $\boldsymbol{\xi}\in\mathbf{V}\times H_0^1(\Omega)$, where $\boldsymbol{\eta} = DS(\boldsymbol{\psi_0})\cdot\boldsymbol{\xi}$ is the Gatêaux derivative of S at ψ_0 along the direction $\boldsymbol{\xi} = (\boldsymbol{\varphi},\phi)$, which satisfy the system (3.7). Now, let $(\mathbf{p},\ q)$ be the solution of the **a**djoint system (3.20) above. Setting $(\boldsymbol{\varphi},\phi) = (\mathbf{p},\ q)$ in system (3.7), $(\boldsymbol{\varphi},\phi) = \boldsymbol{\eta} = (\boldsymbol{\eta_1},\eta_2)$ in adjoint system (3.20), after some manipulations we obtain

$$\begin{aligned}2\partial_t\Big\{(\mathbf{p},\boldsymbol{\eta}_1) + (q,\eta_2)\Big\} &- b_1(\mathbf{p}(t),w(t),\eta_2(t))\\ &+ b_1(\boldsymbol{\eta_1}(t),w(t),q(t)) = -(CS(\boldsymbol{\psi_0})(t) - z(t), C\boldsymbol{\eta}(t))_{Z_1\times Z_2}.\end{aligned}$$

Integrating from 0 to T, we have

$$\begin{aligned}2\Big\{(\mathbf{p}(0),\boldsymbol{\varphi}) + (q(0),\phi)\Big\} + \int_0^T\Big\{b_1(\mathbf{p}(t),w(t),\eta_2(t)) - b_1(\boldsymbol{\eta_1}(t),w(t),q(t))\Big\}dt\\ = \int_0^T (CS(\boldsymbol{\psi_0})(t) - z(t), C\boldsymbol{\eta}(t))_{Z_1\times Z_2}dt\\ = \frac{1}{2}DJ(\boldsymbol{\psi_0})\cdot\boldsymbol{\xi} - \beta((\boldsymbol{\psi_0},\boldsymbol{\xi})),\end{aligned}$$

that is,

$$\begin{aligned}2\Big\{(\mathbf{p}(0),\boldsymbol{\varphi}) + (q(0),\phi)\Big\} + \beta((\boldsymbol{\psi_0},\boldsymbol{\xi}))\\ + \int_0^T\Big\{b_1(\mathbf{p}(t),w(t),\eta_2(t)) - b_1(\boldsymbol{\eta_1}(t),w(t),q(t))\Big\}dt = \frac{1}{2}DJ(\boldsymbol{\psi_0})\cdot\boldsymbol{\xi}.\end{aligned}$$

Using that $DJ(\boldsymbol{\psi_0})\cdot\boldsymbol{\xi} = 0$ for $\boldsymbol{\psi_0}\in U_0$, for all $\boldsymbol{\xi}\in\mathbf{V}\times H_0^1(\Omega)$, we obtain identity (3.21). □

In (ZHOU, 1995), two formulations of the initialization problem for the Navier-Stokes model as an optimal control problem are presented:

I. Assuming that observations are taken continuously in time;

II. Assuming that the final state of system (the state of the system at a terminal time T) is observed.

Moreover, the author shows that the arguments used for the case (I) may be adapted to case (II) analogously with minor modifications. Here, the estimate (2.14) implies that it is possible to get the previous results for the case (II) in the same way as in (ZHOU, 1995) (section 3). Therefore, we will omit the details for the case (II) and only present the corresponding theorems.

Thus, considering that the final state is observed, the problem is to find an initial data so that the corresponding final state is the closest to the observation. We compare the state with the data using the control functional

$$\hat{J}(\boldsymbol{\psi}) = \|CS(\boldsymbol{\psi})(T) - z_d\|^2_{\mathbf{V}\times H_0^1} + \beta\|\boldsymbol{\psi}\|^2_{\mathbf{V}\times H_0^1}, \tag{3.23}$$

where $C = (C_1, C_2) : \mathbf{V} \times H_0^1(\Omega) \longrightarrow \mathbf{V} \times H_0^1(\Omega)$ is a bounded linear mapping, $z_d = (z_d^1, z_d^2) \in \mathbf{V} \times H_0^1(\Omega)$ is the observation at time T and $S(\psi)(T)$ is the evaluation of $S(\psi)$ at time T. As before, the optimal control problem is to find $\psi_0 \in \mathbf{V} \times H_0^1(\Omega)$ such that

$$\hat{J}(\psi_0) = \min\{\hat{J}(\psi) \ ; \ \psi \in \mathbf{V} \times H_0^1(\Omega)\}. \tag{3.24}$$

Theorem 3.8 (Optimality Criterion)**.** *There exists a solution* ψ_0 *of the minimization problem* (3.24)*.*

Theorem 3.9. *Suppose* $\psi_0 \in \hat{U}_0 = \{\psi \ ; \ \psi$ *is a solution of* (3.24)$\}$*. Then, there exist* $u, p \in L^2(0, T; \mathbf{V})$ *and* $w, q \in L^2(0, T; H_0^1(\Omega))$ *such that*

$$\begin{aligned} (\partial_t u, \varphi) + (\nu + \kappa)((u, \varphi)) + b(u, u, \varphi) &= 2\kappa(\textbf{\textit{curl}}\, w, \varphi), \\ (\partial_t w, \phi) + \gamma((w, \phi)) + 4\kappa(w, \phi) + b_1(u, w, \phi) &= 2\kappa(\textit{curl}\, u, \phi), \\ (u(0), w(0)) &= \psi_0, \end{aligned} \tag{3.25}$$

$$\begin{aligned} -(\partial_t p, \varphi) + (\nu + \kappa)((p, \varphi)) + b(\varphi, u, p) + b(u, \varphi, p) &= 2\kappa(\textbf{\textit{curl}}\, q, \varphi), \\ -(\partial_t q, \phi) + \gamma((q, \phi)) + 4\kappa(q, \phi) + b_1(p, w, \phi) + b_1(u, \phi, q) &= 2\kappa(\textit{curl}\, p, \phi), \\ (p(T), q(T)) = (C_1^*(C_1 S(\psi_0)(T) - z_d^1), C_2^*(C_2 S(\psi_0)(T) &- z_d^2)), \end{aligned} \tag{3.26}$$

$$\begin{aligned} &2\Big\{(p(0), \varphi) + (q(0), \phi)\Big\} + \beta((\psi_0, \xi)) \\ &\qquad + \int_0^T \Big\{b_1(p(t), w(t), \eta_2(t)) - b_1(\eta_1(t), w(t), q(t))\Big\} dt = 0, \end{aligned} \tag{3.27}$$

for all $\xi = (\varphi, \phi) \in \mathbf{V} \times H_0^1(\Omega)$*.*

3.3 SUFFICIENT CONDITIONS FOR UNIQUENESS

We now prove, under certain additional conditions of regularity and compactness, that the minimization problem studied above admits at most a finite number of solutions. Moreover, we show that through a suitable relation between T, β and the optimal controls, the solution is indeed unique. We consider the problem (3.17) and, therefore, the system (3.19) - (3.21). We assume the control ψ to be an element of an *a*dmissible set Q_{ad} such that

(i) $Q_{ad} \subset \Big(\mathbf{H}^2(\Omega) \times H^2(\Omega)\Big) \cap \Big(\mathbf{V} \times H_0^1(\Omega)\Big)$ and bounded in $\mathbf{H}^2(\Omega) \times H^2(\Omega)$;

(ii) Q_{ad} is convex and closed in $\mathbf{V} \times H_0^1(\Omega)$.

The existence of a solution to the control problem restricted to Q_{ad} is shown as in the proof of Theorem 3.6, since Q_{ad} is bounded, convex and closed. We define the solution set

$$U = \{\psi_0 \ ; \ \psi_0 \text{ is a solution of problem (3.17)}\}. \tag{3.28}$$

Remark 3.10. *Since* $\Big(\mathbf{H}^2(\Omega)\times H^2(\Omega)\Big)\cap\Big(\mathbf{V}\times H_0^1(\Omega)\Big)\hookrightarrow \mathbf{V}\times H_0^1(\Omega)$ *with compact injection and* $U\subset Q_{ad}$ *is bounded and closed, the set* U *is a compact set in* $\mathbf{V}\times H_0^1(\Omega)$.

Theorem 3.11. *If the time interval* $[0,T]$ *is short enough, then* U *is finite.*

Proof. Initially see that

$$\begin{aligned} D^2J(\psi)\cdot\boldsymbol{\pi}\cdot\boldsymbol{\tau} &= 2(CS(\psi)-Z, CD^2S(\psi)\cdot\boldsymbol{\pi}\cdot\boldsymbol{\tau})_{L^2(0,T;Z_1)\times L^2(0,T;Z_2)} \\ &\quad + 2(C\hat{\eta}, C\eta)_{L^2(0,T;Z_1)\times L^2(0,T;Z_2)} + 2\beta((\boldsymbol{\pi},\boldsymbol{\tau}))_{\mathbf{V}\times H_0^1}, \end{aligned}$$

for all $\boldsymbol{\pi},\boldsymbol{\tau}\in\mathbf{V}\times H_0^1(\Omega)$, where $\hat{\eta}=DS(\psi)\cdot\boldsymbol{\tau}$ and $\eta=DS(\psi)\cdot\boldsymbol{\pi}$. In particular, for $\boldsymbol{\tau}=\boldsymbol{\pi}$ we have

$$\begin{aligned} D^2J(\psi)\cdot\boldsymbol{\pi}\cdot\boldsymbol{\pi} &= 2(CS(\psi)-Z, C\zeta)_{L^2(0,T;Z_1)\times L^2(0,T;Z_2)} \\ &\quad + 2\|C\eta\|^2_{L^2(0,T;Z_1)\times L^2(0,T;Z_2)} + 2\beta\|\boldsymbol{\pi}\|^2_{\mathbf{V}\times H_0^1}, \end{aligned}$$

where η and $\zeta=D^2S(\psi)\cdot\boldsymbol{\pi}\cdot\boldsymbol{\pi}$ satisfy identities (3.7) and (3.14), respectively. Our goal is to show that if $T>0$ is small enough, then $D^2J(\psi)$ is positive definite, that is,

$$D^2J(\psi)\cdot\boldsymbol{\pi}\cdot\boldsymbol{\pi}\geqslant\varepsilon\|\boldsymbol{\pi}\|^2,\ \forall\,\boldsymbol{\pi}\in\mathbf{V}\times H_0^1(\Omega).$$

Then, it will follow by Taylor expansion up to second order that the points of U are isolated. Finally, we use the compactness to prove that U is finite. Indeed, since

$$\begin{aligned} \frac{1}{2}D^2J(\psi)\cdot\boldsymbol{\pi}\cdot\boldsymbol{\pi} &\geqslant \beta\|\boldsymbol{\pi}\|^2+\|C\eta\|^2-\|C\|\|CS(\psi)-Z\|\|\zeta\| \\ &\geqslant \beta\|\boldsymbol{\pi}\|^2-\|C\|\|CS(\psi)-Z\|\|\zeta\|, \end{aligned}$$

we have by the estimate (3.15)

$$D^2J(\psi)\cdot\boldsymbol{\pi}\cdot\boldsymbol{\pi}\geqslant\Big(2\beta-2M\|C\|\|CS(\psi)-Z\|\Big)\|\boldsymbol{\pi}\|^2.$$

By Lemma 3.1, the map $\psi\longmapsto\|CS(\psi)-Z\|_{L^2(0,T;Z_1)\times L^2(0,T;Z_2)}$ is continuous from $\mathbf{V}\times H_0^1(\Omega)$ to $\mathbb{R}$. Then, by compactness of U, we have

$$R_T=\max_{\psi\in U}\|CS(\psi)-Z\|_{L^2(0,T;Z_1)\times L^2(0,T;Z_2)}<+\infty.$$

Therefore,

$$D^2J(\psi)\cdot\boldsymbol{\pi}\cdot\boldsymbol{\pi}\geqslant\Big(2\beta-2M\|C\|R_T\Big)\|\boldsymbol{\pi}\|^2.$$

Since $R_T\to0$ for $T\to0$ and $\beta>0$ fixed, we can choose $T>0$ small enough so that $2\beta-2M\|C\|R_T\geqslant\varepsilon>0$. Therefore, for $T>0$ small enough we have $D^2J(\psi)$ positive definite. □

For the purpose of proving the uniqueness for solution of Problem (3.17), we use

Proposition 3.12. *The solution of the adjoint system satisfies the following inequality*

$$\int_t^T \left\{ \|\mathbf{p}(s)\|^2 + \|q(s)\|^2 \right\} ds \leqslant F(t,T), \tag{3.29}$$

where $\lim_{T\to 0} F(0,T) = 0$.

Proof. Taking $\xi = (\varphi, \phi) = (\mathbf{p}, q)$ in identities (3.20) and adding the resulting expressions,

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\left\{ |\mathbf{p}(t)|^2 + |q(t)|^2 \right\} + \theta\left\{ \|\mathbf{p}(t)\|^2 + \|q(t)\|^2 \right\} \\
&\leqslant |b(p(t), \mathbf{u}(t), \mathbf{p}(t))| + |b_1(\mathbf{p}(t), w(t), q(t))| + 2\kappa\left\{ (\mathbf{curl}\, q(t), \mathbf{p}(t)) + (\mathrm{curl}\, \mathbf{p}(t), q(t)) \right\} \\
&\quad + (C_1 S(\boldsymbol{\psi})(t) - z_1, C_1 \xi(t))_{Z_1} + (C_2 S(\boldsymbol{\psi})(t) - z_2, C_2 \xi(t))_{Z_2} \\
&\leqslant \theta\left\{ \|\mathbf{p}(t)\|^2 + \|q(t)\|^2 \right\} + c_0\left\{ \|\mathbf{u}(t)\|^2 + \|w(t)\|^2 + 1 \right\}\left\{ |\mathbf{p}(t)|^2 + |q(t)|^2 \right\} \\
&\quad + \alpha_1 \|C_1\|^2 \|C_1 S(\boldsymbol{\psi})(t) - z_1(t)\|^2_{Z_1} + \alpha_2 \|C_2\|^2 \|C_2 S(\boldsymbol{\psi})(t) - z_2(t)\|^2_{Z_2} \\
&\leqslant \theta\left\{ \|\mathbf{p}(t)\|^2 + \|q(t)\|^2 \right\} + c_0\left\{ \|\mathbf{u}(t)\|^2 + \|w(t)\|^2 + 1 \right\}\left\{ |\mathbf{p}(t)|^2 + |q(t)|^2 \right\} \\
&\quad + \alpha_3 \|C\|^2 \|C S(\boldsymbol{\psi})(t) - Z(t)\|^2_{Z_1 \times Z_2},
\end{aligned}$$

that is,

$$\begin{aligned}
-\frac{d}{dt}\left\{ |\mathbf{p}(t)|^2 + |q(t)|^2 \right\} \leqslant c_0&\left\{ \|\mathbf{u}(t)\|^2 + \|w(t)\|^2 + 1 \right\}\left\{ |\mathbf{p}(t)|^2 + |q(t)|^2 \right\} \\
&+ \alpha_3 \|C\|^2 \|C S(\boldsymbol{\psi})(t) - Z(t)\|^2_{Z_1 \times Z_2}.
\end{aligned}$$

Integrating from t to T and using that $\mathbf{p}(T) = q(T) = 0$

$$\begin{aligned}
|\mathbf{p}(t)|^2 + |q(t)|^2 \leqslant \alpha_3 \|C\|^2 \int_t^T \exp\left(c_0 \int_\xi^s \left\{ \|\mathbf{u}(\tau)\|^2 + \|w(\tau)\|^2 + 1 \right\} d\tau \right) \\
\times \|C S(\boldsymbol{\psi})(s) - Z(s)\|^2_{Z_1 \times Z_2} ds.
\end{aligned}$$

Hence,

$$|\mathbf{p}(t)|^2 + |q(t)|^2 \leqslant c(t,T) R_T,$$

where $\lim_{T\to 0} c(t,T) > 0$ and R_T is as in the proof of Theorem 3.11. Defining $F_0(t,T) = c(t,T)R_T$, we have

$$|\mathbf{p}(t)|^2 + |q(t)|^2 \leqslant F_0(t,T),$$

with $\lim_{T\to 0} F_0(0,T) = 0$. Similarly as above, we also obtain

$$\begin{aligned}
-\frac{d}{dt}\left\{ |\mathbf{p}(t)|^2 + |q(t)|^2 \right\} + \frac{\theta}{2}\left\{ \|\mathbf{p}(t)\|^2 + \|q(t)\|^2 \right\} \leqslant c_0 &\left\{ \|\mathbf{u}(t)\|^2 + \|w(t)\|^2 + 1 \right\} \\
&\times \left\{ |\mathbf{p}(t)|^2 + |q(t)|^2 \right\} \\
&+ \alpha_3 \|C\|^2 \|C S(\boldsymbol{\psi})(t) - Z(t)\|^2_{Z_1 \times Z_2},
\end{aligned}$$

that is,

$$
\begin{aligned}
\|\mathbf{p}(t)\|^2 + \|q(t)\|^2 \leqslant c_1 \frac{d}{dt}\Big\{|\mathbf{p}(t)|^2 + |q(t)|^2\Big\} \\
+ c_2\Big\{\|\mathbf{u}(t)\|^2 + \|w(t)\|^2 + 1\Big\}\Big\{|\mathbf{p}(t)|^2 + |q(t)|^2\Big\} \\
+ c_3\|CS(\boldsymbol{\psi})(t) - Z(t)\|^2_{Z_1\times Z_2}.
\end{aligned}
$$

Integrating from t to T

$$
\begin{aligned}
\int_t^T \Big\{\|\mathbf{p}(s)\|^2 + \|q(s)\|^2\Big\}ds &\leqslant -c_1\Big\{|\mathbf{p}(t)|^2 + |q(t)|^2\Big\} \\
&\quad + c_2\int_t^T \Big\{\|\mathbf{u}(s)\|^2 + \|w(s)\|^2 + 1\Big\}\Big\{|\mathbf{p}(s)|^2 + |q(s)|^2\Big\}ds \\
&\quad + c_3\int_t^T \|CS(\boldsymbol{\psi})(s) - Z(s)\|^2_{Z_1\times Z_2}ds \\
&\leqslant c_2\int_t^T \Big\{\|\mathbf{u}(s)\|^2 + \|w(s)\|^2 + 1\Big\}\Big\{|\mathbf{p}(s)|^2 + |q(s)|^2\Big\}ds \\
&\quad + c_3\int_t^T \|CS(\boldsymbol{\psi})(s) - Z(s)\|^2_{Z_1\times Z_2}ds \\
&\leqslant c_4(T-t)\max_{t\in[0,T]} F_0(t,T) + c_5R_T := F(t,T).
\end{aligned}
$$

This proves the estimate (3.29). □

Finally, suppose there exist two solutions $\boldsymbol{\psi}_1 = (\mathbf{u}_0^1, w_0^1)$, $\boldsymbol{\psi}_2 = (\mathbf{u}_0^2, w_0^2)$ for the optimal control problem (3.17). If $(\mathbf{u}_1, w_1)$, $(\mathbf{u}_2, w_2)$, $(\mathbf{p}_1, q_1)$, $(\mathbf{p}_2, q_2)$ are the corresponding solutions of the state and adjoint equations as given in Theorem 3.7, we define

$$
\begin{aligned}
\boldsymbol{\delta} &= (\mathbf{u}_1 - \mathbf{u}_2, w_1 - w_2) = (\boldsymbol{\delta}_1, \delta_2), \\
\boldsymbol{\rho} &= (\mathbf{p}_1 - \mathbf{p}_2, q_1 - q_2) = (\boldsymbol{\rho}_1, \rho_2), \\
\boldsymbol{\nu} &= \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2 = (\boldsymbol{\nu}_1, \nu_2).
\end{aligned}
$$

