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CONTROLLABILITY FOR SOME EQUATIONS FROM FLUID MECHANICS

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CONTROLLABILITY FOR SOME EQUATIONS FROM FLUID MECHANICS

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I dedicate this thesis to my parents.

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ABSTRACT

In this thesis we present controllability results for some models of fluid mechanics. More precisely, we investigate the existence of controls that drive the solution of the system from an initial state to a prescribed final state in a given positive time. In the first Chapter, the controllability of the Stokes equation with memory is analyzed. This model is a variant of the well-known Stokes equation, with the addition of a non-local term in time building a memory effect in the equation. This model can also be seen as a linearization around zero of an Oldroyd kind viscoelastic fluid system. We prove that the result of null controllability for this equation is not true, even if the control acts over the whole boundary. To this purpose, it is verified that the corresponding observability inequality is not satisfied. We also build explicit initial data such that, for any control, the corresponding solution is different from zero at final time. The second Chapter is dedicated to the controllability of fluids in which thermal effects are important. We prove the exact controllability to the trajectories of a coupled system of the Boussinesq type, for a fluid satisfying boundary conditions of the Navier kind for the velocity and of the Robin kind for the temperature. The control acts on a part of the boundary. First, using a domain extension procedure, we transform the problem into to distributed controllability problem. Then, we prove an approximate global controllability result, following the strategy of Coron et al [J. EUR. Mathematics. Soc., 22 (2020), pp. 1625-1673]. Through linearization and using appropriate Carleman estimates, we conclude with a local control result.

Keywords: Stokes fluids with memory. Boussinesq systems. Control of fluids.

RESUMO

Nesta tese apresentamos resultados de controlabilidade para alguns modelos da mecânica dos fluidos. Mais precisamente, investigamos a existência de controles que conduzem a solução do sistema de um estado inicial à um estado final prescrito em um tempo positivo dado. No primeiro Capítulo é analisada a controlabilidade da equação de Stokes com memória. Este modelo é uma variante da conhecida equação de Stokes, com o acréscimo de um termo não local em tempo criando um efeito de memória na equação. Este modelo também pode ser visto como uma linearização entorno a zero de um sistema de fluido viscoelástico do tipo Oldroyd. Provamos que o resultado de controlabilidade nula para esta equação não é verdadeiro, mesmo se o controle atuar sobre toda a fronteira. Para isso, verifica-se que a desigualdade de observabilidade correspondente não é satisfeita. Também construimos um dado inicial explícito tal que, para qualquer controle, a solução correspondente é diferente de zero no tempo final. O segundo Capítulo é dedicado à controlabilidade de fluidos nos quais os efeitos térmicos são importantes. Provamos a controlabilidade exata à trajetórias de um sistema acoplado do tipo Boussinesq, para um fluido satisfazendo condições de fronteira do tipo Navier para o campo velocidade e do tipo Robin para a temperatura. O controle atua sobre uma parte da fronteira. Primeiro, usando um argumento de extensão de domínio passamos a um problema de controle distribuído. Então, provamos um resultado global de controlabilidade aproximada, seguindo a estratégia de Coron et al [J. EUR. Mathematics. Soc., 22 (2020), pp. 1625-1673]. Por meio de linearização e usando estimativas de Carleman apropriadas, concluimos com um resultado de controle local.

Palavras-chave: Fluidos de Stokes com memória. Sistemas de Boussinesq. Controle de fluidos.

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1 INTRODUCTION

The main aims of this thesis are the analysis and control (in a sense that we will soon define) of some boundary/initial value problems for partial differential equations originating from fluid mechanics. Throughout this introduction we will present a description of such equations, we will provide a short historical overview of the related control theory and we will describe precisely the problems under study.

1.1 CONTROL THEORY

1.1.1 Generalities

Humanity has always been interested in studying the behavior of certain phenomena in nature. Thus, a question that arises naturally is the possibility of influencing in order to obtain a desired behavior. Once these are understood and represented mathematically, a large number of tools and methods can be applied and it is at this point where the control theory is based.

Looking at history, we realize that already in the design of Roman aqueducts there are elements of the control theory, where they used a system of valves to maintain a constant level of water. At the end of the 17th century, Ch. Huygens and R. Hooke studied the oscillation of the pendulum. These works were then adapted to regulate the speed of windmills and present elements of what we now know as control theory. J. Watt adapted this principle to the steam engine, thus making a crucial contribution to the industrial revolution. In this mechanism, the objective was to control the speed in order to remain approximately constant. When the speed of the spheres increased, one or more of them unblocked some valves, decreasing the pressure and reducing the speed; consequently, the spheres covered the valves again and the speed increased. This self-regulating mechanism is also features of control problems. In 1868, the physicist J. C. Maxwell gave the first mathematical description of the system presented by Watt, explaining some of the somewhat erratic behavior observed in the machines of the time and proposed several control devices.

During the industrial revolution, the ideas of what is now called control theory were taking shape and becoming more and more present. In the 1930s, control engineering formed an important part of the complex systems engineering network. At the same time, there has been an important advance in everything related to automatic control and design and analysis techniques, with several applications. During the Second World War and the years that followed, engineers and scientists had to improve their experience with control mechanisms in the aircraft and anti-aircraft missile segment.

After 1960, the methods described above began to be collected in a systematic way.

This is the origin of "classic" control theory. However, since the Second World War, it is clear that the models considered so far are not sufficiently to completely describe the complexity of real world, which often has a non-linear and non-deterministic behavior.

The fundamental pillars of research in control theory of the last decades have been given by the contributions of R. Bellman in the context of dynamic programming, R. Kalman in the techniques of filtering and algebraic approximation to linear systems, L. Pontryagin with the so called maximum principle for problems of non-linear optimal control and J.-L. Lions with connections of EDPs, numerical analysis and applications.

1.1.2 Controllability problem

Control theory is an area of mathematics that studies the behavior of systems that can be governed through commands (also called controls), applied in practice through actuators. Control theory has become a rich interdisciplinary branch of mathematics, with applications in many areas: engineering, biology, physics, economics, medicine, etc.

A *control system* is a dynamical system, where we have a state (the unknown system variable that we want to control) and a control (the variable that can be freely chosen to act on the system). The goal is to find a control such that the associated state behaves appropriately. In particular, if this appropriate behavior is associated to the value of the state at a given instant of time, we say that we are considering a controllability problem.

To present some concepts of control theory, consider the system

$$\begin{cases} u_t = F(t, u, v), \\ u(x, 0) = u_0, \end{cases} \tag{1.1}$$

where $T > 0$ and represents the time, $u : [0, T] \mapsto \mathcal{H}$ represents state (the variable that we want to control) and the variable $v : [0, T] \mapsto \mathcal{U}$ which represents the control. Here, $\mathcal{H}$ and $\mathcal{U}$ are suitable Banach spaces. Thus, the general question is:

> *Fixed a time $T > 0$ and an initial state u_0, is it possible to find a control v such that the associated solution of* (1.1) *with $u(\cdot, 0) = u_0$ fulfills a desired property?*

Different notions of controllability can be identified and we will present some, among several, present in the literature:

Exact controllability. The control system (1.1) is *exactly controllable* at time $T > 0$ if, for every u_0, $u_T \in \mathcal{H}$, there exists v such that the associated solution to system (1.1) satisfies

$$u(\cdot, T) = u_T.$$

This means that the system can be conducted from any initial state to any target.

Exact controllability to the trajectory. The control system (1.1) is said to be *controllable to the trajectories* at time $T > 0$ if for all $u_0 \in \mathcal{H}$ and any trajectory $(\overline{u}, \overline{v})$ of the system (1.1), there exists a control function v such that the associated solution satisfies

$$u(\cdot, T) = \overline{u}(\cdot, T).$$

This interesting concept means that it is possible to drive any initial state to any prescribed trajectory of the system.

Null controllability. The control system (1.1) is said to be *null controllable* at time $T > 0$ if for all $u_0 \in \mathcal{H}$, there exists a control function v such that the associated solution to system (1.1) satisfies

$$u(\cdot, T) = 0.$$

This means that any initial state can be steered to zero equilibrium.

Approximate controllability. The control system (1.1) is *approximately controllable* at time $T > 0$ if, for every u_0, $u_T \in \mathcal{H}$ and for every $\varepsilon > 0$, there exists v such that the associated solution to system (1.1) satisfies

$$\|u(\cdot, T) - u_T\|_{\mathcal{H}} \leq \varepsilon.$$

This means that, it is possible to steer any initial state to an arbitrarily neighborhood of any target, in some suitable topology.

Let us also remark that for linear systems, null controllability and controllability to the trajectories are equivalents. On the contrary, note that even for linear systems, approximate controllability and exact controllability differ: an affine subspace of an infinite dimensional space can be dense without filling all the space.

The controllability of PDEs is an important area of research and has been the subject of several studies in recent years. The controllability analysis for a given linear PDE is equivalent to obtaining an observability inequality for the adjoint system. This is an estimate that furnishes knowledge of the solution at a given time using only local measurements of it (this property establishes a kind of duality between controllability and observability). However, it is worth mentioning that the proof of such inequalities requires tools adapted to the analyzed PDEs; for example: Ingham-type inequalities, multiplier methods, micro-local analysis or Carleman-like inequalities.

Among the various methodologies, a very flexible tool is Carleman inequalities. These are weighted energy estimates for the solutions to PDEs with weights of exponential growth at final time . They were introduced with the goal to quantify unique continuation. Over the last few years, the field of applications of Carleman inequalities went beyond its original purpose: currently, they are also used in the framework of inverse problems, stabilization of PDEs, numerical calculation of control problem solutions, etc.

However, the arguments for proving controllability results for nonlinear problems are much more complicated. Actually, the techniques are based on one of the following two arguments: rewrite the problem as a fixed-point equation and then apply a Fixed-Point Theorem or linearize at a well chosen control-state pair and then apply an Inverse Function Theorem. A very relevant fact is that most of the control results for nonlinear problems are of the local type.

1.2 SOME EQUATIONS IN FLUID MECHANICS

The mathematical theory of fluid dynamics began in the 17th Century with the work of Newton, who applied his newly developed laws of mechanics to the movement of fluids. Shortly afterwards, in 1755, Euler wrote the partial differential equations that describe the movement of an ideal fluid, that is, a fluid where viscous effects are not present (an inviscid fluid; these equations are known as Euler equations). Later, C.H. Navier (1822) and, independently, G.G. Stokes (1845) considered a viscous fluid and then obtained what we now call the Navier-Stokes equations.

The Euler and Navier-Stokes equations stand out for physically describing a large number of phenomena of natural and industrial interests. For instance, they are used to model the motion of ocean currents, water waves, estuaries, lakes and rivers, of stars inside and outside the galaxy, flow around car and airplane airfoils, smoke spread in fires and in industrial chimneys. They are also used directly in the study of blood flow, in the design of aircraft, cars, hydroelectric plants and marine hydraulics, in the analysis of the effects of water pollution in lakes, oceans and in the study of the dispersion of air pollution, etc.

Some postulates are needed to deduce the Navier-Stokes equations. The first one is that a fluid is a continuous medium, that is, it does not contain voids, such as bubbles dissolved in a gas. Another postulate is that all variables of interest such as pressure, velocity, density, temperature, etc., are differentiable functions of space and time.

Basic principles of conservation of mass and momentum are used to obtain these equations. Sometimes it is necessary to consider an arbitrarily finite volume, called the control volume, over which these principles can be easily applied. This volume is represented by Ω and its containment surface by Γ. The control volume remains fixed in space or can move like the fluid. This leads, however, to special considerations, as we show below (the contents of the following paragraphs have been taken from references are (CHORIN; MARSDEN, 1990; FERNANDEZ-CARA, 2012; JOSEPH, 2013)).

The fundamental problem in continuum mechanics

To fix ideas, let us adopt the viewpoint of a physicist. Thus, let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded connected open set with $\Gamma = \partial\Omega$ and let $T > 0$ be given. It is assumed that a continuum occupies the set Ω during the time interval $(0, T)$. This means that the medium under study is composed of particles and that there exist sufficiently regular functions ρ and $u = (u_1, \cdots, u_n)$ such that the following holds:

1. The mass of the particles of the medium whose positions at time t are points of the open set $W \subset \Omega$ is given by
$$m(W, t) = \int_W \rho(x, t)\, dx.$$
2. The volume of particles of the medium whose positions at time t are points of the open set $W \subset \Omega$ is given by
$$V(W, t) = \int_W dx.$$
3. The linear momentum associated to the particles in $W \subset \Omega$ at time t is
$$L(W, t) = \int_W (\rho u)(x, t)\, dx.$$

The functions ρ and u are the mass density and the velocity field, respectively. For each t, the functions $\rho(\cdot, t)$ and $u(\cdot, t)$ provide a complete description of the mechanical state of the medium at time t. The fundamental problem in continuum mechanics (FPM) is the following:

> *Assume that, for a given medium, the mechanical state at time $t = 0$ and the physical properties for all t are known. Then, determine the mechanical state of this medium for all t.*

In fluid mechanics there are two "canonical" coordinate systems in which the various equations of motion can be written: Lagrangian coordinates are associated with fluid particles (or fluid volume elements) as described below; by contrast, Eulerian coordinates are the coordinates of the particles with respect to fixed reference frame associated with the experiment.

Lagrangian coordinates and the Transport Lemma

In order to be more precise in the formulation of FPM, we need some tools. In particular, we have to first define a trajectory: for any $x \in \Omega$, one considers the Cauchy problem
$$\begin{cases} X_t(x, t) = u(X(x, t), t), \\ X(x, 0) = x. \end{cases}$$

We say that $t \mapsto X(x, t)$ is the trajectory of the fluid particle that was at x at time $t = 0$. We say that $(x, t) \mapsto X(x, t)$ is the flux function of the medium and the components of $X(x, t)$ are the Lagrangian coordinates at time t of the particle that was initially at x.

For any open set $W \subset \Omega$, the set

$$W_t := \{X(x,t) \, : \, x \in W\}$$

must be viewed as the set of the positions at time t of the particles of the medium that were in W at time $t = 0$.

The following result is known as the Transport Lemma.

Lemma 1 (Transport Lemma). *Assume that $h \in C^1(\Omega \times [0,T])$ is given. Let $W \subset \Omega$ and let us set*

$$H(t) := \int_{W_t} h(x,t)\, dx$$

for all $t \in [0,T]$. Then $H : [0,T] \mapsto \mathbb{R}$ is a well defined C^1 function. Moreover,

$$\frac{dH}{dt}(t) = \int_{W_t} \left(\frac{\partial h}{\partial t} + \nabla \cdot (h\, u) \right)(x,t)\, dx \quad \forall\, t \in [0,T].$$

Let us now deduce a set of PDEs that must be satisfied by the density and the velocity field.

The physical principles that lead to these equations are of two kinds: Universal laws, that are common to all media; Particular constitutive laws, only satisfied for some media.

The universal laws

- **Conservation of mass:** Let $W \subset \Omega$ be an open set. Then

$$\frac{d}{dt} \left(\int_{W_t} \rho(x,t)\, dx \right) = 0 \quad \forall\, t \in [0,T].$$

From the Transport Lemma, we easily find that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, u) = 0 \quad \text{in} \quad \Omega \times (0,T).$$

This is a first PDE for ρ and u. It is frequently called the *continuity equation.*

- **Balance of linear momentum:** Let $W \subset \Omega$ be an open set. Then

$$\frac{d}{dt} \left(\int_{W_t} (\rho u)(x,t)\, dx \right) = \mathcal{F}(W_t, t) \quad \forall\, t \in [0,T],$$

where $\mathcal{F}(W_t,t)$ is the resultant of the forces acting on the particles whose positions are in W_t at time t. Actually, this principle is the version in continuum mechanics of the famous Newton‘s second law.

The resultant $\mathcal{F}(W_t,t)$ must be split as the sum of two vectors: $\mathcal{F} = F_{ten} + F_{ext}$, where F_{ten} is the resultant of the tension (or internal) forces and F_{ext} is the resultant of all other external forces. Usually, we have that

$$F_{ten}(W_t,t) = \int_{\partial W_t} \xi \cdot \nu \, d\gamma,$$

where ν is the outward unit normal vector to the boundary, $\xi = \xi(x,t)$ is a C^1 matrix-valued function, usually called the stress tensor and, for some $F = F(x,t)$ one writes

$$F_{ext}(W_t,t) = \int_{W_t} (\rho\, F)(x,t)\, dx.$$

Taking into account the balance of linear momentum and the Transport Lemma, we can deduce the so called *equation of motion*:

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) = \nabla \cdot \xi + \rho F \quad \text{in} \quad \Omega \times (0,T).$$

- **Conservation of energy:** Let $W \subset \Omega$ be an open set. Then

$$\frac{d}{dt}\left(\int_{W_t} \frac{1}{2}\rho |u|^2 + \rho w\, dx\right) = P(W_t,t) \quad \forall\, t \in [0,T],$$

where $P(W_t,t)$ is the instantaneous power of work done by the forces acting on the particles located at the points of W_t at time t and w is the density of internal energy per unit mass.

It is usual to write that

$$\begin{aligned} P(W_t,t) &= \int_{W_t} \rho\, F \cdot u\, dx + \int_{\partial W_t} (\xi \cdot u)\cdot \nu\, d\gamma + \int_{\partial W_t} (-q \cdot \nu)\, d\gamma \\ &= \int_{W_t} \rho\, F \cdot u + \nabla \cdot (\xi \cdot u) - \nabla \cdot q\, dx, \end{aligned}$$

where $q = q(x,t)$ is a new vector-valued unknown, called the heat flux of the medium.

Arguing as before, we find the so called *energy equation*:

$$(\rho w)_t + \nabla \cdot (\rho w u) = \xi \nabla u - \nabla \cdot q \quad \text{in} \quad \Omega \times (0,T).$$

Here, we have a system of $n+2$ equations (the continuity equation, the n equations of motion and the scalar energy equation) for $n^2 + 2n + 2$ unknowns: the scalar ρ and w, the n components of u, the n components of q and the n^2 components of ξ. Of course, they are not enough, by themselves, to provide a complete description of the behavior of the media. We have to particularize and introduce additional specific laws.

A particular set of constitutive laws

- **Incompressibility** (conservation of volume): Assume that the medium is an incompressible fluid. This means that

$$\frac{d}{dt}\left(\int_{W_t} dx\right) = 0 \quad \forall\, t \in [0,T].$$

Then, thanks to the Transport Lemma, we can easily deduce the so called *incompressibility equation* for u:

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0,T).$$

Now, it is convenient to write the stress tensor ξ in the form $\xi = -p\mathrm{Id} + \sigma$, that is, $\xi_{ij} = -p\delta_{ij} + \sigma_{ij}$ for all i and j, where p is called the pressure and σ is the tangential-stress tensor (see (BATCHELOR, 1967) for more details). Thus, the motion equations reads:

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) = -\nabla p + \nabla \cdot \sigma + \rho F \quad \text{in} \quad \Omega \times (0, T). \tag{1.2}$$

• **Newtonian law:** Assume that the fluid is Newtonian. This means that, the velocity field u is of class C^2 and σ depends linearly on the spatial derivatives of u. More precisely, assume that the so called Stokes law

$$\sigma = \mu_0(\nabla u + \nabla u^t) - \frac{2}{3}\mu_0(\nabla \cdot u)\mathrm{Id}, \tag{1.3}$$

is satisfied for some pressure $p \in C^1$ and some positive constant μ_0 (the fluid dynamical viscosity).

Thus, using all the laws described above, we can arrive at the following set of equations:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0 & \text{in} \quad \Omega \times (0, T), \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) - \mu_0 \Delta u + \nabla p = \rho F & \text{in} \quad \Omega \times (0, T), \\ \nabla \cdot u = 0 & \text{in} \quad (0, T) \times \Omega. \end{cases}$$

These are the non-homogeneous incompressible Navier-Stokes equations of a fluid. We find a system of $n + 2$ PDEs for $n + 2$ unknowns.

Note that, if the fluid is homogenous, that is, $\rho(x, t) = \rho_0 > 0$, it is natural to assume that ρ_0 is known and, in this case, the system reduces to the classical Navier-Stokes system

$$\begin{cases} \rho_0(u_t + \nabla \cdot (u \otimes u)) - \mu_0 \Delta u + \nabla p = \rho_0 F & \text{in} \quad \Omega \times (0, T), \\ \nabla \cdot u = 0 & \text{in} \quad (0, T) \times \Omega. \end{cases}$$

• **Fourier law:** Assume that the heat flow q is proportional to the spatial gradient of the temperature. More precisely, assume that for some positive constant κ_0, called the heat diffusion coefficient, one has

$$q = -\kappa_0 \nabla \theta.$$

Usually, w is assumed to be a linear function of the temperature, that is,

$$w = c\,\theta$$

for some positive c, called the specific heat coefficient.

Taking into account these considerations the energy equation now takes the form:

$$c\left((\rho\theta)_t + \nabla \cdot (\rho\theta u)\right) - \kappa_0 \Delta \theta = 2\mu_0 D u \nabla u,$$

where $D = \frac{1}{2}(\nabla + \nabla^t)$ is the symmetric part of the gradient operator.

We will consider a Newtonian fluid, incompressible and homogeneous, governed by the so called Boussinesq systems. This means that the total external force F is given by $f + \theta e_n$, where $f = f(x,t)$ is a prescribed field and e_n is the n-th vector of the canonical basis in $\mathbb{R}^n$. The equations are:

$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \nabla p = f + \theta e_n & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot u = 0 & \text{in} \quad \Omega\times(0,T), \\ \theta_t - \kappa\Delta\theta + u\cdot\nabla\theta = g & \text{in} \quad \Omega\times(0,T), \end{cases}$$

where we have assumed that $\rho_0 = 1$, we have taken $\mu = \mu_0/\rho_0$ (the kinematic viscosity), $\kappa = \kappa_0/\rho_0$ and g represents a heat source.

A fluid can flow in two completely different ways depending on the viscosity. For "large" kinematic viscosity μ, fluid particles follow more or less ordered trajectories. In the case, we say that the fluid is in a laminar regime. On the other hand, for μ small enough, velocity and pressure exhibit extremely rapid variations and oscillations in time and space and we observe "chaotic" behavior in the motion of particles. In this case, we say that the flow is in a turbulent regime.

A particular case of the Navier-Stokes equations appears when the fluid is ideal and incompressible ($\mu = 0$). This leads to the Euler equations:

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla p = f & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot u = 0 & \text{in} \quad \Omega\times(0,T). \end{cases}$$

Another model related to the Navier-Stokes equations is found when we skip the non-linear term. This way, we find the Stokes equations:

$$\begin{cases} u_t - \mu\Delta u + \nabla p = f & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot u = 0 & \text{in} \quad \Omega\times(0,T). \end{cases}$$

This describes the flow of a fluid where inertial forces are small compared to viscous forces and it is a typical situation in flows where fluid velocities are very slow, viscosities are very large or flow length scales are very small. This type of flow was first studied to understand lubrication. In Nature, this behavior is found in the swimming of microorganisms and sperm and, also, in lava flow problems. In technology, it occurs in ink, MEMS devices and in the flow of viscous polymers in general.

The Newtonian fluid model has been used frequently for incompressible viscous fluids allowing the description of moderate velocity flows. However, even before the XIX Century, it was known that there were incompressible and viscous fluids not subject to the Newtonian equations defined in (1.3). There are many fluids with this complex microstructure (biological fluids, polymeris, suspensions and liquid crystals, etc.) frequently used in industrial processes, that show a viscoelastic behavior that cannot be described by the classical Newtonian model.

The first models that took into account the shear rate history of fluids, subsequently called *linear viscosity fluids*, were proposed by Maxwell in (MAXWELL, 1867), Kelvin in (KELVIN, 1875), Voigt in (VOIGT, 1890) and developments analyzed in the 1950's to a considerable extent by Oldroyd in (OLDROYD, 1950) and (OLDROYD, 1953).

The first to find a relation between the stress and strain tensors was Maxwell, which claimed that it is possible to construct a nonlinear invariant theory of fluids based on springs and dampers. In fact, the system formed by a spring and a damper in series is called a Maxwell element, as shown below.

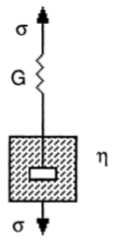

Figure 1 – Maxwell element

The spring constant is called G and the force σ_1 in the spring is $G\lambda_1$, where λ_1 is the displacement of the spring. The force σ_2 in the damper is $\eta\frac{\partial\lambda_2}{\partial t}$, where η is the viscosity and $\frac{\partial\lambda_2}{\partial t}$ is the velocity (the time rate of change of λ_2) and due to the fact that they are in series, one has $\sigma_1 = \sigma_2 = \sigma$.

We also have that the time rate of change of the total displacement is

$$\frac{\partial\lambda}{\partial t} = \frac{\partial\lambda_1}{\partial t} + \frac{\partial\lambda_2}{\partial t} = \frac{1}{G}\frac{\partial\sigma_1}{\partial t} + \frac{\sigma_2}{\eta} = \frac{1}{G}\frac{\partial\sigma}{\partial t} + \frac{\sigma}{\eta}.$$

Hence,

$$\alpha\frac{\partial\sigma}{\partial t} + \sigma = \eta\frac{\partial\lambda}{\partial t}, \tag{1.4}$$

where $\alpha = \frac{\eta}{G}$ is a relaxation time. Another expression for σ is

$$\sigma = \frac{\eta}{\alpha}\int_{-\infty}^{t} \exp\left(\frac{-(t-\tau)}{\alpha}\right)\frac{\partial\lambda(\tau)}{\partial\tau}\,d\tau. \tag{1.5}$$

Obviously (1.4) is a differential equation model for the relation between force and deformation and (1.5) is an integral equation showing that the present value of the force $\sigma(t)$ is determined by the history of λ.

In the one-dimensional case, let $\lambda = \frac{\partial\xi}{\partial x}$ be the strain, where $\frac{\partial\xi}{\partial t} = u$ is the velocity. Then,

$$\frac{\partial\lambda}{\partial t} = \frac{\partial u}{\partial x}$$

and from (1.4) one has

$$\alpha \frac{\partial \sigma}{\partial t} + \sigma = \eta \frac{\partial u}{\partial x}.$$

Solving this differential equation for σ and replacing in (1.2) (considering homogeneous fluid) we have Maxwell's fluid.

Also interesting is the Voigt (or Kelvin) model, where a spring and the damper are in parallel, as in the figure below.

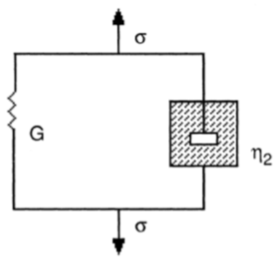

Figure 2 – Voigt element

The force in the elastic element is $G\lambda$ and the force in the viscous element is $\eta_2 \frac{\partial \lambda}{\partial t}$; and when in parallel, they are:

$$\sigma = G\lambda + \eta_2 \frac{\partial \lambda}{\partial t}.$$

This element is instantaneously viscous because, by design, the deformation λ and the rate of deformation $\frac{\partial \lambda}{\partial t}$ of the spring and damper occur simultaneously. The damper must work in each and every deformation. If a constant force is applied, the deformation will be damped by viscosity and the system will come to equilibrium with $\sigma = G\lambda$, as in an elastic body.

The Voigt model is "good" for viscoelastic solids and not for fluids. Thinking about it, Jeffreys (JEFFREYS, 1929) proposed a model which consider a damper and a Voigt model in series, according to the scheme:

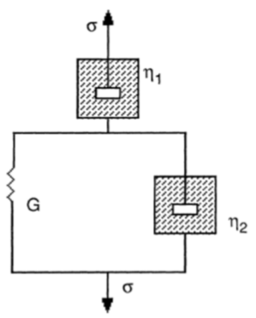

Figure 3 – Jeffreys element

The total displacement λ in a Jeffreys element is the sum of the strains in the damper and in the Voigt element. Hence,

$$\frac{\partial \lambda}{\partial t} = \frac{\partial \lambda_1}{\partial t} + \frac{\partial \lambda_2}{\partial t}.$$

The forces in both elements are the same

$$\sigma = \eta_1 \frac{\partial \lambda_1}{\partial t} = G\lambda_2 + \eta_2 \frac{\partial \lambda_2}{\partial t}.$$

After eliminating λ_1 and λ_2, we get

$$\frac{\eta_1 + \eta_2}{G} \frac{\partial \sigma}{\partial t} + \sigma = \eta_1 \left(\frac{\partial \lambda}{\partial t} + \frac{\eta_2}{G} \frac{\partial^2 \lambda}{\partial t^2} \right).$$

As in the Voigt model, the Jeffreys model is viscous because, by design, purely elastic deformation of the element is impossible. Viscosity is active in each deformation. The Jeffreys element cannot sustain a constant force in equilibrium; the force must relax. The Jeffreys model is good for fluids and not for solids.

We can define a relaxation time and a retardation time, respectively as

$$\gamma_1 = \frac{\eta_1 + \eta_2}{G} \quad \text{and} \quad \gamma_2 = \frac{\eta_2}{G}.$$

Then, recalling the interpretation of force and deformation to stress and strain, as in Maxwell fluids, we have

$$\gamma_1 \frac{\partial \sigma}{\partial t} + \sigma = \eta_1 \left(\frac{\partial u}{\partial x} + \gamma_2 \frac{\partial^2 u}{\partial x \partial t} \right). \tag{1.6}$$

The Maxwell model is recovered with $\gamma_2 = 0$ ($\eta_2 = 0$).

The tensorial generalization of (1.6) follows exactly the same lines established for Maxwell (see (JOSEPH, 2013)). Therefore, one has:

$$\gamma_1 \frac{\partial \tau}{\partial t} + \tau = 2\eta_1 \left[Du + \gamma_2 \frac{\partial Du}{\partial t} \right]. \tag{1.7}$$

We may write previous equation as

$$\tau = \frac{2\eta_1\gamma_2}{\gamma_1} D(u(t)) + \frac{2\eta_1}{\gamma_1} \left(1 - \frac{\gamma_2}{\gamma_1} \right) \int_{-\infty}^{t} \exp[-(t-s)/\gamma_1] D(u(s))\, ds.$$

That may still be split as

$$\tau = \tau_N + \tau_E,$$

where

$$\tau_N = 2\mu_0 Du$$

is the Newtonian contribution and

$$\tau_E = \frac{2\eta_1}{\gamma_1} \left(1 - \frac{\gamma_2}{\gamma_1} \right) \int_{-\infty}^{t} \exp[-(t-s)/\gamma_1] D(u(s))\, ds.$$

is the viscoelastic contribution.

The general development and wide application of the linear theory of viscoelasticity is relatively recent. In fact, the research activity in this field was mainly due to the development and large-scale use of polymeric materials. Many of these newly developed materials exhibit mechanical response features that are outside the scope of such theories of mechanical behavior as elasticity and viscosity. Thus, the importance for a more general theory is evident.

To be more specific, the elasticity theory can explain materials that have the capacity to store mechanical energy without dissipating energy. On the other hand, a viscous Newtonian fluid in a state of non-hydrostatic stress implies ability to dissipate energy, but not to store it. Then, materials that must be outside the scope of these two theories are those that only part of the work done to deform them can be recovered. Such materials have the capacity to store and dissipate mechanical energy.

This type of material has a characteristic that can be described as a memory effect. That is, the material response is not only determined by the current state of stress, but it is also determined by all past states of stress and, in a general sense, the material "remembers" all past states of stress.

In this context, in 1950 Oldroyd (OLDROYD, 1950; OLDROYD, 1953) proposed a model of an incompressible and viscous fluid that satisfied (1.7) in the form

$$\gamma \frac{\partial \sigma}{\partial t} + \sigma = 2\eta D u + 2k \frac{\partial D u}{\partial t}, \tag{1.8}$$

where γ is a time relaxation parameter, η is (again) the kinematic viscosity and k is a constant that represents time delay; it was assumed that γ, η, $k > 0$ and $\eta - \frac{k}{\gamma} > 0$.

Solving (1.8) with initial data $\sigma|_{t=0} = Du|_{t=0} = 0$ in the form of an integral equation and replacing in the equation (1.2), we get the Oldroyd equations of motion, given by

$$\begin{cases} u_t - \mu\Delta u + (u \cdot \nabla)u - \beta\displaystyle\int_0^t \exp\left(-\frac{t-s}{\gamma}\right) \Delta u(x,s)\, ds + \nabla p = f & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \times (0,T), \end{cases}$$

where $\mu = k\gamma^{-1}$, $\beta = \gamma^{-1}(\eta - k\gamma^{-1}) > 0$. For physical details and mathematical modeling, see (ASTARITA; MARRUCCI, 1974; OLDROYD, 1950; OLDROYD, 1953; OSKOLKOV, 1988). For a complete description and analysis, see (JOSEPH, 2013; RENARDY; HRUSA; NOHEL, 1987).

1.2.1 Boundary conditions in fluid mechanics

The most used boundary condition for Navier-Stokes is the full adhesion (or non-slip) requirement. This is often referred to as the *Dirichlet condition*, although it was introduced by Stokes in (STOKES et al., 1851). Under this condition, fluid particles must remain at rest near the boundary, generating boundary layers of large thickness. An alternative is the so-called Navier slip-with-friction condition. When it is imposed, the fluid is allowed to slide along the boundary, but suffers friction near the impermeable walls.

Navier slip-with-friction condition is really suitable for a wide range of applications. For example, it is appropiate for turbulent flow near rough walls (GALDI; LAYTON, 2000; LAUNDER; SPALDING, 1972) and also to model acoustics (GEYMONAT; SÁNCHEZ-PALENCIA, 1981); it is also used for simulations of flows near the rigid boundary, in aerodynamics, weather forecast, hemodynamics, etc.

Historically, the adherence condition was presented after another condition introduced by Navier in (NAVIER, 1823), involving friction:

$$u \cdot \nu = 0, \quad [D(u)\nu + \alpha u]_{\tan} = 0, \tag{1.9}$$

where α is a positive coefficient. For flat boundary, this scalar coefficient measures the amount of friction. When $\alpha = 0$ and the boundary is flat, the fluid slides along the boundary without friction. When $\alpha \longrightarrow +\infty$, the friction is so intense that the fluid is almost at rest near the boundary and, as shown by Kelliher in (KELLIHER, 2006), Navier's condition $[D(u)\nu+\alpha u]_{\tan} = 0$ converges to the usual Dirichlet condition $u = 0$. In (CORON, 1989), the boundary condition (1.9) for the Navier-Stokes equations is deduced rigorously from the boundary condition at the kinetic level (Boltzmann equation).

When the physical friction on the solid boundary is very small, it is interesting to study an asymptotic model describing a situation in which the fluid slips perfectly along the boundary. Unfortunately, a perfect slip situation is not yet fully understood in the mathematical literature.

The situation is simpler in the plane. In 1969, J.-L. Lions introduced the free boundary condition $\omega = 0$ in (LIONS, 1969). This is actually a special case of (1.9) where α depends on the position and $\alpha(x) = 2\kappa(x)$, where $\kappa(x)$ is the curvature of the boundary at $x \in \partial\Omega$.

In space, for flat boundary, the slip is easily described with the usual impermeability restriction $u \cdot \nu = 0$, together with any of the following equivalent conditions:

$$\partial_\nu [u]_{\tan} = 0, \quad [D(u)\nu]_{\tan} = 0, \quad [\nabla \times u]_{\tan} = 0.$$

For general non-flat boundaries, these conditions are no longer equivalent. In fact, this situation leads to some dubiety in the literature as which condition describes correctly a slip situation.

Results of existence, uniqueness and regularity for equations with Navier slip-with-friction condition can be found in (AMROUCHE; REJAIBA, 2014).

1.3 PREVIOUS RESULTS

In the context of fluid mechanics, the main controllability results are related to the Burgers, Stokes, Euler, Navier-Stokes and Boussinesq equations. One tool used to obtain some of these results are Carleman inequalities.

Carleman inequalities were popularized in the works by Fursikov and Imanuvilov, see (FURSIKOV; IMANUVILOV, 1996). In this direction, the work of Lebeau and Robbiano (LEBEAU; ROBBIANO, 1995) stands out; they obtained there the null controllability of the heat equation, using among other tools, a local Carleman inequality for elliptic equations. For the approximate controllability of the semilinear heat equation, see (FABRE; PUEL; ZUAZUA, 1995), by Fabre, Puel and Zuazua (note that, there, instead of an observability inequality, the authors use a weakest property of the adjoint system: unique continuation).

Using Carleman inequalities and applying an Inverse Mapping Theorem, Fursikov and Imanuvilov, in (FURSIKOV; IMANUVILOV, 1999), and Imanuvilov, in (IMANUVILOV, 2001), proved the local exact controllability to the C^∞ trajectories of the Navier-Stokes system. These results were subsequently improved for L^∞ trajectories by Fernández-Cara, Guerrero, Imanuvilov and Puel, in (FERNÁNDEZ-CARA et al., 2004). Then, inspired by (FERNÁNDEZ-CARA et al., 2004; FURSIKOV; IMANUVILOV, 1999), Guerrero proved, in (GUERRERO, 2006a), the local exact controllability to the trajectories to the Boussinesq system. Using similar arguments as in (FERNÁNDEZ-CARA et al., 2004), Fernández-Cara, Guerrero, Imanuvilov and Puel proved, under some geometric conditions, the local exact controllability for Navier-Stokes and Boussinesq with $N-1$, see (FERNÁNDEZ-CARA et al., 2006). We will also mention (CARREÑO, 2012; CARREÑO; GUERRERO, 2013; CORON; GUERRERO, 2009; CORON; LISSY, 2014; FERNÁNDEZ-CARA; SOUZA, 2012; GUERRERO; MONTOYA, 2018), where similar results are obtained with a reduced number of scalar controls.

The small-time global exact null controllability problem for the Navier-Stokes equation was conjectured by Jacques-Louis Lions in the 1990's. In the original Lions question, the boundary condition is of the Dirichlet kind. Since then, the controllability of these equations has been intensively studied, but so far only partial results are known. This is a very challenging open problem, because the Dirichlet boundary condition gives rise to boundary layers that have a large thickness, greater than for Navier slip-with-friction boundary conditions. In (CORON; MARBACH; SUEUR, 2020), the Lions' conjecture is solved when Navier slip-with-friction boundary conditions are imposed. A similar result, controlling strong solutions with regular controls is obtained in (LIAO; SUEUR; ZHANG, 2020). Others controllability results for equations with Navier slip-with-friction boundary condition can be found in (CORON, 1996b; GUERRERO, 2006b; MONTOYA, 2020).

In the literature, most of the controllability results for evolution equations with memory effects are negative. For one-dimensional heat equation, the leak of controllability was proved by Ivanov and Pandolfi in (IVANOV; PANDOLFI, 2009), where it also requires that $\int_0^T u(\cdot,t)\;dt = 0$. In higher dimensions, Guerrero and Imanuvilov proved in (GUERRERO; IMANUVILOV, 2013) that the heat equation with a memory term of the kind $\int_0^t u(\cdot,t)\;dt$ or $\int_0^t \Delta u(\cdot,s)\;ds$ cannot be steered to zero. A similar result was obtained by Zhou and Gao in (ZHOU; GAO, 2014).

In (BOLDRINI et al., 2012), the authors proved that linear systems of Maxwell type large time approximate-finite dimensional and exact controllability results for suitable control domain. Then, Fernández-Cara and Doubova established in (DOUBOVA; FERNÁNDEZ-CARA, 2012) approximate controllability results for linear Jeffreys fluids with controls acting on any part of boundary or any open subset of the physical domain. However, the null controllability property was formulated as an open problem. With an appropriate change of variable, the Jeffreys equation can be rewritten as a Stokes equation with an integral-differential term that can be seen as a memory term.

1.4 MAIN RESULTS OF THE THESIS

In this thesis, we will present positive controllability results for a nonlinear problem and lack of controllability results for some other linear problems. In both cases, the systems have origin in fluid mechanics. All the results are included in accepted or submitted papers.

Chapter 2: Controllability of the Stokes system with memory

Let $T > 0$ be a real number and Ω a regular bounded domain in $\mathbb{R}^n$ ($n = 2$ or $n = 3$). We will consider Stokes equations with memory:

$$\begin{cases} u_t - \Delta u - b\displaystyle\int_0^t e^{-a(t-s)}\Delta u(\cdot\,,s)\;ds + \nabla p = 0 & \text{in} \quad Q := (0,T)\times\Omega, \\ \nabla\cdot u = 0 & \text{in} \quad Q, \\ u = v1_\gamma & \text{on} \quad \Sigma := (0,T)\times\partial\Omega, \\ u(\cdot\,,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.10}$$

where $\gamma \subset \Sigma$ is a non-empty open subset of the boundary on which the control $v \in L^2(\gamma\times(0,T))$ acts. In (1.10), $u = u(t,x)$ can be interpreted as a velocity field of a fluid contained in Ω, p is the associated pressure and $u_0 = u_0(x)$ is an initial velocity. Here, a, $b > 0$ only for physical motivation, a can be used as a “measure” of the elasticity of the media and b depends on the Reynolds number of the fluid and the polymeric viscosity.

Notice that, if $b = 0$ in (1.10), the memory term is eliminated and we find the well-known Stokes equation, which is null controllable. With the presence of the memory term ($b \neq 0$) new difficulties arises, since it is an integro-differential term.

Our first main result proves that the null controllability for Stokes equations with memory is false. Precisely, we have the following result:

Theorem 1. *Let $T > 0$ be given. There exists initial data $u_0 \in H$ such that, for any control $v \in L^2(\gamma\times(0,T))$, the associated solution to* (1.10) *is not identically zero at time T.*

[0] $H := \{\, w \in L^2(\Omega)^n : \nabla\cdot w = 0 \text{ in } \Omega,\ w\cdot\nu = 0 \text{ on } \partial\Omega \,\}$

Some ideas of the proof were adapted from (GUERRERO; IMANUVILOV, 2013), where the heat equation with memory is treated.

Note that the above result ensures that we can construct explicit initial data that cannot be steered to zero, even if the controls act on the whole boundary $\Sigma \times (0,T)$.

It is known that, by a duality argument, the null controllability result for (1.10) is equivalent to an observability inequality for the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi - b\int_t^T e^{-a(s-t)}\Delta\varphi(\cdot\,,s)\;ds + \nabla q = 0 & \text{in} \quad Q, \\ \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(\cdot\,,T) = \varphi_T & \text{in} \quad \Omega. \end{cases} \tag{1.11}$$

The observability inequality for the solution of (1.11) is:

$$\|\varphi(\,\cdot\,,0)\|^2 \leq C \iint_\Sigma \left| \left(-q\mathrm{Id} + \nabla\varphi + b\int_t^T e^{-a(s-t)}\nabla\varphi(\cdot,s)\;ds \right)\cdot\nu \right|^2 d\Gamma dt, \tag{1.12}$$

for all $\varphi_T \in H$, where C is a positive constant.

To prove Theorem 1, first note that, by an extension argument, it is sufficient to prove the result when Ω is a ball of radius R and center at the origin. Then, we compute explicitly a sequence (φ_k, q_k) of radial eigenfunctions and eigenvalues of the Stokes operator in Ω, such that the related pressures q_k are zero.

Our goal is to show that there is no positive constant C such that (1.12) holds. To this propose, we will consider initial data

$$\varphi_T = \sum_{k\geq 1}\beta_k\varphi_k, \quad \{\beta_k\}\in\ell^2.$$

Then, we determine a sequence $\{\varphi^M\}$ of solutions to (1.11), depending on a parameter M. The functions φ^M are taken as follows:

$$\varphi^M(\,\cdot\,,t) = \sum_M \alpha_k(t)\varphi_k \quad \forall t\in(0,T),$$

where the α_k are chosen such that φ^M is a solution to (1.11) and the notation $\sum_M$ stands for a convenient sum (for example, in the three-dimensional case, $k = 8M+j$ with $1\leq j\leq 8$).

We obtain lower and upper estimates of the form:

$$\|\varphi^M(\,\cdot\,,0)\|^2 \geq \frac{C_0}{M^{\gamma_1}}, \tag{1.13}$$

and

$$\iint_\Sigma e^{2(a+b)(T-t)}\left|\frac{\partial\varphi^M}{\partial\nu}\right|^2 d\Gamma dt \leq \frac{C_1}{M^{\gamma_2}}, \tag{1.14}$$

where C_0, C_1 are positive constants and γ_1, γ_2 are suitable positive integers (in the three-dimensional case, $\gamma_1 = 6$ and $\gamma_2 = 10$).

Thanks to (1.13) and (1.14) and taking the parameter M large enough, we ensure that the observability inequality (1.12) is not true. Consequently, (1.10) is not null-controllable. For the sake of completeness, we construct explicitly initial states $u_0 \in H$ such that, for all $v \in L^2(\Sigma)$, the associated solutions to (1.10) do not vanish at $t = T$.

We also have a negative result for distributed controlled systems of the kind

$$\begin{cases} u_t - \Delta u - b\displaystyle\int_0^t e^{-a(t-s)}\Delta u(\cdot\,,s)\,ds + \nabla p = v1_\omega & \text{in} \quad Q, \\ \nabla\cdot u = 0 & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(\cdot\,,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.15}$$

where $\omega \subset \Omega$ is a non-empty open set called the distributed control domain. Specifically, as an immediate consequence of Theorem 1, we get the following negative result of null controllability:

Corollary 1. *System* (1.15) *is not null-controllable, where $\omega \subset \Omega$ is a non-empty open set satisfying $\Omega \setminus \overline{\omega} \neq \emptyset$ and $v \in L^2(\omega \times (0,T))$.*

A different proof is stablished in (MAITY; MITRA; RENARDY, 2019).

Let us consider the following system

$$\begin{cases} u_t - \Delta u + b\displaystyle\int_0^t e^{-a(t-s)}u(\cdot\,,s)\,ds + \nabla p = 0 & \text{in} \quad Q, \\ \nabla\cdot u = 0 & \text{in} \quad Q, \\ u = v1_\gamma & \text{on} \quad \Sigma, \\ u(\cdot\,,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.16}$$

which differs from equation (1.10) in the memory term. Following the same lines as in the proof of Theorem 1, with some adjustments, we obtain an equivalent result:

Theorem 2. *Let $T > 0$. Then, there exist initial conditions $u_0 \in H$ such that for any control function $v \in L^2(\Sigma)$ the associated solution to* (1.16) *is not identically equal to zero at time T.*

On the other hand, the approximate controllability results for (1.15) were established by Fernández-Cara and Doubova in (DOUBOVA; FERNÁNDEZ-CARA, 2012). For that, they proved a unique continuation result. The contribution of this thesis in this direction is the reduction of the numbers of distributed controls. That is, we have the following result:

Theorem 3. *The linear equation* (1.15) *is approximately controllable with $n-1$ scalar controls.*

In general, in the three-dimensional case, the approximate controllability result does not hold using only one control. The approximate controllability with only one control was found to be false in the three-dimensional case for $\Omega = G\times(0,L)$, for some special $G \subset \mathbb{R}^2$.

