Universidade Federal de Pernambuco
Centro de Ciências Exatas e da Natureza
Programa de Pós-Graduação em Matemática

Michele Mendes Novais

FLUIDOS MAGNETO-MICROPOLARES: existência global de solução forte e decaimento na norma L^2 para soluções fracas

Recife
2020

Michele Mendes Novais

FLUIDOS MAGNETO-MICROPOLARES: existência global de solução forte e decaimento na norma L^2 para soluções fracas

Esta tese foi apresentada ao Programa de Pós-graduação em Matemática da Universidade Federal de Pernambuco como requisito parcial para obtenção do título de Doutora em matemática.

Área de concentração: Análise

Orientador: Felipe Wergete Cruz

Recife

2020

Catalogação na fonte
Bibliotecária Monick Raquel Silvestre da S. Portes, CRB4-1217

N935f Novais, Michele Mendes
Fluidos magneto-micropolares: existência global de solução forte e decaimento na norma L^2 para soluções fracas / Michele Mendes Novais. – 2020.
105 f.

Orientador: Felipe Wergete Cruz.
Tese (Doutorado) – Universidade Federal de Pernambuco. CCEN, Matemática, Recife, 2020.
Inclui referências.

1. Análise matemática. 2. Equações diferencias parciais. I. Cruz, Felipe Wergete (orientador). II. Título.

515 CDD (23. ed.) UFPE- CCEN 2020 - 52

MICHELE MENDES NOVAIS

FLUIDOS MAGNETO-MICROPOLARES: EXISTÊNCIA GLOBAL DE SOLUÇÃO FORTE E DECAIMENTO NA NORMA L^2 PARA SOLUÇÕES FRACAS

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

Aprovado em: 07/02/2020

BANCA EXAMINADORA

Prof. Dr. Felipe Wergete Cruz (Orientador)
Universidade Federal de Pernambuco

Prof. Dr. Cilon Valdez Ferreira Perusato (Examinador Interno)
Universidade Federal de Pernambuco

Prof. Dr. Pablo Gustavo Albuquerque Braz e Silva (Examinador Interno)
Universidade Federal de Pernambuco

Prof. Dr. Robert Henrique Rodrigues Guterres (Examinador Externo)
Universidade Federal do Rio Grande do Sul

Prof. Dr. Marcelo Fernandes de Almeida (Examinador Externo)
Universidade Federal de Sergipe

AGRADECIMENTOS

Primeiramente gostaria de agradecer a Deus pela graça de mais uma vitória alcançada.

À minha mãe, Analice, a meus irmãos, Rangel, Rogério e Diôgo, a minhas cunhadas Vaninha, Sthephani e Jô e a meus sobrinhos Everton, Duda, Guga, Thales e Ana Laura, por compreender a minha ausência e me apoiar durante esse processo.

Aos meus avós João e Edite por serem a base forte de nossa família.

Aos meus tios, Nino, Roberto, Dinda, Netinha, Aparecida e Girlania, por me ensinarem a importância dos estudos.

Aos meus primos pelas palavras de incentivo e pelos momentos de lazer que me ajudaram a manter a sanidade. Em especial a Carane, pelas preocupações e palavras de fé nos momentos de dificuldade.

A Fabinho por está sempre do meu lado em minha jornada acadêmica. Me ensinando, me incentivando, me fortalecendo e me fazendo crescer como profissional e como ser humano. Com certeza eu não chegaria até aqui sem você.

Aos professores da UFS pela amizade e pela confiança depositada em mim, em especial Débora, Anderson, Fábio, Bruno, Dirson, Marcelo, Arlúcio e Alysson.

A Gigi como professora e, principalmente, como amiga. Obrigada por ouvir meus choros, minhas reclamações. Por estar ao meu lado em todos os momentos. Por estar presente, mesmo estando distante.

Ao DM-UFRPE, pelo apoio dado nesse processo, oferecendo incentivos para continuar nessa jornada.

A professora Lorena pela presteza em ajudar sempre que precisei.

Ao professor Clessius, por se dispor a tirar as minhas dúvidas, mesmo quando não tinha muito tempo, por me ajudar com latex e pelos cafés da manhã no DM-UFRPE.

Ao professor Fabiano, por me ajudar nas revisões de português e por ouvir meus lamentos toda semana.

Aos meus amigos Carlos, Serginey, Eudes e Joás pelos momentos de lazer.

Ao professor Miguel Loayza por me indicar o melhor caminho a seguir durante o doutorado.

Aos membros da banca examinadora, Robert Henrique, Marcelo Fernandes, Cilon Valdez e Pablo Braz por aceitarem o convite e contribuir com este trabalho.

Finalmente, um agradecimento especial ao meu orientador Felipe Wergete. Obrigada

pelos ensinamentos, pela paciência e pela dedicação. Principalmente, obrigada por compreender minhas dificuldades e não desistir de mim.

Michele Mendes Novais

RESUMO

Estudamos o problema de Cauchy para o sistema de equações que modelam o movimento de um fluido magneto-micropolar incompressível 3D. Tais equações representam uma generalização do clásico modelo de Navier-Stokes e descrevem o comportamento de fluidos com micropartículas levando-se em consideração a presença de um campo magnético. Elas descrevem fenômenos vindo de vários fluidos, tais como sangue humano e de animais, suspensãoes poleméricas, cristais líquidos, librificantes, ferrofluidos, entre outros. Neste trabalho, em um primeiro momento, através de estimativas de energia, obtivemos a existência e unicidade de uma solução forte local do problema em questão. Em seguida, impondo uma condição de pequenez sobre os dados iniciais, mostramos que a solução forte existe globalmente. Em um segundo momento, obtivemos, via o método de decomposição de Fourier (*Fourier splitting method*), taxas de decaimento temporal para as soluções fracas deste sistema. Por fim, através de um argumento mais direto (método da representação integral ou princípio de Duhamel), melhoramos a taxa de decaimento para a velocidade micro-rotacional.

Palavras-chave: Equações diferenciais parciais. Fluidos magneto-micropolares. Método de decomposição de Fourier. Solução forte. Solução fraca. Taxas de decaimento.

ABSTRACT

We study the Cauchy problem for the system of equations that model the motion of a 3D incompressible magneto-micropolar fluid. Such equations represent a generalization of the classic Navier-Stokes model and describe the behavior of fluids with microparticles taking into account the presence of a magnetic field. They describe phenomena coming from various fluids, such as human and animal blood, polymeric suspensions, liquid crystals, lubricants, ferrofluids, among others. In this work, initially, using energy estimates, we obtained the existence and uniqueness of a local strong solution of the problem in question. Next, by imposing a smallness condition on the initial data, we prove that the strong solution exists globally. In a second moment, using the Fourier splitting method, we obtained temporal decay rates for weak solutions of this system. Finally, using a more direct argument (integral representation method or Duhamel's principle), we improved the decay rate to the micro-rotational velocity.

Keywords: Partial differential equations. Magneto-micropolar fluids. Fourier splitting method. Strong solution. Weak solution. Decay rates.

SUMÁRIO

1	**INTRODUÇÃO**	**9**
2	**PRELIMINARES**	**13**
2.1	**Espaços funcionais**	**13**
2.1.1	Os espaços L^p	13
2.1.2	Espaços de Bochner	15
2.1.3	Espaços de Sobolev	16
2.2	**Resultados técnicos**	**17**
3	**SOLUÇÃO FORTE GLOBAL NO TEMPO**	**27**
3.1	**Conceitos fundamentais e resultado principal**	**27**
3.2	**Existência e unicidade de soluções fortes locais no tempo**	**28**
3.2.1	Estimativas *a priori*	28
3.3	**Soluções globais**	**32**
4	**DECAIMENTO DE SOLUÇÕES: O ARGUMENTO FORMAL**	**37**
4.1	**Forças externas nulas**	**37**
4.1.1	Taxa de decaimento através do método de decomposição de Fourier	37
4.1.2	Taxa de decaimento melhorada para a velocidade angular	48
4.2	**Forças externas não nulas**	**60**
4.2.1	Taxa de decaimento melhorada para a micro-rotação	70
5	**DECAIMENTO DE SOLUÇÕES: O ARGUMENTO RIGOROSO**	**83**
5.1	**Soluções aproximadas e resultados auxiliares**	**84**
5.2	**Resultados de decaimento**	**95**
5.2.1	Forças externas nulas	95
5.2.2	Forças externas não nulas	99
	REFERÊNCIAS	**102**

1 INTRODUÇÃO

A teoria dos fluidos micropolares (também conhecidos como fluidos assimétricos) foi estudada primeiramente pelo engenheiro turco-estadunidense Ahmed C. Eringen em 1966 no seu trabalho intitulado *Theory of Micropolar Fluids* (veja [1]). As equações destes fluidos representam uma generalização bem fundamentada das clássicas equações de Navier-Stokes para fluidos newtonianos, levando-se em consideração a presença de micropartículas imersas no fluido. Elas descrevem o comportamento de alguns fluidos complexos, tais como o sangue (humano e animal), cristais líquidos, lubrificantes, entre outros. Se, além da micro-rotação, considerarmos o efeito do campo magnético induzido no movimento do fluido, obtemos o sistema de equações denominado de fluidos magneto-micropolares, a saber

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\mu+\chi)\Delta\boldsymbol{u} + \nabla p = \chi \operatorname{rot}\boldsymbol{w} + \operatorname{rot}\boldsymbol{b}\times\boldsymbol{b} + \boldsymbol{f}, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \gamma\Delta\boldsymbol{w} - \kappa\nabla(\operatorname{div}\boldsymbol{w}) + 2\chi\,\boldsymbol{w} = \chi\operatorname{rot}\boldsymbol{u} + \boldsymbol{g}, \\ \boldsymbol{b}_t - \nu\Delta\boldsymbol{b} = \operatorname{rot}(\boldsymbol{u}\times\boldsymbol{b}), \\ \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{b} = 0. \end{cases} \tag{MMP}$$

Este sistema descreve o movimento de um fluido magneto-micropolar incompressível em todo o espaço $\mathbb{R}^3$ durante um intervalo de tempo $[0,T)$, com $0 < T \leq \infty$. Aqui, $\boldsymbol{u}(\boldsymbol{x},t) \in \mathbb{R}^3$ denota o campo de velocidade incompressível, $p(\boldsymbol{x},t) \in \mathbb{R}$ é a pressão hidrostática, $\boldsymbol{w}(\boldsymbol{x},t) \in \mathbb{R}^3$ descreve a velocidade micro-rotacional e $\boldsymbol{b}(\boldsymbol{x},t) \in \mathbb{R}^3$ é o campo magnético. As constantes positivas μ, χ, γ, κ e ν estão associadas a propriedades do fluido. Mais especificamente, μ é a viscosidade cinemática, χ é a viscosidade do vórtice, γ e κ são as viscosidades de rotação e $1/\nu$ é o número magnético de Reynolds (ν também é conhecida como difusividade magnética). Além disso, $\boldsymbol{f}(\boldsymbol{x},t)$, $\boldsymbol{g}(\boldsymbol{x},t) \in \mathbb{R}^3$ são forças externas dadas. Como o produto vetorial da densidade de corrente $\operatorname{rot}\boldsymbol{b}$ pelo campo magnético $\boldsymbol{b}$ é dado por

$$\operatorname{rot}\boldsymbol{b}\times\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \nabla(|\boldsymbol{b}|^2/2)$$

e

$$\operatorname{rot}(\boldsymbol{u}\times\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} + \operatorname{div}(\boldsymbol{b})\boldsymbol{u} - \operatorname{div}(\boldsymbol{u})\boldsymbol{b},$$

reescrevemos o sistema (MMP) da seguinte maneira:

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\mu+\chi)\Delta\boldsymbol{u} + \nabla\left(p + \dfrac{1}{2}|\boldsymbol{b}|^2\right) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \chi\operatorname{rot}\boldsymbol{w} + \boldsymbol{f}, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \gamma\Delta\boldsymbol{w} - \kappa\nabla(\operatorname{div}\boldsymbol{w}) + 2\chi\,\boldsymbol{w} = \chi\operatorname{rot}\boldsymbol{u} + \boldsymbol{g}, \\ \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - \nu\Delta\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \\ \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{b} = 0. \end{cases} \tag{1.1}$$

A função $|\boldsymbol{b}|^2/2$ é chamada de pressão magnética. Assim, denotamos por $P := p + |\boldsymbol{b}|^2/2$ a pressão total do fluido. Complementamos o sistema (1.1) com condições de Dirichlet homogêneas no infinito, i.e.,

$$\lim_{|\boldsymbol{x}|\to\infty} (\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{w}(\boldsymbol{x},t), \boldsymbol{b}(\boldsymbol{x},t)) = (\mathbf{0},\mathbf{0},\mathbf{0}), \qquad \forall\ t > 0,$$

e com condições iniciais dadas $\boldsymbol{u}_0$, $\boldsymbol{w}_0$ e $\boldsymbol{b}_0$, ou seja,

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{w}(\boldsymbol{x},0) = \boldsymbol{w}_0(\boldsymbol{x}), \quad \boldsymbol{b}(\boldsymbol{x},0) = \boldsymbol{b}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3.$$

Os dados iniciais para os campos velocidade e magnético satisfazem div $\boldsymbol{u}_0 =$ div $\boldsymbol{b}_0 = 0$.

Observe que o sistema (1.1) inclui, como caso particular, as clássicas equações de Navier-Stokes ($\boldsymbol{w} = \boldsymbol{b} = \boldsymbol{g} = \mathbf{0}$ e $\chi = 0$), as equações da magneto-hidrodinâmica (MHD, i.e., $\boldsymbol{w} = \boldsymbol{g} = \mathbf{0}$ e $\chi = 0$) e as equações micropolares ($\boldsymbol{b} = \mathbf{0}$).

Os símbolos $\Delta, \nabla, \mathrm{rot}$ e div denotam, respectivamente, os operadores laplaciano, gradiente, rotacional e divergente; $\boldsymbol{u}_t, \boldsymbol{w}_t$ e $\boldsymbol{b}_t$ são as derivadas com relação ao tempo de $\boldsymbol{u}, \boldsymbol{w}$ e $\boldsymbol{b}$, respectivamente. A i-esima componente de $(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}$ e $(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}$, são dadas, respectivamente, por

$$[(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}]_i = \sum_{j=1}^{3} u_j \frac{\partial v_i}{\partial x_j} \qquad \text{e} \qquad [(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}]_i = \sum_{j=1}^{3} b_j \frac{\partial v_i}{\partial x_j}.$$

Existem muitos resultados de existência, unicidade e regularidade de solução para problemas relacionados a fluidos micropolares (veja, por exemplo, as referências [1–20]).

Com respeito as equações da magneto-hidrodinâmica, vale ressaltar que no artigo [21], J. Wu estabeleceu critérios de regularidade globais para as soluções das equações MHD 3D. Um dos critérios foi o seguinte: se a solução $(\boldsymbol{u}, \boldsymbol{b})$ das equações MHD com dados iniciais $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in \boldsymbol{H}^s \times \boldsymbol{H}^s$, com $s \geq 3$, satisfaz a condição

$$\nabla\boldsymbol{u},\ \nabla\boldsymbol{b} \in L^4\left(0,T;\boldsymbol{L}^2(\mathbb{R}^3)\right),$$

então a solução $(\boldsymbol{u}, \boldsymbol{b})$ permanece suave sobre $[0,T]$. No Capítulo 3, nos inspiramos nesse critério a fim de estudar as soluções globais do problema (1.1).

A seguir, faremos uma breve exposição de alguns trabalhos conhecidos sobre o sistema magneto-micropolar.

Com relação ao problema (1.1), no artigo [22], Ahmadi e Shahinpoor estudaram a estabilidade das soluções. Em 1998, Rojas-Medar e Boldrini [23] estudaram a existência de soluções fracas para o sistema (1.1) considerando domínios limitados em $\mathbb{R}^2$ e $\mathbb{R}^3$. Ademais, no caso 2D, eles provaram a unicidade da solução fraca. Em [24], Guterres *et. al.* provaram, para as soluções globais de Leray, que $\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t)\|_{\boldsymbol{L}^2(\mathbb{R}^n)} \underset{t\to\infty}{\longrightarrow} 0$ com dados iniciais em $\boldsymbol{L}^2(\mathbb{R}^n)$, onde $n = 2$ ou 3. Eles mostraram, também, um decaimento mais rápido para a velocidade micro-rotacional. A saber, $\|\boldsymbol{w}(\cdot,t)\|_{\boldsymbol{L}^2(\mathbb{R}^n)} = o(t^{-1/2})$. No trabalho [25], Braz e

Silva *et. al* obtiveram um resultado de decaimento exponencial em um domínio limitado com fronteira C^2 compacta. Em [26], Ortega-Torres e Rojas-Medar, utilizando argumentos de Serrin, demonstraram a unicidade da solução fraca. No artigo [27], usando o método espectral de Galerkin, Rojas-Medar provou a existência e a unicidade local de soluções fortes. Em 1999, Ortega-Torres e Rojas-Medar [28] provaram, em um domínio limitado tridimensional, um resultado sobre existência global no tempo de soluções fortes assumindo que os dados iniciais são pequenos e que as forças externas decaem com o tempo. Por sua vez, no artigo [29], Ortega-Torres e colaboradores obtiveram estimativas de erro, uniformes no tempo, ótimas nas normas $\boldsymbol{H}^1$ e $\boldsymbol{L}^2$ para a velocidade, a micro-rotação e o campo magnético.

Nesta tese estudamos, primeiramente, a existência e unicidade de solução forte global para o sistema de equações (1.1). Tais resultados foram basedos nos trabalhos de Cruz [16], para as equações dos fluidos micropolares 3D, e Zhong [30], para as equações de Navier-Stokes com amortecimento (*damping*).

Em um segundo momento, estabelecemos uma taxa de decaimento algébrica na norma $\boldsymbol{L}^2$ para as soluções fracas do problema (1.1). Para este fim, começamos aplicando o método de decomposição de Fourier (*"Fourier splitting method"*), e, em seguida, através de um argumento direto usando o princípio de Duhamel, provamos que a velocidade micro-rotacional decai mais rapidamente. Neste caso, obtivemos os mesmos resultados de [15], onde os autores consideraram as equações micropolares. A motivação para este estudo data de 1934 quando Jean Leray propôs o problema de provar que as soluções fracas das equações de Navier-Stokes decaem para zero, em $\boldsymbol{L}^2$, quando o tempo t tende ao infinito (veja [31]), ou seja, provar que

$$\lim_{t\to\infty} \|\boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^2(\mathbb{R}^3)} = 0.$$

Em 1984, [32] provou o problema de Leray usando estimativas de semigrupo. Tosio Kato em [32] provou que

$$\lim_{t\to\infty} t^{\frac{1}{4}-\epsilon}\|\boldsymbol{u}(\cdot,t) - e^{\Delta t}\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\mathbb{R}^3)} = 0,$$

para cada $\epsilon > 0$. Michael Wiegner [33] provou esse problema em 1987, para $\epsilon = 0$. Em 1984, Kyúya Masuda [34] introduziu de forma rudimentar a técnica agora conhecida como método de decomposição de Fourier. Em 1985, María Schonbek em [35] aperfeiçoou essa técnica na qual foi aplicada pela primeira vez no contexto das leis de conservação parabólica [36]. Tal técnica consiste em dividir a região de integração sobre o espaço em dois domínios dependentes do tempo. Vale ressaltar que no artigo [37], utilizando o método de decomposição de Fourier, os autores obtiveram taxas de decaimento para a solução fraca do problema (1.1) quando $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$. Aqui no nosso trabalho, além de considerarmos os casos em que as forças externas não são necessariamente nulas, também provamos que a velocidade angular $\boldsymbol{w}$ decai mais rapidamente que os campos $\boldsymbol{u}$ e $\boldsymbol{b}$.

Neste trabalho, os resultados estão organizados do seguinte modo. No Capítulo 2, apresentamos alguns conceitos importantes de espaço de funções, em particular, os espaços de

Lebesgue L^p e os espaços de Sobolev $W^{m,p}$, bem como algumas desigualdades nesses espaços. Em seguida, apresentamos alguns resultados importantes utilizados no decorrer da tese.

No Capítulo 3 estudamos a existência e unicidade de solução forte global do sistema (1.1),considerando as forças externas nulas. Primeiramente, baseado em estimativas de energia, mostramos a existência e unicidade de solução forte local para o problema em questão. Após isso, provamos que de fato a solução é global.

No Capítulo 4 estabelecemos formalmente as taxas de decaimento temporal para as soluções fracas do sistema (1.1).

Por fim, no Capítulo 5 obtivemos os mesmos resultados do Capítulo 4, entretanto, utilizamos argumentos rigorosos através de funções regularizantes (*mollifiers*).

2 PRELIMINARES

Iniciamos nosso trabalho introduzindo alguns conceitos e resultados auxiliares usados nesta tese. Para mais informações, veja [38], [39], [40], [41] , [42] e [43]. Funções com valores em $\mathbb{R}^m$, bem como seus espaços e pontos, são representados com letras em negrito. No que segue, denotaremos por Ω um subconjunto aberto do $\mathbb{R}^n$.

2.1 Espaços funcionais

Definição 2.1. *Seja $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ um multi-índice. Denotamos por D^α o operador diferencial de ordem*

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

isto é,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Definição 2.2. *Seja $u : \Omega \to \mathbb{R}$ uma função contínua. O suporte de u é o conjunto*

$$\operatorname{supp} u = \overline{\{\boldsymbol{x} \in \Omega;\, u(\boldsymbol{x}) \neq 0\}}.$$

Definição 2.3. *O espaço vetorial $C_0^\infty(\Omega)$ consiste em todas as funções em $C^\infty(\Omega)$ que tem suporte compacto em Ω.*

Definição 2.4. *A sequência $\{\phi_m\}_{m=1}^\infty \subset C_0^\infty(\Omega)$ se diz convergente para zero se as seguintes condições são satisfeitas:*

i. Existe um conjunto compacto K tal que $\operatorname{supp} \phi_m \subset K$, para todo $m \in \mathbb{N}$;

ii. Para cada $\alpha \in \mathbb{N}^n$, a sequência $\left\{D^\alpha \phi_m\right\}_{m=1}^\infty$ converge para zero uniformemente em K.

Para um dado $\phi \in C_0^\infty(\Omega)$, dizemos que a sequência $\{\phi_m\}_{m=1}^\infty \subset C_0^\infty(\Omega)$ converge para ϕ em $C_0^\infty(\Omega)$ se $\{\phi_m - \phi\}_{m=1}^\infty$ converge para zero no sentido acima.

Definição 2.5. *O espaço vetorial $C_0^\infty(\Omega)$ equipado com essa noção de convergência é denotado por $\mathcal{D}(\Omega)$ e é conhecido como espaço das funções testes em Ω.*

Definição 2.6. *Uma distribuição em Ω é um funcional linear contínuo $T : \mathcal{D}(\Omega) \to \mathbb{R}$.*

2.1.1 Os espaços L^p

Definição 2.7. *Seja $u : \Omega \to \mathbb{R}$ uma função mensurável e $1 \leq p < \infty$. Denotamos por $L^p(\Omega)$ o espaço de Banach*

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}; u \text{ é mensurável e } \|u\|_p < \infty\},$$

onde

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u(\boldsymbol{x})|^p \, d\boldsymbol{x} \right)^{1/p}.$$

Observação 2.8. *Se $p = 2$, o espaço $L^2(\Omega)$ é um espaço de Hilbert com produto interno*

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x}, \qquad u,\, v : \Omega \to \mathbb{R}.$$

Definição 2.9. *Uma função $u : \Omega \to \mathbb{R}$ que é mensurável em Ω é dita essencialmente limitada em Ω se existe uma constante $C \in \mathbb{R}^+$ tal que $|u(\boldsymbol{x})| < C$ quase sempre (q.s.) em Ω.*

Observação 2.10. *Usamos a notação q.s. em Ω para indicar que determinada propriedade é válida para todo $\boldsymbol{x} \in \Omega$, exceto, possivelmente, para algum $\boldsymbol{x}$ pertencente a algum subconjunto de Ω com medida nula.*

Definição 2.11. *Chama-se supremo essencial de u ao ínfimo do conjunto*

$$\{C \in \mathbb{R}^+ \,;\, |u(x)| \leq C \ \textit{q.s em } \Omega\}$$

e denotamos por

$$\sup_{\boldsymbol{x}\in\Omega} \operatorname{ess} |u(\boldsymbol{x})|.$$

Definição 2.12. *Para uma função mensurável $u : \Omega \to \mathbb{R}$, denotamos por $L^\infty(\Omega)$ o espaço de Banach*

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \,;\, u \textit{ é mensurável e } \|u\|_\infty < \infty\},$$

onde

$$\|u\|_{L^\infty(\Omega)} = \|u\|_\infty = \inf\{C \in \mathbb{R}^+; |u(\boldsymbol{x})| < C \ \textit{q.s. em } \ \Omega\} = \sup_{\boldsymbol{x}\in\Omega} \operatorname{ess} |u(\boldsymbol{x})|.$$

Definição 2.13. *Seja $1 \leq p \leq \infty$. Dizemos que a função $u : \Omega \to \mathbb{R}$ pertence a $L^p_{loc}(\Omega)$ se $u|_K \in L^p(\Omega)$ para todo compacto $K \subset \Omega$.*

Lema 2.14 (Desigualdade generalizada de Young, [43])**.** *Sejam a, b dois números reais positivos. Considere $1 < p, q < \infty$ tais que $p^{-1} + q^{-1} = 1$ (ou seja, q é o expoente conjugado de p). Então, para todo $\epsilon > 0$, temos*

$$ab \leq \epsilon a^p + C_\epsilon b^q,$$

onde $C_\epsilon = \dfrac{p-1}{p^q}\epsilon^{1-q}$.

Lema 2.15 (Desigualdade de Young para convolução, [44])**.** *Sejam $u \in L^p(\mathbb{R}^n)$ e $v \in L^q(\mathbb{R}^n)$. Então $\|u * v\|_r \leq \|u\|_p \, \|v\|_q$, onde $1 \leq p, q, r \leq \infty$ e $p^{-1} + q^{-1} = r^{-1} + 1$.*

Lema 2.16 (Desigualdade generalizada de Hölder, [45]). *Sejam $1 \leq p_1, \cdots, p_k \leq \infty$. Assuma que $u_1, \cdots, u_k$ são funções tais que, para $i = 1, \cdots, k$, tem-se*

1. $u_i \in L^{p_i}(\Omega)$;
2. $p^{-1} = \sum_{i=1}^{k} p_i^{-1} \leq 1$.

Então $u = \prod_{i=1}^{k} u_i \in L^p(\Omega)$ *e* $\|u\|_p \leq \prod_{i=1}^{k} \|u_i\|_{p_i}$.

Em particular, temos o seguinte Lema

Lema 2.17 (Desigualdade de Hölder, [45]). *Sejam $1 \leq p, q \leq \infty$ tais que $p^{-1} + q^{-1} = 1$. Se $u \in L^p(\Omega)$ e $v \in L^q(\Omega)$, então $uv \in L^1(\Omega)$ e*

$$\|uv\|_1 \leq \|u\|_p \|v\|_q.$$

Como consequencia da desigualdade de Hölder, obtemos a desigualdade de interpolação.

Lema 2.18 (Desigualdade de interpolação, [45]). *Seja $u \in L^p(\Omega) \cap L^q(\Omega)$ com $1 \leq p \leq q \leq \infty$. Então, $u \in L^r(\Omega)$ para todo $r \in [p, q]$ e*

$$\|u\|_r \leq \|u\|_p^{\theta} \|u\|_q^{1-\theta},$$

com

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, \quad \theta \in [0, 1].$$

Usando a desigualdade de Hölder é possível mostrar a desigualdade de Minkowski.

Lema 2.19 (Desigualdade de Minkowski, [45]). *Sejam $u, v \in L^p(\Omega)$ com $1 \leq p \leq \infty$. Então*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

2.1.2 Espaços de Bochner

Definição 2.20. *Sejam $1 \leq p < \infty$ e $\boldsymbol{X}$ um espaço de Banach com norma $\|\cdot\|_{\boldsymbol{X}}$. Denotamos por $L^p(0, T; \boldsymbol{X})$ o conjunto das aplicações $\boldsymbol{u} : [0, T] \to \boldsymbol{X}$ fortemente mensuráveis com norma*

$$\|\boldsymbol{u}\|_{L^p(0,T;\boldsymbol{X})} := \left(\int_0^T \left\| \boldsymbol{u}(t) \right\|_X^p dt \right)^{1/p} < \infty.$$

Definição 2.21. *O espaço $L^\infty(0, T; \boldsymbol{X})$ é definido como o conjunto das aplicações $\boldsymbol{u} : [0, T] \to X$ fortemente mensuráveis com norma*

$$\|\boldsymbol{u}\|_{L^\infty(0,T;\boldsymbol{X})} = \sup_{t \in [0,T]} \operatorname{ess} \|\boldsymbol{u}(t)\|_{\boldsymbol{X}} < \infty.$$

Observação 2.22. *O espaço* $L^p(0,T;\boldsymbol{X})$ *é um espaço de Banach com norma* $\|\boldsymbol{u}\|_{L^p(0,T;\boldsymbol{X})}$, *para todo* $1 \leq p \leq \infty$.

A seguir, enunciaremos dois resultados muito importantes.

Lema 2.23 (Lema 1.2, cap 3, [42]). *Sejam* $\boldsymbol{V}$ *e* $\boldsymbol{H}$ *dois espaços de Hilbert e* $\boldsymbol{V}'$ *e* $\boldsymbol{H}'$ *seus respectivos duais, com* $\boldsymbol{V} \hookrightarrow \boldsymbol{H} \equiv \boldsymbol{H}' \hookrightarrow \boldsymbol{V}'$, *onde as injeções são contínuas. Se a função* $\boldsymbol{u}$ *é tal que* $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ *e* $\boldsymbol{u}_t \in L^2(0,T;\boldsymbol{V}')$, *então* u *é quase sempre igual a uma função contínua de* $[0,T]$ *em* $\boldsymbol{H}$ *e a igualdade*

$$\frac{d}{dt}\|\boldsymbol{u}\|^2 = 2\langle \boldsymbol{u}_t, \boldsymbol{u}\rangle_{\boldsymbol{V}',\boldsymbol{V}}$$

vale no sentido das distribuições sobre $(0,T)$.

Lema 2.24 (Aubin-Lions, Teorema 1.2, capítulo 3, [42]). *Sejam* $\boldsymbol{X}_0, \boldsymbol{X}, \boldsymbol{X}_1$ *espaços de Banach com* $\boldsymbol{X}_0$ *e* $\boldsymbol{X}_1$ *reflexivos. Suponha que as injeções* $\boldsymbol{X}_0 \hookrightarrow \boldsymbol{X} \hookrightarrow \boldsymbol{X}_1$ *sejam contínuas, e que a injeção* $\boldsymbol{X}_0 \hookrightarrow \boldsymbol{X}$ *seja compacta. Para* $T > 0$ *e* $\alpha_0, \alpha_1 \in [1,\infty]$, *seja*

$$\mathcal{Y} := \{\boldsymbol{v} \in L^{\alpha_0}(0,T;\boldsymbol{X}_0) \text{ tal que } \boldsymbol{v}_t \in L^{\alpha_1}(0,T;\boldsymbol{X}_1)\},$$

com norma definida por

$$\|\boldsymbol{v}\|_{\mathcal{Y}} := \|\boldsymbol{v}\|_{L^{\alpha_0}(0,T;\boldsymbol{X}_0)} + \|\boldsymbol{v}_t\|_{L^{\alpha_1}(0,T;\boldsymbol{X}_1)}.$$

Então, $\mathcal{Y}$ *é um espaço de Banach e a injeção* $\mathcal{Y} \hookrightarrow L^{\alpha_0}(0,T;\boldsymbol{X})$ *é compacta.*

2.1.3 Espaços de Sobolev

Definição 2.25. *Sejam* $u, v \in L^1_{loc}(\Omega)$ *e* $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ *um milti-indice de ordem* $|\alpha| = \alpha_1 + \cdots + \alpha_n$. *Dizemos que* v *é uma derivada parcial fraca de* u *de ordem* α, *denotada por*

$$D^\alpha u = v,$$

se

$$\int_\Omega u(\boldsymbol{x})D^\alpha \phi(\boldsymbol{x})\,d\boldsymbol{x} = (-1)^{|\alpha|}\int_\Omega v(\boldsymbol{x})\,\phi(\boldsymbol{x})\,d\boldsymbol{x},$$

para toda função teste $\phi \in C_0^\infty(\Omega)$.

Definição 2.26. *Sejam* $1 \leq p \leq \infty$ *e* m *um número natural. O espaço de Sobolev* $W^{m,p}(\Omega)$ *consiste em todas as distribuições* $u \in \mathcal{D}(\Omega)$ *tal que* $u \in L^p(\Omega)$ *e para cada multi-indice* α *com* $|\alpha| \leq m$, *a derivada* $D^\alpha u$ *existe no sentido fraco e pertence a* $L^p(\Omega)$. *Se* $u \in W^{m,p}(\Omega)$, *definimos sua norma por*

$$\|u\|_{W^{m,p}} := \left(\sum_{|\alpha|\leq m}\int_\Omega |D^\alpha\, u|^p\, d\boldsymbol{x}\right)^{1/p}, \quad 1 \leq p < \infty,$$

e

$$\|u\|_{W^{m,\infty}} = \sum_{|\alpha|\leq m} \sup_{\Omega}\text{ess}\, |D^\alpha u|.$$

Observação 2.27. *Se $p = 2$ e $m \in \mathbb{N}$, o espaço de Sobolev $W^{m,2}(\Omega)$ é usualmente denotado por $H^m(\Omega)$. Pode-se provar que $H^m(\Omega)$ é um espaço de Hilbert com a norma associada ao produto interno*

$$\langle u, v\rangle = \sum_{|\alpha|\leq m} (D^\alpha u, D^\alpha v),$$

onde $(\cdot,\cdot)$ é o produto interno em $L^2(\Omega)$.

Definição 2.28. *Sejam $1 \leq p \leq \infty$ e m um inteiro positivo. O espaço $W_0^{m,p}(\Omega)$ é o fecho de $C_0^\infty(\Omega)$ em $W^{m,p}(\Omega)$. Se $p = 2$, escrevemos $H_0^m(\Omega) = W^{m,2}(\Omega)$.*

Definição 2.29. *Sejam $1 \leq p \leq \infty$, $1 < q \leq \infty$ tais que $p^{-1} + q^{-1} = 1$ e $m \in \mathbb{N}$. O espaço $W^{-m,q}(\Omega)$ é o dual topológico de $W_0^{m,p}(\Omega)$. Se $q = 2$, escrevemos $H^{-m}(\Omega) = W^{-m,2}(\Omega)$.*

Lema 2.30 (Lema 6.7, Cap. 6, [40])**.** *Se Ω é um subconjunto aberto e limitado de $\mathbb{R}^3$ e $\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \cap L^\infty(0,T,\boldsymbol{L}^2(\Omega))$, então $\boldsymbol{u} \in L^4(0,T;\boldsymbol{L}^3(\Omega))$.*

2.2 Resultados técnicos

Definição 2.31. *Se $u \in L^1(\mathbb{R}^n)$, definimos sua transformada de Fourier por*

$$\mathcal{F}\{u\}(\boldsymbol{\xi}) = \widehat{u}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\, u(\boldsymbol{x})\, d\boldsymbol{x}, \qquad \boldsymbol{\xi} \in \mathbb{R}^n,$$

onde $i^2 = -1$.

Teorema 2.32 (Teorema de Plancherel)**.** *Se $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, então $\widehat{u} \in L^2(\mathbb{R}^n)$ e*

$$\|\widehat{u}\|_2 = (2\pi)^{n/2}\, \|u\|_2. \tag{2.1}$$

Demonstração. Observe que se $v \in L^1(\mathbb{R}^n)$, então $\widehat{v} \in L^\infty(\mathbb{R}^n)$. Com efeito,

$$\begin{aligned} |\widehat{v}(\boldsymbol{\xi})| &= \left|\int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} v(\boldsymbol{x})\, d\boldsymbol{x}\right| \\ &\leq \int_{\mathbb{R}^n} |e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} v(\boldsymbol{x})|\, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} |v(\boldsymbol{x})|\, d\boldsymbol{x} < \infty. \end{aligned}$$

Logo, se $v, w \in L^1(\mathbb{R}^n)$, então $\widehat{v}, \widehat{w} \in L^\infty(\mathbb{R}^n)$. Ademais, temos que

$$\int_{\mathbb{R}^n} v(\boldsymbol{x})\, \widehat{w}(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\mathbb{R}^n} \widehat{v}(\boldsymbol{\xi})\, w(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{2.2}$$

Um cálculo direto nos leva a seguinte identidade:

$$\int_{\mathbb{R}^n} e^{i\,\boldsymbol{\xi}\cdot\boldsymbol{x}-t|\boldsymbol{x}|^2}\,d\boldsymbol{x} = \left(\frac{\pi}{t}\right)^{n/2} e^{-\frac{|\boldsymbol{\xi}|^2}{4t}}, \qquad \forall\, t>0.$$

Logo, por (2.2), temos, para cada $\varepsilon > 0$, que

$$\int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi}) e^{-\varepsilon|\boldsymbol{\xi}|^2}\,d\boldsymbol{\xi} = \left(\frac{\pi}{\varepsilon}\right)^{n/2} \int_{\mathbb{R}^n} w(\boldsymbol{x}) e^{-\frac{|\boldsymbol{x}|^2}{4\varepsilon}}\,d\boldsymbol{x}. \tag{2.3}$$

Agora, seja $u \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ e considere a função $v(\boldsymbol{x}) := \overline{u}(-\boldsymbol{x})$, onde $\overline{u}$ é o conjugado complexo de u. Defina $w \in L^1(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ da seguinte maneira:

$$w(\boldsymbol{y}) = (2\pi)^n (u*v)(\boldsymbol{y}) = (2\pi)^n \int_{\mathbb{R}^n} u(\boldsymbol{x})\, v(\boldsymbol{y}-\boldsymbol{x})\,d\boldsymbol{x}.$$

Temos, então, que

$$\begin{aligned}
\widehat{w}(\boldsymbol{\xi}) &= (2\pi)^n \widehat{(u*v)}(\boldsymbol{\xi}) \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \int_{\mathbb{R}^n} u(\boldsymbol{y})\, v(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{x} \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{y}}\, u(\boldsymbol{y}) \left(\int_{\mathbb{R}^n} e^{-i\,(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\xi}}\, v(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{x} \right) d\boldsymbol{y} \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{y}}\, u(\boldsymbol{y})\, \widehat{v}(\boldsymbol{\xi})\,d\boldsymbol{y} \\
&= (2\pi)^n\, \widehat{u}(\boldsymbol{\xi})\, \widehat{v}(\boldsymbol{\xi}).
\end{aligned}$$

Assim, $\widehat{w} \in L^\infty(\mathbb{R}^n)$. Note que,

$$\widehat{v}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\, \overline{u}(-\boldsymbol{x})\,d\boldsymbol{x} = \overline{\widehat{u}}(\boldsymbol{\xi}).$$

Logo, $\widehat{w} = (2\pi)^n|\widehat{u}|^2$. Como w é contínua, então

$$\lim_{\varepsilon\to 0} \left(\frac{\pi}{\varepsilon}\right)^{n/2} \int_{\mathbb{R}^n} w(\boldsymbol{x}) e^{-\frac{|\boldsymbol{x}|^2}{4\varepsilon}}\,d\boldsymbol{x} = (2\pi)^n w(\mathbf{0}).$$

Pela identidade (2.3), concluímos que

$$\int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi})\,d\boldsymbol{\xi} = (2\pi)^n w(\mathbf{0}).$$

Portanto,

$$\int_{\mathbb{R}^n} |\widehat{u}|^2\,d\boldsymbol{\xi} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi})\,d\boldsymbol{\xi} = w(\mathbf{0}) = (2\pi)^n \int_{\mathbb{R}^n} u(\boldsymbol{x})v(-\boldsymbol{x})\,d\boldsymbol{x} = (2\pi)^n \int_{\mathbb{R}^n} |u|^2\,d\boldsymbol{x},$$

obtendo, assim, o resultado desejado. □

Observação 2.33. *A identidade* (2.1) *também é conhecida como identidade de Parseval.*

Lema 2.34 (Desigualdade de Hausdorff-Young, Cap.V, [41])**.** *Se $\varphi \in L^q(\mathbb{R}^n)$, com $1 \leq q \leq 2$, então*

$$\|\widehat{\varphi}\|_r \leq C\,\|\varphi\|_q,$$

onde $q^{-1} + r^{-1} = 1$, para alguma constante $C = C(q,r) \in \mathbb{R}^+$.

Lema 2.35 (Desigualdades de Sobolev-Nirenberg-Gagliardo)**.** *Para todo $v \in H^2(\mathbb{R}^3)$, temos*

$$\|\nabla v\|_2 \leq \|v\|_2^{1/2} \|D^2\, v\|_2^{1/2} \tag{2.4}$$

e

$$\|v\|_\infty \leq \|v\|_2^{1/4} \|D^2\, v\|_2^{3/4}. \tag{2.5}$$

Demonstração. A seguir, demonstraremos apenas a desigualdade (2.4). Usando a definição de transformada de Fourier, o teorema de Plancherel e a desigualdade de Hölder, obtemos

$$\begin{aligned}
\|\nabla v\|_2^2 &= (2\pi)^{-3}\|\widehat{\nabla v}\|_2^2 = (2\pi)^{-3}\int_{\mathbb{R}^3} |\widehat{\nabla v}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \\
&= (2\pi)^{-3}\int_{\mathbb{R}^3} |i\,\boldsymbol{\xi}\,\widehat{v}|^2\, d\boldsymbol{\xi} = (2\pi)^{-3}\int_{\mathbb{R}^3} |\boldsymbol{\xi}|^2\,|\widehat{v}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \\
&= (2\pi)^{-3}\int_{\mathbb{R}^3} |\widehat{v}(\boldsymbol{\xi})|\,|\boldsymbol{\xi}|^2\,|\widehat{v}(\boldsymbol{\xi})|\, d\boldsymbol{\xi} = (2\pi)^{-3}\int_{\mathbb{R}^3} |\widehat{v}(\boldsymbol{\xi})|\,|\widehat{D^2\, v}(\boldsymbol{\xi})|\, d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^3} \Big((2\pi)^{-\frac{3}{2}}|\widehat{v}(\boldsymbol{\xi})|\Big)\Big((2\pi)^{-\frac{3}{2}}|\widehat{D^2\, v}(\boldsymbol{\xi})|\Big)\, d\boldsymbol{\xi} \\
&\leq \left(\int_{\mathbb{R}^3} (2\pi)^{-3}|\widehat{v}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} (2\pi)^{-3}|\widehat{D^2\, v}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}\right)^{\frac{1}{2}} \\
&= \Big((2\pi)^{-\frac{3}{2}}\|\widehat{v}\|_2\Big)\Big((2\pi)^{-\frac{3}{2}}\|\widehat{D^2\, v}\|_2\Big) \\
&= \|v\|_2\,\|D^2\, v\|_2.
\end{aligned}$$

A demonstração da desigualdade (2.5) pode ser encontrada em [46], pág. 51, Teorema 4.5.1 (veja também o artigo [47]). □

Observação 2.36. *Como observado em [48], se uma desigualdade for satisfeita para funções escalares u com uma constante $\widetilde{K} > 0$, então ela também é válida para funções vetoriais $\boldsymbol{u}$ com a mesma constante $\widetilde{K}$ do caso escalar.*

Definição 2.37. *Uma função $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ é denominada um núcleo de Calderón-Zygmund se satisfaz as seguintes condições:*

(i) $|K(\boldsymbol{x},\boldsymbol{y})| \leq \dfrac{C}{\|\boldsymbol{x}-\boldsymbol{y}\|^n}, \quad (\boldsymbol{x} \neq \boldsymbol{y})$;

(ii) $|K(\boldsymbol{x},\boldsymbol{y}) - K(\bar{\boldsymbol{x}},\boldsymbol{y})| + |K(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\bar{\boldsymbol{y}})| \leq \dfrac{C}{\|\boldsymbol{x}-\boldsymbol{y}\|^{n+1}}, \quad (\boldsymbol{x} \neq \boldsymbol{y})$;

(iii) $K(\boldsymbol{x},\boldsymbol{y}) = -K(\boldsymbol{y},\boldsymbol{x})$,

para alguma constante $C \in \mathbb{R}^+$.