We see that $\boldsymbol{\delta}$, $\boldsymbol{\rho}$ and $\boldsymbol{\nu}$ satisfy the system

$$
\begin{aligned}
(\partial_t\boldsymbol{\delta}_1, \boldsymbol{\varphi}) + (\nu+\kappa)((\boldsymbol{\delta}_1, \boldsymbol{\varphi})) + b(\boldsymbol{\delta}_1, \mathbf{u}_1, \boldsymbol{\varphi}) + b(\mathbf{u}_2, \boldsymbol{\delta}_1, \boldsymbol{\varphi}) = 2\kappa(\mathbf{curl}\,\delta_2, \boldsymbol{\varphi}) \\
(\partial_t\delta_2, \phi) + \gamma((\delta_2, \phi)) + 4\kappa(\delta_2, \phi) + b_1(\boldsymbol{\delta}_1, w_1, \phi) + b_1(\mathbf{u}_2, \delta_2, \phi) = 2\kappa(\mathrm{curl}\,\boldsymbol{\delta}_1, \phi) \\
\boldsymbol{\delta}(0) = \boldsymbol{\nu},
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
-(\partial_t\boldsymbol{\rho}_1, \boldsymbol{\varphi}) + (\nu+\kappa)((\boldsymbol{\rho}_1, \boldsymbol{\varphi})) + b(\boldsymbol{\varphi}, \mathbf{u}_1, \boldsymbol{\rho}_1) + b(\boldsymbol{\varphi}, \boldsymbol{\delta}_1, \mathbf{p}_2) + b(\mathbf{u}_1, \boldsymbol{\varphi}, \boldsymbol{\rho}_1) \\
+ b(\boldsymbol{\delta}_1, \boldsymbol{\varphi}, \mathbf{p}_2) = 2\kappa(\mathbf{curl}\,\rho_2, \boldsymbol{\varphi}) + (C_1\boldsymbol{\delta}, C_1\boldsymbol{\xi})_{Z_1}, \\
-(\partial_t\rho_2, \phi) + \gamma((\rho_2, \phi)) + 4\kappa(\rho_2, \phi) + b_1(\boldsymbol{\rho}_1, w_1, \phi) \\
+b_1(\mathbf{p}_2, \delta_2, \phi) + b_1(\mathbf{u}_1, \phi, \rho_2) + b_1(\boldsymbol{\delta}_1, \phi, q_2) = 2\kappa(\mathrm{curl}\,\boldsymbol{\rho}_1, \phi) + (C_2\boldsymbol{\delta}, C_2\boldsymbol{\xi})_{Z_2}, \\
\boldsymbol{\rho}(T) = 0,
\end{aligned}
\tag{3.31}
$$

$$2(\boldsymbol{\rho}(0),\boldsymbol{\xi})+\beta((\boldsymbol{\nu},\boldsymbol{\xi}))$$
$$+\int_0^T\Big\{b_1(\mathbf{p}_1,w_1,\eta_2^1)-b_1(\boldsymbol{\eta}_1^1,w_1,q_1)-b_1(\mathbf{p}_2,w_2,\eta_2^2)+b_1(\boldsymbol{\eta}_1^2,w_2,q_2)\Big\}dt=0, \quad (3.32)$$

for all $\boldsymbol{\xi}=(\boldsymbol{\varphi},\phi)\in\mathbf{V}\times H_0^1(\Omega)$, $C=(C_1,C_2)$ and $\boldsymbol{\eta}^i=(\boldsymbol{\eta}_1^i,\eta_2^i)=DS(\boldsymbol{\psi}_i)\cdot\boldsymbol{\xi}$. We seek estimates for $\boldsymbol{\delta}$, $\boldsymbol{\rho}$ and $\boldsymbol{\nu}$.

Lemma 3.13. *Let $\boldsymbol{\delta}$ and $\boldsymbol{\nu}$ be defined as above. There exists nonnegative functions $G_i(t)$, $i=1,2$, such that*

$$\|\boldsymbol{\delta}(t)\|^2\leqslant \exp\left(\int_0^T G_1(t)dt\right)\left\{\|\boldsymbol{\nu}\|^2+\int_0^T G_2(t)dt\right\}.$$

In particular,

$$\|\boldsymbol{\delta}\|_{L^\infty(0,T;\mathbf{V})\times L^\infty(0,T;H_0^1)}\leqslant C_{\boldsymbol{\delta},1}(T)\|\boldsymbol{\nu}\|+C_{\boldsymbol{\delta},2}(T), \quad (3.33)$$

with $\lim_{T\to 0}C_{\boldsymbol{\delta},i}(T)>0$, $i=1,2$.

Proof. By the estimate (2.15), we have $(\partial_t\boldsymbol{\delta}_1(t),\partial_t\delta_2(t))\in\mathbf{V}\times H_0^1(\Omega)$. Taking $\boldsymbol{\varphi}=\partial_t\boldsymbol{\delta}_1(t)$, $\phi=\partial_t\delta_2(t)$ in system (3.30) and adding the expressions, we get

$$\begin{aligned}|\partial_t\boldsymbol{\delta}_1(t)|^2+|\partial_t\delta_2(t)|^2+\frac{\theta}{2}\frac{d}{dt}\|\boldsymbol{\delta}(t)\|^2\leqslant 2\kappa\Big\{&(\mathbf{curl}\,\delta_2(t),\partial_t\boldsymbol{\delta}_1(t))+(\mathrm{curl}\,\boldsymbol{\delta}_1(t),\partial_t\delta_2(t))\Big\}\\&+|b(\boldsymbol{\delta}_1(t),\mathbf{u}_1(t),\partial_t\boldsymbol{\delta}_1(t))|\\&+|b(\mathbf{u}_2(t),\boldsymbol{\delta}_1(t),\partial_t\boldsymbol{\delta}_1(t))|\\&+|b_1(\boldsymbol{\delta}_1(t),w_1(t),\partial_t\delta_2(t))|\\&+|b_1(\mathbf{u}_2(t),\delta_2(t),\partial_t\delta_2(t))|.\end{aligned}$$

Using Hölder and Young inequalities together with Lemma 2.1, we obtain

$$\frac{d}{dt}\|\boldsymbol{\delta}(t)\|^2\leqslant G_1(t)\|\boldsymbol{\delta}(t)\|^2+G_2(t),$$

where

$$\begin{aligned}G_1(t)&=C_0+C_1\|\mathbf{u}_1(t)\||A\mathbf{u}_1(t)|+C_2\|\mathbf{u}_2(t)\||A\mathbf{u}_2(t)|+C_3\|w_1(t)\|^2+C_4\|\mathbf{u}_2(t)\|^2,\\G_2(t)&=C_5\|\partial_t\delta_2(t)\|^2.\end{aligned}$$

Since $\mathbf{u}_1,\mathbf{u}_2\in L^2(0,T;D(A))\cap C([0,T];\mathbf{V})$ and $w_1,w_2\in L^2(0,T;H_0^1(\Omega))$, we have $\int_0^T G_i(t)dt<\infty$, $i=1,2$. Applying Gronwall's Inequality and using that $\boldsymbol{\delta}(0)=\boldsymbol{\nu}$, we obtain

$$\|\boldsymbol{\delta}(t)\|^2\leqslant \exp\left(\int_0^T G_1(t)dt\right)\left\{\|\boldsymbol{\nu}\|^2+\int_0^T G_2(t)dt\right\}.$$

Then, Theorem 2.5 and Theorem 2.6 give us the estimate (3.33). □

Lemma 3.14. *Given* $T > 0$ *and for* $\boldsymbol{\delta}$, $\boldsymbol{\rho}$ *be defined as above, we have*

$$|\boldsymbol{\rho}(t)|^2 \leqslant \int_t^T \exp\left(K_5 \int_0^s \{\|\mathbf{u}_1(\tau)\|^2+\|w_1(\tau)\|^2+1\} d\tau\right)\left\{K_6+K_7\|\mathbf{p}_2(s)\|^2+K_8\|q_2(t)\|^2\right\}\|\boldsymbol{\delta}(s)\|^2 ds.$$

In particular, there exists $K(t,T) \geqslant 0$ *such that*

$$|\boldsymbol{\rho}(t)|^2 \leqslant K(t,T)\|\boldsymbol{\delta}\|^2_{L^\infty(0,T;\mathbf{V})\times L^\infty(0,T;H^1_0)}, \tag{3.34}$$

where $\lim_{T\to 0} K(0,T) = 0$.

Proof. Taking $\boldsymbol{\xi} = (\boldsymbol{\varphi}, \phi) = \boldsymbol{\rho}$ in system (3.31), adding the resulting expressions and writing $\theta = \min\{\nu+\kappa, \gamma\}$, we have

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}|\boldsymbol{\rho}(t)|^2 + \theta\|\boldsymbol{\rho}(t)\|^2 &\leqslant |b(\boldsymbol{\rho}_1(t), \mathbf{u}_1(t), \boldsymbol{\rho}_1(t))| + |b(\boldsymbol{\rho}_1(t), \boldsymbol{\delta}_1(t), \mathbf{p}_2(t))| \\
&\quad + |b(\boldsymbol{\delta}_1(t), \boldsymbol{\rho}_1(t), \mathbf{p}_2(t))| + |b_1(\boldsymbol{\rho}_1(t), w_1(t), \rho_2(t))| \\
&\quad + |b_1(\mathbf{p}_2(t), \delta_2(t), \rho_2(t))| + |b_1(\boldsymbol{\delta}_1(t), \rho_2(t), q_2(t))| \\
&\quad + 2\kappa\left\{(\mathbf{curl}\,\rho_2(t), \boldsymbol{\rho}_1(t)) + (\mathrm{curl}\,\boldsymbol{\rho}_1(t), \rho_2(t))\right\} + (C\boldsymbol{\delta}(t), C\boldsymbol{\rho}(t)) \\
&\leqslant K_0\|\mathbf{u}_1(t)\||\boldsymbol{\rho}_1(t)|\|\boldsymbol{\rho}_1(t)\| + 2K_1\|\boldsymbol{\rho}_1(t)\|\|\boldsymbol{\delta}_1(t)\|\|\mathbf{p}_2(t)\| \\
&\quad + K_2|\boldsymbol{\rho}_1(t)|^{1/2}\|\boldsymbol{\rho}_1(t)\|^{1/2}\|w_1(t)\||\rho_2(t)|^{1/2}\|\rho_2(t)\|^{1/2} \\
&\quad + K_3\|\mathbf{p}_2(t)\|\|\delta_2(t)\|\|\rho_2(t)\| \\
&\quad + K_4\|\boldsymbol{\delta}_1(t)\|\|\rho_2(t)\|\|q_2(t)\| + 2\kappa\|\rho_2(t)\||\boldsymbol{\rho}_1(t)| \\
&\quad + 2\kappa\|\boldsymbol{\rho}_1(t)\||\rho_2(t)| + \|C\|^2\|\boldsymbol{\delta}(t)\|\|\boldsymbol{\rho}(t)\|,
\end{aligned}$$

that is,

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}|\boldsymbol{\rho}(t)|^2 + \theta\|\boldsymbol{\rho}(t)\|^2 &\leqslant K_5\left\{\|\mathbf{u}_1(t)\|^2 + \|w_1(t)\|^2 + 1\right\}|\boldsymbol{\rho}(t)|^2 + \theta\|\boldsymbol{\rho}(t)\|^2 \\
&\quad + \left\{K_6 + K_7\|\mathbf{p}_2(t)\|^2 + K_8\|q_2(t)\|^2\right\}\|\boldsymbol{\delta}(t)\|^2.
\end{aligned}$$

Through Gronwall's Inequality and the fact that $\boldsymbol{\rho}(T) = 0$, it follows

$$|\boldsymbol{\rho}(t)|^2 \leqslant \int_t^T \exp\left(K_5 \int_0^s \{\|\mathbf{u}_1(\tau)\|^2+\|w_1(\tau)\|^2+1\} d\tau\right)\left\{K_6+K_7\|\mathbf{p}_2(s)\|^2+K_8\|q_2(t)\|^2\right\}\|\boldsymbol{\delta}(s)\|^2 ds.$$

Hence, using the estimate (2.13) and Proposition 3.12, we obtain the estimate (3.34). □

Lemma 3.15. *Let* $\boldsymbol{\nu}$ *and* $\boldsymbol{\rho}$ *be defined as above. If* $\boldsymbol{\nu} \in \mathbf{V} \times H^1_0(\Omega)$ *and* $\boldsymbol{\nu} \neq 0$, *then there exists a constant* $C_{\boldsymbol{\nu},1} > 0$, *independent of* T, *and a nonnegative function* $C_{\boldsymbol{\nu},2}(T)$ *such that*

$$\beta\|\boldsymbol{\nu}\| \leqslant C_{\boldsymbol{\nu},1}|\boldsymbol{\rho}(0)| + C_{\boldsymbol{\nu},2}(T), \tag{3.35}$$

where $\lim_{T\to 0} C_{\boldsymbol{\nu},2}(T) = 0$.

Proof. Taking $\boldsymbol{\xi} = \boldsymbol{\nu}$ in identity (3.32) and using the Cauchy-Schwarz and Poincaré inequalities, we have

$$\begin{aligned}
\beta\|\boldsymbol{\nu}\|^2 = &- 2(\boldsymbol{\rho}(0), \boldsymbol{\nu}) - \int_0^T b_1(\mathbf{p}_1, w_1, \eta_2^1)dt + \int_0^T b_1(\boldsymbol{\eta}_1^1, w_1, q_1)dt \\
&+ \int_0^T b_1(\mathbf{p}_2, w_2, \eta_2^2)dt - \int_0^T b_1(\boldsymbol{\eta}_1^2, w_2, q_2)dt \\
&\leqslant 2d(\Omega)|\boldsymbol{\rho}(0)|\|\boldsymbol{\nu}\| + \int_0^T |b_1(\mathbf{p}_1, w_1, \eta_2^1)|dt + \int_0^T |b_1(\boldsymbol{\eta}_1^1, w_1, q_1)|dt \\
&+ \int_0^T |b_1(\mathbf{p}_2, w_2, \eta_2^2)|dt + \int_0^T |b_1(\boldsymbol{\eta}_1^2, w_2, q_2)|dt \\
&:= C_{\nu,1}|\boldsymbol{\rho}(0)|\|\boldsymbol{\nu}\| + \sum_i I_i.
\end{aligned}$$

It is enough to bound the first integral I_1, since the others can be bounded in an entirely analogous way. To this end, use Lemma 2.7, the fact that $w_i \in C([0,T]; H_0^1(\Omega))$, Hölder inequality and estimate (3.8) to obtain

$$\begin{aligned}
I_1 &= \int_0^T |b_1(\mathbf{p}_1, w_1, \eta_2^1)|dt \leqslant c\int_0^T \|\mathbf{p}_1(t)\|\|w_1(t)\|\|\eta_2^1(t)\|dt \\
&\leqslant c \max_{t\in[0,T]} \|w_1(t)\| \int_0^T \|\mathbf{p}_1(t)\|\|\eta_2^1(t)\|dt \\
&\leqslant c \max_{t\in[0,T]} \|w_1(t)\| \left\{ \int_0^T \|\eta_2^1(t)\|^2 dt \right\}^{1/2} \left\{ \int_0^T \|\mathbf{p}_1(t)\|^2 dt \right\}^{1/2} \\
&\leqslant c_0(T) \max_{t\in[0,T]} \|w_1(t)\|\|\boldsymbol{\nu}\| \left\{ \int_0^T \|\mathbf{p}_1(t)\|^2 dt \right\}^{1/2} \\
&\leqslant c_1(T)\|\boldsymbol{\nu}\|,
\end{aligned}$$

with $\lim_{T\to 0} c_1(T) = 0$. Therefore, defining $C_{\nu,2}(T) = \sum_i c_i(T)$, we obtain

$$\beta\|\boldsymbol{\nu}\|^2 \leqslant \left\{ C_{\nu,1}|\boldsymbol{\rho}(0)| + C_{\nu,2}(T) \right\} \|\boldsymbol{\nu}\|.$$

If $\boldsymbol{\nu} \neq 0$, we have inequality (3.35). □

Theorem 3.16. *Assuming* $U \subset Q_{ad}$ *and the interval* $[0,T]$ *to be short enough, there is only one solution of the initialization problem* (P1).

Proof. Assume that $\boldsymbol{\nu} \neq 0$. By the estimates (3.33), (3.34) and (3.35)

$$\begin{aligned}
\beta\|\boldsymbol{\nu}\| &\leqslant C_{\nu,1}|\boldsymbol{\rho}(0)| + C_{\nu,2}(T) \\
&\leqslant C_{\nu,1}K(0,T)\|\boldsymbol{\delta}\|_{L^\infty(0,T;\mathbf{V})\times L^\infty(0,T;H_0^1)} + C_{\nu,2}(T) \\
&\leqslant C_{\nu,1}K(0,T)C_{\delta,1}(T)\|\boldsymbol{\nu}\| + C_{\nu,1}K(0,T)C_{\delta,2}(T) + C_{\nu,2}(T),
\end{aligned}$$

that is,

$$\beta\|\boldsymbol{\nu}\| \leqslant \hat{K}_1(T)\|\boldsymbol{\nu}\| + \hat{K}_2(T)$$

with $\lim_{T \to 0} \hat{K}_i(T) = 0$, $i = 1, 2$. Thus,

$$0 < \|v\| \leqslant \frac{\hat{K}_2(T)}{\beta - \hat{K}_1(T)}.$$

Letting $T \to 0$, we get $0 < \|v\| \leqslant 0$, which is a contradiction. Therefore, the solution of the Problem (3.17) is unique. $\square$

4 MICROPOLAR FLUIDS WITH SINGULAR INITIAL DATA

The generation of vorticity usually occurs in very small regions, being natural to consider initial vorticity concentrated in a set of zero Lebesgue measure. This motivates us to consider the initial vorticity in L^1 or, more generally, as a finite measure. We call this initial vorticity a **singular initial data**. We would like to understand what is the evolution of this vorticity over time. In the Navier-Stokes case, there are several works with singular initial data, for instance (BEN-ARTZI, 1994; BEN-ARTZI M.; CROISILLE; FISHELOV, 2013; BENFATTO; ESPOSITO; PULVIRENTI, 1985; COTTET, 1986; GALLAGHER; GALLAY, 2005; GALLAGHER; GALLAY; LIONS, 2005; GALLAY; WAYNE, 2005; GIGA; MIYAKAWA; OSADA, 1986), and the references therein. In (BENFATTO; ESPOSITO; PULVIRENTI, 1985), the authors constructed a smooth global solution for the Navier-Stokes model in $\mathbb{R}^2$ by considering the initial vorticity as a linear combination of Dirac masses whose total variation is small when compared to the viscosity. This result was generalized for the case of a finite measure in (COTTET, 1986) and (GIGA; MIYAKAWA; OSADA, 1986). On the other hand, in (BEN-ARTZI, 1994) a global result of well-posedness in $L^1(\mathbb{R}^2)$ was obtained. In (GALLAGHER; GALLAY, 2005), the uniqueness for the initial vorticity being a finite measure was analyzed in bidimensional case. The authors were able to remove the hypothesis about its size. For the three dimensional case, in (COTTET; SOLER, 1988; GIGA; MIYAKAWA, 1989; SOLER, 1994) similar results to the two dimensional case were obtained under the hypothesis of small initial vorticity. In addition, an L^∞ estimate and decay in time of the velocity field were obtained and the physically relevant case of the vortex sheets was also discussed. We recall that the initial data has a vortex sheet structure if the vorticity can be written as $\omega_0 = \alpha\delta_S$, where α is the strength and δ_S is the Dirac measure located on the curve S. In all these works, the parabolic character of the vorticity equation was of great importance so that these singular data are compatible with the considered model.

Turning our attention to the micropolar case, the parabolic character mentioned above was also a key argument in (DONG; ZHANG, 2010), in the particular case of null angular viscosity ($\gamma = 0$), to prove the global existence and uniqueness of smooth solutions for the system (2.6) with initial data in $H^s(\mathbb{R}^2)$, $s > 2$. The authors used an interesting new quantity, namely $Z = \omega - \dfrac{2\kappa}{\nu+\kappa} b$, with the goal of circumventing the recursive relationship that exists between the angular velocity b and the vorticity ω, and then obtain estimates in L^∞ for both quantities (see too (DONG; LI; WU, 2017) and (DONG; CHEN, 2009)).

Motivated by these works, our goal here is to analyze the Cauchy problem associated with the two dimensional micropolar fluids with partial viscosity, namely $\gamma = 0$, in terms of the evolution of the singular initial vorticity. We are also interested in the asymptotic behavior of the micropolar fluid motion with respect to time $t > 0$ as well as in the case of vortex sheets structure of vorticity described above. This type of study is new in the context of micropolar fluids.