The results of this chapter were collected in (FERNÁNDEZ-CARA; MACHADO; SOUZA, 2020) and it was done in collaboration with Enrique Fernández-Cara and Diego A. Souza.

Chapter 3: Controllability of the Boussinesq system

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a smooth bounded domain with $\Gamma := \partial\Omega$ and let $\Gamma_c \subset \Gamma$ be a non-empty open subset which intersects all connected components of Γ.

Let $T > 0$ be a final time. We will consider a Boussinesq system where the fluid velocity field must satisfy a Navier slip-with-friction boundary condition, the temperature fulfills a Robin boundary condition and we assume that the control can act on Γ_c:

$$\begin{cases} u_t - \Delta u + (u\cdot\nabla)u + \nabla p = \theta e_n & \text{in} \quad (0,T)\times\Omega, \\ \theta_t - \Delta\theta + u\cdot\nabla\theta = 0 & \text{in} \quad (0,T)\times\Omega, \\ \nabla\cdot u = 0 & \text{in} \quad (0,T)\times\Omega, \\ u\cdot\nu = 0, \quad N(u) = 0 & \text{on} \quad (0,T)\times(\Gamma\setminus\Gamma_c), \\ R(\theta) = 0 & \text{on} \quad (0,T)\times(\Gamma\setminus\Gamma_c), \\ u(0,\cdot\,) = u_0, \quad \theta(0,\cdot\,) = \theta_0 & \text{in} \quad \Omega. \end{cases} \tag{1.17}$$

The functions $u = u(t,x)$, $\theta = \theta(t,x)$ and $p = p(t,x)$ are, respectively, viewed as the velocity field, the temperature and the pressure of the fluid. The Navier and Robin boundary condition terms are, respectively, given by the following formula:

$$N(u) := [D(u)\nu + Mu]_{tan} \qquad \text{and} \qquad R(\theta) := \frac{\partial\theta}{\partial\nu} + m\theta,$$

where $M = M(t,x)$ is a smooth, symmetric matrix-valued function related to the rugosity of the boundary, called the *friction matrix* and $m = m(t,x)$ is a smooth function again related to the properties of the boundary, known as the *heat transfer coefficient*.

Let us set

$$L^2_c(\Omega)^n := \{u \in L^2(\Omega)^n \ : \ \nabla\cdot u = 0 \text{ in } \Omega,\ u\cdot\nu = 0 \text{ on } \Gamma\setminus\Gamma_c\},$$

$$W_T(\Omega) := [C^0_w([0,T]; L^2_c(\Omega)^n)\cap L^2(0,T;H^1(\Omega)^n)]\times[C^0_w([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))].$$

We have the following main result:

Theorem 4. *Let $T > 0$ be a positive time, let $(u_0,\theta_0) \in L^2_c(\Omega)^n \times L^2(\Omega)$ be an initial state and let $(\overline{u},\overline{\theta}) \in W_T(\Omega)$ be a weak trajectory of* (1.17)*. Then, there exists a weak controlled solution to* (1.17) *in $W_T(\Omega)$ that satisfies*

$$(u,\theta)(T,\cdot\,) = \left(\overline{u},\overline{\theta}\right)(T,\cdot\,).$$

This Theorem generalizes to Boussinesq system (where thermal effects are considered) the main control result in (CORON; MARBACH; SUEUR, 2020), stablished for the Navier-Stokes equations.

We will reduce the task to a *distributed controllability* problem by applying a classical domain extension technique, where we denote by $\mathcal{O}$ the extended domain.

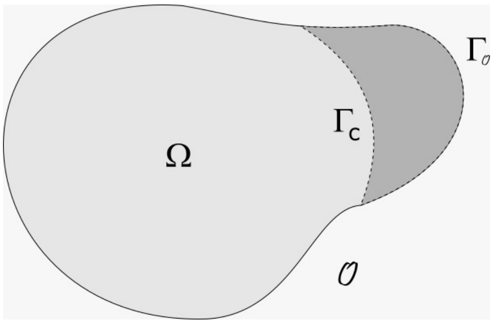

Figure 4 – Extension of the physical domain

Then, we will reduce our considerations to *smooth initial data* by using the smoothing effect of the uncontrolled Boussinesq system. We will consider the nonlinear system

$$\begin{cases} u_t - \Delta u + (u\cdot\nabla)u + \nabla p = \theta e_n + v & \text{in} \quad \mathcal{O}_T := (0,T)\times\mathcal{O}, \\ \theta_t - \Delta\theta + u\cdot\nabla\theta = w & \text{in} \quad \mathcal{O}_T, \\ \nabla\cdot u = \sigma & \text{in} \quad \mathcal{O}_T, \\ u\cdot\nu = 0, \quad N(u) = 0 & \text{on} \quad \varLambda_T := (0,T)\times\partial\mathcal{O}, \\ R(\theta) = 0 & \text{on} \quad \varLambda_T, \\ u(0,\cdot\,) = u_*, \quad \theta(0,\cdot\,) = \theta_* & \text{in} \quad \mathcal{O}, \end{cases} \tag{1.18}$$

where v and w are forcing terms supported in $\overline{\mathcal{O}}\setminus\overline{\Omega}$ and σ is a smooth nonhomogeneous divergence condition also supported in $\overline{\mathcal{O}}\setminus\overline{\Omega}$.

Thus, the proof of Theorem 4 can be divided in two important steps: global approximate controllability and local exact controllability results to the trajectories starting from sufficiently smooth initial data.

The approximate controllability result is the following:

Proposition 1. *Let us assume that* $T>0$ *and* $(\overline{u},\overline{p},\overline{\theta},\overline{v},\overline{w},\overline{\sigma})\in C^\infty([0,T]\times\overline{\mathcal{O}};\mathbb{R}^{2n+4})$ *is a smooth trajectory, with* $\overline{v}$ *and* $\overline{w}$ *supported in* $\overline{\mathcal{O}}\setminus\overline{\Omega}$*. Let* $(u_*,\theta_*)\in[H^3(\mathcal{O})^n\cap L^2_{div}(\mathcal{O})^n]\times H^3(\mathcal{O})$ *be an initial state. Then, for any* $\delta>0$*, there exist regular controls* v*,* w *and* σ*, supported in* $\overline{\mathcal{O}}\setminus\overline{\Omega}$ *and an associated weak solution to* (1.18) *satisfying*

$$\|(u,\theta)(T,\cdot\,) - (\overline{u},\overline{\theta})(T,\cdot\,)\| \le \delta.$$

[0] $L^2_{div}(\mathcal{O})^n := \{u\in L^2(\mathcal{O})^n \;:\; \nabla\cdot u = 0 \text{ in } \mathcal{O}, \;\; u\cdot\nu = 0 \text{ on } \partial\mathcal{O}\}$

For the proof, we follow the strategy introduced by Coron, Marbach, Sueur in (CORON; MARBACH; SUEUR, 2020).

First, a scale change of variable associated to a small parameter $\varepsilon > 0$ is introduced:

$$u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x), \quad p^\varepsilon(t,x) := \varepsilon^2 p(\varepsilon t, x), \quad \theta^\varepsilon(t,x) := \varepsilon^2 \theta(\varepsilon t, x).$$

Then, (1.18) is transformed into a Boussinesq system with small viscosity and heat diffusion ε that must be solved in the (long) time interval $[0, T/\varepsilon]$:

$$\begin{cases} u_t^\varepsilon - \varepsilon \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon = \theta^\varepsilon e_n + v^\varepsilon & \text{in} \quad (0, T/\varepsilon) \times \mathcal{O}, \\ \theta_t^\varepsilon - \varepsilon \Delta \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon = w^\varepsilon & \text{in} \quad (0, T/\varepsilon) \times \mathcal{O}, \\ \nabla \cdot u^\varepsilon = \sigma^\varepsilon & \text{in} \quad (0, T/\varepsilon) \times \mathcal{O}, \\ u^\varepsilon \cdot \nu = 0, \quad N(u^\varepsilon) = 0 & \text{on} \quad (0, T/\varepsilon) \times \partial\mathcal{O}, \\ R(\theta^\varepsilon) = 0 & \text{on} \quad (0, T/\varepsilon) \times \partial\mathcal{O}, \\ u^\varepsilon(0, \cdot) = \varepsilon u_*, \quad \theta^\varepsilon(0, \cdot) = \varepsilon^2 \theta_* & \text{in} \quad \mathcal{O}. \end{cases}$$

The advantage of this scaling is that we can benefit from the roles played by nonlinear terms $(u \cdot \nabla)u$ and $u \cdot \nabla\theta$.

By taking formally $\varepsilon = 0$, we obtain the inviscid Boussinesq system. For this hyperbolic system, we construct a particular nontrivial trajectory that connects $(0,0) \in \mathbb{R}^{n+1}$ to itself and sends any particle outside the physical domain before the final time T.

By linearizing the inviscid Boussinesq system around the previous nontrivial trajectory, we obtain a new hyperbolic linear system that is small-time globally null-controllable. Actually, what we are doing here is to apply the so called return method, due to Coron, see (CORON, 1992). Note that the linearization around the trivial state leads to a non-controllable system.

In the particular case of the special slip boundary condition, that is, M such that $[\nabla \times u]_{\tan} = 0$ on Λ_T and when $m \equiv 0$, we immediately conclude Proposition by estimating the remainder terms. We do not need to use the long interval time $[0, T/\varepsilon]$ to control, since the solution is already small at the intermediate time $T \in (0, T/\varepsilon)$.

Unfortunately, in the general case, a boundary layer appears. This phenomenon was already taken into account in (IFTIMIE; SUEUR, 2011), for the Navier-Stokes equations. Thus, we have to introduce some corrector terms in the asymptotic expansion of the solution depending on ε in order to estimate the residual layers. The boundary layer decays but not enough. Hence, the corrector is not sufficiently small at the final time T/ε and we still cannot conclude the approximate controllability.

In order to overcome this difficulty, we adapt the well-prepared dissipation method, introduced by Marbach in (MARBACH, 2014). The idea is to design a control strategy that reinforces the action of the natural dissipation of the boundary layer after the intermediate time T. A desired small state is obtained at final time and we can finally achieve the proof.

For the local exact controllability to trajectories, we follow some ideas presented by Guerrero in (GUERRERO, 2006b), for the Navier-Stokes equations. We will consider the following Boussinesq system with distributed controls:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = \theta e_n + v\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ \theta_t - \Delta\theta + u \cdot \nabla\theta = w\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot u = 0 & \text{in} \quad \mathcal{O}_T, \\ u \cdot \nu = 0, \quad N(u) + [f(u)]_{tan} = 0 & \text{on} \quad \varLambda_T, \\ R(\theta) + g(\theta) = 0 & \text{on} \quad \varLambda_T, \\ u(0, \cdot\,) = u_*, \quad \theta(0, \cdot\,) = \theta_* & \text{in} \quad \mathcal{O}, \end{cases} \tag{1.19}$$

and we will assume the following regularity for the trajectories:

$$\begin{aligned} &\overline{u} \in X := H^{(3-\ell)/2}(0,T; H^{\vartheta_2+1/2}(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n) \cap H^{1-\ell}(0,T; W^{\vartheta_1+1/2,\vartheta_1+1}(\mathcal{O})^n), \\ &\overline{u}_* \in H^3(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n, \quad N(\overline{u}_*) + [f(\overline{u}_*)]_{tan} = 0 \quad \text{on} \quad \partial\mathcal{O}, \\ &\overline{\theta} \in Y := H^{(3-\ell)/2}(0,T; H^{\vartheta_2+1/2}(\mathcal{O})) \cap H^{1-\ell}(0,T; W^{\vartheta_1+1/2,\vartheta_1+1}(\mathcal{O})), \\ &\overline{\theta}_* \in H^3(\mathcal{O}), \quad R(\overline{\theta}_*) + g(\overline{\theta}_*) = 0 \quad \text{on} \quad \partial\mathcal{O}, \end{aligned} \tag{1.20}$$

with $0 < \ell < 1/2$ arbitrarily close to $1/2$, $\vartheta_2 = (1/2)(3-n) + (1-\ell)(n-2)$ and $\vartheta_1 > 1$ (arbitrarily small) if $n = 3$ and $\vartheta_1 = 1$ if $n = 2$. We introduce the following notation to denote the space where the control is looked for:

$$\mathcal{H} := H^1(0,T; L^2(\mathcal{O})^n) \cap C^0([0,T]; H^1(\mathcal{O})^n) \times H^1(0,T; L^2(\mathcal{O})) \cap C^0([0,T]; H^1(\mathcal{O})).$$

The local controllability result is the following:

Proposition 2. *Let $T > 0$, $f \in C^3(\mathbb{R}^n; \mathbb{R}^n)$, $g \in C^3(\mathbb{R})$, $(\overline{u}_*, \overline{\theta}_*)$ satisfying* (1.20)*. Then, there exists $\delta > 0$ such that, for every $(u_*, \theta_*) \in [H^3(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n] \times H^3(\mathcal{O})$ satisfying the compatibility condition*

$$N(u_*) + [f(u_*)]_{tan} = 0, \qquad R(\theta_*) + g(\theta_*) = 0 \quad \text{on} \quad \partial\mathcal{O}$$

and such that $\|u_* - \overline{u}_*\|_{H^3} \leq \delta$ and $\|\theta_* - \overline{\theta}_*\|_{H^3} \leq \delta$, there exist controls $(v, w) \in \mathcal{H}$ and associated solutions (u, p, θ) to (1.19) *satisfying*

$$u(T, \cdot\,) = \overline{u}(T, \cdot\,) \quad \text{and} \quad \theta(T, \cdot\,) = \overline{\theta}(T, \cdot\,) \quad \text{in} \quad \mathcal{O}.$$

The basic ideas of the proof have been applied to many other nonlinear control problems. The strategy was to obtain appropriate Carleman inequalities and, as a consequence, deduce the null controllability of the linearized system. Then, using a Kakutani fixed-point

Theorem, we reach at the desired result. A technical difficulty found in this proof is to obtain the good Carleman inequality for the heat equation with Robin boundary conditions. And an additional difficulty is to find suitable spaces to apply the fixed point argument.

Notice that, at the end of the approximate controllability step, the obtained estimates are in L^2, however, for the local result the estimates are in H^3. Therefore, we need to prove a smoothing effect result. This was a delicate point that made it necessary to impose the symmetry assumption on M.

The results of this chapter can be found in (CHAVES-SILVA et al., 2020) and it was done in collaboration with Felipe W. Chaves-Silva, Enrique Fernández-Cara, Kévin Le Balc'h and Diego A. Souza.

2 CONTROLLABILITY OF THE STOKES SYSTEM WITH MEMORY

In this chapter, we consider the null controllability problem for the Stokes equations with a memory term. For any positive final time $T > 0$, we construct initial conditions such that the null controllability does not hold even if the controls act on the whole boundary. We also prove that this negative result holds for distributed controls.

2.1 INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain and let $T > 0$ be a prescribed final time. Let us introduce the Hilbert spaces

$$H := \{\, w \in L^2(\Omega)^3 : \nabla \cdot w = 0 \text{ in } \Omega,\ w \cdot \nu = 0 \text{ on } \partial\Omega \,\}$$

and

$$V := \{\, w \in H_0^1(\Omega)^3 : \nabla \cdot w = 0 \text{ in } \Omega \,\},$$

where $\nu = \nu(x)$ is the outward unit normal vector at $x \in \partial\Omega$. It is well known that $V \hookrightarrow H$ with a compact and dense embedding. Consequently, after identification of H and its dual H', we have

$$V \hookrightarrow H \hookrightarrow V',$$

where the second embedding is again dense and compact.

In the sequel, we will use the notation $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$. The usual scalar products and norms in the spaces $L^2(\Omega)^m$ will be denoted by $(\cdot\,,\,\cdot)$ and $\|\cdot\|$, respectively. The symbols $C, C_0, C_1, \dots$ will be used to design generic positive constants.

In this chapter, we will consider the controlled Stokes equations with memory:

$$\begin{cases} u_t - \Delta u - b\displaystyle\int_0^t e^{-a(t-s)}\Delta u(\cdot\,,s)\,ds + \nabla p = 0 & \text{in} \quad Q, \\ \nabla \cdot u = 0 & \text{in} \quad Q, \\ u = v1_\gamma & \text{on} \quad \Sigma, \\ u(\cdot\,,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where a, $b > 0$ and $\gamma \subset \partial\Omega$ is a non-empty open subset of the boundary. Here, $v \in L^2(\gamma \times (0,T))^3$ is a control acting on γ during the whole interval $(0,T)$ and $u_0 \in H$ is an initial state.

For any $u_0 \in H$ and any $v \in L^2(\gamma \times (0,T))^3$, there exists exactly one solution to (2.1), *in the sense of transposition.* This means the following: there exists a unique $u \in L^2(0,T;H) \cap C^0([0,T];V')$ satisfying

$$\int_0^T (u(\cdot\,,t), g(\cdot\,,t))\,dt = (u_0, \psi(\cdot\,,0)) - \iint_{\gamma\times(0,T)} v\left(-\pi\nu + \frac{\partial\psi}{\partial\nu} + b\int_t^T e^{-a(s-t)}\frac{\partial\psi}{\partial\nu}(\cdot\,,s)\,ds\right)d\Gamma dt \tag{2.2}$$

for all $g \in L^2(0,T;H)$, where ψ is, together with some pressure π, the unique (strong) solution to

$$\begin{cases} -\psi_t - \Delta\psi - b\displaystyle\int_t^T e^{-a(s-t)}\Delta\psi(\cdot\,,s)\,ds + \nabla\pi = g & \text{in} \quad Q, \\ \nabla\cdot\psi = 0 & \text{in} \quad Q, \\ \psi = 0 & \text{on} \quad \Sigma, \\ \psi(\cdot\,,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.3}$$

Of course, if $v1_\gamma$ is regular enough (for instance, $v = \overline{u}|_{\gamma\times(0,T)}$ with $\overline{u} \in L^2(0,T;V)$ and $\overline{u}_t \in L^2(0,T;V'))$, then u is, together with some pressure p, the unique weak solution to (2.1).

These assertions are justified in Appendix A.

The boundary null controllability property for (2.1) reads as follows: for each $u_0 \in H$, find a boundary control $v \in L^2(\gamma\times(0,T))^3$ such that the associated solution satisfies $u(\cdot,T) = 0$.

When $b = 0$, (2.1) is the Stokes equations and it is well known that the null controllability holds. In the general case, the presence of the memory term brings difficulties to the analysis of the controllability for (2.1) .

By a duality argument, it is not difficult to see that the null controllability of (2.1) is equivalent to prove an observability inequality for the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi - b\displaystyle\int_t^T e^{-a(s-t)}\Delta\varphi(\cdot\,,s)\,ds + \nabla q = 0 & \text{in} \quad Q, \\ \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(\cdot\,,T) = \varphi_T & \text{in} \quad \Omega. \end{cases} \tag{2.4}$$

The usual way to deduce such an observability estimate is to first prove a global Carleman inequality. But it seems difficult to adapt this approach in the presence of an integro-differential term.

In the last decades, many researchers have been interested by the controllability of systems governed by linear and nonlinear PDEs. For linear PDEs, the first relevant contributions were obtained in (EMANUILOV, 1993; LEBEAU; ROBBIANO, 1995; LIONS, 1988; RUSSELL, 1973; RUSSELL, 1978; SEIDMAN, 1978). For instance, in (RUSSELL, 1978), D.L. Russell presented a rather complete survey on the most relevant results available at that time. There, the author described several tools developed to address controllability problems, in some cases related to other subjects concerning PDEs: multipliers, moment problems, nonharmonic Fourier series, etc. On the other hand, in (LIONS, 1988), J.-L. Lions introduced a very useful technique, the so called *Hilbert Uniqueness Method* (HUM for short). Among other things, this allows to reformulate the solution to an exact controllability problem as a Lax-Milgram problem in an "abstract" Hilbert space that can be identified for instance in the case of the wave equation.

For semilinear systems, one can find the first contributions in (FABRE; PUEL; ZUAZUA, 1995; FURSIKOV; IMANUVILOV, 1996; LASIECKA; TRIGGIANI, 1991; ZUAZUA, 1997) and some other related results can be found in (CORON, 2007; GLOWINSKI; LIONS, 2008).

In the context of fluid mechanics, the main controllability results are related to the Burgers, Stokes, Euler and Navier-Stokes equations. For Stokes equations, the approximate and null controllability with distributed controls have been established in (FABRE, 1996; IMANUVILOV, 2001), respectively. For the Euler equations, global controllability results are proved in (CORON, 1996a; GLASS, 2000a). On the other hand, for the Navier-Stokes equations with initial and Dirichlet boundary conditions, only local exact controllability results are available; see for instance (FERNÁNDEZ-CARA et al., 2004; FERNÁNDEZ-CARA et al., 2006; FURSIKOV; IMANUVILOV, 1999; IMANUVILOV, 2001). For Navier-Stokes equations with Navier-slip (friction) boundary conditions a global exact controllability result is available in (CORON; MARBACH; SUEUR, 2020).

For 1D heat equations with memory, the lack of null controllability for a large class of memory kernels and controls was established in (IVANOV; PANDOLFI, 2009), where the notion of null controllability also requires that $\int_0^T y(\cdot,t)\, dt = 0$. In a higher dimensional situation, Guerrero and Imanuvilov proved in (GUERRERO; IMANUVILOV, 2013) that null controllability does not hold for the following system:

$$\begin{cases} y_t - \Delta y - \displaystyle\int_0^t \Delta y(\cdot\,,s)\,ds = 0 & \text{in} \quad Q, \\ y = v1_\gamma & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{2.5}$$

A similar result was obtained in (ZHOU; GAO, 2014) by Zhou and Gao for

$$\begin{cases} y_t - \Delta y - b\displaystyle\int_0^t e^{-a(t-s)} y(\cdot\,,s)\,ds = 0 & \text{in} \quad Q, \\ y = v & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y_0 & \text{in} \quad \Omega. \end{cases}$$

Our main goal in this work is to prove that the null controllability of (2.1) does not hold. More precisely, we have the following result:

Theorem 2.1. *Let $T > 0$ be given. There exists initial data $u_0 \in H$ such that, for any control $v \in L^2(\gamma \times (0,T))^3$, the associated solution to* (2.1) *is not identically zero at time T.*

The proof of this theorem follows some ideas of (GUERRERO; IMANUVILOV, 2013). Thus, we prove that the required observability inequality does not hold and then, using this fact, we construct explicit initial data that cannot be steered to zero.

We also have a negative result for distributed controlled systems of the kind

$$\begin{cases} u_t - \Delta u - b\displaystyle\int_0^t e^{-a(t-s)}\Delta u(\cdot\,,s)\,ds + \nabla p = v1_\omega & \text{in} \quad Q, \\ \nabla\cdot u = 0 & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(\cdot\,,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{2.6}$$

where $\omega \subset \Omega$ is an open subset. Specifically, as an immediate consequence of Theorem 2.1, we get the following result:

Corollary 2.2. *Let $T > 0$ be given and let ω be a non-empty open set with $\Omega \setminus \overline{\omega} \neq \emptyset$. There exist initial data $u_0 \in H$ such that, for any $v \in L^2(\omega \times (0,T))^3$, the associated solution to* (2.6) *is not identically zero at time T.*

Remark 2.3. Theorem 2.1 and Corollary 2.2 still hold if we replace in (2.1) or (2.6) the integral (memory) term by

$$\int_0^t e^{-a(t-s)}u(\cdot\,,s)\,ds.$$

The analysis of the control of (2.1) and (2.6) is motivated by the interest to understand the limits of controlling *viscoelastic* fluids of the Oldroyd kind. Thus, let us consider the following systems:

$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \nabla p = \nabla\cdot\tau & \text{in} \quad Q, \\ \tau_t + (u\cdot\nabla)\tau + g(\nabla u,\tau) + a\tau = 2bD(u) & \text{in} \quad Q, \\ \nabla\cdot u = 0 & \text{in} \quad Q, \\ u = v1_\gamma & \text{on} \quad \Sigma, \\ u(\cdot\,,0) = u_0, \quad \tau(\cdot\,,0) = \tau_0 & \text{in} \quad \Omega \end{cases} \tag{2.7}$$

and

$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \nabla p = \nabla\cdot\tau + v1_\omega & \text{in} \quad Q, \\ \tau_t + (u\cdot\nabla)\tau + g(\nabla u,\tau) + a\tau = 2bD(u) & \text{in} \quad Q, \\ \nabla\cdot u = 0 & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(\cdot\,,0) = u_0, \quad \tau(\cdot\,,0) = \tau_0 & \text{in} \quad \Omega, \end{cases} \tag{2.8}$$

where $g(\nabla u,\tau) := \tau W(u) - W(u)\tau - k[D(u)\tau + \tau D(u)]$, $k \in [-1,1]$ and we have used the notation $D(u) := \frac{1}{2}(\nabla u + \nabla u^t)$ and $W(u) := \frac{1}{2}(\nabla u - \nabla u^t)$. The functions u, p and τ are respectively the velocity field, the pressure distribution and the elastic extra-stress tensor of the fluid; $u_0 \in H$ and $\tau_0 \in L^2(\Omega; \mathcal{L}_s(\mathbb{R}^3))$.[1] For the physical meaning of these systems, see for instance (JOSEPH, 2013; RENARDY; HRUSA; NOHEL, 1987).

The theoretical analysis of the Oldroyd systems (2.7) and (2.8) has been the subject of considerable work. Note that these systems are more difficult to solve than the usual Navier-Stokes equations. The main reason is the presence of the nonlinear term $g(\nabla u, \tau)$; for details, see (FERNANDEZ-CARA; GUILLEN; ORTEGA, 2002; LIONS; MASMOUDI, 2000; RENARDY, 2009a).

It is worth mentioning that, in (DOUBOVA; FERNÁNDEZ-CARA, 2012), the authors studied a linear version of (2.8):

$$\begin{cases} u_t - \Delta u + \nabla p = \nabla \cdot \tau + v 1_\omega & \text{in} \quad Q, \\ \tau_t + a\tau = 2bD(u) & \text{in} \quad Q, \\ \nabla \cdot u = 0 & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(\cdot\,, 0) = u_0, \quad \tau(\cdot\,, 0) = \tau_0 & \text{in} \quad \Omega. \end{cases} \tag{2.9}$$

Plugging the explicit solution τ of $(2.9)_2$ into $(2.9)_1$, it is easy to see that the previous system can be equivalently rewritten as an integro-differential equation in u:

$$\begin{cases} u_t - \Delta u - b \displaystyle\int_0^t e^{-a(t-s)} \Delta u(\cdot\,, s)\, ds + \nabla p = e^{-at} \nabla \cdot \tau_0 + v 1_\omega & \text{in} \quad Q, \\ \nabla \cdot u = 0 & \text{on} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(\cdot\,, 0) = u_0 & \text{on} \quad \Omega. \end{cases} \tag{2.10}$$

In (DOUBOVA; FERNÁNDEZ-CARA, 2012, Theorem 1.1 and 1.2), approximate controllability results are established for (2.9). Notice that, if τ_0 is the null matrix, then (2.10) and (2.6) are exactly the same.

The system (2.9) governs the behavior of viscoelastic fluids of the so called linear Jeffreys kind. If we neglect the viscosity term, we find a *linear Maxwell fluid,* for which large time controllability results have been established, see (BOLDRINI et al., 2012); see also (RENARDY, 2005; RENARDY, 2009b).

Recall that, in (DOUBOVA; FERNÁNDEZ-CARA, 2012), the null controllability of linear Jeffreys fluids is formulated as an open problem. Hence, Theorem 2.1 and Corollary 2.2 solve this open question proving that the null controllability does not hold. A different proof is stablished in (MAITY; MITRA; RENARDY, 2019).

This chapter is organized as follows. In Section 2.2, we compute the eigenfunctions and eigenvalues of the Stokes operator in a ball and we prove some relevant estimates.

[1] $\mathcal{L}_s(\mathbb{R}^3)$ is the space of symmetric real 3×3 matrices.

In Section 2.3, we prove Theorem 2.1. Finally, in Section 2.4, we present some additional comments and open problems.

2.2 THE RADIALLY SYMMETRIC EIGENFUNCTIONS OF THE STOKES OPERATOR

In this section, we will assume that Ω is the ball of radius R centered at the origin. We will compute explicitly the eigenfunctions and eigenvalues of the Stokes operator and, then, we will deduce some crucial estimates that will be used to prove Theorem 2.1. For simplicity, the coordinates of a generic point in Ω will be denoted by x, y and z.

Let us compute nontrivial couples (φ, q) and positive real numbers λ such that

$$\begin{cases} -\Delta\varphi + \nabla q = \lambda\varphi & \text{in} \quad \Omega, \\ \nabla \cdot \varphi = 0 & \text{in} \quad \Omega, \\ \varphi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.11}$$

Let us look for eigenfunctions as the curl of radial stream functions, i.e. $\varphi = \nabla \times \psi$, for some radial stream function ψ. Setting $w = \nabla \times \varphi$, we can easily deduce that if (w, ψ) solves, together with λ, the eigenvalue problem

$$\begin{cases} -rw'' - 2w' = \lambda rw & \text{in} \quad (0, R), \\ -r\psi'' - 2\psi' = rw & \text{in} \quad (0, R), \\ \psi(R) = 0, \quad \psi'(R) = 0, \quad \lambda > 0 \end{cases} \tag{2.12}$$

then $\varphi = \nabla \times \psi$ is, together with λ and some q, a solution to (2.11). Here, we are using the notation $r = \sqrt{x^2 + y^2 + z^2}$ for any $(x, y, z) \in \Omega$.

In order to compute the solutions to (2.12), let us make the following change of variables: $\zeta = rw$ and $\phi = r\psi$. Then, from (2.12), we see that ζ, ϕ and λ satisfy

$$\begin{cases} -\zeta'' = \lambda\zeta, \quad -\phi'' = \zeta & \text{in} \quad (0, R), \\ \zeta(0) = 0, \quad \phi(0) = 0, \\ \phi(R) = 0, \quad \phi'(R) = 0, \quad \lambda > 0. \end{cases}$$

This way, it is not difficult compute explicitly the eigenvalues λ_n and the corresponding eigenfunctions (φ_n, q_n) for (2.11):

$$\begin{cases} \varphi_n(x, y, z) = \dfrac{1}{\lambda_n^{1/2} r^2} \left(\cos(\lambda_n^{1/2} r) - \dfrac{1}{\lambda_n^{1/2} r} \sin(\lambda_n^{1/2} r) \right) (y - z, z - x, x - y), \\ q_n \equiv 0, \\ \lambda_n^{1/2} R = tg(\lambda_n^{1/2} R). \end{cases} \tag{2.13}$$

Note that

$$\lambda_n = \frac{\pi^2}{R^2}(n+1/2)^2 - \varepsilon_n, \quad \text{for some} \quad \varepsilon_n > 0 \quad \text{with} \quad \varepsilon_n \to 0. \tag{2.14}$$

It is not difficult to see that $\{\varphi_n\}_{n\in\mathbb{N}}$ is an orthogonal family in H. Also, using $(2.13)_3$, we can compute the L^2-norm of φ_n:

$$\begin{aligned}
\|\varphi_n\|^2 &= 8\pi \int_0^R \left(\frac{\cos(\lambda_n^{1/2} r)}{\lambda_n^{1/2}} - \frac{\sin(\lambda_n^{1/2} r)}{\lambda_n r} \right)^2 dr \\
&= \frac{8\pi}{\lambda_n^{3/2}} \left[\frac{\lambda_n^{1/2} R}{2} + \sin(\lambda_n^{1/2} R) \left(\frac{\cos(\lambda_n^{1/2} R)}{2} - \frac{\sin(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R} \right) \right] \\
&= \frac{2\pi R}{\lambda_n}(1 - \cos(2\lambda_n^{1/2} R)).
\end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we see that, if n is large enough, $\cos(2\lambda_n^{1/2} R) < 0$ and, consequently,

$$\|\varphi_n\|^2 \geq \frac{2\pi R}{\lambda_n}. \tag{2.16}$$

On the other hand, we can deduce some estimates for the normal derivatives of φ_n. Indeed, using $(2.13)_1$ and $(2.13)_3$, we get:

$$\begin{aligned}
\left.\frac{\partial \varphi_n^1}{\partial \nu}\right|_{\partial\Omega} &= \left(-\frac{\sin(\lambda_n^{1/2} R)}{R^2} - 3\frac{\cos(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R^3} + 3\frac{\sin(\lambda_n^{1/2} R)}{\lambda_n R^4} \right)(y - z), \\
\left.\frac{\partial \varphi_n^2}{\partial \nu}\right|_{\partial\Omega} &= \left(-\frac{\sin(\lambda_n^{1/2} R)}{R^2} - 3\frac{\cos(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R^3} + 3\frac{\sin(\lambda_n^{1/2} R)}{\lambda_n R^4} \right)(z - x), \\
\left.\frac{\partial \varphi_n^3}{\partial \nu}\right|_{\partial\Omega} &= \left(-\frac{\sin(\lambda_n^{1/2} R)}{R^2} - 3\frac{\cos(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R^3} + 3\frac{\sin(\lambda_n^{1/2} R)}{\lambda_n R^4} \right)(x - y).
\end{aligned}$$

But, thanks to $(2.13)_3$, the following relations hold:

$$-3\frac{\cos(\lambda_n^{1/2} R)}{\lambda_n^{1/2} R^3} + 3\frac{\sin(\lambda_n^{1/2} R)}{\lambda_n R^4} = 0.$$

Therefore,

$$\left.\frac{\partial \varphi_n}{\partial \nu}\right|_{\partial\Omega} = -\frac{\sin(\lambda_n^{1/2} R)}{R^2}(y - z, z - x, x - y). \tag{2.17}$$

2.3 THE LACK OF NULL CONTROLLABILITY

In this section, we prove Theorem 2.1. As already said, we will follow some ideas presented in (GUERRERO; IMANUVILOV, 2013).

Notice that it is sufficient to consider the case where Ω is a ball and the solution is radially symmetric. Indeed, if Ω is a general bounded domain in $\mathbb{R}^3$, we fix an open ball $B \subset \Omega$. If the result is established for any ball, we see that B can be chosen such that, for any $T > 0$, there exist initial states $\hat{u}_0 \in L^2(B)^3$, $\nabla \cdot \hat{u}_0 = 0$ in B and $\hat{u}_0 \cdot \nu = 0$

on ∂B, with the following property: for any boundary control $v \in L^2(\partial B \times (0,T))^3$, the associated solution $\hat{u}$ is not identically equal to zero at time T. Now, by extending $\hat{u}_0$ by zero to the whole domain Ω, considering the extended system (2.1) in Q and arguing by contradiction, we find that the null controllability at time T also fails in $\Omega \times (0,T)$.

Accordingly, we will assume in the sequel that Ω is a ball of radius R.

It is well known that the null controllability of (2.1) is equivalent to the following observability inequality for the solutions to (2.4):

$$\|\varphi(\,\cdot\,,0)\|^2 \le C \iint_\Sigma \left| \left(-q\mathrm{Id} + \nabla\varphi + b\int_t^T e^{-a(s-t)} \nabla\varphi(\cdot,s)\,ds \right) \cdot \nu \right|^2 d\Gamma\,dt \quad \forall \varphi_T \in H. \tag{2.18}$$

Our goal is to show that there is no positive constant C such that (2.18) holds. To this purpose, we will construct a family of solutions to (2.4), denoted φ^M, such that, for all sufficiently large M, one has

$$\|\varphi^M(\,\cdot\,,0)\| \ge \frac{C_1}{M^6} \tag{2.19}$$

and

$$\iint_\Sigma \left| \left(-q\mathrm{Id} + \nabla\varphi^M + b\int_t^T e^{-a(s-t)} \nabla\varphi^M(\cdot,s)\,ds \right) \cdot \nu \right|^2 d\,\Gamma\,dt \le \frac{C_2}{M^{10}}, \tag{2.20}$$

where C_1 and C_2 are independent of M. Then, using these properties of the φ^M, we will be able to construct initial data $\overline{u}_0$ in H such that the solution to (2.1) cannot be steered to zero, no matter the control is.

2.3.1 The structure of the φ^M

For simplicity, the superindex M will be omited in this section (and also in Sections 2.3.2 and 2.3.3).

Let us set

$$\varphi_T := \sum_{n\ge1} \beta_n \varphi_n,$$

where $\{\beta_n\}$ is a real sequence with only a finite amount of non-zero terms, see (2.27). We try to find some particular β_n such that the quotient of (2.26) over (2.33) becomes large, see (2.19) and (2.20).

The solution to (2.4) associated with φ_T can be written in the form

$$\varphi(\,\cdot\,,t) = \sum_{n\ge1} \alpha_n(t)\varphi_n, \quad q \equiv 0, \quad \forall t \in (0,T), \tag{2.21}$$

where the α_n satisfy the following second-order Cauchy problem:

$$\begin{cases} -\alpha_n'' + (\lambda_n + a)\alpha_n' - \lambda_n(a+b)\alpha_n = 0 \quad \text{in} \quad (0,T), \\ \alpha_n(T) = \beta_n, \\ \alpha_n'(T) = \lambda_n\beta_n. \end{cases} \tag{2.22}$$

It is clear that there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$, then $D_n := (\lambda_n + a)^2 - 4(a + b)\lambda_n > 0$. This way, taking $\beta_n = 0$ for $n < n_0$, we have

$$\begin{cases} \alpha_n(t) \equiv 0 \quad \forall n < n_0, \\ \alpha_n(t) \equiv C_{1,n} e^{\mu_n^+(T-t)} + C_{2,n} e^{\mu_n^-(T-t)} \quad \forall n \geq n_0, \end{cases} \tag{2.23}$$

where

$$\mu_n^+ = -\frac{(\lambda_n + a) + \sqrt{D_n}}{2} \quad \text{and} \quad \mu_n^- = -\frac{(\lambda_n + a) - \sqrt{D_n}}{2} \tag{2.24}$$

and the coefficients $C_{1,n}$ and $C_{2,n}$ are given by

$$C_{1,n} = \beta_n \frac{\lambda_n - a + \sqrt{D_n}}{2\sqrt{D_n}} \quad \text{and} \quad C_{2,n} = \beta_n \frac{a - \lambda_n + \sqrt{D_n}}{2\sqrt{D_n}}. \tag{2.25}$$

It is not difficult to check that $\mu_n^+ \to -\infty$ and $\mu_n^- \to -(a+b)$ as $n \to +\infty$. Also, using (2.16), (2.21), (2.23) and the orthogonality of φ_n, we see that

$$\begin{aligned} \|\varphi(\,\cdot\,, 0)\|^2 &= \sum_{n \geq n_0} (C_{1,n} e^{\mu_n^+ T} + C_{2,n} e^{\mu_n^- T})^2 \|\varphi_n\|^2 \\ &\geq \sum_{n \geq n_0} \frac{2\pi R}{\lambda_n} (C_{1,n} e^{\mu_n^+ T} + C_{2,n} e^{\mu_n^- T})^2. \end{aligned} \tag{2.26}$$

Let M be a large integer (such that $8M \geq n_0$) and let us take

$$\beta_n = 0 \quad \forall n \notin \{8M + k : \ 1 \leq k \leq 8\}. \tag{2.27}$$

The coefficients β_n for $n \in \{8M + k : \ 1 \leq k \leq 8\}$ will be chosen below, in Section 2.3.3. Then, one has

$$\varphi(\cdot\,, t) = \sum_M \alpha_n(t) \varphi_n \quad \forall t \in (0, T), \tag{2.28}$$

where $\sum_M$ stands for the sum extended to all indices of the form $n = 8M + k$ with $1 \leq k \leq 8$.

2.3.2 The estimates from below

Let us use (2.26) to prove (2.19). To do this, let us begin with the inequality

$$\sum_M \frac{1}{\lambda_n} \left(C_{1,n} e^{\mu_n^+ T} + C_{2,n} e^{\mu_n^- T} \right)^2 \geq \sum_M \frac{1}{\lambda_n} \left(\frac{3}{4} C_{2,n}^2 e^{2\mu_n^- T} - 3 C_{1,n}^2 e^{2\mu_n^+ T} \right).$$

Let us assume for the moment that the β_{8M+k} and the corresponding $C_{1,8M+k}$ have been chosen bounded independently of M. This choice will be justified below, see Remarks 2.5 and 2.7. Then, from (2.14) and (2.24), we have that

$$C_{1,8M+k}^2 e^{2\mu_{8M+k}^+ T} \leq C e^{-CM^2 T} \quad \forall k = 1, \ldots, 8. \tag{2.29}$$

Here and in the sequel, the generic constant denoted by C is independent of M. On the other hand, using the notations

$$(k-1/2)! = (k-1/2)(k-3/2)\cdots 1/2 \quad \text{and} \quad (-1/2)! = 1,$$

we can expand the quotient $(a-\lambda_n+\sqrt{D_n})/\sqrt{D_n}$ in the definition of $C_{2,n}$ and get:

$$\begin{aligned}\frac{a-\lambda_n+\sqrt{D_n}}{\sqrt{D_n}} &= \left[\frac{2a}{\lambda_n+a} - \frac{2\lambda_n(\lambda_n-a)(a+b)}{(\lambda_n+a)^3} - \frac{\lambda_n-a}{\lambda_n+a}\sum_{k\geq 2}\frac{(k-1/2)!}{k!}\left(\frac{4\lambda_n(a+b)}{(\lambda_n+a)^2}\right)^k\right]\\ &= \left[\frac{2a}{\lambda_n+a} - \frac{2\lambda_n(\lambda_n-a)(a+b)}{(\lambda_n+a)^3} - \frac{6\lambda_n^2(\lambda_n-a)(a+b)^2}{(\lambda_n+a)^5} + \mathcal{O}(\lambda_n^{-3})\right]\\ &\approx \mathcal{O}(\lambda_n^{-1}),\end{aligned} \tag{2.30}$$

for n large enough and, taking into account (2.14), we see that

$$\inf_{1\leq k\leq 8}\left(\frac{a-\lambda_{2,8M+k}+\sqrt{D_{2,8M+k}}}{\sqrt{D_{2,8M+k}}}\right)^2 \geq \frac{C}{M^4} \tag{2.31}$$

for M large enough. Finally, combining (2.26), (2.14), (2.29), (2.31) and the fact that $\mu_n^- \to -(a+b)$, one has:

$$\|\varphi^M(\,\cdot\,,0)\|^2 \geq \frac{C_1}{M^6} \tag{2.32}$$

for M large enough and some positive C_1 independent of M.

2.3.3 The estimates from above

In order to estimate the right hand side of (2.18) from above, it is sufficient to find an upper bound of the integral

$$\iint_\Sigma \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\,\Gamma\,dt.$$

To simplify the computations, let us introduce the weight $e^{2(a+b)(T-t)}$ in the above integral and consider instead this one:

$$\iint_\Sigma e^{2(a+b)(T-t)}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\,\Gamma\,dt.$$

Taking into account (2.17), the following estimate holds:

$$\left|\frac{\partial\varphi}{\partial\nu}\right|^2 \leq 12\left|\sum_{n\geq n_0}\gamma_n\alpha_n(t)\right|^2,$$

where $\gamma_n := \sin(\lambda_n^{1/2}R)/R$ for all n. Therefore,

$$\begin{aligned}\iint_\Sigma e^{2(a+b)(T-t)}\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\,\Gamma\,dt &\leq 48\pi R^2\int_0^T e^{2(a+b)(T-t)}\left|\sum_{n\geq n_0}\alpha_n(t)\gamma_n\right| dt\\ &\leq A_1 + A_2,\end{aligned} \tag{2.33}$$

where we have set

$$A_1 := 96\pi R^2 \int_0^T \left(\sum_M \gamma_n C_{1,n} e^{(a+b+\mu_n^+)(T-t)} \right)^2 dt,$$

$$A_2 := 96\pi R^2 \int_0^T \left(\sum_M \gamma_n C_{2,n} e^{(a+b+\mu_n^-)(T-t)} \right)^2 dt.$$

Let us establish estimates of A_1 and A_2 separately.

Lemma 2.4. *There exists $C > 0$ such that, for M large enough, one has*

$$A_1 \leq \frac{C}{M^{10}}, \tag{2.34}$$

Proof. Let us begin using (2.24) and noting that $e^{(a+b+\mu_n^+)(T-t)} = e^{(a+2b-\lambda_n)(T-t)} e^{B_n(T-t))}$, where $B_n := -\mu_n^- - a - b \to 0$ as $n \to +\infty$. Also, from (2.14), we have

$$e^{(a+2b-\lambda_{8M+k})(T-t)} = e^{\left[a+2b-\frac{\pi^2}{R^2}\left(8M+\frac{1}{2}\right)^2\right](T-t)} e^{\left[-\frac{\pi^2}{R^2}(16Mk+k+k^2)+\varepsilon_{8M+k}\right](T-t)}.$$

Let us rewrite A_1 as follows:

$$A_1 = 96\pi R^2 \int_0^T e^{(2a+4b-2\frac{\pi^2}{R^2}(8M+\frac{1}{2})^2)(T-t)} g_M(t)\, dt,$$

where $g_M(t) := f_M(t)^2$ and f_M is given by

$$f_M(t) := \sum_{k=1}^{8} \gamma_{8M+k} C_{1,8M+k} \exp\left(\left[-\frac{\pi^2}{R^2}(16Mk + k + k^2) + \varepsilon_{8M+k} + B_{8M+k} \right] (T-t) \right).$$

After integrating by parts ten times, we get:

$$\begin{aligned} \int_0^T e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)(T-t)} g_M(t)\ dt \;&=\; \sum_{j=0}^{9} \frac{e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)T} g_M^{(j)}(0) - g_M^{(j)}(T)}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{j+1}} \\ &+ \int_0^T \frac{e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)(T-t)}}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{10}}\, g_M^{(10)}(t)\, dt. \end{aligned} \tag{2.35}$$

The quantities ε_{8M+k}, B_{8M+k} and γ_{8M+k} are bounded independently of M. If the same happens to the $C_{1,8M+k}$, we have $|f_M^{(j)}| = \mathcal{O}(M^j)$ and $g_M^{(j)} = \mathcal{O}(M^j)$ for all $j \geq 1$ and all sufficiently large M, whence

$$\sum_{j=0}^{9} \frac{g_M^{(j)}(T)}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{j+1}} = \mathcal{O}(M^{-2}).$$

Thus, in order to obtain (2.34), we impose the following conditions to the $g_M^{(j)}(T)$:

$$g_M^{(0)}(T) = g_M^{(1)}(T) = \cdots = g_M^{(8)}(T) = g_M^{(9)}(T) = 0. \tag{2.36}$$

Note that these conditions are fulfilled if the constants $C_{1,8M+k}$ $(1 \leq k \leq 8)$ satisfy five linear equations corresponding to the identities $f_M^{(0)}(T) = f_M^{(1)}(T) = f_M^{(2)}(T) = f_M^{(3)}(T) = f_M^{(4)}(T) = 0$. More precisely, the constants $C_{1,8M+k}$ $(1 \leq k \leq 8)$ should satisfy:

$$\begin{cases} \displaystyle\sum_{k=1}^{8} \gamma_{8M+k} \left(-\frac{\pi^2}{R^2}(16Mk + k + k^2) + \varepsilon_{8M+k} + B_{8M+k} \right)^j C_{1,8M+k} = 0, \\ \text{for } j = 0, 1, 2, 3, 4. \end{cases} \tag{2.37}$$

Remark 2.5. In this homogeneous system, there are 5 linear equations for the 8 unknowns $C_{1,8M+k}$. Hence, the space of solutions has, at least, dimension 3 and it is possible to choose a nontrivial solution bounded independently of M. Of course, this is what we do.