Lema 2.38 (Teorema de Calderón-Zygmund, [41]). *Sejam* $u : \mathbb{R}^n \to \mathbb{R}$ *uma função e* $\mathcal{T}$ *um operador integral singular, i.e.,*

$$\mathcal{T}(u)(\boldsymbol{x}) = \int_{\mathbb{R}^n} K(\boldsymbol{x}, \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y},$$

onde K *é um núcleo de Calderón-Zygmund. Se* $u \in L^p(\mathbb{R}^n)$*, com* $1 < p < \infty$*, então* $\mathcal{T}(u) \in L^p(\mathbb{R}^n)$ *e*

$$\|\mathcal{T}(u)\|_p \leq C\, \|u\|_p,$$

onde $C = C(p, n) > 0$.

Lema 2.39 (Proposição 3.2, pág. 1431, [17]). *Seja* M *uma matriz Hermitiana com todos os autovalores* $\{\lambda_j\}_{j=1}^n$ *positivos. Então,*

$$\|e^{-Mt}\| \leq e^{-(min_j \lambda_j)t}, \quad \forall\, t \geq 0.$$

Demonstração. Temos que M é uma matriz diagonalizável, já que M é Hermitiana. Considere, então, $\beta = \{v_1, \cdots, v_n\}$ uma base de autovetores para M com respectivos autovalores $\{\lambda_1, \cdots, \lambda_n\}$. Logo, β é também uma base de autovetores para e^{-Mt} com respectivos autovalores $\{e^{-\lambda_1 t}, \cdots, e^{-\lambda_n t}\}$. Seja $\omega \in \mathbb{C}^n$ tal que

$$\omega = \alpha_1 v_1 + \cdots + \alpha_n v_n,$$

onde $\alpha_j \in \mathbb{C}$ para todo $j = 1, \cdots, n$. Como todas as normas em $\mathbb{C}^n$ são equivalentes, considere a norma do máximo com relação à base β, i.e.,

$$\|\omega\| := \max_{j=1,\cdots,n} |\alpha_j|\,.$$

Logo, para todo $t \geq 0$, temos que

$$\left\|e^{-Mt}\omega\right\| = \left\|\sum_{j=1}^n \alpha_j e^{-\lambda_j t} v_j\right\| \leq \|\omega\| \left(\max_{j=1,\cdots,n} \left|e^{-\lambda_j t}\right|\right) = \|\omega\|\, e^{-(\min_j \lambda_j)t},$$

ou seja,

$$\|e^{-Mt}\| \leq e^{-(min_j \lambda_j)t}, \quad \forall\, t \geq 0 \quad \text{e} \quad j = 1, \cdots, n,$$

que é a desigualdade desejada. □

Definição 2.40. *O semigrupo do calor em* $\mathbb{R}^n$ *é a família de operadores* $\left(e^{\Delta t}\right)_{t \geq 0}$ *definidos da seguinte maneira:*

$$e^{\Delta 0}\, \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n$$

e

$$e^{\Delta t}\, \boldsymbol{v}(\boldsymbol{x}) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}}\, \boldsymbol{v}(\boldsymbol{y})\, d\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^n,\ t > 0.$$

Teorema 2.41. *Seja $\boldsymbol{u}(\boldsymbol{x},t)=e^{\Delta t}\boldsymbol{u}_0(x)$ a solução do PVI*

$$\begin{cases} \boldsymbol{u}_t=\Delta\boldsymbol{u}, & \boldsymbol{x}\in\mathbb{R}^n, \quad t>0, \\ \boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}), & \boldsymbol{x}\in\mathbb{R}^n, \end{cases} \tag{2.6}$$

com $\boldsymbol{u}_0\in\boldsymbol{L}^r(\mathbb{R}^n)$. Então

$$\|D^\alpha e^{\Delta t}\,\boldsymbol{u}_0\|_{\boldsymbol{L}^q}\le C\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}\,t^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}}, \qquad \forall\, t>0, \tag{2.7}$$

para quaisquer $1\le r\le q\le\infty$ e $\alpha\in\mathbb{N}^n$. A constante $C\in\mathbb{R}^+$ depende somente de r, q, n e $|\alpha|$.

Demonstração. Nesta tese, utilizaremos o resultado acima apenas nos casos em que $|\alpha|=0$ e $|\alpha|=1$. Portanto, demonstraremos apenas esses dois casos. O caso geral pode ser encontrado, por exemplo, em [49], Teorema 3.4, pág. 28.

Primeiro provaremos o caso em que $|\alpha|=0$, isto é,

$$\|e^{\Delta t}\,\boldsymbol{u}_0\|_{\boldsymbol{L}^q}\le C\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}\,t^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}, \quad \forall\, t>0. \tag{2.8}$$

Seja $\phi(\boldsymbol{x},t)$ o núcleo do calor, ou seja,

$$\phi(\boldsymbol{x},t)=\frac{1}{(4\pi t)^{n/2}}\,e^{-\frac{|\boldsymbol{x}|^2}{4t}}, \quad \boldsymbol{x}\in\mathbb{R}^n, \quad t>0.$$

Segue, então, que $\boldsymbol{u}(x,t)=\displaystyle\int_{\mathbb{R}^n}\phi(\boldsymbol{x}-\boldsymbol{y},t)\,\boldsymbol{u}_0(\boldsymbol{y})\,d\boldsymbol{y}$ é solução do PVI (2.6). Uma vez que, para cada $t>0$,

$$\int_{\mathbb{R}^n}\phi(\boldsymbol{x},t)\,d\boldsymbol{x}=1, \tag{2.9}$$

obtemos, em particular, que $\displaystyle\int_{\mathbb{R}^n}e^{-\frac{|x|^2}{4t}}\,d\boldsymbol{x}=(4\pi t)^{n/2}$, para todo $t>0$. Assim, da identidade (2.9) e da desigualdade de Young para convolução, temos, para $1\le r\le\infty$, que

$$\|\boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^r}\le\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}, \quad \forall\, t\ge 0. \tag{2.10}$$

De fato,

$$\|\boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^r}=\|\phi(\cdot,t)*\boldsymbol{u}_0\|_{\boldsymbol{L}^r}\le\|\phi(\cdot,t)\|_{\boldsymbol{L}^1}\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}=\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}.$$

Afirmação: Para $1\le r<\infty$, temos

$$\|\boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^\infty}\le(4\pi t)^{-\frac{n}{2r}}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}. \tag{2.11}$$

Com efeito, para todo $t>0$, temos que

$$|\boldsymbol{u}(\boldsymbol{x},t)|\le\int_{\mathbb{R}^n}\phi(\boldsymbol{x}-\boldsymbol{y},t)|\boldsymbol{u}_0(\boldsymbol{y})|\,d\boldsymbol{y}\le(4\pi t)^{-n/2}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^1},$$

pois $\phi(\boldsymbol{x},t) \leq (4\pi t)^{-n/2}$, para todo $(\boldsymbol{x},t) \in \mathbb{R}^n \times \mathbb{R}^+$. Isto prova a desigualdade (2.11) para $r = 1$. Se $1 < r < \infty$, defina p como o expoente conjugado de r, isto é $p^{-1} + r^{-1} = 1$. Assim, da desigualdade de Hölder e de (2.9), temos

$$\begin{aligned} |\boldsymbol{u}(\boldsymbol{x},t)| &\leq \int_{\mathbb{R}^n} \phi(\boldsymbol{x}-\boldsymbol{y},t)|\boldsymbol{u}_0(\boldsymbol{y})|\,d\boldsymbol{y} \\ &= \int_{\mathbb{R}^n} \phi(\boldsymbol{x}-\boldsymbol{y},t)^{\frac{1}{p}}\,\phi(\boldsymbol{x}-\boldsymbol{y},t)^{\frac{1}{r}}\,|\boldsymbol{u}_0(\boldsymbol{y})|\,d\boldsymbol{y} \\ &\leq (4\pi t)^{-n/2r} \int_{\mathbb{R}^n} \phi(\boldsymbol{x}-\boldsymbol{y},t)^{1/p}\,|\boldsymbol{u}_0(\boldsymbol{y})|\,d\boldsymbol{y} \\ &\leq (4\pi t)^{-n/2r} \left(\int_{\mathbb{R}^n} \phi(\boldsymbol{x}-\boldsymbol{y},t)\,d\boldsymbol{y}\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |\boldsymbol{u}_0(\boldsymbol{y})|^r\,d\boldsymbol{y}\right)^{\frac{1}{r}} \\ &= (4\pi t)^{-n/2r}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}, \end{aligned}$$

provando, assim, a Afirmação.

Agora, usando (2.10) e (2.11), para $1 \leq r \leq q < \infty$, temos

$$\begin{aligned} \|\boldsymbol{u}(\cdot,t)\|^q_{\boldsymbol{L}^q} &= \int_{\mathbb{R}^n} |\boldsymbol{u}(\boldsymbol{x},t)|^q\,d\boldsymbol{x} = \int_{\mathbb{R}^n} |\boldsymbol{u}(\boldsymbol{x},t)|^{q-r}|\boldsymbol{u}(\boldsymbol{x},t)|^r\,d\boldsymbol{x} \\ &\leq \|\boldsymbol{u}(\cdot,t)\|^{q-r}_{\boldsymbol{L}^\infty}\|\boldsymbol{u}(\cdot,t)\|^r_{\boldsymbol{L}^r} \\ &\leq (4\pi t)^{-\frac{n}{2r}(q-r)}\|\boldsymbol{u}_0\|^{q-r}_{\boldsymbol{L}^r}\|\boldsymbol{u}_0\|^r_{\boldsymbol{L}^r} \\ &= (4\pi t)^{-\frac{n}{2}(\frac{q}{r}-1)}\|\boldsymbol{u}_0\|^q_{\boldsymbol{L}^r}. \end{aligned}$$

O caso em que $q = \infty$, segue diretamente de (2.11). Isto prova a estimativa (2.8).

Agora, demonstraremos, o caso em que $|\alpha| = 1$, i.e.,

$$\|e^{\Delta t}\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^q} \leq C\|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}\,t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}}, \quad \forall\, t > 0.$$

Inicialmente, provaremos que

$$\int_{\mathbb{R}^n} |D^\alpha\phi(\boldsymbol{x},t)|\,d\boldsymbol{x} = \widetilde{C}\,t^{-|\alpha|/2}, \tag{2.12}$$

onde $\widetilde{C} = \widetilde{C}(\alpha,n) > 0$. Considere a função $g : \mathbb{R}^n \to \mathbb{R}^+$ dada por $g(\boldsymbol{x}) := e^{-|\boldsymbol{x}|^2/4}$. Note que $D^\alpha g(\boldsymbol{x}) = p_{|\alpha|}(\boldsymbol{x})g(\boldsymbol{x})$, onde $p_{|\alpha|}(\boldsymbol{x})$ é um polinômio de grau $|\alpha| > 0$. Como $0 < g(\boldsymbol{x}) \leq 1$, para todo $\boldsymbol{x} \in \mathbb{R}^n$, segue que $D^\alpha g(\boldsymbol{x})$ é uma função limitada e

$$\int_{\mathbb{R}^n} |D^\alpha g(\boldsymbol{x})|\,d\boldsymbol{x} =: K(\alpha,n) < \infty.$$

Como

$$\phi(\boldsymbol{x},t) = (4\pi t)^{-n/2}g(t^{-1/2}\boldsymbol{x}),$$

segue, pela regra da cadeia, que

$$D^\alpha\phi(\boldsymbol{x},t) = (4\pi t)^{-n/2}t^{-|\alpha|/2}D^\alpha g(t^{-1/2}\boldsymbol{x}).$$

Assim,

$$
\begin{aligned}
\int_{\mathbb{R}^n} |D^\alpha \phi(\boldsymbol{x},t)|\, d\boldsymbol{x} &= (4\pi t)^{-n/2}\, t^{-|\alpha|/2} \int_{\mathbb{R}^n} |D^\alpha g(t^{-1/2}\boldsymbol{x})|\, d\boldsymbol{x} \\
&= (4\pi t)^{n/2}\, t^{-|\alpha|/2}\, t^{n/2} \int_{\mathbb{R}^n} |D^\alpha g(\boldsymbol{y})|\, d\boldsymbol{y} \\
&= (4\pi)^{n/2}\, t^{-|\alpha|/2} K(\alpha,n) = \widetilde{C}(\alpha,n)\, t^{-|\alpha|/2},
\end{aligned}
$$

onde $\widetilde{C}(\alpha,n) = (4\pi)^{-n/2} K(\alpha,n)$. Isto prova (2.12).

Fazendo uso da identidade (2.12) e da desigualdade de Young para convolução, temos, para $1 \le r \le \infty$, que

$$
\|D^\alpha \boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^r} \le \widetilde{C}\, t^{-|\alpha|/2} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}, \quad \forall\, t > 0, \tag{2.13}
$$

onde $\widetilde{C} = \widetilde{C}(\alpha,n) > 0$. De fato,

$$
\begin{aligned}
\|D^\alpha \boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^r} &= \|D^\alpha \phi(\cdot,t) * \boldsymbol{u}_0\|_{\boldsymbol{L}^r} \\
&\le \|D^\alpha \phi(\cdot,t)\|_{\boldsymbol{L}^1} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r} = \widetilde{C}\, t^{-|\alpha|/2} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r},
\end{aligned}
$$

para todo $t > 0$, onde $\widetilde{C} = \widetilde{C}(\alpha,n) > 0$. Como

$$
\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}} \boldsymbol{u}_0(y)\, d\boldsymbol{y},
$$

para todo $(\boldsymbol{x},t) \in \mathbb{R}^n \times \mathbb{R}^+$, segue, derivando $\boldsymbol{u}$ em relação a $\boldsymbol{x}_l$ para $l = 1, \cdots, n$, que

$$
\begin{aligned}
D_l \boldsymbol{u}(\boldsymbol{x},t) &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \left(-\frac{1}{4t}\right) \frac{\partial}{\partial x_l}(|\boldsymbol{x}-\boldsymbol{y}|^2)\, e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}} \boldsymbol{u}_0(\boldsymbol{y})\, d\boldsymbol{y} \\
&= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \left(-\frac{1}{4t}\right) 2(\boldsymbol{x}_l - \boldsymbol{y}_l)\, e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}} \boldsymbol{u}_0(\boldsymbol{y})\, d\boldsymbol{y} \\
&= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \left(-\frac{1}{2t}\langle \boldsymbol{x}-\boldsymbol{y}, e_l\rangle\right) e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}} \boldsymbol{u}_0(\boldsymbol{y})\, d\boldsymbol{y}.
\end{aligned}
$$

Logo, pela desigualdade de Cauchy-Schwarz, temos

$$
\begin{aligned}
|D_l \boldsymbol{u}(\boldsymbol{x},t)| &\le \frac{1}{(4\pi t)^{n/2}} \frac{1}{t^{1/2}} \int_{\mathbb{R}^n} \frac{|\boldsymbol{x}-\boldsymbol{y}|}{2t^{1/2}}\, e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}} |\boldsymbol{u}_0(\boldsymbol{y})|\, d\boldsymbol{y} \\
&= \frac{t^{-1/2}}{2(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \frac{|\boldsymbol{x}-\boldsymbol{y}|}{t^{1/2}}\, e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8t}} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8t}} |\boldsymbol{u}_0(\boldsymbol{y})|\, d\boldsymbol{y} \\
&\le \frac{\widetilde{C} t^{-1/2}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8t}} |\boldsymbol{u}_0(\boldsymbol{y})|\, d\boldsymbol{y},
\end{aligned}
$$

onde $\widetilde{C} := \max\{\lambda e^{-\lambda^2/8}; \lambda > 0\} \equiv 2e^{-1/2}$. Portanto, se $p^{-1} + r^{-1} = 1$, então

$$\begin{aligned}
|\nabla \boldsymbol{u}(x,t)| &\leq \frac{\widetilde{C}\, t^{-1/2}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8tp}}\, e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8tr}} |\boldsymbol{u}_0(\boldsymbol{y})|\, d\boldsymbol{y} \\
&\leq \frac{\widetilde{C}\, t^{-1/2}}{(4\pi t)^{n/2}} \left(\int_{\mathbb{R}^n} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8t}}\, d\boldsymbol{y} \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8t}} |\boldsymbol{u}_0(\boldsymbol{y})|^r\, d\boldsymbol{y} \right)^{\frac{1}{r}} \\
&= \frac{\widetilde{C}\, t^{-1/2}}{(4\pi t)^{n/2}} (8\pi t)^{n/2p} \left(\int_{\mathbb{R}^n} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8t}} |\boldsymbol{u}_0(\boldsymbol{y})|^r\, d\boldsymbol{y} \right)^{1/r} \\
&\leq \frac{2^{n/2p}\, \widetilde{C}\, t^{-1/2}}{(4\pi t)^{n/2r}} \left(\int_{\mathbb{R}^n} |\boldsymbol{u}_0(y)|^r\, d\boldsymbol{y} \right)^{1/r} \\
&\leq \frac{2^{n/2}\, \widetilde{C}\, t^{-1/2}}{(4\pi t)^{n/2r}} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r},
\end{aligned}$$

para todo $(\boldsymbol{x},t) \in \mathbb{R}^n \times \mathbb{R}^+$. Assim,

$$\|\nabla \boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^\infty} \leq \frac{2^{n/2}\, \widetilde{C}}{(4\pi)^{n/2r}} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}\, t^{-\frac{n}{2r}-\frac{1}{2}}, \quad \forall\, t > 0. \tag{2.14}$$

Além disso,

$$|\nabla \boldsymbol{u}(\boldsymbol{x},t)|^r \leq \frac{2^{nr/2}\, \widetilde{C}^r\, t^{-r/2}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8t}} |\boldsymbol{u}_0(\boldsymbol{y})|^r\, d\boldsymbol{y}, \quad \forall\, (\boldsymbol{x},t) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Daí, pelo teorema de Fubini, temos, para todo $t > 0$, que

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla \boldsymbol{u}(\boldsymbol{x},t)|^r\, d\boldsymbol{x} &\leq \frac{2^{nr/2}\, \widetilde{C}^r\, t^{-r/2}}{(4\pi\, t)^{n/2}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8t}} |\boldsymbol{u}_0(\boldsymbol{y})|^r\, d\boldsymbol{y} \right) d\boldsymbol{x} \\
&= \frac{2^{nr/2}\, \widetilde{C}^r\, t^{-r/2}}{(4\pi\, t)^{n/2}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8t}}\, d\boldsymbol{x} \right) |\boldsymbol{u}_0(y)|^r\, d\boldsymbol{y} \\
&= \frac{2^{nr/2}\, \widetilde{C}^r\, t^{-r/2}}{(4\pi\, t)^{n/2}} (8\pi t)^{n/2} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}^r = 2^{n/2}\, 2^{nr/2}\, \widetilde{C}^r\, t^{-r/2} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}^r.
\end{aligned}$$

Assim,

$$\|\nabla \boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^r} \;\leq\; 2^n \widetilde{C}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}\, t^{-1/2}, \quad \forall\, t > 0. \tag{2.15}$$

Portanto, por (2.14), (2.15) e pela desigualdade de interpolação (veja o Lema 2.18), temos, para $1 \leq r \leq q \leq \infty$, que

$$\begin{aligned}
\|\nabla \boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^q} &\leq \|\nabla \boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^r}^{r/q} \|\nabla \boldsymbol{u}(\cdot,t)\|_{\boldsymbol{L}^\infty}^{1-r/q} \\
&\leq \left(2^{nr/q}\, \widetilde{C}^{r/q}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}^{r/q}\, t^{-r/2q} \right) \frac{2^{\frac{n}{2}\left(1-\frac{r}{q}\right)}\, \widetilde{C}^{1-\frac{r}{q}}}{(4\pi)^{\frac{n}{2r}\left(1-\frac{r}{q}\right)}} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}^{1-\frac{r}{q}}\, t^{-\left(\frac{n}{2r}+\frac{1}{2}\right)\left(1-\frac{r}{q}\right)} \\
&= \frac{2^{\frac{n}{2}(1+\frac{r}{q})}\, \widetilde{C}}{(4\pi)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^r}\, t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}}, \quad \forall\, t > 0.
\end{aligned}$$

Desta forma, concluímos a demonstração do teorema para os casos em que $|\alpha| \in \{0,1\}$. $\square$

Lema 2.42. *Seja $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(\mathbb{R}^3)$. Então vale as seguintes propriedades:*

i) $\operatorname{rot}(\operatorname{rot}\boldsymbol{v}) = \nabla(\operatorname{div}\boldsymbol{v}) - \Delta\boldsymbol{v}$*;*

ii) $(\operatorname{rot}\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \operatorname{rot}\boldsymbol{v})$.

Demonstração. i)

$$
\begin{aligned}
\operatorname{rot}(\operatorname{rot}\boldsymbol{v}) &= \operatorname{rot}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) \\
&= \left(\frac{\partial}{\partial x_2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right),\right. \\
&\quad \frac{\partial}{\partial x_3}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right), \\
&\quad \left.\frac{\partial}{\partial x_1}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\right) \\
&= \left(\frac{\partial}{\partial x_2}\left(\frac{\partial v_2}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial v_1}{\partial x_3}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial v_3}{\partial x_1}\right),\right. \\
&\quad \frac{\partial}{\partial x_3}\left(\frac{\partial v_3}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial v_2}{\partial x_3}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial v_2}{\partial x_1}\right) + \frac{\partial}{\partial x_1}\left(\frac{\partial v_1}{\partial x_2}\right), \\
&\quad \left.\frac{\partial}{\partial x_1}\left(\frac{\partial v_1}{\partial x_3}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial v_3}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial v_3}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial v_2}{\partial x_3}\right)\right) \\
&= \left(\frac{\partial}{\partial x_1}\left(\frac{\partial v_2}{\partial x_2}\right) + \frac{\partial}{\partial x_1}\left(\frac{\partial v_3}{\partial x_3}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial v_1}{\partial x_3}\right),\right. \\
&\quad \frac{\partial}{\partial x_2}\left(\frac{\partial v_3}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_1}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial v_2}{\partial x_3}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial v_2}{\partial x_1}\right), \\
&\quad \left.\frac{\partial}{\partial x_3}\left(\frac{\partial v_1}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial v_2}{\partial x_2}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial v_3}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial v_3}{\partial x_2}\right)\right) \\
&= \left(\frac{\partial}{\partial x_1}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right), \frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right),\right. \\
&\quad \left.\frac{\partial}{\partial x_3}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)\right) \\
&- \left(\frac{\partial}{\partial x_1}\left(\frac{\partial v_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial v_1}{\partial x_3}\right),\right. \\
&\quad \frac{\partial}{\partial x_1}\left(\frac{\partial v_2}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial v_2}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial v_2}{\partial x_3}\right), \\
&\quad \left.\frac{\partial}{\partial x_1}\left(\frac{\partial v_3}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial v_3}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial v_3}{\partial x_3}\right)\right) \\
&= \nabla \operatorname{div}\boldsymbol{v} - \Delta\boldsymbol{v}.
\end{aligned}
$$

ii)

$$
\begin{aligned}
(\operatorname{rot} \boldsymbol{u}, \boldsymbol{v}) &= \int_{\mathbb{R}^3} \operatorname{rot} \boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \left[\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) v_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) v_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) v_3 \right] d\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \left(\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) u_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) u_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) u_3 \right) d\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \operatorname{rot} \boldsymbol{v} \cdot \boldsymbol{u}\, d\boldsymbol{x} = (\boldsymbol{u}, \operatorname{rot} \boldsymbol{v}).
\end{aligned}
$$

□

Lema 2.43. *Seja* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(\mathbb{R}^3)$, *com* $\operatorname{div} \boldsymbol{v} = 0$. *Então vale as seguintes propriedades:*

i) $((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}, \boldsymbol{u}) = 0$*;*

ii) $((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u}) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) = 0.$

Demonstração. i)

$$
\begin{aligned}
((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}, \boldsymbol{u}) &= \int_{\mathbb{R}^3} (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{u}\, d\boldsymbol{x} = \int_{\mathbb{R}^3} \sum_{k=1}^{3} \left(\sum_{j=1}^{3} v_j \frac{\partial u_k}{\partial x_j} \cdot u_k \right) d\boldsymbol{x} \\
&= \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} v_j \frac{\partial u_k}{\partial x_j} \cdot u_k\, d\boldsymbol{x} = -\sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_k \frac{\partial}{\partial x_j} (v_j \cdot v_k)\, d\boldsymbol{x} \\
&= -\sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_k \cdot \left(\frac{\partial v_j}{\partial x_j} v_k + v_j \frac{\partial v_k}{\partial x_j} \right) d\boldsymbol{x} \\
&= -\sum_{j,k=1}^{3} \int_{\mathbb{R}^3} u_k \cdot v_j \frac{\partial v_k}{\partial x_j}\, d\boldsymbol{x} = -((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}, \boldsymbol{u}).
\end{aligned}
$$

ii)

$$
\begin{aligned}
((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u}) &= \int_{\mathbb{R}^3} (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{u}\, d\boldsymbol{x} = \int_{\mathbb{R}^3} \sum_{k=1}^{3} \left(\sum_{j=1}^{3} v_j \frac{\partial v_k}{\partial x_j} \cdot u_k \right) d\boldsymbol{x} \\
&= \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} v_j \frac{\partial v_k}{\partial x_j} \cdot u_k\, d\boldsymbol{x} = -\sum_{j,k=1}^{3} \int_{\mathbb{R}^3} v_k \frac{\partial}{\partial x_j} (v_j \cdot u_k)\, d\boldsymbol{x} \\
&= -\sum_{j,k=1}^{3} \int_{\mathbb{R}^3} v_k \cdot \left(\frac{\partial v_j}{\partial x_j} u_k + v_j \frac{\partial u_k}{\partial x_j} \right) d\boldsymbol{x} \\
&= -\sum_{j,k=1}^{3} \int_{\mathbb{R}^3} v_k \cdot v_j \frac{\partial u_k}{\partial x_j}\, d\boldsymbol{x} = -((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}).
\end{aligned}
$$

□

3 SOLUÇÃO FORTE GLOBAL NO TEMPO

Neste capítulo, estudamos o problema de valor inicial (PVI) para as equações de um fluido magneto-micropolar incompresível em todo o espaço $\mathbb{R}^3$. Para tanto, assumimos que $\mu = \chi = 1/2$, $\gamma = \kappa = \nu = 1$, $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$ e $P = p + \frac{1}{2}|\boldsymbol{b}|^2$ no sistema de equações (1.1). Ou seja, consideramos em $\mathbb{R}^3 \times (0, T)$, com $0 < T \leq \infty$, o PVI

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla P = \dfrac{1}{2} \operatorname{rot} \boldsymbol{w} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{b}, \\ \boldsymbol{w}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} - \Delta \boldsymbol{w} - \nabla(\operatorname{div} \boldsymbol{w}) + \boldsymbol{w} = \dfrac{1}{2} \operatorname{rot} \boldsymbol{u}, \\ \boldsymbol{b}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - \Delta \boldsymbol{b} = (\boldsymbol{b} \cdot \nabla)\boldsymbol{u}, \\ \operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{b} = 0, \\ (\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, 0) = (\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0)(\cdot), \end{cases} \tag{3.1}$$

complementado com condições de Dirichlet no infinito, i.e.,

$$(\boldsymbol{u}(\boldsymbol{x}, t), \boldsymbol{w}(\boldsymbol{x}, t), \boldsymbol{b}(\boldsymbol{x}, t)) \xrightarrow[|\boldsymbol{x}| \to \infty]{} (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}), \quad \forall\, t > 0.$$

Mostraremos a existência global e a unicidade de soluções fortes do PVI (3.1) para dados iniciais $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{H}^1(\mathbb{R}^3)$ tal que o produto

$$\left(\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2\right)\left(\|\nabla \boldsymbol{u}_0\|_2^2 + \|\nabla \boldsymbol{w}_0\|_2^2 + \|\nabla \boldsymbol{b}_0\|_2^2 + \|\operatorname{rot} \boldsymbol{u}_0 - 2\boldsymbol{w}_0\|_2^2\right)$$

é suficientemente pequeno.

3.1 Conceitos fundamentais e resultado principal

Definição 3.1 (Solução forte)**.** *Sejam $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{H}^1(\mathbb{R}^3)$, com $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$. Uma solução forte do problema* (3.1) *é uma tripla de funções $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ tal que*

$$\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b} \in L^\infty(0, T; \boldsymbol{H}^1(\mathbb{R}^3)) \cap L^2(0, T; \boldsymbol{H}^2(\mathbb{R}^3)),$$

com $\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b}$ satisfazendo as equações $(3.1)_1$–$(3.1)_4$ *q.s. em $\mathbb{R}^3 \times (0, T)$, e as condições iniciais* $(3.1)_5$ *em $\boldsymbol{H}^1(\mathbb{R}^3) \times \boldsymbol{H}^1(\mathbb{R}^3) \times \boldsymbol{H}^1(\mathbb{R}^3)$.*

Definição 3.2. *(Tempo maximal finito) Um número positivo finito T^* é chamado de tempo de existência maximal finito (ou tempo de explosão finito) para a solução forte do problema* (3.1) *se*

$$\sup_{t \in [0, T]} (\|\nabla \boldsymbol{u}(\cdot, t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot, t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot, t)\|_2^2) < \infty, \quad \forall\, T \in [0, T^*)$$

e

$$\lim_{t \uparrow T^*} \sup(\|\nabla \boldsymbol{u}(\cdot, t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot, t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot, t)\|_2^2) = \infty.$$

Neste caso, dizemos que $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ explode em tempo finito.

O principal resultado deste capítulo é o seguinte

Teorema 3.3. *Suponha que os dados iniciais* $(\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0)$ *satisfazem* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{H}^1(\mathbb{R}^3)$, *com* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$. *Então, existe uma constante absoluta* $\varepsilon_0 > 0$ *independente de* $\boldsymbol{u}_0$, $\boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$ *tal que se*

$$\left(\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2\right)\left(\|\nabla \boldsymbol{u}_0\|_2^2 + \|\nabla \boldsymbol{w}_0\|_2^2 + \|\nabla \boldsymbol{b}_0\|_2^2 + \|\operatorname{rot} \boldsymbol{u}_0 - 2\boldsymbol{w}_0\|_2^2\right) \leq \varepsilon_0, \tag{3.2}$$

o problema (3.1) *tem uma única solução forte global.*

Antes de encerrarmos essa seção, enunciaremos um lema que será bastante útil neste capítulo.

Lema 3.4 (Lema 6, pág. 1098 [50])**.** *Suponha que* $g \in W^{1,1}(0,T)$ *e* $k \in L^1(0,T)$ *satisfazem*

$$\frac{dg}{dt} \leq F(g) + k \quad \text{em } [0,T], \quad g(0) \leq g_0$$

onde F *é uma função limitada sobre conjuntos limitados de* $\mathbb{R}$ *em* $\mathbb{R}$, *i.e.,*

$$\forall\, a > 0 \quad \exists\, A > 0 \quad \text{tal que} \quad |x| \leq a \quad \Longrightarrow \quad |F(x)| \leq A.$$

Então, para todo $\varepsilon > 0$, *existe* $T_\varepsilon > 0$ *independente de* g *tal que*

$$g(t) \leq g_0 + \varepsilon \quad \forall\, t \leq T_\varepsilon.$$

3.2 Existência e unicidade de soluções fortes locais no tempo

Nesta seção, provaremos o seguinte resultado de existência local e unicidade para soluções fortes.

Teorema 3.5. *Suponha que* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{H}^1(\mathbb{R}^3)$ *com* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$. *Então, existe* $T_0 \in (0,T]$ *tal que o problema* (3.1) *possui exatamente uma solução forte* $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ *definida em* $\mathbb{R}^3 \times [0, T_0]$.

3.2.1 Estimativas *a priori*

Nesta subseção, obteremos estimativas *a priori* para as soluções do PVI (3.1) em $\mathbb{R}^3 \times (0,T)$. Assuma que $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ é uma solução suficientemente suave. Temos, então, a seguinte estimativa na norma $\boldsymbol{H}^1$.

Proposição 3.6. *Suponha que* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{H}^1(\mathbb{R}^3)$ *com* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$. *Então, existe* $T_0 \in (0,T]$ *tal que*

$$\left\|\boldsymbol{u}(t)\right\|_{\boldsymbol{H}^1}^2 + \left\|\boldsymbol{w}(t)\right\|_{\boldsymbol{H}^1}^2 + \left\|\boldsymbol{b}(t)\right\|_{\boldsymbol{H}^1}^2 \leq C\left(\left\|\boldsymbol{u}_0\right\|_{\boldsymbol{H}^1}^2 + \left\|\boldsymbol{w}_0\right\|_{\boldsymbol{H}^1}^2 + \left\|\boldsymbol{b}_0\right\|_{\boldsymbol{H}^1}^2\right) + 1, \quad \forall\, t \in [0, T_0],$$

onde $C > 0$ *é independente de* $\boldsymbol{u}_0$, $\boldsymbol{w}_0$, $\boldsymbol{b}_0$ *e* T_0.

Demonstração. Multiplicando, em $\boldsymbol{L}^2$, as equações $(3.1)_1$, $(3.1)_2$ e $(3.1)_3$ por $\boldsymbol{u}$, $\boldsymbol{w}$ e $\boldsymbol{b}$, respectivamente, obtemos, após integrarmos por partes e usarmos o Lema 2.43 item i, que

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \|\nabla \boldsymbol{u}\|_2^2 = \frac{1}{2}(\operatorname{rot}\boldsymbol{w},\boldsymbol{u}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{b},\boldsymbol{u}),\\
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_2^2 + \|\nabla \boldsymbol{w}\|_2^2 + \|\operatorname{div}\boldsymbol{w}\|_2^2 + \|\boldsymbol{w}\|_2^2 = \frac{1}{2}(\operatorname{rot}\boldsymbol{u},\boldsymbol{w}),\\
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{b}\|_2^2 + \|\nabla \boldsymbol{b}\|_2^2 = ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u},\boldsymbol{b}).
\end{aligned}$$

Somando essas identidades e usando o Lema 2.43 item ii, obtemos

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{w}\|_2^2 + \|\boldsymbol{b}\|_2^2\right) + \|\nabla\boldsymbol{u}\|_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{b}\|_2^2\\
&+ \|\operatorname{div}\boldsymbol{w}\|_2^2 + \|\boldsymbol{w}\|_2^2 = \frac{1}{2}\left[(\operatorname{rot}\boldsymbol{w},\boldsymbol{u}) + (\operatorname{rot}\boldsymbol{u},\boldsymbol{w})\right]. \qquad (3.3)
\end{aligned}$$

Agora, utilizando o Lema 2.42, concluimos que

$$\begin{aligned}
\|\operatorname{rot}\boldsymbol{u}\|_2^2 &= (\operatorname{rot}\boldsymbol{u},\operatorname{rot}\boldsymbol{u}) = (\boldsymbol{u},\operatorname{rot}(\operatorname{rot}\boldsymbol{u}))\\
&= (\boldsymbol{u},-\Delta\boldsymbol{u}) = \|\nabla\boldsymbol{u}\|_2^2.
\end{aligned}$$

Portanto, usando as desigualdades de Cauchy-Schwarz e Young, obtemos

$$\frac{1}{2}[(\operatorname{rot}\boldsymbol{w},\boldsymbol{u}) + (\operatorname{rot}\boldsymbol{u},\boldsymbol{w})] = (\operatorname{rot}\boldsymbol{u},\boldsymbol{w}) \le \|\operatorname{rot}\boldsymbol{u}\|_2\|\boldsymbol{w}\|_2 \le \frac{1}{2}\|\nabla\boldsymbol{u}\|_2^2 + \frac{1}{2}\|\boldsymbol{w}\|_2^2.$$

Aplicando as estimativas acima em (3.3) e usando a notação $\boldsymbol{z} := (\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})$, obtemos

$$\frac{d}{dt}\|\boldsymbol{z}\|_2^2 + \|\nabla\boldsymbol{z}\|_2^2 \le 0.$$

Finalmente, integrando a desigualdade acima com relação ao tempo, obtemos

$$\|\boldsymbol{z}(\cdot,t)\|_2^2 + \int_0^t \|\nabla\boldsymbol{z}(\cdot,\tau)\|_2^2\,d\tau \le \|\boldsymbol{z}_0\|_2^2. \qquad (3.4)$$

Agora, fazendo o produto escalar da primeira, da segunda e da terceira equações em (3.1) por $\boldsymbol{u}_t$, $\boldsymbol{w}_t$ e $\boldsymbol{b}_t$, respectivamente, somando e integrando o resultado em $\mathbb{R}^3$, obtemos

$$\begin{aligned}
&\|\boldsymbol{z}_t\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{4}\|\nabla\boldsymbol{u}\|_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{b}\|_2^2 + \|\operatorname{div}\boldsymbol{w}\|_2^2 + \frac{1}{4}\|\operatorname{rot}\boldsymbol{u} - 2\boldsymbol{w}\|_2^2\right)\\
&= ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},-\boldsymbol{u}_t) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},-\boldsymbol{w}_t) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{b},-\boldsymbol{b}_t) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{b},\boldsymbol{u}_t) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u},\boldsymbol{b}_t)\\
&\le \frac{1}{2}\|\boldsymbol{z}_t\|_2^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_2^2 + \frac{1}{2}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\|_2^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|_2^2,
\end{aligned}$$

o que nos dá

$$\begin{aligned}
&\|\boldsymbol{z}_t\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{4}\|\nabla\boldsymbol{u}\|_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{b}\|_2^2 + \|\operatorname{div}\boldsymbol{w}\|_2^2 + \frac{1}{4}\|\operatorname{rot}\boldsymbol{u} - 2\boldsymbol{w}\|_2^2\right)\\
&\quad \le 2\left(\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_2^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\|_2^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|_2^2\right). \qquad (3.5)
\end{aligned}$$

A seguir, tomaremos o produto interno $\boldsymbol{L}^2$ das Eqs. $(3.1)_1$, $(3.1)_2$ e $(3.1)_3$ com $-\Delta \boldsymbol{u}$, $-\Delta \boldsymbol{w}$ e $-\Delta \boldsymbol{b}$, respectivamente. Como $(\operatorname{rot} \boldsymbol{u}, -\Delta \boldsymbol{w}) = (\operatorname{rot} \boldsymbol{w}, -\Delta \boldsymbol{u})$, obtemos, apos somar as identidades resultantes, que

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\|\nabla \boldsymbol{z}\|_2^2 + \|\Delta \boldsymbol{z}\|_2^2 + \|\nabla(\operatorname{div} \boldsymbol{w})\|_2^2 + \|\nabla \boldsymbol{w}\|_2^2 \\
&= (\operatorname{rot} \boldsymbol{w}, -\Delta \boldsymbol{u}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, -\Delta \boldsymbol{u}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \Delta \boldsymbol{u}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, \Delta \boldsymbol{w}) \\
&\quad + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, -\Delta \boldsymbol{b}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, \Delta \boldsymbol{b}) \\
&\leq \|\nabla \boldsymbol{w}\|_2^2 + \frac{5}{8}\|\Delta \boldsymbol{z}\|_2^2 + \frac{4}{3}\left(\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|_2^2\right) \\
&\quad + \frac{2}{5}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\|_2^2 + \frac{4}{5}\left(\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|_2^2\right),
\end{aligned}
$$

ou seja,

$$
\begin{aligned}
\frac{d}{dt}&\|\nabla \boldsymbol{z}\|_2^2 + \frac{3}{4}\|\Delta \boldsymbol{z}\|_2^2 + 2\|\nabla(\operatorname{div} \boldsymbol{w})\|_2^2 \\
&\leq 3\left(\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_2^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\|_2^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|_2^2\right). \quad (3.6)
\end{aligned}
$$

Adicionando (3.5) e (3.6), obtemos

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{7}{4}\|\nabla \boldsymbol{u}\|_2^2 + 2\|\nabla \boldsymbol{w}\|_2^2 + 2\|\nabla \boldsymbol{b}\|_2^2 + \|\operatorname{div} \boldsymbol{w}\|_2^2 + \frac{1}{4}\|\operatorname{rot} \boldsymbol{u} - 2\boldsymbol{w}\|_2^2\right) \\
&\quad + \|\boldsymbol{z}_t\|_2^2 + \frac{3}{4}\|\Delta \boldsymbol{z}\|_2^2 + 2\|\nabla(\operatorname{div} \boldsymbol{w})\|_2^2 \\
&\quad \leq 5\left(\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_2^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\|_2^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|_2^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|_2^2\right). \quad (3.7)
\end{aligned}
$$

Por outro lado, pelas desigualdades de Gagliardo-Nirenberg e Sobolev, temos que

$$
\begin{aligned}
5\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_2^2 &\leq C\|\nabla \boldsymbol{u}\|_2^3\|\Delta \boldsymbol{u}\|_2 \leq \frac{3}{8}\|\Delta \boldsymbol{u}\|_2^2 + C\|\nabla \boldsymbol{u}\|_2^6, \\
5\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\|_2^2 &\leq C\|\nabla \boldsymbol{u}\|_2^2\|\nabla \boldsymbol{w}\|_2\|\Delta \boldsymbol{w}\|_2 \leq \frac{3}{8}\|\Delta \boldsymbol{w}\|_2^2 + C\|\nabla \boldsymbol{u}\|_2^4\|\nabla \boldsymbol{w}\|_2^2, \\
5\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|_2^2 &\leq C\|\nabla \boldsymbol{u}\|_2^2\|\nabla \boldsymbol{b}\|_2\|\Delta \boldsymbol{b}\|_2 \leq \frac{1}{8}\|\Delta \boldsymbol{b}\|_2^2 + C\|\nabla \boldsymbol{u}\|_2^4\|\nabla \boldsymbol{b}\|_2^2, \\
5\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|_2^2 &\leq C\|\nabla \boldsymbol{b}\|_2^3\|\Delta \boldsymbol{b}\|_2 \leq \frac{1}{8}\|\Delta \boldsymbol{b}\|_2^2 + C\|\nabla \boldsymbol{b}\|_2^6, \\
5\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|_2^2 &\leq C\|\nabla \boldsymbol{u}\|_2^2\|\nabla \boldsymbol{b}\|_2\|\Delta \boldsymbol{b}\|_2 \leq \frac{1}{8}\|\Delta \boldsymbol{b}\|_2^2 + C\|\nabla \boldsymbol{u}\|_2^4\|\nabla \boldsymbol{b}\|_2^2.
\end{aligned}
$$

Substituindo as estimativas acima na desigualdade (3.7), obtemos

$$
\begin{aligned}
\frac{d}{dt}&\left(\frac{7}{4}\|\nabla \boldsymbol{u}\|_2^2 + 2\|\nabla \boldsymbol{w}\|_2^2 + 2\|\nabla \boldsymbol{b}\|_2^2 + \|\operatorname{div} \boldsymbol{w}\|_2^2 + \frac{1}{4}\|\operatorname{rot} \boldsymbol{u} - 2\boldsymbol{w}\|_2^2\right) \\
&+ \|\boldsymbol{z}_t\|_2^2 + \frac{3}{8}\|\Delta \boldsymbol{z}\|_2^2 + 2\|\nabla(\operatorname{div} \boldsymbol{w})\|_2^2 \\
&\leq C\|\nabla \boldsymbol{u}\|_2^4\|\nabla \boldsymbol{z}\|_2^2 + C\left(\|\nabla \boldsymbol{u}\|_2^4 + \|\nabla \boldsymbol{b}\|_2^4\right)\|\nabla \boldsymbol{b}\|_2^2 \\
&\leq C\left(\|\nabla \boldsymbol{u}\|_2^4 + \|\nabla \boldsymbol{b}\|_2^4\right)\|\nabla \boldsymbol{z}\|_2^2. \quad (3.8)
\end{aligned}
$$

Portanto, pelo Lema 3.4 e pela desigualdade (3.8), concluímos que existe $T_0 \in (0, T]$ tal que, para todo $t \leq T_0$, temos

$$\|\nabla \boldsymbol{z}(t)\|_2^2 \leq \|\nabla \boldsymbol{u}_0\|_2^2 + \frac{12}{7}\|\nabla \boldsymbol{w}_0\|_2^2 + \frac{8}{7}\|\nabla \boldsymbol{b}_0\|_2^2 + \frac{1}{7}\|\mathrm{rot}\, \boldsymbol{u}_0 - 2\boldsymbol{w}_0\|_2^2 + 1.$$

Combinando a estimativa acima com a desigualdade de energia (3.4), obtemos

$$\left\|\boldsymbol{z}(t)\right\|_{\boldsymbol{H}^1}^2 \leq C\left\|\boldsymbol{z}_0\right\|_{\boldsymbol{H}^1}^2 + 1, \quad \forall\, t \leq T_0.$$

Isto finaliza a demonstração da Proposição 3.6. □

Demonstração do Teorema 3.5

Etapa 1: Existência.