Thus, let us consider the following system in $\mathbb{R}^2$:

$$\begin{aligned}\partial_t\omega-(\nu+\kappa)\Delta\omega+(\mathbf{u}\cdot\nabla)\omega&=-2\kappa\Delta b+\operatorname{curl}\mathbf{f},\\ \partial_t b+4\kappa b+(\mathbf{u}\cdot\nabla)b&=2\kappa\omega+g,\\ \mathbf{u}&=\mathbf{K}*\omega,\\ \omega(\cdot,0)&=\omega_0,\\ b(\cdot,0)&=b_0.\end{aligned}\tag{4.1}$$

We assume, without loss of generality, $\mathbf{f}=0$ and $g=0$. We also use the standard norm in the space $L^p(\mathbb{R}^n)$ is denoted by $\|\cdot\|_p$, and for the case of $\Omega\subset\mathbb{R}^n$ bounded open set we denote by $\|\cdot\|_{L^p(\Omega)}$, with $1\leqslant p\leqslant\infty$ and $n\in\mathbb{N}$. On the other hand, for our purposes it is important to mention the well-known Morrey–type space of measures. It is known that these spaces are suitable for the mathematical formulation of phenomena that involve, for example, mass concentration as the initial vorticity (TADMOR, 2001). Let us recall the definition of these spaces. Let $B(x,R)$ be the euclidean ball of center x and radius $R>0$. We define the Morrey space $\mathcal{M}^p(\mathbb{R}^n)=\mathcal{M}^p$ as the set of all measures μ satisfying

$$TV_{B(x,R)}(\mu)\leqslant CR^{n/p'},\quad \frac{1}{p}+\frac{1}{p'}=1,$$

independently of $x\in\mathbb{R}^n$ and $R>0$, where $TV_{B(x,R)}(\mu)$ is the total variation of μ in the $B(x,R)$. The Morrey spaces of functions $M^p(\mathbb{R}^n)=M^p$ are defined as the space of locally integrable functions f such that

$$\int_{B(x,R)}|f(y)|dy<CR^{n/p'}\quad \forall x\in\mathbb{R}^n,\ R>0,$$

where C is independent of x and R. We highlight that $\mathcal{M}^p$ is a Banach space with respect to the norm

$$\|\mu\|_{\mathcal{M}^p}=\sup_{x\in\mathbb{R}^n,R>0}R^{-n/p'}TV_{B(x,R)}(\mu).$$

In the same way, M^p is also a Banach space under the norm

$$\|f\|_{M^p}=\sup_{x\in\mathbb{R}^n,R>0}R^{-n/p'}\int_{B(x,R)}|f(y)|dy.$$

Note that $\mathcal{M}^1$ coincides with the space of finite variation measures, here denoted by $\mathcal{M}$, and $M^1=L^1$. Moreover, $M^\infty=L^\infty$, with equivalent norms, and $L^p\subset M^p$ for $1<p<\infty$, with continuous injection.

In what follows, we use C to denote a generic constant, independent of x and t, but still depending on the general data of the problem. When we need to explicitly indicate certain dependencies, we write $C=C(Z,R,\cdots)$.

We now list the main results of this chapter.

Theorem 4.1. *Let* $\omega_0, b_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *Then, the system* (4.1) *has a unique global mild solution, in the sense of Definition 4.14 (**see page 51 below**), and the inequality*

$$\|\mathbf{u}(\cdot, t)\|_\infty \leqslant C(\nu, \kappa)\{(\nu + \kappa)t\}^{-1/2} \tag{4.2}$$

holds, with $t \in (0, T]$, *for all* $T > 0$. *Moreover, assume that* $(\omega, \mathbf{u}, b)$ *and* $(\hat{\omega}, \hat{\mathbf{u}}, \hat{b})$ *are mild solutions of system* (4.1) *with initial data* (ω_0, b_0) *and* $(\hat{\omega}_0, \hat{b_0})$, *respectively. Then, the following inequalities are verified*

$$\|(\mathbf{u} - \hat{\mathbf{u}})(\cdot, t)\|_\infty \leqslant Ct^{-1/2}, \tag{4.3}$$

$$\|(\omega - \hat{\omega})(\cdot, t)\|_1 + \|(\omega - \hat{\omega})(\cdot, t)\|_\infty + \|(b - \hat{b})(\cdot, t)\|_1 + \|(b - \hat{b})(\cdot, t)\|_\infty \leqslant C\hat{\Pi}, \tag{4.4}$$

where $\hat{\Pi} = \hat{\Pi}(\omega_0, b_0, \hat{\omega}_0, \hat{b_0}) = \max\{\|\omega_0 - \hat{\omega}_0\|_1, \|\omega_0 - \hat{\omega}_0\|_\infty, \|b_0 - \hat{b_0}\|_1, \|b_0 - \hat{b_0}\|_\infty\}$ *and* $C > 0$ *is a constant independent of* $\hat{\Pi}$.

Theorem 4.2. *Let* $\omega_0 \in \mathcal{M}(\mathbb{R}^2)$ *and* $b_0 \in \mathcal{M}(\mathbb{R}^2) \cap \mathcal{M}^p(\mathbb{R}^2)$, *with* $p > 2$. *The system* (4.1) *has a unique global weak solution, in the sense of Definition 4.22 (**see page 63 below**), such that*

$$\|\mathbf{u}(\cdot, t)\|_\infty \leqslant C(\nu, \kappa)\{(\nu + \kappa)t\}^{-1/2}, \tag{4.5}$$

with $t \in (0, T]$, *for all* $T > 0$. *Moreover, the solutions are also stable in an analogous sense to* (4.3)-(4.4), *where in this case* $\hat{\Pi}$ *depends on the norms* $\mathcal{M}(\mathbb{R}^2)$ *and* $\mathcal{M}^p(\mathbb{R}^2)$ *of the initial data.*

Below we summarize the general ideas used to demonstrate the above results. First, we introduce a new quantity relating the vorticity and the angular velocity by

$$W = (\nu + \kappa)\omega - 2\kappa b. \tag{4.6}$$

Then, we have that system (4.1) is formally equivalent to the following system, which will be our main object of study in the sequel,

$$\begin{aligned} \partial_t W - (\nu + \kappa)\Delta W + \frac{4\kappa^2}{\nu + \kappa} W &= -(\mathbf{u} \cdot \nabla)W + \frac{8\kappa^2\nu}{\nu + \kappa} b, \\ \mathbf{u} &= \mathbf{K} * \left(\frac{1}{\nu + \kappa} W + \frac{2\kappa}{\nu + \kappa} b\right), \\ \partial_t b + \frac{4\kappa\nu}{\nu + \kappa} b + (\mathbf{u} \cdot \nabla) b &= \frac{2\kappa}{\nu + \kappa} W \\ W(\cdot, 0) &= W_0 = (\nu + \kappa)\omega_0 - 2\kappa b_0, \\ b(\cdot, 0) &= b_0. \end{aligned} \tag{4.7}$$

The next step will be to consider a regularized problem for (4.7) obtained by regularizing the initial data and introducing a time-delay in the nonlinear terms. We will prove uniform estimates for short times by integral equation techniques. In particular, we prove L^∞ estimates for the velocity field, which allow us to obtain the existence of solutions through compactness arguments. Using the same ideas, uniqueness and stability of solution are obtained.

This chapter is organized as follows: in section 4.1 we introduce some notations and basic results and the definition of solutions. We construct the regularized problem to obtain the existence of a regularized solution. In sections 4.2 and 4.3, we prove our main results.

4.1 PRELIMINARIES AND THE REGULARIZED PROBLEM

The aim of this section is to introduce the notations and summarize the basic results that are necessary to follow the arguments.

We begin with the following result, see (BREZIS, 2010, Proposition 4.16) for the details.

Lemma 4.3. *Let* $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ *and* $h \in L^q(\mathbb{R}^n)$, *with* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Then we have*

$$\int_{\mathbb{R}^n} (f * g) h d\mathbf{x} = \int_{\mathbb{R}^n} g(\check{f} * h) d\mathbf{x},$$

where $\check{f}(\mathbf{x}) = f(-\mathbf{x})$.

The following results will be used extensively in this work to obtain *a priori* estimates for the solutions, see (FISHELOV, 1991; MICHALSKI, 1993).

Lemma 4.4. *If* $\delta, \mu, \tau > 0$ *and* $t > 0$, *then*

$$t^{1-\mu} \int_0^t (t-s)^{\mu-1} s^{\delta-1} e^{-\tau s} ds \leqslant C, \tag{4.8}$$

where $C = C(\mu, \delta, \tau) = \max\{1, 2^{1-\mu}\} G(\delta)(1 + \delta/\mu)\tau^{-\delta}$ *and* G *denotes the Gamma function.*

Lemma 4.5. *Given* $f \in W^{m,p}(\mathbb{R}^n)$ *and* $\theta : \mathbb{R}^n \to \mathbb{R}$ *satisfying:*

1. $\displaystyle\int_{\mathbb{R}^n} \theta(\mathbf{x}) d\mathbf{x} = 1,$

2. $\displaystyle\int_{\mathbb{R}^n} \mathbf{x}^\alpha \theta(\mathbf{x}) d\mathbf{x} = 0, \ \forall \ |\alpha| \leqslant n - 1,$

3. $\displaystyle\int_{\mathbb{R}^n} |\mathbf{x}|^n |\theta(\mathbf{x})| d\mathbf{x} < \infty.$

Set $\theta_\delta(\mathbf{x}) = \frac{1}{\delta^n} \theta\left(\frac{\mathbf{x}}{\delta}\right)$, $\delta > 0$. *Then* $\|f - f * \theta_\delta\|_p \leqslant C\delta^m \|f\|_{W^{m,p}}$.

Lemma 4.6. *Let* $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ *be a bijective map and* $I : \mathbb{R}^2 \to \mathbb{R}^2$ *the identity map. If* $I - \phi \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$, *there exists a constant* C *such that for every* $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, *it holds*

$$\left| \int_{\mathbb{R}^2} (\mathbf{K}(\mathbf{x} - \mathbf{y}) - \mathbf{K}(\mathbf{x} - \phi(\mathbf{y}))) \omega(\mathbf{y})\, d\mathbf{y} \right| \leqslant C \|I - \phi\|_{L^\infty(\mathbb{R}^2)} (1 + |\log \ \|I - \phi\|_{L^\infty(\mathbb{R}^2)}|), \tag{4.9}$$

where $C = C(\|\omega\|_{L^1(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)})$.

We highlight more properties of the Biot-Savart kernel that will be useful. See (GIGA; GIGA; SAAL, 2010, Chapter 6), (GIGA; MIYAKAWA, 1989, p. 588) and (STEIN, 1970, p. 119) for different proofs.

Lemma 4.7. *1. [Hardy-Littlewood-Sobolev inequality] Let* $1 < q < p < \infty$ *with* $\dfrac{1}{p} + \dfrac{1}{n} = \dfrac{1}{q} + 1$ *and* $\mu \in \mathcal{M}^p(\mathbb{R}^n)$*. Then,* $\mathbf{K} * \mu \in M^q(\mathbb{R}^n)$ *and*

$$\|\mathbf{K} * \mu\|_{M^q} \leqslant C(p)\|\mu\|_{\mathcal{M}^p}.$$

2. Consider

$$\frac{1}{p} + \frac{1}{n} < 1 < \frac{1}{q} + \frac{1}{n},$$

and $\mu \in \mathcal{M}^p(\mathbb{R}^n) \cap \mathcal{M}^q(\mathbb{R}^n)$*. Then, there exists a constant* $C > 0$ *so that*

$$\|\mathbf{K} * \mu\|_\infty \leqslant C\|\mu\|_{\mathcal{M}^p}^{\left(\frac{1}{n}-\frac{1}{q'}\right)/\left(\frac{1}{q}-\frac{1}{p}\right)}\|\mu\|_{\mathcal{M}^q}^{\left(\frac{1}{p'}-\frac{1}{n}\right)/\left(\frac{1}{q}-\frac{1}{p}\right)}.$$

System (4.1) couples the heat equation and the non-linear transport equation. Regarding the heat equation, the Duhamel Principle will be widely used here in order to define a mild solution to

$$\partial_t \boldsymbol{\nu} - (\nu + \kappa)\Delta \boldsymbol{\nu} = \mathbf{h},\ x \in \mathbb{R}^2,\ t > 0,$$
$$\boldsymbol{\nu}(\cdot, 0) = \boldsymbol{\nu}_0,$$

which is explicitly given by

$$\boldsymbol{\nu}(\cdot, t) = \Gamma(\cdot, t) * \boldsymbol{\nu}_0 + \int_0^t \Gamma(\cdot, t-s) * \mathbf{h}(\cdot, s)ds,$$

where

$$\Gamma(x, t) = \frac{1}{4\pi(\nu+\kappa)t}\exp\left(-\frac{|x|^2}{4(\nu+\kappa)t}\right)$$

is the Heat kernel.

The following result summarize some properties of the heat kernel Γ in Morrey spaces. See (GIGA; MIYAKAWA, 1989, p. 586 - 590) for the details.

Lemma 4.8. *Let* $1 \leqslant p \leqslant q \leqslant \infty$ *and* $\mu \in \mathcal{M}^p(\mathbb{R}^2)$*. Then, there exists a constant* $C > 0$ *for which the inequality*

$$\|\partial_x^\beta \Gamma(\cdot, t) * \mu\|_{M^q} \leqslant C\{(\nu+\kappa)t\}^{-|\beta|/2-(1/p-1/q)}\|\mu\|_{\mathcal{M}^p}, \tag{4.10}$$

holds, for all $t > 0$*, where* β *is a multi-index. Moreover,* $\Gamma(\cdot, t) * \mu \to \mu$*, as* $t \to 0$*, weakly as measures on each fixed open ball and*

$$\lim_{t\to 0}\|\Gamma(\cdot, t) * \mu\|_{M^p} = \|\mu\|_{\mathcal{M}^p}. \tag{4.11}$$

We recall below the definition of convergence of a function $f(\cdot, t)$, as $t \to 0$, weakly as measures. We begin recalling the definition of weak-$*$ convergence in measures spaces (see (BEN-ARTZI M.; CROISILLE; FISHELOV, 2013, Appendix A)).

Definition 4.9. *We say that the family of measures* $(\mu_\varepsilon)_{\varepsilon>0}$ *converges in the weak-$*$ topology to measure* μ*, if*

$$\lim_{\varepsilon\to 0}\langle \mu_\varepsilon, \varphi\rangle = \langle \mu, \varphi\rangle,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$*, where and* $\langle\cdot,\cdot\rangle$ *denote the pairing between measures and continuous functions, that is,*

$$\langle \mu, \varphi\rangle = \int_{\mathbb{R}^2} \varphi d\mu.$$

For $f(\cdot, t) \in L^1(\mathbb{R}^2)$, with $t > 0$, we can identify this function with a measure $\mu_{f(\cdot,t)}$ by

$$\langle \mu_{f(\cdot,t)}, \varphi\rangle = \int_{\mathbb{R}^2} f(x,t)\varphi(x)dx,$$

where dx is the Lebesgue measure. Thus, $f(\cdot,t) \to \mu$, as $t \to 0$, weakly as measures means

$$\lim_{t\to 0}\langle \mu_{f(\cdot,t)}, \varphi\rangle = \langle \mu, \varphi\rangle, \tag{4.12}$$

or more specifically,

$$\lim_{t\to 0}\int_{\mathbb{R}^2} f(x,t)\varphi(x)dx = \int_{\mathbb{R}^2}\varphi d\mu, \tag{4.13}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$.
If $\mu \in \mathcal{M}^p \cap \mathcal{M}^q$, $1 \leqslant p \leqslant q \leqslant \infty$, then we also have the following result of interpolation in Morrey spaces

$$\|\mu\|_{\mathcal{M}^r} \leqslant \|\mu\|_{\mathcal{M}^p}^{1-\theta}\|\mu\|_{\mathcal{M}^q}^{\theta},$$

with $\dfrac{1}{r} = \dfrac{1-\theta}{p} + \dfrac{\theta}{q}$ and $0 \leqslant \theta \leqslant 1$. In particular, all the previous estimates also hold in L^p spaces.

Now, we recall the classical representation for solutions of the transport equation.

Lemma 4.10. *If* $v \in L^\infty(0,T;\boldsymbol{W}^{1,\infty}(\mathbb{R}^2))$ *and* $h \in L^1(0,T;L^p(\mathbb{R}^2))$*, the following Cauchy problem for* $\theta = \theta(x,t)$ *with initial data* $\theta_0 \in L^p(\mathbb{R}^2)$*,* $1 \leqslant p \leqslant \infty$*, has a unique solution*

$$\begin{aligned} \partial_t\theta - div\ (v\theta) &= h,\ \ x \in \mathbb{R}^2,\ \ t \in [0,T),\\ div\ v &= 0,\\ \theta(\cdot,0) &= \theta_0. \end{aligned}$$

This solution is defined by

$$\theta(x,t) = \theta_0(\mathbf{X}(0;x,t)) + \int_0^t h(\mathbf{X}(s;x,t),s)ds,$$

where the function $\mathbf{X}(t;x,t_0)$*,* $0 \leqslant t_0 < T$*, denotes the characteristics related to* v*, that is, the unique solution of the system*

$$\begin{cases} \dfrac{d}{dt}\mathbf{X}(t;x,t_0) = v(\mathbf{X}(t;x,t_0),t),\ \ t\in[0,T),\\ \mathbf{X}(t_0;x,t_0) = x. \end{cases}$$

In particular,

$$\|\theta(\cdot,t)\|_p \leqslant \|\theta_0\|_p + \int_0^t \|h(\cdot,s)\|_p ds,\ \ t \in [0,T). \tag{4.14}$$

For the proof, see (AMBROSIO, 2008, Chapter 1) and (COTTET, 1988, p. 232). This result can be easily extended to Log-Lipschitz velocity fields

$$|\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{y},t)-\boldsymbol{v}(\boldsymbol{x},t)| \leqslant C|\boldsymbol{y}|(1+|\ln|\boldsymbol{y}||), \tag{4.15}$$

for all $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^2$ and $t>0$, which will be our object of study here.

In order to construct regularized system for (4.7), we consider two C_0^∞ positive functions $\rho(t)$ and $\psi(\boldsymbol{x})$ such that

(i) $\psi(|\boldsymbol{x}|)=\psi(\boldsymbol{x})$ and it is a nonincreasing function of $|\boldsymbol{x}|$,

(ii) $\displaystyle\int \rho dt = 1$ and $\displaystyle\int \psi d\boldsymbol{x} = 1$,

(iii) $\text{supp}(\rho)\subset[1,2]$.

Given $\varepsilon>0$, we set $\rho_\varepsilon(t)=\varepsilon^{-1}\rho(t/\varepsilon)$, $\psi_\varepsilon(\boldsymbol{x})=\varepsilon^{-2}\psi(\boldsymbol{x}/\varepsilon)$, $W_0^\varepsilon=W_0*\psi_\varepsilon$ and $b_0^\varepsilon=b_0*\psi_\varepsilon$, where $*$ denotes the convolution in $\mathbb{R}^2$. Note that $W_0^\varepsilon,\ b_0^\varepsilon\in W^{m,1}(\mathbb{R}^2)\cap W^{m,\infty}(\mathbb{R}^2)$, for all $m\in\mathbb{N}$.

We denote by $\Pi(W_0,b_0)$ a constant depending on the norms of the initial data, either in terms of the L^1 and L^∞ norms, as in Section 4.2, or in terms of the norms in Morrey spaces, as in Section 4.3.

Given a function f defined on $\mathbb{R}^2\times\mathbb{R}$, we define the time-delay mollification of f, denoted by $M^\varepsilon(f)$, by

$$M^\varepsilon(f)(\boldsymbol{x},t)=\int_{2\varepsilon}^{+\infty}\rho_\varepsilon(t-s)f(\boldsymbol{x},s)ds. \tag{4.16}$$

Directly by this definition, it follows

$$\begin{aligned} M^\varepsilon(f)(\boldsymbol{x},t)&=0,\ \text{if}\quad t\leqslant 2\varepsilon,\\ \|M^\varepsilon(f)(\cdot,t)\|_X&\leqslant \max_{t-2\varepsilon<s<t-\varepsilon}\|f(\cdot,s)\|_X\\ &\leqslant \max_{t/2<s<t}\|f(\cdot,s)\|_X, \end{aligned} \tag{4.17}$$

for any normed space X.