Finally, using (2.35), (2.36) and the bounds

$$\frac{e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)T}}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{j+1}}|g_M^{(j)}(0)| \leq Ce^{-CM^2}\frac{1}{M^{j+2}} < \frac{C}{M^{10}} \quad \text{for} \quad 0 \leq j \leq 9$$

and

$$\left|\int_0^T \frac{e^{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)(T-t)}}{(2a+4b-\frac{2\pi^2}{R^2}(8M+\frac{1}{2})^2)^{10}} g_M^{(10)}(t)\ dt\right| \leq \int_0^T \frac{1}{(CM)^{20}} CM^{10}\ dt = \frac{C}{M^{10}}\,,$$

that hold for M large enough, we deduce (2.34). □

Lemma 2.6. *There exists $C > 0$ such that, for M large enough, one has*

$$A_2 \leq \frac{C}{M^{12}}, \tag{2.38}$$

Proof. First, note that

$$\mu_n^- = \frac{\lambda_n + a}{2}\left(-1 + \sqrt{1 - \frac{4\lambda_m(a+b)}{(\lambda_n+a)^2}}\right) = -\frac{\lambda_n+a}{4}\sum_{k\geq 1}\frac{(k-3/2)!}{k!}\left[\frac{4\lambda_n(a+b)}{(\lambda_n+a)^2}\right]^k.$$

On the other hand, the exponent in the expression of A_2 can be split as follows:

$$e^{(a+b+\mu_n^-)(T-t)} = e^{\frac{a(a+b)}{\lambda_n+a}(T-t)}e^{Y_n(T-t)},$$

where

$$Y_n := -\frac{\lambda_n+a}{4}\sum_{k\geq 2}\frac{(k-3/2)!}{k!}\left[\frac{4\lambda_n(a+b)}{(\lambda_n+a)^2}\right]^k.$$

Since $e^x = 1 + x + \mathcal{O}(x^2)$ for $|x| < 1$, we see that

$$e^{\frac{a(a+b)}{\lambda_n+a}(T-t)} = 1 + \frac{a(a+b)}{\lambda_n+a}(T-t) + \mathcal{O}(\lambda_n^{-2}), \tag{2.39}$$

for n large enough.

Now, since $\mu_n^- \to -(a+b)$, we have

$$|Y_n(T-t)| = \left|\left(a+b+\mu_n^- - \frac{a(a+b)}{\lambda_n+a}\right)(T-t)\right| < 1$$

and

$$e^{Y_n(T-t)} = 1 - \frac{\lambda_n^2(a+b)^2}{(\lambda_n+a)^3}(T-t) + \mathcal{O}(\lambda_n^{-2}), \tag{2.40}$$

where we have used that $Y_n = -\frac{\lambda_n^2(a+b)^2}{(\lambda_n+a)^3} + \mathcal{O}(\lambda_n^{-2})$ for n large enough. The following is obtained from (2.39) and (2.40):

$$e^{(a+b+\mu_n^-)(T-t)} = 1 - \frac{\lambda_n^2(a+b)^2}{(\lambda_n+a)^3}(T-t) + \frac{a(a+b)}{\lambda_n+a}(T-t) + \mathcal{O}(\lambda_n^{-2}). \tag{2.41}$$

Using (2.30) and (2.41), we see that

$$\begin{aligned}\gamma_n C_{2,n} e^{(a+b+\mu_n^-)(T-t)} = \gamma_n \frac{\beta_n}{2}\Bigg[&\left(\frac{2a}{\lambda_n+a} - \frac{2\lambda_n(\lambda_n-a)(a+b)}{(\lambda_n+a)^3} - \frac{6\lambda_n^2(\lambda_n-a)(a+b)^2}{(\lambda_n+a)^5}\right)\\ &+ (T-t)\left(-\frac{2\lambda_n^2(a+b)^2a}{(\lambda_n+a)^4} + \frac{2\lambda_n^3(\lambda_n-a)(a+b)^3}{(\lambda_n+a)^6} + \frac{2a^2(a+b)}{(\lambda_n+a)^2}\right.\\ &\left.- \frac{2\lambda_n(\lambda_n-a)a(a+b)^2}{(\lambda_n+a)^4}\right) + \mathcal{O}(\lambda_n^{-3})\Bigg],\end{aligned} \tag{2.42}$$

for n large enough. Thus, in order to deduce (2.38), we impose these two conditions:

$$\sum_M \gamma_n \left(\frac{a}{\lambda_n+a} - \frac{\lambda_n(\lambda_n-a)(a+b)}{(\lambda_n+a)^3} - \frac{3\lambda_n^2(\lambda_n-a)(a+b)^2}{(\lambda_n+a)^5}\right)\beta_n = 0 \tag{2.43}$$

and

$$\sum_M \gamma_n \left(\frac{\lambda_n^2(a+b)^2a}{(\lambda_n+a)^4} - \frac{\lambda_n^3(\lambda_n-a)(a+b)^3}{(\lambda_n+a)^6} - \frac{a^2(a+b)}{(\lambda_n+a)^2} + \frac{\lambda_n(\lambda_n-a)a(a+b)^2}{(\lambda_n+a)^4}\right)\beta_n = 0. \tag{2.44}$$

Remark 2.7. In view of (2.25), we see that (2.37), (2.43) and (2.44) together form a linear homogeneous system of 7 equations for the 8 unknowns $C_{1,8M+k}$. Accordingly, as before, the solution (and also the associated β_{8M+k}) can be chosen bounded independently of M and this will be our choice.

Finally, from (2.42), (2.43) and (2.44), we observe that

$$\left|\sum_M \gamma_n C_{2,n} e^{(a+b+\mu_n^-)(T-t)}\right| \leq \frac{C}{M^6}$$

for M large enough, which leads to (2.38). □

An immediate consequence of the estimates (2.34) and (2.38) is that

$$\iint_\Sigma e^{2(a+b)(T-t)} \left|\frac{\partial \varphi^M}{\partial \nu}\right|^2 d\Gamma\, dt \leq \frac{C}{M^{10}} \tag{2.45}$$

for M large enough.

2.3.4 Construction of non-controllable initial data

From the results obtained in Sections 2.3.1, 2.3.2 and 2.3.3, it becomes clear that there is no C such that (2.18) holds. Consequently, (2.1) is not null-controllable.

For the sake of completeness, let us construct explicitly initial states $u_0 \in H$ such that, for all $v \in L^2(\Sigma)^3$, the associated solutions to (2.1) do not vanish at $t = T$.

Let M be large enough (to be fixed below). In view of (2.26) and (2.32), there exists an integer k_0 with $1 \leq k_0 \leq 8$ and

$$\|\varphi_{8M+k_0}\|^2 \left(C_{1,8M+k_0} e^{\mu^+_{8M+8k_0} T} + C_{2,8M+k_0} e^{\mu^-_{8M+k_0} T} \right)^2 \geq \frac{C_0}{8M^6}. \tag{2.46}$$

Let us introduce

$$u_0 := \sum_{\ell \geq 1} \frac{1}{\ell^{3/4}} \frac{\varphi_{8\ell+k_0}}{\|\varphi_{8\ell+k_0}\|}. \tag{2.47}$$

Then, it is not difficult to see that $u_0 \in H$.

Let us check that u_0 cannot be steered to zero. We will argue by contradiction. Thus, let $v \in L^2(\Sigma)^3$ be such that the solution to (2.1) associated with u_0 satisfies $u(\cdot\,, T) = 0$. Then, we must have

$$\int_\Omega u_0(x) \varphi^M(x,0)\, dx = \iint_\Sigma v \frac{\partial \varphi^M}{\partial \nu}\, d\Gamma\, dt + \ b \int_0^T \!\! \int_0^t e^{-a(t-s)} \left(\int_{\partial\Omega} v(\sigma,s) \frac{\partial \varphi^M}{\partial \nu}(\sigma,t)\, d\Gamma \right) ds\, dt, \tag{2.48}$$

where φ^M is defined in (2.28).

Using (2.47) and the orthogonality of the φ_n, we get the identity

$$\int_\Omega u_0(x) \varphi^M(x,0)\ dx = \frac{1}{M^{3/4}} \|\varphi_{8M+k_0}\| \left(C_{1,8M+k_0} e^{\mu^+_{8M+k_0} T} + C_{2,8M+k_0} e^{\mu^-_{8M+k_0} T} \right)$$

and, in view of (2.46), we find that

$$\left| \int_\Omega u_0(x) \varphi^M(x,0)\, dx \right| \geq \frac{C_1}{M^{15/4}}, \tag{2.49}$$

for some positive constant K_1 independent of M.

On the other hand, taking into account (2.45), we see that the other terms in (2.48) can be bounded as follows

$$\left| \iint_\Sigma v(\sigma,t) \frac{\partial \varphi^M}{\partial \nu}(\sigma,t)\ d\Gamma\ dt \right| \leq \|v\|_{L^2(\Sigma)} \left\| \frac{\partial \varphi^M}{\partial \nu} \right\|_{L^2(\Sigma)} \leq \frac{K_2}{M^5} \tag{2.50}$$

and

$$\left| \int_0^T \!\! \int_0^t e^{-a(t-s)} \left(\int_{\partial\Omega} v(\sigma,s) \frac{\partial \varphi^M}{\partial \nu}(\sigma,t) d\Gamma \right) ds\ dt \right| \leq C \|v\|_{L^2(\Sigma)} \left\| \frac{\partial \varphi^M}{\partial \nu} \right\|_{L^2(\Sigma)} \leq \frac{K_3}{M^5}, \tag{2.51}$$

for some positive K_2 and K_3, again independent of M.

Consequently, (2.49), (2.50) and (2.51) lead to

$$\frac{C_1}{M^{15/4}} \leq \frac{C_4}{M^5},$$

which is an absurd if M is sufficiently large.

2.4 SOME ADDITIONAL COMMENTS AND QUESTIONS

2.4.1 The lack of null controllability for the 2D Stokes equations with a memory term

A result identical to Theorem 2.1 can be established for the two-dimensional Stokes system. As before, it suffices to consider the case where Ω is a ball of radius R centered at the origin. Now, the eigenfunctions (φ_n, q_n) and eigenvalues λ_n are given by

$$\begin{cases} \lambda_n^{1/2} R = j_{1,n} \\ \psi_n(r) = \dfrac{1}{\lambda_n} \displaystyle\int_{\lambda_n^{1/2} r}^{\lambda_n^{1/2} R} J_1(\sigma)\, d\sigma \\ q_n \equiv 0 \\ \varphi_n(x,y) = \dfrac{J_1(\lambda_n^{1/2} r)}{\lambda_n^{1/2} r} \, (-y, x)\,, \end{cases} \tag{2.52}$$

where J_1 is the first order Bessel function of the first kind and $j_{1,n}$ is the n-th positive root of J_1 (for simplicity, x and y denote the coordinates of a generic point in Ω).

Thanks to (LORCH; MULDOON, 2008, Lemma 1), λ_n satisfies the following inequality:

$$\frac{\pi^2}{R^2}\left(n + \frac{1}{8}\right)^2 \le \lambda_n \le \frac{\pi^2}{R^2}\left(n + \frac{1}{4}\right)^2 \quad \forall\, n \ge 1. \tag{2.53}$$

Taking into account $(2.52)_1$, a simple computation gives:

$$\left.\frac{\partial \varphi_n}{\partial \nu}\right|_{\partial\Omega} = J_1'(\lambda_n^{1/2} R)\left(-\frac{y}{R}, \frac{x}{R}\right). \tag{2.54}$$

On the other hand, thanks to (2.53), the following estimates also hold:

$$\begin{aligned} \|\varphi_n\|^2 &= \frac{1}{\lambda_n} \int_\Omega [J_1(\lambda_n^{1/2} r)]^2 \, dx\, dy \\ &= \frac{2\pi}{\lambda_n^2} \int_0^{j_{1,n}} [J_1(s)]^2 s\, ds \\ &\ge \frac{2\pi}{\lambda_n^2} \int_0^1 J_1^2(r) r\, dr \\ &\ge \frac{2\pi C}{\lambda_n^2}. \end{aligned} \tag{2.55}$$

Then, as in the 3D case, we can define $\gamma_n := J_1'(\lambda_n^{1/2} R)$. Thanks to $(2.52)_1$, it is not difficult to see that $\gamma_n = J_0(\lambda_n^{1/2} R)$ and, consequently, it is bounded independently of n. In view of (2.53), (2.54), (2.55) and the boundedness of γ_n, the proof of Theorem 2.1 can be adapted and the desired non-controllability result is deduced.

2.4.2 The heat equation with memory

Using arguments similar to those in the previous sections, the non-controllability results obtained in (GUERRERO; IMANUVILOV, 2013) for (2.5) can be extended to more general situations. More precisely, the following problem for the heat equation with memory can be considered:

$$\begin{cases} y_t - \Delta y - b\displaystyle\int_0^t e^{-a(t-s)}\Delta y(\cdot\,,s)\,ds = 0 & \text{in} \quad Q, \\ y = v & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y_0 & \text{in} \quad \Omega. \end{cases}$$

It would be interesting to investigate which are the most general conditions for a time-dependent memory kernel K under which Theorem 2.1 still holds for the corresponding system

$$\begin{cases} y_t - \Delta y - \displaystyle\int_0^t K(t-s)\Delta y(\cdot\,,s)\,ds = 0 & \text{in} \quad Q, \\ y = v & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = y_0 & \text{in} \quad \Omega. \end{cases}$$

Some results in the one-dimensional case have been obtained in (HALANAY; PANDOLFI, 2014).

2.4.3 Controls with less components

The approximate controllability result for (2.6), proved in (DOUBOVA; FERNÁNDEZ-CARA, 2012), can be improved with respect to the number of scalar controls. More precisely, we will use similar ideas from (LIONS, 1996) to show a new unique continuation property for the solutions to (2.4), i.e.:

$$\varphi_i = 0 \quad \text{in} \quad \omega\times(0,T), \quad \text{for} \quad 1 \le i < n \quad \Rightarrow \quad \varphi = 0 \quad \Omega\times(0,T),$$

where $\omega \subset \Omega$ is an open subset and $n \ge 2$. Indeed, let us first notice that the free divergence condition for φ implies that p is harmonic in Ω with respect to x for all $t \in (0,T)$. On the other hand, since $\varphi_i = 0$ in $\omega\times(0,T)$, for $1 \le i < n$, we deduce that

$$\frac{\partial p}{\partial x_i} = 0 \quad \text{in} \quad \omega\times(0,T) \quad \text{for} \quad 1 \le i < n.$$

Hence, elliptic unique continuation guarantees that

$$\frac{\partial p}{\partial x_i} = 0 \quad \text{in} \quad \Omega\times(0,T) \quad \text{for} \quad 1 \le i < n$$

and then p is a function that only depends of x_n.

Now, one can see that φ_i, for $1 \le i < n$, satisfies the following backward heat equation with memory:

$$\begin{cases} -\varphi_{i,t} - \Delta\varphi_i - b\displaystyle\int_t^T e^{-a(s-t)}\Delta\varphi_i(\cdot\,,s)\,ds = 0 & \text{in} \quad Q, \\ \varphi_i = 0 & \text{on} \quad \Sigma, \\ \varphi_i(\cdot\,,T) = \varphi_{0,i} & \text{in} \quad \Omega. \end{cases}$$

Since the unique continuation result (DOUBOVA; FERNÁNDEZ-CARA, 2012, Lemma 2.3) holds for both heat equation and Stokes equations with memory and the fact that $\varphi_i = 0$ in $\omega \times (0,T)$, for $1 \leq i < n$, we have

$$\varphi_i = 0 \quad \text{in} \quad \Omega \times (0,T) \quad \text{for} \quad 1 \leq i < n.$$

Finally, the free divergence condition and the Dirichlet boundary condition lead to

$$\varphi = 0 \quad \text{in} \quad \Omega \times (0,T).$$

Remark 2.8. *In general, in the three-dimensional case, the approximate controllability result does not hold using only one control. Indeed, let us consider $L > 0$ and $\Omega = G \times (0,L)$, where $G \subset \mathbb{R}^2$ is a bounded domain such that there exists an eigenfunction of the Stokes equations in G in which the corresponding pressure is zero (for example, if G is a ball then this property holds). More precisely, there exist a nontrivial vector field $u = (u_1, u_2)$ and a number $\lambda > 0$ such that*

$$\begin{cases} -\Delta_{(x,y)} u = \lambda u & \text{in} \quad G, \\ \nabla_{(x,y)} \cdot u = 0 & \text{in} \quad G, \\ u = 0 & \text{on} \quad \partial G. \end{cases}$$

To sum up, let us define $\varphi(x,y,z,t) := \alpha(t) \sin(\pi z/L)\,(u_1(x,y), u_2(x,y), 0)$, where α is the solution to (2.22) with $\beta_n = 1$ and $\lambda_n = \lambda + \pi^2/L^2$. Then, φ is a nonzero solution to (2.4) such that $\varphi_3 \equiv 0$.

2.4.4 Hyperbolic equations with memory

Differently to the case of the heat and Stokes equations, the wave equation with memory is exactly controllable if the usual geometric control conditions are satisfied.

This is true, for instance, for a hyperbolic integro-differential equation of the form

$$\begin{cases} y_{tt} - a(t)\Delta y + b(t) y_t + c(t) y - \displaystyle\int_0^t K(t,s)\Delta y(\cdot\,,s)\,ds = 0 & \text{in} \quad \Omega \times (0,T), \\ y = v 1_\gamma & \text{on} \quad \partial\Omega \times (0,T), \\ y(\cdot\,,0) = 0, \quad y_t(\cdot,0) = 0 & \text{in} \quad \Omega, \end{cases}$$

as long as the kernel $K = K(t,s)$ is assumed to belong to $C^2(\mathbb{R}^2_+)$; for details, see (KIM, 1993). It would be interesting to analyze if the exact controllability results obtained there can be extended to the hyperbolic Stokes equation with memory:

$$\begin{cases} y_{tt} - \Delta y - \displaystyle\int_0^t K(t,s)\Delta y(\cdot\,,s)\,ds + \nabla p = 0 & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = v 1_\gamma & \text{on} \quad \Sigma, \\ y(\cdot\,,0) = 0, \quad y_t(\cdot\,,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

2.4.5 Nonlinear systems with memory

Recall that the null and approximate controllability of (2.7) and (2.8) are open questions. It would be very interesting to see whether or not the effect of the nonlinear terms is sufficient to modify the controllability properties of the linearized systems. This is the case, for instance, for the equation studied in (CORON; LISSY, 2014).

2.4.6 Moving control

It would be interesting to see whether or not it is possible to use moving control strategy to obtain a null controllability result for Stokes with memory (2.6).

In (CHAVES-SILVA; ROSIER; ZUAZUA, 2014), it was analyzed the control of a model of viscoelasticity consisting of a wave equation with both viscous Kelvin-Voigt and frictional damping:

$$\begin{cases} y_{tt} - \Delta y - \nu \Delta y_t + \beta(x) y_t = 1_{\omega(t)} v & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y_0, \quad y_t(\cdot, 0) = y_1 & \text{in} \quad \Omega. \end{cases}$$

The strategy relies to allow the control region to move (by the flow of an ODE) in order to cover the whole domain driving the time interval $(0, T)$. At each time t the control support is located in the moving subset $\omega(t)$ of Ω.

This strategy has been extended to other equation involving time integral terms, see (CHAVES-SILVA; SOUZA, 2020; CHAVES-SILVA; ZHANG; ZUAZUA, 2017; KHAPALOV, 1995; KUNISCH; SOUZA, 2018; LÜ; ZHANG; ZUAZUA, 2017).

3 CONTROLLABILITY OF THE BOUSSINESQ SYSTEM

In this chapter, we deal with the global exact controllability to the trajectories of the Boussinesq system. We consider 2D and 3D smooth bounded domains. The velocity field of the fluid must satisfy a Navier slip-with-friction boundary condition and a Robin boundary condition is imposed to the temperature. We assume that one can act on the velocity and the temperature on an arbitrary small part of the boundary. The proof relies on three main arguments. First, we transform the problem into a distributed controllability problem by using a domain extension procedure. Then, we prove a global approximate controllability result by following the strategy of Coron *et al* [J. Eur. Math. Soc., 22 (2020), pp. 1625-1673], which deals with the Navier-Stokes equations. This part relies on the controllability of the inviscid Boussinesq system and asymptotic boundary layer expansions. Finally, we conclude with a local controllability result that we establish with the help of a linearization argument and appropriate Carleman estimates.

3.1 INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a smooth bounded domain with $\Gamma := \partial\Omega$ and let $\Gamma_c \subset \Gamma$ be a non-empty open subset which intersects all connected components of Γ. It will be said that Γ_c is the control boundary. Let us set

$$L^2_c(\Omega)^n := \{u \in L^2(\Omega)^n \ : \ \nabla \cdot u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \Gamma \setminus \Gamma_c\},$$

where $\nu = \nu(x)$ is the outward unit normal vector to Ω at the points $x \in \Gamma$. For a given vector field f, we denote by $[f]_{tan}$ the tangential part of f, $D(f)$ the deformation tensor and $N(f)$ the tangential Navier boundary operator, respectively given by the following formula:

$$\begin{aligned} [f]_{tan} &:= f - (f \cdot \nu)\nu, \\ D(f) &:= \frac{1}{2}\left(\nabla f + \nabla f^t\right), \\ N(f) &:= [D(f)\nu + Mf]_{tan}, \end{aligned} \tag{3.1}$$

where $M = M(t, x)$ is a smooth, symmetric matrix-valued function related to the rugosity of the boundary, called the *friction matrix*. We also set

$$R(\theta) := \frac{\partial \theta}{\partial \nu} + m\theta, \tag{3.2}$$

where $m = m(t, x)$ is a smooth function again related to the properties of the boundary, known as the *heat transfer coefficient*.

Let $T > 0$ be a final time. We will consider the Boussinesq system

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = \theta e_n & \text{in} \quad (0,T) \times \Omega, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0 & \text{in} \quad (0,T) \times \Omega, \\ \nabla \cdot u = 0 & \text{in} \quad (0,T) \times \Omega, \\ u \cdot \nu = 0, \quad N(u) = 0 & \text{on} \quad (0,T) \times (\Gamma \setminus \Gamma_c), \\ R(\theta) = 0 & \text{on} \quad (0,T) \times (\Gamma \setminus \Gamma_c), \\ u(0, \cdot\,) = u_0, \quad \theta(0, \cdot\,) = \theta_0 & \text{in} \quad \Omega, \end{cases} \tag{3.3}$$

where the functions $u = u(t,x)$, $\theta = \theta(t,x)$ and $p = p(t,x)$ must be respectively viewed as the velocity field, the temperature and the pressure of the fluid and e_n is the n-th vector of the canonical basis of $\mathbb{R}^n$, i.e., $e_n = (0,1)$ if $n = 2$ and $e_n = (0,0,1)$ if $n = 3$.

In the controlled system (3.3), at any time $t \in [0,T]$, $(u,\theta)(t, \cdot\,) : \Omega \to \mathbb{R}^n \times \mathbb{R}$ will be interpreted as the *state* of the system and its restriction $(u,\theta)(t, \cdot\,) : \Gamma_c \to \mathbb{R}^n \times \mathbb{R}$ will be regarded as the associated *control*.

3.1.1 Main result

In this section, we state the main result of the paper, which concerns small-time global boundary exact controllability to the trajectories of (3.3).

Let us introduce the following notation:

$$W_T(\Omega) := [C^0_w([0,T]; L^2_c(\Omega)^n) \cap L^2(0,T; H^1(\Omega)^n)] \times C^0_w([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))].$$

We have the following result:

Theorem 3.1. *Let $T > 0$ be a positive time, let $(u_0, \theta_0) \in L^2_c(\Omega)^n \times L^2(\Omega)$ be an initial state and let $(\overline{u}, \overline{\theta}) \in W_T(\Omega)$ be a weak trajectory of* (3.3)*. Then, there exists a weak controlled solution to* (3.3) *in $W_T(\Omega)$ that satisfies*

$$(u, \theta)(T, \cdot\,) = \left(\overline{u}, \overline{\theta}\right)(T, \cdot\,). \tag{3.4}$$

Several comments are in order.

Remark 3.2. F*or the precise notions of weak trajectory and weak controlled solution, see Definition 3.6 below. Essentially, we require to belong to $W_T(\Omega)$ and satisfy the PDEs in* (3.3) *in the weak (distributional) sense.*

Remark 3.3. In Theorem 3.1, we do not indicate explicitly which are the controls. Indeed, once the controlled solution is constructed, we see that the associated controls are the appropriate traces of the solution on $(0,T) \times \Gamma_c$.

Remark 3.4. Theorem 3.1 is stated as an existence result. The lack of uniqueness comes from two main reasons:

- If we do not specify any restriction, there exist many controls that drive the solution to (3.3) to the desired trajectory.
- Even if we select a criterion in order to fix the control without ambiguity, it is not known if weak solutions are unique in the 3-D case (in 2-D, it is known that weak solutions are unique; see (BOYER; FABRIE, 2012; LERAY, 1934) for the Navier-Stokes case).

3.1.2 Bibliographical comments

We now present some existing results in the literature which are related to Theorem 3.1.

There are several papers where the controllability properties of the Boussinesq equations are investigated. Most of them are local results covering boundary conditions of various kinds. For instance, in (FURSIKOV; IMANUVILOV, 1998), the local exact boundary controllability to the trajectories was obtained with boundary controls acting over the whole boundary; in (FURSIKOV; IMANUVILOV, 1999), the exact controllability with distributed controls and periodic boundary conditions was analyzed; in (GUERRERO, 2006a), the author proved the local exact controllability to the trajectories with Dirichlet boundary conditions; this situation is also handled with a reduced number of controls in (CARREÑO, 2012; FERNÁNDEZ-CARA et al., 2005; FERNÁNDEZ-CARA et al., 2006; GONZÁLEZ-BURGOS; GUERRERO; PUEL, 2009). For incompressible ideal fluids, this subject has been investigated by Coron (CORON, 1993; CORON, 1996a) and Glass (GLASS, 1997b; GLASS, 1997a; GLASS, 2000b) and also by Fernández-Cara *et al.* (FERNÁNDEZ-CARA; SANTOS; SOUZA, 2016) when heat effect are considered.

On the other hand, the literature on the Navier-Stokes and Boussinesq equations with Navier-slip boundary conditions is scarce. Let us recall some controllability results obtained for the Navier-Stokes system: in (CORON, 1996b), a small-time global result for the 2D equations has been proved where the exact controllability can be achieved in the interior of the domain and the information about the solution near the boundaries is unknown; the residual boundary layers are too strong to be handled satisfactorily during the control design strategy. Guerrero proved in (GUERRERO, 2006b) the local exact controllability to the trajectories with general nonlinear Navier boundary conditions. Finally, the small-time global exact controllability with Navier slip-with-friction boundary conditions towards weak trajectories was proved in (CORON; MARBACH; SUEUR, 2020) by Coron, Marbach and Sueur. This article answers the famous open question by J.-L. Lions concerning global null-controllability of the Navier-Stokes equations with boundary conditions of this kind. A similar result, controlling strong solutions with regular controls is obtained in (LIAO; SUEUR; ZHANG, 2020). In what concerns the Boussinesq system with Navier-slip boundary conditions, see (IMANUVILOV, 1998; KIM; CAO, 2017; MONTOYA, 2020) for some local results.

3.1.3 Strategy of the proof

We present in this section the main of ideas and results needed for the proof of Theorem 3.1.

- In Section 3.2, we will reduce the task to a *distributed controllability* problem by applying a classical domain extension technique. Then, we will limit our considerations to *smooth initial data* by using the smoothing effect of the uncontrolled Boussinesq system.
- In Section 3.3, starting from a sufficiently smooth initial data, we prove a *global approximate controllability* result. In order to do this, we follow the strategy performed by Coron, Marbach and Sueur in (CORON; MARBACH; SUEUR, 2020) in the Navier-Stokes case.
- In Section 3.4, we prove a *local controllability result* by using Carleman inequalities for the adjoint of the linearized system and a fixed-point strategy.
- In Section 3.5, we combine all these arguments and achieve the proof.

In general, the notation will be abridged. For instance, if $u \in H^2(\Omega)^n$ and $\theta \in H^1(\Omega)$, $\|(u,\theta)\|_{H^2\times H^1}$ will stand for the norm of (u,θ) in the space $H^2(\Omega)^n \times H^1(\Omega)$. The scalar product and norm in L^2 spaces will be denoted by $(\,\cdot\,,\,\cdot\,)$ and $\|\cdot\|$, respectively. The symbol C will stand for a generic positive constant.

3.2 DOMAIN EXTENSION AND SMOOTHING EFFECT

3.2.1 Domain extension

We consider an extended bounded domain $\mathcal{O}$ in such a way that $\Gamma_c \subset \mathcal{O}$ and $\Gamma \setminus \Gamma_c \subset \Gamma_{\mathcal{O}} := \partial\mathcal{O}$. In the sequel, we will denote by $\tilde{\nu} = \tilde{\nu}(x)$ the outward unit normal vector to $\mathcal{O}$ at the points $x \in \partial\mathcal{O}$. We will assume that M and m are extended to $[0,T] \times \partial\mathcal{O}$ as smooth functions such that M is symmetric on $(0,T) \times \partial\mathcal{O}$. This allows to speak of $N(u)$ and $R(\theta)$ on $(0,T) \times \partial\mathcal{O}$.

We will also need the space

$$L^2_{div}(\mathcal{O})^n := \{u \in L^2(\mathcal{O})^n \ : \ \nabla \cdot u = 0 \text{ in } \mathcal{O}, \quad u \cdot \nu = 0 \text{ on } \partial\mathcal{O}\}.$$

The following proposition enables us to extend initial conditions to the whole domain $\mathcal{O}$.

Proposition 3.5. *Let $(u_0,\theta_0) \in L^2_c(\Omega)^n \times L^2(\Omega)$ be given. There exist $(u_*,\theta_*) \in L^2(\mathcal{O})^{n+1}$ and $\sigma_* \in C^\infty_c(\mathcal{O} \setminus \overline{\Omega})$ such that*

$$u_* = u_0 \quad \text{and} \quad \theta_* = \theta_0 \ \text{in } \Omega, \qquad \nabla \cdot u_* = \sigma_* \ \text{in } \mathcal{O}, \qquad u_* \cdot \tilde{\nu} = 0 \ \text{on } \partial\mathcal{O}. \tag{3.5}$$

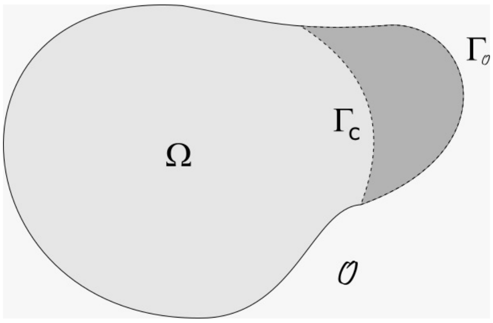

Figure 5 – Extension of the physical domain

Furthermore, (u_*, θ_*) and σ_* can be chosen depending continuously on (u_0, θ_0) in the following sense:

$$\|u_*\| + \|\sigma_*\| \leq C\|u_0\|, \qquad \|\theta_*\| \leq C\|\theta_0\|. \tag{3.6}$$

Proof. Let $\theta_* \in L^2(\mathcal{O})$ be the extension by zero of θ_0 to the whole domain $\mathcal{O}$, then we have

$$\|\theta_*\| \leq \|\theta_0\|.$$

Next, to find an appropriate extension for u_0, we first notice that, since $u_0 \in L^2_c(\Omega)^n$, the normal trace $u_0 \cdot \nu$ has a sense in $H^{-1/2}(\partial\Omega)$, see (BOYER; FABRIE, 2012, Chapter IV, Section 3.2). Let us split

$$\Gamma_c = \bigcup_{i=1}^{k} \Gamma_c^i,$$

where Γ_c^i represent the parts of Γ_c in each connected component of Γ and we stand for $(\mathcal{O} \setminus \overline{\Omega})^i$ its related extension. Also, let $\omega^i \subset\subset (\mathcal{O} \setminus \overline{\Omega})^i$ be a non-empty open subset and $\sigma_*^i \in C_c^\infty(\omega^i)$ such that

$$\int_{(\mathcal{O}\setminus\overline{\Omega})^i} \sigma_*^i = -\langle u_0 \cdot \nu, 1\rangle_{H^{-1/2}(\Gamma_c^i), H^{1/2}(\Gamma_c^i)}.$$

The following non homogeneous elliptic problem admits a unique solution $w^i \in H^1((\mathcal{O}\setminus\overline{\Omega})^i)$:

$$\begin{cases} -\Delta w^i = -\sigma_*^i & \text{in} \quad (\mathcal{O}\setminus\overline{\Omega})^i, \\ \dfrac{\partial w^i}{\partial \nu} = u_0 \cdot \nu & \text{on} \quad \Gamma_c^i, \\ \dfrac{\partial w^i}{\partial \widetilde{\nu}} = 0 & \text{on} \quad \partial(\mathcal{O}\setminus\overline{\Omega})^i \setminus \Gamma_c^i. \end{cases}$$

Let us set

$$u_* := \begin{cases} u_0 & \text{in} \quad \Omega, \\ \nabla w^i & \text{in} \quad (\mathcal{O}\setminus\overline{\Omega})^i, \quad \text{for} \quad i = 1, \ldots, k. \end{cases}$$

It is then clear that $u_* \in L^2(\mathcal{O})^n$, $\nabla \cdot u_* = \sigma_*$ in $\mathcal{O}$ and $u_* \cdot \widetilde{\nu} = 0$ on $\partial\mathcal{O}$. On the other hand, we see that, by construction, (3.5) and (3.6) are satisfied. □

Let us now present the notion of solution used throughout the paper. To this purpose, let us introduce the following notations $\mathcal{O}_T := (0,T) \times \mathcal{O}$ and $\varLambda_T := (0,T) \times \partial\mathcal{O}$. In the sequel, when there is no ambiguity, we will also denote by ν the outward unit normal to $\mathcal{O}$.

Definition 3.6. *Let $T > 0$ and $(u_0, \theta_0) \in L^2_c(\Omega)^n \times L^2(\Omega)$ be given. It will be said that $(u, \theta) \in W_T(\Omega)$ is a weak controlled trajectory of (3.3) if it is the restriction to $(0,T) \times \Omega$ of a weak Leray solution, still denoted by (u, θ), in the space $W_T(\mathcal{O})$, to the nonlinear system*

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = \theta e_n + v & \text{in} \quad \mathcal{O}_T, \\ \partial_t \theta - \Delta\theta + u \cdot \nabla\theta = w & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot u = \sigma & \text{in} \quad \mathcal{O}_T, \\ u \cdot \nu = 0, \quad N(u) = 0 & \text{on} \quad \varLambda_T, \\ R(\theta) = 0 & \text{on} \quad \varLambda_T, \\ u(0, \cdot\,) = u_*, \quad \theta(0, \cdot\,) = \theta_* & \text{in} \quad \mathcal{O}, \end{cases} \tag{3.7}$$

where $v \in H^1(0,T; L^2(\mathcal{O})^n) \cap C^0([0,T]; H^1(\mathcal{O})^n)$ and $w \in H^1(0,T; L^2(\mathcal{O})) \cap C^0([0,T]; H^1(\mathcal{O}))$ are forcing terms supported in $\overline{\mathcal{O}} \setminus \overline{\Omega}$, $\sigma \in C^\infty([0,T] \times \mathcal{O})$ is a nonhomogeneous divergence condition also supported in $\overline{\mathcal{O}} \setminus \overline{\Omega}$ and (u_*, θ_*) is an extension of (u_0, θ_0) furnished by Proposition 3.5, satisfying $\nabla \cdot u_* = \sigma(0, \cdot\,)$.

Let us state an existence result of weak solution to (3.7), whose proof is sketched in Appendix B:

Proposition 3.7. *Let us assume that $T > 0$, $(u_*, \theta_*) \in L^2(\mathcal{O})^n \times L^2(\mathcal{O})$ satisfies $u_* \cdot \nu = 0$ on $\partial\mathcal{O}$, $\sigma \in C^\infty([0,T] \times \mathcal{O})$ satisfies $\sigma(0, \cdot\,) = \nabla \cdot u_*$, $v \in H^1(0,T; L^2(\mathcal{O})^n) \cap C^0([0,T]; H^1(\mathcal{O})^n)$ and $w \in H^1(0,T; L^2(\mathcal{O})) \cap C^0([0,T]; H^1(\mathcal{O}))$. Then there exists at least one weak Leray solution (u, θ) to (3.7).*

3.2.2 Smoothing effect of the uncontrolled Boussinesq system

The goal of this section is to show that, starting from L^2 initial data, for any small time interval, one can find a time such that the solution is sufficiently smooth. More precisely, we have the following result:

Lemma 3.8. *Let us assume that $T > 0$ and $(\overline{u}, \overline{\theta}) \in C^\infty([0,T] \times \overline{\mathcal{O}})^{n+1}$ is such that $\nabla \cdot \overline{u} = 0$ in $\mathcal{O}_T$ and $\overline{u} \cdot \nu = 0$ on $\varLambda_T$. Then there exists a smooth function $\Psi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\Psi(0) = 0$ such that, for any $(r_*, q_*) \in L^2_{div}(\mathcal{O})^n \times L^2(\mathcal{O})$ and any weak Leray-Hopf* *solution* $(r,q) \in W_T(\mathcal{O})$ *to:*

$$\begin{cases} \partial_t r - \Delta r + (r\cdot\nabla)r + (\overline{u}\cdot\nabla)r + (r\cdot\nabla)\overline{u} + \nabla\pi = q e_n & in \quad \mathcal{O}_T, \\ \partial_t q - \Delta q + (r+\overline{u})\cdot\nabla q + r\cdot\nabla\overline{\theta} = 0 & in \quad \mathcal{O}_T, \\ \nabla\cdot r = 0 & in \quad \mathcal{O}_T, \\ r\cdot\nu = 0, \quad N(r) = 0 & on \quad \Lambda_T, \\ R(q) = 0 & on \quad \Lambda_T, \\ r(0,\cdot\,) = r_*, \quad q(0,\cdot\,) = q_* & in \quad \mathcal{O}, \end{cases} \tag{3.8}$$

the following property holds:

$$\exists\ t_0 \in [0,T]; \quad \|(r,q)(t_0,\cdot\,)\|_{H^3} \leq \Psi\left(\|(r_*,q_*)\|\right).$$

The proof of this lemma is quite classical but, for completeness, will be given in Appendix C.

3.3 APPROXIMATE CONTROLLABILITY PROBLEM

In this section, the goal is to prove the following approximate controllability result starting from sufficiently smooth initial data.

Proposition 3.9. *Let us assume that* $T>0$ *and* $(\overline{u},\overline{p},\overline{\theta},\overline{v},\overline{w},\overline{\sigma}) \in C^\infty([0,T]\times\overline{\mathcal{O}};\mathbb{R}^{2n+4})$ *is a smooth trajectory of* (3.7)*, with* $\overline{v}$ *and* $\overline{w}$ *supported in* $\overline{\mathcal{O}}\setminus\overline{\Omega}$*. Let* $(u_*,\theta_*) \in [H^3(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n] \times H^3(\mathcal{O})$ *be an initial state. Then, for any* $\delta>0$ *there exist regular controls* v*,* w *and* σ*, again supported in* $\overline{\mathcal{O}}\setminus\overline{\Omega}$ *and an associated weak solution to* (3.7) *satisfying*

$$\|(u,\theta)(T,\cdot\,) - (\overline{u},\overline{\theta})(T,\cdot\,)\| \leq \delta.$$

For the proof, we will follow the strategy introduced by Coron, Marbach, Sueur in (CORON; MARBACH; SUEUR, 2020). Let us explain how it works:

- First, a scale change associated to a small parameter $\varepsilon>0$ is introduced and (3.7) is transformed into a Boussinesq system with small viscosity ε that must be solved in the (long) time interval $[0,T/\varepsilon]$ starting from a small initial state, see (3.9). The advantage of this scaling is that we can benefit from the nonlinear terms $(u\cdot\nabla)u$ and $u\cdot\nabla\theta$.
- By taking formally $\varepsilon=0$, we obtain the inviscid Boussinesq system. For this hyperbolic system, we construct a particular nontrivial trajectory that connects $(0,0)\in\mathbb{R}^{n+1}$ to itself and sends any particle outside the physical domain before the final time T.

- By linearizing the inviscid Boussinesq system around the previous trajectory, we obtain a new hyperbolic linear system that is small-time globally null-controllable. Actually, what we are doing here is to apply the so called return method, due to Coron, see (CORON, 1992). Note that the linearization around the trivial state leads to a noncontrollable system.

- In the particular case of the special slip boundary condition, that is, M such that $[\nabla \times u]_{\tan} = 0$ on $\varLambda_T$ and $m \equiv 0$, we immediately conclude by estimating the remainder terms. We do not need to use the long interval time $[0, T/\varepsilon]$ to control, since the solution is already small at the intermediate time $T \in (0, T/\varepsilon)$.

- Unfortunately, in the general case, a boundary layer appears. This phenomenon was already taken into account in (IFTIMIE; SUEUR, 2011) for the Navier-Stokes PDEs. Thus, we have to introduce some corrector terms in the asymptotic expansion of the solution depending on ε in order to estimate the residual layers. The boundary layer decays but not enough. Hence, the corrector is not sufficiently small at the final time T/ε and we still cannot conclude.

- In order to overcome this difficulty, we adapt the well-prepared dissipation method, introduced by Marbach in (MARBACH, 2014). The idea is to design a control strategy that reinforces the action of the natural dissipation of the boundary layer after the intermediate time T. A desired small state is obtained at final time and we can finally achieve the proof.

In the sequel, we will frequently need vector functions $(u, p, \theta, v, w, \sigma)$ representing adequate states (u, p, θ), controls (v, w) and auxiliary functions σ, corresponding to some linear or nonlinear systems. In all cases, it will be implicitly assumed that v and w vanish outside $\overline{\mathcal{O}} \setminus \overline{\Omega}$.

3.3.1 Time scaling

Let us introduce u^ε, p^ε, etc., with

$$\begin{aligned} &u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x), \quad p^\varepsilon(t,x) := \varepsilon^2 p(\varepsilon t, x), \quad \theta^\varepsilon(t,x) := \varepsilon^2 \theta(\varepsilon t, x), \\ &v^\varepsilon(t,x) := \varepsilon^2 v(\varepsilon t, x), \quad w^\varepsilon(t,x) := \varepsilon^3 w(\varepsilon t, x) \quad \sigma^\varepsilon(t,x) := \varepsilon \sigma(\varepsilon t, x). \end{aligned}$$

In these new variables, the original system (3.7) reads

$$\begin{cases} \partial_t u^\varepsilon - \varepsilon\Delta u^\varepsilon + (u^\varepsilon\cdot\nabla)u^\varepsilon + \nabla p^\varepsilon = \theta^\varepsilon e_n + v^\varepsilon & \text{in} \quad (0,T/\varepsilon)\times\mathcal{O}, \\ \partial_t\theta^\varepsilon - \varepsilon\Delta\theta^\varepsilon + u^\varepsilon\cdot\nabla\theta^\varepsilon = w^\varepsilon & \text{in} \quad (0,T/\varepsilon)\times\mathcal{O}, \\ \nabla\cdot u^\varepsilon = \sigma^\varepsilon & \text{in} \quad (0,T/\varepsilon)\times\mathcal{O}, \\ u^\varepsilon\cdot\nu = 0, \quad N(u^\varepsilon) = 0 & \text{on} \quad (0,T/\varepsilon)\times\partial\mathcal{O}, \\ R(\theta^\varepsilon) = 0 & \text{on} \quad (0,T/\varepsilon)\times\partial\mathcal{O}, \\ u^\varepsilon(0,\cdot\,) = \varepsilon u_*, \quad \theta^\varepsilon(0,\cdot\,) = \varepsilon^2\theta_* & \text{in} \quad \mathcal{O}. \end{cases} \tag{3.9}$$

Instead of working hard in a small time interval, we now work easily during a large time interval $[0,T/\varepsilon]$. The counterpart is the small viscosity that we find now in (3.9), that can be viewed as a singular perturbation of a nonlinear inviscid system.

To prove Proposition 3.9, it is sufficient to prove that

$$\|u^\varepsilon(T/\varepsilon,\cdot\,) - \varepsilon\overline{u}(T,\cdot\,)\| = o(\varepsilon) \quad \text{and} \quad \left\|\theta^\varepsilon(T/\varepsilon,\cdot\,) - \varepsilon^2\overline{\theta}(T,\cdot\,)\right\| = o(\varepsilon^2).$$

3.3.2 The special case of the slip boundary condition

In this section, we consider a special situation where the fluid perfectly slips. In this case, the proof of Proposition 3.9 is more simple (there is no boundary layer). For the moment, we will also assume that the smooth target trajectory is zero, i.e., $(\overline{u},\overline{p},\overline{\theta},\overline{v},\overline{w},\overline{\sigma})\equiv 0$.