A existência local de soluções fortes para o problema (3.1) pode ser estabelecida pelo método espectral de Galerkin (veja, por exemplo, [27]) baseada nas estimativas apresentadas na Proposição 3.6. Omitiremos, aqui, os detalhes.

Etapa 2: Unicidade.

Sejam $(\boldsymbol{u}_1, \boldsymbol{w}_1, \boldsymbol{b}_1)$ e $(\boldsymbol{u}_2, \boldsymbol{w}_2, \boldsymbol{b}_2)$ duas soluções fortes do sistema (3.1) com os mesmos dados iniciais $(\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0)$. Seja $(\overline{\boldsymbol{u}}, \overline{P}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{b}})$, onde $\overline{\boldsymbol{u}} := \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\overline{P} := P_1 - P_2$, $\overline{\boldsymbol{w}} := \boldsymbol{w}_1 - \boldsymbol{w}_2$ e $\overline{\boldsymbol{b}} := \boldsymbol{b}_1 - \boldsymbol{b}_2$. Então, $(\overline{\boldsymbol{u}}, \overline{P}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{b}})$ satisfazem o seguinte problema:

$$\begin{cases} \overline{\boldsymbol{u}}_t + (\boldsymbol{u}_1 \cdot \nabla)\overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u}_2 - \Delta\overline{\boldsymbol{u}} + \nabla\overline{P} = (\boldsymbol{b}_1 \cdot \nabla)\overline{\boldsymbol{b}} + (\overline{\boldsymbol{b}} \cdot \nabla)\boldsymbol{b}_2 + \dfrac{1}{2}\mathrm{rot}\,\overline{\boldsymbol{w}}, \\ \overline{\boldsymbol{w}}_t + (\boldsymbol{u}_1 \cdot \nabla)\overline{\boldsymbol{w}} + (\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{w}_2 - \Delta\overline{\boldsymbol{w}} - \nabla(\mathrm{div}\,\overline{\boldsymbol{w}}) + \overline{\boldsymbol{w}} = \dfrac{1}{2}\mathrm{rot}\,\overline{\boldsymbol{u}}, \\ \overline{\boldsymbol{b}}_t + (\boldsymbol{u}_1 \cdot \nabla)\overline{\boldsymbol{b}} + (\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{b}_2 - \Delta\overline{\boldsymbol{b}} = (\boldsymbol{b}_1 \cdot \nabla)\overline{\boldsymbol{u}} + (\overline{\boldsymbol{b}} \cdot \nabla)\boldsymbol{u}_2, \\ \mathrm{div}\,\overline{\boldsymbol{u}} = \mathrm{div}\,\overline{\boldsymbol{b}} = 0, \\ \left(\overline{\boldsymbol{u}}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{b}}\right)\big|_{t=0} = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}) \ \text{ em } \mathbb{R}^3. \end{cases} \tag{3.9}$$

Multiplicando, em $\boldsymbol{L}^2$, a Eq. $(3.9)_1$ por $\overline{\boldsymbol{u}}$, a Eq. $(3.9)_2$ por $\overline{\boldsymbol{w}}$ e a Eq. $(3.9)_3$ por $\overline{\boldsymbol{b}}$, obtemos, respectivamente

$$\frac{1}{2}\frac{d}{dt}\|\overline{\boldsymbol{u}}\|_2^2 + \|\nabla\overline{\boldsymbol{u}}\|_2^2 = \frac{1}{2}(\mathrm{rot}\,\overline{\boldsymbol{w}}, \overline{\boldsymbol{u}}) - ((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u}_2, \overline{\boldsymbol{u}}) + ((\boldsymbol{b}_1 \cdot \nabla)\overline{\boldsymbol{b}}, \overline{\boldsymbol{u}}) + ((\overline{\boldsymbol{b}} \cdot \nabla)\boldsymbol{b}_2, \overline{\boldsymbol{u}}), \tag{3.10}$$

$$\frac{1}{2}\frac{d}{dt}\|\overline{\boldsymbol{w}}\|_2^2 + \|\nabla\overline{\boldsymbol{w}}\|_2^2 + \|\mathrm{div}\,\overline{\boldsymbol{w}}\|_2^2 + \|\overline{\boldsymbol{w}}\|_2^2 = \frac{1}{2}(\mathrm{rot}\,\overline{\boldsymbol{u}}, \overline{\boldsymbol{w}}) - ((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{w}_2, \overline{\boldsymbol{w}}) \tag{3.11}$$

e

$$\frac{1}{2}\frac{d}{dt}\|\overline{\boldsymbol{b}}\|_2^2 + \|\nabla\overline{\boldsymbol{b}}\|_2^2 = ((\boldsymbol{b}_1 \cdot \nabla)\overline{\boldsymbol{u}}, \overline{\boldsymbol{b}}) + ((\overline{\boldsymbol{b}} \cdot \nabla)\boldsymbol{u}_2, \overline{\boldsymbol{b}}) - ((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{b}_2, \overline{\boldsymbol{b}}). \tag{3.12}$$

Seja $\overline{\boldsymbol{z}} := (\overline{\boldsymbol{u}}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{b}})$. Somando as identidades (3.10), (3.11) e (3.12), obtemos

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\overline{\boldsymbol{z}}\|_2^2 + \|\nabla\overline{\boldsymbol{z}}\|_2^2 + \|\mathrm{div}\,\overline{\boldsymbol{w}}\|_2^2 + \|\overline{\boldsymbol{w}}\|_2^2 \quad = \quad & (\mathrm{rot}\,\overline{\boldsymbol{u}}, \overline{\boldsymbol{w}}) - ((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u}_2, \overline{\boldsymbol{u}}) \\ & + ((\overline{\boldsymbol{b}} \cdot \nabla)\boldsymbol{b}_2, \overline{\boldsymbol{u}}) - ((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{w}_2, \overline{\boldsymbol{w}}) \\ & + ((\overline{\boldsymbol{b}} \cdot \nabla)\boldsymbol{u}_2, \overline{\boldsymbol{b}}) - ((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{b}_2, \overline{\boldsymbol{b}}). \end{aligned}$$

Usando as desigualdades de Cauchy-Schwarz, Hölder, Sobolev e Young, obtemos

$$
\begin{aligned}
|(\operatorname{rot}\overline{\boldsymbol{u}}, \overline{\boldsymbol{w}})| &\leq \|\overline{\boldsymbol{w}}\|_2^2 + \frac{1}{4}\|\nabla\overline{\boldsymbol{u}}\|_2^2, \\
|((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u}_2, \overline{\boldsymbol{u}})| &\leq \|\overline{\boldsymbol{u}}\|_2 \|\nabla\boldsymbol{u}_2\|_3 \|\overline{\boldsymbol{u}}\|_6 \\
&\leq C\|\overline{\boldsymbol{u}}\|_2 \|\nabla\boldsymbol{u}_2\|_2^{1/2} \|\Delta\boldsymbol{u}_2\|_2^{1/2} \|\nabla\overline{\boldsymbol{u}}\|_2 \\
&\leq \frac{1}{4}\|\nabla\overline{\boldsymbol{u}}\|_2^2 + C\|\nabla\boldsymbol{u}_2\|_2 \|\Delta\boldsymbol{u}_2\|_2 \|\overline{\boldsymbol{u}}\|_2^2, \\
|((\overline{\boldsymbol{b}} \cdot \nabla)\boldsymbol{b}_2, \overline{\boldsymbol{u}})| &\leq \frac{1}{4}\|\nabla\overline{\boldsymbol{u}}\|_2^2 + C\|\nabla\boldsymbol{b}_2\|_2 \|\Delta\boldsymbol{b}_2\|_2 \|\overline{\boldsymbol{b}}\|_2^2, \\
|((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{w}_2, \overline{\boldsymbol{w}})| &\leq \frac{3}{4}\|\nabla\overline{\boldsymbol{w}}\|_2^2 + C\|\nabla\boldsymbol{w}_2\|_2 \|\Delta\boldsymbol{w}_2\|_2 \|\overline{\boldsymbol{u}}\|_2^2, \\
|((\overline{\boldsymbol{b}} \cdot \nabla)\boldsymbol{u}_2, \overline{\boldsymbol{b}})| &\leq \frac{3}{8}\|\nabla\overline{\boldsymbol{b}}\|_2^2 + C\|\nabla\boldsymbol{u}_2\|_2 \|\Delta\boldsymbol{u}_2\|_2 \|\overline{\boldsymbol{b}}\|_2^2, \\
|((\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{b}_2, \overline{\boldsymbol{b}})| &\leq \frac{3}{8}\|\nabla\overline{\boldsymbol{b}}\|_2^2 + C\|\nabla\boldsymbol{b}_2\|_2 \|\Delta\boldsymbol{b}_2\|_2 \|\overline{\boldsymbol{u}}\|_2^2.
\end{aligned}
$$

Assim, temos que

$$
\begin{aligned}
&\frac{d}{dt}\|\overline{\boldsymbol{z}}\|_2^2 + \frac{1}{2}\|\nabla\overline{\boldsymbol{z}}\|_2^2 + 2\|\operatorname{div}\overline{\boldsymbol{w}}\|_2^2 \\
&\qquad \leq C\left(\|\nabla\boldsymbol{u}_2\|_2 \|\Delta\boldsymbol{u}_2\|_2 + \|\nabla\boldsymbol{w}_2\|_2 \|\Delta\boldsymbol{w}_2\|_2 + \|\nabla\boldsymbol{b}_2\|_2 \|\Delta\boldsymbol{b}_2\|_2\right)\|\overline{\boldsymbol{z}}\|_2^2.
\end{aligned}
$$

Integrando a desigualdade acima de 0 até t, obtemos

$$
\begin{aligned}
&\|\overline{\boldsymbol{z}}(\cdot,t)\|_2^2 + \int_0^t \left(\frac{1}{2}\|\nabla\overline{\boldsymbol{z}}(\cdot,\tau)\|_2^2 + 2\|\operatorname{div}\overline{\boldsymbol{w}}(\cdot,\tau)\|_2^2\right) d\tau \\
&\quad \leq \|\overline{\boldsymbol{z}}(\cdot,0)\|_2^2 + C\int_0^t \left(\|\nabla\boldsymbol{u}_2\|_2 \|\Delta\boldsymbol{u}_2\|_2 + \|\nabla\boldsymbol{w}_2\|_2 \|\Delta\boldsymbol{w}_2\|_2 + \|\nabla\boldsymbol{b}_2\|_2 \|\Delta\boldsymbol{b}_2\|_2\right)\|\overline{\boldsymbol{z}}\|_2^2 \, d\tau.
\end{aligned}
$$

Logo, por uma uma desigualdade do tipo Gronwall, obtemos

$$
\begin{aligned}
&\|\overline{\boldsymbol{z}}(\cdot,t)\|_2^2 \\
&\quad \leq \|\overline{\boldsymbol{z}}(\cdot,0)\|_2^2 \times \exp\left\{C\int_0^t \left(\|\nabla\boldsymbol{u}_2\|_2 \|\Delta\boldsymbol{u}_2\|_2 + \|\nabla\boldsymbol{w}_2\|_2 \|\Delta\boldsymbol{w}_2\|_2 + \|\nabla\boldsymbol{b}_2\|_2 \|\Delta\boldsymbol{b}_2\|_2\right) d\tau\right\}.
\end{aligned}
$$

Uma vez que $\boldsymbol{u}_2, \boldsymbol{w}_2, \boldsymbol{b}_2 \in L^\infty(0,T;\boldsymbol{H}^1(\mathbb{R}^3)) \cap L^2(0,T;\boldsymbol{H}^2(\mathbb{R}^3))$, a integral no lado direito da desigualdade acima é finita. Ademais, como $\overline{\boldsymbol{z}}(\cdot,0) = (\overline{\boldsymbol{u}}, \overline{\boldsymbol{w}}, \overline{\boldsymbol{b}})(\cdot,0) = (\boldsymbol{0},\boldsymbol{0},\boldsymbol{0})$, concluímos que

$$
\|\overline{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\overline{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\overline{\boldsymbol{b}}(\cdot,t)\|_2^2 = 0,
$$

ou seja, $(\boldsymbol{u}_1, \boldsymbol{w}_1, \boldsymbol{b}_1) \equiv (\boldsymbol{u}_2, \boldsymbol{w}_2, \boldsymbol{b}_2)$. Isto conclui a prova da unicidade.

3.3 Soluções globais

Ao longo desta seção, C_0 e K_0 denotarão as seguintes constantes:

$$
\begin{aligned}
C_0 &:= \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2, \\
K_0 &:= 2\left(\|\nabla\boldsymbol{u}_0\|_2^2 + \|\nabla\boldsymbol{w}_0\|_2^2 + \|\nabla\boldsymbol{b}_0\|^2 + \|\operatorname{rot}\boldsymbol{u}_0 - 2\boldsymbol{w}_0\|_2^2\right).
\end{aligned}
$$

Ademais, denotaremos por $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ a solução forte do problema (3.1) em $\mathbb{R}^3 \times (0,T)$.

Lema 3.7. *A desigualdade de energia*

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 + \int_0^t \left(\|\nabla\boldsymbol{u}(\cdot,\tau)\|_2^2 + \|\nabla\boldsymbol{w}(\cdot,\tau)\|_2^2 + \|\nabla\boldsymbol{b}(\cdot,\tau)\|_2^2\right) d\tau \le C_0$$

é satisfeita para todo $t \in [0,T)$.

Demonstração. Esta estimativa segue diretamente de (3.4). □

Lema 3.8. *Existe uma constante $C \in \mathbb{R}^+$ independente de T, $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$, tal que a desigualdade*

$$\sup_{0\le s\le t}(\|\nabla\boldsymbol{u}(\cdot,s)\|_2^2 + \|\nabla\boldsymbol{w}(\cdot,s)\|_2^2 + \|\nabla\boldsymbol{b}(\cdot,s)\|_2^2) \le K_0 + C\,C_0 \sup_{0\le s\le t}(\|\nabla\boldsymbol{u}(\cdot,s)\|_2^4 + \|\nabla\boldsymbol{b}(\cdot,s)\|_2^4) \tag{3.13}$$

é válida para todo $t \in [0,T)$.

Demonstração. Integrando a desigualdade (3.8) de 0 até t, obtemos

$$\begin{aligned}
\|\nabla\boldsymbol{z}(\cdot,t)\|_2^2 &+ \frac{4}{7}\int_0^t \|\boldsymbol{z}_t(\cdot,s)\|_2^2\,ds + \frac{3}{14}\int_0^t \|\Delta\boldsymbol{z}(\cdot,s)\|_2^2\,ds \\
&\le \|\nabla\boldsymbol{u}_0\|_2^2 + \frac{12}{7}\|\nabla\boldsymbol{w}_0\|_2^2 + \frac{8}{7}\|\nabla\boldsymbol{b}_0\|_2^2 + \frac{1}{7}\|\mathrm{rot}\,\boldsymbol{u}_0 - 2\boldsymbol{w}_0\|_2^2 \\
&\quad + C\int_0^t \left(\|\nabla\boldsymbol{u}(\cdot,s)\|_2^4 + \|\nabla\boldsymbol{b}(\cdot,s)\|_2^4\right)\|\nabla\boldsymbol{z}(\cdot,s)\|_2^2\,ds.
\end{aligned} \tag{3.14}$$

Disto segue que

$$\begin{aligned}
\|\nabla\boldsymbol{z}(\cdot,t)\|_2^2 &+ \frac{4}{7}\int_0^t \|\boldsymbol{z}_t(\cdot,s)\|_2^2\,ds + \frac{3}{14}\int_0^t \|\Delta\boldsymbol{z}(\cdot,s)\|_2^2\,ds \\
&\le K_0 + C\sup_{0\le s\le t}(\|\nabla\boldsymbol{u}(\cdot,s)\|_2^4 + \|\nabla\boldsymbol{b}(\cdot,s)\|_2^4)\int_0^t \|\nabla\boldsymbol{z}(\cdot,s)\|_2^2\,ds \\
&\le K_0 + C\,C_0\sup_{0\le s\le t}(\|\nabla\boldsymbol{u}(\cdot,s)\|_2^4 + \|\nabla\boldsymbol{b}(\cdot,s)\|_2^4),
\end{aligned}$$

onde, na última desigualdade, usamos a estimativa de energia dada no Lema 3.7. Isto conclui a demonstração do lema. □

Lema 3.9. *Existe uma constante positiva ε_0 independente de T, $\boldsymbol{u}_0$, $\boldsymbol{w}_0$ e $\boldsymbol{b}_0$, tal que se*

$$C_0K_0 \le 2\varepsilon_0, \tag{3.15}$$

então

$$\sup_{0\le t\le T}(\|\nabla\boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla\boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla\boldsymbol{b}(\cdot,t)\|_2^2) \le 2K_0.$$

Demonstração. Inicialmente, definimos a função $E(t)$ por

$$E(t) := \sup_{0\le s\le t} (\|\nabla \boldsymbol{u}(\cdot,s)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,s)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,s)\|_2^2).$$

Observe que

$$E(0) = \|\nabla \boldsymbol{u}_0\|_2^2 + \|\nabla \boldsymbol{w}_0\|_2^2 + \|\nabla \boldsymbol{b}_0\|_2^2 \le K_0 < 4K_0. \tag{3.16}$$

Pelo Lema 3.8, existe uma constante $M > 0$ independente de T, $\boldsymbol{u}_0$, $\boldsymbol{w}_0$ e $\boldsymbol{b}_0$, tal que

$$E(t) \le K_0 + MC_0\,E(t)^2. \tag{3.17}$$

Assuma, agora, que

$$MC_0\,K_0 \le \frac{1}{8} \tag{3.18}$$

e defina o tempo $T_\star$ como sendo

$$T_\star := \sup\big\{t \in [0,T] : E(s) \le 4K_0,\ \forall\, s \in [0,t]\big\}. \tag{3.19}$$

Como $E(t)$ é uma função contínua sobre $[0,T]$, o máximo é atingido.

Afirmação: Temos que $T_\star = T$.

De fato, para efeito de contradição, suponha que $T_\star \ne T$. Por (3.16), temos que $T_\star \in (0,T)$. Segue, de (3.17)–(3.19), que

$$\begin{aligned} E(T_\star) &\le K_0 + MC_0\,E(T_\star)^2 \\ &\le K_0 + 4MC_0\,K_0\,E(T_\star) \\ &\le K_0 + \frac{1}{2}E(T_\star). \end{aligned}$$

Isto implica que $E(T_\star) \le 2K_0 < 4K_0$, o que contradiz (3.19). Portanto $T_\star = T$.

Escolhendo $\varepsilon_0 := \dfrac{1}{16M}$, conclui-se que

$$E(t) \le 2K_0, \quad \forall\, t \in [0,T],$$

desde que a condição (3.15) seja satisfeita. Isto prova o resultado desejado. □

Lema 3.10 (Critério de explosão)**.** *Suponha que* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{H}^1(\mathbb{R}^3)$ *com* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$. *Então* T^* *é o tempo de explosão finito da solução* $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ *se, e somente se,* T^* *é o menor tempo tal que*

$$\int_0^{T^*} \big(\|\nabla \boldsymbol{u}(\cdot,t)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^4\big)dt = \infty.$$

Demonstração. Primeiramente, note, por um argumento tipo Gronwall, que, a partir da desigualdade (3.14), temos

$$E(T) \le K_0 \exp\left\{C\int_0^T \big(\|\nabla \boldsymbol{u}(\cdot,t)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^4\big)\,dt\right\}, \tag{3.20}$$

para todo $T \in [0, T^*)$.

Agora, suponha que T^* é o tempo de explosão finito de $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$. Então, por definição, $E(T) < \infty$, $\forall T \in [0, T^*)$ e $\lim\limits_{t \uparrow T^*} E(t) = \infty$. Note que, por (3.20), temos

$$\lim_{t \uparrow T*} K_0 \exp \left\{ C \int_0^t \left(\|\nabla \boldsymbol{u}(\cdot, s)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot, s)\|_2^4 \right) ds \right\} \geq \lim_{t \uparrow T^*} E(t) = \infty.$$

Logo,

$$\lim_{t \uparrow T*} K_0 \exp \left\{ C \int_0^t \left(\|\nabla \boldsymbol{u}(\cdot, s)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot, s)\|_2^4 \right) ds \right\} = \infty.$$

Desta forma,

$$\int_0^{T^*} \left(\|\nabla \boldsymbol{u}(\cdot, t)\|^4 + \|\nabla \boldsymbol{b}(\cdot, t)\|_2^4 \right) dt = \infty.$$

Reciprocamente, se $T^* \in \mathbb{R}^+$ é tal que $\lim\limits_{t \uparrow T^*} E(t) < \infty$, então

$$\lim_{t \uparrow T^*} \sup_{0 \leq s \leq t} (\|\nabla \boldsymbol{u}(\cdot, s)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot, s)\|_2^2) < \infty.$$

Logo, $\lim\limits_{t \uparrow T^*} \sup\limits_{0 \leq s \leq t} (\|\nabla \boldsymbol{u}(\cdot, s)\|^4 + \|\nabla \boldsymbol{b}(\cdot, s)\|_2^4) =: \widetilde{k} < \infty$. Portanto,

$$\int_0^{T^*} \left(\|\nabla \boldsymbol{u}(\cdot, s)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot, s)\|_2^4 \right) ds = \lim_{t \uparrow T^*} \int_0^t \left(\|\nabla \boldsymbol{u}(\cdot, s)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot, s)\|_2^4 \right) ds \leq \widetilde{k} T^* < \infty,$$

concluindo, então, a demonstração do lema. $\square$

Demonstração do Teorema 3.3

Assuma que $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{H}^1(\mathbb{R}^3)$, com $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$ em $\mathbb{R}^3$, e que

$$\left(\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \right) \left(\|\nabla \boldsymbol{u}_0\|_2^2 + \|\nabla \boldsymbol{w}_0\|_2^2 + \|\nabla \boldsymbol{b}_0\|_2^2 + \|\operatorname{rot} \boldsymbol{u}_0 - 2\boldsymbol{w}_0\|_2 \right) \leq \varepsilon_0,$$

onde $\varepsilon_0 > 0$ é a constante que aparece no Lema 3.9. Seja $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ a única solução forte local do sistema (3.1) dada pelo Teorema 3.5. Além disso, seja $[0, T^*)$ o intervalo maximal de existência para a solução forte do problema (3.1). A seguir, provaremos que $T^* = \infty$. Por contradição, suponha $T^* < \infty$. Então, pelo Lema 3.10, temos que

$$\int_0^{T^*} \left(\|\nabla \boldsymbol{u}(\cdot, t)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot, t)\|_2^4 \right) dt = \infty. \tag{3.21}$$

Por outro lado, pelo Lema 3.9, temos que

$$\sup_{0 \leq t \leq T} \left(\|\nabla \boldsymbol{u}(\cdot, t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot, t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot, t)\|_2^2 \right) \leq 2K_0,$$

para todo $0 \leq T < T^*$. Assim,

$$\begin{aligned}
\int_0^T \left(\|\nabla \boldsymbol{u}(\cdot,t)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^4\right) dt &\leq \int_0^T \left(\|\nabla \boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^2\right)^2 dt \\
&\leq \int_0^T (2K_0)^2\, dt \\
&= 4K_0^2 T.
\end{aligned}$$

Daí, temos que

$$\begin{aligned}
\int_0^{T^*} \left(\|\nabla \boldsymbol{u}(\cdot,t)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^4\right) dt &= \lim_{T \uparrow T^*} \int_0^T \left(\|\nabla \boldsymbol{u}(\cdot,t)\|_2^4 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^4\right) dt \\
&\leq \lim_{T \uparrow T^*} 4K_0^2 T \\
&= 4K_0^2 T^* < \infty,
\end{aligned}$$

contradizendo o critério de explosão (3.21). Portanto, $T^* = \infty$ e a solução é global no tempo.

4 DECAIMENTO DE SOLUÇÕES: O ARGUMENTO FORMAL

Neste capítulo, assumindo hipóteses adequadas para as forças externas $\boldsymbol{f}$ e $\boldsymbol{g}$, provaremos a seguinte estimativa

$$\|\boldsymbol{u}(\cdot,t)\|_2 + \|\boldsymbol{w}(\cdot,t)\|_2 + \|\boldsymbol{b}(\cdot,t)\|_2 \leq C(t+1)^{-3/4}, \quad \forall\, t \geq 0,$$

para a solução do problema de Cauchy em $\mathbb{R}^3 \times (0,\infty)$

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla P = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \dfrac{1}{2}\operatorname{rot}\boldsymbol{w} + \boldsymbol{f}, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \Delta\boldsymbol{w} - \nabla(\operatorname{div}\boldsymbol{w}) + \boldsymbol{w} = \dfrac{1}{2}\operatorname{rot}\boldsymbol{u} + \boldsymbol{g}, \\ \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - \Delta\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \\ \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{b} = 0, \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \boldsymbol{w}(\cdot,0) = \boldsymbol{w}_0, \quad \boldsymbol{b}(\cdot,0) = \boldsymbol{b}_0, \end{cases} \tag{4.1}$$

complementado com condições de Dirichlet homogêas no infinito, ou seja,

$$\lim_{|\boldsymbol{x}|\to\infty} (\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{w}(\boldsymbol{x},t), \boldsymbol{b}(\boldsymbol{x},t)) = (\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}), \qquad \forall\ t > 0.$$

A Seção 4.1 é dedicada ao caso em que $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$. Na Subseção 4.1.2, obtemos uma taxa de decaimento melhorada para a velocidade angular. A saber

$$\|\boldsymbol{w}(\cdot,t)\|_2 \leq C(t+1)^{-5/4}, \quad \forall\, t \geq 0.$$

Na Seção 4.2, obtemos as mesmas estimativas considerando hipóteses adequadas sobre as forças externas.

4.1 Forças externas nulas

4.1.1 Taxa de decaimento através do método de decomposição de Fourier

Lema 4.1. *Seja* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *uma solução suave do problema de Cauchy* (4.1) *com* $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$. *Se* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$, *com* $\operatorname{div}\boldsymbol{u}_0 = \operatorname{div}\boldsymbol{b}_0 = 0$ *então, para todo* $t \geq 0$ *e* $\boldsymbol{\xi} \in \mathbb{R}^3$, *temos*

$$\begin{aligned} & |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| \\ + \ & |\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)||\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t)| \\ \leq \ & [\|\boldsymbol{u}(\cdot,t)\|_2\,(2\|\boldsymbol{u}(\cdot,t)\| + \|\boldsymbol{w}(\cdot,t)\|) + \|\boldsymbol{b}(\cdot,t)\|_2\,(2\|\boldsymbol{u}(\cdot,t)\|_2 + 2\|\boldsymbol{b}(\cdot,t)\|_2)]\,|\boldsymbol{\xi}|. \end{aligned}$$

Em particular,

$$\begin{aligned}&|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)|+|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)|+|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)|+|\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)|\\&+|\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)|+|\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t)|\le C|\boldsymbol{\xi}|,\end{aligned}$$

onde $C\in\mathbb{R}^+$ *depende somente de* $\|\boldsymbol{u}_0\|_2$, $\|\boldsymbol{w}_0\|_2$ *e* $\|\boldsymbol{b}_0\|_2$.

Demonstração. Multiplicando em $\boldsymbol{L}^2$ as Equações $(4.1)_1$ por $\boldsymbol{u}$, $(4.1)_2$ por $\boldsymbol{w}$ e $(4.1)_3$ por $\boldsymbol{b}$, obtemos

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2+\|\nabla\boldsymbol{u}\|_2^2=\frac{1}{2}(\operatorname{rot}\boldsymbol{w},\boldsymbol{u})+((\boldsymbol{b}\cdot\nabla)\boldsymbol{b},\boldsymbol{u}),\\&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_2^2+\|\nabla\boldsymbol{w}\|_2^2+\|\boldsymbol{w}\|_2^2+\|\operatorname{div}\boldsymbol{w}\|_2^2=\frac{1}{2}(\operatorname{rot}\boldsymbol{u},\boldsymbol{w}),\\&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{b}\|_2^2+\|\nabla\boldsymbol{b}\|_2^2=((\boldsymbol{b}\cdot\nabla)\boldsymbol{u},\boldsymbol{b}).\end{aligned}$$

Usando as desigualdades de Cauchy-Shwartz e de Young e somando as três identidades acima, obtemos

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{u}\|_2^2+\|\boldsymbol{w}\|_2^2+\|\boldsymbol{b}\|_2^2\right)\quad&+\quad\left(\|\nabla\boldsymbol{u}\|_2^2+\|\nabla\boldsymbol{w}\|_2^2+\|\nabla\boldsymbol{b}\|_2^2\right)+\|\boldsymbol{w}\|_2^2+\|\operatorname{div}\boldsymbol{w}\|_2^2\\&\le(\operatorname{rot}\boldsymbol{u},\boldsymbol{w})\le\frac{1}{2}\|\boldsymbol{w}\|_2^2+\frac{1}{2}\|\nabla\boldsymbol{u}\|_2^2,\end{aligned}$$

o que implica

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{u}\|_2^2+\|\boldsymbol{w}\|_2^2+\|\boldsymbol{b}\|_2^2\right)+\left(\frac{1}{2}\|\nabla\boldsymbol{u}\|_2^2+\|\nabla\boldsymbol{w}\|_2^2+\|\nabla\boldsymbol{b}\|_2^2\right)+\frac{1}{2}\|\boldsymbol{w}\|_2^2+\|\operatorname{div}\boldsymbol{w}\|_2^2\le 0.$$

Assim, multiplicando por 2 e manipulando os termos do lado esquerdo, obtemos a seguinte *desigualdade de energia*.

$$\frac{d}{dt}\left(\|\boldsymbol{u}\|_2^2+\|\boldsymbol{w}\|_2^2+\|\boldsymbol{b}\|_2^2\right)\le-\left(\|\nabla\boldsymbol{u}\|_2^2+\|\nabla\boldsymbol{w}\|_2^2+\|\nabla\boldsymbol{b}\|_2^2\right). \tag{4.2}$$

Integrando a desigualdade (4.2) em relação ao tempo, obtemos

$$\|\boldsymbol{u}(\cdot,t)\|_2^2+\|\boldsymbol{w}(\cdot,t)\|_2^2+\|\boldsymbol{b}(\cdot,t)\|_2^2\le\|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{w}_0\|_2^2+\|\boldsymbol{b}_0\|_2^2,\ \ \forall\, t\ge 0. \tag{4.3}$$

Temos, de (4.3), que $\boldsymbol{u}(\cdot,t),\boldsymbol{w}(\cdot,t),\boldsymbol{b}(\cdot,t)\in\boldsymbol{L}^2(\mathbb{R}^3)$ para todo $t\ge 0$. Portanto, as transformadas $\widehat{\boldsymbol{u}}(\cdot,t),\widehat{\boldsymbol{w}}(\cdot,t)$ e $\widehat{\boldsymbol{b}}(\cdot,t)$ estão bem definidas. A seguir, estimaremos a k-ésima coordenada de $\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)$. Como

$$[\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)]_k=\int_{\mathbb{R}^3}e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\sum_{j=1}^{3}u_j\frac{\partial u_k}{\partial x_j}\right)\,d\boldsymbol{x},$$

segue, integrando por partes, que

$$\begin{aligned}
[\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)]_k &= -\int_{\mathbb{R}^3} u_k \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} u_j\right)\, d\boldsymbol{x} \\
&= -\int_{\mathbb{R}^3} u_k \sum_{j=1}^{3} \left[\frac{\partial}{\partial x_j}(e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}})u_j + (e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}})\frac{\partial u_j}{\partial x_j}\right]\, d\boldsymbol{x} \\
&= -\int_{\mathbb{R}^3} u_k \sum_{j=1}^{3} \left[e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(-i\xi_j u_j + \frac{\partial u_j}{\partial x_j}\right)\right]\, d\boldsymbol{x}.
\end{aligned}$$

Como div $\boldsymbol{u} = 0$, temos

$$[\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)]_k = i\int_{\mathbb{R}^3} u_k \sum_{j=1}^{3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\xi_j u_j\, d\boldsymbol{x}.$$

Assim,

$$|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| \le \sum_{k,j=1}^{3}\int_{\mathbb{R}^3} |u_j u_k||\xi_j|\, d\boldsymbol{x}.$$

Analogamente, temos

$$|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)| \le \sum_{k,j=1}^{3}\int_{\mathbb{R}^3} |u_j w_k||\xi_j|\, d\boldsymbol{x} \ \text{ e } \ |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| \le \sum_{k,j=1}^{3}\int_{\mathbb{R}^3} |u_j b_k||\xi_j|\, d\boldsymbol{x}.$$

Como div $\boldsymbol{b} = 0$, também temos

$$|\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| \le \sum_{k,j=1}^{3}\int_{\mathbb{R}^3} |b_j u_k||\xi_j|\, d\boldsymbol{x} \quad \text{e} \quad |\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)] \le \sum_{k,j=1}^{3}\int_{\mathbb{R}^3} |b_j b_k||\xi_j|\, d\boldsymbol{x}.$$

Usando a desigualdade Hölder e a estimativa (4.3), obtemos

$$\|u_k u_j(\cdot,t)\|_1 \le \|u_k(\cdot,t)\|_2\|u_j(\cdot,t)\|_2 \le \|\boldsymbol{u}(\cdot,t)\|_2^2 \le \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \le C.$$

Analogamente,

$$\begin{aligned}
&\|u_j w_k(\cdot,t)\|_1 \le \|u_j(\cdot,t)\|_2\|w_k(\cdot,t)\|_2 \le \|\boldsymbol{u}(\cdot,t)\|_2\|\boldsymbol{w}(\cdot,t)\|_2 \le \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \le C,\\
&\|u_j b_k(\cdot,t)\|_1 \le \|u_j(\cdot,t)\|_2\|b_k(\cdot,t)\|_2 \le \|\boldsymbol{u}(\cdot,t)\|_2\|\boldsymbol{b}(\cdot,t)\|_2 \le \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \le C,\\
&\|b_j u_k(\cdot,t)\|_1 \le \|b_j(\cdot,t)\|_2\|u_k(\cdot,t)\|_2 \le \|\boldsymbol{b}(\cdot,t)\|_2\|\boldsymbol{u}(\cdot,t)\|_2 \le \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \le C,\\
&\|b_j b_k(\cdot,t)\|_1 \le \|b_j(\cdot,t)\|_2\|b_k(\cdot,t)\|_2 \le \|\boldsymbol{b}(\cdot,t)\|_2^2 \le \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \le C.
\end{aligned}$$

Assim,

$$\begin{aligned}
&|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| \le \|\boldsymbol{u}\|_2^2\,|\boldsymbol{\xi}| \le C\,|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)| \le \|\boldsymbol{u}\|_2\|\boldsymbol{w}\|_2\,|\boldsymbol{\xi}| \le C\,|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| \le \|\boldsymbol{u}\|_2\|\boldsymbol{b}\|_2\,|\boldsymbol{\xi}| \le C\,|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| \le \|\boldsymbol{b}\|_2\|\boldsymbol{u}\|_2\,|\boldsymbol{\xi}| \le C\,|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| \le \|\boldsymbol{b}\|_2^2\,|\boldsymbol{\xi}| \le C\,|\boldsymbol{\xi}|,
\end{aligned} \tag{4.4}$$

onde $C > 0$ depende apenas da norma de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Agora, mostraremos que $|\mathcal{F}\{\nabla P\}(\boldsymbol{\xi})| \leq (\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{b}\|_2^2)\,|\boldsymbol{\xi}| \leq C\,|\boldsymbol{\xi}|$. Tomando o divergente da Eq. $(4.1)_1$, obtemos

$$\begin{aligned}\Delta P = \operatorname{div}(\nabla P) &= \operatorname{div}(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \operatorname{div}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \\ &= \sum_{j,k=1}^{3} \frac{\partial^2}{\partial x_j \partial x_k}(b_j b_k) - \sum_{j,k=1}^{3} \frac{\partial^2}{\partial x_j \partial x_k}(u_j u_k).\end{aligned} \tag{4.5}$$

Observe que a transformada de Fourier de ΔP é dada por

$$\mathcal{F}\{\Delta P\}(\boldsymbol{\xi},t) = \sum_{k=1}^{3}\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} \frac{\partial^2 P}{\partial x_k^2}\,d\boldsymbol{x} = -\sum_{k=1}^{3}\int_{\mathbb{R}^3} \xi_k^2 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} P\,d\boldsymbol{x} = -|\boldsymbol{\xi}|^2 \widehat{P}(\boldsymbol{\xi},t).$$

Portanto, tomando a transformada de Fourier de (4.5), obtemos

$$-|\boldsymbol{\xi}|^2\widehat{P}(\boldsymbol{\xi},t) = -\sum_{j,k=1}^{3} \xi_j\xi_k \mathcal{F}\{b_j b_k\}(\boldsymbol{\xi},t) + \sum_{j,k=1}^{3} \xi_j\xi_k \mathcal{F}\{u_j u_k\}(\boldsymbol{\xi},t).$$

Como

$$\begin{aligned}|\mathcal{F}\{u_j u_k\}(\boldsymbol{\xi},t)| &\leq \|u_j u_k(\cdot,t)\|_1 \leq \|\boldsymbol{u}(\cdot,t)\|_2^2 \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \leq C, \\ |\mathcal{F}\{b_j b_k\}(\boldsymbol{\xi},t)| &\leq \|b_j b_k(\cdot,t)\|_1 \leq \|\boldsymbol{b}(\cdot,t)\|_2^2 \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \leq C,\end{aligned}$$

temos que

$$|\boldsymbol{\xi}|^2|\widehat{P}(\boldsymbol{\xi},t)| \leq (\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2)|\boldsymbol{\xi}|^2 \leq C|\boldsymbol{\xi}|^2.$$

Portanto,

$$|\widehat{P}(\boldsymbol{\xi},t)| \leq (\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2) \leq C, \quad \text{para todo } \boldsymbol{\xi} \in \mathbb{R}^3\backslash\{\boldsymbol{0}\}.$$

Além disso, temos que $\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t) = i\boldsymbol{\xi}\widehat{P}(\boldsymbol{\xi},t)$. Consequentemente,

$$|\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t)| \leq \left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right)|\boldsymbol{\xi}| \leq C|\boldsymbol{\xi}|. \tag{4.6}$$

Portanto, das estimativas (4.4) e (4.6), obtemos

$$\begin{aligned}&|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| \\ &+|\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t)| \\ &\leq \left[\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{u}\|_2\|\boldsymbol{w}\|_2 + \|\boldsymbol{u}\|_2\|\boldsymbol{b}\|_2 + \|\boldsymbol{b}\|_2\|\boldsymbol{u}\|_2 + \|\boldsymbol{b}\|_2^2\right]|\boldsymbol{\xi}| + [\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{b}\|_2^2]|\boldsymbol{\xi}| \\ &= \left[2\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{u}\|_2\|\boldsymbol{w}\|_2 + 2\|\boldsymbol{b}\|_2^2 + 2\|\boldsymbol{u}\|_2\|\boldsymbol{b}\|_2\right]|\boldsymbol{\xi}| \\ &\leq [\|\boldsymbol{u}(\cdot,t)\|_2\,(2\|\boldsymbol{u}(\cdot,t)\|_2 + \|\boldsymbol{w}(\cdot,t)\|_2) + \|\boldsymbol{b}(\cdot,t)\|_2\,(2\|\boldsymbol{u}(\cdot,t)\|_2 + 2\|\boldsymbol{b}(\cdot,t)\|_2)]\,|\boldsymbol{\xi}| \\ &\leq C\,|\boldsymbol{\xi}|,\end{aligned}$$

onde $C > 0$ depende apenas das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0$ em $\boldsymbol{L}^2(\mathbb{R}^3)$. Isto conclui a demonstração do lema. □

Proposição 4.2. *Seja $\mathcal{K} \subset \mathbb{R}^3$ um conjunto compacto. Assumindo as mesmas hipóteses do Lema 4.1, temos*

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi}, t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi}, t)| + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi}, t)| \leq C|\boldsymbol{\xi}|^{-1},$$

para todo $t \geq 0$ e $\boldsymbol{\xi} \in \mathcal{K}$, com $\boldsymbol{\xi} \neq \boldsymbol{0}$, onde a constante $C > 0$ depende somente do conjunto $\mathcal{K}$ e das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ dos dados iniciais.