Introducing the following regularized and linearized problem

$$\begin{aligned} \partial_t W^\varepsilon-(\nu+\kappa)\Delta W^\varepsilon+\frac{4\kappa^2}{\nu+\kappa}W^\varepsilon&=-M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon)+\frac{8\kappa^2\nu}{\nu+\kappa}M^\varepsilon(b^\varepsilon),\\ \boldsymbol{u}^\varepsilon&=\mathbf{K}*\left(\frac{1}{\nu+\kappa}W^\varepsilon+\frac{2\kappa}{\nu+\kappa}M^\varepsilon(b^\varepsilon)\right),\\ \partial_t b^\varepsilon+\frac{4\kappa\nu}{\nu+\kappa}b^\varepsilon+(\boldsymbol{u}^\varepsilon\cdot\nabla)b^\varepsilon&=\frac{2\kappa}{\nu+\kappa}W^\varepsilon,\\ W^\varepsilon(\cdot,0)&=W_0^\varepsilon,\\ b^\varepsilon(\cdot,0)&=b_0^\varepsilon, \end{aligned} \tag{4.18}$$

one can state the main result of this section:

Proposition 4.11. *For each $\varepsilon > 0$, problem* (4.18) *admits a unique classical solution* $(W^\varepsilon, \mathbf{u}^\varepsilon, b^\varepsilon)$*. This solution is* C^∞ *and* $div\ \mathbf{u}^\varepsilon = 0$.

Proof. We proceed by induction. We consider the time interval $J_m = [m\varepsilon,\ (m+1)\varepsilon],\ m \in \mathbb{N}$. The argument will be as follows: using the definition (4.16), problem (4.18) reduced to a linear problem on each time interval J_m. Through the semigroup theory, the smoothing effect of the heat equation and the classic theory of the transport equation, we obtain smooth solutions W^ε, $\mathbf{u}^\varepsilon$, b^ε in J_m. Since the nonlinear couplings associated to the system for $t \in J_{m+1}$ are delayed on their values in $t \in J_m$, the problem is now linear and we conclude that there exists a unique smooth solution for the above regularized problem. Indeed, using the definition (4.16), for $0 \leqslant t \leqslant \varepsilon$ the system (4.18) transforms into:

$$
\begin{aligned}
\partial_t W_1^\varepsilon - (\nu+\kappa)\Delta W_1^\varepsilon + \frac{4\kappa^2}{\nu+\kappa}W_1^\varepsilon &= 0,\\
\mathbf{u}_1^\varepsilon &= \mathbf{K} * \left(\frac{1}{\nu+\kappa}W_1^\varepsilon\right),\\
\partial_t b_1^\varepsilon + \frac{4\kappa\nu}{\nu+\kappa}b_1^\varepsilon + (\mathbf{u}_1^\varepsilon\cdot\nabla)b_1^\varepsilon &= \frac{2\kappa}{\nu+\kappa}W_1^\varepsilon,\\
W_1^\varepsilon(\cdot,0) &= W_0^\varepsilon,\\
b_1^\varepsilon(\cdot,0) &= b_0^\varepsilon,
\end{aligned}
$$

for $0 \leqslant t \leqslant \varepsilon$. Since $W_0^\varepsilon \in C^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ (in particular), classic theory for heat equation with a potential assures that there exists a unique smooth solution $W_1^\varepsilon(\cdot,t) \in C^\infty(\mathbb{R}^2)$ of the above system. Therefore, $u_1^\varepsilon(\cdot,t) = \dfrac{1}{\nu+\kappa}\mathbf{K} * W^\varepsilon$ is well defined and is $C^\infty(\mathbb{R}^2)$ for $0 \leqslant t \leqslant \varepsilon$. Furthermore, the hypothesis $b_0^\varepsilon \in C^\infty(\mathbb{R}^2)$ implies that there exists a unique $b_1^\varepsilon(\cdot,t) \in C^\infty(\mathbb{R}^2)$ satisfying, in the classic sense, the system

$$
\begin{aligned}
\partial_t b_1^\varepsilon + \frac{4\kappa\nu}{\nu+\kappa}b_1^\varepsilon + (\mathbf{u}_1^\varepsilon\cdot\nabla)b_1^\varepsilon &= \frac{2\kappa}{\nu+\kappa}W_1^\varepsilon,\\
b_1^\varepsilon(\cdot,0) &= b_0^\varepsilon,
\end{aligned}
$$

for $0 \leqslant t \leqslant \varepsilon$. In summary, for $0 \leqslant t \leqslant \varepsilon$ system (4.18) admits a unique C^∞ solution given by $(W_1^\varepsilon, \mathbf{u}_1^\varepsilon, b_1^\varepsilon)$. Now, if $\varepsilon \leqslant t \leqslant 2\varepsilon$, consider

$$
\begin{aligned}
\partial_t W_2^\varepsilon - (\nu+\kappa)\Delta W_2^\varepsilon + \frac{4\kappa^2}{\nu+\kappa}W_2^\varepsilon &= 0,\\
\mathbf{u}_2^\varepsilon &= \mathbf{K} * \left(\frac{1}{\nu+\kappa}W_1^\varepsilon\right),\\
\partial_t b_2^\varepsilon + \frac{4\kappa\nu}{\nu+\kappa}b_2^\varepsilon + (\mathbf{u}_2^\varepsilon\cdot\nabla)b_2^\varepsilon &= \frac{2\kappa}{\nu+\kappa}W_2^\varepsilon,\\
W_2^\varepsilon(\cdot,\varepsilon) &= W_1^\varepsilon(\cdot,\varepsilon),\\
b_2^\varepsilon(\cdot,\varepsilon) &= b_1^\varepsilon(\cdot,\varepsilon),
\end{aligned}
$$

for $\varepsilon \leqslant t \leqslant 2\varepsilon$. As in the previous case, this system admits a unique $C^\infty(\mathbb{R}^2)$ solution given by $(W_2^\varepsilon, \mathbf{u}_2^\varepsilon, b_2^\varepsilon)$. We can repeat the reasonig for each spatial derivative. By gluing the previous solutions, we have that system (4.18) admits a unique C^∞ solution for $0 \leqslant t \leqslant 2\epsilon$. In this way, we construct a sequence $\{(W^\varepsilon_m, \mathbf{u}^\varepsilon_m, b^\varepsilon_m)\}$ given by the solution of the system

$$
\begin{aligned}
\partial_t W^\varepsilon_m - (\nu+\kappa)\Delta W^\varepsilon_m + \frac{4\kappa^2}{\nu+\kappa} W^\varepsilon_m &= G_m,\\
\mathbf{u}^\varepsilon_m &= \mathbf{K} * \left(\frac{1}{\nu+\kappa} W^\varepsilon_m + \frac{2\kappa}{\nu+\kappa} H_m\right),\\
\partial_t b^\varepsilon_m + \frac{4\kappa\nu}{\nu+\kappa} b^\varepsilon_m + (\mathbf{u}^\varepsilon_m \cdot \nabla) b^\varepsilon_m &= \frac{2\kappa}{\nu+\kappa} W^\varepsilon_m,\\
W^\varepsilon_m(\cdot, m\varepsilon) &= W^\varepsilon_{m-1}(\cdot, m\varepsilon),\\
b^\varepsilon_m(\cdot, m\varepsilon) &= b^\varepsilon_{m-1}(\cdot, m\varepsilon),
\end{aligned}
$$

for $(m-1)\varepsilon \leqslant t \leqslant m\varepsilon$. Here, $G_1 = G_2 = G_3 = H_1 = H_2 = H_3 = 0$,

$$
\begin{aligned}
G_4(x,t) &= \int_{2\varepsilon}^{3\varepsilon} \rho_\varepsilon(t-s)\left[\frac{8\kappa^2\nu}{\nu+\kappa} b^\xi_3(x,s) - (\mathbf{u}^\xi_3 \cdot \nabla) W^\xi_3(x,s)\right] ds,\\
H_4(x,t) &= \int_{2\varepsilon}^{3\varepsilon} \rho_\varepsilon(t-s) \frac{2\kappa}{\nu+\kappa} b^\xi_3(x,s) ds,
\end{aligned}
$$

for $3\varepsilon \leqslant t \leqslant 4\varepsilon$. If $m \geqslant 5$,

$$
\begin{aligned}
G_m(x,t) &= \int_{(m-3)\varepsilon}^{(m-2)\varepsilon} \rho_\varepsilon(t-s)\left[\frac{8\kappa^2\nu}{\nu+\kappa} b^\varepsilon_{m-2}(x,s) - (\mathbf{u}^\varepsilon_{m-2} \cdot \nabla) W^\varepsilon_{m-2}(x,s)\right] ds\\
&\quad + \int_{(m-2)\varepsilon}^{(m-1)\varepsilon} \rho_\varepsilon(t-s)\left[\frac{8\kappa^2\nu}{\nu+\kappa} b^\varepsilon_{m-1}(x,s) - (\mathbf{u}^\varepsilon_{m-1} \cdot \nabla) W^\varepsilon_{m-1}(x,s)\right] ds,\\
H_m(x,t) &= \int_{(m-3)\varepsilon}^{(m-2)\varepsilon} \rho_\varepsilon(t-s) \frac{2\kappa}{\nu+\kappa} b^\varepsilon_{m-2}(x,s) ds + \int_{(m-2)\varepsilon}^{(m-1)\varepsilon} \rho_\varepsilon(t-s) \frac{2\kappa}{\nu+\kappa} b^\varepsilon_{m-1}(x,s) ds,
\end{aligned}
$$

for $(m-1)\varepsilon \leqslant t \leqslant m\varepsilon$. By an induction process, we have that $G_m, H_m \in C^\infty$ for every $m \in \mathbb{N}$, which implies, via the Duhamel principle, the existence and uniqueness for the triplet $(W^\varepsilon, \mathbf{u}^\varepsilon, b^\varepsilon)$ defined by $W^\varepsilon(x,t) = W^\varepsilon_m(x,t), \mathbf{u}^\varepsilon(x,t) = \mathbf{u}^\varepsilon_m(x,t), b^\varepsilon(x,t) = b^\varepsilon_m(x,t)$, with m such that $(m-1)\varepsilon \leqslant t \leqslant m\varepsilon$ is the unique C^∞ solution of system (4.18).

Furthermore, $\mathbf{u}^\varepsilon$ is divergence-free as a consequence of $div\ \mathbf{K} = 0$ (for $x \neq 0$) and a classic cutoff argument. Indeed, given $\delta > 0$, we consider a cutoff function $\theta_\delta(x) = \theta(|x|/\delta)$, where $\theta \in C^\infty_0(\mathbb{R}^2)$ is such that $\mathrm{supp}(\theta) \subset [0,2]$ and $\theta(r) = 1$, for all $r \in [0,1]$. If we denote $H^\epsilon = \left(\frac{1}{\nu+\kappa} W^\varepsilon + \frac{2\kappa}{\nu+\kappa} M^\varepsilon(b^\varepsilon)\right)$ and $\overline{\theta}_\delta = 1 - \theta_\delta$, then

$$
\begin{aligned}
\mathbf{u}^\varepsilon(x,t) &= \int_{\mathbb{R}^2} \overline{\theta}_\delta(x-y)\mathbf{K}(x-y) H^\varepsilon(y,t) dy + \int_{\mathbb{R}^2} \theta_\delta(y)\mathbf{K}(y) H^\varepsilon(x-y,t) dy\\
&:= I^\delta_1(x,t) + I^\delta_2(x,t).
\end{aligned}
$$

Since $div(\mathbf{K}(x)) = 0$ for every $x \neq 0$ we deduce

$$
\begin{aligned}
div(I^\delta_1(x,t)) &= \int_{|x-y|>\delta} div(\overline{\theta}_\delta(x-y)\mathbf{K}(x-y)) H^\varepsilon(y,t) dy\\
&= \int_{|x-y|>\delta} \nabla\overline{\theta}_\delta(x-y) \cdot \mathbf{K}(x-y) H^\varepsilon(y,t) dy.
\end{aligned}
\tag{4.19}
$$

Since $\overline{\theta}_\delta(x-y)$ is radial while $\mathbf{K}(x-y)$ is tangential, we have $div(I_1^\delta(x,t)) = 0$. On the other hand, using that $\mathbf{K}$ is integrable in balls centered in 0, the Dominated Convergence Theorem and the induction above, we derive

$$\begin{aligned} div(I_2^\delta(x,t)) &= \int_{\mathbb{R}^2} \theta_\delta(y)\mathbf{K}(y)\cdot\nabla H^\varepsilon(x-y,t)dy \\ &= \int_{|x-y|>\delta} \nabla\overline{\theta}_\delta(x-y)\cdot\mathbf{K}(x-y)H^\varepsilon(x-y,t)dy \overset{\delta\to 0}{\longrightarrow} 0 \end{aligned} \tag{4.20}$$

Therefore, since $div(u^\epsilon(x,t)) = div(I_1^\delta(x,t)) + div(I_2^\delta(x,t))$ for every $\delta > 0$, it follows that $div(u^\epsilon(x,t)) = 0$ and the proof is completed. □

In order to define the mild formulation of system (4.18), note that using $\mathrm{div}\,\mathbf{u}^\varepsilon = 0$, we have $M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon) = M^\varepsilon(\mathrm{div}\ \mathbf{u}^\varepsilon W^\varepsilon)$. Since M^ε commute with the convolution and with the differentiation with respect to x, we obtain

$$\begin{aligned} -\Gamma(\cdot,t-s)*M^\varepsilon(\mathrm{div}\ \mathbf{u}^\varepsilon W^\varepsilon)(\cdot,s) &= -\int_{\mathbb{R}^2}\Gamma(\cdot-y,t-s)\int_{2\varepsilon}^{+\infty}\rho_\varepsilon(s-\tau)\sum_j \frac{\partial(u_j^\varepsilon W^\varepsilon)}{\partial x_j}(y,\tau)d\tau dy \\ &= \int_{2\varepsilon}^{+\infty}\rho_\varepsilon(s-\tau)\int_{\mathbb{R}^2}\sum_j\frac{\partial\Gamma}{\partial x_j}(\cdot-y,t-s)(u_j^\varepsilon W^\varepsilon)(y,\tau)dy d\tau \\ &= \int_{\mathbb{R}^2}\sum_j\frac{\partial\Gamma}{\partial x_j}(\cdot-y,t-s)\int_{2\varepsilon}^{+\infty}\rho_\varepsilon(s-\tau)(u_j^\varepsilon W^\varepsilon)(y,\tau)d\tau dy \\ &= \sum_j\frac{\partial\Gamma}{\partial x_j}(\cdot,t-s)*M^\varepsilon(u_j^\varepsilon W^\varepsilon)(\cdot,s). \end{aligned}$$

Therefore, we can write

$$-\Gamma(\cdot,t-s)*M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s) := \nabla\Gamma(\cdot,t-s)*M^\varepsilon(\mathbf{u}^\varepsilon W^\varepsilon)(\cdot,s).$$

Then, the **mild formulation** of the problem (4.18) can be formulated by using the Duhamel Principle and Lemma 4.10 as follows

$$\begin{aligned} W^\varepsilon(\cdot,t) = {}& e^{-\frac{4\kappa^2}{\nu+\kappa}t}\Gamma(\cdot,t)*W_0^\varepsilon + \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s)*M^\varepsilon(\mathbf{u}^\varepsilon W^\varepsilon)(\cdot,s)ds \\ &+ \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\Gamma(\cdot,t-s)*M^\varepsilon(b^\varepsilon)(\cdot,s)ds, \end{aligned} \tag{4.21}$$

$$\mathbf{u}^\varepsilon(\cdot,t) = \left(\mathbf{K}*\left\{\frac{1}{\nu+\kappa}W^\varepsilon + \frac{2\kappa}{\nu+\kappa}M^\varepsilon(b^\varepsilon)\right\}\right)(\cdot,t), \tag{4.22}$$

$$b^\varepsilon(\cdot,t) = e^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon(\mathbf{X}^\varepsilon(0;\cdot,t)) + \frac{2\kappa}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}W^\varepsilon(\mathbf{X}^\varepsilon(s;\cdot,t),s)ds, \tag{4.23}$$

where

$$\begin{cases} \dfrac{d}{dt}\mathbf{X}^\varepsilon(t;x,t_0) = \mathbf{u}^\varepsilon(\mathbf{X}^\varepsilon(t;x,t_0),t), \ \ t\in[0,T), \\ \mathbf{X}^\varepsilon(t_0;x,t_0) = x. \end{cases}$$

4.2 INITIAL DATA IN $L^1 \cap L^\infty$

In this section, we will present the proof of the Theorem 4.1. It will be divided into several parts dealing with existence, uniqueness and stability of solutions. Firstly, we establish the *a priori* estimates for the sequence $(W^\varepsilon, \mathbf{u}^\varepsilon, b^\varepsilon)$ that was introduced in the previous section and analyze the convergence of this sequence.

4.2.1 *A priori* estimates

We first derive *a priori* bounds for family $(W^\varepsilon, \mathbf{u}^\varepsilon, b^\varepsilon)$ with initial data $W_0, b_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Proposition 4.12. *There exist* $T^* = T^*(\nu, \kappa, \Pi(W_0, b_0)) > 0$ *and a positive constant* $C = C(\nu, \kappa)$ *such that the following estimates*

$$\|\mathbf{u}^\varepsilon(\cdot, t)\|_\infty \leqslant C\{(\nu + \kappa)t\}^{-1/2}, \tag{4.24}$$

$$\|W^\varepsilon(\cdot, t)\|_1 + \|W^\varepsilon(\cdot, t)\|_\infty + \|b^\varepsilon(\cdot, t)\|_1 + \|b^\varepsilon(\cdot, t)\|_\infty \leqslant C\Pi(W_0, b_0), \tag{4.25}$$

hold for any time $t \in (0, T^*]$. *In addition, the function* $\mathbf{u}^\varepsilon$ *is uniformly bounded in* $L^\infty(0, T^*; L^\infty(\mathbb{R}^2))$.

Proof. In view of Lemma 4.7 and identity (4.22), we have

$$\|\mathbf{u}^\varepsilon(\cdot, t)\|_\infty \leqslant \frac{C}{\nu + \kappa}\left\{\|W^\varepsilon(\cdot, t)\|_1 + \|W^\varepsilon(\cdot, t)\|_\infty + 2\kappa\left(\max_{t/2<s<t}\|b^\varepsilon(\cdot, s)\|_1 + \max_{t/2<s<t}\|b^\varepsilon(\cdot, s)\|_\infty\right)\right\}, \tag{4.26}$$

for all $t > 0$.