Thus, the friction coefficient M is assumed to be the Weingarten map (or shape operator) M_w. Thanks to (CORON; MARBACH; SUEUR, 2020, Lemma 1), on the uncontrolled boundary one has

$$u\cdot\nu = 0 \quad \text{and} \quad [\nabla\times u]_{tan} = 0 \quad \text{on} \quad \varLambda_T.$$

3.3.2.1 Ansatz with no correction term

Let us consider an asymptotic expansion of the solution:

$$\begin{aligned} &u^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon r^\varepsilon, \quad p^\varepsilon = p^0 + \varepsilon p^1 + \varepsilon\pi^\varepsilon, \quad \theta^\varepsilon = \theta^0 + \varepsilon^2\theta^1 + \varepsilon^2 q^\varepsilon \\ &v^\varepsilon = v^0 + \varepsilon v^1, \qquad\qquad w^\varepsilon = w^0 + \varepsilon^2 w^1, \qquad\quad \sigma^\varepsilon = \sigma^0. \end{aligned} \tag{3.10}$$

There is some intuition behind (3.10). The first term $(u^0,p^0,\theta^0,v^0,w^0)$ is the solution to an inviscid system, take $\varepsilon = 0$ in (3.9). It models a smooth reference trajectory around which we linearize the original system. This is exactly what we have to do when we apply the return method of Coron, see (CORON, 1992). It will be chosen in such a way that the flow flushes the initial data off the physical domain before time T. The second term $(u^1,p^1,\theta^1,v^1,w^1)$ takes into account the initial data (u_*,θ_*).Then, $(r^\varepsilon,\pi^\varepsilon,q^\varepsilon)$ contains higher order terms. At the end, we need to prove that $\|(r^\varepsilon,q^\varepsilon)(T,\cdot\,)\| = o(1)$, in order to be able to conclude.

3.3.2.2 Inviscid flow

By taking $\varepsilon = 0$ in (3.9), we obtain the following system

$$\begin{cases} \partial_t u^0 + (u^0 \cdot \nabla) u^0 + \nabla p^0 = \theta^0 e_n + v^0 & \text{in} \quad \mathcal{O}_T, \\ \partial_t \theta^0 + u^0 \cdot \nabla \theta^0 = w^0 & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot u^0 = \sigma^0 & \text{in} \quad \mathcal{O}_T, \\ u^0 \cdot \nu = 0 & \text{on} \quad \Lambda_T, \\ u^0(0, \cdot\,) = u^0(T, \cdot\,) = 0 & \text{in} \quad \mathcal{O}, \\ \theta^0(0, \cdot\,) = \theta^0(T, \cdot\,) = 0 & \text{in} \quad \mathcal{O}, \end{cases} \tag{3.11}$$

where v^0, w^0 and σ^0 are smooth forcing terms spatially supported in $\overline{\mathcal{O}} \setminus \overline{\Omega}$. We want to control (3.11) during the shorter time interval $[0, T]$ instead of $[0, T/\varepsilon]$. Let us introduce the flow $\Phi^0 = \Phi^0(s; t, x)$ associated to u^0, ie., for any (t, x), $\Phi^0(\,\cdot\,, t, x)$ solves

$$\begin{cases} \partial_s \Phi^0(s; t, x) & = & u^0(s, \Phi^0(s; t, x)), \\ \Phi^0(s; t, x)|_{s=t} & = & x. \end{cases} \tag{3.12}$$

Hence, we look for trajectories such that:

$$\forall\ x \in \overline{\mathcal{O}},\ \exists\ t_x \in (0, T),\ \Phi^0(t_x; 0, x) \notin \overline{\Omega}. \tag{3.13}$$

This property is obvious for the points x already located in $\overline{\mathcal{O}} \setminus \overline{\Omega}$. For points $x \in \overline{\Omega}$, we use the following result, whose proof can be found in (CORON, 1993; CORON, 1996a; CORON, 1996b; CORON; MARBACH; SUEUR, 2020) in the 2D case and (GLASS, 2000b) in the 3D case:

Lemma 3.10. *There exists a non-zero solution to* (3.11) $(u^0, p^0, \theta^0, v^0, w^0, \sigma^0) \in C^\infty([0, T] \times \overline{\mathcal{O}}; \mathbb{R}^{2n+4})$ *such that the associated flow* Φ_0*, defined in* (3.12)*, satisfies* (3.13)*. Moreover, we can choose* u^0*,* θ^0 *and* w^0 *such that*

$$\theta^0 = w^0 = 0 \quad \text{and} \quad \nabla \times u^0 = 0 \quad \text{in} \quad [0, T] \times \overline{\mathcal{O}} \tag{3.14}$$

and $u^0, p^0, \theta^0, v^0, w^0$ *and* σ^0 *are compactly supported in time in* $(0, T)$*.*

Note that, in the proof of this result, the assumption that Γ_c intersects all connected components of Γ must be used.

In the sequel, if needed, it will be assumed that $u^0, p^0, \theta^0, v^0, w^0$ and σ^0 have been extended by zero after time T.

3.3.2.3 Flushing

Let (u^1, θ^1) be the solution to the linear problem

$$\begin{cases} \partial_t u^1 + (u^0 \cdot \nabla) u^1 + (u^1 \cdot \nabla) u^0 + \nabla p^1 = \Delta u^0 + v^1 & \text{in} \quad \mathcal{O}_T, \\ \partial_t \theta^1 + u^0 \cdot \nabla \theta^1 = w^1 & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot u^1 = 0 & \text{in} \quad \mathcal{O}_T, \\ u^1 \cdot \nu = 0 & \text{on} \quad \varLambda_T, \\ u^1(0, \cdot\,) = u_*, \quad \theta^1(0, \cdot\,) = \theta_* & \text{in} \quad \mathcal{O}, \end{cases} \tag{3.15}$$

where v^1 and w^1 are forcing terms spatially supported in $\overline{\mathcal{O}} \setminus \overline{\Omega}$. Thanks to (3.14), we have $\Delta u^0 = \nabla(\nabla \cdot u^0) = \nabla \sigma^0$. Thus, it is smooth and can be absorbed by the source term v^1. Of course, (3.15) is a linear uncoupled system.

Lemma 3.11. *Let us assume that* $(u_*, \theta_*) \in [H^3(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n] \times H^3(\mathcal{O})$*. There exist forcing terms*

$$v^1 \in C^1([0,T]; H^1(\mathcal{O})^n) \cap C^0([0,T]; H^2(\mathcal{O})^n), \ w^1 \in C^1([0,T]; H^2(\mathcal{O})) \cap C^0([0,T]; H^3(\mathcal{O})), \tag{3.16}$$

with

$$supp(v^1, w^1) \subset\subset \overline{\mathcal{O}} \setminus \overline{\Omega} \tag{3.17}$$

such that the associated solution (u^1, θ^1) *to* (3.15) *satisfies* $(u^1, \theta^1)(T, \cdot\,) = (0, 0)$ *in* $\mathcal{O}$*. Moreover,* (u^1, θ^1) *is bounded (with respect to* ε*) in* $L^\infty(0, T; H^3(\mathcal{O})^n) \times L^\infty(0, T; H^3(\mathcal{O}))$*.*

Proof. First, note that the result for u^1 is proved in (CORON; MARBACH; SUEUR, 2020, Lemma 3).

For θ^1, we have a similar situation and we can apply the same arguments. For completeness, let us sketch the main ideas. We will use the smooth partition of unity η_ℓ for $1 \leq \ell \leq L$ defined in (CORON; MARBACH; SUEUR, 2020, Appendix A) which is related to Φ^0 as follows: thanks to (3.13), we can find $\varepsilon > 0$ and balls B_ℓ for $1 \leq \ell \leq L$ covering $\overline{\mathcal{O}}$ such that

$$\forall \ell, \ \exists t_\ell \in (\varepsilon, T - \varepsilon), \ \exists m_\ell \in \{1, \cdots, \mathcal{M}\} \quad \text{such that} \quad \Phi^0(s; 0, B_\ell) \subset Q_{m_\ell} \quad \forall s \in (t_\ell - \varepsilon, t_\ell + \varepsilon), \tag{3.18}$$

where the Q_{m_ℓ} are squares (or cubes) that never intersect $\overline{\Omega}$; hence, every ball spends a positive amount of time within a given square (cube) where we can use a localized control to act on the θ^1 profile. Here, it is assumed that the η_ℓ satisfy $0 \leq \eta_\ell(x) \leq 1$, $\sum \eta_\ell = 1$ and $supp(\eta_\ell) \subset B_\ell$.

Let us introduce a smooth function $\beta : \mathbb{R} \to [0, 1]$ with $\beta = 1$ on $(-\infty, -\varepsilon)$ and $\beta = 0$ on $(\varepsilon, +\infty)$.

For each ℓ, we consider the solution $\overline{\theta}_\ell$ to

$$\begin{cases} \partial_t \overline{\theta}_\ell + u^0 \cdot \nabla \overline{\theta}_\ell = 0 & \text{in} \quad (0,T) \times \overline{\mathcal{O}}, \\ \overline{\theta}_\ell(0,\cdot\,) = \eta_\ell \theta_* & \text{in} \quad \overline{\mathcal{O}}, \end{cases}$$

and we set $\theta_\ell(t,x) := \beta(t-t_\ell)\overline{\theta}_\ell(t,x)$. Since $\beta(T-t_\ell) = 0$ and $\beta(-t_\ell) = 1$, θ_ℓ solves

$$\begin{cases} \partial_t \theta_\ell + u^0 \cdot \nabla \theta_\ell = w_\ell & \text{in} \quad (0,T) \times \overline{\mathcal{O}}, \\ \theta_\ell(0,\cdot\,) = \eta_\ell \theta_*, \quad \theta_\ell(T,\cdot\,) = 0 & \text{in} \quad \overline{\mathcal{O}}, \end{cases}$$

where $w_\ell(t,x) := \beta'(t-t_\ell)\overline{\theta}_\ell$. Thanks to (3.18), since β' vanish outside $(-\varepsilon,\varepsilon)$, it is easy to see that w_ℓ is supported in Q_{m_ℓ}.

At this point, we take

$$\theta^1 := \sum_\ell \theta_\ell \quad \text{and} \quad w^1 := \sum_\ell w_\ell$$

and we see that the second PDE and the second initial condition in (3.15) are satisfied. Thanks to this explicit construction, the spatial regularity of w^1 and $\overline{\theta}_\ell$ are the same. Then, $w^1 \in C^1([0,T], H^2(\mathcal{O})) \cap C^0([0,T], H^3(\mathcal{O}))$. The fact that θ^1 is bounded in $L^\infty(0,T;H^3(\mathcal{O}))$ readily comes from the fact that each θ_ℓ is bounded in $L^\infty(0,T;H^3(\mathcal{O}))$. This ends the proof.

□

Lemma 3.11 is a null-controllability result. Thanks to the linearity and reversibility of (3.15), it leads to an exact controllability result:

Lemma 3.12. *Let us assume that* $(u_*,\theta_*), (u_T,\theta_T) \in [H^3(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n] \times H^3(\mathcal{O})$. *There exist* v^1 *and* w^1 *as in* (3.16) *and* (3.17) *such that the associated solution to* (3.15) *satisfies* $(u^1,\theta^1)(T,\cdot\,) = (u_T,\theta_T)$. *Moreover,* (u^1,θ^1) *is bounded (with respect to* ε*) in* $L^\infty(0,T;H^3(\mathcal{O})^n) \times L^\infty(0,T;H^3(\mathcal{O}))$.

3.3.2.4 Equations and estimates for the remainder

The equations for r^ε, π^ε and q^ε in the extended domain $\mathcal{O}$ are

$$\begin{cases} \partial_t r^\varepsilon - \varepsilon\Delta r^\varepsilon + (u^\varepsilon\cdot\nabla) r^\varepsilon + \nabla\pi^\varepsilon = f^\varepsilon - A^\varepsilon r^\varepsilon + \varepsilon q^\varepsilon e_n + \varepsilon\theta^1 e_n & \text{in} \quad \mathcal{O}_T, \\ \partial_t q^\varepsilon - \varepsilon\Delta q^\varepsilon + u^\varepsilon\cdot\nabla q^\varepsilon = h^\varepsilon - B^\varepsilon r^\varepsilon & \text{in} \quad \mathcal{O}_T, \\ \nabla\cdot r^\varepsilon = 0 & \text{in} \quad \mathcal{O}_T, \\ r^\varepsilon\cdot\nu = 0, \quad [\nabla\times r^\varepsilon]_{tan} = -[\nabla\times u^1]_{tan} & \text{on} \quad \varLambda_T, \\ R(q^\varepsilon) = -R(\theta^1) & \text{on} \quad \varLambda_T, \\ r^\varepsilon(0,\cdot\,) = 0, \quad q^\varepsilon(0,\cdot\,) = 0 & \text{in} \quad \mathcal{O}, \end{cases} \tag{3.19}$$

where we have introduced

$$f^\varepsilon := \varepsilon\Delta u^1 - \varepsilon(u^1\cdot\nabla)u^1, \quad A^\varepsilon r^\varepsilon := (r^\varepsilon\cdot\nabla)(u^0+\varepsilon u^1),$$
$$h^\varepsilon := \varepsilon\Delta\theta^1 - \varepsilon u^1\cdot\nabla\theta^1, \qquad B^\varepsilon r^\varepsilon := \varepsilon r^\varepsilon\cdot\nabla\theta^1.$$

We can establish energy estimates for the remainder by multiplying $(3.19)_1$ by r^ε and $(3.19)_2$ by q^ε. Indeed, after integration by parts, and thanks to the interpolation inequality in (BOYER; FABRIE, 2012, Theorem $III.2.36$)), we easily obtain the following estimates

$$-\int_{\partial\mathcal{O}} q^\varepsilon \frac{\partial q^\varepsilon}{\partial\nu}\, d\Gamma = \int_{\partial\mathcal{O}} m|q^\varepsilon|^2\, d\Gamma + \int_{\partial\mathcal{O}} q^\varepsilon R(\theta^1)\, d\Gamma, \tag{3.20}$$

$$\left|\int_{\partial\mathcal{O}} q^\varepsilon R(\theta^1)\, d\Gamma\right| \le \|q^\varepsilon\|_{L^2(\partial\mathcal{O})}\left\|R(\theta^1)\right\|_{L^2(\partial\mathcal{O})} \le C\|q^\varepsilon\|_{H^1}\left\|\theta^1\right\|_{H^2},$$
$$\left|\int_{\partial\mathcal{O}} m|q^\varepsilon|^2\, d\Gamma\right| \le C\|q^\varepsilon\|_{L^2}\|q^\varepsilon\|_{H^1}$$

and

$$\begin{aligned}&\frac{d}{dt}(\|r^\varepsilon\|^2+\|q^\varepsilon\|^2)+\varepsilon(\|\nabla\times r^\varepsilon\|^2+\|\nabla q^\varepsilon\|^2)\\ &\le C(\varepsilon+\|\sigma^0\|_{L^\infty}+\|f^\varepsilon\|+\|A^\varepsilon\|_{L^\infty}+\|B^\varepsilon\|_{L^\infty}+\|h^\varepsilon\|)(\|r^\varepsilon\|^2+\|q^\varepsilon\|^2)\\ &\quad+(2\varepsilon\|u^1\|_{H^2}^2+\|f^\varepsilon\|+C\varepsilon\|\theta^1\|_{H^2}^2+\|h^\varepsilon\|),\end{aligned} \tag{3.21}$$

where the boundary term for r^ε is bounded in a similar way as in (CORON; MARBACH; SUEUR, 2020, Section 2.5).

By applying Gronwall's inequality and Lemma 3.11, we deduce that

$$\|r^\varepsilon\|^2_{L^\infty(L^2)}+\|q^\varepsilon\|^2_{L^\infty(L^2)}+\varepsilon\left(\|\nabla\times r^\varepsilon\|^2+\|\nabla q^\varepsilon\|^2\right)=O(\varepsilon).$$

Consequently, at time T, since $(u^0,\theta^0)(T,\cdot)=(u^1,\theta^1)(T,\cdot)=(0,0)$, we find:

$$\|u^\varepsilon(T,\cdot)\| \le \|\varepsilon r^\varepsilon(T,\cdot)\| \le O(\varepsilon^{3/2}) \quad\text{and}\quad \|\theta^\varepsilon(T,\cdot)\| \le \|\varepsilon^2 q^\varepsilon(T,\cdot)\| \le O(\varepsilon^{5/2}).$$

This concludes the proof of Proposition 3.9 in the slip boundary condition case.

Remark 3.13. In the previous proof, we have used in a crucial way the homogeneous Robin boundary conditions satisfied by θ^ε. Indeed, we have used (3.20), among others. Contrarily, with homogeneous Dirichlet boundary conditions on q^ε, we have

$$\left|\int_{\partial\mathcal{O}} q^\varepsilon\frac{\partial q^\varepsilon}{\partial\nu}d\Gamma\right| = \left|\int_{\partial\mathcal{O}} \theta^1\frac{\partial q^\varepsilon}{\partial\nu}d\Gamma\right| \le C\|q^\varepsilon\|_{H^2}\|\theta^1\|_{H^1}.$$

But unfortunately, the norm $\|q^\varepsilon\|_{H^2}$ cannot be absorbed by the left hand side of (3.21).

3.3.3 The case of Navier slip-with-friction boundary conditions

We come back to the general case, i.e. Navier slip-with-friction boundary conditions.

3.3.3.1 Ansatz with correction term

Let us introduce a smooth function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ such that $\varphi = 0$ on $\partial\mathcal{O}$, $\varphi > 0$ in $\mathcal{O}$, $\varphi < 0$ outside of $\overline{\mathcal{O}}$ and $|\varphi(x)| = dist(x, \partial\mathcal{O})$ in a small neighborhood of $\partial\mathcal{O}$. Then, $\nu = -\nabla\varphi$ near $\partial\mathcal{O}$ and ν can be extended smoothly within the full domain $\mathcal{O}$.

Following the original boundary layer expansion for Navier slip-with-friction boundary conditions proved in (IFTIMIE; SUEUR, 2011) by Iftimie and Sueur, we introduce the following expansions of the variables and the forcing terms:

$$\begin{cases} u^\varepsilon(t,x) &= u^0(t,x) + \sqrt{\varepsilon}\rho\left(t, x, \dfrac{\varphi(x)}{\sqrt{\varepsilon}}\right) + \varepsilon u^1(t,x) + \cdots + \varepsilon r^\varepsilon(t,x), \\ p^\varepsilon(t,x) &= p^0(t,x) + \varepsilon p^1(t,x) + \cdots + \varepsilon \pi^\varepsilon(t,x), \\ \theta^\varepsilon(t,x) &= \theta^0(t,x) + \varepsilon^2\theta^1(t,x) + \varepsilon^2 q^\varepsilon(t,x), \end{cases} \tag{3.22}$$

$$\begin{cases} v^\varepsilon(t,x) &= v^0(t,x) + \sqrt{\varepsilon}v^\rho\left(t, x, \dfrac{\varphi(x)}{\sqrt{\varepsilon}}\right) + \varepsilon v_1^1(t,x), \\ w^\varepsilon(t,x) &= w^0(t,x) + \varepsilon^2 w^1(t,x), \\ \sigma^\varepsilon(t,x) &= \sigma^0(t,x). \end{cases}$$

Compared to the previous expansion (3.10), since u^0 cannot satisfy the Navier slip-with-friction boundary condition on $\partial\mathcal{O}$, the expansion (3.22) introduces a boundary correction ρ. This profile is expressed in terms of both the slow space variable $x \in \mathcal{O}$ and a fast scalar variable $z = \varphi(x)/\sqrt{\varepsilon}$. In the equations of (3.22), the missing terms will help us to prove that the remainder is small; the details are given in Section 3.3.3.4. We use the profiles (u^0, θ^0) and (u^1, θ^1) (extended by zero for $t > T$) introduced in the previous sections, see Sections 3.3.2.2 and 3.3.2.3. The following sections are devoted to analyze and estimate the terms of the expansion in (3.22).

The boundary layer corrector will be given as the solution to an initial boundary value problem with a boundary condition associated to the extra variable. As in (IFTIMIE; SUEUR, 2011), the boundary layer correction will be described by a tangential vector field $\rho = \rho(t,x,z)$ satisfying the equation:

$$\begin{cases} \partial_t\rho + [(u^0\cdot\nabla)\rho + (\rho\cdot\nabla)u^0]_{tan} + u^0_\flat z\partial_z\rho - \partial_{zz}\rho = v^\rho & \text{in } \mathbb{R}_+\times\overline{\mathcal{O}}\times\mathbb{R}_+, \\ \partial_z\rho(t,x,0) = g^0(t,x) & \text{in } \mathbb{R}_+\times\overline{\mathcal{O}}, \\ \rho(0,x,z) = 0 & \text{in } \overline{\mathcal{O}}\times\mathbb{R}_+, \end{cases} \tag{3.23}$$

where we have used the following notation:

$$\begin{aligned} u^0_\flat(t,x) &:= -\frac{u^0(t,x)\cdot\nu(x)}{\varphi(x)} \quad \text{in } \mathbb{R}_+\times\mathcal{O}, \\ g^0(t,x) &:= 2\chi(x)N(u^0)(t,x) \quad \text{in } \mathbb{R}_+\times\mathcal{O}, \end{aligned} \tag{3.24}$$

with a smooth cut-off function χ satisfying $\chi = 1$ in a neighbourhood of the boundary $\partial\mathcal{O}$.

We can formally obtain (3.23) by plugging the expansion $u^0 + \sqrt{\varepsilon}\rho(t, x, \varphi(x)/\sqrt{\varepsilon})$ into (3.9) and keeping the terms of order $\sqrt{\varepsilon}$.

The following points are in order:

- v^ρ must be viewed as a smooth control whose spatial support is located outside of $\overline{\Omega}$. With the help of the transport term, this control will enable us to modify the behavior of ρ inside the physical domain Ω.

- ρ depends on $n+1$ spatial variables (n slow variables x_i and one fast variable z); it is thus not set in curvilinear coordinates. It is implicitly assumed that ν actually refers to the extension $-\nabla\varphi$ of the normal and, in turn, this furnishes extensions of the identities in (3.1).

- We will check that the construction above satisfies $v^\rho \cdot \nu = 0$. Since the equation is linear, it preserves the relation $\rho(0, x, z) \cdot \nu(x) = 0$ at initial time. Thus, the boundary profile will be tangential, even inside the domain. Actually, this is the reason why the equation (3.23) is linear; see (IFTIMIE; SUEUR, 2011, Section 2) for more details.

- In (3.24), the role of the function χ is to ensure that ρ is compactly supported near $\partial\mathcal{O}$.

- Since u^0 is smooth and tangent to the boundary, a Taylor expansion proves that $u^0_\flat$ is smooth in $\overline{\mathcal{O}}$.

- The boundary layer profile ρ does not depend on ε.

3.3.3.2 Well-prepared dissipation method

Unlike in the previous section, where T is the fixed time control, we will use here virtually long time intervals $[0, T/\varepsilon]$ to dissipate the boundary layer.

The most natural strategy would be to use that u^0 is equal to 0 after time T. Then (3.23) would be reduced to a heat equation posed on the half line $\mathbb{R}^+$ with homogeneous Neumann boundary conditions and the boundary layer would decay. Unfortunately, this decay is too slow: one can prove that $\sqrt{\varepsilon}\rho(T/\varepsilon, \cdot, \varphi(\cdot)/\sqrt{\varepsilon}) = O(\varepsilon)$, see (CORON; MARBACH; SUEUR, 2020, Section 3.2). Therefore, by dividing by ε, $u(T, \cdot) = O(1)$ and this is not enough for using the local result at the end.

This is why we use the source v^ρ to prepare the dissipation of the boundary layer.

Let us define the following weighted Sobolev spaces

$$H^{s,k}(\mathbb{R}) := \left\{ f \in H^s(\mathbb{R}) \; ; \; \sum_{|\alpha|=0}^{s} \int_{\mathbb{R}} (1 + |z|^2)^k |\partial^\alpha f(z)|^2 dz < +\infty \right\},$$

endowed with the corresponding (natural) norms. In (CORON; MARBACH; SUEUR, 2020, Lemma 7), the following result is proved:

Lemma 3.14. *Let $k \geq 1$ and $u^0 \in C^\infty([0,T] \times \overline{\mathcal{O}})$ be a fixed reference flow in (3.11). There exists $v^\rho \in C^\infty(\mathbb{R}_+ \times \overline{\mathcal{O}} \times \mathbb{R}_+)$ with $v^\rho \cdot \nu = 0$, such that the x-support is included in $\overline{\mathcal{O}} \setminus \overline{\Omega}$, the time support is compact in $(0,T)$ and, for any $s, p \in \mathbb{N}$ and any $0 \leq m \leq k$, the associated boundary layer profile satisfies:*

$$|\rho(t,\cdot,\cdot)|_{H^p_x(H^{s,m}_z)} \leq C \left| \frac{\log(2+t)}{2+t} \right|^{\frac{1}{4}+\frac{k}{2}-\frac{m}{2}}, \tag{3.25}$$

where the positive constant C depends on p, s, m and u^0 but not on t.

The interest of Lemma 3.14 is twofold.

- The estimate (3.25) will be used to show that the source terms generated by the boundary layer are integrable in long time and the equation satisfied by the remainder term is well-posed.
- It will be also used to prove that the boundary layer is sufficiently small at time T/ε.

Remark 3.15. A more ambitious idea would be to design a control strategy to get exactly $\rho(T,\cdot,\varphi(\cdot)/\sqrt{\varepsilon}) \equiv 0$. But, unfortunately, it can be proved that (3.23) is not null-controllable at time T, see (CORON; MARBACH; SUEUR, 2020, Section 3.5).

3.3.3.3 Technical profiles

For a function $f = f(t,x,z)$, we will use the notation $\{f\}$ to denote its values at points (t,x,z) with $z = \varphi(x)/\sqrt{\varepsilon}$. The full decomposition will be the following

$$\begin{aligned}
u^\varepsilon &= u^0 + \sqrt{\varepsilon}\{\rho\} + \varepsilon u^1 + \varepsilon \nabla \zeta^\varepsilon + \varepsilon\{\beta\} + \varepsilon r^\varepsilon, \\
p^\varepsilon &= p^0 + \varepsilon\{\psi\} + \varepsilon p^1 + \varepsilon \mu^\varepsilon + \varepsilon \pi^\varepsilon, \\
\theta^\varepsilon &= \theta^0 + \varepsilon^2 \theta^1 + \varepsilon^2 q^\varepsilon, \\
v^\varepsilon &= v^0 + \sqrt{\varepsilon}\{v^\rho\} + \varepsilon v^1, \\
w^\varepsilon &= w^0 + \varepsilon^2 w^1, \\
\sigma^\varepsilon &= \sigma^0.
\end{aligned} \tag{3.26}$$

The functions β, ζ^ε and ψ are given as follows:

$$\beta(t,x,z) = -2e^{-z} N(\rho)(t,x,0) - \nu(x) \int_z^{+\infty} \nabla_x \cdot \rho(t,x,z')dz', \tag{3.27}$$

$$\begin{cases} \Delta\zeta^\varepsilon = -\{\nabla\cdot\beta\} & \text{in} \quad \mathcal{O}, \\ \partial_\nu\zeta^\varepsilon = -\beta(t,\cdot,0)\cdot\nu & \text{on} \quad \partial\mathcal{O}, \end{cases} \tag{3.28}$$

$$\psi = \psi(t,x,z) \text{ satisfies } [(u^0\cdot\nabla)\rho+(\rho\cdot\nabla)u^0]\cdot\nu = \partial_z\psi \text{ and } \psi(t,x,z) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow +\infty. \tag{3.29}$$

It is proved in (CORON; MARBACH; SUEUR, 2020, Section 4.2) that the definitions (3.27), (3.28) and (3.29) are compatible with (3.9) and, furthermore, the following estimates hold:

$$\|\beta(t,\cdot,\cdot)\|_{H^p_x(H^{s,k}_z)} \le C\|\rho(t,\cdot,\cdot)\|_{H^{p+1}_x(H^{s+1,k+2}_z)}, \tag{3.30}$$

$$\|\zeta^\varepsilon(t,\cdot)\|_{H^4} \le C\left(\varepsilon^{-3/4}\|\beta(t,\cdot,\cdot)\|_{H^4_x(H^{2,0}_z)} + \|\rho(t,\cdot,\cdot)\|_{H^3_x(H^{0,1}_z)}\right), \tag{3.31}$$

$$\|\zeta^\varepsilon(t,\cdot)\|_{H^3} \le C\left(\varepsilon^{-1/4}\|\beta(t,\cdot,\cdot)\|_{H^3_x(H^{1,0}_z)} + \|\rho(t,\cdot,\cdot)\|_{H^2_x(H^{0,1}_z)}\right), \tag{3.32}$$

$$\|\zeta^\varepsilon(t,\cdot)\|_{H^2} \le C\left(\varepsilon^{1/4}\|\beta(t,\cdot,\cdot)\|_{H^2_x(H^{0,0}_z)} + \|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{0,1}_z)}\right), \tag{3.33}$$

$$\|\psi(t,\cdot,\cdot)\|_{H^1_x(H^{0,0}_z)} \le C\|\rho(t,\cdot,\cdot)\|_{H^2_x(H^{0,2}_z)}. \tag{3.34}$$

3.3.3.4 Equation and estimates of the remainder

We will now analyze the remainder defined in (3.26), which is in fact a solution in the extended domain $\mathcal{O}$ to

$$\begin{cases} \partial_t r^\varepsilon - \varepsilon\Delta r^\varepsilon + (u^\varepsilon\cdot\nabla)r^\varepsilon + \nabla\pi^\varepsilon = \{f^\varepsilon\} - \{A^\varepsilon r^\varepsilon\} + \varepsilon q^\varepsilon e_n + \varepsilon\theta^1 e_n & \text{in} \quad (0,+\infty)\times\mathcal{O}, \\ \partial_t q^\varepsilon - \varepsilon\Delta q^\varepsilon + u^\varepsilon\cdot\nabla q^\varepsilon = \{h^\varepsilon\} - B^\varepsilon r^\varepsilon & \text{in} \quad (0,+\infty)\times\mathcal{O}, \\ \nabla\cdot r^\varepsilon = 0 & \text{in} \quad (0,+\infty)\times\mathcal{O}, \\ r^\varepsilon\cdot\nu = 0, \quad N(r^\varepsilon) = -N(g^\varepsilon) & \text{on} \quad (0,+\infty)\times\partial\mathcal{O}, \\ R(q^\varepsilon) = -R(\theta^1) & \text{on} \quad (0,+\infty)\times\partial\mathcal{O}, \\ r^\varepsilon(0,\cdot) = 0, \quad q^\varepsilon(0,\cdot) = 0 & \text{in} \quad \mathcal{O}, \end{cases} \tag{3.35}$$

where $g^\varepsilon := u^1 + \nabla\zeta^\varepsilon + \beta|_{z=0}$. Let us introduce the amplification operators A^ε and B^ε, given by

$$A^\varepsilon r^\varepsilon := (r^\varepsilon\cdot\nabla)(u^0 + \sqrt{\varepsilon}\rho + \varepsilon u^1 + \varepsilon\nabla\zeta^\varepsilon + \varepsilon\beta) - (r^\varepsilon\cdot\nu)(\partial_z\rho + \sqrt{\varepsilon}\partial_z\beta)$$

and

$$B^\varepsilon r^\varepsilon := \varepsilon r^\varepsilon\cdot\nabla\theta^1 \tag{3.36}$$

and the forcing terms f^ε and h^ε, with

$$\begin{aligned}
f^\varepsilon := &(\Delta\varphi\partial_z\rho - 2(\nu\cdot\nabla)\partial_z\rho + \partial_{zz}\beta) + \sqrt{\varepsilon}(\Delta\rho + \Delta\varphi\partial_z\beta - 2(\nu\cdot\nabla)\partial_z\beta)\\
&+\varepsilon(\Delta\beta + \Delta u^1 + \Delta\nabla\zeta^\varepsilon) - ((\rho + \sqrt{\varepsilon}(\beta + u^1 + \nabla\zeta^\varepsilon))\cdot\nabla)(\rho + \sqrt{\varepsilon}(\beta + u^1 + \nabla\zeta^\varepsilon))\\
&-(u^0\cdot\nabla)\beta - (\beta\cdot\nabla)u^0 - u^0_\flat z\partial_z\beta + (\beta + u^1 + \nabla\zeta^\varepsilon)\cdot\nu\partial_z(\rho + \sqrt{\varepsilon}\beta)\\
&-\nabla\psi - \partial_t\beta
\end{aligned} \tag{3.37}$$

and

$$h^\varepsilon := \varepsilon\Delta\theta^1 - (\sqrt{\varepsilon}\rho + \varepsilon(u^1 + \nabla\zeta^\varepsilon + \beta))\cdot\nabla\theta^1. \tag{3.38}$$

We have to estimate the size of the remainder $(r^\varepsilon, q^\varepsilon)$ at final time and check that it is small. We begin by establishing an energy estimate. Thus, we multiply equation $(3.35)_1$ by r^ε and the equation $(3.35)_2$ by q^ε and integrate by parts. We proceed as before, term by term, the unique different being the terms coming from the boundary.

We recall the following identity, see (IFTIMIE; SUEUR, 2011, Lemma 2.2) which will be used throughout the paper

$$\int_{\mathcal{O}}(-\Delta u)\cdot v = 2\int_{\mathcal{O}} D(u)\cdot D(v) - 2\int_{\partial\mathcal{O}}[D(u)\nu]_{tan}\cdot v\, d\Gamma_{\mathcal{O}},$$

where u and v are smooth vector fields such that v is divergence free and tangent to the boundary.

Therefore, it follows that

$$-\varepsilon\int_{\mathcal{O}}\Delta r^\varepsilon\cdot r^\varepsilon = 2\varepsilon\|D(r^\varepsilon)\|^2 + 2\varepsilon\int_{\partial\mathcal{O}}([Mr^\varepsilon]_{tan} - N(g^\varepsilon))\cdot r^\varepsilon\, d\Gamma_{\mathcal{O}},$$

and we estimate the boundary term as follows

$$\begin{aligned}
2\left|\int_{\partial\mathcal{O}}([Mr^\varepsilon]_{tan} - N(g^\varepsilon))\cdot r^\varepsilon\, d\Gamma_{\mathcal{O}}\right| &= 2\left|\int_{\partial\mathcal{O}} Mr^\varepsilon\cdot r^\varepsilon - N(g^\varepsilon)\cdot r^\varepsilon\, d\Gamma_{\mathcal{O}}\right|\\
&\le \lambda\|\nabla r^\varepsilon\|^2 + C_\lambda(\|r^\varepsilon\|^2 + \|N(g^\varepsilon)\|^2_{L^2(\partial\mathcal{O})})\\
&\le \lambda\|\nabla r^\varepsilon\|^2 + C_\lambda(\|r^\varepsilon\|^2 + \|g^\varepsilon\|^2_{L^2(\partial\mathcal{O})} + \|D(g^\varepsilon)\|^2_{L^2(\partial\mathcal{O})})\\
&\le \lambda\|\nabla r^\varepsilon\|^2 + C_\lambda(\|r^\varepsilon\|^2 + \|g^\varepsilon\|^2_{H^2}),
\end{aligned} \tag{3.39}$$

for any $\lambda > 0$, where C_λ is a constant depending on λ. Let us absorb the term $\|\nabla r^\varepsilon\|^2$ in (3.39). Thanks to the classical Korn's inequality (see Lemma B.1), since $\nabla\cdot r^\varepsilon = 0$ in $\mathcal{O}$ and $r^\varepsilon\cdot\nu = 0$ on $\partial\mathcal{O}$, we have

$$\|r^\varepsilon\|^2_{H^1} \le C_K\|r^\varepsilon\|^2 + C_K\|D(r^\varepsilon)\|^2.$$

for some $C_K > 0$. Choosing $\lambda = 1/(2C_K)$, we get:

$$\begin{aligned}
\frac{d}{dt}\|r^\varepsilon\|^2 + \varepsilon\|D(r^\varepsilon)\|^2 \le &\Big(\|\sigma^0\|_\infty + C\varepsilon + \|\{f^\varepsilon\}\| + 2\|\{A^\varepsilon\}\|_\infty\Big)\|r^\varepsilon\|^2\\
&+\Big(C\varepsilon\|g^\varepsilon\|^2_{H^2} + \|\{f^\varepsilon\}\| + \varepsilon\|\theta^1\|^2\Big) + \varepsilon\|q^\varepsilon\|^2
\end{aligned}$$

and

$$\frac{d}{dt}\|q^\varepsilon\|^2 + \varepsilon\|\nabla q^\varepsilon\|^2 \le \Big(\|\sigma^0\|_\infty + \|\{h^\varepsilon\}\| + \|\{B^\varepsilon\}\|_\infty + C\varepsilon\Big)\|q^\varepsilon\|^2 + \Big(\|\{h^\varepsilon\}\| + C\varepsilon\|\theta^1\|^2_{H^2}\Big) + \|\{B^\varepsilon\}\|_\infty\|r^\varepsilon\|^2.$$

Adding the two estimates above, we get

$$\begin{aligned}&\frac{d}{dt}(\|r^\varepsilon\|^2 + \|q^\varepsilon\|^2) + \varepsilon(\|D(r^\varepsilon)\|^2 + \|\nabla q^\varepsilon\|^2)\\ &\quad\le \Big(\|\sigma^0\|_\infty + C\varepsilon + \|\{f^\varepsilon\}\| + 2\|\{A^\varepsilon\}\|_\infty + \|\{h^\varepsilon\}\| + \|B^\varepsilon\|_\infty\Big)\\ &\quad\times\Big(\|r^\varepsilon\|^2 + \|q^\varepsilon\|^2\Big) + \Big(C\varepsilon\|g^\varepsilon\|^2_{H^2} + \|\{f^\varepsilon\}\| + \|\{h^\varepsilon\}\| + C\varepsilon\|\theta^1\|^2_{H^2}\Big).\end{aligned}$$

Applying Gronwall's inequality in the interval $(0, T/\varepsilon)$, and using the fact that the initial datum is equal to 0 and the estimates

$$\|\{A^\varepsilon\}\|_{L^1(L^\infty)} + \|B^\varepsilon\|_{L^1(L^\infty)} = O(1), \tag{3.40}$$

$$\varepsilon\|\theta^1\|^2_{L^2(H^2)} + \varepsilon\|g^\varepsilon\|^2_{L^2(H^2)} = O(\varepsilon^{\frac{1}{4}}), \tag{3.41}$$

$$\|\{f^\varepsilon\}\|_{L^1(L^2)} + \|\{h^\varepsilon\}\|_{L^1(L^2)} = O(\varepsilon^{\frac{1}{4}}), \tag{3.42}$$

we obtain:

$$\|r^\varepsilon\|^2_{L^\infty(L^2)} + \|q^\varepsilon\|^2_{L^\infty(L^2)} + \varepsilon\|D(r^\varepsilon)\|^2_{L^2(L^2)} + \varepsilon\|\nabla q^\varepsilon\|^2_{L^2(L^2)} = O(\varepsilon^{\frac{1}{4}}). \tag{3.43}$$

The estimates (3.40), (3.41) and (3.42) hold on the whole interval $[0, +\infty)$. The estimates for $\{A^\varepsilon\}$, g^ε and $\{f^\varepsilon\}$ can be found in (CORON; MARBACH; SUEUR, 2020, Section 4.4). Here, we give some details to obtain the estimates for B^ε, θ^1 and $\{h^\varepsilon\}$, which are new.

First, the estimates of B^ε and θ^1 are straightforward by using (3.36), and Lemma 3.11. Indeed, we easily get that $\|B^\varepsilon\|_{L^1(L^\infty)} = O(1)$ and that $\varepsilon\|\theta^1\|^2_{L^2(H^2)} = O(\varepsilon)$.

Now, let us justify the estimate of $\{h^\varepsilon\}$. Note that the fast scaling variable enables us to win a factor $\varepsilon^{1/4}$, see (IFTIMIE; SUEUR, 2011, Lemma 3). In what follows, we estimate each one of the terms in (3.38). The first term is $O(\varepsilon)$, thanks to the regularity of θ^1. The second one can be treated as follows

$$\begin{aligned}\|\sqrt{\varepsilon}\{\rho(t,\cdot)\}\cdot\nabla\theta^1(t,\cdot)\| &\le C\sqrt{\varepsilon}\|\{\rho(t,\cdot)\}\|_{H^1}\|\nabla\theta^1(t,\cdot)\|_{H^1}\\ &\le C\left(\sqrt{\varepsilon}\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{0,0}_z)} + \|\{\partial_z\rho(t,\cdot)\}\|\right)\|\nabla\theta^1(t,\cdot)\|_{H^1}\\ &\le C\left(\sqrt{\varepsilon}\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{0,0}_z)} + \varepsilon^{1/4}\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{1,0}_z)}\right)\|\nabla\theta^1(t,\cdot)\|_{H^1}\\ &\le C\varepsilon^{1/4}\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{1,0}_z)}\|\nabla\theta^1(t,\cdot)\|_{H^1}.\end{aligned}$$

Then, integrating by parts this last inequality with respect to time over $(0, T/\varepsilon)$, using Lemma 3.14 for $k = 4$ and also the fact that θ^1 is bounded in $L^\infty(0,T;H^3(\mathcal{O}))$, we get:

$$\|\sqrt{\varepsilon}\{\rho(\cdot,\cdot)\}\cdot\nabla\theta^1\|_{L^1(L^2)} = O(\varepsilon^{1/4}).$$

Now, for the third term, by using (3.30) and (3.33) we have the following:

$$\begin{aligned}\|\varepsilon\nabla_x\zeta^\varepsilon(t,\cdot)\cdot\nabla\theta^1(t,\cdot)\| &\le C\varepsilon\|\nabla_x\zeta^\varepsilon(t,\cdot)\|_{H^1}\|\nabla\theta^1(t,\cdot)\|_{H^1}\\ &\le C\varepsilon\|\zeta^\varepsilon(t,\cdot)\|_{H^2}\|\nabla\theta^1(t,\cdot)\|_{H^1}\\ &\le C\varepsilon\left(\varepsilon^{1/4}\|\beta(t,\cdot,\cdot)\|_{H^2_x(H^{0,0}_z)}+\|\rho(t,\cdot,\cdot)\|_{H^1_x(H^{0,1}_z)}\right)\|\nabla\theta^1(t,\cdot)\|_{H^1}\\ &\le C\varepsilon\|\rho(t,\cdot,\cdot)\|_{H^3_x(H^{1,2}_z)}\|\nabla\theta^1(t,\cdot)\|_{H^1}.\end{aligned}$$

Integrating by parts this last inequality, with respect to time, and using again Lemma 3.14 for $k=3$ and the fact that θ^1 is bounded in $L^\infty(0,T;H^3(\mathcal{O}))$, we find that

$$\|\varepsilon\nabla_x\zeta^\varepsilon(\cdot,\cdot)\cdot\nabla\theta^1(\cdot,\cdot)\|_{L^1(L^2)}=O(\varepsilon^{1/4}).$$

The last term can be estimated in a similar way, using (3.30).

3.3.4 Towards the trajectory

In this section, we deduce a small-time global approximate controllability result to smooth trajectories. For that, we will use Lemma 3.14 and the estimates on the remainder term (3.43).

Let $(u^\varepsilon,p^\varepsilon,\theta^\varepsilon)$ be the solution to the equation (3.9) on the time interval $[0,T/\varepsilon]$. First, during the interval $[0,T]$, we use the expansions

$$\begin{aligned}u^\varepsilon &= u^0+\sqrt{\varepsilon}\{\rho\}+\varepsilon u^{1,\varepsilon}+\varepsilon\nabla\zeta^\varepsilon+\varepsilon\{\beta\}+\varepsilon r^\varepsilon,\\ \theta^\varepsilon &= \theta^0+\varepsilon^2\theta^{1,\varepsilon}+\varepsilon^2 q^\varepsilon,\end{aligned}\tag{3.44}$$

where $u^{1,\varepsilon}(0,\cdot)=u_*$, $\theta^{1,\varepsilon}(0,\cdot)=\theta_*$ and $u^{1,\varepsilon}(T,\cdot)=\overline{u}(\varepsilon T,\cdot)$, $\theta^{1,\varepsilon}(T,\cdot)=\overline{\theta}(\varepsilon T,\cdot)$. The couple $(u^{1,\varepsilon},\theta^{1,\varepsilon})$ solves, together with some $p^{1,\varepsilon}$, the usual first-order system (3.15) and the profiles $u^{1,\varepsilon}$ and $\theta^{1,\varepsilon}$ depend on ε. However, since the reference trajectory belongs to C^∞, all the required estimates can be made independent of ε. In a second step, for large times $t\ge T$, we change our expansions and set:

$$\begin{aligned}u^\varepsilon &= \sqrt{\varepsilon}\{\rho\}+\varepsilon\overline{u}(\varepsilon t,\cdot)+\varepsilon\nabla\zeta^\varepsilon+\varepsilon\{\beta\}+\varepsilon r^\varepsilon,\\ p^\varepsilon &= \varepsilon^2\overline{p}(\varepsilon t,\cdot)+\varepsilon\mu^\varepsilon+\varepsilon\pi^\varepsilon,\\ v^\varepsilon &= \sqrt{\varepsilon}v^\rho+\varepsilon^2\overline{v},\\ \theta^\varepsilon &= \varepsilon^2\overline{\theta}(\varepsilon t,\cdot)+\varepsilon^2 q^\varepsilon,\\ w^\varepsilon &= \varepsilon^3\overline{w}.\end{aligned}\tag{3.45}$$

Note that, for $t\ge T$, we have $u^0=0$ and the profile (u^1,θ^1) is the main trajectory and changing (3.44) by (3.45) allow us to get rid of some terms in the equation satisfied by the remainder. Indeed, terms such as $\varepsilon\Delta u^1$, $\varepsilon(u^1\cdot\nabla)u^1$, $\varepsilon u^1\cdot\nabla\theta^1$ and $\varepsilon\Delta\theta^1$ will not appear any more in (3.37) and (3.38) because they are already taken into account by $\left(\overline{u},\overline{\theta}\right)$. Actually, despite the presence of the profile (u^1, θ^1) in both steps, the estimates obtained for the remainder profile are as in Section 3.3.3.4.