Demonstração. Inicialmente, vamos aplicar a transformada de Fourier nas Eqs. $(4.1)_1$, $(4.1)_2$ e $(4.1)_3$.

i. Transformada de Fourier para $\boldsymbol{u}_t$, $\boldsymbol{w}_t$ e $\boldsymbol{b}_t$.

Sejam $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{w} = (w_1, w_2, w_3)$ e $\boldsymbol{b} = (b_1, b_2, b_3)$. Assim,

$$\boldsymbol{u}_t = (\partial_t u_1, \partial_t u_2, \partial_t u_3),\ \boldsymbol{w}_t = (\partial_t w_1, \partial_t w_2, \partial_t w_3)\ \text{ e }\ \boldsymbol{b}_t = (\partial_t b_1, \partial_t b_2, \partial_t b_3).$$

Então,

$$\begin{aligned}
\mathcal{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi}, t) &= (\mathcal{F}\{\partial_t u_1\}(\boldsymbol{\xi}, t), \mathcal{F}\{\partial_t u_2\}(\boldsymbol{\xi}, t), \mathcal{F}\{\partial_t u_3\}(\boldsymbol{\xi}, t)) \\
\mathcal{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi}, t) &= (\mathcal{F}\{\partial_t w_1\}(\boldsymbol{\xi}, t), \mathcal{F}\{\partial_t w_2\}(\boldsymbol{\xi}, t), \mathcal{F}\{\partial_t w_3\}(\boldsymbol{\xi}, t)) \\
\mathcal{F}\{\boldsymbol{b}_t\}(\boldsymbol{\xi}, t) &= (\mathcal{F}\{\partial_t b_1\}(\boldsymbol{\xi}, t), \mathcal{F}\{\partial_t b_2\}(\boldsymbol{\xi}, t), \mathcal{F}\{\partial_t b_3\}(\boldsymbol{\xi}, t)).
\end{aligned}$$

Sejam $[\mathcal{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi}, t)]_j$, $[\mathcal{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi}, t)]_j$ e $[\mathcal{F}\{\boldsymbol{b}_t\}(\boldsymbol{\xi}, t)]_j$ a j-ésima coordenada de $\mathcal{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi}, t)$, $\mathcal{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi}, t)$ e $\mathcal{F}\{\boldsymbol{b}_t\}(\boldsymbol{\xi}, t)$, respectivamente. Observe que

$$\begin{aligned}
[\mathcal{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi}, t)]_j &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} \partial_t u_j \, d\boldsymbol{x} = \frac{\partial}{\partial t}\left[\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} u_j \, d\boldsymbol{x}\right] = \partial_t \widehat{u_j}(\boldsymbol{\xi}, t), \\
[\mathcal{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi}, t)]_j &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} \partial_t w_j \, d\boldsymbol{x} = \frac{\partial}{\partial t}\left[\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} w_j \, d\boldsymbol{x}\right] = \partial_t \widehat{w_j}(\boldsymbol{\xi}, t), \\
[\mathcal{F}\{\boldsymbol{b}_t\}(\boldsymbol{\xi}, t)]_j &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} \partial_t b_j \, d\boldsymbol{x} = \frac{\partial}{\partial t}\left[\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} b_j \, d\boldsymbol{x}\right] = \partial_t \widehat{b_j}(\boldsymbol{\xi}, t).
\end{aligned}$$

Desta forma,

$$\mathcal{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi}, t) = \widehat{\boldsymbol{u}}_t(\boldsymbol{\xi}, t), \quad \mathcal{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi}, t) = \widehat{\boldsymbol{w}_t}(\boldsymbol{\xi}, t), \quad \mathcal{F}\{\boldsymbol{b}_t\}(\boldsymbol{\xi}, t) = \widehat{\boldsymbol{b}}_t(\boldsymbol{\xi}, t). \tag{4.7}$$

ii. Transformada de Fourier de $\Delta\boldsymbol{u}$, $\Delta\boldsymbol{w}$ e $\Delta\boldsymbol{b}$.

Note que $\Delta\boldsymbol{u} = (\Delta u_1, \Delta u_2, \Delta u_3)$, $\Delta\boldsymbol{w} = (\Delta w_1, \Delta w_2, \Delta w_3)$ e $\Delta\boldsymbol{b} = (\Delta b_1, \Delta b_2, \Delta b_3)$. Então,

$$\begin{aligned}
\mathcal{F}\{\Delta\boldsymbol{u}\}(\boldsymbol{\xi}, t) &= (\mathcal{F}\{\Delta u_1\}(\boldsymbol{\xi}, t), \mathcal{F}\{\Delta u_2\}(\boldsymbol{\xi}, t), \mathcal{F}\{\Delta u_3\}(\boldsymbol{\xi}, t)), \\
\mathcal{F}\{\Delta\boldsymbol{w}\}(\boldsymbol{\xi}, t) &= (\mathcal{F}\{\Delta w_1\}(\boldsymbol{\xi}, t), \mathcal{F}\{\Delta w_2\}(\boldsymbol{\xi}, t), \mathcal{F}\{\Delta w_3\}(\boldsymbol{\xi}, t)), \\
\mathcal{F}\{\Delta\boldsymbol{b}\}(\boldsymbol{\xi}, t) &= (\mathcal{F}\{\Delta b_1\}(\boldsymbol{\xi}, t), \mathcal{F}\{\Delta b_2\}(\boldsymbol{\xi}, t), \mathcal{F}\{\Delta b_3\}(\boldsymbol{\xi}, t)).
\end{aligned}$$

Sejam $[\mathcal{F}\{\Delta \boldsymbol{u}\}(\boldsymbol{\xi},t)]_j$, $[\mathcal{F}\{\Delta \boldsymbol{w}\}(\boldsymbol{\xi},t)]_j$ e $[\mathcal{F}\{\Delta \boldsymbol{b}\}(\boldsymbol{\xi},t)]_j$ a j-ésima coordenada de $\mathcal{F}\{\Delta \boldsymbol{u}\}(\boldsymbol{\xi},t)$, $\mathcal{F}\{\Delta \boldsymbol{w}\}(\boldsymbol{\xi},t)$ e $\mathcal{F}\{\Delta \boldsymbol{b}\}(\boldsymbol{\xi},t)$, respectivamente. Como visto no Lema 4.1,

$$[\mathcal{F}\{\Delta \boldsymbol{u}\}(\boldsymbol{\xi},t)]_j = \sum_{k=1}^{3}\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\frac{\partial^2 u_j}{\partial x_k^2}\,d\boldsymbol{x} = -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\xi_k^2 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}u_j\,dx = -|\boldsymbol{\xi}|^2\widehat{u}_j(\boldsymbol{\xi},t),$$
$$[\mathcal{F}\{\Delta \boldsymbol{w}\}(\boldsymbol{\xi},t)]_j = \sum_{k=1}^{3}\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\frac{\partial^2 w_j}{\partial x_k^2}\,d\boldsymbol{x} = -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\xi_k^2 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}w_j\,d\boldsymbol{x} = -|\boldsymbol{\xi}|^2\widehat{w}_j(\boldsymbol{\xi},t),$$
$$[\mathcal{F}\{\Delta \boldsymbol{b}\}(\boldsymbol{\xi},t)]_j = \sum_{k=1}^{3}\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\frac{\partial^2 b_j}{\partial x_k^2}\,d\boldsymbol{x} = -\sum_{k=1}^{3}\int_{\mathbb{R}^3}\xi_k^2 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}b_j\,d\boldsymbol{x} = -|\boldsymbol{\xi}|^2\widehat{b}_j(\boldsymbol{\xi},t).$$

Consequentemente,

$$\mathcal{F}\{\Delta \boldsymbol{u}\}(\boldsymbol{\xi},t) = -|\boldsymbol{\xi}|^2\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t),\ \mathcal{F}\{\Delta \boldsymbol{w}\}(\boldsymbol{\xi},t) = -|\boldsymbol{\xi}|^2\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t),\ \mathcal{F}\{\Delta \boldsymbol{b}\}(\boldsymbol{\xi},t) = -|\boldsymbol{\xi}|^2\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t). \tag{4.8}$$

iii. Transformada de Fourier de $\operatorname{rot}\boldsymbol{u}$ e $\operatorname{rot}\boldsymbol{w}$.

Sejam $\operatorname{rot}\boldsymbol{u}$ e $\operatorname{rot}\boldsymbol{w}$ dados, respectivamente, por

$$\begin{aligned}
\operatorname{rot}\boldsymbol{u} &= \left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)\\
\operatorname{rot}\boldsymbol{w} &= \left(\frac{\partial w_3}{\partial x_2}-\frac{\partial w_2}{\partial x_3}, \frac{\partial w_1}{\partial x_3}-\frac{\partial w_3}{\partial x_1}, \frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right).
\end{aligned}$$

Então, calculando a transformada de Fourier e integrando por partes, obtemos

$$\begin{aligned}
[\mathcal{F}\{\operatorname{rot}\boldsymbol{u}\}(\boldsymbol{\xi},t)]_1 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_3}{\partial x_2}\right)d\boldsymbol{x} - \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_2}{\partial x_3}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} i\xi_2 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}u_3\,d\boldsymbol{x} - \int_{\mathbb{R}^3} i\xi_3 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}u_2\,d\boldsymbol{x} = i(\xi_2\widehat{u_3}-\xi_3\widehat{u_2});
\end{aligned}$$

$$\begin{aligned}
[\mathcal{F}\{\operatorname{rot}\boldsymbol{u}\}(\boldsymbol{\xi},t)]_2 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_1}{\partial x_3}\right)d\boldsymbol{x} - \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_3}{\partial x_1}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} i\xi_3 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}u_1\,d\boldsymbol{x} - \int_{\mathbb{R}^3} i\xi_1 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}u_3\,d\boldsymbol{x} = i(\xi_3\widehat{u_1}-\xi_1\widehat{u_3});
\end{aligned}$$

$$\begin{aligned}
[\mathcal{F}\{\operatorname{rot}\boldsymbol{u}\}(\boldsymbol{\xi},t)]_3 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_2}{\partial x_1}\right)d\boldsymbol{x} - \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial u_1}{\partial x_2}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} i\xi_1 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}u_2\,d\boldsymbol{x} - \int_{\mathbb{R}^3} i\xi_2 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}u_1\,d\boldsymbol{x} = i(\xi_1\widehat{u_2}-\xi_2\widehat{u_1}).
\end{aligned}$$

Similarmente, temos, para $\operatorname{rot}\boldsymbol{w}$, que

$$\begin{aligned}
[\mathcal{F}\{\operatorname{rot}\boldsymbol{w}\}(\boldsymbol{\xi},t)]_1 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_3}{\partial x_2}-\frac{\partial w_2}{\partial x_3}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_3}{\partial x_2}\right)d\boldsymbol{x}-\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_2}{\partial x_3}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} i\xi_2 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} w_3\,d\boldsymbol{x}-\int_{\mathbb{R}^3} i\xi_3 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} w_2\,d\boldsymbol{x}=i(\xi_2\widehat{w_3}-\xi_3\widehat{w_2});
\end{aligned}$$

$$\begin{aligned}
[\mathcal{F}\{\operatorname{rot}\boldsymbol{w}\}(\boldsymbol{\xi},t)]_2 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_1}{\partial x_3}-\frac{\partial w_3}{\partial x_1}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_1}{\partial x_3}\right)d\boldsymbol{x}-\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_3}{\partial x_1}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} i\xi_3 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} w_1\,d\boldsymbol{x}-\int_{\mathbb{R}^3} i\xi_1 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} w_3\,d\boldsymbol{x}=i(\xi_3\widehat{w_1}-\xi_1\widehat{w_3});
\end{aligned}$$

$$\begin{aligned}
[\mathcal{F}\{\operatorname{rot}\boldsymbol{w}\}(\boldsymbol{\xi},t)]_3 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_2}{\partial x_1}\right)d\boldsymbol{x}-\int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_1}{\partial x_2}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} i\xi_1 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} w_2\,d\boldsymbol{x}-\int_{\mathbb{R}^3} i\xi_2 e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} w_1\,d\boldsymbol{x}=i(\xi_1\widehat{w_2}-\xi_2\widehat{w_1}).
\end{aligned}$$

Desta forma, concluímos que

$$\mathcal{F}\{\operatorname{rot}\boldsymbol{u}\}(\boldsymbol{\xi},t)=i\begin{pmatrix}0&-\xi_3&\xi_2\\ \xi_3&0&-\xi_1\\ -\xi_2&\xi_1&0\end{pmatrix}\begin{pmatrix}\widehat{u_1}\\ \widehat{u_2}\\ \widehat{u_3}\end{pmatrix} \tag{4.9}$$

e

$$\mathcal{F}\{\operatorname{rot}\boldsymbol{w}\}(\boldsymbol{\xi},t)=i\begin{pmatrix}0&-\xi_3&\xi_2\\ \xi_3&0&-\xi_1\\ -\xi_2&\xi_1&0\end{pmatrix}\begin{pmatrix}\widehat{w_1}\\ \widehat{w_2}\\ \widehat{w_3}\end{pmatrix}. \tag{4.10}$$

iv. Transformada de Fourier de $\nabla(\operatorname{div}\boldsymbol{w})$.

Um cálculo direto nos mostra que $\nabla(\operatorname{div}\boldsymbol{w})$ é dado por

$$\left(\frac{\partial^2 w_1}{\partial x_1^2}+\frac{\partial^2 w_2}{\partial x_1\partial x_2}+\frac{\partial^2 w_3}{\partial x_1\partial x_3},\frac{\partial^2 w_1}{\partial x_2\partial x_1}+\frac{\partial^2 w_2}{\partial x_2^2}+\frac{\partial^2 w_3}{\partial x_2\partial x_3},\frac{\partial^2 w_1}{\partial x_3\partial x_1}+\frac{\partial^2 w_2}{\partial x_3\partial x_2}+\frac{\partial^2 w_3}{\partial x_3^2}\right).$$

Assim,

$$\begin{aligned}
[\mathcal{F}\{\nabla(\operatorname{div}\boldsymbol{w})\}(\boldsymbol{\xi},t)]_1 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial^2 w_1}{\partial x_1^2}+\frac{\partial^2 w_2}{\partial x_1\partial x_2}+\frac{\partial^2 w_3}{\partial x_1\partial x_3}\right)d\boldsymbol{x}\\
&= -\xi_1^2\widehat{w_1}-\xi_1\xi_2\widehat{w_2}-\xi_1\xi_3\widehat{w_3},
\end{aligned}$$

$$
\begin{aligned}
[\mathcal{F}\{\nabla(\operatorname{div}\boldsymbol{w})\}(\boldsymbol{\xi},t)]_2 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial^2 w_1}{\partial x_2\partial x_1}+\frac{\partial^2 w_2}{\partial x_2^2}+\frac{\partial^2 w_3}{\partial x_2\partial x_3}\right)d\boldsymbol{x}\\
&= -\xi_2\xi_1\widehat{w_1}-\xi_2^2\widehat{w_2}-\xi_2\xi_3\widehat{w_3}
\end{aligned}
$$

$$
\begin{aligned}
[\mathcal{F}\{\nabla(\operatorname{div}\boldsymbol{w})\}(\boldsymbol{\xi},t)]_3 &= \int_{\mathbb{R}^3} e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial^2 w_1}{\partial x_3\partial x_1}+\frac{\partial^2 w_2}{\partial x_3\partial x_2}+\frac{\partial^2 w_3}{\partial x_3^2}\right)d\boldsymbol{x}\\
&= -\xi_3\xi_1\widehat{w_1}-\xi_3\xi_2\widehat{w_2}-\xi_3^2\widehat{w_3}.
\end{aligned}
$$

Assim,

$$
\mathcal{F}\{\nabla(\operatorname{div}\boldsymbol{w})\}(\boldsymbol{\xi},t) = -\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3\\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3\\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2\end{pmatrix}\begin{pmatrix}\widehat{w_1}\\ \widehat{w_2}\\ \widehat{w_3}\end{pmatrix}. \tag{4.11}
$$

Agora, tomando as transformadas de Fourier das Eqs. $(4.1)_1$, $(4.1)_2$ e $(4.1)_3$ e usando as identidades (4.7)–(4.11), obtemos

$$
\begin{cases}
\widehat{\boldsymbol{u}}_t+|\boldsymbol{\xi}|^2\widehat{\boldsymbol{u}}-\dfrac{i}{2}L(\boldsymbol{\xi})\widehat{\boldsymbol{w}}=\boldsymbol{G}_1(\boldsymbol{\xi},t), \quad \boldsymbol{\xi}\in\mathbb{R}^3,\ t>0,\\
\widehat{\boldsymbol{w}}_t+(|\boldsymbol{\xi}|^2+1)\widehat{\boldsymbol{w}}-\dfrac{i}{2}L(\boldsymbol{\xi})\widehat{\boldsymbol{u}}-P(\boldsymbol{\xi})\widehat{\boldsymbol{w}}=\boldsymbol{G}_2(\boldsymbol{\xi},t), \quad \boldsymbol{\xi}\in\mathbb{R}^3,\ t>0,\\
\widehat{\boldsymbol{b}}_t+|\boldsymbol{\xi}|^2\widehat{\boldsymbol{b}}=\boldsymbol{G}_3(\boldsymbol{\xi},t), \quad \boldsymbol{\xi}\in\mathbb{R}^3,\ t>0,
\end{cases} \tag{4.12}
$$

onde

$$
\begin{cases}
\boldsymbol{G}_1(\boldsymbol{\xi},t):=\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)-\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)-\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t),\\
\boldsymbol{G}_2(\boldsymbol{\xi},t):=-\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t),\\
\boldsymbol{G}_3(\boldsymbol{\xi},t):=\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)-\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)
\end{cases} \tag{4.13}
$$

e

$$
L(\boldsymbol{\xi}):=\begin{pmatrix}0 & -\xi_3 & \xi_2\\ \xi_3 & 0 & -\xi_1\\ -\xi_2 & \xi_1 & 0\end{pmatrix} \ \text{ e } \ P(\boldsymbol{\xi}):=-\begin{pmatrix}\xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3\\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3\\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2\end{pmatrix}. \tag{4.14}
$$

Agora, sejam $\boldsymbol{z} := (\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})^T$ e $\boldsymbol{G} := (\boldsymbol{G}_1,\boldsymbol{G}_2,\boldsymbol{G}_3)^T$. Então, podemos reescrever o sistema (4.12) como segue:

$$
\widehat{\boldsymbol{z}}_t+A(\boldsymbol{\xi})\widehat{\boldsymbol{z}}=\boldsymbol{G}(\boldsymbol{\xi},t), \tag{4.15}
$$

onde $A(\boldsymbol{\xi})$ é a matriz Hermitiana

$$
A(\boldsymbol{\xi}):=\begin{pmatrix}|\boldsymbol{\xi}|^2 I & B(\boldsymbol{\xi}) & \mathcal{O}\\ B(\boldsymbol{\xi}) & R(\boldsymbol{\xi})+(|\boldsymbol{\xi}|^2+1)I & \mathcal{O}\\ \mathcal{O} & \mathcal{O} & |\boldsymbol{\xi}|^2 I\end{pmatrix}, \tag{4.16}
$$

com $B(\boldsymbol{\xi}) := -\dfrac{i}{2}L(\boldsymbol{\xi})$, $R(\boldsymbol{\xi}) := -P(\boldsymbol{\xi})$, I é a identidade 3×3 e $\mathcal{O}$ é a matriz nula 3×3. Então, pelo Lema 2.39, existe uma constante $\sigma > 0$ independente de $\boldsymbol{\xi}$ e t, tal que

$$
\|e^{-tA(\boldsymbol{\xi})}\| \le e^{-\sigma|\boldsymbol{\xi}|^2 t}, \quad \forall\, t\ge 0 \ \text{ e } \ \boldsymbol{\xi}\in\mathbb{R}^3, \tag{4.17}
$$

onde $\|\cdot\|$ denota a norma euclidiana da matriz. Multiplicando a equação (4.15) pelo fator integrante $e^{tA(\boldsymbol{\xi})}$, obtemos

$$\frac{d}{dt}\left(e^{tA(\boldsymbol{\xi})}\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)\right) = e^{tA(\boldsymbol{\xi})}\boldsymbol{G}(\boldsymbol{\xi},t).$$

Integrando com relação ao tempo, obtemos

$$\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t) = e^{-tA(\boldsymbol{\xi})}\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi}) + \int_0^t e^{-(t-s)A(\boldsymbol{\xi})}\boldsymbol{G}(\boldsymbol{\xi},s)\,ds,$$

onde $\widehat{\boldsymbol{z}_0} := \widehat{\boldsymbol{z}}(\cdot,0) = (\widehat{\boldsymbol{u}_0}, \widehat{\boldsymbol{w}_0}, \widehat{\boldsymbol{b}_0})^T$. Usando (4.17), temos:

$$\begin{aligned}|\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| &\leq \|e^{-tA(\boldsymbol{\xi})}\||\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| + \int_0^t \|e^{-(t-s)A(\boldsymbol{\xi})}\||G(\boldsymbol{\xi},s)|\,ds \\ &\leq e^{-\sigma|\boldsymbol{\xi}|^2 t}|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| + \int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}|\boldsymbol{G}(\boldsymbol{\xi},s)|\,ds. \end{aligned} \tag{4.18}$$

Pelo Lema 4.1, $|\boldsymbol{G}(\boldsymbol{\xi},t)| \leq C\,|\boldsymbol{\xi}|, \forall\, t \geq 0$ e $\boldsymbol{\xi} \in \mathbb{R}^3$, onde $C > 0$ depende somente das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Da desigualdade (4.18), concluímos que

$$|\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| \leq e^{-\sigma|\boldsymbol{\xi}|^2 t}|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| + C\int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}|\boldsymbol{\xi}|\,ds. \tag{4.19}$$

Note que,

$$|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| \leq |\widehat{\boldsymbol{u}_0}(\boldsymbol{\xi})| + |\widehat{\boldsymbol{w}_0}(\boldsymbol{\xi})| + |\widehat{\boldsymbol{b}_0}(\boldsymbol{\xi})| \leq \|\boldsymbol{u}_0\|_1 + \|\boldsymbol{w}_0\|_1 + \|\boldsymbol{b}_0\|_1 \leq C,$$

para todo $\boldsymbol{\xi} \in \mathbb{R}^3$ e alguma constante positiva C dependendo das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0$ em $\boldsymbol{L}^1$. Observe ainda que

$$\int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}|\boldsymbol{\xi}|\,ds = e^{-\sigma|\boldsymbol{\xi}|^2 t}|\boldsymbol{\xi}|\int_0^t e^{\sigma|\boldsymbol{\xi}|^2 s}\,ds = e^{-\sigma|\boldsymbol{\xi}|^2 t}|\boldsymbol{\xi}|\frac{1}{\sigma|\boldsymbol{\xi}|^2}\left[e^{\sigma|\boldsymbol{\xi}|^2 s}\right]_{s=0}^{s=t} = \frac{1}{\sigma|\boldsymbol{\xi}|}\left[1 - e^{-\sigma|\boldsymbol{\xi}|^2 t}\right].$$

Portanto, da desigualdade (4.19), obtemos, para todos $t \geq 0$ e $\boldsymbol{\xi} \neq \boldsymbol{0}$, que

$$\begin{aligned}|\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| &\leq Ce^{-\sigma|\boldsymbol{\xi}|^2 t} + \frac{C}{\sigma|\boldsymbol{\xi}|}\left(1 - e^{-\sigma|\boldsymbol{\xi}|^2 t}\right) \\ &\leq \frac{C}{|\boldsymbol{\xi}|} + Ce^{-\sigma|\boldsymbol{\xi}|^2 t}\left(1 - \frac{1}{|\boldsymbol{\xi}|}\right),\end{aligned}$$

onde $C \in \mathbb{R}^+$ dependendo somente das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Uma vez que $\mathcal{K} \in \mathbb{R}^3$ é compacto, encontramos, para todos $t \geq 0$ e $\boldsymbol{\xi} \in \mathcal{K}\setminus\{\boldsymbol{0}\}$, que

$$|\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1},$$

onde $C > 0$ depende somente do compacto $\mathcal{K}$ e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. □

Teorema 4.3. *Seja* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *uma solução suave do problema de Cauchy* (4.1) *com* $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$. *Se* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$, *com* $\operatorname{div}\boldsymbol{u}_0 = \operatorname{div}\boldsymbol{b}_0 = 0$, *então existe uma constante* $C > 0$ *tal que*

$$\|\boldsymbol{u}(\cdot,t)\|_2 + \|\boldsymbol{w}(\cdot,t)\|_2 + \|\boldsymbol{b}(\cdot,t)\|_2 \leq C(t+1)^{-\frac{3}{4}}, \quad \forall\, t \geq 0.$$

A constante $C > 0$ *depende somente das normas em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$ *de* $\boldsymbol{u}_0$, $\boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$.

Demonstração. Relembremos que, da desigualdade (4.3), $\boldsymbol{u}, \boldsymbol{w}$ e $\boldsymbol{b} \in \boldsymbol{L}^2(\mathbb{R}^3)$ para todo $t \geq 0$. Logo, as transformadas $\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{w}}$ e $\widehat{\boldsymbol{b}}$ estão bem definidas. Aplicando a transformada de Fourier em (4.2) e usando o teorema de Plancherel, obtemos

$$\begin{aligned} \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right) &\leq -\left(\|\widehat{\nabla \boldsymbol{u}}\|_2^2 + \|\widehat{\nabla \boldsymbol{w}}\|_2^2 + \|\widehat{\nabla \boldsymbol{b}}\|_2^2\right) \\ &= -\int_{\mathbb{R}^3}\left(|\widehat{\nabla \boldsymbol{u}}|^2 + |\widehat{\nabla \boldsymbol{w}}|^2 + |\widehat{\nabla \boldsymbol{b}}|^2\right) d\boldsymbol{\xi}. \end{aligned}$$

Note que $\widehat{\nabla \boldsymbol{u}}(\boldsymbol{\xi}, t) = i\, \boldsymbol{\xi}\, \widehat{\boldsymbol{u}}(\boldsymbol{\xi}, t)$. Assim, $|\widehat{\nabla \boldsymbol{u}}|^2 = |\boldsymbol{\xi}|^2 |\widehat{\boldsymbol{u}}|^2$. Analogamente, temos $|\widehat{\nabla \boldsymbol{w}}|^2 = |\boldsymbol{\xi}|^2 |\widehat{\boldsymbol{w}}|^2$ e $|\widehat{\nabla \boldsymbol{b}}|^2 = |\boldsymbol{\xi}|^2 |\widehat{\boldsymbol{b}}|^2$. Portanto,

$$\frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right) \leq -\int_{\mathbb{R}^3} |\boldsymbol{\xi}|^2 (|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2) d\boldsymbol{\xi}.$$

Agora, defina

$$S(t) := \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| \leq r(t)\}, \tag{4.20}$$

onde $r(t) := \sqrt{\frac{3}{t+1}}$. Note que $S(t)^c = \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| > r(t)\} = \left\{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| > \sqrt{\frac{3}{t+1}}\right\}$. Como $\mathbb{R}^3 \equiv S(t) \cup S(t)^c$, temos que

$$\begin{aligned} \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right) &\leq -\int_{S(t)^c} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2\right) d\boldsymbol{\xi} \\ &\quad -\int_{S(t)} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2\right) d\boldsymbol{\xi} \\ &\leq -\int_{S(t)^c} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2\right) d\boldsymbol{\xi} \\ &\leq -\frac{3}{t+1}\int_{S(t)^c} \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2\right) d\boldsymbol{\xi}. \end{aligned}$$

Logo,

$$\begin{aligned} \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right) &\leq -\frac{3}{t+1}\int_{\mathbb{R}^3} \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2\right) d\boldsymbol{\xi} \\ &\quad +\frac{3}{t+1}\int_{S(t)} \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2\right) d\boldsymbol{\xi}. \end{aligned}$$

Isto é,

$$\begin{aligned} \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right) &+ \frac{3}{t+1}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right) \\ &\leq \frac{3}{t+1}\int_{S(t)} \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2\right) d\boldsymbol{\xi}. \end{aligned}$$

Multiplicando a desigualdade acima pelo fator integrante $(t+1)^3$, obtemos

$$\begin{aligned} (t+1)^3 \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right) &+ 3(t+1)^2\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right) \\ &\leq 3(t+1)^2\int_{S(t)} \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2 + |\widehat{\boldsymbol{b}}|^2\right) d\boldsymbol{\xi}, \end{aligned}$$

concluindo, assim, que

$$\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}\|_2^2+\|\widehat{\boldsymbol{w}}\|_2^2+\|\widehat{\boldsymbol{b}}\|_2^2\right)\right]\leq 3(t+1)^2\int_{S(t)}\left(|\widehat{\boldsymbol{u}}|^2+|\widehat{\boldsymbol{w}}|^2+|\widehat{\boldsymbol{b}}|^2\right)d\boldsymbol{\xi}. \quad (4.21)$$

Da Proposição 4.2, temos

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2+|\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2+|\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\leq C|\boldsymbol{\xi}|^{-2},\;\;\forall\,\boldsymbol{\xi}\in S(t)\backslash\{\mathbf{0}\}\;\text{ e }\;\forall\,t\geq 0, \quad (4.22)$$

onde a constante $C>0$ depende somente das normas de $\boldsymbol{u}_0,\boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. De (4.21) e (4.22), obtemos

$$\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}\|_2^2+\|\widehat{\boldsymbol{w}}\|_2^2+\|\widehat{\boldsymbol{b}}\|_2^2\right)\right]\leq C(t+1)^2\int_{S(t)}|\boldsymbol{\xi}|^{-2}d\boldsymbol{\xi}.$$

Usando coordenadas esféricas, é fácil obter que

$$\int_{S(t)}|\boldsymbol{\xi}|^{-2}d\boldsymbol{\xi}=4\pi r(t)=4\sqrt{3}\pi(t+1)^{-1/2},$$

onde $r(t)$ é o raio da bola $S(t)$. Segue, então, que

$$\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}\|_2^2+\|\widehat{\boldsymbol{w}}\|_2^2+\|\widehat{\boldsymbol{b}}\|_2^2\right)\right]\leq C(t+1)^{3/2}.$$

Integrando com respeito ao tempo, obtemos

$$(t+1)^3\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2+\|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2+\|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right)\leq\|\widehat{\boldsymbol{u}_0}\|_2^2+\|\widehat{\boldsymbol{w}_0}\|_2^2+\|\widehat{\boldsymbol{b}_0}\|_2^2+C[(t+1)^{\frac{5}{2}}-1],$$

para todo $t\geq 0$, onde $\widehat{\boldsymbol{u}_0}=\widehat{\boldsymbol{u}}(\cdot,0),\widehat{\boldsymbol{w}_0}=\widehat{\boldsymbol{w}}(\cdot,0)$ e $\widehat{\boldsymbol{b}_0}=\widehat{\boldsymbol{b}}(\cdot,0)$. Usando o teorema de Plancherel, encontramos

$$\begin{aligned}(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2+\|\boldsymbol{w}(\cdot,t)\|_2^2+\|\boldsymbol{b}(\cdot,t)\|_2^2\right) &\leq \|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{w}_0\|_2^2+\|\boldsymbol{b}_0\|_2^2+C[(t+1)^{5/2}-1]\\ &\leq \|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{w}_0\|_2^2+\|\boldsymbol{b}_0\|_2^2+C(t+1)^{5/2}\\ &\leq C(t+1)^{5/2},\end{aligned}$$

ou seja,

$$\|\boldsymbol{u}(\cdot,t)\|_2^2+\|\boldsymbol{w}(\cdot,t)\|_2^2+\|\boldsymbol{b}(\cdot,t)\|_2^2\leq C(t+1)^{-1/2},\quad\forall\,t\geq 0, \quad (4.23)$$

onde a constante $C>0$ depende das normas de $\boldsymbol{u}_0,\boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Esta estimativa pode ser melhorada como segue. Pelo Lema 4.1 e pelas estimativas (4.19) e (4.23), temos, para alguma constante positiva σ (independente $t,\boldsymbol{\xi},\boldsymbol{u}_0,\boldsymbol{w}_0,\boldsymbol{b}_0$) e C (dependendo somente das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0,\boldsymbol{w}_0,\boldsymbol{b}_0$) que

$$\begin{aligned}&|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|+|\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|+|\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|\\ \leq\;& C\left\{\|\boldsymbol{u}_0\|_1+\|\boldsymbol{w}_0\|_1+\|\boldsymbol{b}_0\|_1+\int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}|\boldsymbol{\xi}|(\|\boldsymbol{u}(\cdot,s)\|_2^2+\|\boldsymbol{w}(\cdot,s)\|_2^2+\|\boldsymbol{b}(\cdot,s)\|_2^2)\,ds\right\}\\ \leq\;& C\left\{\|\boldsymbol{u}_0\|_1+\|\boldsymbol{w}_0\|_1+\|\boldsymbol{b}_0\|_1+\int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}|\boldsymbol{\xi}|(s+1)^{-1/2}\,ds\right\}\\ \leq\;& C\left\{\|\boldsymbol{u}_0\|_1+\|\boldsymbol{w}_0\|_1+\|\boldsymbol{b}_0\|_1+\int_0^t [2e\sigma(t-s)]^{-1/2}(s+1)^{-1/2}\,ds\right\}\\ \leq\;& C\left\{\|\boldsymbol{u}_0\|_1+\|\boldsymbol{w}_0\|_1+\|\boldsymbol{b}_0\|_1+\int_0^t (t-s)^{-1/2}(s+1)^{-1/2}\,ds\right\},\end{aligned}$$

onde, na terceira desigualdade acima, usamos o fato que $e^{-a^2\tau}\, a \leq (2e\tau)^{-1/2}$, para todo $a \geq 0, \tau > 0$. Como $\displaystyle\int_0^t (t-s)^{-1/2}(s+1)^{-1/2}ds < \pi$ para todo $t > 0$, obtemos

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2 \leq C, \;\; \forall\, \boldsymbol{\xi} \in \mathbb{R}^3, t > 0$$

(compare com a estimativa (4.22)), onde a constante $C > 0$ depende somentes das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Da desigualdade (4.21), obtemos, para todo $t > 0$, que

$$\begin{aligned}\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right)\right] &\leq C(t+1)^2 r(t)^3 \\ &\leq C(t+1)^{1/2},\end{aligned}$$

pois $r(t) = \sqrt{3}\,(t+1)^{-1/2}$. Itegrando em relação a t e usando novamente o teorema de Plancherel, obtemos

$$\begin{aligned}(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right) &\leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \\ &\quad + C[(t+1)^{\frac{3}{2}} - 1] \\ &\leq C(t+1)^{3/2}.\end{aligned}$$

Portanto,

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 \leq C(t+1)^{-3/2}, \quad \forall\, t \geq 0,$$

onde a constante $C > 0$ depende apenas das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Isto finaliza a demonstração do teorema. $\square$

4.1.2 Taxa de decaimento melhorada para a velocidade angular

Vamos mostrar agora que a taxa de decaimento para $\|\boldsymbol{w}(\cdot,t)\|_2$ apresentada no Teorema 4.3 pode ser melhorada. Para isso, defina o tempo t_0 por

$$t_0 := 2^{20}\left(1 + \left\|\boldsymbol{u}_0\right\|_2^2 + \left\|\boldsymbol{w}_0\right\|_2^2 + \left\|\boldsymbol{b}_0\right\|_2^2\right)^2. \tag{4.24}$$

Primeiro, provaremos o seguinte Lema.

Lema 4.4. *Seja t_0 dado por* (4.24). *Assumindo as mesmas hipóteses do Teorema 4.3, temos*

$$\|\nabla\boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla\boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla\boldsymbol{b}(\cdot,t)\|_2^2 \leq \frac{2^{-20}}{1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2}, \quad \forall\, t \geq t_0.$$

Demonstração. Integrando a desigualdade de energia (4.2) com respeito a variável temporal, obtemos

$$\int_0^t \left(\|\nabla\boldsymbol{u}(\cdot,\tau)\|_2^2 + \|\nabla\boldsymbol{w}(\cdot,\tau)\|_2^2 + \|\nabla\boldsymbol{b}(\cdot,\tau)\|_2^2\right) d\tau \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2, \tag{4.25}$$

para todo $t \geq 0$. Então, tomando $t = t_0$ em (4.25), do teorema do valor médio para integrais de Lebesgue, concluímos que existe um conjunto $E \subseteq (0, t_0)$ com medida positiva tal que

$$\begin{aligned} &t_0 \left(\|\nabla \boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^2\right) \\ &\quad \leq \int_0^{t_0} \left(\|\nabla \boldsymbol{u}(\cdot,\tau)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,\tau)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,\tau)\|_2^2\right) d\tau, \quad \forall\, t \in E. \end{aligned}$$

Logo, por (4.25), temos

$$\|\nabla \boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^2 \leq \frac{\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2}{t_0}, \quad \forall\, t \in E. \tag{4.26}$$

A seguir, usaremos as seguintes notações:

$$\begin{aligned} &\boldsymbol{z}(\cdot,t) := (\boldsymbol{u}(\cdot,t), \boldsymbol{w}(\cdot,t), \boldsymbol{b}(\cdot,t)), \\ &\boldsymbol{z}_0 := (\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0) \\ &\|\boldsymbol{z}_0\|_2^2 := \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2, \\ &\|\boldsymbol{z}(\cdot,t)\|_2^2 := \|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2, \\ &\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 := \|\nabla \boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^2, \\ &\|D^2 \boldsymbol{z}(\cdot,t)\|_2^2 := \|D^2 \boldsymbol{u}(\cdot,t)\|_2^2 + \|D^2 \boldsymbol{w}(\cdot,t)\|_2^2 + \|D^2 \boldsymbol{b}(\cdot,t)\|_2^2, \end{aligned}$$

onde

$$\|D^2 \boldsymbol{v}(\cdot,t)\|_2^2 := \sum_{j,k=1}^{3} \|D_j D_k \boldsymbol{v}(\cdot,t)\|_2^2, \quad \text{com } D_l = \partial/\partial x_l.$$

Derivando as equações $(4.1)_1, (4.1)_2$ e $(4.1)_3$ em relação a x_l, obtemos

$$\begin{cases} \partial_t D_l \boldsymbol{u} + (D_l \boldsymbol{u} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) D_l \boldsymbol{u} - \Delta D_l \boldsymbol{u} + \nabla D_l P \\ \qquad\qquad\qquad\qquad = \dfrac{1}{2} \operatorname{rot} D_l \boldsymbol{w} + (D_l \boldsymbol{b} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla) D_l \boldsymbol{b}, \\ \partial_t D_l \boldsymbol{w} + (D_l \boldsymbol{u} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{u} \cdot \nabla) D_l \boldsymbol{w} - \Delta D_l \boldsymbol{w} - \nabla(\operatorname{div} D_l \boldsymbol{w}) + D_l \boldsymbol{w} = \dfrac{1}{2} \operatorname{rot} D_l \boldsymbol{u}, \\ \partial_t D_l \boldsymbol{b} + (D_l \boldsymbol{u} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{u} \cdot \nabla) D_l \boldsymbol{b} - \Delta D_l \boldsymbol{b} = (D_l \boldsymbol{b} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{b} \cdot \nabla) D_l \boldsymbol{u}. \end{cases} \tag{4.27}$$

Multiplicando as equações $(4.27)_1$ por $D_l \boldsymbol{u}$, $(4.27)_2$ por $D_l \boldsymbol{w}$ e $(4.27)_3$ por $D_l \boldsymbol{b}$ em $\boldsymbol{L}^2$, obtemos:

$$\begin{cases} \dfrac{1}{2} \dfrac{d}{dt} \|D_l \boldsymbol{u}\|_2^2 + \|\nabla D_l \boldsymbol{u}\|_2^2 = \dfrac{1}{2} (\operatorname{rot} D_l \boldsymbol{w}, D_l \boldsymbol{u}) + ((D_l \boldsymbol{b} \cdot \nabla)\boldsymbol{b}, D_l \boldsymbol{u}) \\ \qquad\qquad\qquad\qquad + ((\boldsymbol{b} \cdot \nabla) D_l \boldsymbol{b}, D_l \boldsymbol{u}) - ((D_l \boldsymbol{u} \cdot \nabla)\boldsymbol{u}, D_l \boldsymbol{u}) \\ \dfrac{1}{2} \dfrac{d}{dt} \|D_l \boldsymbol{w}\|_2^2 + \|\nabla D_l \boldsymbol{w}\|_2^2 + \|\operatorname{div} D_l \boldsymbol{w}\|_2^2 + \|D_l \boldsymbol{w}\|_2^2 = \dfrac{1}{2} (\operatorname{rot} D_l \boldsymbol{u}, D_l \boldsymbol{w}) \\ \qquad\qquad\qquad\qquad - ((D_l \boldsymbol{u} \cdot \nabla)\boldsymbol{w}, D_l \boldsymbol{w}) \\ \dfrac{1}{2} \dfrac{d}{dt} \|D_l \boldsymbol{b}\|_2^2 + \|\nabla D_l \boldsymbol{b}\|_2^2 = -((D_l \boldsymbol{u} \cdot \nabla)\boldsymbol{b}, D_l \boldsymbol{b}) + ((D_l \boldsymbol{b} \cdot \nabla)\boldsymbol{u}, D_l \boldsymbol{b}) \\ \qquad\qquad\qquad\qquad + ((\boldsymbol{b} \cdot \nabla) D_l \boldsymbol{u}, D_l \boldsymbol{b}). \end{cases} \tag{4.28}$$

Como $\operatorname{div} \boldsymbol{b} = 0$ então $((\boldsymbol{b}\cdot\nabla)D_l\boldsymbol{b}, D_l\boldsymbol{u}) + ((\boldsymbol{b}\cdot\nabla)D_l\boldsymbol{u}, D_l\boldsymbol{b}) = 0$. Portanto, somando as equações de (4.28), obtemos

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|D_l\boldsymbol{u}\|_2^2 + \|D_l\boldsymbol{w}\|_2^2 + \|D_l\boldsymbol{b}\|_2^2\right] + \|D_l(\nabla\boldsymbol{u})\|_2^2 + \|D_l(\nabla\boldsymbol{w})\|_2^2 + \|D_l(\nabla\boldsymbol{b})\|_2^2\\ &+\|D_l(\operatorname{div}\boldsymbol{w})\|_2^2 + \|D_l\boldsymbol{w}\|_2^2 = (\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w}) + ((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{u})\\ &-((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{u}) - ((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, D_l\boldsymbol{w}) - ((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{b})\\ &+((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{b}). \qquad (4.29)\end{aligned}$$