Let $q = 1$ or ∞. Using identity (4.23), we have

$$\|b^\varepsilon(\cdot, t)\|_q \leqslant \Pi(W_0, b_0) + \frac{2\kappa}{\nu + \kappa}\int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\|W^\varepsilon(\cdot, s)\|_q ds, \tag{4.27}$$

for all $t > 0$. Now, combining property (4.17), Lemma 4.8 and identity (4.21), we have

$$\begin{aligned}
\|W^\varepsilon(\cdot, t)\|_q \leqslant\ & e^{-\frac{4\kappa^2}{\nu+\kappa}t}\|\Gamma(\cdot, t) * W_0^\varepsilon\|_q + \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\nabla\Gamma(\cdot, t-s) * M^\varepsilon(W^\varepsilon\mathbf{u}^\varepsilon)(\cdot, s)\|_q ds \\
& + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\Gamma(\cdot, t-s) * M^\varepsilon(b^\varepsilon)(\cdot, s)\|_q ds \\
\leqslant\ & \|W_0^\varepsilon\|_q + C(\nu+\kappa)^{-1/2}\int_0^t (t-s)^{-1/2} e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|M^\varepsilon(W^\varepsilon\mathbf{u}^\varepsilon)(\cdot, s)\|_q ds \\
& + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|M^\varepsilon(b^\varepsilon)(\cdot, s)\|_q ds.
\end{aligned}$$

Then, we find

$$\begin{aligned}
\|W^\varepsilon(\cdot, t)\|_q \leqslant\ & \Pi(W_0, b_0) + C(\nu+\kappa)^{-1/2}\int_0^t (t-s)^{-1/2} e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \max_{s/2<\tau<s}\|\mathbf{u}^\varepsilon(\cdot, \tau)\|_\infty\|W^\varepsilon(\cdot, \tau)\|_q ds \\
& + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \max_{s/2<\tau<s}\|b^\varepsilon(\cdot, \tau)\|_q ds,
\end{aligned} \tag{4.28}$$

for all $t > 0$. Combining the estimates (4.26), (4.27) and (4.28), and setting

$$\lambda(t) = \sup_{s\leqslant t}\left\{\|W^\varepsilon(\cdot,s)\|_1, \|W^\varepsilon(\cdot,s)\|_\infty, \|b^\varepsilon(\cdot,s)\|_1, \|b^\varepsilon(\cdot,s)\|_\infty\right\},$$

we find

$$\lambda(t) \leqslant C_0\Pi(W_0,b_0) + C_1(\nu,\kappa)t\lambda(t) + C_2(\nu,\kappa)\sqrt{t}\lambda(t)^2,$$

or equivalently

$$0 \leqslant C_0\Pi(W_0,b_0) + \big(C_1 t - 1\big)\lambda(t) + C_2\sqrt{t}\lambda(t)^2,$$

where $C_0 > 1$, $C_1(\nu,\kappa) = \dfrac{8\kappa^2\nu + 2\kappa}{\nu+\kappa}$ and $C_2(\nu,\kappa) = C(\nu+\kappa)^{-3/2}(2\kappa+1)$. Choosing $T^* > 0$ small enough such that

$$\begin{cases} T^* < \dfrac{1}{C_1}, \\ \sigma = (C_1T^* - 1)^2 - 4C_0\Pi(W_0,b_0)C_2\sqrt{T^*} > 0, \end{cases} \tag{4.29}$$

and by the continuity of λ, we obtain

$$0 \leqslant \lambda(t) \leqslant \frac{1 - C_1 t - \sqrt{\sigma}}{2C_2\sqrt{t}} \leqslant \frac{1}{2C_2\sqrt{t}},$$

for $t \in (0,T^*]$. From this we deduce $\|\mathbf{u}^\varepsilon(\cdot,t)\|_\infty \leqslant \dfrac{C(2\kappa+1)}{\nu+\kappa}\lambda(t)$, $t > 0$, and the estimate (4.24) holds. Now, replacing the estimate (4.24) into (4.28) and adding the estimate (4.27), we obtain

$$\|W^\varepsilon(\cdot,t)\|_q + \|b^\varepsilon(\cdot,t)\|_q \leqslant C_0\Pi(W_0,b_0) + C(\nu,\kappa)\int_0^t\left\{(t-s)^{-1/2}s^{-1/2} + e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} + e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\right\}$$
$$\times\left\{\max_{s/2<\tau<s}\|W^\varepsilon(\cdot,\tau)\|_q + \max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_q\right\}ds.$$

If $\lambda_1(t) = \max_{s\leqslant t}\left\{\|W^\varepsilon(\cdot,s)\|_q, \|b^\varepsilon(\cdot,s)\|_q\right\}$, we have

$$\begin{aligned}\lambda_1(t) &\leqslant C_0\Pi(W_0,b_0) + C(\nu,\kappa)\lambda_1(t)\int_0^t\Big((t-s)^{-1/2}s^{-1/2} + e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} + e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\Big)\,ds\\ &\leqslant C_0\Pi(W_0,b_0) + C(\nu,\kappa)M(t)\lambda_1(t),\end{aligned}$$

with $M(t) = 2t + C$. Then, if $C(\nu,\kappa)M(T^*) < 1$, we have proved the estimate (4.25). Since we know that $\|\mathbf{u}^\varepsilon(\cdot,t)\|_{L^\infty} \leqslant \dfrac{C(2\kappa+1)}{\nu+\kappa}\lambda(t)$ and we have just proved that $\lambda(t)$ is uniformly bounded, we deduce that $\mathbf{u}^\varepsilon(x,t)$ is uniformly bounded in L^∞. □

Remark 4.13. *Since W_0^ε, $b_0^\varepsilon \in W^{m,1}(\mathbb{R}^2)\cap W^{m,\infty}(\mathbb{R}^2)$, for all $m \in \mathbb{N}$, repeating the previous arguments for each spatial derivative of $\mathbf{u}^\varepsilon$, W^ε, b^ε, we can obtain estimates for $\|W^\varepsilon(\cdot,t)\|_{W^{m,p}}$, and $\|b^\varepsilon(\cdot,t)\|_{W^{m,p}}$, this time depending on initial conditions and on a negative power of ε.*

4.2.2 Existence of solution

The objective of this section is to prove the existence of mild solution for system (4.7). We prove that the estimates given in Proposition 4.12 provide compactness properties which in turn allow us to pass to the limit in the regularized problem. The existence of the limit is based on the Ascoli-Arzelá Theorem (see (GIGA; GIGA; SAAL, 2010, Chapter 5)). We begin with the definition of mild solution.

Definition 4.14. *The triplet* $(W, \mathbf{u}, b)$ *is a mild solution for system* (4.7) *if it satisfies the initial data in the sense of weak-⋆ convergence as* $t \to 0$ *(see Definition 4.9, identity* (4.13)*), and*

$$W(x,t) = e^{-\frac{4\kappa^2}{\nu+\kappa}t}(\Gamma(\cdot,t) * W_0)(x) + \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * (\mathbf{u}W)(x,s)ds$$
$$+ \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\Gamma(\cdot,t-s) * b(x,s)ds,$$

$$b(x,t) = e^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0(\mathbf{X}(0;x,t)) + \frac{2\kappa}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}W(\mathbf{X}(s;x,t),s)ds,$$
$$\mathbf{u}(x,t) = \frac{1}{\nu+\kappa}\mathbf{K} * W(x,t) + \frac{2\kappa}{\nu+\kappa}\mathbf{K} * b(x,t),$$

hold, where the equality is understood in the sense of $L^\infty(0,T^*;\ L^1(\mathbb{R}^2))$ *functions.*

Proposition 4.15. *The family of solutions* $(W^\varepsilon, \mathbf{u}^\varepsilon, b^\varepsilon)_{\varepsilon>0}$ *to the system* (4.18) *admit a subfamily, still denoted by* $(W^\varepsilon, \mathbf{u}^\varepsilon, b^\varepsilon)_{\varepsilon>0}$*, such that*

$$W^\varepsilon \rightharpoonup W \text{ weak-}* \text{ in } L^\infty(0,T^*;L^\infty(\mathbb{R}^2)), \tag{4.30}$$
$$b^\varepsilon \rightharpoonup b \text{ weak-}* \text{ in } L^\infty(0,T^*;L^\infty(\mathbb{R}^2)), \tag{4.31}$$
$$\mathbf{u}^\varepsilon \longrightarrow \mathbf{u} \text{ uniformly in compact set of } \mathbb{R}^2\times[0,T^*]. \tag{4.32}$$

Proof. Through the estimate (4.25), one gets $\{W^\varepsilon\}_{\varepsilon>0}$ and $\{b^\varepsilon\}_{\varepsilon>0}$ are bounded in $L^\infty(0,T^*;L^\infty(\mathbb{R}^2))$. Therefore, there exists a subfamily and W, b such that the convergences (4.30) and (4.31) hold.

In order to obtain the convergence (4.32), we use the estimates (4.24) and (4.25) to find that $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(0,T^*;\ L^\infty(\mathbb{R}^2))$. Now, we need to ensure the family $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ to be equicontinuous in time and space. By inequality (4.15), we obtain the spatial equicontinuity for fixed time, that is, for all $\beta > 0$, there exist $\delta > 0$ such that if $|(x,t)-(y,t)| < \delta$, then $|\mathbf{u}^\varepsilon(x,t) - \mathbf{u}^\varepsilon(y,t)| < \beta$, for $x, y \in \mathbb{R}^2$ and $t \in [0,T^*]$. For the equicontinuity in time, let $t, t+h \in [0,T^*]$ with $h > 0$. We can write

$$\mathbf{u}^\varepsilon(x,t+h) - \mathbf{u}^\varepsilon(x,t) = \frac{1}{\nu+\kappa}\mathbf{K} * \Big\{W^\varepsilon(x,t+h) - W^\varepsilon(x,t)\Big\}$$
$$+\frac{2\kappa}{\nu+\kappa}\mathbf{K} * \Big\{M^\varepsilon(b^\varepsilon)(x,t+h) - M^\varepsilon(b^\varepsilon)(x,t)\Big\}. \tag{4.33}$$

The definition of M^ε combined with identities (4.21) and (4.23), give

$$\begin{aligned}\mathbf{K}*\Big\{W^\varepsilon(x,t+h)-W^\varepsilon(x,t)\Big\} &= \mathbf{K}*\Big\{\Gamma(x,h)*W^\varepsilon(x,t)-W^\varepsilon(x,t)\Big\}\\ &\quad+\int_t^{t+h} e^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\nabla\Gamma(x,t+h-s)*\mathbf{K}*M^\varepsilon(W^\varepsilon\mathbf{u}^\varepsilon)(x,s)ds\\ &\quad+\frac{8\kappa^2\nu}{\nu+\kappa}\int_t^{t+h} e^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\Gamma(x,t+h-s)*\mathbf{K}*M^\varepsilon(b^\varepsilon)(x,s)ds\end{aligned}$$

and

$$\mathbf{K}*\Big\{M^\varepsilon(b^\varepsilon)(x,t+h)-M^\varepsilon(b^\varepsilon)(x,t)\Big\}=\int_{2\varepsilon}^{+\infty}\rho_\varepsilon(t-s)\mathbf{K}*\Big\{b^\varepsilon(x,s+h)-b^\varepsilon(x,s)\Big\}ds,$$

with

$$\begin{aligned}\mathbf{K}*\Big\{b^\varepsilon(x,s+h)-b^\varepsilon(x,s)\Big\} &= \mathbf{K}*\Big\{\ e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon(X^\varepsilon(s,x,s+h),s)-b^\varepsilon(x,s)\Big\}\\ &\quad+\frac{2\kappa}{\nu+\kappa}\int_s^{s+h} e^{-\frac{4\kappa\nu}{\nu+\kappa}(\tau-s)}\mathbf{K}*W^\varepsilon(X^\varepsilon(\tau,x,\tau+h),\tau)d\tau.\end{aligned}$$

Then, we obtain

$$\begin{aligned}|\mathbf{K}*\{W^\varepsilon(x,t+h)-W^\varepsilon(x,t)\}| &\leqslant |\mathbf{K}*\{\Gamma(x,h)*W^\varepsilon(x,t)-W^\varepsilon(x,t)\}|\\ &\quad+\left|\mathbf{K}*\left\{\int_t^{t+h}\ e^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\nabla\Gamma(x,t+h-s)*M^\varepsilon(W^\varepsilon\mathbf{u}^\varepsilon)(x,s)ds\right\}\right|\\ &\quad+\left|\mathbf{K}*\left\{\frac{8\kappa^2\nu}{\nu+\kappa}\int_t^{t+h}\ e^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\Gamma(x,t+h-s)*M^\varepsilon(b^\varepsilon)(x,s)ds\right\}\right|\\ &:= W_1+W_2+\frac{8\kappa^2\nu}{\nu+\kappa}W_3.\end{aligned}$$

In order to estimate W_1, applying Lemma 4.5 with $\theta=\Gamma$, we obtain

$$W_1=|\mathbf{K}*\{\Gamma(x,h)*W^\varepsilon(x,t)-W^\varepsilon(x,t)\}|\leqslant C\sqrt{h}, \tag{4.34}$$

where, for small ε, we have chosen the ε^{-1} associated with the $W^{1,p}$ norm of $W^\varepsilon(\cdot,t)$ in the order of $\varepsilon=\sqrt{h}$, while for ε large it is enough to apply the mean value theorem. Due to Proposition 4.12, W_2 and W_3 can be estimated as

$$\begin{aligned}W_2 &= \left|\mathbf{K}*\left\{\int_t^{t+h} e^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\nabla\Gamma(x,t+h-s)*M^\varepsilon(W^\varepsilon\mathbf{u}^\varepsilon)(x,s)ds\right\}\right|\\ &\leqslant \int_t^{t+h}\|\nabla\Gamma(\cdot,t+h-s)\|_1\|M^\varepsilon(W^\varepsilon\mathbf{u}^\varepsilon)(\cdot,s)\|_\infty^{1/2}\|M^\varepsilon(W^\varepsilon\mathbf{u}^\varepsilon)(\cdot,s)\|_1^{1/2}ds\\ &\leqslant C\int_t^{t+h}(t+h-s)^{-1/2}ds\leqslant Ch^{1/2},\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}W_3 &= \left|\mathbf{K}*\left\{\int_t^{t+h} e^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\Gamma(x,t+h-s)*M^\varepsilon(b^\varepsilon)(x,s)ds\right\}\right|\\ &\leqslant \int_t^{t+h}\|\mathbf{K}*M^\varepsilon(b^\varepsilon)(\cdot,s)\|_\infty ds\\ &\leqslant \int_t^{t+h}\max_{s/2\leqslant\tau\leqslant s}\|b^\varepsilon(\cdot,\tau)\|_1^{1/2}\|b^\varepsilon(\cdot,\tau)\|_\infty^{1/2}ds\leqslant Ch.\end{aligned} \tag{4.36}$$

On the other hand, in order to estimate the second term of identity (4.33), note that

$$\begin{aligned}|\mathbf{K}*\{b^\varepsilon(x,s+h)-b^\varepsilon(x,s)\}| \leqslant& \left|\mathbf{K}*\left\{e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon\left(X^\varepsilon(s,x,s+h),s\right)-b^\varepsilon(x,s)\right\}\right| \\ &+\left|\frac{2\kappa}{\nu+\kappa}\int_s^{s+h} e^{-\frac{4\kappa\nu}{\nu+\kappa}(\tau-s)}\mathbf{K}*W^\varepsilon(X^\varepsilon(\tau,x,\tau+h),\tau)d\tau\right| \\ :=& \ B_1+\frac{2\kappa}{\nu+\kappa}B_2.\end{aligned}$$

Now, in order to bound the term B_1, we write

$$\begin{aligned}B_1 =& \left|\mathbf{K}*\left\{e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon\left(X^\varepsilon(s,x,s+h),s\right)-b^\varepsilon(x,s)\right.\right. \\ &\left.\left.+e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon(x,s)-e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon(x,s)\right\}\right| \\ \leqslant& \left|\mathbf{K}*\left\{e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon\left(X^\varepsilon(s,x,s+h),s\right)-e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon(x,s)\right\}\right| \\ &+\left|\mathbf{K}*\left\{e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon(x,s)-b^\varepsilon(x,s)\right\}\right| \\ :=& B_{11}+B_{12}.\end{aligned}$$

Then, we have by Lemma 4.6,

$$\begin{aligned}B_{11} =& \left|\int_{\mathbb{R}^2}\mathbf{K}(x-y)e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon\left(X^\varepsilon(s,y,s+h),s\right)dy-\int_{\mathbb{R}^2}\mathbf{K}(x-z)e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon(z,s)dz\right| \\ =& \left|\int_{\mathbb{R}^2}[\mathbf{K}(x-y)-\mathbf{K}(x-X^\varepsilon(s,y,s+h))]e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon\left(X^\varepsilon(s,y,s+h),s\right)dy\right| \\ \leqslant& C\|I(\cdot)-X^\varepsilon(s,\cdot,s+h)\|_\infty\left(1+|\log\|I(\cdot)-X^\varepsilon(s,\cdot,s+h)\|_\infty|\right),\end{aligned}$$

where $C=C(\|b^\varepsilon(\cdot,s)\|_1+\|b^\varepsilon(\cdot,s)\|_\infty)>0$. Since

$$\begin{aligned}\|I(\cdot)-X^\varepsilon(s,\cdot,s+h)\|_\infty =& \sup_y|X^\varepsilon(s,y,s+h)-I(y)| \\ =& \sup_y|X^\varepsilon(s,y,s+h)-X^\varepsilon(s+h,y,s+h)| \\ =& \sup_y\left|\int_s^{s+h}u^\varepsilon(X^\varepsilon(\xi,y,s+h),\xi)d\xi\right| \leqslant Ch,\end{aligned}$$

we obtain

$$B_{11}\leqslant Ch\left[1+\log\left(Ch\right)\right], \tag{4.37}$$

and

$$\begin{aligned}B_{12} =& \left|\mathbf{K}*\left\{\left(e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}-1\right)b^\varepsilon(y,s)\right\}\right| \\ \leqslant& C\left|e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}-1\right|\|b^\epsilon(\cdot,s)\|_1^{1/2}\|b^\epsilon(\cdot,s)\|_\infty^{1/2} \\ \leqslant& Ch.\end{aligned}\tag{4.38}$$

Permuting integrals in time with the convolution, taking into account the estimates for W^ε and the area preservation properties of the mapping flow X^ε, we also deduce

$$B_2=\left|\mathbf{K}*\left\{\int_s^{s+h}e^{-\frac{4\kappa\nu}{\nu+\kappa}(\tau-s)}W^\varepsilon(X^\varepsilon(\tau,x,\tau+h),\tau)d\tau\right\}\right|\leqslant Ch. \tag{4.39}$$

Hence, combining the estimates (4.34)-(4.39), the family $t \mapsto \mathbf{u}^\varepsilon(x,t)$ is equicontinuous in time allowing to apply Ascoli-ArzelÃ¡ theorem to deduce the existence of a subfamily of $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ such that convergence (4.32) hold. □

Proposition 4.16. *The triplet* $(W, \mathbf{u}, b)$ *given by Proposition 4.15 is a mild solution of system* (4.7).