Let us introduce

$$u^{(\varepsilon)}(t,x) := \frac{1}{\varepsilon} u^\varepsilon \left(\frac{t}{\varepsilon}, x\right) \quad \text{and} \quad \theta^{(\varepsilon)}(t,x) := \frac{1}{\varepsilon^2} \theta^\varepsilon \left(\frac{t}{\varepsilon}, x\right).$$

Then, thanks to (3.30), (3.33) and (3.43), we see that

$$\begin{aligned}
\left\| u^{(\varepsilon)}(T,\cdot) - \overline{u}(T,\cdot) \right\| &= \left\| \varepsilon^{-1/2} \left\{\rho\left(T/\varepsilon,\cdot\right)\right\} + \nabla\zeta^\varepsilon(T/\varepsilon,\cdot) + \left\{\beta\left(T/\varepsilon,\cdot\right)\right\} + r^\varepsilon\left(T/\varepsilon,\cdot\right) \right\| \\
&\le \varepsilon^{-1/2} \left\|\left\{\rho\left(T/\varepsilon,\cdot\right)\right\}\right\| + \varepsilon^{1/4} \|\beta(T/\varepsilon,\cdot,\cdot)\|_{H^2_x(H^{0,0}_z)} \\
&\quad + \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^1_x(H^{0,1}_z)} + \left\|\left\{\beta\left(T/\varepsilon,\cdot\right)\right\}\right\| + \left\|r^\varepsilon\left(T/\varepsilon,\cdot\right)\right\| \\
&\le \varepsilon^{-1/2} \left\|\left\{\rho\left(T/\varepsilon,\cdot\right)\right\}\right\| + \varepsilon^{1/4} \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^3_x(H^{1,2}_z)} \\
&\quad + \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^1_x(H^{0,1}_z)} + \left\|\rho\left(T/\varepsilon,\cdot,\cdot\right)\right\|_{H^1_x(H^{1,2}_z)} + O(\varepsilon^{\frac{1}{8}}).
\end{aligned}$$

We can use (3.25) to estimate the terms containing ρ in the estimates above. First, note that

$$\lim_{s\to+\infty} \frac{\log s}{s^{1/2}} = 0$$

and, consequently, there exists a positive constant $C > 0$ such that

$$\frac{\log s}{s} \le C s^{-1/2} \quad \forall s \ge 1.$$

Then, by taking ε sufficiently small, the following is found for $k \ge 2$:

$$\begin{aligned}
&\varepsilon^{-\frac{1}{2}} \left\|\left\{\rho\left(T/\varepsilon,\cdot\right)\right\}\right\| = \varepsilon^{-\frac{1}{2}} \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^0_x(H^{0,0}_z)} \le C\varepsilon^{-\frac{1}{2}} \left|\frac{\log(2+T/\varepsilon)}{2+T/\varepsilon}\right|^{\frac{1}{4}+\frac{k}{2}} \le C\varepsilon^{-\frac{1}{2}+\frac{1}{8}+\frac{k}{4}}, \\
&\varepsilon^{\frac{1}{4}} \|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^3_x(H^{1,2}_z)} \le C\varepsilon^{\frac{1}{4}} \left|\frac{\log(2+T/\varepsilon)}{2+T/\varepsilon}\right|^{\frac{1}{4}+\frac{k}{2}-1} \le C\varepsilon^{\frac{1}{4}+\frac{1}{8}+\frac{k}{4}-\frac{1}{2}}, \\
&\|\rho(T/\varepsilon,\cdot,\cdot)\|_{H^1_x(H^{0,1}_z)} \le C \left|\frac{\log(2+T/\varepsilon)}{2+T/\varepsilon}\right|^{\frac{1}{4}+\frac{k}{2}-\frac{1}{2}} \le C\varepsilon^{\frac{1}{8}+\frac{k}{4}-\frac{1}{4}}, \\
&|\rho(T/\varepsilon,\cdot,\cdot)|_{H^1_x(H^{1,2}_z)} \le C \left|\frac{\log(2+T/\varepsilon)}{2+T/\varepsilon}\right|^{\frac{1}{4}+\frac{k}{2}-1} \le C\varepsilon^{\frac{1}{8}+\frac{k}{4}-\frac{1}{2}}.
\end{aligned}$$

Finally, we choosing k large enough, we conclude that

$$\left\| u^{(\varepsilon)}(T,\cdot) - \overline{u}(T,\cdot) \right\| = O(\varepsilon^{\frac{1}{8}})$$

and, from (3.43), we have

$$\left\| \theta^{(\varepsilon)}(T,\cdot) - \overline{\theta}(T,\cdot) \right\| = \left\| q^\varepsilon\left(T/\varepsilon,\cdot\right)\right\| = O(\varepsilon^{\frac{1}{8}}).$$

This concludes the proof of Proposition 3.9 holds.

3.4 LOCAL CONTROLLABILITY OF THE BOUSSINESQ SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

Let ω_c and ω be two non-empty open subsets such that $\omega_c \subset\subset \omega \subset\subset \mathcal{O} \setminus \overline{\Omega}$ and let χ_ω be a cut-off function such that $\chi_\omega = 0$ outside ω and $\chi_\omega = 1$ in ω_c.

The goal of this section is to prove the local exact controllability to trajectories for the Boussinesq system with distributed controls:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = \theta e_n + v\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = w\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot u = 0 & \text{in} \quad \mathcal{O}_T, \\ u \cdot \nu = 0, \quad N(u) + [f(u)]_{tan} = 0 & \text{on} \quad \Lambda_T, \\ R(\theta) + g(\theta) = 0 & \text{on} \quad \Lambda_T, \\ u(0,\cdot) = u_*, \quad \theta(0,\cdot) = \theta_* & \text{in} \quad \mathcal{O}, \end{cases} \tag{3.46}$$

where $f \in C^3(\mathbb{R}^n;\mathbb{R}^n)$ with a symmetric Jacobian matrix (or equivalently, f is an irrotational field) and $g \in C^3(\mathbb{R})$. Note that, to prove Theorem 3.1, we only need to prove a local-controllability result for (3.7), with linear Navier boundary conditions $N(u)$ as in (3.1) and linear Robin boundary conditions $R(\theta)$ as (3.2). However, for sake of completeness, we establish a local controllability result for the Boussinesq system with nonlinear Navier boundary conditions on the velocity field and nonlinear Fourier boundary conditions on the temperate.

Since (3.46) is nonlinear, we first begin by proving a (global) null-controllability result for the following system

$$\begin{cases} \partial_t z - \Delta z + ((a+b) \cdot \nabla) z + (z \cdot \nabla) b + \nabla q = h e_n + v\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ \partial_t h - \Delta h + (a+b) \cdot \nabla h + z \cdot \nabla c = w\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot z = 0 & \text{in} \quad \mathcal{O}_T, \\ z \cdot \nu = 0, \quad [D(z)\nu + Az]_{tan} = 0 & \text{on} \quad \Lambda_T, \\ \dfrac{\partial h}{\partial \nu} + Bh = 0 & \text{on} \quad \Lambda_T, \\ z(0,\cdot) = z_*, \quad h(0,\cdot) = h_* & \text{in} \quad \mathcal{O}, \end{cases} \tag{3.47}$$

where the vector fields a, b, the scalar function c, the symmetric matrix A and the scalar function B satisfy the following assumptions:

$$a, b \in L^\infty(0,T; L^2_{div}(\mathcal{O})^n \cap L^\infty(\mathcal{O})^n), \;\; a_t, b_t \in L^2(0,T; L^r(\mathcal{O})^n), \tag{3.48}$$

$$c \in L^\infty(0,T; L^\infty(\mathcal{O})), \quad c_t \in L^2(0,T; L^r(\mathcal{O})), \tag{3.49}$$

$$A \in P := H^{1-\ell}(0,T; W^{\vartheta_1,\vartheta_1+1}(\partial\mathcal{O})^{n\times n}) \cap H^{(3-\ell)/2}(0,T; H^{\vartheta_2}(\partial\mathcal{O})^{n\times n}), \tag{3.50}$$

$$B \in Q := H^{1-\ell}(0,T; W^{\vartheta_1,\vartheta_1+1}(\partial\mathcal{O})) \cap H^{(3-\ell)/2}(0,T; H^{\vartheta_2}(\partial\mathcal{O})), \tag{3.51}$$

with $0 < \ell < 1/2$ arbitrarily close to $1/2$, $r = 2n$, $\vartheta_2 = (1/2)(3-n) + (1-\ell)(n-2)$ and $\vartheta_1 > 1$ (arbitrarily small) if $n = 3$ and $\vartheta_1 = 1$ if $n = 2$. By Sobolev embeddings, we readily have $P \hookrightarrow L^\infty((0,T)\times\partial\mathcal{O})^{n\times n}$ and $Q \hookrightarrow L^\infty((0,T)\times\partial\mathcal{O})$.

It is well-known, that the null-controllability of system (3.47) is equivalent to prove an observability estimate for the adjoint system:

$$\begin{cases} -\partial_t\varphi - \Delta\varphi - (a\cdot\nabla)\varphi - D(\varphi)b + \nabla\pi = c\nabla\psi & \text{in} \quad \mathcal{O}_T, \\ -\partial_t\psi - \Delta\psi - (a+b)\cdot\nabla\psi = \varphi\cdot e_n & \text{in} \quad \mathcal{O}_T, \\ \nabla\cdot\varphi = 0 & \text{in} \quad \mathcal{O}_T, \\ \varphi\cdot\nu = 0, \quad [D(\varphi)\nu + A\varphi]_{tan} = 0 & \text{on} \quad \Lambda_T, \\ \dfrac{\partial\psi}{\partial\nu} + B\psi = 0 & \text{on} \quad \Lambda_T, \\ \varphi(T,\cdot) = \varphi_*, \quad \psi(T,\cdot) = \psi_* & \text{in} \quad \mathcal{O}. \end{cases} \tag{3.52}$$

The desired observability inequality will be a consequence of a global Carleman inequality for (3.52), see Proposition 3.16 below.

3.4.1 Carleman estimates

Before stating the required Carleman inequality, let us introduce several classical weights in the study of Carleman inequalities for parabolic equations, see (FURSIKOV; IMANUVILOV, 1996). The basic weight will be a function $\eta^0 \in C^2(\overline{\mathcal{O}})$ verifying

$$\eta^0 > 0 \quad \text{in} \quad \mathcal{O}, \quad \eta^0 \equiv 0 \quad \text{on} \quad \partial\mathcal{O}, \quad |\nabla\eta^0| > 0 \quad \text{in} \quad \overline{\mathcal{O}\setminus\omega'}, \tag{3.53}$$

where $\omega' \subset\subset \omega_c$ is a non-empty open set.

Thus, for any $\lambda > 0$ we set:

$$\alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t^4(T-t)^4}, \quad \xi(x,t) = \frac{e^{\lambda\eta^0(x)}}{t^4(T-t)^4},$$
$$\alpha^*(t) = \max_{x\in\overline{\mathcal{O}}} \alpha(x,t), \qquad \widehat{\alpha}(t) = \min_{x\in\overline{\mathcal{O}}} \alpha(x,t),$$
$$\xi^*(t) = \min_{x\in\overline{\mathcal{O}}} \xi(x,t), \qquad \widehat{\xi}(t) = \max_{x\in\overline{\mathcal{O}}} \xi(x,t).$$

We also introduce the following notation:

$$\begin{aligned} I(s,\lambda;\varphi) \quad = \quad & s^3\lambda^4 \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi^3|\varphi|^2 + s\lambda^2 \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi|\nabla\varphi|^2 \\ & + s^{-1} \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi^{-1}(|\varphi_t|^2 + |\Delta\varphi|^2), \end{aligned}$$

where s and λ are positive real numbers and $\varphi = \varphi(t,x)$.

We have the following Carleman inequality for (3.52):

Proposition 3.16. *Assume that the assumptions* (3.48)*,* (3.49)*,* (3.50)*,* (3.51) *are fulfilled. There exist positive constants* $\widetilde{\lambda}$*,* $\widetilde{s}$ *and* $C = C(\mathcal{O}, \omega_c)$ *such that, for any* $(\varphi_*, \psi_*) \in L^2_{div}(\mathcal{O}) \times L^2(\mathcal{O})$*, the corresponding solution to* (3.52) *verifies:*

$$I(s,\lambda;\varphi) + I(s,\lambda;\psi) \le C(1+T^2)s^{15/2}\lambda^8 \iint_{(0,T)\times\omega_c} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^{15/2}(|\varphi|^2 + |\psi|^2), \quad (3.54)$$

for all $\lambda \ge \widetilde{\lambda}$ *and* $s \ge \widetilde{s}$*. Furthermore,* $\widetilde{\lambda}$ *and* $\widetilde{s}$ *have the form* $\widetilde{\lambda} = \widetilde{\lambda}_0 e^{\widetilde{\lambda}_1 T}$ *and* $\widetilde{s} = \widetilde{s}_0 e^{\lambda \widetilde{s}_1}(T^4 + T^8)$*, where* $\widetilde{\lambda}_0$*,* $\widetilde{\lambda}_1$ *and* $\widetilde{s}_0$ *only depend on* $\|a\|_\infty$*,* $\|b\|_\infty$*,* $\|c\|_\infty$*,* $\|a_t\|_{L^2(L^r)}$*,* $\|b_t\|_{L^2(L^r)}$*,* $\|c_t\|_{L^2(L^r)}$*,* $\|A\|_P$ *and* $\|B\|_Q$*, and* $\widetilde{s}_1$ *only depend on* $\mathcal{O}$ *and* ω_c*.*

The proof of Proposition 3.16 follows the arguments of (GUERRERO, 2006b). For completeness and because of the presence of the equation satisfied by ψ, we provide its details in Appendices D and E.

3.4.2 Null controllability of the linearized system

In this section we will deduce the null-controllability of the linear system (3.47) as a consequence of the inequality (3.54). We introduce the following notation for denoting the space where the control is found:

$$\mathcal{H} := H^1(0,T;L^2(\mathcal{O})^n)\cap C^0([0,T];H^1(\mathcal{O})^n)\times H^1(0,T;L^2(\mathcal{O}))\cap C^0([0,T];H^1(\mathcal{O})). \quad (3.55)$$

Proposition 3.17. *Let* $(z_*, h_*) \in L^2_{div}(\mathcal{O})^n \times L^2(\mathcal{O})$ *and let us suppose that* (3.48)*,* (3.49)*,* (3.50) *and* (3.51) *holds. Then, there exist controls* $(v, w) \in \mathcal{H}$ *such that the corresponding solution to* (3.47) *satisfies*

$$z(T,\cdot) = 0 \quad \textit{and} \quad h(T,\cdot) = 0. \quad (3.56)$$

Moreover, the following estimate holds

$$\|\kappa^{1/2}v\chi_\omega\| + \|\kappa^{1/2}w\chi_\omega\| + \|v\|_{H^1(L^2)} + \|v\|_{L^\infty(H^1)} + \|w\|_{H^1(L^2)} + \|w\|_{L^\infty(H^1)} \le C(\|z_*\| + \|h_*\|),$$

where the positive constant C, depending only on $\mathcal{O}$*,* ω*,* T*,* $\|a\|_\infty$*,* $\|b\|_\infty$*,* $\|c\|_\infty$*,* $\|a_t\|_{L^2(L^r)}$*,* $\|b_t\|_{L^2(L^r)}$*,* $\|c_t\|_{L^2(L^r)}$*,* $\|A\|_P$ *and* $\|B\|_Q$*, and* $\kappa(t) = e^{4s\hat{\alpha}-2s\alpha^*}\hat{\xi}^{-15/2}$ *and* s*,* λ*,* $\hat{\alpha}$*,* α^* *and* $\hat{\xi}$ *are defined at the beginning of the previous section.*

Proof. It follows the ideas of (GUERRERO, 2006b). It is based on a penalized Hilbert Uniqueness Method. Thus, let $(z_*, h_*) \in L^2_{\text{div}}(\mathcal{O})^n \times L^2(\mathcal{O})$, and for each $\varepsilon > 0$, let consider the extremal problem

$$\begin{cases} \text{Minimize } \dfrac{1}{2}\displaystyle\iint_{\mathcal{O}_T} \kappa(t)\left(|v|^2 + |w|^2\right)\chi_\omega + \dfrac{1}{2\varepsilon}\left(\|z(T,\cdot)\|^2_{L^2} + \|h(T,\cdot)\|^2_{L^2}\right) \\ \text{Subject to } v \in L^2(\mathcal{O}_T),\ w \in L^2(\mathcal{O}_T) \text{ and } (z,h,v,w) \text{ solves } (3.47). \end{cases} \quad (3.57)$$

There exists a (unique) solution to (3.57), denoted by $(z^\varepsilon, h^\varepsilon, v^\varepsilon, w^\varepsilon)$, with $\kappa(t)^{1/2}(v^\varepsilon, w^\varepsilon) \in L^2(\mathcal{O}_T)^{n+1}$, since the functional in (3.57) is coercive, strictly convex and C^1 in the Hilbert space $L^2(\mathcal{O}_T)^{n+1}$. The associated Euler-Lagrange equation yields

$$\iint_{\mathcal{O}_T} \kappa(t)\,(v^\varepsilon \cdot v + w^\varepsilon w)\,\chi_\omega \;+ \frac{1}{\varepsilon}\int_{\mathcal{O}} [z^\varepsilon(T,\cdot)\cdot Z(T,\cdot) + h^\varepsilon(T,\cdot)H(T,\cdot)] = 0 \qquad (3.58)$$

for all $(v, w) \in L^2(\mathcal{O}_T)^n \times L^2(\mathcal{O}_T)$, where (Z, H) is, together with some Π, the solution to the system

$$\begin{cases} \partial_t Z - \Delta Z + ((a+b)\cdot\nabla)Z + (Z\cdot\nabla)b + \nabla\Pi = He_n + v\chi_\omega & \text{in } \mathcal{O}_T, \\ \partial_t H - \Delta H + (a+b)\cdot\nabla H + Z\cdot\nabla c = w\chi_\omega & \text{in } \mathcal{O}_T, \\ \nabla\cdot Z = 0 & \text{in } \mathcal{O}_T, \\ Z\cdot\nu = 0, \quad [D(Z)\nu + AZ]_{tan} = 0 & \text{on } \Lambda_T, \\ \dfrac{\partial H}{\partial\nu} + BH = 0 & \text{on } \Lambda_T, \\ Z(0,\cdot) = 0, \quad H(0,\cdot) = 0 & \text{in } \mathcal{O}. \end{cases}$$

Let us now introduce the solution $(\varphi^\varepsilon, \pi^\varepsilon, \psi^\varepsilon)$ to the following homogeneous adjoint system:

$$\begin{cases} -\partial_t\varphi^\varepsilon - \Delta\varphi^\varepsilon - (a\cdot\nabla)\varphi^\varepsilon - D\varphi^\varepsilon b + \nabla\pi^\varepsilon = c\nabla\psi^\varepsilon & \text{in } \mathcal{O}_T, \\ -\partial_t\psi^\varepsilon - \Delta\psi^\varepsilon - (a+b)\cdot\nabla\psi^\varepsilon = \varphi^\varepsilon\cdot e_n & \text{in } \mathcal{O}_T, \\ \nabla\cdot\varphi^\varepsilon = 0 & \text{in } \mathcal{O}_T, \\ \varphi^\varepsilon\cdot\nu = 0, \quad [D(\varphi^\varepsilon)\nu + A\varphi^\varepsilon]_{tan} = 0 & \text{on } \Lambda_T, \\ \dfrac{\partial\psi^\varepsilon}{\partial\nu} + B\psi^\varepsilon = 0 & \text{on } \Lambda_T, \\ \varphi^\varepsilon(T,\cdot) = -\dfrac{1}{\varepsilon}z^\varepsilon(T,\cdot), \quad \psi^\varepsilon(T,\cdot) = -\dfrac{1}{\varepsilon}h^\varepsilon(T,\cdot) & \text{in } \mathcal{O}. \end{cases}$$

The duality between $(\varphi^\varepsilon, \psi^\varepsilon)$ and (Z, H) give

$$-\frac{1}{\varepsilon}\int_{\mathcal{O}} [z^\varepsilon(T,\cdot)\cdot Z(T,\cdot) + h^\varepsilon(T,\cdot)H(T,\cdot)] \;=\; \iint_{\mathcal{O}_T} (v\cdot\varphi^\varepsilon + w\,\psi^\varepsilon)\chi_\omega$$

which, combined with (3.58), yields

$$\iint_{\mathcal{O}_T} (v\,\cdot\,\varphi^\varepsilon + w\,\psi^\varepsilon)\chi_\omega \;=\; \iint_{\mathcal{O}_T} \kappa(t)\,(v^\varepsilon\cdot v + w^\varepsilon w)\,\chi_\omega$$

for all $(v, w) \in L^2(\mathcal{O}_T)^n \times L^2(\mathcal{O}_T)$. Consequently, we have the following identify

$$v^\varepsilon = \kappa^{-1}(t)\varphi^\varepsilon \quad \text{and} \quad w^\varepsilon = \kappa^{-1}(t)\psi^\varepsilon$$

The duality between the systems fulfilled by $(z^\varepsilon, h^\varepsilon)$ and $(\varphi^\varepsilon, \psi^\varepsilon)$, gives

$$-\frac{1}{\varepsilon}\left(\|z^\varepsilon(T,\cdot)\|^2 + \|w^\varepsilon(T,\cdot)\|^2\right) = \int_{\mathcal{O}} z_*\cdot\varphi^\varepsilon(0,\cdot) + h_*\psi^\varepsilon(0,\cdot) + \iint_{\mathcal{O}_T} \kappa^{-1}(t)\left(|\varphi^\varepsilon|^2 + |\psi^\varepsilon|^2\right)\chi_\omega.$$

Moreover, the Carleman inequality (3.54) applied to $(\varphi^\varepsilon, \psi^\varepsilon)$ gives

$$\|\varphi^\varepsilon(0,\cdot)\|^2 + \|\psi^\varepsilon(0,\cdot)\|^2 \leq C(\mathcal{O},\omega,T,a,b,c,A,B) \iint_{\mathcal{O}_T} \kappa^{-1}(t)(|\varphi^\varepsilon|^2 + |\psi^\varepsilon|^2)\chi_\omega.$$

Hence, we conclude that

$$\frac{1}{\varepsilon}\|z^\varepsilon(T,\cdot)\|^2 + \frac{1}{\varepsilon}\|w^\varepsilon(T,\cdot)\|^2 + \|\kappa^{1/2}v^\varepsilon\chi_\omega\|^2 + \|\kappa^{1/2}w^\varepsilon\chi_\omega\|^2 \leq C\left(\|z_*\|^2 + \|h_*\|^2\right) \quad \forall \varepsilon > 0. \tag{3.59}$$

Let us now estimate the norms of $(v^\varepsilon, w^\varepsilon)$ in $\mathcal{H}$. To this purpose, let us introduce the functions $(\widetilde{\varphi}^\varepsilon, \widetilde{\pi}^\varepsilon, \widetilde{\psi}^\varepsilon) := \kappa(t)^{-1}(\varphi^\varepsilon, \pi^\varepsilon, \psi^\varepsilon)$, which satisfy

$$\begin{cases} -\partial_t\widetilde{\varphi}^\varepsilon - \Delta\widetilde{\varphi}^\varepsilon - (a\cdot\nabla)\widetilde{\varphi}^\varepsilon - D(\widetilde{\varphi}^\varepsilon)b + \nabla\widetilde{\pi}^\varepsilon = c\nabla\widetilde{\psi}^\varepsilon - \partial_t[\kappa(t)^{-1}]\varphi^\varepsilon & \text{in} \quad \mathcal{O}_T, \\ -\partial_t\widetilde{\psi}^\varepsilon - \Delta\widetilde{\psi}^\varepsilon - (a+b)\cdot\nabla\widetilde{\psi}^\varepsilon = \widetilde{\varphi}^\varepsilon\cdot e_n - \partial_t[\kappa(t)^{-1}]\psi^\varepsilon & \text{in} \quad \mathcal{O}_T, \\ \nabla\cdot\widetilde{\varphi}^\varepsilon = 0 & \text{in} \quad \mathcal{O}_T, \\ \widetilde{\varphi}^\varepsilon\cdot\nu = 0, \quad [D(\widetilde{\varphi}^\varepsilon)\nu + A\widetilde{\varphi}^\varepsilon]_{tan} = 0 & \text{on} \quad \varLambda_T, \\ \dfrac{\partial\widetilde{\psi}^\varepsilon}{\partial\nu} + B\widetilde{\psi}^\varepsilon = 0 & \text{on} \quad \varLambda_T, \\ \widetilde{\varphi}^\varepsilon(T,\cdot) = 0, \quad \widetilde{\psi}^\varepsilon(T,\cdot) = 0 & \text{in} \quad \mathcal{O}. \end{cases}$$

Then, it is not difficult to deduce the following energy estimate

$$\|\widetilde{\varphi}^\varepsilon\|_{L^2(H^1)} + \|\widetilde{\psi}^\varepsilon\|_{L^2(H^1)} \leq Ce^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|A\|_\infty^2+\|B\|_\infty^2)}\left(\|(\kappa^{-1})_t\varphi^\varepsilon\| + \|(\kappa^{-1})_t\psi^\varepsilon\|\right).$$

Now, applying strong energy estimates for the Stokes equations with Navier slip boundary conditions, see (GUERRERO, 2006b, Proposition 1.1), and for the heat equation with Robin boundary conditions, see (FERNÁNDEZ-CARA et al., 2006a, Proposition 2), we deduce that

$$\|\widetilde{\varphi}_t^\varepsilon\| + \|\widetilde{\psi}_t^\varepsilon\| + \|\widetilde{\varphi}^\varepsilon\|_{L^2(H^2)} + \|\widetilde{\psi}^\varepsilon\|_{L^2(H^2)} + \|\widetilde{\varphi}^\varepsilon\|_{C^0(H^1)} + \|\widetilde{\psi}^\varepsilon\|_{C^0(H^1)} \leq C\Big(\|(\kappa^{-1})_t\varphi^\varepsilon\| + \|(\kappa^{-1})_t\psi^\varepsilon\|\Big),$$

where $C = C(\mathcal{O},\omega,T,a,b,c,A,B)$. Combining the previous estimate and the Carleman estimate (3.54), we get

$$\|v_t^\varepsilon\| + \|w_t^\varepsilon\| + \|v^\varepsilon\|_{L^2(H^2)} + \|w^\varepsilon\|_{L^2(H^2)} + \|v^\varepsilon\|_{C^0(H^1)} + \|w^\varepsilon\|_{C^0(H^1)} \leq C\Big(\|\kappa^{1/2}v^\varepsilon\chi_\omega\| + \|\kappa^{1/2}w^\varepsilon\chi_\omega\|\Big).$$

Finally, thanks to (3.59), there exists a control pair (v, w) such that

$$(v,w) \in H^1(0,T;L^2(\mathcal{O})^{n+1}) \cap L^2(0,T;H^2(\mathcal{O})^{n+1}) \cap C^0(0,T;H^1(\mathcal{O})^{n+1}),$$

$$\|v\|_{H^1(L^2)} + \|w\|_{H^1(L^2)} + \|v\|_{L^2(H^2)} + \|w\|_{L^2(H^2)} + \|v\|_{C^0(H^1)} + \|w\|_{C^0(H^1)} \leq C\left(\|z_*\|_H^2 + \|h_*\|^2\right).$$

and the associated solution to (3.47), denoted by (z, q, h), satisfies (3.56) and, moreover,

$$\|\kappa^{1/2}v\chi_\omega\|^2 + \|\kappa^{1/2}w\chi_\omega\|^2 \leq C(\|z_*\|^2 + \|h_*\|^2).$$

This concludes the proof. □

3.4.3 Local exact controllability to the trajectories of the Boussinesq system

This section is devoted to prove the local exact controllability to the trajectories of (3.46). Let $(\overline{u}, \overline{p}, \overline{\theta})$ an uncontrolled solution of (3.46), that is, a solution of

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} + (\overline{u} \cdot \nabla)\overline{u} + \nabla \overline{p} = \overline{\theta} e_n & \text{in} \quad \mathcal{O}_T, \\ \partial_t \overline{\theta} - \Delta \overline{\theta} + \overline{u} \cdot \nabla \overline{\theta} = 0 & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot \overline{u} = 0 & \text{in} \quad \mathcal{O}_T, \\ \overline{u} \cdot \nu = 0, \quad N(\overline{u}) + [f(\overline{u})]_{tan} = 0 & \text{on} \quad \Lambda_T, \\ R(\overline{\theta}) + g(\overline{\theta}) = 0 & \text{on} \quad \Lambda_T, \\ \overline{u}(0, \cdot\,) = \overline{u}_*, \quad \overline{\theta}(0, \cdot\,) = \overline{\theta}_* & \text{in} \quad \mathcal{O}. \end{cases}$$

We will assume the following regularity for the trajectories:

$$\begin{aligned} &\overline{u} \in X := H^{(3-\ell)/2}(0, T; H^{\vartheta_2+1/2}(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n) \cap H^{1-\ell}(0, T; W^{\vartheta_1+1/2, \vartheta_1+1}(\mathcal{O})^n), \\ &\overline{u}_* \in H^3(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n, \quad N(\overline{u}_*) + [f(\overline{u}_*)]_{tan} = 0 \quad \text{on} \quad \partial\mathcal{O}, \\ &\overline{\theta} \in Y := H^{(3-\ell)/2}(0, T; H^{\vartheta_2+1/2}(\mathcal{O})) \cap H^{1-\ell}(0, T; W^{\vartheta_1+1/2, \vartheta_1+1}(\mathcal{O})), \\ &\overline{\theta}_* \in H^3(\mathcal{O}), \quad R(\overline{\theta}_*) + g(\overline{\theta}_*) = 0 \quad \text{on} \quad \partial\mathcal{O}. \end{aligned} \tag{3.60}$$

We have the following result:

Proposition 3.18. *Let $f \in C^3(\mathbb{R}^n; \mathbb{R}^n)$, $g \in C^3(\mathbb{R})$, $T > 0$, $(\overline{u}_*, \overline{\theta}_*)$ satisfying* (3.60)*. Then, there exists $\delta > 0$ such that, for every $(u_*, \theta_*) \in [H^3(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n] \times H^3(\mathcal{O})$ satisfying the compatibility condition*

$$N(u_*) + [f(u_*)]_{tan} = 0, \qquad R(\theta_*) + g(\theta_*) = 0 \quad \text{on} \quad \partial\mathcal{O} \tag{3.61}$$

and such that $\|u_* - \overline{u}_*\|_{H^3} \leq \delta$ and $\|\theta_* - \overline{\theta}_*\|_{H^3} \leq \delta$, there exist controls $(v, w) \in \mathcal{H}$ and associated solutions (u, p, θ) to (3.46) *satisfying*

$$u(T, \cdot\,) = \overline{u}(T, \cdot\,) \quad \text{and} \quad \theta(T, \cdot\,) = \overline{\theta}(T, \cdot\,) \quad \text{in } \mathcal{O}.$$

Proof. Let us denote $z = u - \overline{u}$ and $h = \theta - \overline{\theta}$. Making the difference between the system fulfilled by $\overline{u}$ and system (3.46), we have:

$$\begin{cases} \partial_t z - \Delta z + (z \cdot \nabla)z + (z \cdot \nabla)\overline{u} + (\overline{u} \cdot \nabla)z + \nabla q = h e_n + v\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ \partial_t h - \Delta h + z \cdot \nabla h + \overline{u} \cdot \nabla h + z \cdot \nabla \overline{\theta} = w\chi_\omega & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot z = 0 & \text{in} \quad \mathcal{O}_T, \\ z \cdot \nu = 0, \quad N(z) + [F(\overline{u}, z)z]_{tan} = 0 & \text{on} \quad \Lambda_T, \\ R(h) + G(\overline{\theta}, h)h = 0 & \text{on} \quad \Lambda_T, \\ z(0, \cdot\,) = u_* - \overline{u}_* = z_*, \quad h(0, \cdot\,) = \theta_* - \overline{\theta}_* = h_* & \text{in} \quad \mathcal{O}, \end{cases}$$

where

$$F_i(\overline{u},z)=\int_0^1 \nabla f_i(\overline{u}+sz)\,ds,\quad F(\overline{u},z)=(F_1(\overline{u},z),\dots,F_n(\overline{u},z))$$

and

$$G(\overline{\theta},h)=\int_0^1 g'(\overline{\theta}+sh)\,ds.$$

Now, our goal is to find controls (v,w) such that $z(T,\cdot\,)\equiv 0$ and $h(T,\cdot\,)\equiv 0$. To this purpose, we will use Proposition 3.17 and a fixed-point argument.

First, we introduce the following closed linear manifolds

$$X_0=\{\Phi\in X:\ \Phi(0,\cdot\,)=z_*\ \text{ in }\ \mathcal{O}\}\quad\text{and}\quad Y_0=\{\Psi\in Y:\ \Psi(0,\cdot\,)=h_*\ \text{ in }\ \mathcal{O}\}.$$

Then, for each $(\Phi,\Psi)\in X_0\times Y_0$, we can apply Proposition 3.17 to guarantee the existence of controls $(v_{(\Phi,\Psi)},w_{(\Phi,\Psi)})\in\mathcal{H}$ such that the associated solution to

$$\begin{cases}\partial_t z-\Delta z+(\Phi\cdot\nabla)z+(z\cdot\nabla)\overline{u}+(\overline{u}\cdot\nabla)z+\nabla q=he_n+v\chi_\omega & \text{in}\quad \mathcal{O}_T,\\ \partial_t h-\Delta h+\Phi\cdot\nabla h+\overline{u}\cdot\nabla h+z\cdot\nabla\overline{\theta}=w\chi_\omega & \text{in}\quad \mathcal{O}_T,\\ \nabla\cdot z=0 & \text{in}\quad \mathcal{O}_T,\\ z\cdot\nu=0,\quad N(z)+[F(\overline{u},\Phi)z]_{tan}=0 & \text{on}\quad \Lambda_T,\\ R(h)+G(\overline{\theta},\Psi)h=0 & \text{on}\quad \Lambda_T,\\ z(0,\cdot\,)=u_*-\overline{u}_*=z_*,\quad h(0,\cdot\,)=\theta_*-\overline{\theta}_*=h_* & \text{in}\quad \mathcal{O},\end{cases}\tag{3.62}$$

verifies $z_{(\Phi,\Psi)}(T,\cdot\,)=0$, $h_{(\Phi,\Psi)}(T,\cdot\,)=0$. Since $F\in C^2(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^{n\times n})$ and $G\in C^2(\mathbb{R}^2;\mathbb{R})$ and $\overline{u}$ and $\overline{\theta}$ verify (3.60), we have $F_i(\overline{u},\Phi)\in X$ for all $\Phi\in X_0$, $i=1,\dots,n$, and $G(\overline{\theta},\Psi)\in Y$ for all $\Psi\in Y_0$. Moreover, since f is an irrotational field, we have that $F(\overline{u},\Phi)$ is symmetric.

Furthermore, these controls can be chosen satisfying

$$\|v_{(\Phi,\Psi)}\|_{H^1(L^2)}+\|w_{(\Phi,\Psi)}\|_{H^1(L^2)}+\|v_{(\Phi,\Psi)}\|_{L^\infty(H^1)}+\|w_{(\Phi,\Psi)}\|_{L^\infty(H^1)}\le C\left(\|z_*\|^2+\|h_*\|^2\right),\tag{3.63}$$

for some positive constant $C=C(\mathcal{O},\omega,T,\overline{u},\overline{\theta},\|\Phi\|_X,\|A\|_P,\|B\|_Q,\|F(\overline{u},\Phi)\|_X,\|G(\overline{\theta},\Psi)\|_Y)$.

Next, since we can prove that the terms $(\Phi\cdot\nabla)z_{(\Phi,\Psi)}$, $(z_{(\Phi,\Psi)}\cdot\nabla)\overline{u}$, $(\overline{u}\cdot\nabla)z_{(\Phi,\Psi)}$, $h_{(\Phi,\Psi)}e_n$ and $v_{(\Phi,\Psi)}\chi_\omega$ belong to $L^\infty(0,T;H^1(\mathcal{O})^n)\cap H^1(0,T;L^2(\mathcal{O})^n)$. Thanks to (3.60) and (3.61), we see that $z_*\in H^3(\mathcal{O})^n\cap L^2_{div}(\mathcal{O})^n$ satisfies the compatibility condition $N(z_*)+[F(\overline{u},\Phi)(0,\cdot)z_*]_{tan}=0$.Hence, we can apply (GUERRERO, 2006b, Proposition 1.2) to deduce that

$$z_{(\Phi,\Psi)}\in\widetilde{X}:=H^2(0,T;L^2_{div}(\mathcal{O}))\cap H^1(0,T;H^2(\mathcal{O})^n\cap L^2_{div}(\mathcal{O})).$$

Likewise, one can prove that the terms $\Phi\cdot\nabla h_{(\Phi,\Psi)}$, $\overline{u}\cdot\nabla h_{(\Phi,\Psi)}$, $z_{(\Phi,\Psi)}\cdot\nabla\overline{\theta}$ and $w_{(\Phi,\Psi)}\chi_\omega$ belong to $L^\infty(0,T;H^1(\mathcal{O}))\cap H^1(0,T;L^2(\mathcal{O}))$ and, thanks to equations (3.60) and (3.61),

$h_* \in H^3(\mathcal{O})$ satisfies the compatibility condition $R(h_*) + G(\overline{\theta}, \Psi)(0,\cdot)h_* = 0$. Therefore, we have

$$h_{(\Phi,\Psi)} \in \widetilde{Y} := H^2(0,T;L^2(\mathcal{O})) \cap H^1(0,T;H^2(\mathcal{O})).$$

Moreover, there exists $\widetilde{C} = \widetilde{C}(\mathcal{O}, \omega, T, \overline{u}, \overline{\theta}, \|\Phi\|_X, \|A\|_P, \|B\|_Q, \|F(\overline{u},\Phi)\|_X, \|G(\overline{\theta},\Psi)\|_Y)$ a positive constant such that

$$\|(z_{(\Phi,\Psi)}, h_{(\Phi,\Psi)})\|_{\widetilde{X}\times\widetilde{Y}} \leq \widetilde{C}\left(\|z_*\|_{H^3\cap W} + \|h_*\|_{H^3}\right). \tag{3.64}$$

From well-known interpolation arguments, one has $\widetilde{X} \subset\subset X$ and $\widetilde{Y} \subset\subset Y$, where the notation $\subset\subset$ stands for compact embedding.

For each $(\Phi,\Psi) \in X \times Y$, let the set of admissible controls $\Lambda(\Phi,\Psi)$ be, by definition, the family of controls $(v_{(\Phi,\Psi)}, w_{(\Phi,\Psi)}) \in \mathcal{H}$ that drive the solution $(z_{(\Phi,\Psi)}, h_{(\Phi,\Psi)})$ to zero at time T and such that (3.63) holds. On the other hand, let us set

$$\mathcal{E}(\Phi,\Psi) := \left\{ (z_{(\Phi,\Psi)}, h_{(\Phi,\Psi)}) \in \widetilde{X}\times\widetilde{Y} \colon \begin{array}{c} (z_{(\Phi,\Psi)}, h_{(\Phi,\Psi)}) \text{ solves (3.62) with} \\ (v_{(\Phi,\Psi)}, w_{(\Phi,\Psi)}) \in \Lambda(\Phi,\Psi) \end{array} \right\}.$$

Notice that $\mathcal{E}(\Phi,\Psi) \subset \widetilde{X}\times\widetilde{Y} \subset\subset X \times Y$.

In what follows, we will prove that the set-valued mapping $\mathcal{E}\colon X_0 \times Y_0 \mapsto 2^{X_0\times Y_0}$ possesses at least one fixed point. We will use the additional hypothesis $\|(z_*, h_*)\|_{H^3\times H^3} \leq \delta$ for some sufficiently small δ depending on $\mathcal{O}$, ω and T and we will apply Kakutani's Theorem. More precisely, we will check that the mapping $\mathcal{E}$ satisfies the following assumptions:

i) $\mathcal{E}(\Phi,\Psi)$ is a non-empty closed and convex set of $X_0 \times Y_0$ for all $(\Phi,\Psi) \in X_0 \times Y_0$;

ii) There exists a convex compact set $K \subset X_0 \times Y_0$ such that $\mathcal{E}((\Phi,\Psi)) \subset K$ for all $(\Phi,\Psi) \in K$;

iii) $\mathcal{E}(\Phi,\Psi)$ is upper-hemicontinuous in $X_0 \times Y_0$, i.e. for any $\Upsilon \in X_0' \times Y_0'$ the mapping

$$(\Phi,\Psi) \mapsto \sup_{(\overline{\Phi},\overline{\Psi})\in\mathcal{E}(\Phi,\Psi)} \left\langle \Upsilon, (\overline{\Phi},\overline{\Psi}) \right\rangle_{X_0'\times Y_0', X_0\times Y_0}$$

is upper semicontinuous.

Then, in view of Kakutani's Theorem, there exists $(z,h) \in K$ such that $(z,h) \in \mathcal{E}(z,h)$.

Proof of assumption i) **of Kakutani's Theorem.** This is easy. Indeed, for every $(\Phi,\Psi) \in X_0 \times Y_0$, $\mathcal{E}(\Phi,\Psi)$ is a non-empty set because of the null controllability property of (3.62). On the other hand, since (3.62) is linear, we readily have that $\mathcal{E}(\Phi,\Psi)$ is closed and convex.

Proof of assumption ii) **of Kakutani's Theorem.** Let $R > 0$ be given and let us introduce

$$C(R) := \sup_{\|(\Phi,\Psi)\|_{X\times Y}\leq R} \widetilde{C}(\mathcal{O}, \omega, T, \overline{u}, \overline{\theta}, \|\Phi\|_X, \|A\|_P, \|B\|_Q, \|F(\overline{u}, \Phi)\|_X, \|G(\overline{\theta}, \Psi)\|_Y),$$

where $\widetilde{C}$ is the constant arising in (3.64). If we choose $\delta \leq R/C(R)$ and from (3.64) and the fact that $\widetilde{X} \times \widetilde{Y} \subset\subset X \times Y$, we see that $\mathcal{E}$ maps the closed convex set

$$\widetilde{K} = \{(\Phi, \Psi) \in X_0 \times Y_0; \|(\Phi, \Psi)\|_{X\times Y} \leq R\}$$

into a compact set $K \subset \widetilde{K}$.

Proof of assumption iii) **of Kakutani's Theorem.** Let us prove that $\mathcal{E}$ is upper-hemicontinuous. In fact, let $\{(\Phi_k, \Psi_k)\}$ be such that

$$(\Phi_k, \Psi_k) \to (\Phi, \Psi) \quad \text{in} \quad X_0 \times Y_0.$$

From the compactness of $\mathcal{E}(\Phi_k, \Psi_k)$ into $X \times Y$, we deduce that there exist $(z_k, h_k) \in \mathcal{E}(\Phi_k, \Psi_k)$ for $k = 1, 2, \cdots$ such that

$$\sup_{(\overline{\Phi},\overline{\Psi})\in\mathcal{E}(\Phi_k,\Psi_k)} \left\langle \Upsilon, (\overline{\Phi}, \overline{\Psi}) \right\rangle_{X'\times Y', X\times Y} = \langle \Upsilon, (z_k, h_k)\rangle_{X'\times Y', X\times Y} \quad \forall\, k \geq 1.$$

We can choose a subsequence $\{(\Phi_{k'}, \Psi_{k'})\}$ such that

$$\limsup_{k\to\infty} \sup_{(\overline{\Phi},\overline{\Psi})\in\mathcal{E}(\Phi_k,\Psi_k)} \left\langle \Upsilon, (\overline{\Phi}, \overline{\Psi}) \right\rangle_{X'\times Y', X\times Y} = \lim_{k'\to\infty} \langle \Upsilon, (z_{k'}, h_{k'})\rangle_{X'\times Y', X\times Y}.$$

Denote by $(v_{k'}, w_{k'}) \in \Lambda(\Phi_{k'}, \Psi_{k'})$ controls associated to $(z_{k'}, h_{k'})$ solution of following systems:

$$\begin{cases}
\partial_t z_{k'} - \Delta z_{k'} + (\Phi_{k'} \cdot \nabla) z_{k'} + (z_{k'} \cdot \nabla)\overline{u} + (\overline{u} \cdot \nabla) z_{k'} + \nabla q_{k'} = h_{k'} e_n + v_{k'} \chi_\omega & \text{in} \quad \mathcal{O}_T, \\
\partial_t h_{k'} - \Delta h_{k'} + \Phi_{k'} \cdot \nabla h_{k'} + \overline{u} \cdot \nabla h_{k'} + z_{k'} \cdot \nabla \overline{\theta} = w_{k'} \chi_\omega & \text{in} \quad \mathcal{O}_T, \\
\nabla \cdot z_{k'} = 0 & \text{in} \quad \mathcal{O}_T, \\
z_{k'} \cdot \nu = 0, \quad N(z_{k'}) + [F(\overline{u}, \Phi_{k'}) z_{k'}]_{tan} = 0 & \text{on} \quad \Lambda_T, \\
R(h_{k'}) + G(\overline{\theta}, \Psi_{k'}) h_{k'} = 0 & \text{on} \quad \Lambda_T, \\
z_{k'}(0, \cdot) = z_*, \quad h_{k'}(0, \cdot) = h_* & \text{in} \quad \mathcal{O}.
\end{cases}$$

Then, using the fact the $F(\overline{u}, \Phi_{k'}) \to F(\overline{u}, \Phi)$ in X and $G(\overline{\theta}, \Psi_{k'}) \to G(\overline{\theta}, \Phi)$ in Y, we find that the constants in (3.63) and (3.64) can be chosen independent of k'. Therefore, the compact embedding $\widetilde{X} \times \widetilde{Y} \subset\subset X \times Y$, together with the estimates (3.63) and (3.64), guarantees that, at least for a subsequence, we have

$$(z_{k'}, h_{k'}) \to (z, h) \quad \text{in} \quad X \times Y,$$

$$v_{k'} \to v \quad \text{weakly in} \quad H^1(0,T;L^2(\mathcal{O})^n) \cap L^\infty(0,T;H^1(\mathcal{O})^n),$$

$$w_{k'} \to w \quad \text{weakly in} \quad H^1(0,T;L^2(\mathcal{O})) \cap L^\infty(0,T;H^1(\mathcal{O})).$$

It is not difficult to conclude that $(v,w) \in \Lambda(\Phi,\Psi)$ and that $(z,h) \in \mathcal{E}(\Phi,\Psi)$. Therefore, one has:

$$\begin{aligned}\limsup_{k\to\infty} \sup_{(\overline{\Phi},\overline{\Psi})\in\mathcal{E}(\Phi_k,\Psi_k)} \left\langle \Upsilon, (\overline{\Phi},\overline{\Psi})\right\rangle_{X'\times Y',X\times Y} &= \langle \Upsilon,(z,h)\rangle_{X'\times Y',X\times Y} \\ &\leq \sup_{(\bar z,\bar h)\in\mathcal{E}(\Phi,\Psi)} \left\langle \Upsilon,(\bar z,\bar h)\right\rangle_{X'\times Y',X\times Y}.\end{aligned}$$

This proves the upper-hemicontinuity of $\mathcal{E}$.