Agora, multiplicando por 2 e somando a equação (4.29) sobre $l = 1, 2, 3$, obtemos:

$$\begin{aligned}&\frac{d}{dt}\left[\sum_{l=1}^{3}\|D_l\boldsymbol{u}\|_2^2 + \sum_{l=1}^{3}\|D_l\boldsymbol{w}\|_2^2 + \sum_{l=1}^{3}\|D_l\boldsymbol{b}\|_2^2\right] + 2\sum_{l=1}^{3}\|D_l(\nabla\boldsymbol{u})\|_2^2 + 2\sum_{l=1}^{3}\|D_l(\nabla\boldsymbol{w})\|_2^2\\ &+2\sum_{l=1}^{3}\|D_l(\nabla\boldsymbol{b})\|_2^2 + 2\sum_{l=1}^{3}\|D_l(\operatorname{div}\boldsymbol{w})\|_2^2 + 2\sum_{l=1}^{3}\|D_l\boldsymbol{w}\|_2^2 = 2\sum_{l=1}^{3}(\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w})\\ &+2\sum_{l=1}^{3}((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{u}) - 2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{u}) - 2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, D_l\boldsymbol{w})\\ &-2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{b}) + 2\sum_{l=1}^{3}((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{b}). \qquad (4.30)\end{aligned}$$

Observe que pelas desigualdades de Cauchy-Schwarz e Young, temos

$$\begin{aligned}2\sum_{l=1}^{3}(\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w}) &\le 2\sum_{l=1}^{3}|(\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w})| \le 2\sum_{l=1}^{3}\|D_l\boldsymbol{w}\|_2\|\nabla D_l\boldsymbol{u}\|_2\\ &\le 2\sum_{l=1}^{3}\|D_l\boldsymbol{w}\|_2^2 + \frac{1}{2}\sum_{l=1}^{3}\|\nabla(D_l\boldsymbol{u})\|_2^2 = 2\|\nabla\boldsymbol{w}\|_2^2 + \frac{1}{2}\|D^2\boldsymbol{u}\|_2^2.\end{aligned}$$

Note ainda que

$$-2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{u}) = 2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)D_l\boldsymbol{u}, \boldsymbol{u}) \le 2\|\boldsymbol{u}\|_\infty\|\nabla\boldsymbol{u}\|_2\|D^2\boldsymbol{u}\|_2.$$

Analogamente,

$$\begin{aligned}-2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, D_l\boldsymbol{w}) &= 2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)D_l\boldsymbol{w}, \boldsymbol{w}) \le 2\|\boldsymbol{w}\|_\infty\|\nabla\boldsymbol{u}\|_2\|D^2\boldsymbol{w}\|_2,\\ -2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{b}) &= 2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)D_l\boldsymbol{b}, \boldsymbol{b}) \le 2\|\boldsymbol{b}\|_\infty\|\nabla\boldsymbol{u}\|_2\|D^2\boldsymbol{b}\|_2,\\ 2\sum_{l=1}^{3}((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{u}) &= -2\sum_{l=1}^{3}((D_l\boldsymbol{b}\cdot\nabla)D_l\boldsymbol{u}, \boldsymbol{b}) \le 2\|\boldsymbol{b}\|_\infty\|\nabla\boldsymbol{b}\|_2\|D^2\boldsymbol{u}\|_2,\\ 2\sum_{l=1}^{3}((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{b}) &= 2\sum_{l=1}^{3}((D_l\boldsymbol{b}\cdot\nabla)D_l\boldsymbol{b}, \boldsymbol{u}) \le 2\|\boldsymbol{u}\|_\infty\|\nabla\boldsymbol{b}\|_2\|D^2\boldsymbol{b}\|_2,\end{aligned}$$

onde $\|\boldsymbol{v}(\cdot,\tau)\|_\infty = \sup\{|\boldsymbol{v}(\boldsymbol{x},\tau)| : \boldsymbol{x}\in\mathbb{R}^3\}$. Assim, pela identidade (4.30) e pelas estimativas acima, obtemos

$$\frac{d}{dt}\|\nabla\boldsymbol{z}\|_2^2 + \|D^2\boldsymbol{z}\|_2^2 \le 10\|\boldsymbol{z}\|_\infty\|\nabla\boldsymbol{z}\|_2\|D^2\boldsymbol{z}\|_2.$$

Usando as desigualdades de Sobolev-Nirenberg-Gagliardo (2.4)–(2.5) (veja o Lema 2.35), obtemos

$$\|\boldsymbol{v}\|_\infty \|\nabla \boldsymbol{v}\|_2 \leq \|\boldsymbol{v}\|_2^{1/2} \|\nabla \boldsymbol{v}\|_2^{1/2} \|D^2 \boldsymbol{v}\|_2, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^2(\mathbb{R}^3). \tag{4.31}$$

Portanto

$$\frac{d}{dt}\|\nabla \boldsymbol{z}\|_2^2 + \|D^2 \boldsymbol{z}\|_2^2 \leq 10 \|\boldsymbol{z}\|_2^{1/2} \|\nabla \boldsymbol{z}\|_2^{1/2} \|D^2 \boldsymbol{z}\|_2^2. \tag{4.32}$$

Escolha $t_1 \in E$ (fixo, porém arbitrário). Integrando a desigualdade (4.32) sobre o intervalo (t_1, t), obtemos

$$\begin{aligned} & \|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + \int_{t_1}^t \|D^2 \boldsymbol{z}(\cdot,\tau)\|_2^2 \, d\tau \\ \leq\ & \|\nabla \boldsymbol{z}(\cdot,t_1)\|_2^2 + 10 \int_{t_1}^t \|\boldsymbol{z}(\cdot,\tau)\|_2^{1/2} \|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^{1/2} \|D^2 \boldsymbol{z}(\cdot,\tau)\|_2^2 \, d\tau \\ \leq\ & \|\nabla \boldsymbol{z}(\cdot,t_1)\|_2^2 + 10 \|\boldsymbol{z}_0\|_2^{1/2} \int_{t_1}^t \|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^{1/2} \|D^2 \boldsymbol{z}(\cdot,\tau)\|_2^2 \, d\tau, \end{aligned} \tag{4.33}$$

para todo $t \geq t_1$, onde a segunda desigualdade vem de (4.3). Da definição de t_0 em (4.24) e da desigualdade (4.26), temos que

$$\begin{aligned} 10\|z_0\|_2^{1/2} \|\nabla \boldsymbol{z}(\cdot,t_1)\|_2^{1/2} &\leq 10 \|z_0\|_2^{1/2} \frac{\|z_0\|_2^{1/2}}{t_0^{1/4}} = \frac{10\|z_0\|_2}{2^5(1+\|\boldsymbol{z}_0\|_2^2)^{1/2}} \\ &\leq \frac{5}{16}\left(\frac{\|z_0\|_2^2}{1+\|\boldsymbol{z}_0\|_2^2}\right)^{1/2} \\ &< \frac{1}{2}. \end{aligned}$$

Então, por continuidade, existe um $\delta > 0$ tal que $10\|z_0\|_2^{1/2}\|\nabla \boldsymbol{z}(\cdot,t)\|_2^{1/2} \leq 1$ para todo $t \in [t_1, t_1+\delta]$. Logo, para todo $t \in [t_1, t_1 + \delta]$, temos que

$$10\|\boldsymbol{z}_0\|_2^{1/2} \|\nabla \boldsymbol{z}(\cdot,t)\|_2^{1/2} \|D^2 \boldsymbol{z}(\cdot,t)\|_2^2 \leq \|D^2 \boldsymbol{z}(\cdot,t)\|_2^2.$$

Portanto, da desigualdade (4.33), segue que

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 \leq \|\nabla \boldsymbol{z}(\cdot,t_1)\|_2^2, \quad \forall\, t \in [t_1, t_1+\delta].$$

Isto prova que $[t_1, t_1+\delta] \subseteq E$, de modo que o argumento pode ser repetido escolhendo $t_1 + \delta$ como "ponto de partida". Prosseguindo assim, concluiremos que $t \in E$ para todo $t \geq t_1$. Uma vez que $t_0 \geq t_1$, temos que $\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 \leq \|z_0\|_2^2/t_0$ para todo $t \geq t_0$. Portanto, da definição de t_0, temos

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 \leq \frac{\|z_0\|_2^2}{t_0} = \frac{\|z_0\|_2^2}{2^{20}(1+\|\boldsymbol{z}_0\|_2^2)^2} = \frac{1}{2^{20}(1+\|\boldsymbol{z}_0\|_2^2)} \cdot \frac{\|z_0\|_2^2}{1+\|\boldsymbol{z}_0\|_2^2} < \frac{2^{-20}}{1+\|\boldsymbol{z}_0\|_2^2},$$

para todo $t \geq t_0$, concluindo, assim, a demonstração do lema. □

Observação 4.5. *A demonstração apresentada acima mostra também que* $\|\nabla \boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^2$ *é decrescente para todo* $t \in [t_0, \infty)$*, onde* t_0 *é o tempo definido em* (4.24).

Proposição 4.6. *Seja $t_0 > 0$ definido em (4.24). Assumindo as mesmas hipóteses do Teorema 4.3, temos que*

$$\|\nabla \boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^2 \leq C(t+1)^{-5/2}, \quad \forall\, t \geq t_0,$$

para alguma constante $C > 0$ dependendo apenas das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$.

Demonstração. Multiplicando a equação (4.2) por $(t+1)^2$, obtemos

$$(t+1)^2 \frac{d}{dt}\|\boldsymbol{z}(\cdot,t)\|_2^2 + (t+1)^2\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 \leq 0.$$

Assim,

$$\frac{d}{dt}\left[(t+1)^2\|\boldsymbol{z}(\cdot,t)\|_2^2\right] + (t+1)^2\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 \leq 2(t+1)\|\boldsymbol{z}(\cdot,t)\|_2^2.$$

Integrando de 0 até t, obtemos

$$(t+1)^2\|\boldsymbol{z}(\cdot,t)\|_2^2 + \int_0^t (\tau+1)^2\|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau \leq \|\boldsymbol{z}_0\|_2^2 + 2\int_0^t (\tau+1)\|\boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau,$$

para todo $t \geq 0$. Em particular,

$$\int_0^t (\tau+1)^2\|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau \leq \|\boldsymbol{z}_0\|_2^2 + 2\int_0^t (\tau+1)\|\boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau, \quad \forall\, t \geq 0.$$

Sabemos do Teorema 4.3 que $\|z(\cdot,t)\|_2^2 \leq C(t+1)^{-3/2}$, para todo $t \geq 0$. Portanto,

$$\begin{aligned}\int_0^t (\tau+1)^2\|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau &\leq \|\boldsymbol{z}_0\|_2^2 + 2C\int_0^t (\tau+1)^{-1/2}\, d\tau \\ &= \|\boldsymbol{z}_0\|_2^2 + 4C[(t+1)^{1/2} - 1] \\ &\leq C(t+1)^{1/2}, \end{aligned} \tag{4.34}$$

para todo $t \geq 0$ e alguma constante $C > 0$ que depende somente das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Agora, multiplicando a equação (4.32) por $(t+1)^3$, obtemos

$$\begin{aligned}&(t+1)^3\frac{d}{dt}\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + (t+1)^3\|D^2\boldsymbol{z}(\cdot,t)\|_2^2 \\ &\leq 10(t+1)^3\|\boldsymbol{z}(\cdot,t)\|_2^{1/2}\|\nabla \boldsymbol{z}(\cdot,t)\|_2^{1/2}\|D^2\boldsymbol{z}(\cdot,t)\|_2^2,\end{aligned}$$

isto é,

$$\begin{aligned}&\frac{d}{dt}\left[(t+1)^3\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2\right] + (t+1)^3\|D^2\boldsymbol{z}(\cdot,t)\|_2^2 \\ &\leq 3(t+1)^2\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + 10(t+1)^3\|\boldsymbol{z}(\cdot,t)\|_2^{1/2}\|\nabla \boldsymbol{z}(\cdot,t)\|_2^{1/2}\|D^2\boldsymbol{z}(\cdot,t)\|_2^2.\end{aligned}$$

Integrando em (t_0, t), onde t_0 é dado por (4.24), obtemos

$$\begin{aligned}&(t+1)^3\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + \int_{t_0}^t (\tau+1)^3\|D^2\boldsymbol{z}(\cdot,\tau)\|_2^2 \leq (t_0+1)^3\|\nabla \boldsymbol{z}(\cdot,t_0)\|_2^2 \\ &+ 3\int_{t_0}^t (\tau+1)^2\|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau + 10\int_{t_0}^t (\tau+1)^3\|\boldsymbol{z}(\cdot,\tau)\|_2^{1/2}\|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^{1/2}\|D^2\boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau,\end{aligned}$$

para todo $t \geq t_0$. Agora estimaremos os dois primeiros termos do lado direito da desigualdade acima. Utilizando o Lema 4.4 e a definição de t_0 em (4.24), obtemos

$$(t_0+1)^3 \|\nabla \boldsymbol{z}(\cdot, t_0)\|_2^2 \leq (t_0+1)^3 \frac{2^{-20}}{1+\|\boldsymbol{z}_0\|_2^2} \leq C,$$

onde a constante $C > 0$ depende somente das normas em $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Além disso, da equação (4.34), temos

$$3 \int_{t_0}^{t} (\tau+1)^2 \|\nabla \boldsymbol{z}(\cdot, \tau)\|_2^2 \, d\tau \leq C(t+1)^{1/2}.$$

Assim, pelas estiamtivas acima e por (4.3) temos

$$\begin{aligned} & (t+1)^3 \|\nabla \boldsymbol{z}(\cdot, t)\|_2^2 + \int_{t_0}^{t} (\tau+1)^3 \|D^2 \boldsymbol{z}(\cdot, \tau)\|_2^2 \, d\tau \\ \leq \; & C(t+1)^{\frac{1}{2}} + 10\|\boldsymbol{z}_0\|^{1/2} \int_{t_0}^{t} (\tau+1)^3 \|\nabla \boldsymbol{z}(\cdot, \tau)\|_2^{1/2} \|D^2 \boldsymbol{z}(\cdot, \tau)\|_2^2 \, d\tau, \end{aligned}$$

para alguma constante $C \in \mathbb{R}^+$ dependendo apenas da norma de $\boldsymbol{z}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Pelo Lema 4.4, temos

$$10\|\boldsymbol{z}_0\|_2^{1/2} \|\nabla \boldsymbol{z}(\cdot, \tau)\|_2^{1/2} \leq 10\|\boldsymbol{z}_0\|_2^{1/2} \frac{2^{-5}}{(1+\|\boldsymbol{z}_0\|_2^2)^{1/4}} = \frac{5}{16} \left(\frac{\|\boldsymbol{z}_0\|_2^2}{1+\|\boldsymbol{z}_0\|_2^2} \right)^{1/4} < \frac{1}{2}, \quad \forall \, \tau \geq t_0.$$

Portanto,

$$(t+1)^3 \|\nabla \boldsymbol{z}(\cdot, t)\|_2^2 + \frac{1}{2} \int_{t_0}^{t} (\tau+1)^3 \|D^2 \boldsymbol{z}(\cdot, \tau)\|_2^2 \, d\tau \leq C(t+1)^{\frac{1}{2}}, \quad \forall \, t \geq t_0,$$

o que implica,

$$\|\nabla \boldsymbol{z}(\cdot, t)\|_2^2 \leq C(t+1)^{-5/2},$$

para todo $t \geq t_0$ e para alguma constante $C > 0$ dependendo somente das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. □

Obteremos, a seguir, a taxa de decaimento algébrica melhorada para a velocidade micro-rotacional $\boldsymbol{w}$.

Teorema 4.7. *Assumindo as mesmas hipóteses do Teorema 4.3, temos*

$$\|\boldsymbol{w}(\cdot, t)\|_2 \leq C(t+1)^{-\frac{5}{4}}, \quad \forall \, t \geq 0,$$

onde a constante $C > 0$ *depende somente das normas de* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$ *em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$.

Demonstração. Seja t_0 dado por (4.24). Observe que t_0 é uma constante que depende apenas da norma de $\boldsymbol{z}_0$ em $\boldsymbol{L}^2$. Assim, por (4.3), temos que

$$(t_0+1)^{5/4} \|\boldsymbol{w}(\cdot, t)\|_2 \leq (t_0+1)^{5/4} \|\boldsymbol{z}_0\|_2 \leq C,$$

onde $C > 0$ depende somente da norma de $\boldsymbol{z}_0$ em $\boldsymbol{L}^2$. Logo, se $0 \leq t < t_0$, então

$$\|\boldsymbol{w}(\cdot, t)\|_2 \leq C(t_0 + 1)^{-5/4} < C(t + 1)^{-5/4}.$$

Basta portanto, provar o mesmo resultado para $t \geq t_0$.

Reescrevendo a equação $(4.1)_2$ como

$$\boldsymbol{w}_t(\cdot, t) + \boldsymbol{w}(\cdot, t) = \Delta \boldsymbol{w}(\cdot, t) + \boldsymbol{Q}(\cdot, t),$$

onde $\boldsymbol{Q}(\cdot, t) := \dfrac{1}{2}\mathrm{rot}\,\boldsymbol{u} + \nabla(\mathrm{div}\,\boldsymbol{w}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{w}$ e multiplicando a equação acima pelo fator integrante e^t, obtemos

$$\frac{\partial}{\partial t}[e^t \boldsymbol{w}(\cdot, t)] = e^t \Delta \boldsymbol{w}(\cdot, t) + e^t \boldsymbol{Q}(\cdot, t).$$

Fazendo a mudança de variável

$$\boldsymbol{W}(\cdot, t) := e^t \boldsymbol{w}(\cdot, t) \quad \text{e} \quad \widetilde{\boldsymbol{Q}}(\cdot, t) := e^t \boldsymbol{Q}(\cdot, t),$$

obtemos

$$\frac{\partial}{\partial t}\boldsymbol{W}(\cdot, t) = \Delta \boldsymbol{W}(\cdot, t) + \widetilde{\boldsymbol{Q}}(\cdot, t). \tag{4.35}$$

Pelo princípio de Duhamel aplicado a $\boldsymbol{W}(\cdot, t)$ a partir de t_0, obtemos

$$\boldsymbol{W}(\cdot, t) = e^{\Delta(t - t_0)}\boldsymbol{W}(\cdot, t_0) + \int_{t_0}^{t} e^{\Delta(t-s)}\widetilde{\boldsymbol{Q}}(\cdot, s)\, ds,$$

onde $\left(e^{\Delta t}\right)_{t \geq 0}$ é o semigrupo do calor (veja a Definição 2.40). Portanto,

$$\begin{aligned} \boldsymbol{w}(\cdot, t) &= e^{-(t-t_0)} e^{\Delta(t-t_0)} \boldsymbol{w}(\cdot, t_0) \\ &+ \int_{t_0}^{t} e^{-(t-s)} e^{\Delta(t-s)} \left\{ \frac{1}{2}\mathrm{rot}\,\boldsymbol{u} + \nabla(\mathrm{div}\,\boldsymbol{w}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} \right\}(\cdot, s) ds, \end{aligned} \tag{4.36}$$

Tomando a norma $\boldsymbol{L}^2$ de (4.36) e usando a desigualdade de Minkowski, obtemos

$$\begin{aligned} \|\boldsymbol{w}(\cdot, t)\|_2 \leq e^{-(t-t_0)} \|e^{\Delta(t-t_0)} \boldsymbol{w}(\cdot, t_0)\|_2 &+ \frac{1}{2}\int_{t_0}^{t} e^{-(t-s)} \|e^{\Delta(t-s)} \mathrm{rot}\,\boldsymbol{u}(\cdot, s)\|_2\, ds \\ &+ \int_{t_0}^{t} e^{-(t-s)} \|e^{\Delta(t-s)} \nabla(\mathrm{div}\,\boldsymbol{w})(\cdot, s)\|_2\, ds \\ &+ \int_{t_0}^{t} e^{-(t-s)} \|e^{\Delta(t-s)} (\boldsymbol{u} \cdot \nabla)\boldsymbol{w}(\cdot, s)\|_2\, ds. \end{aligned}$$

Assim, pelo Teorema 2.41, pela estimativa (4.3) e pela desigualdade de Hölder, temos

$$\begin{aligned} \frac{1}{2}\|e^{\Delta(t-s)}\mathrm{rot}\,\boldsymbol{u}(\cdot, s)\|_2 &\leq C\|\mathrm{rot}\,\boldsymbol{u}(\cdot, s)\|_2 = C\|\nabla \boldsymbol{u}(\cdot, s)\|_2, \\ \|e^{\Delta(t-s)}\nabla \mathrm{div}\,\boldsymbol{w}(\cdot, s)\|_2 &\leq C(t-s)^{-1/2}\|\mathrm{div}\,\boldsymbol{w}(\cdot, s)\|_2 \leq C(t-s)^{-1/2}\|\nabla \boldsymbol{w}(\cdot, s)\|_2, \\ \|e^{\Delta(t-s)}(\boldsymbol{u} \cdot \nabla)\boldsymbol{w}(\cdot, s)\|_2 &\leq (t-s)^{-3/4}\|(\boldsymbol{u} \cdot \nabla)\boldsymbol{w}(\cdot, s)\|_1 \\ &\leq C(t-s)^{-3/4}\|\boldsymbol{u}(\cdot, s)\|_2\|\nabla \boldsymbol{w}(\cdot, s)\|_2 \\ &\leq C(t-s)^{-3/4}\|\nabla \boldsymbol{w}(\cdot, s)\|_2, \end{aligned}$$

onde a constante $C > 0$ da última estimativa depende somente das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$.

Por outro lado, sabemos que $\boldsymbol{v}(\boldsymbol{x}, t) = e^{\Delta(t-t_0)}\boldsymbol{w}(\boldsymbol{x}, t_0)$ é solução do PVI

$$\begin{cases} \boldsymbol{v}_t = \Delta \boldsymbol{v}, \quad \boldsymbol{x} \in \mathbb{R}^3, \quad t > t_0, \\ \boldsymbol{v}(\cdot, t_0) = \boldsymbol{w}(\cdot, t_0) \in \boldsymbol{L}^2(\mathbb{R}^3). \end{cases} \tag{4.37}$$

Assim, aplicando a transformada de Fourier ao PVI (4.37), obtemos o problema de Cauchy

$$\widehat{\boldsymbol{v}}_t = -|\boldsymbol{\xi}|^2 \widehat{\boldsymbol{v}}, \qquad \widehat{\boldsymbol{v}}(\cdot, t_0) = \widehat{\boldsymbol{w}}(\cdot, t_0),$$

cuja solução é $\widehat{\boldsymbol{v}}(\boldsymbol{\xi}, t) = e^{-|\boldsymbol{\xi}|^2(t-t_0)}\,\widehat{\boldsymbol{w}}(\boldsymbol{\xi}, t_0)$. Logo, pelo teorema de Plancherel, segue que

$$\begin{aligned} \|\boldsymbol{v}(\cdot, t)\|_2^2 &= (2\pi)^{-3}\|\widehat{\boldsymbol{v}}(\cdot, t)\|_2^2 = (2\pi)^{-3}\int_{\mathbb{R}^3} |\widehat{\boldsymbol{v}}(\boldsymbol{\xi}, t)|^2\, d\boldsymbol{\xi} \\ &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{-2|\boldsymbol{\xi}|^2(t-t_0)}|\widehat{\boldsymbol{w}}(\boldsymbol{\xi}, t_0)|^2\, d\boldsymbol{\xi} \\ &\leq (2\pi)^{-3}\int_{\mathbb{R}^3} |\widehat{\boldsymbol{w}}(\boldsymbol{\xi}, t_0)|^2\, d\boldsymbol{\xi} = (2\pi)^{-3}\|\widehat{\boldsymbol{w}}(\cdot, t_0)\|_2^2 = \|\boldsymbol{w}(\cdot, t_0)\|_2^2 \leq \|\boldsymbol{z}_0\|_2^2, \end{aligned}$$

onde, na última estimativa, usamos a desigualdade (4.3). Portanto,

$$\|e^{\Delta(t-t_0)}\boldsymbol{w}(\cdot, t_0)\|_2 = \|\boldsymbol{v}(\cdot, t)\|_2 \leq \|\boldsymbol{z}_0\|_2, \quad \forall\, t \geq t_0.$$

Desta forma,

$$\begin{aligned} \|\boldsymbol{w}(\cdot, t)\|_2 \leq e^{-(t-t_0)}\|\boldsymbol{z}_0\|_2 \;&+\; C\int_{t_0}^{t} e^{-(t-s)}\|\nabla \boldsymbol{u}(\cdot, s)\|_2\, ds \\ &+\; C\int_{t_0}^{t} e^{-(t-s)}(t-s)^{-1/2}\|\nabla \boldsymbol{w}(\cdot, s)\|_2\, ds \\ &+\; C\int_{t_0}^{t} e^{-(t-s)}(t-s)^{-3/4}\|\nabla \boldsymbol{w}(\cdot, s)\|_2\, ds. \end{aligned}$$

Da Proposição 4.6, temos

$$\begin{aligned} \|\boldsymbol{w}(\cdot, t)\|_2 \leq e^{-(t-t_0)}\|\boldsymbol{z}_0\|_2 \;&+\; C\int_{t_0}^{t} e^{-(t-s)}(s+1)^{-5/4}\, ds \\ &+\; C\int_{t_0}^{t} e^{-(t-s)}(t-s)^{-1/2}(s+1)^{-5/4}\, ds \\ &+\; C\int_{t_0}^{t} e^{-(t-s)}(t-s)^{-3/4}(s+1)^{-5/4}\, ds, \end{aligned} \tag{4.38}$$

para todo $t \geq t_0$, onde a constante $C > 0$ depende apenas das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. A seguir, estimaremos os quatro termos do lado direito da desigualdade (4.38).

A) Seja $h(t) := (t+1)^{5/4}\,e^{-t}$, para $t \geq -1$. Então, $h(t) \leq h(1/4) < 1.03$, para todo $t \geq -1$. Assim,

$$(t+1)^{5/4}e^{-(t-t_0)}\|\boldsymbol{z}_0\|_2 = (t+1)^{5/4}e^{-t}e^{t_0}\|\boldsymbol{z}_0\|_2 \leq K e^{t_0}\|\boldsymbol{z}_0\|_2 \leq C,$$

onde $C > 0$ depende somente da norma de $\boldsymbol{z}_0$ em $\boldsymbol{L}^2$. Portanto,

$$e^{-(t-t_0)}\|\boldsymbol{z}_0\|_2 \leq C(t+1)^{-5/4}. \tag{4.39}$$

B) Considere $\displaystyle\int_{t_0}^{t} e^{-(t-s)}(s+1)^{-5/4}\,ds =: I_1(t) + I_2(t)$, onde

$$I_1(t) := \int_{t_0}^{(t+t_0)/2} e^{-(t-s)}(s+1)^{-5/4}\,ds \quad \text{e} \quad I_2(t) := \int_{(t+t_0)/2}^{t} e^{-(t-s)}(s+1)^{-5/4}\,ds.$$

i. Estimativa para $I_1(t)$.

Note que

$$s \leq \frac{t_0+t}{2} \Rightarrow -\frac{t_0+t}{2} \leq -s \Rightarrow \frac{t-t_0}{2} \leq t-s \Rightarrow -(t-s) \leq -\frac{(t-t_0)}{2}.$$

Logo,

$$\begin{aligned} I_1(t) &\leq e^{-\left(\frac{t-t_0}{2}\right)} \int_{t_0}^{\frac{t+t_0}{2}} (s+1)^{-5/4}\,ds \leq e^{-\left(\frac{t-t_0}{2}\right)} \int_0^{\infty} (s+1)^{-5/4}\,ds \\ &= 4e^{-\left(\frac{t-t_0}{2}\right)} = 4(t+1)^{5/4}e^{-t/2}e^{t_0/2}(t+1)^{-5/4} \leq Ke^{t_0/2}(t+1)^{-5/4} \\ &\leq C(t+1)^{-5/4}, \end{aligned}$$

onde $C > 0$ depende somente da norma de $\boldsymbol{z}_0$ em $\boldsymbol{L}^2$ e a terceira desigualdade se justifica pelo fato de $(t+1)^{5/4}e^{-t/2} < 1.49$.

ii. Estimativa para $I_2(t)$.

Observe que

$$\frac{t+t_0}{2} \leq s \Rightarrow (s+1)^{-5/4} \leq \left(\frac{t+t_0}{2}+1\right)^{-5/4}.$$

Assim,

$$I_2(t) \leq \left(\frac{t+t_0}{2}+1\right)^{-5/4} \int_{\frac{t+t_0}{2}}^{t} e^{-(t-s)}\,ds.$$

Como $t+1 \leq t+t_0+2$, segue que

$$(t+t_0+2)^{-5/4} \leq (t+1)^{-5/4}.$$

Logo,

$$\left(\frac{t+t_0}{2}+1\right)^{-5/4} = 2^{5/4}(t+t_0+2)^{-5/4} \leq 2^{5/4}(t+1)^{-5/4}.$$

Assim,

$$\begin{aligned} I_2(t) &\leq 2^{5/4}(t+1)^{-5/4} \int_0^t e^{-(t-s)}\,ds \\ &= 2^{5/4}(t+1)^{-5/4} \int_0^t e^{-\tau}\,d\tau \\ &\leq 2^{5/4}(t+1)^{-5/4} \int_0^{\infty} e^{-\tau}\,d\tau \leq 2^{5/4}(t+1)^{-5/4}. \end{aligned}$$

Portanto,

$$C\int_{t_0}^{t} e^{-(t-s)}(s+1)^{-5/4}\,ds \leq C(t+1)^{-5/4}, \tag{4.40}$$

onde $C>0$ depende apenas das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{z}_0$.

C) Considere $\displaystyle\int_{t_0}^{t} e^{-(t-s)}(s+1)^{-5/4}(t-s)^{-1/2}\,ds =: \widetilde{I}_1(t)+\widetilde{I}_2(t)$, onde

$$\widetilde{I}_1(t) := \int_{t_0}^{(t+t_0)/2} e^{-(t-s)}(s+1)^{-5/4}(t-s)^{-1/2}\,ds$$

e

$$\widetilde{I}_2(t) := \int_{(t+t_0)/2}^{t} e^{-(t-s)}(s+1)^{-5/4}(t-s)^{-1/2}\,ds.$$

i. Estimativa para $\widetilde{I}_1(t)$.

$$\widetilde{I}_1(t) \leq e^{-\left(\frac{t-t_0}{2}\right)}\int_0^t (s+1)^{-5/4}(t-s)^{-1/2}\,ds.$$

Observe que $s\leq \frac{t}{2} \Rightarrow -s\geq -\frac{t}{2} \Rightarrow t-s\geq \frac{t}{2} \Rightarrow (t-s)^{-1/2}\leq \left(\frac{t}{2}\right)^{-1/2}$. Portanto,

$$\begin{aligned}\int_0^{t/2}(s+1)^{-5/4}(t-s)^{-1/2}\,ds &\leq \left(\frac{t}{2}\right)^{-1/2}\int_0^{t/2}(s+1)^{-5/4}\,ds\\ &\leq \frac{\sqrt{2}}{\sqrt{t}}\int_0^{\infty}(s+1)^{-5/4}\,ds = \frac{4\sqrt{2}}{\sqrt{t}} < 4\sqrt{2},\end{aligned}$$

pois $t>1$, já que $t\geq t_0$.

Note também que $\frac{t}{2}\leq s \Rightarrow \frac{t}{2}+1\leq s+1 \Rightarrow (s+1)^{-5/4}\leq \left(\frac{t}{2}+1\right)^{-5/4}$. Portanto,

$$\begin{aligned}\int_{t/2}^{t}(s+1)^{-5/4}(t-s)^{-1/2}\,ds &\leq \left(\frac{t}{2}+1\right)^{-5/4}\int_{t/2}^{t}(t-s)^{-1/2}\,ds\\ &= 2^{5/4}\,(t+2)^{-5/4}\,t^{-1/2}\int_{\frac{t}{2}}^{t}\left(1-\frac{s}{t}\right)^{-1/2}\,ds\\ &= 2^{5/4}\,(t+2)^{-5/4}\,t^{1/2}\int_{\frac{1}{2}}^{1}(1-\tau)^{-1/2}\,d\tau\\ &= 2^{7/4}\left(\frac{t}{t+2}\right)^{1/2}\left(\frac{1}{t+2}\right)^{3/4} < 2^{7/4}.\end{aligned}$$

Segue que

$$\widetilde{I}_1(t)\leq Ke^{-\left(\frac{t-t_0}{2}\right)} = K(t+1)^{5/4}\,e^{-t/2}\,e^{t_0/2}(t+1)^{-5/4}.$$

Como $(t+1)^{5/4}e^{-t/2}<1.49$, então

$$\widetilde{I}_1(t)\leq C(t+1)^{-5/4},$$

onde $C>0$ depende apenas da norma de $\boldsymbol{z}_0$ em $\boldsymbol{L}^2$.

ii. Estimativa para $\widetilde{I}_2(t)$.

Como $\dfrac{t+t_0}{2} \leq s$, então

$$(s+1)^{-5/4} \leq \left(\frac{t+t_0}{2}+1\right)^{-5/4} = 2^{5/4}\,(t+t_0+2)^{-5/4} \leq 2^{5/4}(t+1)^{-5/4},$$

pois $t+1 \leq t+t_0+2$. Logo,

$$\begin{aligned}
\widetilde{I}_2(t) &\leq 2^{5/4}(t+1)^{-5/4}\int_{t_0}^{t} e^{-(t-s)}(t-s)^{-1/2}\,ds \\
&= 2^{5/4}\,(t+1)^{-5/4}\int_0^{t-t_0} e^{-\tau}\tau^{-1/2}\,d\tau \\
&\leq 2^{5/4}(t+1)^{-5/4}\int_0^{\infty} e^{-\tau}\tau^{-1/2}\,d\tau \\
&= 2^{5/4}\,\Gamma\left(\frac{1}{2}\right)(t+1)^{-5/4} = 2^{5/4}\sqrt{\pi}\,(t+1)^{-5/4},
\end{aligned}$$

onde Γ é a função gama de Euler. Portanto,

$$C\int_{t_0}^{t} e^{-(t-s)}(s+1)^{-5/4}(t-s)^{-1/2}\,ds \leq C(t+1)^{-5/4}, \tag{4.41}$$

onde $C>0$ depende somente das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{z}_0$.

D) Considere $\displaystyle\int_{t_0}^{t} e^{-(t-s)}(s+1)^{-5/4}(t-s)^{-3/4}\,ds =: J_1(t)+J_2(t)$, onde

$$J_1(t) := \int_{t_0}^{(t+t_0)/2} e^{-(t-s)}(s+1)^{-5/4}(t-s)^{-3/4}\,ds$$

e

$$J_2(t) := \int_{(t+t_0)/2}^{t} e^{-(t-s)}(s+1)^{-5/4}(t-s)^{-3/4}\,ds.$$

i. Estimativa para $J_1(t)$.

$$J_1(t) \leq e^{-\left(\frac{t-t_0}{2}\right)}\int_0^{t}(s+1)^{-5/4}(t-s)^{-3/4}\,ds.$$

Observe que,

$$\begin{aligned}
\int_0^{t/2}(s+1)^{-5/4}(t-s)^{-3/4}\,ds &\leq \left(\frac{t}{2}\right)^{-3/4}\int_0^{t/2}(s+1)^{-5/4}\,ds \\
&\leq \frac{2^{3/4}}{t^{3/4}}\int_0^{\infty}(s+1)^{-5/4}\,ds = \frac{2^{11/4}}{t^{3/4}} < 2^{11/4},
\end{aligned}$$

pois $t>1$, já que $t \geq t_0$.

Observe ainda que

$$\begin{aligned}
\int_{t/2}^{t} (s+1)^{-5/4}(t-s)^{-3/4}\,ds &\leq \left(\frac{t}{2}+1\right)^{-5/4} \int_{t/2}^{t} (t-s)^{-3/4}\,ds \\
&= 2^{5/4}\,(t+2)^{-5/4}\,t^{1/4} \int_{1/2}^{1} (1-\tau)^{-3/4}\,d\tau \\
&= 8\left(\frac{t}{t+2}\right)^{1/4}\left(\frac{1}{t+2}\right) < 8.
\end{aligned}$$

Portanto,

$$J_1(t) \leq K e^{-\left(\frac{t-t_0}{2}\right)} = K(t+1)^{5/4} e^{-t/2} e^{t_0/2} (t+1)^{-5/4}.$$

Ou seja,

$$J_1(t) \leq C(t+1)^{-5/4},$$

onde $C > 0$ depende somente da norma de $\boldsymbol{z}_0$ em $\boldsymbol{L}^2$.

ii. Estimativa para $J_2(t)$.

$$\begin{aligned}
J_2(t) &\leq 2^{5/4}(t+1)^{-5/4} \int_{t_0}^{t} e^{-(t-s)}(t-s)^{-3/4}\,ds \\
&= 2^{5/4}\,(t+1)^{-5/4} \int_{0}^{t-t_0} e^{-\tau}\,\tau^{-3/4}\,d\tau \\
&\leq 2^{5/4}\,(t+1)^{-5/4} \int_{0}^{\infty} e^{-\tau}\tau^{-3/4}\,d\tau \\
&= 2^{5/4}\,\Gamma\left(\frac{1}{4}\right)(t+1)^{-5/4} \leq K(t+1)^{-5/4}.
\end{aligned}$$

Portanto,

$$C \int_{t_0}^{t} e^{-(t-s)}(s+1)^{-5/4}(t-s)^{-3/4}\,ds \leq C(t+1)^{-5/4}, \tag{4.42}$$

onde $C > 0$ depende apenas das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{z}_0$.

Utilizando as estimativas (4.39), (4.40), (4.41) e (4.42) na desigualdade (4.38), concluímos que

$$\|\boldsymbol{w}(\cdot,t)\|_2 \leq C(t+1)^{-5/4}, \quad \forall\, t \geq t_0,$$

onde $C \in \mathbb{R}^+$ depende apenas das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ dos dados inciais. Isto completa a demonstração do Teorema 4.7. $\square$

4.2 Forças externas não nulas

Agora, examinaremos o caso em que as forças externas $\boldsymbol{f}$ e $\boldsymbol{g}$ estão presentes. Para obtermos os mesmo resultados anteriores, assumiremos que $\boldsymbol{f}$ e $\boldsymbol{g}$ satisfazem:

$$\operatorname{div} \boldsymbol{f}(\cdot,t) = 0, \quad \forall\, t \geq 0, \tag{4.43}$$

$$(t+1)^{1/2}\|\boldsymbol{f}(\cdot,t)\|_2 + \|\boldsymbol{g}(\cdot,t)\|_2 \leq K_1(t+1)^{-5/4}, \quad \forall\, t \geq 0, \tag{4.44}$$

$$|\widehat{\boldsymbol{f}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{g}}(\boldsymbol{\xi},t)| \leq K_2(t+1)^{-1/2}|\boldsymbol{\xi}|, \quad \forall\, t \geq 0 \;\; e \;\; |\boldsymbol{\xi}| \leq \; 1, \tag{4.45}$$

para algumas constantes positivas K_1 e K_2.

Lema 4.8. *Seja* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *uma solução suave do problema de Cauchy* (4.1) *com* $\boldsymbol{f}$ *e* $\boldsymbol{g}$ *satisfazendo* (4.43)–(4.45). *Se* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$ *com* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$*, então, para todo* $t \geq 0$ *e* $\boldsymbol{\xi} \in \mathbb{R}^3$*, temos*

$$\begin{aligned}
& |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| \\
+ \;\; & |\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t)| \\
\leq \;\; & \left[\|\boldsymbol{u}(\cdot,t)\|_2\,(2\|\boldsymbol{u}(\cdot,t)\| + \|\boldsymbol{w}(\cdot,t)\|) + \|\boldsymbol{b}(\cdot,t)\|_2\,(2\|\boldsymbol{u}(\cdot,t)\|_2 + 2\|\boldsymbol{b}(\cdot,t)\|_2)\right]|\boldsymbol{\xi}|.
\end{aligned}$$

Em particular,

$$\begin{aligned}
& |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| \\
& + |\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t)| \leq C\,|\boldsymbol{\xi}|,
\end{aligned}$$

onde $C > 0$ *depende somente de* K_1 , $\|\boldsymbol{u}_0\|_2, \|\boldsymbol{w}_0\|_2$ *e* $\|\boldsymbol{b}_0\|_2$.