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0,T^*))$ and $\Omega = \operatorname{supp}(\varphi)$. The identity (4.21) leads to

$$
\begin{aligned}
\int_0^{T^*}\int_{\mathbb{R}^2} W^\varepsilon(x,t)\varphi(x,t)\,dx\,dt &= \int_0^{T^*}\int_{\mathbb{R}^2} e^{-\frac{4\kappa^2}{\nu+\kappa}t}\Gamma(x,t) * W_0^\varepsilon(x)\varphi(x,t)\,dx\,dt \\
&+ \int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * M^\varepsilon(\mathbf{u}^\varepsilon W^\varepsilon)(x,s)ds\right]\varphi(x,t)\,dx\,dt \\
&+ \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\Gamma(\cdot,t-s) * M^\varepsilon(b^\varepsilon)(x,s)ds\right]\varphi(x,t)\,dx\,dt.
\end{aligned}
$$

The convergence of the left hand side follows directly from weak-$*$ convergence of W^ε. For the first term of right hand side it's possible to take the limit by definition of mollifier sequence. Now, we prove the convergence of the second term of the right hand side, and the same idea can be used to prove the convergence for the last term. Thus, we consider

$$
\int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * M^\varepsilon(\mathbf{u}^\varepsilon W^\varepsilon)(x,s)ds\right]\varphi(x,t)\,dx\,dt := I_1 + I_2,
$$

where

$$
I_1 = \int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * (M^\varepsilon(\mathbf{u}^\varepsilon W^\varepsilon) - \mathbf{u}^\varepsilon W^\varepsilon)(x,s)ds\right]\varphi(x,t)\,dx\,dt
$$

and

$$
I_2 = \int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * (\mathbf{u}^\varepsilon W^\varepsilon)(x,s)ds\right]\varphi(x,t)\,dx\,dt.
$$

We begin showing that I_1 converges to 0. Indeed, since

$$
\begin{aligned}
|I_1| &= \left|\int_0^{T^*}\int_0^t\int_{\mathbb{R}^2} e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * (M^\varepsilon(\mathbf{u}^\varepsilon W^\varepsilon) - \mathbf{u}^\varepsilon W^\varepsilon)(x,s)\varphi(x,t)\,dx\,ds\,dt\right| \\
&\leqslant C(\nu,\kappa,T^*)\|\varphi\|_{L^\infty(0,T^*;L^\infty(\Omega))}\|M^\varepsilon(\mathbf{u}^\varepsilon W^\varepsilon) - \mathbf{u}^\varepsilon W^\varepsilon\|_{L^\infty(0,T^*;L^1(\Omega))},
\end{aligned}
$$

and $\|M^\varepsilon(\mathbf{u}^\varepsilon W^\varepsilon) - \mathbf{u}^\varepsilon W^\varepsilon\|_{L^\infty(0,T^*;L^1(\Omega))} \leqslant C\varepsilon$, we have $I_1 \to 0$. Now, we focus on I_2. Let us write $I_2 = I_{21} + I_{22}$, where

$$
I_{21} = \int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * (\mathbf{u}^\varepsilon W^\varepsilon - \mathbf{u}W^\varepsilon)(x,s)ds\right]\varphi(x,t)\,dx\,dt.
$$

and

$$
I_{22} = \int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * \mathbf{u}W^\varepsilon(x,s)ds\right]\varphi(x,t)\,dx\,dt.
$$

We have

$$
\begin{aligned}
|I_{21}| &= \left| \int_0^{T^*} \int_{\mathbb{R}^2} \left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) * (\mathbf{u}^\varepsilon W^\varepsilon - \mathbf{u} W^\varepsilon)(x, s) ds \right] \varphi(x, t)\, dx\, dt \right| \\
&\leqslant \|\varphi\|_{L^\infty(0,T^*;L^\infty(\mathbb{R}^2))} \int_0^{T^*} \int_0^t \|\nabla\Gamma(\cdot, t-s) * (\mathbf{u}^\varepsilon - \mathbf{u}) W^\varepsilon)(\cdot, s)\|_{L^1(\Omega)}\, ds\, dt \\
&\leqslant C(\nu, \kappa, T^*) \|\varphi\|_{L^\infty(0,T^*;L^\infty(\Omega))} \|\mathbf{u}^\varepsilon - \mathbf{u}\|_{L^\infty(0,T^*;L^\infty(\Omega))} \|W^\varepsilon\|_{L^\infty(0,T^*;L^1(\Omega))} \to 0.
\end{aligned}
$$

On the other hand, using Fubini theorem and the fact that $W^\varepsilon \rightharpoonup W$ weak-$*$ in $L^\infty(0, T^*; L^\infty(\mathbb{R}^2))$, we find

$$
I_{22} \to \int_0^{T^*} \int_{\mathbb{R}^2} \left[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) * (\mathbf{u} W)(x, s) ds \right] \varphi(x, t)\, dx\, dt.
$$

These arguments together with the Fundamental Lemma of Calculus of Variations lead to

$$
\begin{aligned}
W(x, t) = e^{-\frac{4\kappa^2}{\nu+\kappa} t} \Gamma(\cdot, t) * W_0(x) + \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) * (\mathbf{u} W)(x, s)\, ds \\
+ \frac{8\kappa^2\nu}{\nu+\kappa} \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \Gamma(\cdot, t-s) * b(x, s)\, ds.
\end{aligned} \tag{4.40}
$$

The same strategy applied to identity (4.23) yields

$$
\begin{aligned}
\int_0^{T^*} \int_{\mathbb{R}^2} b^\varepsilon(x, t) \varphi(x, t)\, dx\, dt &= \int_0^{T^*} \int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa} t} b_0^\varepsilon(\mathbf{X}^\varepsilon(0; x, t)) \varphi(x, t)\, dx\, dt \\
&+ \frac{2\kappa}{\nu+\kappa} \int_0^{T^*} \int_{\mathbb{R}^2} \left[\int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)} W^\varepsilon(\mathbf{X}^\varepsilon(s; x, t), s) ds \right] \varphi(x, t)\, dx\, dt.
\end{aligned}
$$

We analysis the term

$$
J = \int_0^{T^*} \int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa} t} b_0^\varepsilon(\mathbf{X}^\varepsilon(0; x, t)) \varphi(x, t)\, dx\, dt.
$$

For the rest of the terms, the argument is analogous. Using the change of variables $\mathbf{X}^\varepsilon(0; x, t) = \mathbf{y}$, we can write it as

$$
\begin{aligned}
\int_0^{T^*} \int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa} t} b_0^\varepsilon(\mathbf{X}^\varepsilon(0; x, t)) \varphi(x, t) dx dt &= \int_0^{T^*} \int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa} t} b_0^\varepsilon(\mathbf{y}) \varphi(\mathbf{X}^\varepsilon(0; \mathbf{y}, t), t)\, d\mathbf{y}\, dt \\
&:= J_1 + J_2,
\end{aligned}
$$

where

$$
J_1 = \int_0^{T^*} \int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa} t} b_0^\varepsilon(\mathbf{y}) (\varphi(\mathbf{X}^\varepsilon(0; \mathbf{y}, t), t) - \varphi(\mathbf{X}(0; \mathbf{y}, t), t))\, d\mathbf{y}\, dt,
$$

and

$$
J_2 = \int_0^{T^*} \int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa} t} b_0^\varepsilon(\mathbf{y}) \varphi(\mathbf{X}(0; \mathbf{y}, t), t)\, d\mathbf{y}\, dt.
$$

The term J_1 converges to 0 due to the continuity of φ (uniform on compact sets):

$$
\begin{aligned}
|J_1| &= \left| \int_0^{T^*} \int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa} t} b_0^\varepsilon(\mathbf{y}) (\varphi(\mathbf{X}^\varepsilon(0; \mathbf{y}, t), t) - \varphi(\mathbf{X}(0; \mathbf{y}, t), t))\, d\mathbf{y}\, dt \right| \\
&\leqslant \|b_0^\varepsilon\|_{L^\infty} \int_0^{T^*} \int_\Omega |\varphi(\mathbf{X}^\varepsilon(0; \mathbf{y}, t), t) - \varphi(\mathbf{X}(0; \mathbf{y}, t), t)|\, d\mathbf{y}\, dt \to 0.
\end{aligned}
$$

Since $\|b_0^\varepsilon - b_0\|_1 \to 0$, we find

$$J_2 \to \int_0^{T^*}\int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa}t} b_0(\mathbf{y})\varphi(\mathbf{X}(0;\mathbf{y},t),t)\, d\mathbf{y}\, dt.$$

By change of variable, we obtain

$$J \to \int_0^{T^*}\int_{\mathbb{R}^2} e^{-\frac{4\kappa\nu}{\nu+\kappa}t} b_0(\mathbf{X}(0;\mathbf{x},t))\varphi(\mathbf{x},t)\, d\mathbf{x}\, dt.$$

Then, we conclude

$$b(\mathbf{x},t) = e^{-\frac{4\kappa\nu}{\nu+\kappa}t} b_0(\mathbf{X}(0;\mathbf{x},t)) + \frac{2\kappa}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)} W(\mathbf{X}(s;\mathbf{x},t),s)\, ds. \tag{4.41}$$

Finally, the identity (4.22) leads to

$$\begin{aligned}\int_0^{T^*}\int_{\mathbb{R}^2} \mathbf{u}^\varepsilon(\mathbf{x},t)\varphi(\mathbf{x},t)\, d\mathbf{x}\, dt &= \int_0^{T^*}\int_{\mathbb{R}^2} \frac{1}{\nu+\kappa}\mathbf{K} * W^\varepsilon(\mathbf{x},t)\varphi(\mathbf{x},t)\, d\mathbf{x}\, dt \\ &+ \int_0^{T^*}\int_{\mathbb{R}^2} \frac{2\kappa}{\nu+\kappa}\mathbf{K} * M^\varepsilon(b^\varepsilon)(\mathbf{x},t)\varphi(\mathbf{x},t)\, d\mathbf{x}\, dt := N_1 + N_2.\end{aligned}$$

We prove the convergence of N_1 directly from Lemma 4.3 and definition of weak-$*$ convergence of $(W^\varepsilon)_\varepsilon$. For N_2, we proceed in the same way as above, that is, adding and subtracting b^ε in the second member of the convolution. Then, the proof is completed by combining the above inequalities. □

In order to obtain global existence for the mild solution, we can consider the initial time to $t = \delta < T^*$, and take $T^* > 0$ as in Proposition 4.12. Then, we have the estimates (4.24) and (4.25) hold in $[\delta, T^* + \delta]$ and so on. We have proved the following result

Corollary 4.17. *Let* $W_0, b_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *For any* $T > 0$, *the system* (4.7) *has a global mild solution, such that*

$$\{(\nu+\kappa)t\}^{1/2}\mathbf{u} \in L^\infty(0,T;L^\infty(\mathbb{R}^2)).$$

Remark 4.18. *Note that in each step described above, the initial data is more regular and thus we have regularity gain for the obtained solution (see (CAFFARELLI; KOHN; NIRENBERG, 1982)).*

4.2.3 Uniqueness and stability of the solution

Let us take advantage of the results in the previous section to show uniqueness and asymptotic stability for the solution of the problem (4.7) with respect to disturbances of the initial data.

Proposition 4.19. *Let* $W_0, b_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *There exists a unique global solution of system* (4.7) *satisfying the estimate* (4.2).

Proof. Suppose $(W^1, \mathbf{u}^1, b^1)$ and $(W^2, \mathbf{u}^2, b^2)$ are solutions of system (4.7). We will show that $W^1 = W^2$ and $b^1 = b^2$ on $\mathbb{R}^2 \times (0, T]$, for all $T > 0$. We set

$$Z(\cdot, t) = (W^1 - W^2)(\cdot, t),$$
$$E(\cdot, t) = (b^1 - b^2)(\cdot, t).$$

Note that (Z, E) satisfy the system

$$\begin{aligned}
\partial_t Z - (\nu+\kappa)\Delta Z + \frac{4\kappa^2}{\nu+\kappa} Z &= -\left(\left(\mathbf{K} * \left(\frac{1}{\nu+\kappa} Z + \frac{2\kappa}{\nu+\kappa} E\right)\right) \cdot \nabla\right) W^1 - (u^2 \cdot \nabla) Z + \frac{8\kappa^2\nu}{\nu+\kappa} E \\
\partial_t E + \frac{4\kappa\nu}{\nu+\kappa} E + (\mathbf{u}^2 \cdot \nabla) E &= -\left(\left(\mathbf{K} * \left(\frac{1}{\nu+\kappa} Z + \frac{2\kappa}{\nu+\kappa} E\right)\right) \cdot \nabla\right) b^1 + \frac{2\kappa}{\nu+\kappa} Z \\
Z(\cdot, 0) &= 0, \\
E(\cdot, 0) &= 0,
\end{aligned}$$

and

$$\begin{aligned}
Z(\cdot, t) = & \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) * \left\{ W^1 \left(\mathbf{K} * \left(\frac{1}{\nu+\kappa} Z + \frac{2\kappa}{\nu+\kappa} E \right) \right) \right\} (\cdot, s) ds \\
& + \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) * (Z\mathbf{u}^2)(\cdot, s) ds \\
& + \frac{8\kappa^2\nu}{\nu+\kappa} \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \Gamma(\cdot, t-s) * E(\cdot, s) ds, \\
E(\cdot, t) = & \int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) * \left\{ b^1 \left(\mathbf{K} * \left(\frac{1}{\nu+\kappa} Z + \frac{2\kappa}{\nu+\kappa} E \right) \right) \right\} (\cdot, s) ds \\
& + \frac{2\kappa}{\nu+\kappa} \int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)} Z(\cdot, s) ds.
\end{aligned}$$

Then,

$$\begin{aligned}
\|Z(\cdot, t)\|_1 \leqslant & \int_0^t \left\| \nabla\Gamma(\cdot, t-s) * \left\{ W^1 \left(\mathbf{K} * \left(\frac{1}{\nu+\kappa} Z + \frac{2\kappa}{\nu+\kappa} E \right) \right) \right\} (\cdot, s) \right\|_1 ds \\
& + \int_0^t \|\nabla\Gamma(\cdot, t-s) * (Z\mathbf{u}^2)(\cdot, s)\|_1 ds \\
& + \frac{8\kappa^2\nu}{\nu+\kappa} \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \|\Gamma(\cdot, t-s) * E(\cdot, s)\|_1 ds \\
\leqslant & \; C_0 (\nu+\kappa)^{-3/2} \int_0^t (t-s)^{-1/2} s^{-1/2} \left\{ \|Z(\cdot, s)\|_1 + 2\kappa \|E(\cdot, s)\|_1 \right\} ds \\
& + C_0 \int_0^t (t-s)^{-1/2} s^{-1/2} \|Z(\cdot, s)\|_1 ds \\
& + \frac{4\kappa\nu}{\nu+\kappa} \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} 2\kappa \|E(\cdot, s)\|_1 \, ds,
\end{aligned}$$

that is,

$$\|Z(\cdot, t)\|_1 \leqslant C_1 \int_0^t \left\{ (t-s)^{-1/2} s^{-1/2} + e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \right\} \left\{ \|Z(\cdot, s)\|_1 + 2\kappa \|E(\cdot, s)\|_1 \right\} ds,$$

with $C_1 = \max\left\{C_0(\nu+\kappa)^{-3/2} + C_0,\ \dfrac{4\kappa\nu}{\nu+\kappa}\right\}$. Similarly, we have

$$2\kappa\|E(\cdot,t)\|_1 \leqslant C_2\int_0^t \left\{(t-s)^{-1/2}s^{-1/2} + e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\right\}\left\{\|Z(\cdot,s)\|_1 + 2\kappa\|E(\cdot,s)\|_1\right\}ds,$$

with $C_2 = \max\left\{2\kappa C_0(\nu+\kappa)^{-3/2},\ \dfrac{4\kappa^2}{\nu+\kappa}\right\}$. Therefore,

$$\begin{aligned}&\|Z(\cdot,t)\|_1 + 2\kappa\|E(\cdot,t)\|_1\\ &\qquad\leqslant C_3\int_0^t \left\{(t-s)^{-1/2}s^{-1/2} + e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} + e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\right\}\left\{\|Z(\cdot,s)\|_1 + 2\kappa\|E(\cdot,s)\|_1\right\}ds,\end{aligned}$$

where $C_3 = \max\{C_1,\ C_2\}$. Define $\Upsilon(t) = \sup_{s\leqslant t}\left\{\|Z(\cdot,s)\|_1 + 2\kappa\|E(\cdot,s)\|_1\right\}$, $t > 0$. Then, $\Upsilon(t) \leqslant C(\nu,\kappa,T)\Upsilon(t)$, with $C(\nu,\kappa,T) < 1$, see Proposition 4.12. This estimate implies $\Upsilon(t) = 0$ and, therefore, the solution of problem (4.7) is unique. □

Finally, let us state the stability result in the following proposition.

Proposition 4.20. *Let $W_0, b_0, \hat{W}_0, \hat{b_0} \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Assume that the $(W,\mathbf{u},b)$ and $(\hat{W},\hat{\mathbf{u}},\hat{b})$ are solution of problem (4.7) with initial data (W_0,b_0) and $(\hat{W}_0,\hat{b_0})$, respectively. Then, the inequalities below are satisfied for $t>0$*

$$\|(\mathbf{u}-\hat{\mathbf{u}})(\cdot,t)\|_\infty \leqslant Ct^{-1/2}, \tag{4.42}$$

$$\|(W-\hat{W})(\cdot,t)\|_1 + \|(W-\hat{W})(\cdot,t)\|_\infty + \|(b-\hat{b})(\cdot,t)\|_1 + \|(b-\hat{b})(\cdot,t)\|_\infty \leqslant C\hat{\Pi}, \tag{4.43}$$

where $\hat{\Pi} = \hat{\Pi}(W_0,b_0,\hat{W}_0,\hat{b_0}) = \max\{\|W_0-\hat{W}_0\|_1, \|W_0-\hat{W}_0\|_\infty, \|b_0-\hat{b_0}\|_1, \|b_0-\hat{b_0}\|_\infty\}$ and $C>0$ is a constant independent of $\hat{\Pi}$.

Proof. Defining $Z = W-\hat{W}$, $E = b-\hat{b}$ and $\mathbf{U} = \mathbf{u}-\hat{\mathbf{u}}$, they satisfy the system

$$\begin{aligned}\partial_t Z - (\nu+\kappa)\Delta Z + \frac{4\kappa^2}{\nu+\kappa}Z &= -(\mathbf{U}\cdot\nabla)W - (\hat{\mathbf{u}}\cdot\nabla)Z + \frac{8\kappa^2\nu}{\nu+\kappa}E,\\ \mathbf{U} &= \mathbf{K}*\left(\frac{1}{\nu+\kappa}Z + \frac{2\kappa}{\nu+\kappa}E\right),\\ \partial_t E + \frac{4\kappa\nu}{\nu+\kappa}E + (\hat{\mathbf{u}}\cdot\nabla)E &= -(\mathbf{U}\cdot\nabla)b + \frac{2\kappa}{\nu+\kappa}Z,\\ Z(\cdot,0) &= Z_0 = W_0-\hat{W}_0,\\ E(\cdot,0) &= E_0 = b_0-\hat{b_0}.\end{aligned}$$

Then, we can proceed as Proposition 4.19. □

4.3 MEASURES AS INITIAL DATA

In this section we deal with *a priori* estimates considering the initial data $W_0 \in \mathcal{M}(\mathbb{R}^2)$ and $b_0 \in \mathcal{M}(\mathbb{R}^2) \cap \mathcal{M}^p(\mathbb{R}^2)$, with $p > 2$ fixed. This case lead us to a theory of existence of weak solutions for system (4.7). The main difference with respect to the case in which initial data are in $L^1 \cap L^\infty$ is how one derives the appropriated *a priori* estimates.

First, observe that by $W^\varepsilon(\cdot,t), b^\varepsilon(\cdot,t) \in C^\infty(\mathbb{R})^2 \cap L^1(\mathbb{R}^2)$ and $\mathbf{u}^\varepsilon = \mathbf{K} * \left(\frac{1}{\nu+\kappa} W^\varepsilon + \frac{2\kappa}{\nu+\kappa} M^\varepsilon(b^\varepsilon)\right)$. We have

$$\operatorname{curl} \mathbf{u}^\varepsilon(\cdot,t) = \frac{1}{\nu+\kappa} W^\varepsilon(\cdot,t) + \frac{2\kappa}{\nu+\kappa} M^\varepsilon(b^\varepsilon)(\cdot,t),$$

that is,

$$W^\varepsilon(\cdot,t) = (\nu+\kappa)\operatorname{curl} \mathbf{u}^\varepsilon(\cdot,t) - 2\kappa M^\varepsilon(b^\varepsilon)(\cdot,t).$$

Therefore, for $0 < s < t$, we can write

$$-\Gamma(\cdot,t-s) * M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s) = \nabla\Gamma(\cdot,t-s) * M^\varepsilon(W^\varepsilon \mathbf{u}^\varepsilon)(\cdot,s) \tag{4.44}$$

and

$$\begin{aligned}-\Gamma(\cdot,t-s) * M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s) &= -(\nu+\kappa)\operatorname{curl}\nabla\Gamma(\cdot,t-s) * M^\varepsilon(\mathbf{u}^\varepsilon\mathbf{u}^\varepsilon)(\cdot,s)\\ &\quad - 2\kappa\nabla\Gamma(\cdot,t-s) * M^\varepsilon(M^\varepsilon(b^\varepsilon)\mathbf{u}^\varepsilon)(\cdot,s),\end{aligned} \tag{4.45}$$

where $\mathbf{u}^\varepsilon\mathbf{u}^\varepsilon := \sum_{i,j}^{2} u_i^\varepsilon u_j^\varepsilon$. We have the following result:

Proposition 4.21. *There exists* $T^* = T^*(\nu,\kappa,\Pi(W_0,b_0)) > 0$ *and a positive constant* $C = C(\nu,\kappa)$ *such that*

$$\begin{aligned}\|\mathbf{u}^\varepsilon(\cdot,t)\|_\infty &\leqslant C\{(\nu+\kappa)t\}^{-1/2},\\ \|W^\varepsilon(\cdot,t)\|_1 + \|b^\varepsilon(\cdot,t)\|_1 &\leqslant C\Pi(W_0,b_0),\\ \{(\nu+\kappa)t\}^{1-\frac{1}{p}}\|W^\varepsilon(\cdot,t)\|_{M^p} + \|b^\varepsilon(\cdot,t)\|_{M^p} &\leqslant C\Pi(W_0,b_0),\end{aligned}$$

hold for all $t \in (0,T^*]$.