Thus, we have proved that $\mathcal{E}$ has a fixed-point (z,h) and this achieves the proof of Proposition 3.18. □

3.5 GLOBAL CONTROLLABILITY TO THE TRAJECTORIES

This section is devoted to explain how the previous arguments can be chained in order to prove our main result, that is, Theorem 3.1.

First, we reduce the controllability to weak trajectories to controllability smooth trajectories as follows.

Despite that $(\overline{u},\overline{p},\overline{\theta})$ is only a weak solution in $[0,T]$, there exists an interval time $[T_1,T_2] \subset (0,T)$ such that $(\overline{u},\overline{p},\overline{\theta})$ is smooth in $[T_1,T_2]$. Then, we can start our control strategy by doing nothing in $[0,T_1]$, that is, taking $v = w = \sigma = 0$ in (3.7), and wait for the reference trajectory to be regularized. Thus, the weak trajectory will move from (u_*,θ_*) to some $(u,\theta)(T_1,\cdot\,)$, that will be considered the new initial data. Hence, without loss of generality, we can work with a smooth reference trajectory.

We split the control strategy into four steps.

Step 1: Regularization of the data. We begin by extending Ω to a new domain $\mathcal{O}$, as explained in Section 3.2.1. We also use Proposition 3.5 to guarantee the existence of $(u_*,\theta_*,\sigma_*) \in L^2(\mathcal{O})^n \times L^2(\mathcal{O}) \times C_c^\infty(\omega_0)$ satisfying (3.5). We set $\sigma(t,x) := \beta(t/T)\sigma_*(x)$ with β a smooth decreasing function such that $\beta \equiv 1$ near 0 and $\beta \equiv 0$ near 1/8. The function σ must satisfy the compatibility condition $\nabla \cdot u_* = \sigma(0,\cdot\,)$. Then, we let system (3.7) evolve with $v = w = 0$ in the time interval $(0,T/8)$ in order to reach some data $(u,\theta)(T/8,\cdot\,) \in L^2_{div}(\mathcal{O})^n \times L^2(\mathcal{O})$. Next, by using the smoothing effect of the uncontrolled Boussinesq system starting from divergence free data (see Lemma 3.8), we deduce that there exists $T_1 \in (0,T/4)$ such that $(u,\theta)(T_1,\cdot\,) \in H^3(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n \times H^3(\mathcal{O})$. Accordingly, we can apply Lemma 3.12.

Step 2: Global approximate controllability result in $L^2(\mathcal{O})$. Let us set $T_2 := T/2$. Starting from the new initial data $(u,\theta)(T_1,\cdot\,)$, we use the global approximate controllability result stated in Proposition 3.9 in a time interval of size $T_2 - T_1 \geq T/4$. Thus, for any $\delta > 0$, we can build a trajectory starting from $(u,\theta)(T_1,\cdot\,)$ and such that

$$\|(u,\theta)(T_2,\cdot\,) - (\overline{u},\overline{\theta})(T_2,\cdot\,)\| \leq \delta.$$

In particular, we can find δ small enough such that

$$\Psi_{T/4}(\delta) \leq \delta_{T/4},$$

where $\delta_{T/4}$ is the radius of local controllability result given in Proposition 3.18 , for $f = 0$ and $g = 0$, and the function $\Psi_{T/4}$ appears in the regularity result for the free Boussinesq system; see Lemma 3.8.

Step 3: Regularizing argument. Now, we use again Lemma 3.8 to obtain the existence of a time $T_3 \in (T/2, 3T/4)$ such that

$$\|(u,\theta)(T_3,\cdot\,) - (\overline{u},\overline{\theta})(T_3,\cdot\,)\|_{H^3} \leq \Psi_{T/4}(\delta) \leq \delta_{T/4}.$$

Step 4: Local controllability in $H^3(\mathcal{O})$. Finally, we use the local controllability result in $[T_3, T_3 + T/4]$, and get

$$(u,\theta)(T_3 + T/4,\cdot\,) = (\overline{u},\overline{\theta})(T_3 + T/4,\cdot\,).$$

Then, extending the control by zero for $t \in [T_3 + T/4]$, we obtain (3.4) and the proof is complete.

3.6 COMMENTS AND OPEN QUESTIONS

3.6.1 Controlling with less controls

A natural extension of our main result would be the global exact controllability with a reduced number of controls acting on a small part of the boundary. Unfortunately, in this situation, one cannot use the extension domain technique.

However, in the spirit of (FERNÁNDEZ-CARA et al., 2006; GUERRERO; MONTOYA, 2018), one could try to establish a small-time global null controllability for the internal control system (3.7) in 2-D by acting only on the temperature. Roughly speaking, the intuition behind a result of this kind is the following: the temperature θ is directly controlled by w, then θ acts as an indirect control through the coupling term θe_2 to control the component u_2, then u_2 acts also as an indirect control through the incompressibility condition to control the component u_1. Results of this kind will be analyzed in a forthcoming paper.

One could also try get the local control result acting only on the motion equation, that is, with $w = 0$ in (3.46). However, at least in the case of Neumann boundary conditions for θ, that is, with $m \equiv 0$ and $g \equiv 0$, the system does not seem to be controllable. To justify this assertion, note that, by integrating in $\mathcal{O}$ the equation satisfied by θ, integrating by parts and using the incompressibility and impermeability conditions, we find that the total mass of θ is conserved:

$$\int_{\mathcal{O}} \theta(T, \cdot\,) = \int_{\mathcal{O}} \theta_*.$$

Therefore, we cannot expect general null controllability.

3.6.2 Other boundary conditions

Another natural question is if Theorem 3.1 holds with u and θ subject to other boundary conditions.

For Dirichlet boundary conditions on the temperature, this is an interesting open problem. As noticed for the slip case in Remark 3.13, the main difficulty is to obtain good estimates for the remainder terms.

When we consider Dirichlet boundary conditions for the velocity, we face a challenging open problem. This is related to a well know conjecture by Jacques-Louis Lions. As pointed in (CORON; MARBACH; SUEUR, 2020), the boundary layer has a behavior which is not good as in the case of Navier boundary conditions. This implies many difficulties to estimate the boundary layer profiles and the remainder terms.

3.6.3 Controlling for strong solutions

Recently, in (LIAO; SUEUR; ZHANG, 2020), the small-time global null controllability for strong solutions of Navier-Stokes was obtained. An interesting question would be to extend this result to strong solutions of the Boussinesq system (then improving the Theorem 3.1).

3.6.4 Fluid-structure

Some controllability results for fluid-structure problem have been obtained in the past years. For instance, local null controllability of a two-dimensional viscous fluid-structure was obtained in (BOULAKIA; OSSES, 2008); on the other hand, boundary controllability for the motion of a rigid body immersed in a perfect two-dimensional fluid was proved in (GLASS; KOLUMBÁN; SUEUR, 2020). It would be interesting to analyze the global controllability for fluid-structure problem with Navier-slip boundary conditions.

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APPENDIX A – EXISTENCE AND UNIQUENESS OF A SOLUTION TO STOKES EQUATIONS WITH MEMORY

Let us denote by A the usual Stokes operator, with domain $D(A) := H^2(\Omega)^3 \cap V$. Recall that $D(A) \hookrightarrow V \hookrightarrow H$, with dense and compact embeddings. Consequently, after identification of H and its dual space, we also have $H \hookrightarrow V' \hookrightarrow D(A)'$, where the embeddings are again dense and compact.

Let us prove that, for each $g \in L^2(0,T;H)$, there exists exactly one *strong* solution to (2.3). This can be seen (for example) as follows.

Let us introduce the change of variables

$$\varphi = \int_t^T e^{-as}\psi(\cdot\,,s)\,ds, \quad \eta = e^{-at}\pi(\cdot\,,t). \tag{A.1}$$

Then, at least formally, we see that (ψ,π) solves (2.3) if and only if (φ,η) solves the system

$$\begin{cases} \varphi_{tt} + a\varphi_t + b\Delta\varphi_t - \Delta\varphi + \nabla\eta = \tilde{g} & \text{in} \quad Q, \\ \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(\cdot\,,T) = 0, \quad \varphi_t(\cdot\,,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{A.2}$$

where $\tilde{g}(\cdot\,,t) := e^{-at}g(\cdot\,,t)$.

The existence and uniqueness of a solution to (A.2) can be deduced in a completely standard way, for instance via the Galerkin method. Thus, we first introduce an orthogonal basis in V (for instance, the basis formed by the eigenfunctions of the Stokes operator), we solve the associated finite dimensional problems, we deduce uniform estimates for the corresponding solutions in $L^\infty(0,T;D(A))$, for their first-order time derivatives in $L^\infty(0,T;V)$ and $L^2(0,T;D(A))$ and also for their second-order time derivatives in $L^2(0,T;H)$, we extract convergent subsequences and we finally take limits and check that (A.2) is satisfied for some $\eta \in L^2(0,T;H^1(\Omega))$. We also get estimates in these spaces that prove linear and continuous dependence of g. The process is described with detail for general second-order in time systems for instance in (EVANS, 1998, Chapter 7, pp. 380–394); see also (TEMAM, 1977, Chapter 3, pp. 255–265).

With the help of (A.1), we deduce that there exists exactly one solution to (2.3), with

$$\psi \in L^\infty(0,T;V) \cap L^2(0,T;D(A)), \quad \psi_t \in L^2(0,T;H), \quad \pi \in L^2(0,T;H^1(\Omega))$$

and, consequently,

$$\psi \in C^0([0,T];V) \;\text{ and }\; \left(-\pi n + \frac{\partial\psi}{\partial\nu} + b\int_\cdot^T e^{-a(s-t)}\frac{\partial\psi}{\partial\nu}(\cdot\,,s)\,ds\right)\Big|_\Sigma \in L^2(0,T;H^{1/2}(\partial\Omega)^3),$$

with appropriate estimates.

Now, let $u_0 \in H$ and $v \in L^2(\gamma \times (0,T))^3$ be given. For any $g \in L^2(0,T;H)$, the right hand side of (2.2) (where (ψ,π) solves the corresponding system (2.3)) makes sense and is linearly and continuously dependent of g. Consequently, there exists a unique $u \in L^2(0,T;H)$ satisfying (2.2) for all $g \in L^2(0,T;H)$ (by definition, this is the solution by trasposition to (2.1)).

Note that u solves, together with some p, $(2.1)_1$ in the distributional sense in Q (this is immediate if we first compute the action of the left hand side of $(2.1)_1$ on a test function in Q with zero divergence and, then, we apply De Rham's Lemma). Therefore, $u_t \in L^2(0,T;D(A)')$, whence we deduce that $u \in C^0([0,T];V')$.

Finally, note that the solution by transposition to (2.1) can actually be defined for more general u_0 and v: in view of the previous argument, it suffices $u_0 \in V'$ and $v \in L^2(0,T;H^{-1/2}(\gamma)^3)$.

APPENDIX B – THE EXISTENCE OF SOLUTION TO BOUSSINESQ EQUATIONS

In this section, we give a sketch of the proof of Proposition 3.7.

First, since the divergence source term is smooth, we start by solving a Stokes problem in order to lift the non homogeneous divergence condition. To do that, we define (u_σ, p_σ) as the solution to:

$$\begin{cases} \partial_t u_\sigma - \Delta u_\sigma + \nabla p_\sigma = 0 & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot u_\sigma = \sigma & \text{in} \quad \mathcal{O}_T, \\ u_\sigma \cdot \nu = 0, \quad N(u_\sigma) = 0 & \text{on} \quad \Lambda_T, \\ u_\sigma(0, \cdot\,) = 0 & \text{in} \quad \mathcal{O}. \end{cases}$$

Smoothness (in time and space) of σ immediately gives smoothness on u_σ. These are standard maximal regularity estimates for the Stokes problem in the case of the Dirichlet boundary condition. For Navier slip-with-friction boundary conditions, we refer to (SHIMADA, 2007), (SHIBATA; SHIMADA, 2007) and (SHIBATA; SHIMADA et al., 2007). Then, by using Sobolev embeddings, we get that there exists a positive constant $C > 0$ depending on σ such that

$$\|u_\sigma\|_{L^\infty(0,T;W^{1,\infty}(\mathcal{O}))} \leq C. \tag{B.1}$$

Decomposing $u = u_\sigma + u_h$ and $p = p_\sigma + p_h$, we obtain the following system for (u_h, p_h, θ):

$$\begin{cases} \partial_t u_h - \Delta u_h + (u_\sigma \cdot \nabla) u_h + (u_h \cdot \nabla) u_\sigma + (u_h \cdot \nabla) u_h + \nabla p_h = \theta e_n + v - (u_\sigma \cdot \nabla) u_\sigma & \text{in} \quad \mathcal{O}_T, \\ \partial_t \theta - \Delta \theta + (u_h + u_\sigma) \cdot \nabla \theta = w & \text{in} \quad \mathcal{O}_T, \\ \nabla \cdot u_h = 0 & \text{in} \quad \mathcal{O}_T, \\ u_h \cdot \nu = 0, \quad N(u_h) = 0 & \text{on} \quad \Lambda_T, \\ R(\theta) = 0 & \text{on} \quad \Lambda_T, \\ u_h(0, \cdot\,) = u_*, \quad \theta(0, \cdot\,) = \theta_* & \text{in} \quad \mathcal{O}. \end{cases} \tag{B.2}$$

So, it is sufficient to obtain the existence result for the system (B.2). We define weak solutions to (B.2) as follows.

Recall that

$$W_T(\mathcal{O}) = [C_w^0([0,T]; L^2_{div}(\mathcal{O})^n) \cap L^2(0,T; H^1(\mathcal{O})^n)] \times [C_w^0([0,T]; L^2(\mathcal{O})) \cap L^2(0,T; H^1(\mathcal{O}))].$$

We say that $(u_h, \theta) \in W_T(\mathcal{O})$ is a weak solution to (B.2) if it satisfies the following:

$$\begin{aligned}
&-\iint_{\mathcal{O}_T} u_h \partial_t \phi + \iint_{\mathcal{O}_T} ((u_\sigma \cdot \nabla) u_h + (u_h \cdot \nabla) u_\sigma + (u_h \cdot \nabla) u_h)\phi + 2\iint_{\mathcal{O}_T} D(u_h) \cdot D(\phi) \\
&= \int_{\mathcal{O}} u_* \cdot \phi(0,x) - 2\iint_{\partial\mathcal{O}_T} (Mu_h) \cdot \phi + \iint_{\mathcal{O}_T} (v - (u_\sigma \cdot \nabla) u_\sigma)\phi + \iint_{\mathcal{O}_T} \theta e_n \phi,
\end{aligned}$$

and

$$\begin{aligned}
-\iint_{\mathcal{O}_T} \theta \partial_t \psi + \iint_{\mathcal{O}_T} ((u_h \cdot \nabla\theta) + (u_\sigma \cdot \nabla\theta))\psi + \iint_{\mathcal{O}_T} \nabla\theta \cdot \nabla\psi \\
= \int_{\mathcal{O}} \theta_* \psi(0,x) - \iint_{\partial\mathcal{O}_T} m\theta\psi + \iint_{\mathcal{O}_T} w\psi,
\end{aligned}$$

for any which is divergence free and tangent to $\partial\mathcal{O}$ function $\phi \in C_c^\infty([0,T) \times \overline{\mathcal{O}})^n$ and any $\psi \in C_c^\infty([0,T) \times \overline{\mathcal{O}})$. We moreover require that they satisfy the so-called *strong energy inequality* for almost every $t \in (0,T)$

$$\begin{aligned}
\|u_h(t,\cdot)\|^2 + \|\theta(t,\cdot)\|^2 + 4\iint_{\mathcal{O}_t} |D(u_h)|^2 + 2\iint_{\mathcal{O}_t} |\nabla\theta|^2 \leq \|u_h(0,\cdot)\|^2 + \|\theta(0,\cdot)\|^2 \\
-4\iint_{\partial\mathcal{O}_t} (Mu_h) \cdot u_h - 2\iint_{\partial\mathcal{O}_t} m|\theta|^2 \\
+2\iint_{\mathcal{O}_t} \left[\sigma|u_h|^2 - (u_h \cdot \nabla) u_\sigma \cdot u_h + (v - (u_\sigma \cdot \nabla) u_\sigma + \theta e_n) \cdot u_h + \sigma|\theta|^2 + w\theta\right].
\end{aligned} \tag{B.3}$$

Proof of the existence of solutions to (B.2). We recall the following identity, which will be used throughout the paper:

$$-\int_{\mathcal{O}} \Delta\tilde{u} \cdot \tilde{v} = 2\int_{\mathcal{O}} D(\tilde{u}) \cdot D(\tilde{v}) - 2\int_{\partial\mathcal{O}} [D(\tilde{u})\nu]_{tan} \cdot \tilde{v},$$

where $\tilde{u}$ and $\tilde{v}$ are smooth vector fields such that $\tilde{v}$ is divergence free and tangent to the boundary. Therefore, using above $\phi = u_h$ and $\psi = \theta$, we obtain formally the energy equality (B.3) replacing $\leq$ by $=$. We can get a bound of the right hand side term of (B.3) by using a L^∞ bound of σ and (B.1). Thus, we deduce that there exists a positive constant $C > 0$ depending on σ, v and w such that

$$\begin{aligned}
&\|u_h(t,\cdot)\|^2 + \|\theta(t,\cdot)\|^2 + 4\iint_{\mathcal{O}_t} |D(u_h)|^2 + 2\iint_{\mathcal{O}_t} |\nabla\theta|^2 \\
&\leq C\left(\|u_h(0,\cdot)\|^2 + \|\theta(0,\cdot)\|^2 + \iint_{\mathcal{O}_t} |u_h|^2 + |\theta|^2\right) - 4\iint_{\partial\mathcal{O}_t} (Mu_h) \cdot u_h - 2\iint_{\partial\mathcal{O}_t} m|\theta|^2.
\end{aligned} \tag{B.4}$$

From (B.4), and Gronwall Lemma, we obtain an a priori bound for (u_h, θ) in $L^\infty(0,T; L^2(\mathcal{O})^{n+1})$. Before continuing, let us recall the following Korn inequality.

Lemma B.1. *[Second Korn inequality] There exist two positive constants $C_1, C_2 > 0$ such that, for every $u \in H^1(\mathcal{O})^n$, one has*

$$C_1 (\|u\| + \|D(u)\|) \leq \|u\|_{H^1} \leq C_2 (\|u\| + \|D(u)\|).$$

By using the previous a priori bound for (u_h, θ) in $L^\infty(0,T;L^2(\mathcal{O})^{n+1})$, the second Korn inequality and the estimate (B.4), we also obtain an a priori bound in $L^2(0,T;H^1(\mathcal{O})^{n+1})$. A standard Galerkin procedure implies the existence of a solution with this regularity.

We next justify that this solution can be assumed to verify the energy inequality. We recall the standard argument to justify the energy inequality. Let (u_h^N, θ^N) be the approximate solution obtained via the Galerkin method. We write the energy inequality (B.3) that holds true for (u_h^N, θ^N) and pass to the limit as $N \to +\infty$. We observe that the right-hand side converges, because (u_h^N, θ^N) converges strongly to (u_h, θ) in $L^2(\mathcal{O}_T)$ as $N \to +\infty$; this is a consequence the two previous bounds and, for instance, Aubin-Lions Lemma. For the left-hand side, it is enough to use convexity, lower semicontinuity of the norms and weak convergence.

Although uniqueness of weak Leray solutions is still an open question, adapting the classic Leray-Hopf theory proves the global existence of weak solution for the case of Navier boundary conditions (see (CLOPEAU; MIKELIC; ROBERT, 1998) and (FILHO; LOPES; PLANAS, 2005) for 2D or (IFTIMIE; PLANAS, 2006) for 3D). Once forcing terms v, w and σ are fixed, there exists thus at least one weak Leray solution (u, θ) to (3.7).

APPENDIX C – SMOOTHING EFFECT FOR THE UNCONTROLLED BOUSSINESQ SYSTEM

Let us present the proof of Lemma 3.8. In the following, we will use Korn's inequality recurrently, see Lemma B.1. We will also need the following results:

Lemma C.1. *There exist positive constants $C_l, C_r, K > 0$ such that, for every $u \in H^1(\mathcal{O})^n$, we have*

$$C_l\|u\|_{K,M} \leq \|u\|_{H^1} \leq C_r\|u\|_{K,M},$$

where $\|u\|_{K,M} := \left(K\|u\|^2 + \int_{\partial\mathcal{O}} Mu\cdot u + \|D(u)\|^2\right)^{1/2}$.

Lemma C.2. *There exist positive constants $C_l, C_r, \gamma > 0$ such that, for every $\theta \in H^1(\mathcal{O})$, we have*

$$C_l\|\theta\|_{\gamma,m} \leq \|\theta\|_{H^1} \leq C_r\|\theta\|_{\gamma,m},$$

where $\|\theta\|_{\gamma,m} := \left(\gamma\|\theta\|^2 + \int_{\partial\mathcal{O}} m|\theta|^2 + \|\nabla\theta\|^2\right)^{1/2}$.

The proofs of the two above Lemmas rely on the interpolation inequality (BOYER; FABRIE, 2012, Theorem $III.2.36$). In particular, it is used that there exists a positive constant C such that

$$\|u\|_{L^2(\partial\mathcal{O})} \leq C\|u\|^{1/2}\|u\|_{H^1}^{1/2} \quad \forall u \in H^1(\mathcal{O}).$$

Lemma C.3 (Proposition $III.2.35$, (BOYER; FABRIE, 2012))**.** *Let $p \in [1, +\infty]$ and $q \in [p, p^*]$, where p^* is the critical exponent associated with p. Then, there exists $C > 0$ such that*

$$\|u\|_{L^q} \leq C\|u\|_{L^p}^{1+n/q-n/p}\|u\|_{W^{1,p}}^{n/p-n/q} \quad \forall u \in W^{1,p}(\mathcal{O}).$$

Lemma C.4 (Pages 490-494, (GUERRERO, 2006b))**.** *Let $f \in L^2(\mathcal{O})^n$ and $g \in H^{1/2}(\partial\mathcal{O})^n$. Then, there exists a unique strong solution $(u,p) \in H^2(\mathcal{O})^n \times H^1(\mathcal{O})$ to the Stokes problem*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in} \quad \mathcal{O}, \\ \nabla\cdot u = 0 & \text{in} \quad \mathcal{O}, \\ u\cdot\nu = 0, \quad N(u) = g & \text{on} \quad \partial\mathcal{O}, \end{cases}$$

and there exists a positive constant $C > 0$ such that

$$\|u\|_{H^2} + \|p\|_{H^1} \leq C(\|f\| + \|g\|_{H^{1/2}}).$$

Moreover, if $f \in H^k(\mathcal{O})^n$ and $g \in H^{k+1/2}(\partial\mathcal{O})^n$ for some $k \geq 0$, then $(u,p) \in H^{k+2}(\mathcal{O})^n \times H^{k+1}(\mathcal{O})$ and we have

$$\|u\|_{H^{k+2}} + \|p\|_{H^{k+1}} \leq C(\|f\|_{H^k} + \|g\|_{H^{k+1/2}}).$$

Lemma C.5. *Let $S : D(S) \to L^2_{div}(\mathcal{O})^n$ be the Stokes operator, where $D(S) = \{v \in H^2(\mathcal{O})^n \cap L^2_{div}(\mathcal{O})^n : N(v) = 0\}$ and $S := -\mathbb{P}\Delta$. There exists a positive constant $C > 0$ such that, for every $u \in D(S)$, we have*

$$\|u\|_{H^2} \leq C\left(\|Su\| + \|u\|_{H^1}\right).$$

Moreover, if $Su \in H^k(\mathcal{O})^n$ for some $k \geq 0$, then $u \in H^{k+2}(\mathcal{O})^n$ and we have

$$\|u\|_{H^{k+2}} \leq C(\|Su\|_{H^k} + \|u\|_{H^{k+1}}).$$

Lemma C.6. *Let $u \in H^1(\mathcal{O})$ satisfy $\Delta u \in L^2(\mathcal{O})$ and*

$$\frac{\partial u}{\partial \nu} + mu = 0 \quad on \quad \partial\mathcal{O},$$

where $m \in L^\infty(\partial\mathcal{O})$. Then, there exists a constant $C > 0$, only depending on $\mathcal{O}$, such that

$$\|u\|_{H^2} \leq C(\|\Delta u\| + \|mu\|_{H^{1/2}(\partial\mathcal{O})}).$$

Moreover, if $\Delta u \in H^k(\mathcal{O})$ for some $k \geq 0$, then $u \in H^{k+2}(\mathcal{O})$ and we have

$$\|u\|_{H^{k+2}} \leq C(\|\Delta u\|_{H^k} + \|mu\|_{H^{k+1/2}(\partial\mathcal{O})}).$$

The proof this Lemma is consequence of (BOYER; FABRIE, 2012, Theorem III.4.3).

Throughout the proof of Lemma 3.8, the constants C can increase from line to line and depend on T and the trajectory $(\overline{u}, \overline{\theta})$. For simplicity, we consider the case $n = 3$.

Step 1: Weak estimates in $(0, T/3)$. Let us first multiply $(3.8)_1$ by r and $(3.8)_2$ by q, integrate by parts, and sum. We get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|r\|^2 + \|q\|^2\right) + 2\|Dr\|^2 + \|\nabla q\|^2 + 2\int_{\partial\mathcal{O}} Mr\cdot r + \int_{\partial\mathcal{O}} m|q|^2 \\ = (qe_n, r) - \int_{\mathcal{O}} (r\cdot\nabla)\overline{u}\cdot r - \int_{\mathcal{O}} r\cdot\nabla\overline{\theta}\, q.\end{aligned}$$

By Cauchy-Schwarz and Young inequalities, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left(\|r\|^2 + \|q\|^2\right) + 2\|Dr\|^2 + \|\nabla q\|^2 + 2\int_{\partial\mathcal{O}} Mr\cdot r + \int_{\partial\mathcal{O}} m|q|^2 \leq C(\|r\|^2 + \|q\|^2).$$

Using Lemmas C.1 and C.2, we deduce

$$\frac{1}{2}\frac{d}{dt}\left(\|r\|^2 + \|q\|^2\right) + \frac{2}{C_l^2}(\|r\|^2_{H^1} + \|q\|^2_{H^1}) \leq (C + 2K)\|r\|^2 + (C + 2\gamma)\|q\|^2. \qquad \text{(C.1)}$$

By applying Gronwall Lemma, we have for a.e. $t \in [0, T]$ that

$$\|r(t,\cdot)\|^2 + \|q(t,\cdot)\|^2 + \int_0^t \left(\|r(s,\cdot)\|^2_{H^1} + \|q(s,\cdot)\|^2_{H^1}\right) ds \leq e^{Ct}\left(\|r_*\|^2 + \|q_*\|^2\right).$$

Therefore, from the Mean Value Theorem, we deduce by contradiction that there exists $0 \leq t_1 \leq T/3$ such that

$$\|r(t_1, \cdot)\|_{H^1}^2 + \|q(t_1, \cdot)\|_{H^1}^2 \leq C_1 \left(\|r_*\|^2 + \|q_*\|^2\right), \tag{C.2}$$

for a positive constant C_1 independent of t_1.

Step 2: Strong estimates in $(t_1, 2T/3)$**.** Let $\mathbb{P}$ be the classical Leray projector. We multiply $(3.8)_1$ and $(3.8)_2$ by $-Sr$ and $-\Delta q$, respectively, then integrate by parts. Since M is symmetric, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|Dr\|^2 + \int_{\partial\mathcal{O}} Mr \cdot r\right) + \|Sr\|^2 \\
&\quad = \int_{\partial\mathcal{O}} (M_t) r \cdot r + \int_{\mathcal{O}} \left((r\cdot\nabla)r\cdot Sr + (\overline{u}\cdot\nabla)r\cdot Sr + (r\cdot\nabla)\overline{u}\cdot Sr - (qe_n, Sr)\right) \\
&\quad \leq C\|r\|_{H^1}^2 + \tfrac{1}{2}\|Sr\|^2 + C\|q\|^2 + \|r\|_{L^6}^2\|\nabla r\|_{L^3}^2.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|\nabla q\|^2 + \int_{\partial\mathcal{O}} m|q|^2\right) + \|\Delta q\|^2 \;&=\; \frac{1}{2}\int_{\partial\mathcal{O}} (m_t) q\cdot q + (r\cdot\nabla q, \Delta q) \\
&\quad + (\overline{u}\cdot\nabla q, \Delta q) + (r\cdot\nabla\overline{\theta}, \Delta q) \\
&\leq C\|q\|_{H^1}^2 + \tfrac{1}{2}\|\Delta q\|^2 + C\|r\|^2 + \|r\|_{L^6}^2\|\nabla q\|_{L^3}^2.
\end{aligned}$$

Multiplying (C.1) by $\varsigma = \max\{K, \gamma\}$, adding the above inequalities and using Lemmas C.1 – C.6, we deduce the following:

$$\begin{aligned}
\frac{d}{dt}\left(\|r\|_{\varsigma,M}^2 + \|q\|_{\varsigma,m}^2\right) + \|r\|_{H^2}^2 + \|q\|_{H^2}^2 &\leq C(\|r\|_{\varsigma,M}^2 + \|q\|_{\varsigma,m}^2 + \|r\|_{L^6}^2\|\nabla r\|_{L^3}^2 + \|r\|_{L^6}^2\|\nabla q\|_{L^3}^2) \\
&\leq C\left[(\|r\|_{\varsigma,M}^2 + \|q\|_{\varsigma,m}^2) + (\|r\|_{\varsigma,M}^2 + \|q\|_{\varsigma,m}^2)^3\right].
\end{aligned} \tag{C.3}$$

Introducing $Y(t) := \|r(t,\cdot)\|_{\varsigma,M}^2 + \|q(t,\cdot)\|_{\varsigma,m}^2$, we see that Y is a.e. differentiable and, from (C.3), we have that

$$Y' \leq C(Y^3 + Y). \tag{C.4}$$

In view of (C.4), we obtain

$$Y(t)^2 \leq \frac{e^{C(t-t_1)}Y(t_1)^2}{Y(t_1)^2 + 1 - e^{C(t-t_1)}Y(t_1)^2}.$$

Let us take $t - t_1 \leq \tau_1$ small enough and such that $e^{C(t-t_1)} \leq 1 + \frac{1}{2Y(t_1)^2}$. Then, $Y(t)^2 \leq 2e^{C(t-t_1)}Y(t_1)^2$ and, from (C.2), we deduce that $Y(t) \leq CY_*$, where $Y_* := \|r_*\|^2 + \|q_*\|^2$. Therefore,

$$\|r(t,\cdot)\|_{\varsigma,M}^2 + \|q(t,\cdot)\|_{\varsigma,m}^2 + \int_{t_1}^{t} (\|r(s,\cdot)\|_{H^2}^2 + \|q(s,\cdot)\|_{H^2}^2)ds \leq CY_* + C(Y_* + Y_*^3)\tau_1.$$

Taking τ_1 small enough such that $\tau_1 \leq (1+Y_*^2)^{-1}$, we have that $CY_*+C(Y_*+Y_*^3)\tau_1 \leq C_2Y_*$. Therefore, one has

$$\|r(t,\cdot)\|^2_{\varsigma,M}+\|q(t,\cdot)\|^2_{\varsigma,m}+\int_{t_1}^t(\|r(s,\cdot)\|^2_{H^2}+\|q(s,\cdot)\|^2_{H^2})ds \leq C_2\left(\|r_*\|^2+\|q_*\|^2\right) \quad \text{(C.5)}$$

for $t_1 \leq t \leq t_1+\tau_1$. This ensures the existence of $t_1 \leq t_2 < \min\{2T/3, t_1+\tau_1\}$ such that

$$\|r(t_2,\cdot)\|^2_{H^2}+\|q(t_2,\cdot)\|^2_{H^2} \leq \frac{C_2}{\tau_1}\left(\|r_*\|^2+\|q_*\|^2\right).$$

Step 3: Third energy estimate in (t_2, T)**.** At this point, we differentiate (3.8) with respect to time and multiply by $\partial_t r$ and $\partial_t q$. Then, we integrate by parts to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|r_t\|^2+2\|Dr_t\|^2+2\int_{\partial\mathcal{O}}Mr_t\cdot r_t\\
&=-2\int_{\partial\mathcal{O}}M_tr\cdot r_t+(q_te_n,r_t)-(r_t\cdot\nabla)r\cdot r_t-(\overline{u}_t\cdot\nabla)r\cdot r_t-(r_t\cdot\nabla)\overline{u}\cdot r_t-(r\cdot\nabla)\overline{u}_t\cdot r_t\\
&\leq C\left(\|r\|_{H^1}\|r_t\|_{H^1}+\|q_t\|^2+\|r_t\|^2+\|r_t\|_3\|\nabla r\|\|r_t\|_6+\|r\|^2_{H^1}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|q_t\|^2+\|\nabla q_t\|^2+\int_{\partial\mathcal{O}}m|q_t|^2 &= -\int_{\partial\mathcal{O}}m_tqq_t-((r_t+\overline{u})\cdot\nabla q,q_t)\\
&\quad -(r_t\cdot\nabla\overline{\theta},q_t)-(r\cdot\nabla\overline{\theta}_t,q_t)\\
&\leq C\,(\|q\|_{H^1}\|q_t\|_{H^1}+\|q\|^2_{H^1}+\|q_t\|^2\\
&\quad +\|r_t\|^2+\|r_t\|_3\|\nabla q\|\|r_t\|_6)\,.\end{aligned}$$

Consequently, using Lemmas C.1 – C.3 and adding the two above inequalities, we have

$$\begin{aligned}&\frac{d}{dt}\left(\|r_t\|^2+\|q_t\|^2\right)+\|r_t\|^2_{H^1}+\|q_t\|^2_{H^1}\\
&\qquad\leq C\left(\left(\|r\|^4_{H^1}+\|q\|^4_{H^1}+1\right)\|r_t\|^2+\|q_t\|^2+\|r\|^2_{H^1}+\|q\|^2_{H^1}\right).\end{aligned}$$

Now, introducing $Z(t) := \|r_t(t,\cdot)\|^2+\|q_t(t,\cdot)\|^2$, we find from (C.5) that

$$Z' \leq C[(1+Y_*^2)Z+Y_*]$$

for $t_2 \leq t \leq t_1+\tau_1$. By applying Gronwall's Lemma, we have for a.e. $t \in [t_2, t_1+\tau_1]$

$$Z(t) \leq e^{C(1+Y_*^2)(t-t_2)}\left(Z(t_2)+CY_*(t-t_2)\right).$$

Since we have $Z(t_2) \leq \Psi_1(Y_*)$ for some nonnegative regular Ψ_1 with $\Psi_1(0)=0$, we find that $Z(t) \leq \Psi_2(Y_*)$, with

$$\Psi_2(s) := e^{C(1+s^2)}(\Psi_1(s)+Cs) \quad \forall s \geq 0.$$

Therefore,

$$\|r_t(t,\cdot)\|^2+\|q_t(t\cdot)\|^2+\int_{t_2}^t\left(\|r_t(s,\cdot)\|^2_{H^1}+\|q_t(s,\cdot)\|^2_{H^1}\right)ds \leq \Psi_3(Y_*) \quad \forall t\in[t_2,t_1+\tau_1], \quad \text{(C.6)}$$

where $\Psi_3(s) := C[(1+s^2)\Psi_2(s)+s]$. In particular, this yields the existence of $t_3 \in (t_2, t_1+\tau_1)$ such that

$$\|r_t(t_3,\,\cdot\,)\|_{H^1}^2 + \|q_t(t_3,\,\cdot\,)\|_{H^1}^2 \leq \frac{\Psi_3(Y_*)}{(t_1-t_2+\tau_1)}. \tag{C.7}$$

Actually, it is not difficult to check that the set of times $t_3 \in (t_2, t_1+\tau_1)$ satisfying (C.7) has a positive measure.

Step 4: Conclusion. Using (C.5) and (C.6), we deduce an estimate of r in $L^\infty(H^2)$. It suffices to view $(3.8)_1$ as a family of Stokes problems (see Lemma C.4 and the arguments presented in (TEMAM, 1977, Theorem 3.8)). Then, looking $(3.8)_2$ as a family of elliptic problems, we also find $L^\infty(H^2)$ estimates for q, see Lemma C.6. Both estimates depend on Y_* continuously. Therefore, repeating the procedure, we see that $(r(t_3), q(t_3)) \in H^3 \times H^3$ with an estimate of the form $\Psi(Y_*)$.

APPENDIX D – CARLEMAN INEQUALITY FOR HEAT EQUATION WITH ROBIN BOUNDARY CONDITION

Let us consider the following heat equation

$$\begin{cases} -\psi_t - \Delta\psi = f & \text{in } \mathcal{O}_T := (0,T)\times\mathcal{O}, \\ \dfrac{\partial\psi}{\partial\nu} + B\psi = 0 & \text{on } \Lambda_T := (0,T)\times\partial\mathcal{O}, \\ \psi(T,x) = \psi_T(x) & \text{in } \mathcal{O}, \end{cases} \tag{D.1}$$

with $B \in Q$ (see (3.51) for the definition of Q).

Before stating the required Carleman inequality, let us introduce several classical weights in the study of Carleman inequalities for parabolic equations. The basic weight will be a function $\eta^0 \in C^2(\overline{\mathcal{O}})$ defined in (3.53).

Thus, for any $\lambda > 0$ we set:

$$\alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t^4(T-t)^4}, \quad \xi(x,t) = \frac{e^{\lambda\eta^0(x)}}{t^4(T-t)^4},$$
$$\tilde{\alpha}(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{-\lambda\eta^0(x)}}{t^4(T-t)^4}, \quad \tilde{\xi}(x,t) = \frac{e^{-\lambda\eta^0(x)}}{t^4(T-t)^4},$$

We also introduce the following notation:

$$\begin{aligned} I(s,\lambda;\psi) \;=\; & s^3\lambda^4 \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi^3|\psi|^2 + s\lambda^2 \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi|\nabla\psi|^2 \\ & + s^{-1} \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2), \end{aligned}$$

where s and λ are positive real numbers and $\psi = \psi(t,x)$.

The following Carleman inequality holds:

Lemma D.1. *Let $f \in L^2(\mathcal{O})$ be given. There exist $\lambda_0 = \lambda_0(\mathcal{O},\omega)$, $s_0 = s_0(\mathcal{O},\omega')$ and $C = C(\mathcal{O},\omega)$ such that, for every $\lambda \geq \lambda_0 e^{CT\|B\|_Q^2}(1+\|B\|_Q^5)$, any $s \geq s_0 e^{4\lambda\|\eta^0\|_\infty}(T^4+T^8)$ and any $\psi_T \in L^2(\mathcal{O})$, the weak solution of* (D.1) *satisfies*

$$I(s,\lambda;\psi) \leq C\left(\lambda \iint_{\mathcal{O}_T} e^{-2s\alpha}|f|^2 + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2\right).$$

Proof. We begin by noticing that the following inequality is holds:

$$\begin{aligned} & s^3\lambda^4 \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi^3|\psi|^2 + s\lambda^2 \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi|\nabla\psi|^2 + s^2\lambda^3 \iint_{\Lambda_T} e^{-2s\alpha}\xi^2|\psi|^2 \\ & \leq C\left(\iint_{\mathcal{O}_T} e^{-2s\alpha}|f|^2 + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2\right) \end{aligned} \tag{D.2}$$

In fact, this inequality follows from the Carleman inequality given in (FERNÁNDEZ-CARA et al., 2006b, Theorem 1).

For an easier comprehension the proof will be divided into several steps.