Demonstração. Multiplicando as Eqs. $(4.1)_1$ por $\boldsymbol{u}$, $(4.1)_2$ por $\boldsymbol{w}$ e $(4.1)_3$ por $\boldsymbol{b}$ em $\boldsymbol{L}^2$, obtemos

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \|\nabla\boldsymbol{u}\|_2^2 = \frac{1}{2}(\operatorname{rot}\boldsymbol{w},\boldsymbol{u}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{b},\boldsymbol{u}) + (\boldsymbol{f},\boldsymbol{u}), \\
& \frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\operatorname{div}\boldsymbol{w}\|_2^2 + \|\boldsymbol{w}\|_2^2 = \frac{1}{2}(\operatorname{rot}\boldsymbol{u},\boldsymbol{w}) + (\boldsymbol{g},\boldsymbol{w}), \\
& \frac{1}{2}\frac{d}{dt}\|\boldsymbol{b}\|_2^2 + \|\nabla\boldsymbol{b}\|_2^2 = ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u},\boldsymbol{b}).
\end{aligned}$$

Somando as três identidades acima e usando o fato que $((\boldsymbol{b}\cdot\nabla)\boldsymbol{b},\boldsymbol{u}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u},\boldsymbol{b}) = 0$, pois $\operatorname{div}\boldsymbol{b} = 0$, temos, pelas desigualdades de Cauchy-Schwarz e Young, que

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{w}\|_2^2 + \|\boldsymbol{b}\|_2^2\right) + \|\nabla\boldsymbol{u}\|_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{b}\|_2^2 + \|\operatorname{div}\boldsymbol{w}\|_2^2 + \|\boldsymbol{w}\|_2^2 \\
&= (\operatorname{rot}\boldsymbol{u},\boldsymbol{w}) + (\boldsymbol{f},\boldsymbol{u}) + (\boldsymbol{g},\boldsymbol{w}) \\
&\leq \frac{3}{4}\|\nabla\boldsymbol{u}\|_2^2 + \frac{7}{12}\|\boldsymbol{w}\|_2^2 + \|\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2 + \|\boldsymbol{g}\|_2^2.
\end{aligned} \tag{4.46}$$

Multiplicando por 2 e manipulando os termos adequadamente, obtemos

$$\begin{aligned}
\frac{d}{dt}\left(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{w}\|_2^2 + \|\boldsymbol{b}\|_2^2\right) + \frac{1}{2}\left(\|\nabla\boldsymbol{u}\|_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{b}\|_2^2\right) + \frac{1}{2}\|\boldsymbol{w}\|_2^2 \\
\leq 2\|\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2 + 2\|\boldsymbol{g}\|_2^2 =: H(t).
\end{aligned} \tag{4.47}$$

A seguir, usando um argumento tipo Gronwall, provaremos que

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 \leq C, \quad \forall\, t \geq 0, \tag{4.48}$$

onde $C > 0$ depende apenas de K_1 e das normas em $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Para isto, seja

$$J(t) := \|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2.$$

De (4.47), temos

$$\begin{aligned}\frac{d}{dt}J(t) \leq H(t) &= 2\|\boldsymbol{f}(\cdot,t)\|_2\|\boldsymbol{u}(\cdot,t)\|_2 + 2\|\boldsymbol{g}(\cdot,t)\|_2^2 \\ &\leq 2\|\boldsymbol{f}(\cdot,t)\|_2 J(t)^{1/2} + 2\|\boldsymbol{g}(\cdot,t)\|_2^2.\end{aligned}$$

Agora, seja $\widetilde{J}(t) := 1 + J(t)$. Note que $\widetilde{J}(t) \geq 1$, para todo $t \geq 0$, pois $J(t) \geq 0$. Segue que

$$\frac{d}{dt}\widetilde{J}(t) = \frac{d}{dt}J(t) \leq 2\|\boldsymbol{f}(\cdot,t)\|_2\widetilde{J}(t)^{1/2} + 2\|\boldsymbol{g}(\cdot,t)\|_2^2. \tag{4.49}$$

Dividindo (4.49) por $2\widetilde{J}(t)^{1/2}$, obtemos

$$\frac{1}{2}\widetilde{J}(t)^{-1/2}\frac{d}{dt}\widetilde{J}(t) \leq \|\boldsymbol{f}(\cdot,t)\|_2 + \|\boldsymbol{g}(\cdot,t)\|_2^2\widetilde{J}(t)^{-1/2} \leq \|\boldsymbol{f}(\cdot,t)\|_2 + \|\boldsymbol{g}(\cdot,t)\|_2^2,$$

pois $\widetilde{J}(t) \geq 1$. Ou seja,

$$\frac{d}{dt}\left(\widetilde{J}(t)^{1/2}\right) \leq \|\boldsymbol{f}(\cdot,t)\|_2 + \|\boldsymbol{g}(\cdot,t)\|_2^2, \quad \forall\, t \geq 0.$$

Integrando a desigualdade acima no intervalo $[0,t]$, resulta

$$\begin{aligned}\widetilde{J}(t)^{1/2} &\leq \widetilde{J}(0)^{1/2} + \int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\,ds + \int_0^t \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds \\ &= (1 + \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2)^{1/2} + \int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\,ds + \int_0^t \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds \\ &\leq 1 + \|\boldsymbol{u}_0\|_2 + \|\boldsymbol{w}_0\|_2 + \|\boldsymbol{b}_0\|_2 + \int_0^\infty \|\boldsymbol{f}(\cdot,t)\|_2\,dt + \int_0^\infty \|\boldsymbol{g}(\cdot,t)\|_2^2\,dt.\end{aligned}$$

Como $J(t)^{1/2} \leq \widetilde{J}(t)^{1/2}$, para todo $t \geq 0$, então

$$J(t)^{1/2} \leq 1 + \|\boldsymbol{u}_0\|_2 + \|\boldsymbol{w}_0\|_2 + \|\boldsymbol{b}_0\|_2 + \int_0^\infty \|\boldsymbol{f}(\cdot,t)\|_2\,dt + \int_0^\infty \|\boldsymbol{g}(\cdot,t)\|_2^2\,dt, \tag{4.50}$$

para todo $t \geq 0$. Observe, ainda, que da desigualdade (4.44), temos

$$\begin{aligned}\int_0^\infty \|\boldsymbol{f}(\cdot,t)\|\,dt &\leq K_1\int_0^\infty (t+1)^{-7/4}\,dt = \frac{4}{3}K_1, \\ \int_0^\infty \|\boldsymbol{g}(\cdot,t)\|^2\,dt &\leq K_1^2\int_0^\infty (t+1)^{-5/2}\,dt = \frac{2}{3}K_1^2.\end{aligned}$$

Portanto, de (4.50), concluímos que $J(t) \leq C$, onde a constante $C > 0$ depende apenas de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$, provando, assim, a estimativa (4.48).

Note que de (4.48), temos $\boldsymbol{u}(\cdot,t), \boldsymbol{w}(\cdot,t), \boldsymbol{b}(\cdot,t) \in \boldsymbol{L}^2(\mathbb{R}^3)$ para todo $t \geq 0$. Logo, as transformadas $\widehat{\boldsymbol{u}}(\cdot,t), \widehat{\boldsymbol{w}}(\cdot,t), \widehat{\boldsymbol{b}}(\cdot,t)$ estão bem definidas. Repetindo os argumentos do Lema 4.1 e usando (4.48), obtemos

$$\begin{aligned}
&|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{u}\|_2^2|\boldsymbol{\xi}| \leq C|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{u}\|_2\|\boldsymbol{w}\|_2|\boldsymbol{\xi}| \leq C|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{u}\|_2\|\boldsymbol{b}\|_2|\boldsymbol{\xi}| \leq C|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{b}\|_2\|\boldsymbol{u}\|_2|\boldsymbol{\xi}| \leq C|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{b}\|_2^2|\boldsymbol{\xi}| \leq C|\boldsymbol{\xi}|,
\end{aligned} \tag{4.51}$$

onde $C > 0$ depende somente de K_1 das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$.

Ademais, aplicando o operador divergente na Eq. $(4.1)_1$ e usando que div $\boldsymbol{f} = 0$ (veja a hipótese (4.43)), obtemos

$$\Delta P = \operatorname{div}(\nabla P) = \operatorname{div}(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \operatorname{div}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}. \tag{4.52}$$

Portanto, tomando a tranformada de Fourier de (4.52) e usando (4.48), concluímos, como em (4.6), que

$$|\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t)| \leq \left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right)|\boldsymbol{\xi}| \leq C|\boldsymbol{\xi}|, \tag{4.53}$$

onde $C > 0$ depende somente de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Portanto, das estimativas (4.51) e (4.53), obtemos o resultado desejado. □

Proposição 4.9. *Assumindo as mesmas hipóteses do Lema 4.8, temos*

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1},$$

para todo $t \geq 0$ e $0 < |\boldsymbol{\xi}| \leq 1$, onde a constante $C \in \mathbb{R}^+$ depende somente de K_1, K_2 e das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ dos dados iniciais.

Demonstração. Da desigualdade (4.48), concluímos que $\boldsymbol{u}(\cdot,t), \boldsymbol{w}(\cdot,t), \boldsymbol{b}(\cdot,t) \in \boldsymbol{L}^2(\mathbb{R}^3)$ para todo $t \geq 0$. Portanto, as transformadas $\widehat{\boldsymbol{u}}(\cdot,t), \widehat{\boldsymbol{w}}(\cdot,t), \widehat{\boldsymbol{b}}(\cdot,t)$ estão bem definidas. Tomando a transformada de Fourier das Eqs. $(4.1)_1, (4.1)_2$ e $(4.1)_3$ e usando as identidades (4.7)–(4.11), obtemos

$$\begin{cases}
\widehat{\boldsymbol{u}}_t + |\boldsymbol{\xi}|^2\widehat{\boldsymbol{u}} - \dfrac{i}{2}L(\boldsymbol{\xi})\widehat{\boldsymbol{w}} = \boldsymbol{G}_1(\boldsymbol{\xi},t) + \widehat{\boldsymbol{f}}(\boldsymbol{\xi},t),\\
\widehat{\boldsymbol{w}}_t + (|\boldsymbol{\xi}|^2+1)\widehat{\boldsymbol{w}} - \dfrac{i}{2}L(\boldsymbol{\xi})\widehat{\boldsymbol{u}} - P(\boldsymbol{\xi})\widehat{\boldsymbol{w}} = \boldsymbol{G}_2(\boldsymbol{\xi},t) + \widehat{\boldsymbol{g}}(\boldsymbol{\xi},t),\\
\widehat{\boldsymbol{b}}_t + |\boldsymbol{\xi}|^2\widehat{\boldsymbol{b}} = \boldsymbol{G}_3(\boldsymbol{\xi},t),
\end{cases} \tag{4.54}$$

onde $\boldsymbol{G}_1, \boldsymbol{G}_2$ e $\boldsymbol{G}_3$ estão definidas em (4.13) e $L(\boldsymbol{\xi})$ e $P(\boldsymbol{\xi})$ são as matrizes dadas em (4.14).

Agora, sejam $\boldsymbol{z} := (\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})^T$ e $\boldsymbol{G} := (\boldsymbol{G}_1,\boldsymbol{G}_2,\boldsymbol{G}_3)^T$ e $\boldsymbol{F} := (\widehat{\boldsymbol{f}},\widehat{\boldsymbol{g}},\boldsymbol{0})^T$. Temos, então, que

$$\widehat{\boldsymbol{z}}_t + A(\boldsymbol{\xi})\widehat{\boldsymbol{z}} = \boldsymbol{G}(\boldsymbol{\xi},t) + \boldsymbol{F}(\boldsymbol{\xi},t), \tag{4.55}$$

onde $A(\boldsymbol{\xi})$ é a matriz Hermitiana definida em (4.16). Multiplicando (4.55) pelo fator integrante $e^{tA(\boldsymbol{\xi})}$, obtemos

$$\frac{d}{dt}\left(e^{tA(\boldsymbol{\xi})}\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)\right) = e^{tA(\boldsymbol{\xi})}\left[\boldsymbol{G}(\boldsymbol{\xi},t) + \boldsymbol{F}(\boldsymbol{\xi},t)\right].$$

Integrando com relação ao tempo, temos

$$\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t) = e^{-tA(\boldsymbol{\xi})}\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi}) + \int_0^t e^{-(t-s)A(\boldsymbol{\xi})}\left[\boldsymbol{G}(\boldsymbol{\xi},t) + \boldsymbol{F}(\boldsymbol{\xi},t)\right] ds,$$

onde $\widehat{\boldsymbol{z}_0} := \widehat{\boldsymbol{z}}(\cdot,0) = (\widehat{\boldsymbol{u}_0}, \widehat{\boldsymbol{w}_0}, \widehat{\boldsymbol{b}_0})^T$. Usando (4.17), temos

$$\begin{aligned} |\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| &\leq \|e^{-tA(\boldsymbol{\xi})}\||\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| + \int_0^t \|e^{-(t-s)A(\boldsymbol{\xi})}\|\left[|\boldsymbol{G}(\boldsymbol{\xi},s)| + |\boldsymbol{F}(\boldsymbol{\xi},s)|\right] ds \\ &\leq e^{-\sigma|\boldsymbol{\xi}|^2 t}|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| + \int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}\left[|\boldsymbol{G}(\boldsymbol{\xi},s)| + |\boldsymbol{F}(\boldsymbol{\xi},s)|\right] ds. \qquad (4.56) \end{aligned}$$

Pelo Lema 4.8, temos que $|\boldsymbol{G}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|$, para todo $t \geq 0$ e $\boldsymbol{\xi} \in \mathbb{R}^3$, onde $C > 0$ depende somente de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Observe também que de (4.45), $|\boldsymbol{F}(\boldsymbol{\xi},t)| \leq C\,|\boldsymbol{\xi}|$, para todo $t \geq 0$ e $|\boldsymbol{\xi}| \leq 1$, onde $C \in \mathbb{R}^+$ depende apenas de K_2. Assim,

$$|\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| \leq e^{-\sigma|\boldsymbol{\xi}|^2 t}|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| + C\int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}|\boldsymbol{\xi}|\, ds,$$

para todo $t \geq 0$ e $|\boldsymbol{\xi}| \leq 1$, onde $C > 0$ depende somente de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Uma vez que $\boldsymbol{z}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \times \boldsymbol{L}^1(\mathbb{R}^3) \times \boldsymbol{L}^1(\mathbb{R}^3)$, segue que $\widehat{\boldsymbol{z}_0} \in \boldsymbol{L}^\infty(\mathbb{R}^3) \times \boldsymbol{L}^\infty(\mathbb{R}^3) \times \boldsymbol{L}^\infty(\mathbb{R}^3)$ e

$$|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| \leq |\widehat{\boldsymbol{u}_0}(\boldsymbol{\xi})| + |\widehat{\boldsymbol{w}_0}(\boldsymbol{\xi})| + |\widehat{\boldsymbol{b}_0}(\boldsymbol{\xi})| \leq \|\boldsymbol{u}_0\|_1 + \|\boldsymbol{w}_0\|_1 + \|\boldsymbol{b}_0\|_1 \leq C,$$

para todo $\boldsymbol{\xi} \in \mathbb{R}^3$ e alguma constante positiva C dependendo das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$. Como

$$\int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}\, ds = \frac{1}{\sigma|\boldsymbol{\xi}|^2}\left(1 - e^{-\sigma|\boldsymbol{\xi}|^2 t}\right),$$

para todo $t \geq 0$ e $\boldsymbol{\xi} \neq \boldsymbol{0}$, temos

$$|\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| \leq Ce^{-\sigma|\boldsymbol{\xi}|^2 t} + \frac{C}{|\boldsymbol{\xi}|}\left(1 - e^{-\sigma|\boldsymbol{\xi}|^2 t}\right) = \frac{C}{|\boldsymbol{\xi}|} + Ce^{-\sigma|\boldsymbol{\xi}|^2 t}\left(1 - \frac{1}{|\boldsymbol{\xi}|}\right) \leq \frac{C}{|\boldsymbol{\xi}|}, \qquad (4.57)$$

pois $0 < |\boldsymbol{\xi}| \leq 1$. Logo,

$$|\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1},\ \forall\, t \geq 0 \ \text{ e } \ 0 < |\boldsymbol{\xi}| \leq 1, \qquad (4.58)$$

onde $C > 0$ depende apenas das constantes K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$, e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Isto finaliza a demonstração da proposição. □

Teorema 4.10. *Seja* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *uma solução suave do problema de Cauchy* (4.1). *Se* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$, *com* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$ *e* $\boldsymbol{f}$, $\boldsymbol{g}$ *satisfazem* (4.43)–(4.45), *então*

$$\|\boldsymbol{u}(\cdot,t)\|_2 + \|\boldsymbol{w}(\cdot,t)\|_2 + \|\boldsymbol{b}(\cdot,t)\|_2 \leq C(t+1)^{-\frac{3}{4}}, \quad \forall\, t \geq 0,$$

onde $C > 0$ *depende somente de* K_1, K_2 *e das normas de* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$ *em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$.

Demonstração. Pela desigualdade de energia (4.47) e pelo teorema de Plancherel, obtemos

$$\begin{aligned} \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right) \;\leq\; & -\frac{1}{2}\int_{\mathbb{R}^3} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} \\ & + (2\pi)^3 H(t), \end{aligned} \tag{4.59}$$

para todo $t \geq 0$.

Agora, seja $B(t) := \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| \leq R(t)\}$, onde $R(t) := \sqrt{6}\,(t+1)^{-1/2}$. Observe que $B(t)^C = \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| > \sqrt{6}\,(t+1)^{-1/2}\}$. Como $\mathbb{R}^3 \equiv B(t)^C \cup B(t)$ então, de (4.59), obtemos

$$\begin{aligned} & \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right) \\ & \quad \leq -\frac{1}{2}\int_{B(t)^C} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} \\ & \qquad -\frac{1}{2}\int_{B(t)} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} + (2\pi)^3 H(t) \\ & \quad \leq -\frac{1}{2}\int_{B(t)^C} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} + (2\pi)^3 H(t) \\ & \quad \leq -\frac{3}{t+1}\int_{B(t)^C} \left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} + (2\pi)^3 H(t). \end{aligned}$$

Disto segue que

$$\begin{aligned} & \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right) + \frac{3}{t+1}\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right) \\ & \quad \leq \frac{3}{t+1}\int_{B(t)} \left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} + (2\pi)^3 H(t). \end{aligned}$$

Multiplicando a desigualdade acima pelo fator integrante $(t+1)^3$, obtemos

$$\begin{aligned} & \frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right)\right] \\ & \quad \leq 3(t+1)^2\int_{B(t)} \left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} + 8\pi^3(t+1)^3 H(t). \end{aligned} \tag{4.60}$$

Observe que de (4.44) e (4.48), temos

$$\begin{aligned} H(t) &= 2\|\boldsymbol{f}(\cdot,t)\|_2\|\boldsymbol{u}(\cdot,t)\|_2 + 2\|\boldsymbol{g}(\cdot,t)\|_2^2 \\ &\leq 2C\|\boldsymbol{f}(\cdot,t)\|_2 + 2\|\boldsymbol{g}(\cdot,t)\|_2^2 \\ &\leq 2CK_1(t+1)^{-7/4} + 2K_1^2(t+1)^{-5/2} \\ &\leq C(t+1)^{-7/4}, \end{aligned}$$

para todo $t \geq 0$, onde $C > 0$ depende somente de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Assim,

$$\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right)\right]$$
$$\leq 3(t+1)^2 \int_{B(t)} \left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} + C(t+1)^{5/4}, \tag{4.61}$$

para todo $t \geq 0$, onde $C \in \mathbb{R}^+$ depende apenas de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$.

Pela Proposição 4.9, sabemos que

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2 \leq C|\boldsymbol{\xi}|^{-2},$$

para todo $0 < |\boldsymbol{\xi}| \leq 1$ e $t \geq 0$, onde a constante C depende somente de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Note que para $t \geq 5$, temos que $\frac{\sqrt{6}}{\sqrt{t+1}} \leq 1$. Isto é, $R(t) \leq 1$. Logo, para $t \geq 5$, se $\boldsymbol{\xi} \in B(t)$ então $0 < |\boldsymbol{\xi}| \leq 1$. Portanto, de (4.61), segue que

$$\begin{aligned}
\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right)\right] &\leq C(t+1)^2 \int_{B(t)} |\boldsymbol{\xi}|^{-2}\, d\boldsymbol{\xi} + C(t+1)^{5/4} \\
&= C(t+1)^2\,[4\sqrt{6}\,\pi(t+1)^{-1/2}] + C(t+1)^{5/4} \\
&\leq C(t+1)^{3/2} + C(t+1)^{5/4} \\
&\leq \leq C(t+1)^{3/2},
\end{aligned}$$

i.e.,

$$\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2|\right)\right] \leq C(t+1)^{3/2}, \quad \forall\, t \geq 5.$$

Integrando esta desigualdade com respeito ao tempo no intervalo $[5,t]$, obtemos

$$\begin{aligned}
(t+1)^3\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right) &\leq 6^3\left(\|\widehat{\boldsymbol{u}}(\cdot,5)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,5)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,5)\|_2^2\right) \\
&\quad + C\int_5^t (s+1)^{3/2}\, ds \\
&\leq 6^3\left(\|\widehat{\boldsymbol{u}}(\cdot,5)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,5)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,5)\|_2^2\right) \\
&\quad + C(t+1)^{5/2}.
\end{aligned}$$

Usando o teorema de Plancherel, temos

$$\begin{aligned}
(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right) &\leq 6^3\left(\|\boldsymbol{u}(\cdot,5)\|_2^2 + \|\boldsymbol{w}(\cdot,5)\|_2^2 + \|\boldsymbol{b}(\cdot,5)\|_2^2\right) \\
&\quad + C(t+1)^{5/2}.
\end{aligned}$$

Portanto,

$$\begin{aligned}
\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 &\leq C(t+1)^{-3}\left(\|\boldsymbol{u}(\cdot,5)\|_2^2 + \|\boldsymbol{w}(\cdot,5)\|_2^2 + \|\boldsymbol{b}(\cdot,5)\|_2^2\right) \\
&\quad + C(t+1)^{-1/2}.
\end{aligned}$$

Pela desigualdade (4.48), temos que

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|_2^2+\|\boldsymbol{w}(\cdot,t)\|_2^2+\|\boldsymbol{b}(\cdot,t)\|_2^2 &\leq C(t+1)^{-3}+C(t+1)^{-1/2}\\ &\leq C(t+1)^{-1/2},\end{aligned}$$

para todo $t \geq 5$ e para alguma constante $C > 0$ dependendo somente de K_1 e K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$.

Por outro lado, para $0 \leq t < 5$, temos que $1 \leq t+1 < 6$ o que implica $1 \leq (t+1)^{1/2} < \sqrt{6}$. Logo, por (4.48), temos

$$(t+1)^{1/2}\left(\|\boldsymbol{u}(\cdot,t)\|_2^2+\|\boldsymbol{w}(\cdot,t)\|_2^2+\|\boldsymbol{b}(\cdot,t)\|_2^2\right) < \sqrt{6}\left(\|\boldsymbol{u}(\cdot,t)\|_2^2+\|\boldsymbol{w}(\cdot,t)\|_2^2+\|\boldsymbol{b}(\cdot,t)\|_2^2\right) \leq C.$$

Ou seja,

$$\|\boldsymbol{u}(\cdot,t)\|_2^2+\|\boldsymbol{w}(\cdot,t)\|_2^2+\|\boldsymbol{b}(\cdot,t)\|_2^2 \leq C(t+1)^{-1/2},\ \forall\, t \in [0,5),$$

para alguma constante $C > 0$ dependendo apenas de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Portanto,

$$\|\boldsymbol{u}(\cdot,t)\|_2^2+\|\boldsymbol{w}(\cdot,t)\|_2^2+\|\boldsymbol{b}(\cdot,t)\|_2^2 \leq C_0\,(t+1)^{-1/2}, \quad \forall\, t \geq 0, \tag{4.62}$$

com a constante $C_0 > 0$ dependendo somente de K_1 e K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$.

Como antes, a próxima etapa é melhorar as estimativas (4.58) e (4.62), que podem ser feitas como segue. Pelo Lema 4.8 e pela desigualdade (4.45), temos que $|\boldsymbol{G}(\cdot,t)| \leq C\,|\boldsymbol{\xi}|\,\|\boldsymbol{z}(\cdot,t)\|_2^2$ e $|\boldsymbol{F}(\cdot,t)| \leq C\,|\boldsymbol{\xi}|$, para todo $t \geq 0$. Segue, de (4.56) e (4.62), que

$$|\widehat{\boldsymbol{z}}(\boldsymbol{\xi},t)| \;\leq\; e^{-\sigma|\boldsymbol{\xi}|^2 t}\,|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})| + C\int_0^t e^{-\sigma|\boldsymbol{\xi}|^2(t-s)}\,|\boldsymbol{\xi}|\,(s+1)^{-1/2}\,ds,$$

para todo $t \geq 0$ e $|\boldsymbol{\xi}| \leq 1$, onde a constante $C > 0$ depende somente de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Portanto, usando o fato de que $e^{-a^2\tau}\,a \leq (2e\tau)^{-1/2}$, para todo $a \geq 0$ e $\tau > 0$, temos

$$\begin{aligned}&|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|+|\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|+|\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|\\ &\quad\leq C\left\{\|\boldsymbol{u}_0\|_1+\|\boldsymbol{w}_0\|_1+\|\boldsymbol{b}_0\|_1+\int_0^t[2e\sigma(t-s)]^{-1/2}(s+1)^{-1/2}ds\right\}\\ &\quad\leq C\left\{\|\boldsymbol{u}_0\|_1+\|\boldsymbol{w}_0\|_1+\|\boldsymbol{b}_0\|_1+\int_0^t(t-s)^{-1/2}(s+1)^{-1/2}ds\right\}\\ &\quad\leq C,\end{aligned}$$

pois $\displaystyle\int_0^t (t-s)^{-1/2}(s+1)^{-1/2}ds < \pi$ para todo $t > 0$. Logo,

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2+|\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2+|\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2 \leq C, \quad \forall\, t \geq 0,\ |\boldsymbol{\xi}| \leq 1, \tag{4.63}$$

(compare com (4.58)), onde a cosntante $C > 0$ depende somente de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$.

Das desigualdades (4.60) e (4.63) e observando que, para $t \geq 5$, se $\boldsymbol{\xi} \in B(t)$ então $|\boldsymbol{\xi}| \leq 1$, obtemos

$$\begin{aligned}
\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2 + \|\widehat{\boldsymbol{b}}\|_2^2\right)\right] &\leq C(t+1)^2\int_{B(t)} d\boldsymbol{\xi} + (2\pi)^3(t+1)^3 H(t) \\
&\leq C(t+1)^2\, 4\pi/3\, R(t)^3 + (2\pi)^3(t+1)^3 H(t) \\
&\leq C(t+1)^{1/2} + (2\pi)^3(t+1)^3 H(t),
\end{aligned}$$

para todo $t \geq 5$, onde $R(t) = \sqrt{6}\,(t+1)^{-1/2}$ é o raio da bola fechada $B(t)$. Portanto, usando o teorema de Plancherel, obtemos, para todo $t \geq 5$, que

$$\frac{d}{dt}\left[(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right)\right] \leq C(t+1)^{1/2} + (t+1)^3 H(t), \quad (4.64)$$

onde $C > 0$ depende apenas de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$.

Por outro lado, note que, se $0 \leq t < 5$, então

$$\begin{aligned}
(t+1)^{3/2}&\int_{B(t)}\left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)\|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} \\
&< 6^{3/2}\int_{B(t)}\left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} \\
&\leq C\int_{\mathbb{R}^3}\left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} \\
&= C\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + |\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right) \\
&= C(2\pi)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right) \\
&\leq C,
\end{aligned}$$

onde na última igualdade usamos o teorema de Plancherel e na última desigualdade utilizamos a estimativa (4.48). Aqui, $C \in \mathbb{R}^+$ depende apenas de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Multiplicando a desigualdade acima por $(t+1)^{1/2}$, obtemos

$$(t+1)^2\int_{B(t)}\left(|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)|^2\right) d\boldsymbol{\xi} \leq C(t+1)^{1/2},$$

para todo $0 \leq t < 5$. Segue, de (4.60), que

$$\frac{d}{dt}\left[(t+1)^3\left(\|\widehat{\boldsymbol{u}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{w}}(\cdot,t)\|_2^2 + \|\widehat{\boldsymbol{b}}(\cdot,t)\|_2^2\right)\right] \leq C(t+1)^{1/2} + (2\pi)^3(t+1)^3 H(t),$$

para todo $t \in [0,5)$. Aplicando o teorema de Plancherel, obtemos, para todo $t \in [0,5)$, que

$$\frac{d}{dt}\left[(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right)\right] \leq C(t+1)^{1/2} + (t+1)^3 H(t), \quad (4.65)$$

onde a constante $C > 0$ depende somente de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Portanto, das desigualdades (4.64) e (4.65), concluímos que

$$\frac{d}{dt}\left[(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right)\right] \leq C_*(t+1)^{1/2} + (t+1)^3 H(t), \quad (4.66)$$

para todo $t \geq 0$, onde a constante $C_* \in \mathbb{R}^+$ depende somente de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$.

Agora, estamos em um bom momento para melhorar a taxa de decaimento apresentada em (4.62) através de um método iterativo que será descrito a seguir. Uma vez que

$$H(t) = 2\|\boldsymbol{f}(\cdot,t)\|_2\|\boldsymbol{u}(\cdot,t)\|_2 + 2\|\boldsymbol{g}(\cdot,t)\|_2^2,$$

para todo $t \geq 0$, segue da hipótese (4.44), que

$$H(t) \leq 2K_1(t+1)^{-7/4}\|\boldsymbol{u}(\cdot,t)\|_2 + 2K_1^2(t+1)^{-5/2}, \quad \forall\, t \geq 0. \tag{4.67}$$

Logo, por (4.62), segue que

$$H(t) \leq 2K_1C_0^{1/2}(t+1)^{-2} + 2K_1^2(t+1)^{-5/2}, \quad \forall\, t \geq 0,$$

de modo que, por (4.66), temos, para todo $t \geq 0$,

$$\begin{aligned}\frac{d}{dt}\left[(t+1)^3\left(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{w}\|_2^2 + \|\boldsymbol{b}\|_2^2\right)\right] &\leq C_*(t+1)^{1/2} + 2K_1C_0^{1/2}(t+1) + 2K_1^2(t+1)^{1/2} \\ &= (C_* + 2K_1^2)(t+1)^{1/2} + 2K_1C_0^{1/2}(t+1) \\ &\leq (C_* + 2K_1^2 + 2K_1C_0^{1/2})(t+1).\end{aligned}$$

Integrando a desigualdade acima com respeito a variável temporal, obtemos

$$\begin{aligned}&(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right) \\ &\quad\leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + (C_* + 2K_1^2 + 2K_1C_0^{1/2})\int_0^t (s+1)\,ds \\ &\quad= \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + (C_* + 2K_1^2 + 2K_1C_0^{1/2})\,\frac{1}{2}\left[(t+1)^2 - 1\right] \\ &\quad\leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + (C_* + 2K_1^2 + 2K_1C_0^{1/2})(t+1)^2, \quad \forall\, t \geq 0.\end{aligned}$$

Desta forma,

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 &\leq \left(\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2\right)(t+1)^{-3} \\ &\quad + (C_* + 2K_1^2 + 2K_1C_0^{1/2})(t+1)^{-1}, \quad \forall\, t \geq 0.\end{aligned}$$

Portanto,

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 \leq C_1(t+1)^{-1}, \quad \forall\, t \geq 0, \tag{4.68}$$

(compare com (4.62)) onde $C_1 := \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + C_* + 2K_1^2 + 2K_1C_0^{1/2}$. Note que $C_1 > 0$ depende de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$.

Afirmação: Temos que

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 \leq C_m(t+1)^{-2\alpha_m}, \quad \forall\, t \geq 0, \quad \forall\, m \in \mathbb{N}\cup\{0\}, \tag{4.69}$$

onde

$$\alpha_m := \frac{3}{4} - 2^{-(m+1)}, \quad \forall\, m \geq 0 \tag{4.70}$$

e

$$C_m := \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + C_* + 2K_1^2 + 2K_1 C_{m-1}^{1/2}, \quad \forall\, m \geq 1, \tag{4.71}$$

e C_0 e C_* são as constantes que aparecem em (4.62) e (4.66), respectivamente.

De fato, como $\alpha_0 = \dfrac{3}{4} - 2^{-1} = \dfrac{1}{4}$ e $\alpha_1 = \dfrac{3}{4} - 2^{-2} = \dfrac{1}{2}$, os casos $m = 0$ e $m = 1$ foram provados em (4.62) e (4.68), respectivamente. Agora, suponha que (*hipótese de indução*)

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 \leq C_{m-1}(t+1)^{-2\alpha_{m-1}}, \quad \forall\, t \geq 0, \tag{4.72}$$

para algum $m \geq 1$ e algum α_{m-1} (note que $1/4 \leq \alpha_{m-1} < 3/4$). Segue, de (4.67) e (4.72), que

$$H(t) \leq 2K_1 C_{m-1}^{1/2}(t+1)^{-\frac{7}{4}-\alpha_{m-1}} + 2K_1^2(t+1)^{-5/2}.$$

Portanto, por (4.66), temos

$$\begin{aligned}
&\frac{d}{dt}\left[(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right)\right] \\
&\qquad \leq C_*(t+1)^{1/2} + 2K_1 C_{m-1}^{1/2}(t+1)^{\frac{5}{4}-\alpha_{m-1}} + 2K_1^2(t+1)^{1/2} \\
&\qquad = (C_* + 2K_1^2)(t+1)^{1/2} + 2K_1 C_{m-1}^{1/2}(t+1)^{\frac{5}{4}-\alpha_{m-1}} \\
&\qquad \leq (C_* + 2K_1^2 + 2K_1 C_{m-1}^{1/2})(t+1)^{\frac{5}{4}-\alpha_{m-1}},
\end{aligned}$$

pois $\dfrac{1}{2} < \dfrac{5}{4} - \alpha_{m-1}$, uma vez que $\alpha_{m-1} < \dfrac{3}{4}$. Integrando de 0 até t, obtemos

$$\begin{aligned}
&(t+1)^3\left(\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2\right) \\
&\qquad \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + (C_* + 2K_1^2 + 2K_1 C_{m-1}^{1/2})\int_0^t (s+1)^{\frac{5}{4}-\alpha_{m-1}}\,ds \\
&\qquad \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + (C_* + 2K_1^2 + 2K_1 C_{m-1}^{1/2})(t+1)^{\frac{9}{4}-\alpha_{m-1}}.
\end{aligned}$$

Assim,

$$\begin{aligned}
&\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 \\
&\qquad \leq (\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2)(t+1)^{-3} + (C_* + 2K_1^2 + 2K_1 C_{m-1}^{1/2})(t+1)^{-\frac{3}{4}-\alpha_{m-1}} \\
&\qquad \leq (\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + C_* + 2K_1^2 + 2K_1 C_{m-1}^{1/2})(t+1)^{-\frac{3}{4}-\alpha_{m-1}},
\end{aligned} \tag{4.73}$$

pois $\dfrac{3}{4} + \alpha_{m-1} < 3$. Como $\alpha_{m-1} = \dfrac{3}{4} - 2^{-m}$, temos que

$$\frac{1}{2}\left(\frac{3}{4} + \alpha_{m-1}\right) = \frac{1}{2}\left(\frac{6}{4} - 2^{-m}\right) = \frac{3}{4} - 2^{-m-1} = \alpha_m,$$

ou seja, $2\alpha_m = \dfrac{3}{4} + \alpha_{m-1}$. Portanto, por (4.73), temos

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 \leq C_m(t+1)^{-2\alpha_m}, \quad \forall\, t \geq 0, \quad \forall\, m = 0, 1, 2, \cdots,$$

onde $\alpha_m = \dfrac{3}{4} - 2^{-m-1}$ e $C_m = \widetilde{B} + 2K_1 C_{m-1}^{1/2}$, com $\widetilde{B} := \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + C_* + 2K_1^2$, provando, assim, a afirmação.

Ademais, sejam $\widetilde{L} := \max\{2\widetilde{B}, 4K_1, C_0, 1\}$ e C_m dado em (4.71). Então

$$C_m \leq \widetilde{L}^2, \text{ para todo } m \in \mathbb{N} \cup \{0\}.$$

Com efeito, como $1 \leq \widetilde{L}$ então $\widetilde{L} \leq \widetilde{L}^2$. Logo, sendo $C_0 \leq \widetilde{L}$, segue que $C_0 \leq \widetilde{L}^2$. Uma vez que $C_1 = \widetilde{B} + 2K_1 C_0^{1/2}$, $2\widetilde{B} \leq \widetilde{L}$, $4K_1 \leq \widetilde{L}$ e $C_0^{1/2} \leq \widetilde{L}$ (pois $C_0 \leq \widetilde{L}^2$), temos que

$$C_1 \leq \frac{\widetilde{L}}{2} + \frac{\widetilde{L}}{2} \cdot \widetilde{L} = \frac{\widetilde{L}}{2} + \frac{\widetilde{L}^2}{2} \leq \frac{\widetilde{L}^2}{2} + \frac{\widetilde{L}^2}{2} = \widetilde{L}^2.$$

Por fim, suponha que $C_{m-1} \leq \widetilde{L}^2$, para algum $m \geq 1$. Em particular, temos que $C_{m-1}^{1/2} \leq \widetilde{L}$. Portanto,

$$C_m = \widetilde{B} + 2K_1 C_{m-1}^{1/2} \leq \frac{\widetilde{L}}{2} + \frac{\widetilde{L}}{2} \cdot \widetilde{L} = \frac{\widetilde{L}}{2} + \frac{\widetilde{L}^2}{2} \leq \frac{\widetilde{L}^2}{2} + \frac{\widetilde{L}^2}{2} = \widetilde{L}^2,$$

como afirmado anteriormente. Como $\alpha_m \underset{m\to\infty}{\longrightarrow} \dfrac{3}{4}$, obtemos o resultado desejado fazendo $m \to \infty$ em (4.69), i.e.,

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2 \leq \widetilde{L}^2 \,(t+1)^{-3/2}, \quad \forall\, t \geq 0.$$

Isto conclui a prova do Teorema 4.10. □

4.2.1 Taxa de decaimento melhorada para a micro-rotação

Finalmente, observamos que, de forma similar ao caso onde as forças externas são nulas, podemos obter uma taxa de decaimento mais rápida para $\|\boldsymbol{w}(\cdot,t)\|_2$, assumindo a seguinte condição adicional:

$$(t+1)\|\nabla \boldsymbol{f}(\cdot,t)\|_2 + \|\nabla \boldsymbol{g}(\cdot,t)\|_2 \leq K_3 (t+1)^{-7/4}, \quad \forall\, t \geq \tilde{t}_1, \tag{4.74}$$

para algum $\tilde{t}_1 \geq 0$ e alguma constante $K_3 > 0$.

A fim de obter a estimativa melhorada para a velocidade micro-rotacional, definimos o tempo $\widetilde{T} > 0$ por

$$\widetilde{T} := 2^{24}(1 + M_0)^4, \tag{4.75}$$

onde M_0 é dado por

$$M_0 := \left\{ \|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,t)\|_2^2 \,dt \right\}^{1/2} + \int_0^\infty \|\boldsymbol{f}(\cdot,t)\|_2 \,dt. \tag{4.76}$$

Lema 4.11. *Seja* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *uma solução suave do PVI* (4.1). *Se* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$, *com* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$, *e* $\boldsymbol{f}$, $\boldsymbol{g}$ *satisfazem* (4.43)–(4.45) *e* (4.74), *então*

$$\|\nabla \boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^2 \leq \frac{2^{-20}}{(1+M_0)^2}, \quad \forall\, t \geq t_2,$$

onde $t_2 \geq \tilde{t}_1$ *e* M_0 *é dado por* (4.76).