Proof. Note that

$$
\begin{aligned}
\|\mathbf{u}^\varepsilon(\cdot,t)\|_\infty \leqslant{}& \frac{e^{-\frac{4\kappa^2}{\nu+\kappa}t}}{\nu+\kappa}\|(\mathbf{K}*\Gamma(\cdot,t))*W_0^\varepsilon\|_\infty \\
&+\frac{1}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\mathbf{K}*(\Gamma(\cdot,t-s)*M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon))(\cdot,s)\|_\infty ds \\
&+\frac{8\kappa^2\nu}{(\nu+\kappa)^2}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\Gamma(\cdot,t-s)*(\mathbf{K}*M^\varepsilon(b^\varepsilon))(\cdot,s)\|_\infty ds \\
&+\frac{2\kappa}{\nu+\kappa}\|\mathbf{K}*M^\varepsilon(b^\varepsilon)(\cdot,t)\|_\infty \\
\leqslant{}& \frac{C}{\nu+\kappa}\{(\nu+\kappa)t\}^{-1/2}\|W_0\|_{\mathcal{M}} \\
&+\frac{C}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\Gamma(\cdot,t-s)*M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s)\|_1^{1/2} \\
&\qquad\qquad\times\|\Gamma(\cdot,t-s)*M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s)\|_\infty^{1/2}ds \\
&+\frac{8\kappa^2\nu}{(\nu+\kappa)^2}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\mathbf{K}*M^\varepsilon(b^\varepsilon)(\cdot,s)\|_\infty ds \\
&+\frac{2\kappa}{\nu+\kappa}\|\mathbf{K}*M^\varepsilon(b^\varepsilon)(\cdot,t)\|_\infty.
\end{aligned}
$$

Applying Lemma 4.8, we obtain the estimates

$$
\begin{aligned}
\|\Gamma(\cdot,t-s)*M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s)\|_1 &= \|\nabla\Gamma(\cdot,t-s)*M^\varepsilon(W^\varepsilon\mathbf{u}^\varepsilon)(\cdot,s)\|_1 \\
&\leqslant C\{(\nu+\kappa)(t-s)\}^{-1/2}\max_{s/2<\tau<s}\|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\Gamma(\cdot,t-s)*M^\varepsilon((\mathbf{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s)\|_\infty &\leqslant (\nu+\kappa)\|\operatorname{curl}\nabla\Gamma(\cdot,t-s)*M^\varepsilon(\mathbf{u}^\varepsilon\mathbf{u}^\varepsilon)(\cdot,s)\|_\infty \\
&\quad+2\kappa\|\nabla\Gamma(\cdot,t-s)*M^\varepsilon(M^\varepsilon(b^\varepsilon)\mathbf{u}^\varepsilon)(\cdot,s)\|_\infty \\
&\leqslant C\{(\nu+\kappa)(t-s)\}^{-1}\max_{s/2<\tau<s}\|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty^2 \\
&\quad+2\kappa C\{(\nu+\kappa)(t-s)\}^{-1/2-1/p} \\
&\qquad\times\max_{s/2<\tau<s}\left\{\|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty\max_{\tau/2<\eta<\tau}\|b^\varepsilon(\cdot,\eta)\|_{M^p}\right\}.
\end{aligned}
$$

On the other hand, by Lemma 4.7 we have

$$
\begin{aligned}
\|\mathbf{K}*M^\varepsilon(b^\varepsilon)(\cdot,t)\|_\infty &\leqslant C\|M^\varepsilon(b^\varepsilon)(\cdot,s)\|_1^\alpha\|M^\varepsilon(b^\varepsilon)(\cdot,s)\|_{M^p}^{1-\alpha} \\
&\leqslant C\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_1^\alpha\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_{M^p}^{1-\alpha},
\end{aligned}
$$

where $\alpha = \dfrac{p-2}{2(p-1)} \in (0,1)$. Therefore, combining the above estimates we obtain

$$\begin{aligned}
\|\mathbf{u}^\varepsilon(\cdot,t)\|_\infty \leqslant{}& \frac{C}{\nu+\kappa}\{(\nu+\kappa)t\}^{-1/2}\|W_0\|_{\mathcal{M}} \\
&+ \frac{C}{(\nu+\kappa)^{1/2}}\int_0^t \{(\nu+\kappa)(t-s)\}^{-3/4} \max_{s/2<\tau<s} \|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty^{3/2}\|W^\varepsilon(\cdot,\tau)\|_1^{1/2} ds \\
&+ \frac{(2\kappa)^{1/2}C}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\{(\nu+\kappa)(t-s)\}^{-1/2-1/2p} \\
&\qquad \times \max_{s/2<\tau<s}\left\{\|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty \|W^\varepsilon(\cdot,\tau)\|_1^{1/2} \max_{\tau/2<\eta<\tau}\|b^\varepsilon(\cdot,\eta)\|_{M^p}^{1/2}\right\} ds \\
&+ \frac{8\kappa^2\nu C}{(\nu+\kappa)^2}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_1^\alpha \max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_{M^p}^{1-\alpha} ds \\
&+ \frac{2\kappa C}{\nu+\kappa}\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_1^\alpha \max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_{M^p}^{1-\alpha}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\{(\nu+\kappa)t\}^{1/2}\|\mathbf{u}^\varepsilon(\cdot,t)\|_\infty \leqslant{}& C(\nu+\kappa)^{-1}\|W_0\|_{\mathcal{M}} \\
&+ C(\nu+\kappa)^{-3/2}t^{1/2}\int_0^t (t-s)^{-3/4}s^{-3/4} \\
&\qquad \times \max_{s/2<\tau<s}(\{(\nu+\kappa)\tau\}^{1/2}\|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty)^{3/2}\|W^\varepsilon(\cdot,\tau)\|_1^{1/2} ds \\
&+ (2\kappa)^{1/2}C(\nu+\kappa)^{-3/2-1/2p}t^{1/2}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}(t-s)^{-1/2-1/2p}s^{-1/2} \\
&\qquad \times \max_{s/2<\tau<s}\Big\{\{(\nu+\kappa)\tau\}^{1/2}\|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_1^{1/2} \\
&\qquad\qquad \times \max_{\tau/2<\eta<\tau}\|b^\varepsilon(\cdot,\eta)\|_{M^p}^{1/2}\Big\} ds \\
&+ 8\kappa^2\nu C(\nu+\kappa)^{-3/2}t^{1/2}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_1^\alpha \\
&\qquad \times \max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_{M^p}^{1-\alpha} ds \\
&+ 2\kappa C(\nu+\kappa)^{-1/2}t^{1/2}\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_1^\alpha \max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_{M^p}^{1-\alpha}.
\end{aligned} \tag{4.46}$$

Analogously, we obtain the following estimates for W^ε and b^ε, where we assume $1 < 2p(p+2)^{-1} < r < 2$ and $0 \leqslant \beta \leqslant 1$ such that $\dfrac{1}{r} = \beta + \dfrac{1-\beta}{p}$:

$$\|b^\varepsilon(\cdot,t)\|_1 \leqslant \|b_0\|_{\mathcal{M}} + 2\kappa(\nu+\kappa)^{-1}\int_0^t \|W^\varepsilon(\cdot,s)\|_1 ds, \tag{4.47}$$

$$\|b^\varepsilon(\cdot,t)\|_{M^p} \leqslant \|b_0\|_{\mathcal{M}^p} + 2\kappa(\nu+\kappa)^{-2+1/p}\int_0^t s^{-1+1/p}\{(\nu+\kappa)s\}^{1-1/p}\|W^\varepsilon(\cdot,s)\|_{M^p} ds, \tag{4.48}$$

$$\|W^\varepsilon(\cdot,t)\|_1 \leqslant \|W_0\|_{\mathcal{M}} + C(\nu+\kappa)^{-1}\int_0^t (t-s)^{-1/2}s^{-1/2}\max_{s/2<\tau<s}\{(\nu+\kappa)\tau\}^{1/2}\|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_1 ds + 8\kappa^2\nu(\nu+\kappa)^{-1}\int_0^t \max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_1 ds, \tag{4.49}$$

and

$$\{(\nu+\kappa)t\}^{1-1/p}\|W^\varepsilon(\cdot,t)\|_{M^p} \leqslant C\|W_0\|_{\mathcal{M}} + C(\nu+\kappa)^{-1}t^{1-1/p}\int_0^t (t-s)^{-1/2-1/r+1/p}s^{-3/2+1/r} \times \max_{s/2<\tau<s}\Big\{\{(\nu+\kappa)\tau\}^{1/2}\|\mathbf{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_1^\beta \times (\{(\nu+\kappa)\tau\}^{1-1/p}\|W^\varepsilon(\cdot,\tau)\|_{M^p})^{1-\beta}\Big\}ds + 8\kappa^2\nu(\nu+\kappa)^{-1/p}t^{1-1/p}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_{M^p}ds. \tag{4.50}$$

Setting $\Pi(W_0,b_0) = \max\{\|W_0\|_{\mathcal{M}}, \|b_0\|_{\mathcal{M}}, \|b_0\|_{\mathcal{M}^p}\}$,

$$\lambda(t) = \sup_{s\leqslant t}\Big\{\{(\nu+\kappa)s\}^{1/2}\|\mathbf{u}^\varepsilon(\cdot,s)\|_\infty, \|W^\varepsilon(\cdot,s)\|_1, \{(\nu+\kappa)s\}^{1-1/p}\|W^\varepsilon(\cdot,s)\|_{M^p}, \|b^\varepsilon(\cdot,s)\|_1, \|b^\varepsilon(\cdot,s)\|_{M^p}\Big\},$$

and taking into account Lemma 4.4, we can rewrite the estimates (4.46) - (4.50)

$$\{(\nu+\kappa)t\}^{1/2}\|\mathbf{u}^\varepsilon(\cdot,t)\|_\infty \leqslant C(\nu+\kappa)^{-1}\Pi(W_0,b_0) + C\Big((\nu+\kappa)^{-3/2} + (2\kappa)^{1/2}(\nu+\kappa)^{-3/2-1/2p}\Big)\lambda(t)^2 + C(\nu+\kappa)^{1/2}t^{1/2}\lambda(t),$$

$$\|b^\varepsilon(\cdot,t)\|_1 \leqslant \Pi(W_0,b_0) + 2\kappa(\nu+\kappa)^{-1}t\lambda(t),$$
$$\|b^\varepsilon(\cdot,t)\|_{M^p} \leqslant \Pi(W_0,b_0) + 2\kappa(\nu+\kappa)^{-2+1/p}t^{1/p}\lambda(t),$$

$$\|W^\varepsilon(\cdot,t)\|_1 \leqslant \Pi(W_0,b_0) + C(\nu+\kappa)^{-1}\lambda(t)^2 + 8\kappa^2\nu(\nu+\kappa)^{-1}t\lambda(t),$$

and

$$\{(\nu+\kappa)t\}^{1-1/p}\|W^\varepsilon(\cdot,t)\|_{M^p} \leqslant C\Pi(W_0,b_0) + C(\nu+\kappa)^{-1}\lambda(t)^2 + 2\nu(\nu+\kappa)^{1-1/p}t^{1-1/p}\lambda(t).$$

Therefore,

$$0 \leqslant \lambda(t) \leqslant C_0(\nu,\kappa)\Pi(W_0,b_0) + C_1(\nu,\kappa,p)\Big[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\Big]\lambda(t) + C_2(\nu,\kappa,p)\lambda(t)^2,$$

or equivalently

$$0 \leqslant C_0\Pi(W_0, b_0) + \left\{C_1\left[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\right] - 1\right\}\lambda(t) + C_2\lambda(t)^2,$$

where C_0, C_1 and C_2 depend on $\nu + \kappa$ and are given by

$$C_0 = C(\nu + \kappa)^{-1} + C + 3,$$

$$C_1 = \max\left\{C(\nu+\kappa)^{1/2}, 2\kappa(\nu+\kappa)^{-1} + 8\kappa^2\nu(\nu+\kappa)^{-1}, 2\kappa(\nu+\kappa)^{-2+1/p}, 2\nu(\nu+\kappa)^{1-1/p}\right\}$$

and

$$C_2 = C((\nu+\kappa)^{-3/2} + (2\kappa)^{1/2}(\nu+\kappa)^{-3/2-1/2p}) + 2C(\nu+\kappa)^{-1}.$$

Thus, for initial data such that $4C_0C_2\Pi(W_0, b_0) < 1$, there exists $T^* > 0$ for which

$$\begin{cases} C_1\left[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\right] - 1 < 0, \\ \sigma = \left\{C_1\left[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\right] - 1\right\}^2 - 4C_0C_2\Pi(W_0, b_0) > 0, \end{cases} \tag{4.51}$$

hold for all $t \in [0, T^*]$. Indeed, since $\lim_{t\to 0^+} C_1\left[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\right] = 0$, there exists $\delta > 0$ such that $t \in (0, \delta)$ implies $C_1\left[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\right] < 1 - 2\sqrt{C_0C_2\Pi(W_0, b_0)}$, proving inequality (4.51) for $0 < T^* \leqslant \delta$. Therefore, since λ is continuous, we conclude that $0 \leqslant \lambda \leqslant \dfrac{1}{2C_2}$. This complete the proof. □

With these estimates, all the results from the previous section can be extended to the case under discussion considering $t > 0$. However, this solution must be understood in a weak sense, see Definition 4.22 below. However, a bootstrap *a* posteriori argument provides that the solutions will be regular, see (CAFFARELLI; KOHN; NIRENBERG, 1982).

Definition 4.22. *Given* $W_0, b_0 \in \mathcal{M}(\mathbb{R}^2)$ *and* $T > 0$, *we say that* $(W, \mathbf{u}, b)$ *is a weak solution of system* (4.7) *if it satisfies the initial data in weak-∗ sense, see Definition 4.9, identity* (4.12), *and*

(i) $\mathbf{u} = \mathbf{K} * \left(\dfrac{1}{\nu+\kappa}W + \dfrac{2\kappa}{\nu+\kappa}b\right)$,

(ii) The following equalities

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^2}\left(\partial_t\varphi - (\nu+\kappa)\Delta\varphi + \frac{4\kappa^2}{\nu+\kappa}\varphi + (\mathbf{u}\cdot\nabla)\varphi\right)\,dW \\ &\qquad = -\int_{\mathbb{R}^2}\varphi(\cdot,0)\,dW_0 + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^T\int_{\mathbb{R}^2}\varphi\,db, \\ &\int_0^T\int_{\mathbb{R}^2}\left(\partial_t\phi + \frac{4\kappa\nu}{\nu+\kappa}\phi + (\mathbf{u}\cdot\nabla)\phi\right)\,db \\ &\qquad = -\int_{\mathbb{R}^2}\phi(\cdot,0)\,db_0 + \frac{2\kappa}{\nu+\kappa}\int_0^T\int_{\mathbb{R}^2}\phi\,dW. \end{aligned} \tag{4.52}$$

hold for every $\varphi \in C_0^2(\mathbb{R}^2 \times [0,T))$ *and* $\phi \in C_0^1(\mathbb{R}^2 \times [0,T))$.

Indeed, taking into account that every uniformly bounded family in L^1 admits a subfamily which converges weak-$*$ as measure in $L^\infty(0,T^*;\mathcal{M}(\mathbb{R}^2))$ together with Proposition 4.21, then an identical argument to that of Proposition 4.15 allows to conclude the following result.

Proposition 4.23. *The family* $(W^\varepsilon,\mathbf{u}^\varepsilon,b^\varepsilon)_{\varepsilon>0}$ *of solutions system* (4.18) *admits a subfamily, still denoted by* $(W^\varepsilon,\mathbf{u}^\varepsilon,b^\varepsilon)_{\varepsilon>0}$*, such that*

$$W^\varepsilon \rightharpoonup W \text{ weak-}* \text{ as measures in } L^\infty(0,T^*;\mathcal{M}(\mathbb{R}^2)), \tag{4.53}$$

$$b^\varepsilon \rightharpoonup b \text{ weak-}* \text{ as measures in } L^\infty(0,T^*;\mathcal{M}(\mathbb{R}^2)), \tag{4.54}$$

$$\mathbf{u}^\varepsilon \longrightarrow \mathbf{u} \text{ uniformly in compact set of } \mathbb{R}^2\times(0,T^*). \tag{4.55}$$

Proposition 4.24. *The triplet* $(W,\mathbf{u},b)$ *given by Proposition 4.23 is a weak solution of system* (4.7).

Proof. The regularized solutions $(W^\varepsilon,\mathbf{u}^\varepsilon,b^\varepsilon)_{\varepsilon>0}$ satisty the system (4.18) in the weak sense

$$\begin{aligned}\int_0^{T^*}\int_{\mathbb{R}^2}\left(\partial_t\varphi-(\nu+\kappa)\Delta\varphi+\frac{4\kappa^2}{\nu+\kappa}\varphi\right)W^\varepsilon\,d\mathbf{x}\,dt+\int_0^{T^*}\int_{\mathbb{R}^2}M^\varepsilon\Big(\big((\mathbf{u}^\varepsilon\cdot\nabla)\varphi(\mathbf{x},t)\big)W^\varepsilon\Big)d\mathbf{x}\,dt\\ =-\int_{\mathbb{R}^2}\varphi(\cdot,0)W_0^\varepsilon\,d\mathbf{x}+\frac{8\kappa^2\nu}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2}M^\varepsilon(b^\varepsilon)\varphi\,d\mathbf{x}\,dt,\end{aligned} \tag{4.56}$$

$$\int_0^{T^*}\int_{\mathbb{R}^2}\left(\partial_t\phi+\frac{4\kappa\nu}{\nu+\kappa}\phi+(\mathbf{u}^\varepsilon\cdot\nabla)\phi\right)b^\varepsilon\,d\mathbf{x}\,dt=-\int_{\mathbb{R}^2}\phi(\cdot,0)b_0^\varepsilon\,d\mathbf{x}+\frac{2\kappa}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2}W^\varepsilon\phi\,d\mathbf{x}\,dt, \tag{4.57}$$

for every $\varphi\in C_0^2(\mathbb{R}^2\times[0,T))$ and $\phi\in C_0^1(\mathbb{R}^2\times[0,T))$. Note that the explicit dependence of the variables $(\mathbf{x},t)$ in identity (4.56) for φ indicates that the function is not affected by the time-delay of M^ε. From the estimates for W^ε and b^ε in Morrey spaces (Proposition 4.21), we deduce that the velocity is equicontinuous in space and time for $t>0$. Next, considering the non-linear term in identity (4.56)

$$I^\varepsilon:=\int_0^{T^*}\int_{\mathbb{R}^2}M^\varepsilon\Big(\big((\mathbf{u}^\varepsilon\cdot\nabla)\varphi(\mathbf{x},t)\big)W^\varepsilon\Big)d\mathbf{x}\,dt,$$

and using identity (4.22), we obtain

$$\begin{aligned}I^\varepsilon&=\frac{1}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2}M^\varepsilon\Big((\mathbf{K}*W^\varepsilon)\cdot\nabla\varphi(\mathbf{x},t)W^\varepsilon\Big)(\mathbf{x},t)d\mathbf{x}dt\\&\quad+\frac{2\kappa}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2}M^\varepsilon\Big((\mathbf{K}*M^\varepsilon(b^\varepsilon))\cdot\nabla\varphi(\mathbf{x},t)W^\varepsilon\Big)(\mathbf{x},t)d\mathbf{x}dt\\&:=\int_0^{T^*}\int_{\mathbb{R}^2\times\mathbb{R}^2}M^\varepsilon\Big(\mathbf{K}(\mathbf{x}-\mathbf{y})\nabla\varphi(\mathbf{x},t)W_{\mathbf{y}}^\varepsilon W_{\mathbf{x}}^\varepsilon\Big)(\mathbf{x},t)\\&\quad+\int_0^{T^*}\int_{\mathbb{R}^2\times\mathbb{R}^2}M^\varepsilon\Big(\mathbf{K}(\mathbf{x}-\mathbf{y})\nabla\varphi(\mathbf{x},t)M^\varepsilon(b^\varepsilon)_{\mathbf{y}}W_{\mathbf{x}}^\varepsilon\Big)(\mathbf{x},t).\end{aligned}$$

By the estimates in Morrey spaces, there are no concentrations in the diagonal $(x = y)$ towards Dirac masses at the limit of $M^{\varepsilon}\bigg((\mathbf{K}(x-y)\nabla\varphi(x,t)W_y^{\varepsilon}W_x^{\varepsilon}\bigg)(x,t)$, in compact sets of $\mathbb{R}^2 \times \mathbb{R}^2 \times (0, T^*)$, as $\varepsilon \to 0$. The same idea applies to the second term of I^{ε}. Then, the following convergence property for the non-linear terms

$$\int_0^{T^*}\int_{\mathbb{R}^2} M^{\varepsilon}\Big(((\mathbf{u}^{\varepsilon}\cdot\nabla)\varphi(x,t))W^{\varepsilon}\Big)\, dx\, dt \to \int_0^{T^*}\int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla)\varphi\, dW$$

hold true, see (NIETO; POUPAUD; SOLER, 2001; SCHOCHET, 1995) for similar argument. Passing to the limit in the linear terms does not present additional difficulty. Thus, we have proved the proposition. □

The nonconcentration argument from the preceding proof and the convolution properties in Morrey spaces with singular kernels allow us to deduce the following result.