Step. 1. For $s > 0$ and $\lambda > 0$, we make the change of variable

$$w(t,x) = e^{-s\alpha}\psi(t,x),$$

which implies

$$w(T,x) = w(0,x) = 0.$$

Notice that the boundary condition becomes

$$\frac{\partial w}{\partial \nu} + Bw = s\lambda\xi w\frac{\partial \eta^0}{\partial \nu}.$$

We have

$$L_1w + L_2w = f_s, \tag{D.3}$$

where

$$L_1w = -w_t + 2s\lambda\xi\nabla\eta^0\cdot\nabla w + 2s\lambda^2\xi|\nabla\eta^0|^2 w,$$

$$L_2w = -\Delta w - s^2\lambda^2\xi^2|\nabla\eta^0|^2 w - s\alpha_t w$$

and

$$f_s = e^{-s\alpha}f + s\lambda^2\xi|\nabla\eta^0|^2 w - s\lambda\xi\Delta\eta^0 w.$$

From (D.3), we have that

$$||L_1w||^2 + ||L_2w||^2 + 2(L_1w, L_2w) = ||f_s||^2. \tag{D.4}$$

In this step we analyze the terms appearing in (L_1w, L_2w). First, we write

$$(L_1w, L_2w) = \sum_{i,j=1}^{3} I_{ij}.$$

We have

$$I_{11} = \iint_{\mathcal{O}_T} w_t\Delta w \; = \iint_{\Lambda_T}\frac{\partial w}{\partial \nu}w_t.$$

Then,

$$\begin{aligned} I_{12} &= \frac{1}{2}\iint_{\mathcal{O}_T}[s^2\lambda^2\xi^2|\nabla\eta^0|^2\frac{d}{dt}|w|^2] = -\iint_{\mathcal{O}_T} s^2\lambda^2\xi\xi_t|\nabla\eta^0|^2|w|^2 \\ &\leq Cs^2\lambda^2T^4\iint_{\mathcal{O}_T}\xi^3|w|^2 \end{aligned}$$

and

$$I_{13} = \iint_{\mathcal{O}_T} s\alpha_t w w_t \leq Ce^{\lambda||\eta^0||_\infty}sT^8\iint_{\mathcal{O}_T}\xi^3|w|^2,$$

since

$$|\alpha_{tt}| \leq Ce^{\lambda||\eta^0||_\infty}\xi^2(1+T^8\xi) \leq Ce^{\lambda||\eta^0||_\infty}\xi^3.$$

Next,

$$
\begin{aligned}
I_{21} =& -2\iint_{\mathcal{O}_T} s\lambda\xi\nabla\eta^0\nabla w\Delta w\\
=& -2s\lambda\iint_{\Lambda_T}\xi\frac{\partial\eta^0}{\partial\nu}\Big|\frac{\partial w}{\partial\nu}\Big|^2 + 2s\lambda\iint_{\mathcal{O}_T}\frac{\partial^2\eta^0}{\partial x_i x_j}\xi\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\\
&+ 2s\lambda^2\iint_{\mathcal{O}_T}\xi|\nabla\eta^0\cdot\nabla w|^2 + s\lambda\iint_{\mathcal{O}_T}\xi\nabla\eta^0\cdot\nabla|\nabla w|^2\\
:=& A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

We keep the boundary term A_1, the term $A_3 \geq 0$ and estimate the other two terms:

$$
|A_2| \leq Cs\lambda\iint_{\mathcal{O}_T}\xi|\nabla w|^2 .
$$

For A_4, we first observe that

$$
\begin{aligned}
A_4 =& s\lambda\iint_{\Lambda_T}\xi\frac{\partial\eta^0}{\partial\nu}|\nabla w|^2 - s\lambda^2\iint_{\mathcal{O}_T}\xi|\nabla\eta^0|^2|\nabla w|^2\\
&- s\lambda\iint_{\mathcal{O}_T}\xi\Delta\eta^0|\nabla w|^2\\
:=& A_{41} + A_{42} + A_{43}.
\end{aligned}
$$

We keep A_{41} and A_{42} and estimate the last term as

$$
|A_{43}| \leq Cs\lambda\iint_{\mathcal{O}_T}\xi|\nabla w|^2 .
$$

Consequently,

$$
\begin{aligned}
I_{21} \geq& -2s\lambda\iint_{\Lambda_T}\xi\frac{\partial\eta^0}{\partial\nu}\Big|\frac{\partial w}{\partial\nu}\Big|^2 + 2s\lambda^2\iint_{\mathcal{O}_T}\xi|\nabla\eta^0\cdot\nabla w|^2\\
&+ s\lambda\iint_{\Lambda_T}\xi\frac{\partial\eta^0}{\partial\nu}|\nabla w|^2 - s\lambda^2\iint_{\mathcal{O}_T}\xi|\nabla\eta^0|^2|\nabla w|^2\\
&- Cs\lambda\iint_{\mathcal{O}_T}\xi|\nabla w|^2 .
\end{aligned}
$$

Next,

$$
\begin{aligned}
I_{22} =& -2s^3\lambda^3\iint_{\mathcal{O}_T}\nabla\eta^0\cdot\nabla w\xi^3|\nabla\eta^0|^2 w\\
=& 3s^3\lambda^4\iint_{\mathcal{O}_T}\xi^3|\nabla\eta^0|^4|w|^2 + s^3\lambda^3\iint_{\mathcal{O}_T}\Delta\eta^0|\nabla\eta^0|^2\xi^3|w|^2\\
&+ 2s^3\lambda^3\iint_{\mathcal{O}_T}\frac{\partial\eta^0}{\partial x_i}\frac{\partial\eta^0}{\partial x_i x_j}\frac{\partial\eta^0}{\partial x_j}\xi^3|w|^2\\
&- s^3\lambda^3\iint_{\Lambda_T}\xi^3|\nabla\eta^0|^2\frac{\partial\eta^0}{\partial\nu}|w|^2\\
:=& B_1 + B_2 + B_3 + B_4.
\end{aligned}
$$

We keep B_1 and B_4. For the other two terms, we have

$$
|B_2| + |B_3| \leq Cs^3\lambda^3\iint_{\mathcal{O}_T}\xi^3|w|^2.
$$

Consequently,

$$I_{22} \geq 3s^3\lambda^4 \iint_{\mathcal{O}_T} \xi^3|\nabla\eta^0|^4|w|^2 - s^3\lambda^3 \iint_{\Lambda_T} \xi^3|\nabla\eta^0|^2\frac{\partial\eta^0}{\partial\nu}|w|^2 - Cs^3\lambda^3 \iint_{\mathcal{O}_T} \xi^3|w|^2.$$

We also have

$$\begin{aligned}
I_{23} =& -2s^2\lambda \iint_{\mathcal{O}_T} \nabla\eta^0 \cdot \nabla w \xi \alpha_t w \\
=& -s^2\lambda \iint_{\Lambda_T} \xi\alpha_t\frac{\partial\eta^0}{\partial\nu}|w|^2 + s^2\lambda^2 \iint_{\mathcal{O}_T} |\nabla\eta^0|^2\xi\alpha_t|w|^2 \\
&+ s^2\lambda \iint_{\mathcal{O}_T} \xi\nabla\eta^0 \cdot \nabla\alpha_t|w|^2 + s^2\lambda \iint_{\mathcal{O}_T} \xi\Delta\eta^0\alpha_t|w|^2 \\
:=& D_1 + D_2 + D_3 + D_4.
\end{aligned}$$

Notice that

$$|D_2| + |D_3| + |D_4| \leq Ce^{2\lambda||\eta^0||_\infty}s^2\lambda^2T^4 \iint_{\mathcal{O}_T} |\xi|^3|w|^2,$$

which gives

$$I_{23} \geq -s^2\lambda \iint_{\Lambda_T} \xi\alpha_t\frac{\partial\eta^0}{\partial\nu}|w|^2 - Ce^{2\lambda||\eta^0||_\infty}s^2\lambda^2T^4 \iint_{\mathcal{O}_T} |\xi|^3|w|^2.$$

Next,

$$\begin{aligned}
I_{31} =& -2s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla\eta^0|^2 w\Delta w \\
=& -2s\lambda^2 \iint_{\Lambda_T} \xi|\nabla\eta^0|^2\frac{\partial w}{\partial\nu}w + 2s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla\eta^0|^2|\nabla w|^2 \\
&+ 4s\lambda^2 \iint_{\mathcal{O}_T} \xi\frac{\partial\eta^0}{\partial x_i}\frac{\partial\eta^0}{\partial x_i x_j}\frac{\partial w}{\partial x_j}w \ + s\lambda^3 \iint_{\mathcal{O}_T} \xi|\nabla\eta^0|^2\nabla\eta^0 \cdot \nabla|w|^2 \\
=& E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

We keep E_1 and E_2. For E_3 and E_4, we have

$$E_3 \leq Cs\lambda^4 \iint_{\mathcal{O}_T} \xi|w|^2 + Cs \iint_{\mathcal{O}_T} \xi|\nabla w|^2$$

and

$$E_4 \leq Cs^2\lambda^4 \iint_{\mathcal{O}_T} \xi^2|w|^2 + C\lambda^2 \iint_{\mathcal{O}_T} |\nabla w|^2.$$

Hence,

$$\begin{aligned}
I_{31} \geq& -2s\lambda^2 \iint_{\Lambda_T} \xi|\nabla\eta^0|^2\frac{\partial w}{\partial\nu}w + 2s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla\eta^0|^2|\nabla w|^2 \\
&- Cs\lambda^4 \iint_{\mathcal{O}_T} \xi^2|w|^2 - Cs \iint_{\mathcal{O}_T} |\nabla w|^2 \\
&- Cs^2\lambda^4 \iint_{\mathcal{O}_T} \xi^2|w|^2 - C\lambda^2 \iint_{\mathcal{O}_T} |\nabla w|^2
\end{aligned}$$

and

$$I_{32} = -2s^3\lambda^4 \iint_{\mathcal{O}_T} |\nabla\eta^0|^4\xi^3|w|^2.$$

Finally,

$$I_{33} = -2s^2\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla\eta^0|^2\alpha_t|w|^2 \le Ce^{2\lambda||\eta^0||_\infty}s^2\lambda^2T^4 \iint_{\mathcal{O}_T} \xi^3|w|^2.$$

Therefore,

$$\begin{aligned}(L_1w, L_2w) \ge & \iint_{\Lambda_T} \frac{\partial w}{\partial\nu}w_t - s^2\lambda \iint_{\Lambda_T} \xi\alpha_t\frac{\partial\eta^0}{\partial\nu}|w|^2 \\ & - 2s\lambda \iint_{\Lambda_T} \xi\frac{\partial\eta^0}{\partial\nu}\Big|\frac{\partial w}{\partial\nu}\Big|^2 + s\lambda \iint_{\Lambda_T} \xi\frac{\partial\eta^0}{\partial\nu}|\nabla w|^2 \\ & - 2s\lambda^2 \iint_{\Lambda_T} \xi|\nabla\eta^0|^2\frac{\partial w}{\partial\nu}w - s^3\lambda^3 \iint_{\Lambda_T} \xi^3|\nabla\eta^0|^2\frac{\partial\eta^0}{\partial\nu}|w|^2 \\ & + s^3\lambda^4 \iint_{\mathcal{O}_T} \xi^3|\nabla\eta^0|^4|w|^2 + s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla\eta^0|^2|\nabla w|^2 \\ & - Cs^2\lambda^4 \iint_{\mathcal{O}_T} \xi^2|w|^2 - C\lambda^2 \iint_{\mathcal{O}_T} |\nabla w|^2 \\ & - Cs\lambda^4 \iint_{\mathcal{O}_T} \xi^2|w|^2 - Cs \iint_{\mathcal{O}_T} |\nabla w|^2 \\ & - Ce^{2\lambda||\eta^0||_\infty}s^2\lambda^2T^4 \iint_{\mathcal{O}_T} \xi^3|w|^2 - Cs^2\lambda^2T^4 \iint_{\mathcal{O}_T} \xi^3|w|^2 \\ & - Cs\lambda \iint_{\mathcal{O}_T} \xi|\nabla w|^2 - Ce^{\lambda||\eta^0||_\infty}sT^8 \iint_{\mathcal{O}_T} \xi^3|w|^2 \\ & - Ce^{2\lambda||\eta^0||_\infty}s^2\lambda^2T^4 \iint_{\mathcal{O}_T} \xi^3|w|^2 - Cs^3\lambda^3 \iint_{\mathcal{O}_T} \xi^3|w|^2 .\end{aligned} \tag{D.5}$$

As a consequence of the properties of η^0, we have that

$$\begin{aligned} s^3\lambda^4 & \iint_{\mathcal{O}_T} \xi^3|\nabla\eta^0|^4|w|^2 + s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla\eta^0|^2|\nabla w|^2 \\ & \ge Cs^3\lambda^4 \iint_{\mathcal{O}_T} \xi^3|w|^2 + Cs\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla w|^2 \\ & - Cs^3\lambda^4 \iint_{\omega'\times(0,T)} \xi^3|w|^2 - Cs\lambda^2 \iint_{\omega'\times(0,T)} \xi|\nabla w|^2. \end{aligned}$$

Taking $\lambda \ge \lambda_0(\mathcal{O},\omega)$ and $s \ge s_0(\mathcal{O},\omega)(e^{2\lambda||\eta^0||_\infty}T^4 + T^8)$, we can absorb the lower order terms in (D.5) and in f_s and obtain, from (D.4), that

$$\begin{aligned} & ||L_1w||^2 + ||L_2w||^2 + s^3\lambda^4 \iint_{\mathcal{O}_T} \xi^3|w|^2 + s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla w|^2 \\ & + \iint_{\Lambda_T} \frac{\partial w}{\partial\nu}w_t - s^2\lambda \iint_{\Lambda_T} \xi\alpha_t\frac{\partial\eta^0}{\partial\nu}|w|^2 \\ & - 2s\lambda \iint_{\Lambda_T} \xi\frac{\partial\eta^0}{\partial\nu}\Big|\frac{\partial w}{\partial\nu}\Big|^2 + s\lambda \iint_{\Lambda_T} \xi\frac{\partial\eta^0}{\partial\nu}|\nabla w|^2 \\ & - 2s\lambda^2 \iint_{\Lambda_T} \xi|\nabla\eta^0|^2\frac{\partial w}{\partial\nu}w - s^3\lambda^3 \iint_{\Lambda_T} \xi^3|\nabla\eta^0|^2\frac{\partial\eta^0}{\partial\nu}|w|^2 \\ & \le C\Big(||e^{-s\alpha}f||^2 + s^3\lambda^4 \iint_{\omega'\times(0,T)} \xi^3|w|^2 + s\lambda^2 \iint_{\omega'\times(0,T)} \xi|\nabla w|^2\Big). \end{aligned} \tag{D.6}$$

Let us now add integrals of Δw and w_t in the left-hand side of (D.6). This can be done since

$$s^{-1} \iint_{\mathcal{O}_T} \xi^{-1}|w_t|^2 \le C\left(s\lambda^4 \iint_{\mathcal{O}_T} \xi|w|^2 + s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla w|^2 + ||L_1w||^2\right)$$

and

$$s^{-1} \iint_{\mathcal{O}_T} \xi^{-1}|\Delta w|^2 \le C \left(s^3\lambda^4 \iint_{\mathcal{O}_T} \xi|w|^2 + sT^8 e^{4\lambda||\eta^0||_\infty} \iint_{\mathcal{O}_T} \xi^3|w|^2 + ||L_2 w||^2 \right),$$

for $s \ge CT^8$. Accordingly, we deduce from (D.6) that

$$\begin{aligned}
& s^{-1} \iint_{\mathcal{O}_T} \xi^{-1}(|w_t|^2 + |\Delta w|^2) + s^3\lambda^4 \iint_{\mathcal{O}_T} \xi^3|w|^2 + s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla w|^2 \\
& + \iint_{\Lambda_T} \frac{\partial w}{\partial \nu} w_t - s^2\lambda \iint_{\Lambda_T} \xi\alpha_t \frac{\partial \eta^0}{\partial \nu}|w|^2 \\
& - 2s\lambda \iint_{\Lambda_T} \xi \frac{\partial \eta^0}{\partial \nu}\left|\frac{\partial w}{\partial \nu}\right|^2 + s\lambda \iint_{\Lambda_T} \xi \frac{\partial \eta^0}{\partial \nu}|\nabla w|^2 \\
& - 2s\lambda^2 \iint_{\Lambda_T} \xi|\nabla \eta^0|^2 \frac{\partial w}{\partial \nu} w - s^3\lambda^3 \iint_{\Lambda_T} \xi^3|\nabla \eta^0|^2 \frac{\partial \eta^0}{\partial \nu}|w|^2 \\
& \le C \left(||e^{-s\alpha} f||^2 + s^3\lambda^4 \iint_{\omega' \times (0,T)} \xi^3|w|^2 + s\lambda^2 \iint_{\omega' \times (0,T)} \xi|\nabla w|^2 \right).
\end{aligned} \tag{D.7}$$

Now, we estimate the local integral of ∇w in (D.7). To this end, let us introduce a function $\theta = \theta(x)$, with

$$\theta \in C_0^\infty(\omega), \quad \theta \equiv 1 \text{ in } \omega', \quad 0 \le \theta \le 1$$

and we make some computations:

$$\begin{aligned}
s\lambda^2 \iint_{\omega' \times (0,T)} \xi|\nabla w|^2 & \le s\lambda^2 \iint_{\omega \times (0,T)} \theta\xi|\nabla w|^2 \\
& = - s\lambda^2 \iint_{\omega \times (0,T)} \theta\xi w\Delta w - s\lambda^2 \iint_{\omega \times (0,T)} \xi(\nabla w \cdot \nabla\theta)w \\
& \quad - s\lambda^3 \iint_{\omega \times (0,T)} \theta\xi(\nabla w \cdot \nabla \eta^0)w \\
& \le \delta s^{-1} \iint_{\omega \times (0,T)} \xi^{-1}|\Delta w|^2 + s^3\lambda^4 \iint_{\omega \times (0,T)} \xi^3|w|^2 .
\end{aligned}$$

In view of this last estimate, we deduce that the integral on ∇w in the right-hand side of (D.7) can be supressed if we enlarge slightly the control domain. We have the following estimate:

$$\begin{aligned}
& s^{-1} \iint_{\mathcal{O}_T} \xi^{-1}(|w_t|^2 + |\Delta w|^2) + s^3\lambda^4 \iint_{\mathcal{O}_T} \xi^3|w|^2 + s\lambda^2 \iint_{\mathcal{O}_T} \xi|\nabla w|^2 \\
& + \iint_{\Lambda_T} \frac{\partial w}{\partial \nu} w_t - s^2\lambda \iint_{\Lambda_T} \xi\alpha_t \frac{\partial \eta^0}{\partial \nu}|w|^2 \\
& - 2s\lambda \iint_{\Lambda_T} \xi \frac{\partial \eta^0}{\partial \nu}\left|\frac{\partial w}{\partial \nu}\right|^2 + s\lambda \iint_{\Lambda_T} \xi \frac{\partial \eta^0}{\partial \nu}|\nabla w|^2 \\
& - 2s\lambda^2 \iint_{\Lambda_T} \xi|\nabla \eta^0|^2 \frac{\partial w}{\partial \nu} w - s^3\lambda^3 \iint_{\Lambda_T} \xi^3|\nabla \eta^0|^2 \frac{\partial \eta^0}{\partial \nu}|w|^2 \\
& \le C \left(||e^{-s\alpha} f||^2 + s^3\lambda^4 \iint_{\omega \times (0,T)} \xi^3|w|^2 \right).
\end{aligned} \tag{D.8}$$

Step. 2. We perform the same computations as in *step 1*, but now with the weights $\tilde{\alpha}$ and $\tilde{\xi}$ instead of α and ξ.

For $s>0$ and $\lambda>0$, we consider the change of variable

$$\tilde{w}(t,w)=e^{-s\tilde{\alpha}}\psi(t,w),$$

which implies

$$\tilde{w}(T,x)=\tilde{w}(0,x)=0.$$

By the boundary condition, we have

$$\frac{\partial\tilde{w}}{\partial\nu}+B\tilde{w}=-s\lambda\tilde{\xi}\tilde{w}\frac{\partial\eta^0}{\partial\nu}.$$

We have

$$\tilde{L}_1w+\tilde{L}_2w=\tilde{f}_s, \tag{D.9}$$

where

$$\tilde{L}_1\tilde{w}=-\tilde{w}_t-2s\lambda\tilde{\xi}\nabla\eta^0\cdot\nabla\tilde{w}+2s\lambda^2\tilde{\xi}|\nabla\eta^0|^2\tilde{w},$$
$$\tilde{L}_2\tilde{w}=-\Delta\tilde{w}-s^2\lambda^2\tilde{\xi}^2|\nabla\eta^0|^2\tilde{w}-s\tilde{\alpha}_t\tilde{w}$$

and

$$\tilde{f}_s=e^{-s\tilde{\alpha}}f+s\lambda^2\tilde{\xi}|\nabla\eta^0|^2\tilde{w}+s\lambda\tilde{\xi}\Delta\eta^0\tilde{w}.$$

From (D.9), we have that

$$||\tilde{L}_1\tilde{w}||^2+||\tilde{L}_2\tilde{w}||^2+2(\tilde{L}_1\tilde{w},\tilde{L}_2\tilde{w})=||\tilde{f}_s||^2.$$

In this step we analyze the terms appearing in (L_1w,L_2w). First, we write

$$(\tilde{L}_1\tilde{w},\tilde{L}_2\tilde{w})=\sum_{i,j=1}^{3}\tilde{I}_{ij}.$$

As before, the following inequality can be proved

$$\begin{aligned}
&s^{-1}\iint_{\mathcal{O}_T}\tilde{\xi}^{-1}(|\tilde{w}_t|^2+|\Delta\tilde{w}|^2)+s^3\lambda^4\iint_{\mathcal{O}_T}\tilde{\xi}^3|\tilde{w}|^2+s\lambda^2\iint_{\mathcal{O}_T}\tilde{\xi}|\nabla\tilde{w}|^2\\
&+\iint_{\Lambda_T}\frac{\partial\tilde{w}}{\partial\nu}\tilde{w}_t+s^2\lambda\iint_{\Lambda_T}\tilde{\xi}\tilde{\alpha}_t\frac{\partial\eta^0}{\partial\nu}|\tilde{w}|^2\\
&+2s\lambda\iint_{\Lambda_T}\tilde{\xi}\frac{\partial\eta^0}{\partial\nu}\Big|\frac{\partial\tilde{w}}{\partial\nu}\Big|^2-s\lambda\iint_{\Lambda_T}\tilde{\xi}\frac{\partial\eta^0}{\partial\nu}|\nabla\tilde{w}|^2\\
&-2s\lambda^2\iint_{\Lambda_T}\tilde{\xi}|\nabla\eta^0|^2\frac{\partial\tilde{w}}{\partial\nu}\tilde{w}+s^3\lambda^3\iint_{\Lambda_T}\tilde{\xi}^3|\nabla\eta^0|^2\frac{\partial\eta^0}{\partial\nu}|\tilde{w}|^2\\
&\leq C\left(||e^{-s\tilde{\alpha}}f||^2+s^3\lambda^4\iint_{\omega\times(0,T)}\tilde{\xi}^3|\tilde{w}|^2\right),
\end{aligned}\tag{D.10}$$

for $\lambda\geq\lambda_0(\mathcal{O},\omega)$ and $s\geq s_0(\mathcal{O},\omega)(e^{2\lambda||\eta^0||_\infty}T^4+T^8)$.

Step. 3. Let us now add the inequalities (D.8) and (D.10) and let us check that the integrals on Λ_T can be simplified, so that there it will only remain integrals in $\mathcal{O}_T$.

First, observe that, since $\eta^0 = 0$ on $\partial\mathcal{O}$, we have

$$\xi = \tilde{\xi},\ \alpha = \tilde{\alpha} \quad \text{and} \quad w = \tilde{w} \text{ on } \Lambda_T.$$

Let us set

$$\alpha^*(t) := \alpha|_{\Lambda_T}, \quad \xi^*(t) := \xi|_{\Lambda_T} \quad \text{and} \quad w^* := e^{-s\alpha^*}\psi.$$

However, we know that

$$\frac{\partial \tilde{w}}{\partial \nu} + B\tilde{w} = -s\lambda\tilde{\xi}\tilde{w}\frac{\partial \eta^0}{\partial \nu} \qquad \text{and} \qquad \frac{\partial w}{\partial \nu} + Bw = s\lambda\xi w\frac{\partial \eta^0}{\partial \nu},$$

and therefore

$$\frac{\partial w}{\partial \nu} + \frac{\partial \tilde{w}}{\partial \nu} = -2Bw \qquad \text{and} \qquad \frac{\partial w}{\partial \nu} - \frac{\partial \tilde{w}}{\partial \nu} = 2s\lambda\xi w\frac{\partial \eta^0}{\partial \nu}.$$

We now add (D.8) and (D.10) and analyze each one of the boundary terms.

For the first one, we have

$$S_1 := \iint_{\Lambda_T} \frac{\partial w}{\partial \nu} w_t + \iint_{\Lambda_T} \frac{\partial \tilde{w}}{\partial \nu}\tilde{w}_t = -2\iint_{\Lambda_T} Bw\, w_t = -2\iint_{\Lambda_T} Bw^*\, w_t^*.$$

For the moment, assume that $w^* \in H^{(1+\ell)/2}(H^{1/2-\ell}(\partial\mathcal{O}))$, then by product of functions and since $B \in Q \subset H^{1-\ell}(W^{\vartheta_1,\vartheta_1+1}(\partial\mathcal{O})) \subset H^{(1-\ell)/2}(H^{-3/2+n/2+2\ell}(\partial\mathcal{O}))$ we have that $Bw^* \in H^{(1-\ell)/2}(H^{-1/2+\ell}(\partial\mathcal{O}))$. Note that $w_t^* \in H^{(-1+\ell)/2}(H^{1/2-\ell}(\partial\mathcal{O}))$, thus by duality we obtain

$$\begin{aligned} -2\iint_{\Lambda_T} Bw^*\, w_t^* &\leq C\|Bw^*\|_{H^{(1-\ell)/2}(H^{-1/2+\ell}(\partial\mathcal{O}))}\|w_t^*\|_{H^{(-1+\ell)/2}(H^{1/2-\ell}(\partial\mathcal{O}))} \\ &\leq C\|B\|_Q\|w^*\|^2_{H^{(1+\ell)/2}(H^{1/2-\ell}(\partial\mathcal{O}))}. \end{aligned}$$

In order to estimate the last expression, note that w^* solves

$$\begin{cases} -w_t^* - \Delta w^* = f^* & \text{in } \mathcal{O}_T, \\ \dfrac{\partial w^*}{\partial \nu} + Bw^* = 0 & \text{on } \Lambda_T, \\ w^*(T,\cdot) = 0 & \text{in } \mathcal{O}, \end{cases}$$

where

$$f^* := e^{-s\alpha^*}f - (e^{-s\alpha^*})_t\psi.$$

One can perform a similar estimate like of the (GUERRERO, 2006b, Proposition 1.1), that is

$$\|w^*\|^2_{H^1(L^2(\mathcal{O}))} + \|w^*\|^2_{L^2(H^2(\mathcal{O}))} \leq Ce^{CT\|B\|^2_Q}(1 + \|B\|^4_Q)\|f^*\|^2_{L^2(\mathcal{O}_T)}.$$

Besides, by interpolation inequality $w^* \in H^{(1+\ell)/2}(H^{1/2-\ell}(\partial\mathcal{O}))$ and can be estimated as follow

$$\|w^*\|^2_{H^{(1+\ell)/2}(H^{1/2-\ell}(\partial\mathcal{O}))} \leq C\|w^*\|^{1+\ell}_{H^1(L^2(\mathcal{O}))}\|w^*\|^{1-\ell}_{L^2(H^2(\mathcal{O}))} \leq C(\|w^*\|^2_{H^1(L^2(\mathcal{O}))} + \|w^*\|^2_{L^2(H^2(\mathcal{O}))}).$$

From the properties of α^*, we have

$$|\alpha_t^*| \leq CTe^{2s\|\eta^0\|_\infty}(\xi^*)^{5/4}.$$

Combining the above estimates, we find

$$\begin{aligned} S_1 &\leq C\|B\|_Q e^{CT\|B\|_Q^2}(1+\|B\|_Q^4)\|f^*\|^2 \\ &\leq Ce^{CT\|B\|_Q^2}(1+\|B\|_Q^5)\left(\iint_{\mathcal{O}_T} e^{-2s\alpha^*}|f|^2 + s^2T^2e^{4\lambda\|\eta^0\|_\infty}\iint_{\mathcal{O}_T}(\xi^*)^{5/2}|w^*|^2\right) \\ &\leq \lambda\iint_{\mathcal{O}_T} e^{-2s\alpha}|f|^2 + \lambda s^3\iint_{\mathcal{O}_T}\xi^3|w|^2, \end{aligned}$$

taking $\lambda \geq Ce^{CT\|B\|_Q^2}(1+\|B\|_Q^5)$ and $s \geq Ce^{4\lambda\|\eta^0\|_\infty}T^6$.

Also

$$s^2\lambda\iint_{\Lambda_T}\tilde{\xi}\tilde{\alpha}_t\frac{\partial\eta^0}{\partial\nu}|\tilde{w}|^2 - s^2\lambda\iint_{\Lambda_T}\xi\alpha_t\frac{\partial\eta^0}{\partial\nu}|w|^2 = 0.$$

Next,

$$\begin{aligned} &2s\lambda\iint_{\Lambda_T}\tilde{\xi}\frac{\partial\eta^0}{\partial\nu}\left|\frac{\partial\tilde{w}}{\partial\nu}\right|^2 - 2s\lambda\iint_{\Lambda_T}\xi\frac{\partial\eta^0}{\partial\nu}\left|\frac{\partial w}{\partial\nu}\right|^2 \\ &= 2s\lambda\iint_{\Lambda_T}\xi\frac{\partial\eta^0}{\partial\nu}\left(\left|\frac{\partial\tilde{w}}{\partial\nu}\right|^2 - \left|\frac{\partial w}{\partial\nu}\right|^2\right) \\ &= 8s^2\lambda^2\iint_{\Lambda_T}\xi^2\left|\frac{\partial\eta^0}{\partial\nu}\right|^2B|w|^2. \end{aligned}$$

For the other terms, we have

$$\begin{aligned} &s\lambda\iint_{\Lambda_T}\xi\frac{\partial\eta^0}{\partial\nu}|\nabla w|^2 - s\lambda\iint_{\Lambda_T}\tilde{\xi}\frac{\partial\eta^0}{\partial\nu}|\nabla\tilde{w}|^2 \\ &= s\lambda\iint_{\Lambda_T}\xi\frac{\partial\eta^0}{\partial\nu}\left(\left|\frac{\partial w}{\partial\nu}\right|^2 - \left|\frac{\partial\tilde{w}}{\partial\nu}\right|^2\right) \\ &= -4s^2\lambda^2\iint_{\Lambda_T}\xi^2\left|\frac{\partial\eta^0}{\partial\nu}\right|^2B|w|^2, \end{aligned}$$

$$\begin{aligned} &-2s\lambda^2\iint_{\Lambda_T}\xi|\nabla\eta^0|^2\frac{\partial w}{\partial\nu}w - 2s\lambda^2\iint_{\Lambda_T}\tilde{\xi}|\nabla\eta^0|^2\frac{\partial\tilde{w}}{\partial\nu}\tilde{w} \\ &= -2s\lambda^2\iint_{\Lambda_T}\xi|\nabla\eta^0|^2\left(\frac{\partial w}{\partial\nu} + \frac{\partial\tilde{w}}{\partial\nu}\right)w \\ &= 4s\lambda^2\iint_{\Lambda_T}\xi|\nabla\eta^0|^2B|w|^2 \end{aligned}$$

and also

$$s^3\lambda^3\iint_{\Lambda_T}\tilde{\xi}^3|\nabla\eta^0|^2\frac{\partial\eta^0}{\partial\nu}|\tilde{w}|^2 - s^3\lambda^3\iint_{\Lambda_T}\xi^3|\nabla\eta^0|^2\frac{\partial\eta^0}{\partial\nu}|w|^2 = 0.$$

Thus, notice that all the boundary terms can be estimated by

$$|\text{Boundary terms}| \leq Cs^2\lambda^2\|B\|_{L^\infty}\iint_{\Lambda_T}\xi^2|w|^2 + \lambda\iint_{\mathcal{O}_T}e^{-2s\alpha}|f|^2 + \lambda s^3\iint_{\mathcal{O}_T}\xi^3|w|^2,$$

for $\lambda \geq Ce^{CT\|B\|_Q^2}(1+\|B\|_Q^5)$ and $s \geq Ce^{4\lambda\|\eta^0\|_\infty}T^6$.

Let us now finish the proof. To do this, using the previous estimates, in the sum of (D.8) and (D.10), we conclude that

$$
\begin{aligned}
&s^{-1}\iint_{\mathcal{O}_T}\tilde{\xi}^{-1}(|\tilde{w}_t|^2+|\Delta\tilde{w}|^2)+s^3\lambda^4\iint_{\mathcal{O}_T}\tilde{\xi}^3|\tilde{w}|^2+s\lambda^2\iint_{\mathcal{O}_T}\tilde{\xi}|\nabla\tilde{w}|^2\\
&s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}(|w_t|^2+|\Delta w|^2)+s^3\lambda^4\iint_{\mathcal{O}_T}\xi^3|w|^2+s\lambda^2\iint_{\mathcal{O}_T}\xi|\nabla w|^2\\
&\leq C\Big(||e^{-s\alpha}f||^2+||e^{-s\tilde{\alpha}}f||^2+s^3\lambda^4\iint_{\omega\times(0,T)}\xi^3|w|^2+\tilde{\xi}^3|\tilde{w}|^2\Big) \qquad \text{(D.11)}\\
&+Cs^2\lambda^2\|B\|_{L^\infty}\iint_{\Lambda_T}\xi^2|w|^2+\lambda\iint_{\mathcal{O}_T}e^{-2s\alpha}|f|^2+\lambda s^3\iint_{\mathcal{O}_T}\xi^3|w|^2,
\end{aligned}
$$

for $\lambda \geq \lambda_0(\mathcal{O},\omega)e^{CT\|B\|_Q^2}(1+\|B\|_Q^5)$ and $s \geq s_0(\mathcal{O},\omega)e^{4\lambda||\eta^0||_\infty}(T^4+T^8)$.

From the definitions os ξ, $\tilde{\xi}$, α and $\tilde{\alpha}$, we have

$$
\tilde{\xi}\leq\xi,\quad e^{-2s\tilde{\alpha}}\leq e^{-2s\alpha}\ \text{ in }\ \mathcal{O}_T.
$$

Hence, (D.11) yields

$$
\begin{aligned}
&s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}(|w_t|^2+|\Delta w|^2)+s\lambda^2\iint_{\mathcal{O}_T}\xi|\nabla w|^2+s^3\lambda^4\iint_{\mathcal{O}_T}\xi^3|w|^2\\
&\leq C\left(\lambda\iint_{\mathcal{O}_T}e^{-2s\alpha}|f|^2+s^3\lambda^4\iint_{\omega\times(0,T)}\xi^3|w|^2+s^2\lambda^2\|B\|_{L^\infty}\iint_{\Lambda_T}\xi^2|w|^2\right), \qquad \text{(D.12)}
\end{aligned}
$$

for $\lambda \geq \lambda_0(\mathcal{O},\omega)e^{CT\|B\|_Q^2}(1+\|B\|_Q^5)$ and $s \geq s_0(\mathcal{O},\omega)e^{4\lambda||\eta^0||_\infty}(T^4+T^8)$.

We finally turn back to ψ. For the moment, we have

$$
\begin{aligned}
&s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}(|w_t|^2+|\Delta w|^2)+s\lambda^2\iint_{\mathcal{O}_T}\xi|\nabla w|^2+s^3\lambda^4\iint_{\mathcal{O}_T}\xi^3e^{-2s\alpha}|\psi|^2 \qquad \text{(D.13)}\\
&\leq C\left(\lambda\iint_{\mathcal{O}_T}e^{-2s\alpha}|f|^2+s^3\lambda^4\iint_{\omega\times(0,T)}\xi^3e^{-2s\alpha}|\psi|^2+s^2\lambda^2\|B\|_{L^\infty}\iint_{\Lambda_T}\xi^2e^{-2s\alpha}|\psi|^2.\right)
\end{aligned}
$$

Thus, we find that

$$
s\lambda^2\iint_{\mathcal{O}_T}\xi e^{-2s\alpha}|\nabla\psi|^2\leq C\Big(s\lambda^2\iint_{\mathcal{O}_T}\xi|\nabla w|^2+s^3\lambda^4\iint_{\mathcal{O}_T}\xi^3e^{-2s\alpha}|\psi|^2\Big).
$$

Accordingly, the previous integral in $\nabla\psi$ can be added to the left-hand side of (D.13):

$$
\begin{aligned}
&s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}(|w_t|^2+|\Delta w|^2)+s\lambda^2\iint_{\mathcal{O}_T}\xi e^{-2s\alpha}|\nabla\psi|^2+s^3\lambda^4\iint_{\mathcal{O}_T}\xi^3e^{-2s\alpha}|\psi|^2\\
&\leq C\left(\lambda\iint_{\mathcal{O}_T}e^{-2s\alpha}|f|^2+s^3\lambda^4\iint_{\omega\times(0,T)}\xi^3e^{-2s\alpha}|\psi|^2+s^2\lambda^2\|B\|_{L^\infty}\iint_{\Lambda_T}\xi^2e^{-2s\alpha}|\psi|^2.\right)
\end{aligned}
$$

For $\Delta\psi$, we use the identity

$$
\Delta w=e^{-s\alpha}(\Delta\psi-s\lambda\Delta\eta^0\xi\psi+s\lambda^2|\nabla\eta^0|^2\xi\psi-2s\lambda\nabla\eta^0\cdot\nabla\psi+s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi)
$$

and we obtain

$$
\begin{aligned}
s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}e^{-2s\alpha}|\Delta\psi|^2\leq C\Big(&s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}|\Delta w|^2\\
&+s^3\lambda^4\iint_{\mathcal{O}_T}\xi^3e^{-2s\alpha}|\psi|^2+s\lambda^2\iint_{\mathcal{O}_T}\xi e^{-2s\alpha}|\nabla\psi|^2\Big).
\end{aligned}
$$

Finally, for ψ_t, we get

$$s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}e^{-2s\alpha}|\psi_t|^2 \leq s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}|w_t|^2 + se^{4\lambda\|\eta^0\|_\infty}T^8\iint_{\mathcal{O}_T}\xi^3e^{-2s\alpha}|\psi|^2,$$

since

$$\psi_t = e^{s\alpha}(w_t + s\alpha_t w).$$

Therefore, if $\lambda \geq \lambda_0(\mathcal{O},\omega)e^{CT\|B\|_Q^2}(1+\|B\|_Q^5)$ and $s \geq s_0(\mathcal{O},\omega)e^{4\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

$$s^{-1}\iint_{\mathcal{O}_T}\xi^{-1}e^{-2s\alpha}(|\psi_t|^2+|\Delta\psi|^2) + s\lambda^2\iint_{\mathcal{O}_T}\xi e^{-2s\alpha}|\nabla\psi|^2 + s^3\lambda^4\iint_{\mathcal{O}_T}\xi^3e^{-2s\alpha}|\psi|^2$$
$$\leq C\left(\lambda\iint_{\mathcal{O}_T}e^{-2s\alpha}|f|^2 + s^3\lambda^4\iint_{\omega\times(0,T)}\xi^3e^{-2s\alpha}|\psi|^2 + s^2\lambda^2\|B\|_{L^\infty}\iint_{\Lambda_T}\xi^2e^{-2s\alpha}|\psi|^2\right)$$

and the proof of Lemma D.1 follows directly from (D.2).

□

APPENDIX E – GLOBAL CARLEMAN ESTIMATE FOR BOUSSINESQ SYSTEM WITH NAVIER-SLIP AND ROBIN BOUNDARY CONDITIONS

This appendix is dedicated to the proof of Proposition 3.16.

The proof is divided into eight steps and is inspired by the ideas of (GUERRERO, 2006b). In the following, the positive constants C vary from line to line and depend only on $\mathcal{O}$ and ω.

Let the non-empty open sets ω' and ω_c be given, with $\omega' \subset\subset \omega_0 \subset\subset \omega_c$.

Step 1: Global Carleman estimates for ϕ and ψ and absorption of global terms.

We apply the Carleman estimate (GUERRERO, 2006b, Proposition 2.1) for the heat system $(3.52)_1$ with source term $G := c\nabla\psi - \nabla\pi + D\varphi b + (a\cdot\nabla)\varphi$, to get

$$I(s,\lambda;\varphi) \leq C\bigg(s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\varphi|^2 + \lambda(\|a\|_\infty^2 + \|b\|_\infty^2)\iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\varphi|^2 + \lambda\iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\pi|^2 + \lambda\|c\|_\infty^2 \iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\psi|^2\bigg), \tag{E.1}$$

for $\lambda \geq \widehat{\lambda}e^{\widehat{\lambda}T\|A\|_P^2}(1+\|A\|_P^5)$ and $s \geq \widehat{s}e^{4\lambda\|\eta^0\|_\infty}(T^6+T^8)$ and $\widehat{\lambda}$, $\widehat{s}$ only depend on $\mathcal{O}$ and ω. Thanks to the definition of ξ, we have $1 \leq CT^8\xi \leq Cs\xi$ and we can eliminate the second term in the right-hand side of (E.1) with the term in $s\lambda^2$ that appears in the expression of $I(s,\lambda;\varphi)$. Indeed, if we take $\lambda \geq \widehat{\lambda}(\|a\|_\infty^2 + \|b\|_\infty^2)$, we get

$$I(s,\lambda;\varphi) \leq C\bigg(s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\varphi|^2 + \lambda\iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\pi|^2 + \lambda\|c\|_\infty^2 \iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\psi|^2\bigg), \tag{E.2}$$

for any $\lambda \geq \widehat{\lambda}e^{\widehat{\lambda}T\|A\|_P^2}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^5)$ and any $s \geq \widehat{s}e^{4\lambda\|\eta^0\|_\infty}(T^6+T^8)$.

Next, we apply the known Carleman estimates for the heat equation with homogeneous Robin boundary condition fulfilled by ψ (see Appendx D), which gives

$$I(s,\lambda;\psi) \leq C\bigg(s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\psi|^2 + (\|a\|_\infty^2 + \|b\|_\infty^2)\lambda\iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\psi|^2 + \lambda\iint_{\mathcal{O}_T} e^{-2s\alpha}|\varphi\cdot e_n|^2\bigg),$$

for $\lambda \geq \widehat{\lambda}e^{\widehat{\lambda}T(\|A\|_P^2+\|B\|_Q^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|B\|_Q^5+\|A\|_P^5)$ and $s \geq \widehat{s}e^{4\lambda\|\eta^0\|_\infty}(T^4+T^8)$. The same argument above yields

$$I(s,\lambda;\psi) \leq C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\psi|^2 + \lambda\iint_{\mathcal{O}_T} e^{-2s\alpha}|\varphi\cdot e_n|^2\right), \tag{E.3}$$

for any $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|A\|_P^2+\|B\|_Q^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|B\|_Q^5+\|A\|_P^5)$ and $s \geq \widehat{s} e^{4\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

From (E.2) and (E.3), we get

$$\begin{aligned} I(s,\lambda;\psi)+I(s,\lambda;\varphi) \leq C\Bigg(& s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\varphi|^2 + s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\psi|^2 \\ & +\lambda \iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\pi|^2 + \lambda\|c\|_\infty^2 \iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\psi|^2 \\ & +\lambda \iint_{\mathcal{O}_T} e^{-2s\alpha}|\varphi\cdot e_n|^2 \Bigg), \end{aligned} \tag{E.4}$$

for any $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|A\|_P^2+\|B\|_Q^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|B\|_Q^5+\|A\|_P^5)$ and any $s \geq \widehat{s} e^{4\lambda\|\eta^0\|_\infty}(T^4+T^8)$. Using the parameters $s^3\lambda^4$, $s\lambda^2$ appearing in $I(s,\lambda;\varphi)$ and $I(s,\lambda;\psi)$ we can absorb the lower order terms on the right-hand side of (E.4). This way, we have

$$\begin{aligned} I(s,\lambda;\psi)+I(s,\lambda;\varphi) \leq C\Bigg(& s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\varphi|^2 + s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\psi|^2 \\ & +\lambda \iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla\pi|^2 \Bigg), \end{aligned} \tag{E.5}$$

for every $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|A\|_P^2+\|B\|_Q^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|B\|_Q^5+\|A\|_P^5)$ and any $s \geq \widehat{s} e^{4\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Step 2: Localization of the pressure term by a global elliptic Carleman estimate.

We estimate the integral on the pressure term in (E.5). To do that, let us take the divergence operator in the equation verified by φ, thus

$$\Delta\pi(t,\cdot) = \nabla\cdot((a\cdot\nabla)\varphi + D\varphi b + c\nabla\psi) \quad \text{in} \quad \mathcal{O} \quad \text{a.e.} \quad t\in(0,T). \tag{E.6}$$

Now, since the right-hand side of (E.6) is a H^{-1} term, we can apply the elliptic Carleman inequality given in (IMANUVILOV; PUEL, 2002, Theorem 0.1). Hence, there exist two positive constants $\tilde{\tau} \geq 1$ and $\tilde{\lambda} \geq 1$, such that

$$\begin{aligned} \int_{\mathcal{O}} e^{2\tau\eta}|\nabla\pi(t,\cdot)|^2 + \tau^2\lambda^2 \int_{\mathcal{O}} e^{2\tau\eta}\eta^2|\pi(t,\cdot)|^2 & \\ \leq C\Bigg(& \tau^{\frac{1}{2}} e^{2\tau}\|\pi(t,\cdot)\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 + \tau \int_{\mathcal{O}} e^{2\tau\eta}\eta|(a\cdot\nabla)\varphi + D\varphi b + c\nabla\psi|^2 \\ & +\tau^2\lambda^2 \int_{\omega'} e^{2\tau\eta}\eta^2|\pi(t,\cdot)|^2 + \int_{\omega'} e^{2\tau\eta}|\nabla\pi(t,\cdot)|^2 \Bigg) \\ \leq C\Bigg(& \tau(\|a\|_\infty^2+\|b\|_\infty^2) \int_{\mathcal{O}} e^{2\tau\eta}\eta|\nabla\varphi(t,\cdot)|^2 + \tau^{\frac{1}{2}} e^{2\tau}\|\pi(t,\cdot)\|_{H^{\frac{1}{2}}(\partial\mathcal{O})}^2 \\ & +\tau\|c\|_\infty^2 \int_{\mathcal{O}} e^{2\tau\eta}\eta|\nabla\psi(t,\cdot)|^2 + \tau^2\lambda^2 \int_{\omega'} e^{2\tau\eta}\eta^2|\pi(t,\cdot)|^2 \\ & + \int_{\omega'} e^{2\tau\eta}|\nabla\pi(t,\cdot)|^2 \Bigg) \end{aligned}$$

for $\tau \geq \widehat{\tau}$ and $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|A\|_P^2+\|B\|_Q^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|B\|_Q^5+\|A\|_P^5)$. Here, for each $\lambda > 0$, the function η is given by $\eta(x) = e^{\lambda \eta^0(x)}$ where the function η^0 is defined in (3.53). Let us now set $\tau = s/(t^4(T-t)^4)$. We multiply the previous inequality by $\exp(-2se^{2\lambda\|\eta^0\|_\infty}/(t^4(T-t)^4))$ and integrate between $t=0$ and $t=T$. It is not difficult to see that

$$\begin{aligned}
&\iint_{\mathcal{O}_T} e^{-2s\alpha}|\nabla \pi|^2 + s^2\lambda^2 \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi^2|\pi|^2 \\
&\qquad \leq C\Bigg(s(\|a\|_\infty^2+\|b\|_\infty^2)\iint_{\mathcal{O}_T} e^{-2s\alpha}\xi|\nabla\varphi|^2 + s\|c\|_\infty^2 \iint_{\mathcal{O}_T} e^{-2s\alpha}\xi|\nabla\psi|^2 \\
&\qquad\quad + s^{\frac{1}{2}}\int_0^T e^{-2s\alpha^*}(\xi^*)^{\frac{1}{2}}\|\pi(t,\cdot)\|^2_{H^{\frac{1}{2}}(\partial\mathcal{O})} + s^2\lambda^2 \iint_{(0,T)\times\omega'} e^{-2s\alpha}\xi^2|\pi|^2 \\
&\qquad\quad + \iint_{(0,T)\times\omega'} e^{-2s\alpha}|\nabla\pi|^2 \Bigg)
\end{aligned} \tag{E.7}$$

for $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|A\|_P^2+\|B\|_Q^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|B\|_Q^5+\|A\|_P^5)$ and $s \geq \widehat{s} e^{4\lambda\|\eta^0\|_\infty}(T^4+T^8)$. Combining (E.7) with (E.5), we can absorb the first and second terms in the right hand side of (E.7) to get

$$\begin{aligned}
I(s,\lambda;\psi)+I(s,\lambda;\varphi) &\leq C\Bigg(s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\varphi|^2 + s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\psi|^2 \\
&\quad + s^{\frac{1}{2}}\lambda \int_0^T e^{-2s\alpha^*}(\xi^*)^{\frac{1}{2}}\|\pi(t,\cdot)\|^2_{H^{\frac{1}{2}}(\partial\mathcal{O})} + s^2\lambda^3 \iint_{(0,T)\times\omega'} e^{-2s\alpha}\xi^2|\pi|^2 \\
&\quad + \lambda \iint_{(0,T)\times\omega'} e^{-2s\alpha}|\nabla\pi|^2 \Bigg)
\end{aligned} \tag{E.8}$$

for $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|A\|_P^2+\|B\|_Q^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|B\|_Q^5+\|A\|_P^5)$ and $s \geq \widehat{s} e^{4\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Step 3: Estimate of the trace of the pressure.

We introduce the followings functions:

$$\beta(t) = s^{\frac{1}{4}} e^{-s\alpha^*}(\xi^*)^{\frac{1}{4}}, \quad \tilde{\varphi} = \beta\varphi \quad \text{and} \quad \tilde{\pi} = \beta\pi,$$

which satisfy

$$\begin{cases}
-\partial_t\tilde{\varphi} - \Delta\tilde{\varphi} - (a\cdot\nabla)\tilde{\varphi} - D\tilde{\varphi}b + \nabla\tilde{\pi} = \beta c\nabla\psi - \beta_t\varphi & \text{in } \mathcal{O}_T, \\
\nabla\cdot\tilde{\varphi} = 0 & \text{in } \mathcal{O}_T, \\
\tilde{\varphi}\cdot\nu = 0, \quad [D(\tilde{\varphi})\nu + A\tilde{\varphi}]_{tan} = 0 & \text{on } \mathit{\Lambda}_T, \\
\tilde{\varphi}(T,\cdot) = 0 & \text{in } \mathcal{O}.
\end{cases} \tag{E.9}$$

Let us regard $\tilde{\varphi}$ as a weak solution to (E.9). In particular, $\tilde{\varphi}$ satisfies, by well-known energy estimates for the Stokes equation (see the beginning of the proof of (GUERRERO, 2006b, Proposition 1.1)), the following:

$$\|\tilde{\varphi}\|^2_{L^2(H^1)} \leq e^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_\infty^2)}\|\beta c\nabla\psi - \beta_t\varphi\|^2.$$

Again from energy estimates, using the fact that $P \hookrightarrow L^\infty(\Lambda_T)^{n\times n}$, we have

$$\|\tilde{\pi}\|^2_{L^2(H^1)} \leq Ce^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P)}(1+\|A\|^4_P)(1+\|a\|^2_\infty+\|b\|^2_\infty) \times \left(s^{\frac{5}{2}}e^{4\lambda\|\eta^0\|_\infty}T^2\iint_{\mathcal{O}_T}e^{-2s\alpha^*}(\xi^*)^3|\varphi|^2+\|c\|^2_\infty s^{\frac{1}{2}}\iint_{\mathcal{O}_T}e^{-2s\alpha^*}(\xi^*)^{\frac{1}{2}}|\nabla\psi|^2\right),$$

where we have used that $\|\alpha^*_t\|+\|\xi^*_t\| \leq CTe^{2\lambda\|\eta^0\|_\infty}(\xi^*)^{5/4}$.