Demonstração. Integrando a desigualdade de energia (4.47) no intervalo $[0,t]$, obtemos

$$\begin{aligned}\|\boldsymbol{z}(\cdot,t)\|_2^2 + \frac{1}{2}\int_0^t \|\nabla \boldsymbol{z}(\cdot,s)\|_2^2\,ds + \frac{1}{2}\int_0^t \|\boldsymbol{w}(\cdot,s)\|_2^2\,ds \\ \leq \|\boldsymbol{z}_0\|_2^2 + 2\int_0^t \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds + 2\int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\,\|\boldsymbol{u}(\cdot,s)\|_2\,ds, \quad \forall\, t \geq 0. \end{aligned} \tag{4.77}$$

Em particular,

$$\|\boldsymbol{z}(\cdot,t)\|_2^2 \leq \|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds + 2\int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\|\boldsymbol{z}(\cdot,s)\|_2\,ds, \quad \forall\, t \geq 0. \tag{4.78}$$

Seja $\widetilde{A} := \|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds$. Então de (4.78), temos que

$$\|\boldsymbol{z}(\cdot,t)\|_2^2 \leq \widetilde{A} + 2\int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\|\boldsymbol{z}(\cdot,s)\|_2\,ds, \quad \forall\, t \geq 0. \tag{4.79}$$

Usando um argumento tipo Gronwall, provaremos, a seguir, que

$$\|\boldsymbol{z}(\cdot,t)\|_2 \leq M_0, \quad \forall\, t \geq 0, \tag{4.80}$$

onde M_0 foi definido em (4.76). Dado $\epsilon > 0$, defina

$$J_\epsilon(t) := \widetilde{A} + \epsilon^2 + 2\int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\|\boldsymbol{z}(\cdot,s)\|_2\,ds,$$

para todo $t \geq 0$. Então, de (4.79) temos

$$\|\boldsymbol{z}(\cdot,t)\|_2^2 < J_\epsilon(t), \quad \forall\, t \geq 0. \tag{4.81}$$

Observe que derivando $J_\epsilon(t)$ e usando (4.81), obtemos

$$\frac{d}{dt}J_\epsilon(t) = 2\,\|\boldsymbol{f}(\cdot,t)\|_2\,\|\boldsymbol{z}(\cdot,t)\|_2 < 2J_\epsilon(t)^{1/2}\|\boldsymbol{f}(\cdot,t)\|_2.$$

Como $J_\epsilon(t) > 0$, para todo $t \geq 0$, temos que

$$\frac{1}{2}J_\epsilon(t)^{-1/2}\frac{d}{dt}J_\epsilon(t) < \|\boldsymbol{f}(\cdot,t)\|_2.$$

Isto é,

$$\frac{d}{dt}\left[J_\epsilon(t)^{1/2}\right] < \|\boldsymbol{f}(\cdot,t)\|_2.$$

Integrando a desigualdade acima no intervalo $[0,t]$, obtemos

$$\|\boldsymbol{z}(\cdot,t)\|_2 < J_\epsilon(t)^{1/2} < J_\epsilon(0)^{1/2} + \int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\,ds,$$

ou seja,

$$\begin{aligned}\|\boldsymbol{z}(\cdot,t)\|_2 &< (\widetilde{A} + \epsilon^2)^{1/2} + \int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\,ds \\ &\leq \widetilde{A}^{1/2} + \epsilon + \int_0^\infty \|\boldsymbol{f}(\cdot,t)\|_2\,dt.\end{aligned}$$

Como $\epsilon > 0$ é arbitrário, segue, da definição de $\widetilde{A}$, que

$$\|\boldsymbol{z}(\cdot,t)\|_2 \leq \left\{\|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds\right\}^{1/2} + \int_0^\infty \|\boldsymbol{f}(\cdot,s)\|_2\,ds = M_0, \quad \forall\, t \geq 0. \tag{4.82}$$

Além disso, note que, da desigualdade (4.77), temos

$$\begin{aligned}
\frac{1}{2}\int_0^t \|\nabla \boldsymbol{z}(\cdot,s)\|_2^2\,ds &\leq \|\boldsymbol{z}_0\|_2^2 + 2\int_0^t \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds + 2\int_0^t \|\boldsymbol{f}(\cdot,s)\|_2\,\|\boldsymbol{u}(\cdot,s)\|_2\,ds \\
&\leq \|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds + 2\int_0^\infty \|\boldsymbol{f}(\cdot,s)\|_2\|\boldsymbol{z}(\cdot,s)\|_2\,ds \\
&\leq \|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds + 2M_0\int_0^\infty \|\boldsymbol{f}(\cdot,s)\|_2\,ds \\
&\leq \|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds + \left(\int_0^\infty \|\boldsymbol{f}(\cdot,s)\|_2\,ds\right)^2 + M_0^2 \\
&= \left\{\left[\|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds + \left(\int_0^\infty \|\boldsymbol{f}(\cdot,s)\|_2\,ds\right)^2\right]^{1/2}\right\}^2 + M_0^2 \\
&\leq \left\{\left[\|\boldsymbol{z}_0\|_2^2 + 2\int_0^\infty \|\boldsymbol{g}(\cdot,s)\|_2^2\,ds\right]^{1/2} + \int_0^\infty \|\boldsymbol{f}(\cdot,s)\|_2\,ds\right\}^2 + M_0^2 \\
&= M_0^2 + M_0^2 = 2M_0^2,
\end{aligned}$$

onde na terceira desigualdade usamos (4.82) e na quarta desigualdade usamos Young. Portanto, da estimativa acima, temos que

$$\int_0^t \|\nabla \boldsymbol{z}(\cdot,s)\|_2^2\,ds \leq 4M_0^2, \quad \forall\, t \geq 0. \tag{4.83}$$

Agora, derivando as equações $(4.1)_1$, $(4.1)_2$ e $(4.1)_3$ em relação a x_l, obtemos

$$\left\{\begin{aligned}
&\partial_t D_l\boldsymbol{u} + (D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)D_l\boldsymbol{u} - \Delta D_l\boldsymbol{u} + \nabla D_l P \\
&\qquad\qquad = \frac{1}{2}\operatorname{rot} D_l\boldsymbol{w} + (D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)D_l\boldsymbol{b} + D_l\boldsymbol{f}, \\
&\partial_t D_l\boldsymbol{w} + (D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)D_l\boldsymbol{w} - \Delta D_l\boldsymbol{w} - \nabla(\operatorname{div} D_l\boldsymbol{w}) + D_l\boldsymbol{w} \\
&\qquad\qquad = \frac{1}{2}\operatorname{rot} D_l\boldsymbol{u} + D_l\boldsymbol{g}, \\
&\partial_t D_l\boldsymbol{b} + (D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{u}\cdot\nabla)D_l\boldsymbol{b} - \Delta D_l\boldsymbol{b} = (D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{b}\cdot\nabla)D_l\boldsymbol{u}.
\end{aligned}\right. \tag{4.84}$$

Multiplicando as equações $(4.84)_1$ por $D_l\boldsymbol{u}$, $(4.84)_2$ por $D_l\boldsymbol{w}$ e $(4.84)_3$ por $D_l\boldsymbol{b}$ em $\boldsymbol{L}^2$, obtemos:

$$\left\{\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|D_l\boldsymbol{u}\|_2^2 + \|\nabla D_l\boldsymbol{u}\|_2^2 = \frac{1}{2}(\operatorname{rot} D_l\boldsymbol{w}, D_l\boldsymbol{u}) + ((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{u}) \\
&\qquad +((\boldsymbol{b}\cdot\nabla)D_l\boldsymbol{b}, D_l\boldsymbol{u}) - ((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{u}) + (D_l\boldsymbol{f}, D_l\boldsymbol{u}), \\
&\frac{1}{2}\frac{d}{dt}\|D_l\boldsymbol{w}\|_2^2 + \|\nabla D_l\boldsymbol{w}\|_2^2 + \|\operatorname{div} D_l\boldsymbol{w}\|_2^2 + \|D_l\boldsymbol{w}\|_2^2 = \frac{1}{2}(\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w}) \\
&\qquad -((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, D_l\boldsymbol{w}) + (D_l\boldsymbol{g}, D_l\boldsymbol{w}), \\
&\frac{1}{2}\frac{d}{dt}\|D_l\boldsymbol{b}\|_2^2 + \|\nabla D_l\boldsymbol{b}\|_2^2 = -((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{b}) + ((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{b}) \\
&\qquad +((\boldsymbol{b}\cdot\nabla)D_l\boldsymbol{u}, D_l\boldsymbol{b}).
\end{aligned}\right. \tag{4.85}$$

Como div $\boldsymbol{b} = 0$, então, após integrarmos por partes, concluímos que $((\boldsymbol{b} \cdot \nabla)D_l\boldsymbol{b}, D_l\boldsymbol{u}) + ((\boldsymbol{b} \cdot \nabla)D_l\boldsymbol{u}, D_l\boldsymbol{b}) = 0$. Portanto, somando as identidades de (4.85), obtemos

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|D_l\boldsymbol{u}\|_2^2 + \|D_l\boldsymbol{w}\|_2^2 + \|D_l\boldsymbol{b}\|_2^2\right] + \|D_l(\nabla\boldsymbol{u})\|_2^2 + \|D_l(\nabla\boldsymbol{w})\|_2^2 + \|D_l(\nabla\boldsymbol{b})\|_2^2 \\
&+\|D_l(\operatorname{div}\boldsymbol{w})\|_2^2 + \|D_l\boldsymbol{w}\|_2^2 = (\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w}) + ((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{u}) \\
&-((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{u}) - ((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, D_l\boldsymbol{w}) - ((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{b}) \\
&+((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{b}) + (D_l\boldsymbol{f}, D_l\boldsymbol{u}) + (D_l\boldsymbol{g}, D_l\boldsymbol{w}).
\end{aligned} \tag{4.86}$$

Agora, multiplicando por 2 e somando a equação (4.86) sobre $l = 1, 2, 3$, obtemos

$$\begin{aligned}
&\frac{d}{dt}\left[\sum_{l=1}^{3}\|D_l\boldsymbol{u}\|_2^2 + \sum_{l=1}^{3}\|D_l\boldsymbol{w}\|_2^2 + \sum_{l=1}^{3}\|D_l\boldsymbol{b}\|_2^2\right] + 2\sum_{l=1}^{3}\|D_l(\nabla\boldsymbol{u})\|_2^2 + 2\sum_{l=1}^{3}\|D_l(\nabla\boldsymbol{w})\|_2^2 \\
&+2\sum_{l=1}^{3}\|D_l(\nabla\boldsymbol{b})\|_2^2 + 2\sum_{l=1}^{3}\|D_l(\operatorname{div}\boldsymbol{w})\|_2^2 + 2\sum_{l=1}^{3}\|D_l\boldsymbol{w}\|_2^2 = 2\sum_{l=1}^{3}(\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w}) \\
&+2\sum_{l=1}^{3}((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{u}) - 2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{u}) - 2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{w}, D_l\boldsymbol{w}) \\
&-2\sum_{l=1}^{3}((D_l\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, D_l\boldsymbol{b}) + 2\sum_{l=1}^{3}((D_l\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, D_l\boldsymbol{b}) + 2\sum_{l=1}^{3}(D_l\boldsymbol{f}, D_l\boldsymbol{u}) \\
&+2\sum_{l=1}^{3}(D_l\boldsymbol{g}, D_l\boldsymbol{w}).
\end{aligned} \tag{4.87}$$

Pelas desigualdades de Cauchy-Schwarz e Young, obtemos

$$\begin{aligned}
2\sum_{l=1}^{3}(\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w}) &\leq 2\sum_{l=1}^{3}|(\operatorname{rot} D_l\boldsymbol{u}, D_l\boldsymbol{w})| \\
&\leq 2\sum_{l=1}^{3}\|\nabla D_l\boldsymbol{u}\|_2\|D_l\boldsymbol{w}\|_2 \\
&\leq \frac{7}{8}\sum_{l=1}^{3}\|\nabla D_l\boldsymbol{u}\|_2^2 + \frac{8}{7}\sum_{l=1}^{3}\|D_l\boldsymbol{w}\|_2^2 = \frac{7}{8}\|D^2\boldsymbol{u}\|_2^2 + \frac{8}{7}\|\nabla\boldsymbol{w}\|_2^2
\end{aligned}$$

e

$$\begin{aligned}
2\sum_{l=1}^{3}(D_l\boldsymbol{g}, D_l\boldsymbol{w}) &\leq 2\sum_{l=1}^{3}|(D_l\boldsymbol{g}, D_l\boldsymbol{w})| \\
&\leq 2\sum_{l=1}^{3}\|D_l\boldsymbol{g}\|_2\|D_l\boldsymbol{w}\|_2 \\
&\leq 2\sum_{l=1}^{3}\|D_l\boldsymbol{g}\|_2^2 + \frac{1}{2}\sum_{l=1}^{3}\|D_l\boldsymbol{w}\|_2^2 = 2\|\nabla\boldsymbol{g}\|_2^2 + \frac{1}{2}\|\nabla\boldsymbol{w}\|_2^2.
\end{aligned}$$

Das desigualdades de Cauchy- Schwarz, de Gagliardo-Niremberg e de Young, temos

$$
\begin{aligned}
2\sum_{l=1}^{3}(D_l\boldsymbol{f}, D_l\boldsymbol{u}) &\leq 2\sum_{l=1}^{3}|(D_l\boldsymbol{f}, D_l\boldsymbol{u})| \leq 2\sum_{l=1}^{3}\|D_l\boldsymbol{f}\|_2\|D_l\boldsymbol{u}\|_2 \leq 2\|\nabla\boldsymbol{f}\|_2\|\nabla\boldsymbol{u}\|_2 \\
&\leq 2\,\|\nabla\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2^{1/2}\|D^2\boldsymbol{u}\|_2^{1/2} \leq 2\,\|\nabla\boldsymbol{f}\|_2\left(\|\boldsymbol{u}\|_2 + \frac{1}{4}\|D^2\boldsymbol{u}\|_2\right) \\
&= 2\|\nabla\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2 + \frac{1}{2}\|\nabla\boldsymbol{f}\|_2\|D^2\boldsymbol{u}\|_2 \leq 2\,\|\nabla\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2 + \frac{1}{4}\|\nabla\boldsymbol{f}\|_2^2 + \frac{1}{4}\|D^2\boldsymbol{u}\|_2^2 \\
&< 2\,\|\nabla\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2 + 2\,\|\nabla\boldsymbol{f}\|_2^2 + \frac{1}{2}\|D^2\boldsymbol{u}\|_2^2.
\end{aligned}
$$

Assim, pelas estimativas acima e de forma analóga ao Lema 4.4, obtemos, por (4.87), que

$$
\begin{aligned}
&\frac{d}{dt}\left[\|\nabla\boldsymbol{u}\|_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\boldsymbol{b}\|_2^2\right] + 2(\|D^2\boldsymbol{u}\|_2^2 + \|D^2\boldsymbol{w}\|_2^2 + \|D^2\boldsymbol{b}\|_2^2) + 2\|\nabla\mathrm{div}\,\boldsymbol{w}\|_2^2 + 2\|\nabla\boldsymbol{w}\|_2^2 \\
&\leq 10\|\boldsymbol{z}\|_\infty\|\nabla\boldsymbol{z}\|_2\|D^2\boldsymbol{z}\|_2 + \frac{11}{8}\|D^2\boldsymbol{u}\|_2^2 + \frac{23}{14}\|\nabla\boldsymbol{w}\|_2^2 + 2\|\nabla\boldsymbol{g}\|_2^2 + 2\,\|\nabla\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2 + 2\|\nabla\boldsymbol{f}\|_2^2.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\boldsymbol{z}\|_2^2 + \frac{5}{8}\|D^2\boldsymbol{z}\|_2^2 + \frac{1}{4}\|\nabla\boldsymbol{w}\|_2^2 &\leq 10\|\boldsymbol{z}\|_\infty\|\nabla\boldsymbol{z}\|_2\|D^2\boldsymbol{z}\|_2 \\
&\quad + 2\|\nabla\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2 + 2\,\|\nabla\boldsymbol{f}\|_2^2 + 2\,\|\nabla\boldsymbol{g}\|_2^2.
\end{aligned}
$$

Segue, da desigualdade (4.31), que

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\boldsymbol{z}\|_2^2 + \frac{5}{8}\|D^2\boldsymbol{z}\|_2^2 + \frac{1}{4}\|\nabla\boldsymbol{w}\|_2^2 &\leq 10\|\boldsymbol{z}\|_2^{1/2}\|\nabla\boldsymbol{z}\|_2^{1/2}\|D^2\boldsymbol{z}\|_2^2 \\
&\quad + 2\|\nabla\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2 + 2\|\nabla\boldsymbol{f}\|_2^2 + 2\,\|\nabla\boldsymbol{g}\|_2^2.
\end{aligned}
\tag{4.88}
$$

Agora, escolha $t_* \geq \tilde{t}_1$ tal que

$$
M_0\int_{t_*}^{\infty}\|\nabla\boldsymbol{f}(\cdot,s)\|_2\,ds + \int_{t_*}^{\infty}\left(\|\nabla\boldsymbol{f}(\cdot,s)\|_2^2 + \|\nabla\boldsymbol{g}(\cdot,s)\|_2^2\right)ds \leq \frac{2^{-22}}{(1+M_0)^2}. \tag{4.89}
$$

Observe que, de (4.83), temos

$$
\int_0^{\infty}\|\nabla\boldsymbol{z}(\cdot,s)\|_2^2\,ds \leq 4M_0^2.
$$

Então, em particular, temos que

$$
\int_{t_*}^{t_*+\widetilde{T}}\|\nabla\boldsymbol{z}(\cdot,s)\|_2^2\,ds \leq 4M_0^2, \tag{4.90}
$$

onde $\widetilde{T}$ foi definido em (4.75). Pelo Teorema do valor médio para integrais de Lebesgue, existe um conjunto $\widetilde{E} \subseteq (t_*, t_* + \widetilde{T})$ com medida positiva tal que

$$
\widetilde{T}\,\|\nabla\boldsymbol{z}(\cdot,t)\|_2^2 \leq \int_{t_*}^{t_*+\widetilde{T}}\|\nabla\boldsymbol{z}(\cdot,s)\|_2^2\,ds, \quad \forall\, t \in \widetilde{E}.
$$

Logo, de (4.90), temos que

$$
\|\nabla\boldsymbol{z}(\cdot,t)\|_2^2 \leq \frac{4\,M_0^2}{\widetilde{T}}, \quad \forall\, t \in \widetilde{E}.
$$

Da definição de $\widetilde{T}$ em (4.75), segue que

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 \leq \frac{2^{-22}}{(1+M_0)^2}, \quad \forall\, t \in \widetilde{E}. \tag{4.91}$$

Agora, escolha $\tilde{t}_0 \in \widetilde{E}$ (fixo, porém arbitrário). Integrando a desigualdade (4.88) sobre $(\tilde{t}_0, t)$ e usando a estimativa (4.82), obtemos

$$\begin{aligned}
&\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + \frac{5}{8}\int_{\tilde{t}_0}^{t} \|D^2\boldsymbol{z}(\cdot,s)\|_2^2\,ds + \frac{1}{4}\int_{\tilde{t}_0}^{t} \|\nabla \boldsymbol{w}(\cdot,s)\|_2^2\,ds \\
&\quad \leq \|\nabla \boldsymbol{z}(\cdot,\tilde{t}_0)\|_2^2 + 10M_0^{1/2}\int_{\tilde{t}_0}^{t} \|\nabla \boldsymbol{z}(\cdot,s)\|_2^{1/2}\|D^2\boldsymbol{z}(\cdot,s)\|_2^2\,ds \\
&\qquad + 2M_0\int_{\tilde{t}_0}^{t} \|\nabla \boldsymbol{f}(\cdot,s)\|_2\,ds + 2\int_{\tilde{t}_0}^{t} (\|\nabla \boldsymbol{f}(\cdot,s)\|_2^2 + \|\nabla \boldsymbol{g}(\cdot,s)\|_2^2)\,ds,
\end{aligned} \tag{4.92}$$

para todo $t \geq \tilde{t}_0$. Como $\tilde{t}_0 \in \widetilde{E} \subseteq (t_*, t_* + \widetilde{T})$, segue, de (4.91), que

$$\|\nabla \boldsymbol{z}(\cdot,\tilde{t}_0)\|_2 \leq \frac{2^{-11}}{1+M_0}. \tag{4.93}$$

Da desigualdade (4.93), temos que

$$10M_0^{1/2}\|\nabla \boldsymbol{z}(\cdot,\tilde{t}_0)\|_2^{1/2} \leq 10M_0^{1/2}\left(\frac{2^{-11}}{1+M_0}\right)^{1/2} = \frac{10}{32\sqrt{2}}\left(\frac{M_0}{1+M_0}\right)^{1/2} < \frac{5}{16\sqrt{2}} < \frac{5}{16}. \tag{4.94}$$

Então, por continuidade, existe um $\delta > 0$ tal que $10M_0^{1/2}\|\nabla \boldsymbol{z}(\cdot,t)\|_2^{1/2} \leq 5/8$, para todo $t \in [\tilde{t}_0, \tilde{t}_0 + \delta]$. Logo, por (4.92), temos que

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 \leq \|\nabla \boldsymbol{z}(\cdot,\tilde{t}_0)\|_2^2 + 2M_0\int_{\tilde{t}_0}^{t} \|\nabla \boldsymbol{f}(\cdot,s)\|_2\,ds + 2\int_{\tilde{t}_0}^{t}(\|\nabla \boldsymbol{f}(\cdot,s)\|_2^2 + \|\nabla \boldsymbol{g}(\cdot,s)\|_2^2)\,ds,$$

para todo $t \in [\tilde{t}_0, \tilde{t}_0 + \delta]$. Assim, de (4.89) e (4.93), obtemos, para todo $t \in [\tilde{t}_0, \tilde{t}_0 + \delta]$, que

$$\begin{aligned}
\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 &\leq \frac{2^{-22}}{(1+M_0)^2} + 2\cdot\frac{2^{-22}}{(1+M_0)^2} = 3\cdot\frac{2^{-22}}{(1+M_0)^2} \\
&< 4\cdot\frac{2^{-22}}{(1+M_0)^2} = \frac{2^{-20}}{(1+M_0)^2}.
\end{aligned}$$

Portanto,

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2 < \frac{2^{-10}}{1+M_0}, \quad \forall\, t \in [\tilde{t}_0, \tilde{t}_0 + \delta].$$

Afirmação:

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2 < \frac{2^{-10}}{1+M_0}, \quad \forall\, t \geq \tilde{t}_0. \tag{4.95}$$

Com efeito, se (4.95) fosse falso, existiria $\bar{t} > \tilde{t}_0 + \delta$ tal que

$$\|\nabla \boldsymbol{z}(\cdot,\bar{t})\|_2 = \frac{2^{-10}}{1+M_0} \tag{4.96}$$

e

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2 < \frac{2^{-10}}{1+M_0}, \quad \forall\, t \in [\tilde{t}_0, \bar{t}\,). \tag{4.97}$$

Uma vez que $\bar{t} > \tilde{t}_0$, da desigualdade (4.92), temos que

$$\begin{aligned}
&\|\nabla \boldsymbol{z}(\cdot,\bar{t})\|_2^2 + \frac{5}{8}\int_{\tilde{t}_0}^{\bar{t}} \|D^2\boldsymbol{z}(\cdot,s)\|_2^2\,ds + \frac{1}{4}\int_{\tilde{t}_0}^{\bar{t}} \|\nabla \boldsymbol{w}(\cdot,s)\|_2^2\,ds \\
&\qquad \le \|\nabla \boldsymbol{z}(\cdot,\tilde{t}_0)\|_2^2 + 10M_0^{1/2}\int_{\tilde{t}_0}^{\bar{t}} \|\nabla \boldsymbol{z}(\cdot,s)\|_2^{1/2}\|D^2\boldsymbol{z}(\cdot,s)\|_2^2\,ds \\
&\qquad\quad + 2M_0\int_{\tilde{t}_0}^{\bar{t}} \|\nabla \boldsymbol{f}(\cdot,s)\|_2\,ds + 2\int_{\tilde{t}_0}^{\bar{t}} (\|\nabla \boldsymbol{f}(\cdot,s)\|_2^2 + \|\nabla \boldsymbol{g}(\cdot,s)\|_2^2)\,ds.
\end{aligned}$$

Observe que, de (4.97), temos $10M_0^{1/2}\|\nabla \boldsymbol{z}(\cdot,s)\|_2^{1/2} < 5/16$, para todo $s \in [\tilde{t}_0, \bar{t}\,)$. Como $\tilde{t}_0 > t_*$, das estimativas (4.89) e (4.93) e da desigualdade acima, segue que

$$\begin{aligned}
\|\nabla \boldsymbol{z}(\cdot,\bar{t})\|_2^2 &\le \frac{2^{-22}}{(1+M_0)^2} + 2\cdot\frac{2^{-22}}{(1+M_0)^2} = 3\cdot\frac{2^{-22}}{(1+M_0)^2} \\
&< 4\cdot\frac{2^{-22}}{(1+M_0)^2} = \frac{2^{-20}}{(1+M_0)^2}.
\end{aligned}$$

Isto é,

$$\|\nabla \boldsymbol{z}(\cdot,\bar{t})\|_2 < \frac{2^{-10}}{1+M_0},$$

contradizendo a identidade (4.96). Portanto,

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2 < \frac{2^{-10}}{1+M_0}, \quad \forall\, t \ge \tilde{t}_0.$$

Em particular, uma vez que $t_* + \widetilde{T} > \tilde{t}_0$, temos que

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2 \le \frac{2^{-10}}{1+M_0}, \quad \forall\, t \ge t_* + \widetilde{T} =: t_2, \tag{4.98}$$

concluindo, assim, a demonstração do lema. □

Proposição 4.12. *Sejam* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *uma solução suave do problema de Cauchy* (4.1) *e* $t_2 \ge \tilde{t}_1$ *definido em* (4.98)*. Se* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$*, com* $\operatorname{div}\boldsymbol{u}_0 = \operatorname{div}\boldsymbol{b}_0 = 0$*, e* $\boldsymbol{f}$*,* $\boldsymbol{g}$ *satisfazem* (4.43)–(4.45) *e* (4.74)*, então*

$$\|\nabla \boldsymbol{u}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{w}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{b}(\cdot,t)\|_2^2 \le C(t+1)^{-5/2}, \quad \forall\, t \ge t_2,$$

onde $C > 0$ *depende somente de* $\tilde{t}_1, K_1, K_2, K_3$ *e das normas de* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$ *em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$*.*

Demonstração. Multiplicando a desigualdade de energia (4.47) por $(t+1)^2$, obtemos

$$\begin{aligned}
(t+1)^2\frac{d}{dt}\|\boldsymbol{z}(\cdot,t)\|_2^2 + \frac{1}{2}(t+1)^2\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 &\le 2(t+1)^2\|\boldsymbol{f}(\cdot,t)\|_2\|\boldsymbol{u}(\cdot,t)\|_2 \\
&\quad + 2(t+1)^2\|\boldsymbol{g}(\cdot,t)\|_2^2,
\end{aligned}$$

que implica em

$$\begin{aligned}
\frac{d}{dt}\left[(t+1)^2\|\boldsymbol{z}\|_2^2\right] + \frac{1}{2}(t+1)^2\|\nabla \boldsymbol{z}\|_2^2 &\le 2(t+1)\|\boldsymbol{z}(\cdot,t)\|_2^2 + 2(t+1)^2\|\boldsymbol{f}(\cdot,t)\|_2\|\boldsymbol{u}(\cdot,t)\|_2 \\
&\quad + 2(t+1)^2\|\boldsymbol{g}(\cdot,t)\|_2^2.
\end{aligned}$$

Integrando a desigualdade acima sobre (t_2, t), obtemos

$$
\begin{aligned}
(t+1)^2\|\boldsymbol{z}(t)\|_2^2 + \frac{1}{2}\int_{t_2}^{t}(\tau+1)^2\|\nabla \boldsymbol{z}(\tau)\|_2^2\,d\tau \;\le\; & (t_2+1)^2\|\boldsymbol{z}(t_2)\|_2^2 + 2\int_{t_2}^{t}(\tau+1)\|\boldsymbol{z}(\tau)\|_2^2\,d\tau \\
& + 2\int_{t_2}^{t}(\tau+1)^2\|\boldsymbol{f}(\cdot,\tau)\|_2\|\boldsymbol{u}(\cdot,\tau)\|_2\,d\tau \\
& + 2\int_{t_2}^{t}(\tau+1)^2\|\boldsymbol{g}(\cdot,\tau)\|_2^2\,d\tau,
\end{aligned}
$$

para todo $t \ge t_2$. Em particular, temos que

$$
\begin{aligned}
\frac{1}{2}\int_{t_2}^{t}(\tau+1)^2\|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^2\,d\tau \;\le\; & (t_2+1)^2\|\boldsymbol{z}(\cdot,t_2)\|_2^2 + 2\int_{t_2}^{t}(\tau+1)\|\boldsymbol{z}(\cdot,\tau)\|_2^2\,d\tau \\
& + 2\int_{t_2}^{t}(\tau+1)^2\|\boldsymbol{f}(\cdot,\tau)\|_2\|\boldsymbol{u}(\cdot,\tau)\|_2\,d\tau \\
& + 2\int_{t_2}^{t}(\tau+1)^2\|\boldsymbol{g}(\cdot,\tau)\|_2^2\,d\tau, \quad \forall\, t \ge t_2. \qquad (4.99)
\end{aligned}
$$

Agora, vamos estimar os termos do lado direito de (4.99). Do Teorema 4.10, sabemos que $\|\boldsymbol{z}(\cdot,t)\|_2^2 \le C(t+1)^{-3/2}$, para todo $t \ge 0$. Então,

$$
\begin{aligned}
2\int_{t_2}^{t}(\tau+1)\|\boldsymbol{z}(\cdot,\tau)\|_2^2\,d\tau \;\le\; & 2C\int_{t_2}^{t}(\tau+1)^{-1/2}\,d\tau \\
\le\; & 4C\left[(t+1)^{1/2} - (t_2+1)^{1/2}\right] \\
\le\; & C(t+1)^{1/2}, \qquad (4.100)
\end{aligned}
$$

onde a constante $C > 0$ depende somente de K_1, K_2 e das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0$. De (4.44), inferimos que

$$
\|\boldsymbol{f}(\cdot,t)\|_2 \le K_1(t+1)^{-7/4} \quad \text{e} \quad \|\boldsymbol{g}(\cdot,t)\|_2^2 \le K_1^2(t+1)^{-5/2}, \quad \forall\, t \ge 0.
$$

Também do Teorema 4.10, temos $\|\boldsymbol{u}(\cdot,t)\|_2 \le C(t+1)^{-3/4}$, para todo $t \ge 0$. Logo,

$$
\begin{aligned}
& 2(t+1)^2\|\boldsymbol{f}(\cdot,t)\|_2\|\boldsymbol{u}(\cdot,t)\|_2 + 2(t+1)^2\|\boldsymbol{g}(\cdot,t)\|_2^2 \\
&\qquad \le 2K_1C(t+1)^2(t+1)^{-7/4}(t+1)^{-3/4} + 2K_1^2(t+1)^2(t+1)^{-5/2} \\
&\qquad \le C(t+1)^{-1/2},
\end{aligned}
$$

para todo $t \ge 0$, onde a constante $C > 0$ depende somente de K_1, K_2 e das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Assim,

$$
\begin{aligned}
& 2\int_{t_2}^{t}(\tau+1)^2\|\boldsymbol{f}(\cdot,\tau)\|_2\|\boldsymbol{u}(\cdot,\tau)\|_2\,d\tau + 2\int_{t_2}^{t}(\tau+1)^2\|\boldsymbol{g}(\cdot,\tau)\|_2^2\,d\tau \\
& = 2\int_{t_2}^{t}\left[(\tau+1)^2\|\boldsymbol{f}(\cdot,\tau)\|_2\|\boldsymbol{u}(\cdot,\tau)\|_2 + (\tau+1)^2\|\boldsymbol{g}(\cdot,\tau)\|_2^2\right]d\tau \\
& \le C\int_{t_2}^{t}(\tau+1)^{-1/2}\,d\tau \le C[(t+1)^{1/2} - (t_2+1)^{1/2}] \le C(t+1)^{1/2}, \qquad (4.101)
\end{aligned}
$$

para todo $t \geq t_2$, onde a constante $C > 0$ depende somente de K_1, K_2 e das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Portanto, usando as estimativas (4.100) e (4.101) em (4.99), concluímos que

$$\int_{t_2}^{t} (\tau+1)^2 \|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^2 \, d\tau \leq (t_2+1)^2 \|\boldsymbol{z}(\cdot,t_2)\|_2^2 + C(t+1)^{1/2} \leq C(t+1)^{1/2}, \tag{4.102}$$

para todo $t \geq t_2$, onde a constante $C > 0$ depende somente de $\tilde{t}_1, K_1, K_2$ e das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$.

Agora, multiplicando a desigualdade (4.88) por $(t+1)^3$, obtemos

$$\begin{aligned}
&(t+1)^3 \frac{d}{dt} \|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + \frac{5}{8}(t+1)^3 \|D^2 \boldsymbol{z}(\cdot,t)\|_2^2 \\
&\quad \leq 10(t+1)^3 \|\boldsymbol{z}(\cdot,t)\|_2^{1/2} \|\nabla \boldsymbol{z}(\cdot,t)\|_2^{1/2} \|D^2 \boldsymbol{z}(\cdot,t)\|_2^2 \\
&\qquad + 2(t+1)^3 \|\nabla \boldsymbol{f}(\cdot,t)\|_2 \|\boldsymbol{u}(\cdot,t)\|_2 + 2(t+1)^3 \|\nabla \boldsymbol{f}(\cdot,t)\|_2^2 + 2(t+1)^3 \|\nabla \boldsymbol{g}(\cdot,t)\|_2^2,
\end{aligned}$$

ou seja,

$$\begin{aligned}
&\frac{d}{dt}\left[(t+1)^3 \|\nabla \boldsymbol{z}(\cdot,t)\|_2^2\right] + \frac{5}{8}(t+1)^3 \|D^2 \boldsymbol{z}(\cdot,t)\|_2^2 \\
&\quad \leq 3(t+1)^2 \|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + 10(t+1)^3 \|\boldsymbol{z}(\cdot,t)\|_2^{1/2} \|\nabla \boldsymbol{z}(\cdot,t)\|_2^{1/2} \|D^2 \boldsymbol{z}(\cdot,t)\|_2^2 \\
&\qquad + 2(t+1)^3 \|\nabla \boldsymbol{f}(\cdot,t)\|_2 \|\boldsymbol{u}(\cdot,t)\|_2 + 2(t+1)^3 \|\nabla \boldsymbol{f}(\cdot,t)\|_2^2 + 2(t+1)^3 \|\nabla \boldsymbol{g}(\cdot,t)\|_2^2.
\end{aligned}$$

Integrando a desigualdade acima de t_2 até t, onde t_2 foi definido em (4.98), obtemos

$$\begin{aligned}
&(t+1)^3 \|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + \frac{5}{8} \int_{t_2}^{t} (\tau+1)^3 \|D^2 \boldsymbol{z}(\cdot,\tau)\|_2^2 \, d\tau \leq (t_2+1)^3 \|\nabla \boldsymbol{z}(\cdot,t_2)\|_2^2 \\
&+ 3 \int_{t_2}^{t} (\tau+1)^2 \|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^2 \, d\tau + 10 \int_{t_2}^{t} (\tau+1)^3 \|\boldsymbol{z}(\cdot,\tau)\|_2^{1/2} \|\nabla \boldsymbol{z}(\cdot,\tau)\|_2^{1/2} \|D^2 \boldsymbol{z}(\cdot,\tau)\|_2^2 \, d\tau \\
&+ 2 \int_{t_2}^{t} (\tau+1)^3 \|\nabla \boldsymbol{f}(\cdot,\tau)\|_2 \|\boldsymbol{u}(\cdot,\tau)\|_2 \, d\tau + 2 \int_{t_2}^{t} (\tau+1)^3 \|\nabla \boldsymbol{f}(\cdot,\tau)\|_2^2 \, d\tau \\
&+ 2 \int_{t_2}^{t} (\tau+1)^3 \|\nabla \boldsymbol{g}(\cdot,\tau)\|_2^2 \, d\tau,
\end{aligned} \tag{4.103}$$

para todo $t \geq t_2$.

A seguir, estimaremos os termos do lado direito da desigualdade acima. Do Lema 4.11, temos que

$$(t_2+1)^3 \|\nabla \boldsymbol{z}(\cdot,t_2)\|_2^2 \leq (t_2+1)^3 \frac{2^{-20}}{(1+M_0)^2} \leq C, \tag{4.104}$$

onde $C > 0$ é uma constante que só depende $\tilde{t}_1, K_1$ e das normas em $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Da hipótese (4.74), concluímos que

$$\|\nabla \boldsymbol{f}(\cdot,t)\|_2 \leq K_3(t+1)^{-11/4}, \quad \|\nabla \boldsymbol{f}(\cdot,t)\|_2^2 \leq K_3^2(t+1)^{-11/2}, \quad \|\nabla \boldsymbol{g}(\cdot,t)\|_2^2 \leq K_3^2(t+1)^{-7/2},$$

para todo $t \geq \tilde{t}_1$. Do Teorema 4.10, temos $\|\boldsymbol{u}(\cdot,t)\|_2 \leq \|\boldsymbol{z}(\cdot,t)\|_2 \leq C(t+1)^{-3/4}$. Assim,

$$\begin{aligned}
&2(t+1)^3 \left[\|\nabla \boldsymbol{f}(\cdot,t)\|_2 \|\boldsymbol{u}(\cdot,t)\|_2 + \|\nabla \boldsymbol{f}(\cdot,t)\|_2^2 + \|\nabla \boldsymbol{g}(\cdot,t)\|_2^2\right] \\
&\leq C(t+1)^3 \left[(t+1)^{-3/4}(t+1)^{-11/4} + (t+1)^{-11/2} + (t+1)^{-7/2}\right] \\
&\leq C(t+1)^3 (t+1)^{-7/2} = C(t+1)^{-1/2},
\end{aligned}$$

para todo $t \geq \tilde{t}_1$ (em particular, $t \geq t_2$), onde a constante $C > 0$ depende de $\tilde{t}_1$, de K_1, K_2, K_3 e das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Daí,

$$\begin{aligned} &2\int_{t_2}^{t} \left[(\tau+1)^3 \left(\|\nabla \boldsymbol{f}(\cdot,\tau)\|_2 \|\boldsymbol{u}(\cdot,\tau)\|_2 + \|\nabla \boldsymbol{f}(\cdot,\tau)\|_2^2 + \|\nabla \boldsymbol{g}(\cdot,\tau)\|_2^2\right)\right] d\tau \\ &\leq C\int_{t_2}^{t} (\tau+1)^{-1/2}\, d\tau \leq C[(t+1)^{1/2} - (t_2+1)^{1/2}] \leq C(t+1)^{1/2}, \end{aligned} \tag{4.105}$$

para todo $t \geq t_2$, onde a constante $C > 0$ depende de $\tilde{t}_1$, de K_1, K_2, K_3 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Além disso, pelo Lema 4.11, temos que

$$10M_0^{1/2}\|\nabla \boldsymbol{z}(\cdot,t)\|^{1/2} \leq 10M_0^{1/2}\frac{2^{-5}}{(1+M_0)^{1/2}} = \frac{10}{32}\left(\frac{M_0}{1+M_0}\right)^{1/2} < \frac{5}{16}, \tag{4.106}$$

para todo $t \geq t_2$. Assim, usando as estimativas (4.80), (4.102) e (4.104)–(4.106) em (4.103), obtemos

$$\begin{aligned} (t+1)^3\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + \frac{5}{8}\int_{t_2}^{t}(\tau+1)^3\|D^2\boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau \quad &\leq \quad C + C(t+1)^{1/2} \\ &\quad + \frac{5}{16}\int_{t_2}^{t}(\tau+1)^3\|D^2\boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau, \end{aligned}$$

ou ainda,

$$(t+1)^3\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 + \frac{5}{16}\int_{t_2}^{t}(\tau+1)^3\|D^2\boldsymbol{z}(\cdot,\tau)\|_2^2\, d\tau \leq C(t+1)^{1/2}.$$

Portanto,

$$\|\nabla \boldsymbol{z}(\cdot,t)\|_2^2 \leq C(t+1)^{-5/2}, \quad \forall\, t \geq t_2, \tag{4.107}$$

onde a constante $C > 0$ depende apenas de $\tilde{t}_1, K_1, K_2, K_3$ e das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Isto conclui a demonstração da Proposição 4.12. $\square$

Teorema 4.13. *Seja* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *uma solução suave do PVI* (4.1)*. Se* $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$*, com* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{b}_0 = 0$*, e* $\boldsymbol{f}$*,* $\boldsymbol{g}$ *satisfazem* (4.43)–(4.45) *e* (4.74)*, então*

$$\|\boldsymbol{w}(\cdot,t)\|_2 \leq C(t+1)^{-\frac{5}{4}}, \quad \forall\, t \geq 0,$$

onde $C \in \mathbb{R}^+$ *depende somente de* $\tilde{t}_1, K_1, K_2, K_3$ *e das normas de* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$ *em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$*.*

Demonstração. Da desigualdade (4.48) temos que

$$\|\boldsymbol{w}(\cdot,t)\|_2^2 \leq C, \quad \forall\, t \geq 0,$$

onde a cosntante $C > 0$ depende somente de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Assim,

$$(t_2+1)^{5/4}\|\boldsymbol{w}(\cdot,t)\|_2 \leq C(t_2+1)^{5/4} \leq C,$$

onde $C > 0$ depende somente de $\tilde{t}_1$, de K_1 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^2$. Logo, se $0 \leq t < t_2$, então

$$\|\boldsymbol{w}(\cdot,t)\|_2 \leq C(t_2+1)^{-5/4} < C(t+1)^{-5/4},$$

pois $t < t_2$. Portanto, basta provar o resultado para $t \geq t_2$.

Reescrevendo a equação $(4.1)_2$ como

$$\boldsymbol{w}_t(\cdot,t) + \boldsymbol{w}(\cdot,t) = \Delta\boldsymbol{w}(\cdot,t) + \boldsymbol{Q}(\cdot,t),$$

onde $\boldsymbol{Q}(\cdot,t) := \dfrac{1}{2}\operatorname{rot}\boldsymbol{u} + \nabla(\operatorname{div}\boldsymbol{w}) - (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} + \boldsymbol{g}$ e multiplicando a equação acima pelo fator integrante e^t, obtemos

$$\frac{\partial}{\partial t}[e^t\boldsymbol{w}(\cdot,t)] = e^t\Delta\boldsymbol{w}(\cdot,t) + e^t\boldsymbol{Q}(\cdot,t).$$

Fazendo a mudança de variável

$$\boldsymbol{W}(\cdot,t) := e^t\boldsymbol{w}(\cdot,t) \quad \text{e} \quad \widetilde{\boldsymbol{Q}}(\cdot,t) := e^t\boldsymbol{Q}(\cdot,t),$$

obtemos

$$\frac{\partial}{\partial t}\boldsymbol{W}(\cdot,t) = \Delta\boldsymbol{W}(\cdot,t) + \widetilde{\boldsymbol{Q}}(\cdot,t). \tag{4.108}$$

Aplicando o princípio de Duhamel a $\boldsymbol{W}(\cdot,t)$ a partir do tempo t_2, encontramos

$$\boldsymbol{W}(\cdot,t) = e^{\Delta(t-t_2)}\boldsymbol{W}(\cdot,t_2) + \int_{t_2}^{t} e^{\Delta(t-s)}\widetilde{\boldsymbol{Q}}(\cdot,s)\,ds,$$

onde $\left(e^{\Delta t}\right)_{t\geq 0}$ é o semigrupo do calor. Disto segue que

$$\begin{aligned}\boldsymbol{w}(\cdot,t) &= e^{-(t-t_2)}e^{\Delta(t-t_2)}\boldsymbol{w}(\cdot,t_2)\\ &\quad + \int_{t_2}^{t} e^{-(t-s)}e^{\Delta(t-s)}\left\{\frac{1}{2}\operatorname{rot}\boldsymbol{u} + \nabla(\operatorname{div}\boldsymbol{w}) - (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} + \boldsymbol{g}\right\}(\cdot,s)\,ds.\end{aligned}$$

Tomando a norma $\boldsymbol{L}^2$ e usando a desigualdade de Minkowski, obtemos

$$\begin{aligned}\|\boldsymbol{w}(\cdot,t)\|_2 \leq e^{-(t-t_2)}\|e^{\Delta(t-t_2)}\boldsymbol{w}(\cdot,t_2)\|_2 &+ \frac{1}{2}\int_{t_2}^{t} e^{-(t-s)}\|e^{\Delta(t-s)}\operatorname{rot}\boldsymbol{u}(\cdot,s)\|_2\,ds\\ &+ \int_{t_2}^{t} e^{-(t-s)}\|e^{\Delta(t-s)}\,\nabla(\operatorname{div}\boldsymbol{w})(\cdot,s)\|_2\,ds\\ &+ \int_{t_2}^{t} e^{-(t-s)}\|e^{\Delta(t-s)}\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}(\cdot,s)\|_2\,ds\\ &+ \int_{t_2}^{t} e^{-(t-s)}\|e^{\Delta(t-s)}\,\boldsymbol{g}(\cdot,s)\|_2\,ds.\end{aligned} \tag{4.109}$$

Agora, estimaremos os cinco termos do lado direito da desigualdade acima.