Corollary 4.25. *If* $W_0 \in \mathcal{M}(\mathbb{R}^2)$ *and* $b_0 \in \mathcal{M}(\mathbb{R}^2) \cap \mathcal{M}^p(\mathbb{R}^2)$, *with* $p > 2$, *then the system* (4.7) *has a global weak solution such that for any* $T > 0$,

$$\{(\nu+\kappa)t\}^{1/2}\mathbf{u} \in L^{\infty}(0,T; L^{\infty}(\mathbb{R}^2)).$$

The uniqueness and stability of solution are entirely analogous to the previous case. This complete the proof of Theorem 4.2.

5 ADDITIONAL COMMENTS AND FUTURE WORKS

We briefly present some ongoing projects and future works that naturally arise from what was discussed in the previous chapters.

It is of great importance for the formulation of the initialization problem as an optimal control problem that the model be well posed, in the sense of Hadamard, in order to ensure the map S, defined in section 3.1, to be well defined. In this sense, one may seek conditions ensuring good properties for the map S in order to extend the results from chapter 3 for the variable density case. Obtaining analogous results for the three dimensional case is also a natural question. We intend to address these in a near future.

In the introduction of chapter 4, we have mentioned the case of vortex sheets as a motivation to consider initial data in Morrey spaces. Classically, this configuration is associated with vorticity. Indeed, as mentioned before, let δ_S be the Dirac measure located on the curve S in $\mathbb{R}^2$, with no end points, parametrized by a piecewise C^1 function $\zeta : I \to \mathbb{R}^2$. Here, I is an open real interval and α a function on the curve S which represents its density or strength. In the case of a vortex sheet structure of vorticity, the initial data is a measure of the type $\alpha\delta_S$. These measures belongs to $\mathcal{M}^s(\mathbb{R}^2)$, where s depends on the regularity of α and on the regularity of the tangent τ to S according to the following result.

Lemma 5.1. *If the initial vorticity ω_0 is given by $\alpha\delta_S$ and $\tau\alpha \in L^p(S)^2$, with $p \geqslant 1$, then $\omega_0 \in \mathcal{M}^s(\mathbb{R}^2)$, with $s = \dfrac{2p'}{2p'-1}$ and p' is defined as $\frac{1}{p'} + \frac{1}{p} = 1$.*

Proof. By the definition of the Dirac delta on a curve, we have

$$\langle \alpha\delta_S, \boldsymbol{\varphi} \rangle = \int_I \boldsymbol{\varphi}(\zeta(\xi)) \cdot \tau(\zeta(\xi))\alpha(\zeta(\xi))\, d\xi, \ \forall\ \boldsymbol{\varphi} \in C_0(\mathbb{R}^2)^2,$$

where $\cdot$ is the inner product in $\mathbb{R}^2$. Therefore

$$\begin{aligned} TV_{B(x,R)}(\alpha\delta_S) &\leqslant \int_{\zeta(I)\cap B(x,R)} |\boldsymbol{\varphi}(\zeta(\xi))|\,|\tau(\zeta(\xi))\alpha(\zeta(\xi))|\, d\xi \\ &\leqslant \|\boldsymbol{\varphi}\|_{L^\infty}\|\tau\alpha\|_{L^p}\, |\zeta(I)\cap B(x,R)|^{1/p'} \leqslant \|\boldsymbol{\varphi}\|_{L^\infty}\|\tau\alpha\|_{L^p} R^{1/p'}. \end{aligned}$$

Then, for $s' = 2p'$, we have $\dfrac{1}{s} + \dfrac{1}{s'} = 1$ and conclude that

$$TV_{B(x,R)}(\alpha\delta_S) \leqslant \|\boldsymbol{\varphi}\|_{L^\infty}\|\tau\alpha\|_{L^p} R^{2/s'}$$

and $\alpha\delta_S \in \mathcal{M}^s(\mathbb{R}^2)$. $\square$

Note that in the particular case of Lipschitz sheets, $p = \infty$, the initial vorticity belongs to $\mathcal{M}^2(\mathbb{R}^2)$. Also, we can consider extending the structure of vortex sheet to the initial configuration of W, which is included in the functional framework studied above. This possibility involves not only the vorticity, but also affects the microstructure represented by b. Our goal is to analize if there are persistent conditions in time for this structure and then to study its dynamics in the microscale associated with b. This study is ongoing.

In the context discussed in chapter 4, a natural research line is the evolution of configurations of constant vorticity in a domain (vortex patches). Despite the complexity of the motion and the deformation process that the vorticity undergoes, these special vortices subsist without any deformation and keep their shape during the motion. In the framework of bidimensional Euler equations, the first known example in the literature goes back to Kirchhoff who discovered that a vorticity uniformly distributed inside an elliptic shape performs a uniform rotation about its center with constant angular velocity. Notice that the solutions of vortex patch type have motivated important mathematical achievements in recent years (BURBEA, 1982; CHEMIN, 1993; DEEM; ZABUSKY, 1978). The vanishing viscosity limit has been one of the key tools in the analysis of Euler's equations from Navier-Stokes results. In this scenario, we can study the evolution of vortex patches. This a very typical problem in the framework of Euler's equations in fluid mechanics (SUEUR, 2015). The question to be raised concerns the existence of vortex patch-type structures in micropolar fluids, of course in the vanishing viscosity regime. In this context, the objective is related with the very challenging problem on the inviscid limit of incompressible Navier-Stokes equations. In our case, the occurring spin up of the flow created by the rotating particles may lead to a decrease of the effective viscosity. So, the additional microrotation viscosity may be negative and could compensate the viscosity of the fluid[1].

[1] It seems that, at least, in the case of ferrofluids the microrotation viscosity could be negative and cancel the overall viscosity.

REFERENCES

AMBROSIO, L. Transport equation and cauchy problem for non-smooth vector fields. In: ______. *Calculus of Variations and Nonlinear Partial Differential Equations: With a historical overview by Elvira Mascolo*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008. p. 1–41. ISBN 978-3-540-75914-0. Available at: <https://doi.org/10.1007/978-3-540-75914-0_1>.

BÉJAR-LÓPEZ A.; CUNHA, C.; SOLER, J. Two-dimensional incompressible micropolar fluids model with singular initial data, to appear.

BEN-ARTZI, M. Global solutions of two-dimensional navier-stokes and euler equations. *Arch. Rational Mech. Anal.*, v. 128, n. 4, p. 329–358, 1994. Available at: <https://doi.org/10.1007/BF00387712>.

BEN-ARTZI M.; CROISILLE, J.-P.; FISHELOV, D. *Navier-Stokes equations in planar domains*. [S.l.]: Imperial College Press, London, 2013. ISBN 978-1-84816-275-4.

BENFATTO, G.; ESPOSITO, R.; PULVIRENTI, M. Planar navier-stokes flow for singular initial data. *Nonlinear Analysis: Theory, Methods & Applications*, Elsevier, v. 9, n. 6, p. 533–545, 1985.

BOLDRINI, J. L.; DURAN, M.; ROJAS-MEDAR, M. A. Existence and uniqueness of strong solution for the incompressible micropolar fluid equations in domains of $\mathbb{R}^3$. *ANNALI DELL UNIVERSITA DI FERRARA*, Springer, v. 56, n. 1, p. 37–51, 2010.

BREZIS, H. *Functional analysis, Sobolev spaces and partial differential equations*. [S.l.]: Springer Science & Business Media, 2010.

BURBEA, J. Motions of vortex patches. *Letters in Mathematical Physics*, Springer, v. 6, n. 1, p. 1–16, 1982.

CAFFARELLI, L.; KOHN, R.; NIRENBERG, L. Partial regularity of suitable weak solutions of the navier-stokes equations. *Communications on pure and applied mathematics*, Wiley Online Library, v. 35, n. 6, p. 771–831, 1982.

CARA, E. F. A review of basic theoretical results concerning the navier-stokes and other similar equations. *Boletín de la Sociedad Española de Matemática Aplicada, 32, 45-73.*, Sociedad Española de Matemática Aplicada, 2005.

CHEMIN, J.-Y. Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. In: *Annales scientifiques de l'Ecole normale supérieure*. [S.l.: s.n.], 1993. v. 26, n. 4, p. 517–542.

CHORIN, A. J. The evolution of a turbulent vortex. *Communications in Mathematical Physics*, Springer, v. 83, n. 4, p. 517–535, 1982.

CHORIN, A. J.; MARSDEN, J. E. *A mathematical introduction to fluid mechanics*. [S.l.]: Springer, 1990.

COTTET, G.-H. Équations de navier-stokes dans le plan avec tourbillon initial mesure. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, v. 303, n. 4, p. 105–108, 1986.

COTTET, G.-H. A new approach for the analysis of vortex methods in two and three dimensions. In: ELSEVIER. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*. [S.l.], 1988. v. 5, n. 3, p. 227–285.

COTTET, G.-H.; SOLER, J. Three-dimensional navier-stokes equations for singular filament initial data. *Journal of differential equations*, Elsevier, v. 74, n. 2, p. 234–253, 1988.

DEEM, G. S.; ZABUSKY, N. J. Vortex waves: Stationary" v states," interactions, recurrence, and breaking. *Physical Review Letters*, APS, v. 40, n. 13, p. 859–862, 1978.

DONG, B.-Q.; CHEN, Z.-M. Asymptotic profiles of solutions to the 2d viscous incompressible micropolar fluid flows. *Discrete & Continuous Dynamical Systems-A*, American Institute of Mathematical Sciences, v. 23, n. 3, p. 765–784, 2009.

DONG, B.-Q.; LI, J.; WU, J. Global well-posedness and large-time decay for the 2d micropolar equations. *Journal of Differential Equations*, Elsevier, v. 262, n. 6, p. 3488–3523, 2017.

DONG, B.-Q.; ZHANG, Z. Global regularity of the 2d micropolar fluid flows with zero angular viscosity. *Journal of Differential Equations*, Academic Press, v. 249, n. 1, p. 200–213, 2010.

ELLAHI, R.; RAHMAN, S.; GULZAR, M. M.; NADEEM, S.; VAFAI, K. A mathematical study of non-newtonian micropolar fluid in arterial blood flow through composite stenosis. *Applied Mathematics & Information Sciences*, Citeseer, v. 8, n. 4, p. 1567——1573, 2014.

ERINGEN, A. C. Simple microfluids. *International Journal of Engineering Science*, Elsevier, v. 2, n. 2, p. 205–217, 1964.

ERINGEN, A. C. Theory of micropolar fluids. *Journal of Mathematics and Mechanics*, JSTOR, p. 1–18, 1966.

FERRARI, C. On lubrification with structured fluid. *Appl. Anal*, v. 15, p. 127—-146, 1983.

FISHELOV, D. Vortex schemes for viscous flow'. *Lectures in Applied Mathematics, Vortex Dynamics and Vortex Methods*, v. 28, p. 153–164, 1991.

GAL-CHEN, T. A method for the initialization of the anelastic equations: Implications for matching models with observations. *Monthly Weather Review*, v. 106, n. 5, p. 587–606, 1978.

GALDI, G. P.; RIONERO, S. A note on the existence and uniqueness of solutions of the micropolar fluid equations. *International Journal of Engineering Science*, Pergamon, v. 15, n. 2, p. 105–108, 1977.

GALLAGHER, I.; GALLAY, T. Uniqueness for the two-dimensional navier–stokes equation with a measure as initial vorticity. *Mathematische Annalen*, Springer, v. 332, n. 2, p. 287–327, 2005.

GALLAGHER, I.; GALLAY, T.; LIONS, P.-L. On the uniqueness of the solution of the two-dimensional navier–stokes equation with a dirac mass as initial vorticity. *Mathematische Nachrichten*, Wiley Online Library, v. 278, n. 14, p. 1665–1672, 2005.

GALLAY, T.; WAYNE, C. E. Global stability of vortex solutions of the two-dimensional navier-stokes equation. *Communications in mathematical physics*, Springer, v. 255, n. 1, p. 97–129, 2005.

GIGA, M.-H.; GIGA, Y.; SAAL, J. *Nonlinear partial differential equations: Asymptotic behavior of solutions and self-similar solutions*. [S.l.]: Springer Science & Business Media, 2010.

GIGA, Y.; MIYAKAWA, T. Navier-stokes flow in r3 with measures as initial vorticity and morrey spaces. *Communications in partial differential equations*, Taylor & Francis, v. 14, n. 5, p. 577–618, 1989.

GIGA, Y.; MIYAKAWA, T.; OSADA, H. Two dimensional navier stokes flow with measures as initial vorticity. *Arch. Rational Mech. Anal.*, v. 104, n. 3, p. 223—–250, 1986.

JACOBSON, M. Z. *Fundamentals of atmospheric modeling*. [S.l.]: Cambridge university press, 2005.

LEDIMET, F.-X.; TALAGRAND, O. Variational algorithms for analysis and assimilation of meteorological observations: theoretical aspects. *Tellus A: Dynamic Meteorology and Oceanography*, Taylor & Francis, v. 38, n. 2, p. 97–110, 1986.

LIONS, J. L. Optimal control of systems governed by partial differential equations. Springer Verlag, v. 170, 1971.

LORENC, A. C. Analysis methods for numerical weather prediction. *Quarterly Journal of the Royal Meteorological Society*, v. 112, p. 1177—–1194, 1986.

LUKASZEWICZ, G. On nonstationary flows of incompressible asymmetric fluids. *Math. Methods Apply. Sci.*, v. 13, n. 3, p. 219—–232, 1990.

LUKASZEWICZ, G. *Micropolar Fluids: Theory and Applications*. [S.l.]: Birkhauser, Boston, 1999.

LUKASZEWICZ, G. Long time behavior of 2d micropolar fluid flows. *Math. Comput. Modelling*, v. 34, n. 5-6, p. 487—–509, 2001.

LUKASZEWICZ, G. Asymptotic behavior of micropylar fluid flows. *Internat. J. Engrg. Sci.*, v. 41, n. 3-5, p. 259–269, 2003.

LUKASZEWICZ, G.; SADOWSKI, W. Uniform attractor for 2d magneto-micropolar fluid flow in some unbounded domains. *Z. Angew. Math. Phys.*, v. 55, n. 2, p. 247—–257, 2004.

MAJDA, A. Vorticity and the mathematical theory of incompressible fluid flow. *Communications on Pure and Applied Mathematics*, Wiley Online Library, v. 39, n. S1, p. 187–220, 1986.

MAJDA, A. J.; BERTOZZI, A. L.; OGAWA, A. Vorticity and incompressible flow. cambridge texts in applied mathematics. *Appl. Mech. Rev.*, v. 55, n. 4, p. B77–B78, 2002.

MICHALSKI, M. W. *Derivatives of noninteger order and their applications*. [S.l.]: Polska Akademia Nauk, Institut Matematyczny Warszawa, Poland, 1993.

NIETO, J.; POUPAUD, F.; SOLER, J. High-field limit for the vlasov-poisson-fokker-planck system. *Archive for rational mechanics and analysis*, Springer, v. 158, n. 1, p. 29–59, 2001.

NOTTE-CUELLO, E.; ROJAS-MEDAR, M. A.; SANTOS, M. *A study of the initialization problem for the magnetohydrodynamic type equations*. [S.l.: s.n.], 2002. 39–49 p.

ORTEGA-TORRES, E.; ROJAS-MEDAR, M. A.; CABRALES, R. A uniform error estimate in time for spectral galerkin approximations of the magneto-micropolar fluid equations. *Numerical Methods for Partial Differential Equations*, Wiley Online Library, v. 28, n. 2, p. 689–706, 2012.

ORTEGA-TORRES, E. E.; ROJAS-MEDAR, M. A. Magneto-micropolar fluid motion: global existence of strong solutions. In: HINDAWI. *Abstract and Applied Analysis*. [S.l.], 1999. v. 4.

ROJAS-MEDAR, M. A. Magneto-micropolar fluid motion: existence and uniqueness of strong solution. *Mathematische Nachrichten*, Wiley Online Library, v. 188, n. 1, p. 301–319, 1997.

ROJAS-MEDAR, M. A. Magneto-micropolar fluid motion: existence of weak solution. *Rev. Mat. Complut.*, v. 11, n. 2, p. 443——460, 1998.

ROJAS-MEDAR, M. A.; BOLDRINI, J. L. Magneto-micropolar fluid motion: On the convergence rate of the spectral galerkin approximation. *ZAMM - Journal of Applied Mathematics and Mechanics*, v. 77, n. 10, p. 723——732, 1997.

SCHOCHET, S. The weak vorticity formulation of the 2-d euler equations and concentration-cancellation. (english summary). *Comm. Partial Differential Equations*, v. 20, n. 5-6, p. 1077——1104, 1995.

SILVA, P. B. e; CRUZ, F. W.; FREITAS, L. B. S.; ZINGANO, P. R. On the l^2 decay of weak solutions for the 3d asymmetric fluids equations. *Journal of Differential Equations*, Elsevier, v. 267, n. 6, p. 3578–3609, 2019.

SILVA, P. B. e; CRUZ, F. W.; ROJAS-MEDAR, M. A. Vanishing viscosity for non-homogeneous asymmetric fluids in $\mathbb{R}^3$: the l^2 case. *Journal of Mathematical Analysis and Applications*, Elsevier, v. 420, n. 1, p. 207–221, 2014.

SILVA, P. B. e; CUNHA, C.; ROJAS-MEDAR, M. A. The initialization problem for the equations of incompressible asymmetric fluids. *Applied Mathematics & Optimization*, Springer, p. 1–24, 2020.

SILVA, P. B. e; FERNANDEZ-CARA, E.; ROJAS-MEDAR, M. A. Vanishing viscosity for non-homogeneous asymmetric fluids in $\mathbb{R}^3$, journal=Journal of mathematical analysis and applications, volume=332, number=2, pages=833–845, year=2007, publisher=Elsevier.

SILVA, P. B. e; FRIZ, L.; ROJAS-MEDAR, M. Exponential stability for magneto-micropolar fluids. *Nonlinear Analysis: Theory, Methods & Applications*, Elsevier, v. 143, p. 211–223, 2016.

SOLER, J. stability for weak solutions of the navier-stokes equations in r3 with sin- gular initial data in morrey spaces. *Journal of mathematical Analysis and Applications*, v. 187, p. 513——525, 1994.

STEIN, E. M. *Singular integrals and differentiability properties of functions. Princeton Mathematical Series*. [S.l.]: Princeton University Press, 1970.

SUEUR, F. Viscous profiles of vortex patches. *Séminaire Équations aux dérivées partielles (Polytechnique)*, p. 1–11, 2015.

SZOPA, P. On existence and regularity of solutions for 2-d micropolar fluid equations with periodic boundary conditions. *Mathematical methods in the applied sciences*, Wiley Online Library, v. 30, n. 3, p. 331–346, 2007.

TADMOR, E. On a new scale of regularity spaces with applications to euler's equations. *Nonlinearity*, IOP Publishing, v. 14, n. 3, p. 513, 2001.

TEMAM, R. Navier-stokes equations and nonlinear functional analysis (book). In: *Philadelphia, PA, Society for Industrial and Applied Mathematics(CBMS-NSF Regional Conference Series in Applied Mathematics*. [S.l.: s.n.], 1983.

TEMAM, R.; FORSTE, J. Navier-stokes equations: Amsterdam-new york. north-holland publishing company (studies in mathematics and its applications 2). *ZaMM*, v. 59, n. 9, 1979.

WHITE, L. W. A study of uniqueness for the initialization problem for burgers equation. *Journal of mathematical analysis and applications*, Elsevier, v. 172, n. 2, p. 412–431, 1993.

YAMAGUCHI, N. Existence of global strong solution to the micropolar fluid system in a bounded domain. *Mathematical Methods in the Applied Sciences*, Wiley Online Library, v. 28, n. 13, p. 1507–1526, 2005.

ZHOU, J. A study of the initialization problem for the navier-stokes equations. *Journal of mathematical analysis and applications*, Academic Press, v. 190, n. 3, p. 678–696, 1995.