Taking $\lambda \geq \widehat{\lambda}e^{\widehat{\lambda}T(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P+\|B\|^2_Q)}(1+\|a\|^2_\infty+\|b\|^2_\infty)(1+\|A\|^5_P+\|B\|^5_Q)(1+\|c\|^2_\infty)$ and $s \geq \widehat{s}e^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$, from this last estimate and (E.7), we get:

$$\begin{aligned}\iint_{\mathcal{O}_T}e^{-2s\alpha}|\nabla\pi|^2+s^2\lambda^2\iint_{\mathcal{O}_T}e^{-2s\alpha}\xi^2|\pi|^2 &\leq C\Big(s(\|a\|^2_\infty+\|b\|^2_\infty)\iint_{\mathcal{O}_T}e^{-2s\alpha}\xi|\nabla\varphi|^2\\ &+s\|c\|^2_\infty\iint_{\mathcal{O}_T}e^{-2s\alpha}\xi|\nabla\psi|^2+s^2\lambda^2\iint_{(0,T)\times\omega'}e^{-2s\alpha}\xi^2|\pi|^2\\ &+\iint_{(0,T)\times\omega'}e^{-2s\alpha}|\nabla\pi|^2+s^3\lambda\iint_{\mathcal{O}_T}e^{-2s\alpha}\xi^3|\varphi|^2\Big).\end{aligned}$$

Combining this and (E.8) and absorbing the lower order terms, we also get the estimates

$$\begin{aligned}I(s,\lambda;\psi)+I(s,\lambda;\varphi) \leq C\Big(&s^3\lambda^4\iint_{(0,T)\times\omega_0}e^{-2s\alpha}\xi^3|\varphi|^2\\ &+s^3\lambda^4\iint_{(0,T)\times\omega_0}e^{-2s\alpha}\xi^3|\psi|^2\\ &+s^2\lambda^3\iint_{\omega'\times(0,T)}e^{-2s\alpha}\xi^2|\pi|^2+\lambda\iint_{(0,T)\times\omega'}e^{-2s\alpha}|\nabla\pi|^2\Big)\end{aligned} \tag{E.10}$$

for $\lambda \geq \widehat{\lambda}e^{\widehat{\lambda}T(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P+\|B\|^2_Q)}(1+\|a\|^2_\infty+\|b\|^2_\infty)(1+\|A\|^5_P+\|B\|^5_Q)(1+\|c\|^2_\infty)$ and $s \geq \widehat{s}e^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Step 4: Local estimates of the pressure.

We now follow the ideas of (FERNÁNDEZ-CARA et al., 2004) to estimate the local terms on the pressure. Indeed, we assume that the pressure π has mean-value zero in ω':

$$\int_{\omega'}\pi(t,\cdot\,)=0 \quad \text{a.e.} \quad t\in(0,T).$$

Then, using that $e^{-2s\alpha}\xi^2 \leq e^{-2s\hat{\alpha}}\hat{\xi}^2$ and the Poincaré-Wirtinger's inequality, we have

$$s^2\lambda^3\iint_{(0,T)\times\omega'}e^{-2s\alpha}\xi^2|\pi|^2 \leq Cs^2\lambda^3\iint_{(0,T)\times\omega'}e^{-2s\hat{\alpha}}\hat{\xi}^2|\nabla\pi|^2$$

and

$$\lambda\iint_{(0,T)\times\omega'}e^{-2s\alpha}|\nabla\pi|^2 \leq Cs^2\lambda^3\iint_{(0,T)\times\omega'}e^{-2s\hat{\alpha}}\hat{\xi}^2|\nabla\pi|^2.$$

Now, using that

$$\nabla\pi = \partial_t\varphi+\Delta\varphi+(a\cdot\nabla)\varphi+D\varphi b+c\nabla\psi,$$

the estimate (E.10) gives

$$\begin{aligned} I(s,\lambda;\psi)+I(s,\lambda;\varphi) \le C\Bigg(& s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\varphi|^2 + s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\psi|^2 \\ & + s^2\lambda^3 \iint_{(0,T)\times\omega'} e^{-2s\hat\alpha}\hat\xi^2|\varphi_t|^2 + s^2\lambda^3 \iint_{(0,T)\times\omega'} e^{-2s\hat\alpha}\hat\xi^2|\Delta\varphi|^2 \\ & + s^2\lambda^3(\|a\|_\infty^2+\|b\|_\infty^2) \iint_{(0,T)\times\omega'} e^{-2s\hat\alpha}\hat\xi^2|\nabla\varphi|^2 \\ & + s^2\lambda^3\|c\|_\infty^2 \iint_{(0,T)\times\omega'} e^{-2s\hat\alpha}\hat\xi^2|\nabla\psi|^2 \Bigg) \end{aligned} \tag{E.11}$$

for $\lambda \ge \widehat{\lambda} e^{\widehat{\lambda} T(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2+\|B\|_Q^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2)(1+\|A\|_P^5+\|B\|_Q^5)(1+\|c\|_\infty^2)$ and $s \ge \widehat{s} e^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Step 5: Local estimate of the term on $\Delta\varphi$.

Now, we present a local estimate of the integral on $\Delta\varphi$ in the right-hand side of (E.11); this follows the ideas included in (FERNÁNDEZ-CARA et al., 2004, Step 4 of the proof of Theorem 1).

Let us introduce an additional open set ω_1 such that $\omega' \subset\subset \omega_1 \subset\subset \omega_0 \subset\subset \omega_c$, $dist(\partial\omega', \partial\omega^1) \ge dist(\partial\omega^1, \partial\omega^0)$ and a positive function $\zeta \in \mathcal{D}(\omega_0)$ satisfying $\zeta = 1$ in ω_1. Let $\widehat{\eta}(t) := s\lambda^{\frac{3}{2}} e^{-s\hat\alpha(t)}\hat\xi(t)$ and

$$\tilde u(t,x) := \hat\eta(t)\zeta(x)\Delta\varphi(T-t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}^n,$$

where $\tilde u$ has been extended by zero outside ω_0.

Applying Laplace operator to $(3.52)_1$, we get

$$(\Delta\varphi(T-t,\cdot))_t - \Delta(\Delta\varphi(T-t,\cdot)) = \tilde f \quad \text{in} \quad Q, \tag{E.12}$$

where

$$\begin{aligned} \tilde f \;:=\; & \Delta((a\cdot\nabla)\varphi)(T-t,\cdot) + \Delta(D\varphi b)(T-t,\cdot) + \Delta(c\nabla\psi)(T-t,\cdot) \\ & -\nabla\left(\nabla\cdot((a\cdot\nabla)\varphi)(T-t,\cdot)\right) - \nabla(\nabla\cdot(D\varphi b))(T-t,\cdot) - \nabla(\nabla\cdot(c\nabla\psi))(T-t,\cdot). \end{aligned}$$

From (E.12), we deduce that $\tilde u$ solves

$$\begin{cases} \partial_t\tilde u - \Delta\tilde u = \tilde F & \text{in} \quad (0,T)\times\mathbb{R}^n, \\ \tilde u(0,\cdot) = 0 & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{E.13}$$

with

$$\tilde F = \hat\eta\zeta\tilde f + \hat\eta'\zeta\Delta\varphi(T-t,\cdot) - 2\hat\eta\nabla\zeta\cdot\nabla\Delta\varphi(T-t,\cdot) - \hat\eta\Delta\zeta\Delta\varphi(T-t,\cdot).$$

Notice that $\tilde F \in L^2(0,T;H^{-2}(\mathbb{R}^n)^n)$ and we a priori know that $\tilde u \in L^2((0,T)\times\mathbb{R}^n)^n$ (from its definition). From (E.13), we have that $\tilde u_t \in L^2(0,T;H^{-2}(\mathbb{R}^n)^n)$, so that $u(0,\cdot)$ makes sense. Now, we rewrite $\tilde{F}$ in a more appropriate way, so that it is given by the sum of two functions: in the first one, we include all the terms with derivatives of second order of $(a\cdot\nabla)\varphi$, $D\varphi b$, $c\nabla\psi$ and φ; in the second one, we consider all the other terms. Notice that this second function has a support contained in $\omega_0 \setminus \overline{\omega}_1$ (because derivatives of ζ appear everywhere). More precisely, we set $\tilde{F} = \tilde{F}_1 + \tilde{F}_2$, with

$$\begin{aligned}\tilde{F}_1 = \hat{\eta}\Delta\left(\zeta((a\cdot\nabla)\varphi)(T-t,\cdot\,)\right) + \hat{\eta}\Delta\left(\zeta(D\varphi b)(T-t,\cdot\,)\right) + \hat{\eta}\Delta\left(\zeta(c\nabla\psi)(T-t,\cdot\,)\right)\\ -\hat{\eta}\nabla\left(\nabla\cdot(\zeta((a\cdot\nabla)\varphi)(T-t,\cdot\,))\right) - \hat{\eta}\nabla\left(\nabla\cdot(\zeta(D\varphi b)(T-t,\cdot\,))\right)\\ -\hat{\eta}\nabla\left(\nabla\cdot(\zeta(c\nabla\psi)(T-t,\cdot\,))\right) + \hat{\eta}'\Delta(\zeta\varphi(T-t,\cdot\,)),\end{aligned}$$

and

$$\begin{aligned}\tilde{F}_2 = -2\hat{\eta}\nabla\zeta\cdot\nabla((a\cdot\nabla)\varphi)(T-t,\cdot) - \hat{\eta}\Delta\zeta((a\cdot\nabla)\varphi)(T-t,\cdot) - 2\hat{\eta}\nabla\zeta\cdot\nabla(D\varphi b)(T-t,\cdot)\\ -\hat{\eta}\Delta\zeta(D\varphi b)(T-t,\cdot\,) - 2\hat{\eta}\nabla\zeta\cdot\nabla(c\nabla\psi)(T-t,\cdot\,) - \hat{\eta}\Delta\zeta(c\nabla\psi)(T-t,\cdot\,)\\ +\hat{\eta}\nabla\left(\nabla\zeta\cdot((a\cdot\nabla)\varphi)(T-t,\cdot\,))\right) + \hat{\eta}\nabla\zeta(\nabla\cdot((a\cdot\nabla)\varphi)(T-t,\cdot\,))\\ +\hat{\eta}\nabla\left(\nabla\zeta\cdot(D\varphi b)(T-t,\cdot\,))\right) + \hat{\eta}\nabla\zeta(\nabla\cdot(D\varphi b)(T-t,\cdot\,))\\ +\hat{\eta}\nabla\left(\nabla\zeta\cdot(c\nabla\psi)(T-t,\cdot\,))\right) + \hat{\eta}\nabla\zeta(\nabla\cdot(c\nabla\psi)(T-t,\cdot\,))\\ -2\hat{\eta}'\nabla\zeta\cdot\nabla\varphi(T-t,\cdot) - \hat{\eta}'\Delta\zeta\varphi(T-t,\cdot) - 2\hat{\eta}\nabla\zeta\cdot\nabla\Delta\varphi(T-t,\cdot) - \hat{\eta}\Delta\zeta\Delta\varphi(T-t,\cdot).\end{aligned}$$

Notice that $\tilde{F}$, $\tilde{F}_1 \in L^2(0,T;H^{-2}(\mathbb{R}^n)^n)$, while $\tilde{F}_2 \in L^2(0,T;H^{-1}(\mathbb{R}^n)^n)$.

Next, we introduce two functions $\tilde{u}^1$ and $\tilde{u}^2$ in $L^2((0,T)\times\mathbb{R}^n)^n$ satisfying

$$\begin{cases}\partial_t\tilde{u}^i - \Delta\tilde{u}^i = \tilde{F}_i & \text{in} \quad (0,T)\times\mathbb{R}^n,\\ \tilde{u}^i(0,\cdot\,) = 0 & \text{in} \quad \mathbb{R}^n,\end{cases} \tag{E.14}$$

for $i = 1,2$. It is clear that $\tilde{u} = \tilde{u}^1 + \tilde{u}^2$ then

$$\iint_{(0,T)\times\omega'} |\tilde{u}|^2 \le 2\left(\iint_{(0,T)\times\omega'} |\tilde{u}^1|^2 + \iint_{(0,T)\times\omega'} |\tilde{u}^2|^2\right).$$

Step 5.a: Estimates of $\tilde{u}^1$.

We see $\tilde{u}^1$ as the transposition solution of the Cauchy problem for the heat equation (E.14) for $i = 1$. This means that $\tilde{u}^1$ is the unique function in $L^2((0,T)\times\mathbb{R}^n)^n$ that, for each $h \in L^2((0,T)\times\mathbb{R}^n)^n$, one has

$$\begin{aligned}\iint_{(0,T)\times\mathbb{R}^n} \tilde{u}^1\cdot h \quad = \quad & \iint_{(0,T)\times\mathbb{R}^n} (\hat{\eta}\zeta((a\cdot\nabla)\varphi + D\varphi b + c\nabla\psi)(T-t,\cdot\,))\cdot\Delta z\\ & -\iint_{(0,T)\times\mathbb{R}^n} \hat{\eta}\zeta((a\cdot\nabla)\varphi + D\varphi b + c\nabla\psi)(T-t,\cdot\,)\cdot\nabla(\nabla\cdot z)\\ & +\iint_{(0,T)\times\mathbb{R}^n} \hat{\eta}'\zeta\varphi(T-t,\cdot\,)\cdot\Delta z\,,\end{aligned}$$

where z is the solution of

$$\begin{cases} -\partial_t z - \Delta z = h & \text{in} \quad (0,T)\times\mathbb{R}^n, \\ z(T,\cdot\,) = 0 & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{E.15}$$

Remark that, for every $h \in L^2((0,T)\times\mathbb{R}^n)^n$, equation (E.15) possesses exactly one solution $z \in L^2(0,T;H^2(\mathbb{R}^n)^n)$ that depends continuously on h. Therefore, $\tilde{u}^1$ is well defined and

$$\|\tilde{u}^1\|_{L^2((0,T)\times\mathbb{R}^n)^n} \le C\|\tilde{F}_1\|_{L^2(0,T;H^{-2}(\mathbb{R}^n)^n)}. \tag{E.16}$$

Furthermore, it is not difficult to show that $\tilde{u}^1 \in C^0([0,T];H^{-2}(\mathbb{R}^n)^n)$ and solves (E.14) for $i = 1$ in the distributional sense. Moreover, from (E.16) it follows that

$$\begin{aligned} \iint_{(0,T)\times\mathbb{R}^n} |\tilde{u}^1|^2 \;\; &\le \;\; C\bigg(\iint_{(0,T)\times\mathbb{R}^n} |\hat{\eta}\zeta(a\cdot\nabla)\varphi|^2 + \iint_{(0,T)\times\mathbb{R}^n} |\hat{\eta}\zeta D\varphi b|^2 \\ &\qquad + \iint_{(0,T)\times\mathbb{R}^n} |\hat{\eta}\zeta c\nabla\psi|^2 + \iint_{(0,T)\times\mathbb{R}^n} |\hat{\eta}'\zeta\varphi|^2 \bigg). \end{aligned}$$

Here, we have used the fact that $\hat{\eta}(T-t,\cdot\,) = \hat{\eta}(t,\cdot\,) \quad \forall t \in (0,T)$. Thanks to the properties of ζ, we finally get

$$\begin{aligned} \iint_{(0,T)\times\omega'} |\tilde{u}^1|^2 \;\; &\le \;\; \iint_{(0,T)\times\mathbb{R}^n} |\tilde{u}^1|^2 \\ &\le \;\; C\bigg(\iint_{(0,T)\times\omega_0} |\hat{\eta}(a\cdot\nabla)\varphi|^2 + \iint_{(0,T)\times\omega_0} |\hat{\eta}D\varphi b|^2 \\ &\qquad + \iint_{(0,T)\times\omega_0} |\hat{\eta}c\nabla\psi|^2 + \iint_{(0,T)\times\omega_0} |\hat{\eta}'\varphi|^2 \bigg). \end{aligned} \tag{E.17}$$

Step 5.b: Estimates of $\tilde{u}^2$.

Now, we deal with the Cauchy problem (E.14) for $i = 2$, where the right-hand side is in $L^2(0,T;H^{-1}(\mathbb{R}^n)^n)$. The existence and uniqueness of a solution $\tilde{u}^2 \in L^2(0,T;H^1(\mathbb{R}^n)^n)$ is classical. Recall that $\tilde{F}_2(t,\cdot\,)$ has support in $\omega_0 \setminus \overline{\omega}_1$ for almos every t, while we would like to estimate the L^2-norm of the solution in ω' and ω' is disjoint of $\omega_0 \setminus \overline{\omega}_1$. We will start by writing $\tilde{u}^2$ in terms of the fundamental solution $G = G(t,x)$ of the heat equation. To do this, we first notice that $\tilde{F}_2$ can be written in the form

$$\tilde{F}_2 = \tilde{F}_{21} + \nabla\cdot\tilde{F}_{22},$$

where $\tilde{F}_{21}$ and $\tilde{F}_{22}$ are L^2 functions supported in $[0,T]\times(\omega_0\setminus\overline{\omega}_1)$ which can be written as sums of derivatives up to the second order of products $\hat{\eta}D^\beta\zeta\varphi$, $\hat{\eta}D^\beta\zeta(a\cdot\nabla)\varphi$, $\hat{\eta}D^\beta\zeta D\varphi b$, $\hat{\eta}D^\beta\zeta c\nabla\psi$ and $\hat{\eta}'D^\beta\zeta\varphi$ with $1 \le |\beta| \le 4$. Thus, we have:

$$\tilde{u}^2(t,x) = \int_0^t\!\!\int_{\omega_0\setminus\overline{\omega}_1} G(t-s,x-y)\tilde{F}_{21}(s,y)\,dy\,ds - \int_0^t\!\!\int_{\omega_0\setminus\overline{\omega}_1} \nabla_y G(t-s,x-y)\cdot\tilde{F}_{22}(s,y)\,dy\,ds, \tag{E.18}$$

for all $(t,x) \in (0,T) \times \omega'$, where G is the fundamental solution for the heat operator given by

$$G(t,x) = \frac{e^{-|x|^2/2t}}{(4\pi t)^{n/2}} \quad \forall x \in \mathbb{R}^n, \quad \forall t > 0.$$

Notice that the above formula makes sense because the integration is over a region far from the singularity of G, i.e. for any $y \in \omega_0 \setminus \overline{\omega}_1$ and any $x \in \omega'$, one has $|x-y| \geq dist(\partial\omega_1, \partial\omega_0) > 0$. Integrating by parts with respect to y in (E.18) and passing all the derivatives from $\tilde{F}_{21}$ and $\tilde{F}_{22}$ to G and $\nabla_y G$, we obtain an expression for $\tilde{u}^2$ of the form

$$\tilde{u}^2(t,x) = \iint_{(0,t)\times(\omega_0\setminus\overline{\omega}_1)} \sum_{\alpha\in I, \beta\in J} D_y^\alpha G(t-s, x-y) D_y^\beta \zeta(y) z_{\alpha,\beta}(s,y) dy\, ds,$$

where all $\alpha \in I$ satisfy $|\alpha| \leq 3$, all $\beta \in J$ satisfy $1 \leq |\beta| \leq 4$ and

$$\begin{aligned} z_{\alpha,\beta}(s,y) &= \hat{\eta}(s)\left(C_{\alpha,\beta}\varphi(s,y) + D_{\alpha,\beta}((a\cdot\nabla)\varphi)(s,y) + E_{\alpha,\beta}(D\varphi b)(s,y)\right) \\ &\quad + F_{\alpha,\beta}(c\nabla\psi)(s,y) + L_{\alpha,\beta}\hat{\eta}'(s)\varphi(s,y), \end{aligned}$$

with $C_{\alpha,\beta}$, $D_{\alpha,\beta}$, $E_{\alpha,\beta}$, $F_{\alpha,\beta}$, $L_{\alpha,\beta} \in \mathbb{R}$. The expression for $\tilde{u}^2$ yields

$$|\tilde{u}^2(t,x)| \leq \iint_{(0,t)\times(\omega_0\setminus\overline{\omega}_1)} \sum_{\alpha\in I} |D_y^\alpha G(t-s,x-y)||z(s,y)| dy\, ds$$

for all $(t,x) \in (0,T) \times \omega'$, where

$$\begin{aligned} z(s,y) &= \hat{\eta}(s)\left(C_1\varphi(s,y) + C_2((a\cdot\nabla)\varphi)(s,y) + C_3(D\varphi b)(s,y)\right) \\ &\quad + C_4(c\nabla\psi)(s,y) + C_5\hat{\eta}'(s)\varphi(s,y) \end{aligned}$$

Now, for every $0 < \delta < dist(\partial\omega_1, \partial\omega_0)$ there exists a positive constant $C(\delta, \omega_c)$ such that

$$|D^\alpha G(t-s, x-y)| \leq C \exp\left(\frac{-\delta^2}{2(t-s)}\right), \forall \alpha \in I,\ (t,x) \in (0,T)\times\omega',\ \forall (s,y) \in (0,t)\times(\omega_0\setminus\overline{\omega}_1).$$

Thus, we have that

$$|\tilde{u}^2(t,x)| \leq C \iint_{(0,t)\times(\omega_0\setminus\overline{\omega}_1)} \exp\left(\frac{-\delta^2}{2(t-s)}\right) |z(s,y)|\, dy\, ds.$$

Next, we integrate this last estimate in $(0,T)\times\omega'$ and use Cauchy-Scwharz inequality to obtain

$$\begin{aligned} \iint_{(0,T)\times\omega'} |\tilde{u}^2(t,x)|^2 &\leq C \int_0^T \left(\int_0^t \int_{\omega_0\setminus\overline{\omega}_1} \exp\left(\frac{-\delta^2}{2(t-s)}\right) |z(s,y)| dy ds \right)^2 dt \\ &\leq CT \int_0^T \left(\int_0^t \exp\left(\frac{-\delta^2}{2(t-s)}\right) \|z(s)\|^2_{L^2(\omega_0)} ds \right) dt. \end{aligned}$$

Finally, observe that we can write the last term of the previous estimate as a convolution, i.e.

$$\int_0^T \left(\int_0^t \exp\left(\frac{-\delta^2}{2(t-s)}\right) \|z(s,\cdot)\|^2_{L^2(\omega_0)} ds \right) dt = \int_0^T (f_1 * f_2)(t) dt,$$

where

$$f_1(t) := e^{-\delta^2/2t} 1_{(0,+\infty)}(t) \quad \text{and} \quad f_2(t) := \|z(t,\cdot)\|^2_{L^2(\omega_0)} 1_{[0,T]}(t),$$

that is, f_1, $f_2 \in L^1(\mathbb{R})$. From Young's inequality, we obtain

$$\iint_{(0,T)\times\omega'} |\tilde{u}^2(t,x)|^2 \le CT^2 \iint_{(0,T)\times\omega_0} |z(t,x)|^2$$

and the definition of z gives

$$\iint_{(0,T)\times\omega'} |\tilde{u}^2(t,x)|^2 \le CT^2 \left(\iint_{(0,T)\times\omega_0} |\hat{\eta}'\varphi|^2 + |\hat{\eta}|^2 \left(|\varphi|^2 + |(a\cdot\nabla)\varphi)|^2 + |D\varphi b|^2 + |c\nabla\psi|^2 \right) \right).$$

Hence, from (E.17), and the previous estimates of $\tilde{u}^1$ and $\tilde{u}^2$, we deduce the following

$$\begin{aligned}
\iint_{(0,T)\times\omega'} |\hat{\eta}|^2 |\Delta\varphi|^2 \;&\le\; C(1+T^2) \left(\iint_{(0,T)\times\omega_0} |\hat{\eta}'\varphi|^2 + |\hat{\eta}|^2 \left(|\varphi|^2 + |(a\cdot\nabla)\varphi|^2 \right.\right. \\
&\qquad \left.\left. + |D\varphi b|^2 + |c\nabla\psi|^2 \right) \right) \\
&\le\; C(1+T^2) \Bigg(s^{9/2}\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}} \hat{\xi}^{9/2} |\varphi|^2 \\
&\qquad + s^2\lambda^3 (\|a\|^2_\infty + \|b\|^2_\infty) \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}} \hat{\xi}^2 |\nabla\varphi|^2 \\
&\qquad + s^2\lambda^3 \|c\|^2_\infty \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}} \hat{\xi}^2 |\nabla\psi|^2 \Bigg) \\
&\le\; C(1+T^2) \Bigg(s^{9/2}\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}} \hat{\xi}^{9/2} |\varphi|^2 \\
&\qquad + s^2\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}} \hat{\xi}^2 (|\nabla\varphi|^2 + |\nabla\psi|^2) \Bigg),
\end{aligned} \tag{E.19}$$

for $\lambda \ge \widehat{\lambda} e^{\widehat{\lambda} T(\|a\|^2_\infty + \|b\|^2_\infty + \|A\|^2_P + \|B\|^2_Q)} (1 + \|a\|^2_\infty + \|b\|^2_\infty)(1 + \|A\|^5_P + \|B\|^5_Q)(1 + \|c\|^2_\infty)$ and $s \ge \widehat{s} e^{8\lambda\|\eta^0\|_\infty} (T^4 + T^8)$.

Step 6: Local estimate of φ_t.

In this step, we estimate the local term on φ_t in (E.11). First, integration by parts gives

$$\begin{aligned}
s^2\lambda^3 \iint_{(0,T)\times\omega'} e^{-2s\hat{\alpha}} \hat{\xi}^2 |\varphi_t|^2 \;&=\; \frac{1}{2} s^2\lambda^3 \iint_{(0,T)\times\omega'} \left(e^{-2s\hat{\alpha}} \hat{\xi}^2 \right)_{tt} |\varphi|^2 \\
&\quad - s^2\lambda^3 \iint_{(0,T)\times\omega'} e^{-2s\hat{\alpha}} \hat{\xi}^2 \varphi \cdot \varphi_{tt}\,.
\end{aligned}$$

Now, since there exists $C > 0$ such that

$$\left| \left(e^{-2s\hat{\alpha}} \hat{\xi}^2 \right)_{tt} \right| \le C s^2 T^2 e^{-2s\hat{\alpha}} \hat{\xi}^{9/2} \quad \text{and} \quad e^{-2s\hat{\alpha}} \le C e^{-4s\hat{\alpha} + 2s\alpha^*}$$

we have that

$$\begin{aligned}
s^2\lambda^3 \iint_{(0,T)\times\omega'} e^{-2s\hat{\alpha}} \hat{\xi}^2 |\varphi_t|^2 \;&\le C s^{15/2}\lambda^8 \iint_{(0,T)\times\omega'} e^{-4s\hat{\alpha} + 2s\alpha^*} \hat{\xi}^{15/2} |\varphi|^2 \\
&\quad + \iint_{(0,T)\times\omega'} |\eta^*|^2 |\varphi_{tt}|^2\,,
\end{aligned} \tag{E.20}$$

with

$$\eta^* := s^{-7/4}\lambda^{-1}e^{-s\alpha^*}\hat{\xi}^{-7/4}.$$

In what follows, we estimate the second term in the right-hand side of (E.20). To do this, we set $(y, q, \phi) := (\eta^*\varphi_t, \eta^*\pi_t, \eta^*\psi_t)$, and note that (y, q, ϕ) solves

$$\begin{cases} -\partial_t y - \Delta y - (a\cdot\nabla)y - Dyb + \nabla q = c\nabla\phi + G_1 & \text{in} \quad \mathcal{O}_T, \\ -\partial_t\phi - \Delta\phi - (a+b)\cdot\nabla\phi = y\cdot e_n + G_2 & \text{in} \quad \mathcal{O}_T, \\ \nabla\cdot y = 0 & \text{in} \quad \mathcal{O}_T, \\ y\cdot\nu = 0, \quad [D(y)\nu + Ay]_{tan} = -\eta^* A_t\varphi & \text{on} \quad \Lambda_T, \\ \dfrac{\partial\phi}{\partial\nu} + B\phi = -\eta^* B_t\psi & \text{on} \quad \Lambda_T, \\ y(T,\cdot\,) = 0, \quad \phi(T,\cdot\,) = 0 & \text{in} \quad \mathcal{O}, \end{cases} \tag{E.21}$$

where

$$G_1 = -\eta^*_t\varphi_t + \eta^*(a_t\cdot\nabla)\varphi + \eta^* D\varphi b_t + \eta^* c_t\nabla\psi$$

and

$$G_2 = -\eta^*_t\psi_t + \eta^*(a+b)_t\cdot\nabla\psi.$$

To see that (y, q, ϕ) solves (E.21), one can take a sequence of regular functions (a^k, b^k, c^k) such that

$$(a^k, b^k, c^k) \longrightarrow (a, b, c) \quad \text{weakly star in} \quad L^\infty(0,T;L^\infty(\mathcal{O})^{2n+1})$$

and

$$(a^k_t, b^k_t, c^k_t) \longrightarrow (a_t, b_t, c_t) \quad \text{weakly in} \quad L^2(0,T;L^r(\mathcal{O})^{2n+1}).$$

Since there exists a unique solution (y^k, q^k, ϕ^k) to (E.21) with (a, b, c) replaced by (a^k, b^k, c^k), one can take limits and conclude that (y, q, ϕ) solves (E.21).

Next, using the hypothesis on a, b, c, A and B, the fact that $\varphi \in L^2(0,T;H^1(\mathcal{O})^n) \cap H^1(0,T;H^{-1}(\mathcal{O})^n)$ and $\psi \in L^2(0,T;H^1(\mathcal{O})) \cap H^1(0,T;H^{-1}(\mathcal{O}))$, we see that $G_1 \in L^2(0,T;H^{-1}(\mathcal{O})^n)$, $G_2 \in L^2(0,T;H^{-1}(\mathcal{O}))$, $\eta^* A_t\varphi \in L^2(0,T;H^{-1/2}(\partial\mathcal{O})^n)$ and $\eta^* B_t\psi \in L^2(0,T;H^{-1/2}(\partial\mathcal{O}))$. Moreover, the following estimate holds

$$\begin{aligned} &\|y\|^2_{L^2(H^1)} + \|\phi\|^2_{L^2(H^1)} \\ &\quad \le Ce^{CT(\|a\|^2_\infty + \|b\|^2_\infty + \|c\|^2_\infty + \|A\|^2_\infty)}\left(\|(G_1, G_2)\|^2_{L^2(H^{-1})} + \|\eta^*(A_t\varphi, B_t\psi)\|^2_{L^2(H^{-1/2})}\right). \end{aligned} \tag{E.22}$$

Notice that this is still not enough to absorb the local term on φ_{tt} in (E.20). Thus, we must show that y is actually a strong solution of $(\text{E.21})_{1,3,4,6}$, which will be true if we prove that $G_1 \in L^2(\mathcal{O}_T)^n$ and $\eta^* A_t\varphi \in H^{(1-l)/2}(0,T;H^{(l-1/2)}(\partial\mathcal{O})^n) \cap L^2(0,T;H^{1/2}(\partial\mathcal{O})^N)$.

To see that $G_1 \in L^2(\mathcal{O}_T)^n$, we must verify that $\eta^*(a_t\cdot\nabla)\varphi$, $\eta^* D\varphi b_t$ and $\eta^* c_t\nabla\psi$ belong to $L^2(\mathcal{O}_T)^n$. In fact, since $y \in L^2(0,T;H^1(\mathcal{O})^n)$, we have that $\eta^*\nabla\varphi \in H^1(0,T;L^2(\mathcal{O})^{n\times n})$

and, using that $\eta^*\nabla\varphi \in L^2(0,T;H^1(\mathcal{O})^{n\times n})$ and (BOYER; FABRIE, 2012, Theorem II.5.14), we conclude that

$$\eta^*\nabla\varphi \in C^0([0,T];H^{1/2}(\mathcal{O})^{n\times n}).$$

Analogously, we have that

$$\eta^*\nabla\psi \in C^0([0,T];H^{1/2}(\mathcal{O})^{n}).$$

Hence, from the assumptions on a_t, b_t and c_t, we readily see that $\eta^*(a_t\cdot\nabla)\varphi \in L^2(\mathcal{O}_T)^n$, $\eta^*D\varphi b_t \in L^2(\mathcal{O}_T)^n$ and $\eta^* c_t\nabla\psi \in L^2(\mathcal{O}_T)^n$. Moreover, the following estimate holds

$$\begin{aligned}
\|\eta^*(a_t\cdot\nabla)\varphi\|^2_{L^2(\mathcal{O}_T)^n} &+ \|\eta^*D\varphi b_t\|^2_{L^2(\mathcal{O}_T)^n} + \|\eta^*c_t\nabla\psi\|^2_{L^2(\mathcal{O}_T)^n}\\
&\leq C\left(\|a_t\|^2_{L^2(L^r)} + \|b_t\|^2_{L^2(L^r)} + \|c_t\|^2_{L^2(L^r)}\right)\\
&\times\Bigg(\|\eta^*\varphi\|^2_{L^2(H^2)} + \|\eta^*_t\varphi\|^2_{L^2(H^1)} + \|y\|^2_{L^2(H^1)}\\
&+\|\eta^*\psi\|^2_{L^2(H^2)} + \|\eta^*_t\psi\|^2_{L^2(H^1)} + \|\phi\|^2_{L^2(H^1)}\Bigg).
\end{aligned}$$

Let us now prove that $\eta^*A_t\varphi \in H^{(1-l)/2}(0,T;H^{(l-1/2)}(\partial\mathcal{O})^n) \cap L^2(0,T;H^{1/2}(\partial\mathcal{O})^n)$. Indeed, from estimate (E.22) we see that $\eta^*\varphi \in H^1(0,T;H^{1/2}(\partial\mathcal{O})^n)$ and, together with assumption (3.50) on A, we obtain

$$\eta^*A_t\varphi \in H^{(1-l)/2}(0,T;H^{\vartheta_2}(\partial\mathcal{O})^n) \subset H^{(1-l)/2}(0,T;H^{l-1/2}(\partial\mathcal{O})^n),$$

with the following estimate

$$\|\eta^*(\partial_t A)\varphi\|^2_{H^{(1-l)/2}(H^{l-1/2})} \leq C\|A\|^2_{H^{(3-l)/2}(H^{\vartheta_2})}\left(\|\eta^*_t\varphi\|^2_{L^2(H^1)} + \|y\|^2_{L^2(H^1)} + \|\eta^*\varphi\|^2_{L^2(H^1)}\right).$$

Also, since $\eta^*\varphi \in L^2(0,T;H^2(\mathcal{O})^n)\cap H^1(0,T;H^1(\mathcal{O})^n)$, by interpolation we have that $\eta^*\varphi \in H^{1/4}(0,T;H^{5/4}(\partial\mathcal{O})^n)$, which gives $\eta^*A_t\varphi \in L^2(0,T;H^{1/2}(\partial\mathcal{O})^n)$, because $A_t \in H^{(1-l)/2}(0,T;H^{\vartheta_2}(\partial\mathcal{O}^{n\times n}))$. Moreover,

$$\|\eta^*A_t\varphi\|^2_{L^2(H^{1/2})} \leq C\|A\|^2_{H^{(3-l)/2}(H^{\vartheta_2})}\left(\|\eta^*\varphi\|^2_{L^2(H^2)} + \|\eta^*_t\varphi\|^2_{L^2(H^1)} + \|y\|^2_{L^2(H^1)}\right).$$

Thus, we have proved that y is a strong solution of $(\text{E.21})_{1,3,4,6}$. Recalling (GUERRERO, 2006b, Proposition 1.1), we deduce in particular that $y_t \in L^2(\mathcal{O}_T)$ and

$$\begin{aligned}
\|y_t\|^2_{L^2(\mathcal{O}_T)} &\leq Ce^{CT\|A\|^2_P}(1+\|A\|^4_P)\Bigg(\|G_1\|^2_{L^2(\mathcal{O}_T)^n} + \|(a\cdot\nabla)y\|^2_{L^2(\mathcal{O}_T)^n}\\
&+\|Dyb\|^2_{L^2(\mathcal{O}_T)^n} + \|c\nabla\phi\|^2_{L^2(\mathcal{O}_T)^n} + \|\eta^*A_t\varphi\|^2_{L^2(H^{1/2})} + \|\eta^*A_t\varphi\|^2_{H^{(1-l)/2}(H^{l-1/2})}\Bigg)\\
&\leq Ce^{CT\|A\|^2_P}(1+\|A\|^4_P)(1+\|A\|^2_P)\Bigg[\left(1+\|a_t\|^2_{L^2(L^r)} + \|b_t\|^2_{L^2(L^r)} + \|c_t\|^2_{L^2(L^r)}\right)\\
&\times\left(\|\eta^*_t\varphi_t\|^2_{L^2(\mathcal{O}_T)^n} + \|\eta^*\varphi\|^2_{L^2(H^2)} + \|\eta^*_t\varphi\|^2_{L^2(H^1)} + \|y\|^2_{L^2(H^1)} + \|\eta^*\psi\|^2_{L^2(H^2)}\right.\\
&\left.+\|\eta^*_t\psi\|^2_{L^2(H^1)} + \|\phi\|^2_{L^2(H^1)}\right) + \left(\|a\|^2_\infty + \|b\|^2_\infty + \|c\|^2_\infty\right)\left(\|y\|^2_{L^2(H^1)} + \|\phi\|^2_{L^2(H^1)}\right)\\
&+\left(\|\eta^*\varphi\|^2_{L^2(H^2)} + \|\eta^*_t\varphi\|^2_{L^2(H^1)} + \|y\|^2_{L^2(H^1)}\right)\Bigg].
\end{aligned}$$

Taking now $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2+\|B\|_Q^2)}(1+\|A\|_P^2)(1+\|A\|_P^5+\|B\|_Q^5)(1+\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|a_t\|_{L^2(L^r)}^2+\|b_t\|_{L^2(L^r)}^2+\|c_t\|_{L^2(L^r)}^2)(1+\|c\|_\infty^2)$ and $s \geq \widehat{s} e^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$, from (E.22) we obtain

$$\begin{aligned}\|\eta^*\varphi_{tt}\|_{L^2(\mathcal{O}_T)}^2 \;\leq\; & C\lambda^2 \Big(\|\eta_t^*\varphi_t\|_{L^2(\mathcal{O}_T)^n}^2 + \|\eta_t^*\psi_t\|_{L^2(\mathcal{O}_T)}^2 + \|\eta^*\varphi\|_{L^2(H^2)}^2 + \|\eta_t^*\varphi\|_{L^2(H^1)}^2 \\ & + \|\eta^*\psi\|_{L^2(H^2)}^2 + \|\eta_t^*\psi\|_{L^2(H^1)}^2 + \|y\|_{L^2(\mathcal{O}_T)}^2 + \|\phi\|_{L^2(\mathcal{O}_T)}^2\Big).\end{aligned}$$

Since $|\eta_t^*| \leq \varepsilon\lambda^{-1}s^{-1/2}\hat{\xi}^{-1/2}e^{-s\alpha^*}$, for ε sufficiently small, the following is found:

$$\begin{aligned}\|\eta^*\varphi_{tt}\|_{L^2(\mathcal{O}_T)}^2 \;\leq\; & C\varepsilon\Bigg(s^{-1}\iint_{\mathcal{O}_T} e^{-2s\alpha^*}\hat{\xi}^{-1}(|\varphi_t|^2+|\psi_t|^2) \\ & + s^{-1}\iint_{\mathcal{O}_T} e^{-2s\alpha^*}\hat{\xi}^{-1}(|\nabla\varphi|^2+|\nabla\psi|^2)\Bigg) \\ & + C\lambda^2\left(\|\eta^*\varphi\|_{L^2(H^2)}^2 + \|\eta^*\psi\|_{L^2(H^2)}^2\right).\end{aligned} \tag{E.23}$$

We need to estimate the terms $\|\eta^*\varphi\|_{L^2(H^2)}^2$ and $\|\eta^*\psi\|_{L^2(H^2)}^2$. Thus, let us set $(\hat{\varphi},\hat{\pi},\hat{\psi}) := \eta^*(\varphi,\pi,\psi)$. One has:

$$\begin{cases} -\partial_t\hat{\varphi} - \Delta\hat{\varphi} - (a\cdot\nabla)\hat{\varphi} - D\hat{\varphi}b + \nabla\hat{\pi} = \eta^* c\nabla\psi - \eta_t^*\varphi & \text{in} \quad \mathcal{O}_T, \\ -\partial_t\hat{\psi} - \Delta\hat{\psi} - (a+b)\cdot\nabla\hat{\psi} = \eta^*\varphi\cdot e_n - \eta_t^*\psi & \text{in} \quad \mathcal{O}_T, \\ \nabla\cdot\hat{\varphi} = 0 & \text{in} \quad \mathcal{O}_T, \\ \hat{\varphi}\cdot\nu = 0, \quad [D(\hat{\varphi})\nu + A\hat{\varphi}]_{tan} = 0 & \text{on} \quad \Lambda_T, \\ \dfrac{\partial\hat{\psi}}{\partial\nu} + B\hat{\psi} = 0 & \text{on} \quad \Lambda_T, \\ \hat{\varphi}(T,\cdot) = 0, \quad \hat{\psi}(T,\cdot) = 0 & \text{in} \quad \mathcal{O}. \end{cases}$$

Again, from energy estimates (GUERRERO, 2006b, Proposition 1.1), we find that

$$\|\hat{\varphi}\|_{L^2(H^2)}^2 \leq Ce^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2)}(1+\|A\|_P^4)(1+\|a\|_\infty^2+\|b\|_\infty^2)\left(\|\eta_t^*\varphi\|^2 + \|c\|_\infty^2\|\eta^*\nabla\psi\|^2\right),$$

and, from maximal L^2-regularity estimates for the heat equation with homogeneous Robin boundary conditions (similar arguments as in (FERNÁNDEZ-CARA et al., 2006a, Proposition 2)), we deduce that

$$\|\hat{\psi}\|_{L^2(H^2)}^2 \leq Ce^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|B\|_Q^2)}(1+\|B\|_Q^4)(1+\|a\|_\infty^2+\|b\|_\infty^2)\left(\|\eta^*\varphi\|^2 + \|\eta_t^*\psi\|^2\right).$$

Adding the last two inequalities, we have:

$$\begin{aligned}\|\hat{\varphi}\|_{L^2(H^2)}^2 + \|\hat{\psi}\|_{L^2(H^2)}^2 \;\leq\; & \lambda\left(\|\eta_t^*\varphi\|^2 + \|\eta_t^*\psi\|^2 + \|\eta^*\nabla\psi\|^2 + \|\eta^*\varphi\|^2\right) \\ \leq\; & \lambda\Bigg(\varepsilon s^{-1}\lambda^{-2}\iint_{\mathcal{O}_T} e^{-2s\alpha^*}\hat{\xi}^{-1}(|\varphi|^2+|\psi|^2) \\ & + s^{-7/2}\lambda^{-2}\iint_{\mathcal{O}_T} e^{-2s\alpha^*}\hat{\xi}^{-7/2}|\nabla\psi|^2\Bigg).\end{aligned}$$

From this last estimate, (E.23) and (E.20), we see that

$$s^2\lambda^3 \iint_{(0,T)\times\omega'} e^{-2s\hat{\alpha}}\hat{\xi}^2|\varphi_t|^2 \leq Cs^{15/2}\lambda^8 \iint_{(0,T)\times\omega'} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^{15/2}|\varphi|^2 + \varepsilon I(s,\lambda;\varphi) + \varepsilon I(s,\lambda;\psi), \tag{E.24}$$

for $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2+\|B\|_Q^2)}(1+\|A\|_P^2)(1+\|A\|_P^5)(1+\|B\|_Q^5)(1+\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|a_t\|_{L^2(L^r)}^2+\|b_t\|_{L^2(L^r)}^2+\|c_t\|_{L^2(L^r)}^2+\|B\|_\infty^2)(1+\|c\|_\infty^2)$ and $s \geq \widehat{s} e^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Step 7: Arrangements.

Combining (E.11), (E.19) and (E.24), it follows that

$$I(s,\lambda;\psi)+I(s,\lambda;\varphi) \leq C(1+T^2)\Bigg(s^{15/2}\lambda^8 \iint_{(0,T)\times\omega_0} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^{15/2}|\varphi|^2 + s^3\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^3|\psi|^2 + s^2\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}}\hat{\xi}^2(|\nabla\varphi|^2+|\nabla\psi|^2)\Bigg), \tag{E.25}$$

for $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2+\|B\|_Q^2)}(1+\|A\|_P^2)(1+\|A\|_P^5)(1+\|B\|_Q^5)(1+\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|a_t\|_{L^2(L^r)}^2+\|b_t\|_{L^2(L^r)}^2+\|c_t\|_{L^2(L^r)}^2+\|B\|_\infty^2)(1+\|c\|_\infty^2)$ and $s \geq \widehat{s} e^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Step 8: Estimates of the local gradient terms.

Let us consider a cut-off function $\rho \in C^1(\overline{\omega}_c)$ with $\rho = 1$ in ω_0, supp $\rho \subset\subset \omega_c$. Then,

$$s^2\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}}\hat{\xi}^2|\nabla\varphi|^2 \leq s^2\lambda^4 \iint_{\mathcal{O}_T} e^{-2s\hat{\alpha}}\hat{\xi}^2\rho|\nabla\varphi|^2\,.$$

After integration by parts, thanks to Hölder and Young inequalities, we deduce that

$$s^2\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}}\hat{\xi}^2|\nabla\varphi|^2 \leq \varepsilon s^{-1} \iint_{\mathcal{O}_T} e^{-2s\alpha^*}\hat{\xi}^{-1}(|\Delta\varphi|^2+|\nabla\varphi|^2) + Cs^5\lambda^8 \iint_{(0,T)\times\omega_c} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^5|\varphi|^2\,,$$

where ε is a small enough constant.

Similar computations yield

$$s^2\lambda^4 \iint_{(0,T)\times\omega_0} e^{-2s\hat{\alpha}}\hat{\xi}^2|\nabla\psi|^2 \leq \varepsilon s^{-1} \iint_{\mathcal{O}_T} e^{-2s\alpha^*}\hat{\xi}^{-1}(|\Delta\psi|^2+|\nabla\psi|^2) + Cs^5\lambda^8 \iint_{(0,T)\times\omega_c} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^5|\psi|^2\,.$$

Then, using (E.25), and these inequalities, we obtain

$$\begin{aligned} I(s,\lambda;\psi)+I(s,\lambda;\varphi) \leq C(1+T^2)\bigg(& s^{15/2}\lambda^8 \iint_{(0,T)\times\omega_c} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^{15/2}|\varphi|^2 \\ & + s^5\lambda^8 \iint_{(0,T)\times\omega_c} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^5|\varphi|^2 \\ & + s^3\lambda^4 \iint_{(0,T)\times\omega_c} e^{-2s\alpha}\xi^3|\psi|^2 \\ & + s^5\lambda^8 \iint_{(0,T)\times\omega_c} e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^5|\psi|^2 \bigg) \\ & + \varepsilon I(s,\lambda;\varphi)+\varepsilon I(s,\lambda;\psi), \end{aligned}$$

for $\lambda \geq \widehat{\lambda} e^{\widehat{\lambda} T(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2+\|B\|_Q^2)}(1+\|A\|_P^2)(1+\|A\|_P^5)(1+\|B\|_Q^5)(1+\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|a_t\|_{L^2(L^r)}^2+\|b_t\|_{L^2(L^r)}^2+\|c_t\|_{L^2(L^r)}^2+\|B\|_\infty^2)(1+\|c\|_\infty^2)$ and $s \geq \widehat{s} e^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$. Finally, we easily obtain the desired Carleman estimate (3.54) by taking ε sufficiently small.

This concludes the proof.