Sabemos que $\boldsymbol{v}(\boldsymbol{x},t) = e^{\Delta(t-t_2)}\boldsymbol{w}(\boldsymbol{x},t_2)$ é solução do PVI

$$\begin{cases}\boldsymbol{v}_t = \Delta\boldsymbol{v}, \quad \boldsymbol{x}\in\mathbb{R}^3, \quad t > t_2,\\ \boldsymbol{v}(\cdot,t_2) = \boldsymbol{w}(\cdot,t_2) \in \boldsymbol{L}^2(\mathbb{R}^3).\end{cases} \tag{4.110}$$

Assim, aplicando a transformada de Fourier ao PVI (4.110), obtemos o problema de Cauchy

$$\widehat{\boldsymbol{v}}_t = -|\boldsymbol{\xi}|^2\widehat{\boldsymbol{v}}, \qquad \widehat{\boldsymbol{v}}(\cdot,t_2) = \widehat{\boldsymbol{w}}(\cdot,t_2),$$

cuja solução é $\widehat{\boldsymbol{v}}(\boldsymbol{\xi},t) = e^{-|\boldsymbol{\xi}|^2(t-t_2)}\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t_2)$. Logo, pelo teorema de Plancherel, segue que

$$\begin{aligned}
\|\boldsymbol{v}(\cdot,t)\|_2^2 &= (2\pi)^{-3}\|\widehat{\boldsymbol{v}}(\cdot,t)\|_2^2 = (2\pi)^{-3}\int_{\mathbb{R}^3}|\widehat{\boldsymbol{v}}(\boldsymbol{\xi},t)|^2\,d\boldsymbol{\xi}\\
&= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{-2|\boldsymbol{\xi}|^2(t-t_2)}|\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t_2)|^2\,d\boldsymbol{\xi}\\
&\le (2\pi)^{-3}\int_{\mathbb{R}^3}|\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t_2)|^2\,d\boldsymbol{\xi} = (2\pi)^{-3}\|\widehat{\boldsymbol{w}}(\cdot,t_2)\|_2^2 = \|\boldsymbol{w}(\cdot,t_2)\|_2^2 \le M_0^2,
\end{aligned}$$

onde, na última estimativa, usamos a desigualdade (4.80). Portanto,

$$\|e^{\Delta(t-t_2)}\boldsymbol{w}(\cdot,t_2)\|_2 = \|\boldsymbol{v}(\cdot,t)\|_2 \le M_0, \quad \forall\, t \ge t_2.$$

Por outro lado, pela desigualdade de Hölder e pelas estimativas (2.7) e (4.48), obtemos

$$\begin{aligned}
\frac{1}{2}\|e^{\Delta(t-s)}\mathrm{rot}\,\boldsymbol{u}(\cdot,s)\|_2 &\le C\|\mathrm{rot}\,\boldsymbol{u}(\cdot,s)\|_2 = C\|\nabla\boldsymbol{u}(\cdot,s)\|_2,\\
\|e^{\Delta(t-s)}\nabla(\mathrm{div}\,\boldsymbol{w})(\cdot,s)\|_2 &\le C(t-s)^{-1/2}\|\mathrm{div}\,\boldsymbol{w}(\cdot,s)\|_2 \le C(t-s)^{-1/2}\|\nabla\boldsymbol{w}(\cdot,s)\|_2,\\
\|e^{\Delta(t-s)}(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}(\cdot,s)\|_2 &\le C(t-s)^{-3/4}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}(\cdot,s)\|_1\\
&\le C(t-s)^{-3/4}\|\boldsymbol{u}(\cdot,s)\|_2\|\nabla\boldsymbol{w}(\cdot,s)\|_2\\
&\le C(t-s)^{-3/4}\|\nabla\boldsymbol{w}(\cdot,s)\|_2,\\
\|e^{\Delta(t-s)}\boldsymbol{g}(\cdot,s)\|_2 &\le C\,\|\boldsymbol{g}(\cdot,s)\|_2.
\end{aligned}$$

Desta forma, segue de (4.109), que

$$\begin{aligned}
\|\boldsymbol{w}(\cdot,t)\|_2 \le e^{-(t-t_2)}M_0 &+ C\int_{t_2}^{t} e^{-(t-s)}\|\nabla\boldsymbol{u}(\cdot,s)\|_2\,ds\\
&+ C\int_{t_2}^{t} e^{-(t-s)}(t-s)^{-1/2}\|\nabla\boldsymbol{w}(\cdot,s)\|_2\,ds\\
&+ C\int_{t_2}^{t} e^{-(t-s)}(t-s)^{-3/4}\|\nabla\boldsymbol{w}(\cdot,s)\|_2\,ds\\
&+ C\int_{t_2}^{t} e^{-(t-s)}\|\boldsymbol{g}(\cdot,s)\|_2\,ds.
\end{aligned}$$

Da Proposição 4.12 e da hipótese (4.44), obtemos

$$\begin{aligned}
\|\boldsymbol{w}(\cdot,t)\|_2 \le e^{-(t-t_2)}M_0 &+ C\int_{t_2}^{t} e^{-(t-s)}(s+1)^{-5/4}\,ds\\
&+ C\int_{t_2}^{t} e^{-(t-s)}(t-s)^{-1/2}(s+1)^{-5/4}\,ds\\
&+ C\int_{t_2}^{t} e^{-(t-s)}(t-s)^{-3/4}(s+1)^{-5/4}\,ds\\
&+ C\int_{t_2}^{t} e^{-(t-s)}(s+1)^{-5/4}\,ds,
\end{aligned}$$

para todo $t \geq t_2$, onde $C > 0$ depende somente de $\tilde{t}_1, K_1, K_2, K_3$ e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Portanto,

$$\begin{aligned} \|\boldsymbol{w}(\cdot,t)\|_2 \leq e^{-(t-t_2)}M_0 \;&+\; C\int_{t_2}^{t} e^{-(t-s)}(s+1)^{-5/4}\,ds \\ &+\; C\int_{t_2}^{t} e^{-(t-s)}(t-s)^{-1/2}(s+1)^{-5/4}\,ds \\ &+\; C\int_{t_2}^{t} e^{-(t-s)}(t-s)^{-3/4}(s+1)^{-5/4}\,ds, \end{aligned} \tag{4.111}$$

para todo $t \geq t_2$, onde $C \in \mathbb{R}^+$ depende apenas de $\tilde{t}_1, K_1, K_2, K_3$ e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Estimando os termos do lado direito de (4.111) de forma completamente análoga às desigualdades (4.39), (4.40), (4.41) e (4.42) no Teorema 4.7, obtemos

$$\begin{aligned} e^{-(t-t_2)}M_0 &\leq C(t+1)^{-5/4}, \quad \forall\, t \geq t_2, \\ C\int_{t_2}^{t} e^{-(t-s)}(s+1)^{-5/4}\,ds &\leq C(t+1)^{-5/4}, \quad \forall\, t \geq t_2, \\ C\int_{t_2}^{t} e^{-(t-s)}(t-s)^{-1/2}(s+1)^{-5/4}\,ds &\leq C(t+1)^{-5/4}, \quad \forall\, t \geq t_2, \\ C\int_{t_2}^{t} e^{-(t-s)}(t-s)^{-3/4}(s+1)^{-5/4}\,ds &\leq (t+1)^{-5/4}, \quad \forall\, t \geq t_2, \end{aligned}$$

onde $C > 0$ depende somente de $\tilde{t}_1, K_1, K_2, K_3$ e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Portanto, utilizando as estimativas acima na desigualdade (4.111), concluímos que

$$\|\boldsymbol{w}(\cdot,t)\|_2 \leq C(t+1)^{-5/4}, \quad \forall\, t \geq t_2,$$

onde $C > 0$ depende somente de $\tilde{t}_1, K_1, K_2, K_3$ e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Isto finaliza a demonstração do teorema. $\square$

5 DECAIMENTO DE SOLUÇÕES: O ARGUMENTO RIGOROSO

Neste capítulo, obteremos, de maneira rigorosa, o decaimento na norma $\boldsymbol{L}^2$ de uma solução fraca do problema

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla P = \dfrac{1}{2}\operatorname{rot}\boldsymbol{w} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \boldsymbol{f}, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \Delta\boldsymbol{w} - \nabla(\operatorname{div}\boldsymbol{w}) + \boldsymbol{w} = \dfrac{1}{2}\operatorname{rot}\boldsymbol{u} + \boldsymbol{g}, \\ \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - \Delta\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \\ \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{b} = 0, \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \boldsymbol{w}(\cdot,0) = \boldsymbol{w}_0, \quad \boldsymbol{b}(\cdot,0) = \boldsymbol{b}_0, \end{cases} \tag{5.1}$$

complementado com condições de Dirichlet no infinito, isto é,

$$(\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{w}(\boldsymbol{x},t), \boldsymbol{b}(\boldsymbol{x},t)) \xrightarrow[|\boldsymbol{x}|\to\infty]{} (\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}), \quad \forall\, t > 0.$$

Para isto, seja T um número positivo arbitrário. A seguir, usaremos a seguinte notação:

$$\mathcal{Q}_T := \mathbb{R}^3 \times (0,T),$$

$$\boldsymbol{\mathcal{V}} := \left\{ \boldsymbol{v} \in \boldsymbol{C}_0^\infty(\mathbb{R}^3) \,/\, \operatorname{div}\boldsymbol{v} = 0 \text{ em } \mathbb{R}^3 \right\},$$

$$\boldsymbol{H} := \text{o fecho de } \boldsymbol{\mathcal{V}} \text{ em } \boldsymbol{L}^2(\mathbb{R}^3),$$

$$\boldsymbol{V} := \text{o fecho de } \boldsymbol{\mathcal{V}} \text{ em } \boldsymbol{H}^1(\mathbb{R}^3),$$

$$\boldsymbol{V}' = \text{o dual topológico de } \boldsymbol{V}.$$

Definição 5.1. *Sejam $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \boldsymbol{H}$, $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$, $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$ e $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$. Dizemos que a tripla $(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})$ é uma solução fraca do problema* (5.1) *se*

$$\boldsymbol{u},\boldsymbol{b} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H}),$$
$$\boldsymbol{w} \in L^2(0,T,\boldsymbol{H}^1(\mathbb{R}^3)) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)),$$

e $(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})$ satisfaz

$$\begin{cases} (\boldsymbol{u}_t,\boldsymbol{\varphi}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{\varphi}) + (\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}) = \dfrac{1}{2}(\operatorname{rot}\boldsymbol{w},\boldsymbol{\varphi}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{b},\boldsymbol{\varphi}) + \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle, \\ (\boldsymbol{w}_t,\boldsymbol{\phi}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{\phi}) + (\nabla\boldsymbol{w},\nabla\boldsymbol{\phi}) + (\operatorname{div}\boldsymbol{w},\operatorname{div}\boldsymbol{\phi}) + (\boldsymbol{w},\boldsymbol{\phi}) \\ \qquad\qquad = \dfrac{1}{2}(\operatorname{rot}\boldsymbol{u},\boldsymbol{\phi}) + \langle\boldsymbol{g},\boldsymbol{\phi}\rangle, \\ (\boldsymbol{b}_t,\boldsymbol{\eta}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{b},\boldsymbol{\eta}) + (\nabla\boldsymbol{b},\nabla\boldsymbol{\eta}) = ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u},\boldsymbol{\eta}), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \boldsymbol{w}(\cdot,0) = \boldsymbol{w}_0, \quad \boldsymbol{b}(\cdot,0) = \boldsymbol{b}_0 \quad \textit{fracamente em } \boldsymbol{L}^2(\mathbb{R}^3), \end{cases}$$

para todo $\boldsymbol{\varphi},\boldsymbol{\eta} \in \boldsymbol{V}$ e $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\mathbb{R}^3)$.

Na próxima seção, construiremos soluções aproximadas para o sistema (5.1) e apresentaremos alguns resultados auxiliares. Por fim, na Seção 5.2, provaremos o decaimento na norma $\boldsymbol{L}^2$ para as soluções aproximadas do problema (5.1).

5.1 Soluções aproximadas e resultados auxiliares

Seguindo as ideias introduzidas por L. Caffarelli, R. Kohn e L. Nirenberg em [51], definimos uma função regularizante ("retarded mollifier") da seguinte forma:

Definição 5.2. *Para $\boldsymbol{r} \in L^2(0,T;\boldsymbol{V})$ e $\delta > 0$, a função regularizante (retarded mollifier) de $\boldsymbol{r}$ é definida por*

$$\Psi_\delta(\boldsymbol{r})(\boldsymbol{x},t) := \frac{1}{\delta^4} \iint_{\mathbb{R}^4} \psi\left(\frac{\boldsymbol{y}}{\delta}, \frac{\tau}{\delta}\right) \widetilde{\boldsymbol{r}}(\boldsymbol{x}-\boldsymbol{y}, t-\tau)\, d\boldsymbol{y}\, d\tau,$$

onde $\psi(\boldsymbol{x},t) \in C^\infty(\mathbb{R}^4)$ satisfaz

$$\psi \geq 0, \qquad \iint_{\mathbb{R}^4} \psi(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt = 1,$$

$$\operatorname{supp}\psi \subseteq \left\{(\boldsymbol{x},t) \in \mathbb{R}^4 \,:\, |\boldsymbol{x}|^2 < t,\ 1 < t < 2\right\},$$

e $\widetilde{\boldsymbol{r}} : \mathbb{R}^4 \to \mathbb{R}^3$ é a extensão por zero de $\boldsymbol{r}$, ou seja,

$$\widetilde{\boldsymbol{r}}(\boldsymbol{x},t) := \begin{cases} \boldsymbol{r}(\boldsymbol{x},t), & \text{se} \quad (\boldsymbol{x},t) \in \mathbb{R}^3 \times (0,T), \\ \boldsymbol{0}, & \quad\text{caso contrário}. \end{cases}$$

Observação 5.3. *Note que $\Psi_\delta(\boldsymbol{r})$ é uma função suave, isto é, $\Psi_\delta(\boldsymbol{r}) \in \boldsymbol{C}^\infty(\mathbb{R}^3 \times [0,T];\mathbb{R}^3)$, cujo valor no tempo t depende somente dos valores de $\boldsymbol{r}$ no tempo $\tau \in (t-2\delta, t-\delta)$.*

Observação 5.4. *Observe que $\Psi_\delta(\boldsymbol{r}) = \widetilde{\boldsymbol{r}} * \Phi_\delta$, onde $*$ denota o operador de convolução e $\Phi_\delta : \mathbb{R}^4 \to \mathbb{R}$ é definida por $\Phi_\delta(\boldsymbol{x},t) := \delta^{-4}\psi(\delta^{-1}\boldsymbol{x}, \delta^{-1}t)$. Em particular, pela desigualdade de Young para convolução, a saber*

$$\|f * g\|_q \leq \|f\|_p \|g\|_r, \quad \text{onde} \quad 1 \leq p,q,r \leq \infty \quad \text{e} \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1,$$

obtemos, para $\boldsymbol{r} \in \boldsymbol{X} = L^q\left(0,T;\boldsymbol{L}^p(\mathbb{R}^3)\right), 1 \leq p,q \leq \infty$, que

$$\|\Psi_\delta(\boldsymbol{r})\|_{\boldsymbol{X}} \leq C\|\boldsymbol{r}\|_{\boldsymbol{X}}. \tag{5.2}$$

A constante $C \in \mathbb{R}^+$ depende somente da norma de $\boldsymbol{r}$ no espaço produto.

Agora, seguindo os argumentos de L. Caffarelli *et al.* [51] e G. Łukaszewicz [18], definimos, para $N \in \mathbb{N}^*$ e para algum $T > 0$ fixo, soluções aproximadas $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$ para o problema (5.1) como as soluções do sistema

$$\begin{cases} \boldsymbol{u}_t^N + (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N - \Delta\boldsymbol{u}^N + \nabla P^N = \dfrac{1}{2}\operatorname{rot}\boldsymbol{w}^N + (\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\boldsymbol{b}^N + \boldsymbol{f} \ \text{ em } \mathcal{Q}_T, \\ \boldsymbol{w}_t^N + (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{w}^N - \Delta\boldsymbol{w}^N - \nabla(\operatorname{div}\boldsymbol{w}^N) + \boldsymbol{w}^N = \dfrac{1}{2}\operatorname{rot}\boldsymbol{u}^N + \boldsymbol{g} \ \text{ em } \mathcal{Q}_T, \\ \boldsymbol{b}_t^N + (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{b}^N - \Delta\boldsymbol{b}^N = (\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\boldsymbol{u}^N \ \text{ em } \mathcal{Q}_T, \\ \operatorname{div}\boldsymbol{u}^N = \operatorname{div}\boldsymbol{b}^N = 0 \ \text{ em } \mathcal{Q}_T, \\ \boldsymbol{u}^N(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{w}^N(\boldsymbol{x},0) = \boldsymbol{w}_0(\boldsymbol{x}), \quad \boldsymbol{b}^N(\boldsymbol{x},0) = \boldsymbol{b}_0(\boldsymbol{x}) \ \text{ em } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} P^N(\boldsymbol{x},t)\, d\boldsymbol{x} = 0 \ \text{ q.s. em } [0,T]. \end{cases} \tag{5.3}$$

No sistema de equações acima, $\delta = T/N$. A seguir, apresentaremos alguns resultados similares aos lemas e teoremas estabelecidos em [18, 51, 52].

Lema 5.5. *Suponha que P é uma distribuição, $\widetilde{\boldsymbol{F}} \in L^2(0,T;\boldsymbol{V}')$, $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ e*

$$\boldsymbol{u}_t - \Delta \boldsymbol{u} + \nabla P = \widetilde{\boldsymbol{F}}, \tag{5.4}$$

no sentido das distribuições sobre Q_T. Considere $\widetilde{\boldsymbol{G}} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$, $\boldsymbol{w} \in L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3))$ e

$$\boldsymbol{w}_t - \Delta \boldsymbol{w} - \nabla(\operatorname{div} \boldsymbol{w}) = \widetilde{\boldsymbol{G}}. \tag{5.5}$$

Além disso, assuma $\widetilde{\boldsymbol{D}} \in L^2(0,T;\boldsymbol{V}'), \boldsymbol{b} \in L^2(0,T;\boldsymbol{V})$ e

$$\boldsymbol{b}_t - \Delta \boldsymbol{b} = \widetilde{\boldsymbol{D}}, \tag{5.6}$$

Então,

$$\boldsymbol{u}_t \in L^2(0,T;\boldsymbol{V}'), \quad \boldsymbol{w}_t \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3)), \qquad \boldsymbol{b}_t \in L^2(0,T;\boldsymbol{V}'), \tag{5.7}$$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(\cdot,t)\|_2^2 = (\boldsymbol{u}_t, \boldsymbol{u}), \quad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}(\cdot,t)\|_2^2 = (\boldsymbol{w}_t, \boldsymbol{w}) \qquad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{b}(\cdot,t)\|_2^2 = (\boldsymbol{b}_t, \boldsymbol{b}), \tag{5.8}$$

no sentido das distribuições sobre $(0,T)$, e

$$\boldsymbol{u}, \boldsymbol{b} \in C([0,T];\boldsymbol{H}), \quad \boldsymbol{w} \in C([0,T];\boldsymbol{L}^2(\mathbb{R}^3)), \tag{5.9}$$

após uma modificação em um conjunto de medida nula. As soluções de (5.4) e (5.6) são únicas no espaço $L^2(0,T;\boldsymbol{V})$ para dados iniciais $\boldsymbol{u}_0 \in \boldsymbol{H}$ e $\boldsymbol{b}_0 \in \boldsymbol{H}$, respctivamente. Similarmente, soluções de (5.5) são únicas no espaço $L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3))$ para o dado inicial $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$.

Demonstração. Multiplicando a equação (5.4) por $\boldsymbol{v} \in \boldsymbol{V}$ em $\boldsymbol{L}^2(\mathbb{R}^3)$, obtemos

$$\frac{d}{dt}(\boldsymbol{u}, \boldsymbol{v}) + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \langle \widetilde{\boldsymbol{F}}, \boldsymbol{v} \rangle, \tag{5.10}$$

pois $\operatorname{div} \boldsymbol{v} = 0$. Como

$$V \ni \boldsymbol{v} \mapsto (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$$

é uma aplicação linear e contínua sobre $\boldsymbol{V}$, existe um elemento $A\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}')$ tal que

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}).$$

Assim, podemos reescrever a identidade (5.10) do seguinte modo:

$$\frac{d}{dt}\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \widetilde{\boldsymbol{F}} - A\boldsymbol{u}, \boldsymbol{v} \rangle \tag{5.11}$$

Uma vez que $\widetilde{\boldsymbol{F}}, A\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}')$, concluímos que

$$\boldsymbol{u}_t \in L^2(0,T;\boldsymbol{V}'). \tag{5.12}$$

Dessa forma, de (5.12) e usando o Lema 2.23 para $\boldsymbol{V}, \boldsymbol{V}'$ e $\boldsymbol{H}$, obtemos (5.8) e (5.9) para $\boldsymbol{u}$. Para provar a unicidade de $\boldsymbol{u}$, assuma que $(\boldsymbol{u}^1, P^1)$ e $(\boldsymbol{u}^2, P^2)$ são duas soluções de (5.4) com a mesma condição inicial $\boldsymbol{u}_0$ e a mesma força externa $\widetilde{\boldsymbol{F}}$, ou seja,

$$\boldsymbol{u}^1_t - \Delta \boldsymbol{u}^1 + \nabla P^1 = \widetilde{\boldsymbol{F}}, \quad \boldsymbol{u}^1(\cdot, 0) = \boldsymbol{u}_0(\cdot),$$
$$\boldsymbol{u}^2_t - \Delta \boldsymbol{u}^2 + \nabla P^2 = \widetilde{\boldsymbol{F}}, \quad \boldsymbol{u}^2(\cdot, 0) = \boldsymbol{u}_0(\cdot).$$

Defina $\boldsymbol{u} := \boldsymbol{u}^1 - \boldsymbol{u}^2$ e $P := P^1 - P^2$. Então $\boldsymbol{u}$ pertence ao mesmo espaço de $\boldsymbol{u}^1$ e $\boldsymbol{u}^2$ e satisfaz

$$\boldsymbol{u}_t - \Delta \boldsymbol{u} + \nabla P = \boldsymbol{0}, \quad \boldsymbol{u}(\cdot, 0) = \boldsymbol{0}. \tag{5.13}$$

Multiplicando (5.13) po $\boldsymbol{u}$, da primeira identidade de (5.8), obtemos

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \|\nabla \boldsymbol{u}\|_2^2 = 0.$$

Agora, integrando a desigualdade acima de 0 até t, com $0 < t \leq T$, obtemos

$$\|\boldsymbol{u}(\cdot, t)\|_2^2 \leq \|\boldsymbol{u}(\cdot, 0)\|_2^2 = 0. \tag{5.14}$$

Desta forma, $\boldsymbol{u}^1 \equiv \boldsymbol{u}^2$. De forma completamente análoga, é possível mostrar o resultado para $\boldsymbol{w}$ e $\boldsymbol{b}$. Omitiremos os detalhes aqui. $\square$

Teorema 5.6. *Sejam $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \boldsymbol{H}$, $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$, $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$ e $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$. Existe uma solução fraca $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ do sistema (5.1) tal que*

$$\boldsymbol{u}, \boldsymbol{b} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H}) \tag{5.15}$$

e

$$\boldsymbol{w} \in L^2(0,T,\boldsymbol{H}^1(\mathbb{R}^3)) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)). \tag{5.16}$$

Demonstração. Para mostrar a existência de uma solução fraca para o problema (5.1) usaremos o método de Faedo-Galerkin. Considere $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset \boldsymbol{V}$ uma base de autofunções do operador de Stokes $A := -\mathbb{P}_H \Delta$, onde $\mathbb{P}_H$ é a projeção ortogonal de $\boldsymbol{L}^2(\mathbb{R}^3)$ em $\boldsymbol{H}$ (denominada de projeção de Helmholtz) e $(\boldsymbol{\phi}_k)_{k\in\mathbb{N}}$ é uma base de autofunções do operador de Lamé $L : -\Delta - \nabla\mathrm{div}$ (veja [19]). Para cada $m \in \mathbb{N}$, definimos uma solução aproximada do problema (5.1) do seguinte modo:

$$\begin{aligned}
\boldsymbol{u}_m(\boldsymbol{x}, t) &= \sum_{k=1}^m c_{km}(t)\boldsymbol{\varphi}_k(\boldsymbol{x}), \quad c_{km}(t) \in \mathbb{R},\\
\boldsymbol{w}_m(\boldsymbol{x}, t) &= \sum_{k=1}^m d_{km}(t)\boldsymbol{\phi}_k(\boldsymbol{x}), \quad d_{km}(t) \in \mathbb{R},\\
\boldsymbol{b}_m(\boldsymbol{x}, t) &= \sum_{k=1}^m e_{km}(t)\boldsymbol{\varphi}_k(\boldsymbol{x}), \quad e_{km}(t) \in \mathbb{R},
\end{aligned}$$

onde a tripla $(\boldsymbol{u}_m, \boldsymbol{w}_m, \boldsymbol{b}_m)$ é a solução do problema aproximado

$$\begin{cases} (\partial_t \boldsymbol{u}_m, \boldsymbol{\varphi}_k) + ((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{\varphi}_k) - (\Delta \boldsymbol{u}_m, \boldsymbol{\varphi}_k) = \dfrac{1}{2}(\text{rot}\, \boldsymbol{w}_m, \boldsymbol{\varphi}_k) + ((\boldsymbol{b}_m \cdot \nabla)\boldsymbol{b}_m, \boldsymbol{\varphi}_k) \\ \qquad + (\boldsymbol{f}, \boldsymbol{\varphi}_k), \quad k = 1, \cdots, m, \\ (\partial_t \boldsymbol{w}_m, \boldsymbol{\phi}_k) + ((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{w}_m, \boldsymbol{\phi}_k) - (\Delta \boldsymbol{w}_m, \boldsymbol{\phi}_k) - (\nabla \text{div}\, \boldsymbol{w}_m, \boldsymbol{\phi}_k) + (\boldsymbol{w}_m, \boldsymbol{\phi}_k) \\ \qquad = \dfrac{1}{2}(\text{rot}\, \boldsymbol{u}_m, \boldsymbol{\phi}_k) + (\boldsymbol{g}, \boldsymbol{\phi}_k), \quad k = 1, \cdots, m, \\ (\partial_t \boldsymbol{b}_m, \boldsymbol{\varphi}_k) + ((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{b}_m, \boldsymbol{\varphi}_k) - (\Delta \boldsymbol{b}_m, \boldsymbol{\varphi}_k) = ((\boldsymbol{b}_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{\varphi}_k), \quad k = 1, \cdots, m, \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m}, \quad \boldsymbol{w}_m(0) = \boldsymbol{w}_{0m}, \quad \boldsymbol{b}_m(0) = \boldsymbol{b}_{0m}. \end{cases} \tag{5.17}$$

Não é difícil provar que o sistema de equações diferenciais ordinárias não linear (5.17) tem uma única solução $(\boldsymbol{u}_m, \boldsymbol{w}_m, \boldsymbol{b}_m)$ definida no intervalo $[0, T_m)$, com $T_m \in (0, T)$. Na verdade, é possível estender essa solução para $[0, T]$ (para mais detalhes, veja o artigo de Boldrini *et. al.* [19]). Agora, multiplicando $(5.17)_1$ por c_{km}, $(5.17)_2$ por d_{km} e $(5.17)_3$ por e_{km}, somando as equações correspondentes de $k = 1, \cdots, m$ e usando as desigualdades de Cauchy-Schwarz e Young, obtemos

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|_2^2 + \|\nabla \boldsymbol{u}_m\|_2^2 \leq \frac{1}{4}\|\boldsymbol{w}_m\|_2^2 + \frac{1}{4}\|\nabla \boldsymbol{u}_m\|_2^2 + \|\boldsymbol{f}(t)\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + ((\boldsymbol{b}_m \cdot \nabla)\boldsymbol{b}_m, \boldsymbol{u}_m),$$
$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_m\|_2^2 + \|\nabla \boldsymbol{w}_m\|_2^2 + \|\text{div}\, \boldsymbol{w}_m\|_2^2 + \|\boldsymbol{w}_m\|_2^2 \leq \frac{1}{4}\|\boldsymbol{w}_m\|_2^2 + \frac{1}{4}\|\nabla \boldsymbol{u}_m\|_2^2 + \|\boldsymbol{g}(t)\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1},$$
$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{b}_m\|_2^2 + \|\nabla \boldsymbol{b}_m\|_2^2 = ((\boldsymbol{b}_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{b}_m),$$

ou seja,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{u}_m\|_2^2 + \|\boldsymbol{w}_m\|_2^2 + \|\boldsymbol{b}_m\|_2^2\right) + \frac{1}{2}\|\nabla \boldsymbol{u}_m\|_2^2 + \|\nabla \boldsymbol{w}_m\|_2^2 + \|\nabla \boldsymbol{b}_m\|_2^2 \\ &\qquad + \|\text{div}\, \boldsymbol{w}_m\|_2^2 + \frac{1}{2}\|\boldsymbol{w}_m\|_2^2 \leq \left(\|\boldsymbol{f}(t)\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}(t)\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}\right). \end{aligned} \tag{5.18}$$

Multiplicando por 2 , manipulando os termos e integrando a desigualdade (5.18) com respeito ao tempo no intervalo $[0, s]$, com $s > 0$, obtemos

$$\begin{aligned} \|\boldsymbol{u}_m(s)\|_2^2 + \|\boldsymbol{w}_m(s)\|_2^2 + \|\boldsymbol{b}_m(s)\|_2^2 &\leq \|\boldsymbol{u}_{0m}\|_2^2 + \|\boldsymbol{w}_{0m}\|_2^2 + \|\boldsymbol{b}_{0m}\|_2^2 \\ &\quad + 2\int_0^s \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}\right) dt \\ &\leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 \\ &\quad + 2\int_0^s \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}\right) dt. \end{aligned}$$

Segue, da desigualdade de de Young e das hipóteses sobre $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \boldsymbol{f}$ e $\boldsymbol{g}$, que

$$\|\boldsymbol{u}_m(s)\|_2^2 + \|\boldsymbol{w}_m(s)\|_2^2 + \|\boldsymbol{b}_m(s)\|_2^2 \leq C + 2\int_0^s \left(\|\boldsymbol{u}_m\|_{\boldsymbol{V}}^2 + \|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}^2\right) dt.$$

Aplicando a desigualdade de Gronwall, obtemos

$$\sup_{s \in [0,T]} \left(\|\boldsymbol{u}_m(s)\|_2^2 + \|\boldsymbol{w}_m(s)\|_2^2 + \|\boldsymbol{b}_m(s)\|_2^2\right) \leq C. \tag{5.19}$$

Agora, novamente multiplicando a estimativa (5.18) por 2, manipulando os termos adequadamente e integrando a desigualdade resultante de 0 a T, obtemos

$$\begin{aligned}
&\|\boldsymbol{u}_m(T)\|_2^2 + \|\boldsymbol{w}_m(T)\|_2^2 + \|\boldsymbol{b}_m(T)\|_2^2 + \int_0^T \left(\|\nabla \boldsymbol{u}_m(t)\|_2^2 + \|\nabla \boldsymbol{w}_m(t)\|_2^2 + \|\nabla \boldsymbol{b}_m(t)\|_2^2\right) dt \\
&\quad\leq \|\boldsymbol{u}_{0m}\|_2^2 + \|\boldsymbol{w}_{0m}\|_2^2 + \|\boldsymbol{b}_{0m}\|_2^2 + 2\int_0^T (\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1})\, dt \\
&\quad\leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \|\boldsymbol{b}_0\|_2^2 + 2\int_0^T (\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1})\, dt. \qquad (5.20)
\end{aligned}$$

A seguir, provaremos a existência de $\boldsymbol{u}, \boldsymbol{b}$ satisfazendo (5.15). Para isso, observe que, da desigualdade (5.19), temos que

$$\{\boldsymbol{u}_m\}, \{\boldsymbol{b}_m\} \text{ são limitadas em } L^\infty(0,T;\boldsymbol{H}), \qquad (5.21)$$

e que

$$\{\boldsymbol{w}_m\} \text{ é limitada em } L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)). \qquad (5.22)$$

Da estimativa (5.20), segue que

$$\{\boldsymbol{u}_m\}, \{\boldsymbol{b}_m\} \text{ são limitadas em } L^2(0,T;\boldsymbol{V}), \qquad (5.23)$$

e que

$$\{\boldsymbol{w}_m\} \text{ é limitada em } L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)). \qquad (5.24)$$

As afirmações (5.21) e (5.22) garantem a existência de $\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b}$, e uma subsequência $m' \to \infty$, tal que

$$\boldsymbol{u}_{m'} \overset{*}{\rightharpoonup} \boldsymbol{u}_* \text{ em } L^\infty(0,T;\boldsymbol{H}), \qquad (5.25)$$

$$\boldsymbol{w}_{m'} \overset{*}{\rightharpoonup} \boldsymbol{w}_* \text{ em } L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)) \qquad (5.26)$$

e

$$\boldsymbol{b}_{m'} \overset{*}{\rightharpoonup} \boldsymbol{b}_* \text{ em } L^\infty(0,T;\boldsymbol{H}). \qquad (5.27)$$

Por outro lado, das afirmações (5.23) e (5.24), temos que

$$\{\boldsymbol{u}_{m'}\}, \{\boldsymbol{b}_{m'}\} \text{ são limitadas em } L^2(0,T;\boldsymbol{V}) \qquad (5.28)$$

e

$$\{\boldsymbol{w}_{m'}\} \text{ é limitada em } L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)). \qquad (5.29)$$

Logo, existe $\boldsymbol{u}_*, \boldsymbol{b}_* \in L^2(0,T;\boldsymbol{V})$, $\boldsymbol{w}_* \in L^2(0,T;\boldsymbol{H}_0^1(\mathbb{R}^3))$, e uma subsequência, que por simplicidade denotaremos novamente por $\boldsymbol{u}_{m'}, \boldsymbol{w}_{m'}, \boldsymbol{b}_{m'}$, tal que

$$\boldsymbol{u}_{m'} \rightharpoonup \boldsymbol{u} \text{ em } L^2(0,T;\boldsymbol{V}), \qquad (5.30)$$

$$\boldsymbol{w}_{m'} \overset{*}{\rightharpoonup} \boldsymbol{w}_* \text{ em } L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)) \qquad (5.31)$$

e

$$\boldsymbol{b}_{m'} \rightharpoonup \boldsymbol{b} \text{ em } L^2(0,T;\boldsymbol{V}). \tag{5.32}$$

Em particular, para cada $\boldsymbol{\nu} \in L^2(0,T;\boldsymbol{V})$, temos que

$$\int_0^T (\boldsymbol{u}_{m'}(t), \boldsymbol{\nu}(t))\,dt \to \int_0^T (\boldsymbol{u}_*(t), \boldsymbol{\nu}(t))\,dt, \tag{5.33}$$

$$\int_0^T (\boldsymbol{b}_{m'}(t), \boldsymbol{\nu}(t))\,dt \to \int_0^T (\boldsymbol{b}_*(t), \boldsymbol{\nu}(t))\,dt. \tag{5.34}$$

E, para cada $\widetilde{\boldsymbol{\nu}} \in L^2(0,T;\boldsymbol{H}_0^1(\mathbb{R}^3))$, temos

$$\int_0^T (\boldsymbol{w}_{m'}(t), \widetilde{\boldsymbol{\nu}}(t))\,dt \to \int_0^T (\boldsymbol{w}_*(t), \widetilde{\boldsymbol{\nu}}(t))\,dt. \tag{5.35}$$

Então, de (5.25)–(5.27), temos

$$\int_0^T (\boldsymbol{u}(t) - \boldsymbol{u}_*(t), \boldsymbol{\nu}(t))\,dt = 0, \quad \forall\, \boldsymbol{\nu} \in L^2(0,T;\boldsymbol{V}), \tag{5.36}$$

$$\int_0^T (\boldsymbol{w}(t) - \boldsymbol{w}_*(t), \widetilde{\boldsymbol{\nu}}(t))\,dt = 0, \quad \forall\, \widetilde{\boldsymbol{\nu}} \in L^2(0,T;\boldsymbol{H}_0^1(\mathbb{R}^3)) \tag{5.37}$$

e

$$\int_0^T (\boldsymbol{b}(t) - \boldsymbol{b}_*(t), \boldsymbol{\nu}(t))\,dt = 0, \quad \forall\, \boldsymbol{\nu} \in L^2(0,T;\boldsymbol{V}). \tag{5.38}$$

Portanto,

$$\boldsymbol{u} = \boldsymbol{u}_*, \quad \boldsymbol{b} = \boldsymbol{b}_* \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$$

e

$$\boldsymbol{w} = \boldsymbol{w}_* \in L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)).$$

Assim, fica demonstrado a existência de $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$. Finalmente, mostraremos a existência da pressão P. Considere $\boldsymbol{u}, \boldsymbol{w}$ e $\boldsymbol{b}$ satisfazendo $\boldsymbol{u}, \boldsymbol{b} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$, $\boldsymbol{w} \in L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3))$ e satisfazendo, também, para cada $\boldsymbol{v} \in \boldsymbol{V}$,

$$\frac{d}{dt}(\boldsymbol{u}, \boldsymbol{v}) - (\Delta \boldsymbol{u}, \boldsymbol{v}) - \frac{1}{2}(\operatorname{rot} \boldsymbol{w}, \boldsymbol{v}) - \langle \widetilde{\boldsymbol{f}}, \boldsymbol{v} \rangle = 0, \tag{5.39}$$

onde $\widetilde{\boldsymbol{f}} = \boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{b}$. Sejam

$$\boldsymbol{U}(t) := \int_0^t \boldsymbol{u}(s)\,ds, \quad \boldsymbol{W}(t) := \int_0^t \boldsymbol{w}(s)\,ds, \quad \boldsymbol{F}(t) := \int_0^t \widetilde{\boldsymbol{f}}(s)\,ds. \tag{5.40}$$

Então, $\boldsymbol{U} \in C([0,T];\boldsymbol{V})$, $\boldsymbol{W} \in C([0,T];\boldsymbol{H}^1(\mathbb{R}^3))$ e $\boldsymbol{F} \in C([0,T],\boldsymbol{V}')$. Assim, integrando (5.39) com respeito ao tempo t, obtemos para $\boldsymbol{v} \in \boldsymbol{V}$, que

$$\left\langle \boldsymbol{u}(t) - \boldsymbol{u}_0 - \Delta \boldsymbol{U}(t) - \frac{1}{2}\operatorname{rot} \boldsymbol{W}(t) - \boldsymbol{F}(t), \boldsymbol{v} \right\rangle = 0, \quad \forall\, t \in [0,T]. \tag{5.41}$$

Segue, das Proposições 1.1 e 1.2 no Capítulo I de [42], que existe uma função $\widetilde{P}(t) \in L^2(\mathbb{R}^3)$ tal que

$$\boldsymbol{u}(t) - \boldsymbol{u}_0 - \Delta \boldsymbol{U}(t) - \frac{1}{2}\operatorname{rot} \boldsymbol{W}(t) - \boldsymbol{F}(t) = -\nabla \widetilde{P}(t). \tag{5.42}$$

Desta forma, $\widetilde{P} \in C([0,T]; L^2(\mathbb{R}^3))$. Logo, derivando a equação (5.42) com respeito a t, no sentido das distribuições em Q_T e escolhendo

$$P = \frac{\partial \widetilde{P}}{\partial t}, \tag{5.43}$$

obtemos

$$\boldsymbol{u}_t - \Delta \boldsymbol{u} - \frac{1}{2}\operatorname{rot} \boldsymbol{w} - \boldsymbol{f} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} = -\nabla P \quad \text{em} \quad Q_T.$$

Isto finaliza a prova da existência da pressão total P. □

Lema 5.7. *Suponha que $\boldsymbol{v}(\cdot,t), \boldsymbol{w}(\cdot,t), \overline{\boldsymbol{v}}(\cdot,t), \boldsymbol{b}(\cdot,t) \in \boldsymbol{C}^\infty(\mathbb{R}^3)$, $\operatorname{div} \boldsymbol{v} = \operatorname{div} \overline{\boldsymbol{v}} = \operatorname{div} \boldsymbol{b} = 0$, $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$ e $\boldsymbol{u}_0 \in \boldsymbol{H}$. Então existe uma única função $\boldsymbol{u}$,*

$$\boldsymbol{u} \in C([0,T];\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}), \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot), \tag{5.44}$$

e uma distribuição P sobre Q_T tal que a equação

$$\boldsymbol{u}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla P - \frac{1}{2}\operatorname{rot} \boldsymbol{w} - (\overline{\boldsymbol{v}} \cdot \nabla)\boldsymbol{b} = \boldsymbol{f} \tag{5.45}$$

é satisfeita no sentido das distribuições sobre Q_T.

Demonstração. A existência de $\boldsymbol{u}$ e P foi demonstrada no Teorema 5.6 e a afirmação (5.44) segue diretamente repetindo-se os mesmos argumentos do Lema 5.5 e pelo Lema 2.23. □

Corolário 5.8. *Assumindo as mesmas hipóteses do Lema 5.7, segue que*

$$\boldsymbol{u}_t,\ (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}').$$

Lema 5.9. *Suponha $\boldsymbol{u}(\cdot,t) \in \boldsymbol{C}^\infty(\mathbb{R}^3)$, $\operatorname{div} \boldsymbol{u} = 0$, $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$ e $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$. Então o problema de Cauchy*

$$\begin{cases} \boldsymbol{w}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} - \Delta \boldsymbol{w} - \nabla(\operatorname{div} \boldsymbol{w}) + \boldsymbol{w} - \dfrac{1}{2}\operatorname{rot} \boldsymbol{u} = \boldsymbol{g} \quad \text{em} \quad Q_T, \\ \boldsymbol{w}(\cdot,0) = \boldsymbol{w}_0(\cdot) \quad \text{em} \quad \mathbb{R}^3, \end{cases}$$

tem uma única solução $\boldsymbol{w} \in C([0,T];\boldsymbol{L}^2(\mathbb{R}^3)) \cap L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$.

Demonstração. Basta usar o Lema 2.23 e repetir os mesmos argumentos do Lema 5.7 □

Corolário 5.10. *Sob as mesmas hipóteses do Lema 5.9, temos que*

$$\boldsymbol{w}_t,\ (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3)).$$

Lema 5.11. *Suponha que* $\boldsymbol{b}_0 \in \boldsymbol{H}$ *e* $\boldsymbol{u}(\cdot,t), \overline{\boldsymbol{v}}(\cdot,t) \in \boldsymbol{C}^\infty(\mathbb{R}^3)$, *com* $\operatorname{div} \boldsymbol{u} = \operatorname{div} \overline{\boldsymbol{v}} = 0$. *Então o PVI*

$$\begin{cases} \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - \Delta\boldsymbol{b} = (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u} \quad \text{em} \quad \mathcal{Q}_T, \\ \boldsymbol{b}(\cdot,0) = \boldsymbol{b}_0(\cdot) \quad \text{em} \quad \mathbb{R}^3, \end{cases} \tag{5.46}$$

possui uma única solução $\boldsymbol{b} \in C([0,T];\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$.

Demonstração. Segue do Lema 2.23 e dos argumentos usados no Lema 5.7. □

Corolário 5.12. *Assumindo as mesmas hipóteses do Lema 5.11, segue que*

$$\boldsymbol{b}_t,\ (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} \in L^2(0,T;\boldsymbol{V}').$$

Lema 5.13. *Para todo* $\boldsymbol{r} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$, *temos que*

$$\operatorname{div} \Psi_\delta(\boldsymbol{r}) = 0 \tag{5.47}$$

e

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |\Psi_\delta(\boldsymbol{r})|^2(\boldsymbol{x},t)\,d\boldsymbol{x} \le C \operatorname*{sup\,ess}_{t\in(0,T)} \int_{\mathbb{R}_3} |\boldsymbol{r}|^2\,d\boldsymbol{x}, \tag{5.48}$$

onde $C \in \mathbb{R}^+$ *é uma constante universal.*

Demonstração. Temos que

$$\operatorname{div} \widetilde{\boldsymbol{r}} = \begin{cases} \operatorname{div} \boldsymbol{r}, & \text{se} \quad (\boldsymbol{x},t) \in \mathbb{R}^3 \times (0,T), \\ 0, & \text{caso contrário.} \end{cases}$$

Como $\operatorname{div} \boldsymbol{r}$=0, segue que $\operatorname{div} \widetilde{\boldsymbol{r}} \equiv 0$. Isto prova a identidade (5.47). A desigualdade (5.48) segue diretamente da Observação 5.4. □

Lema 5.14. *Suponha* $\boldsymbol{u}_0 \in \boldsymbol{H}$ *e* $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$ *com* $\operatorname{div} \boldsymbol{f} = 0$. *Então, no contexto do Lema 5.7, a pressão total* P *satisfaz*

$$\Delta P = -\operatorname{div}\left[(\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - \operatorname{div}(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b}\right] = -\sum_{j,k=1}^{3} \frac{\partial^2}{\partial x_j \partial x_k}(v_j u_k - \overline{v}_j b_k),$$

e P *está limitada em* $L^{5/3}(\mathcal{Q}_T)$, *para todo* $T > 0$.

Demonstração. Aplicando o operador divergente na equação

$$\boldsymbol{u}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla P - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b} - \frac{1}{2}\operatorname{rot}\boldsymbol{w} = \boldsymbol{f},$$

obtemos

$$\Delta P = -\operatorname{div}\left[(\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - \operatorname{div}(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b}\right] = -\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(v_i u_j - \overline{v}_i b_j).$$

Seja $\sigma(\boldsymbol{x},t) := \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(v_i u_j - \overline{v}_i b_j)(\boldsymbol{x},t)$. Segue, então, que a função P satisfaz $-\Delta P = \sigma$ (uma equação do tipo Poisson). Logo,

$$P(\boldsymbol{x},t) = \int_{\mathbb{R}^3} \phi(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{y},t)\,d\boldsymbol{y},$$

onde ϕ é a solução fundamental da equação de Laplace. É bem conhecido que, para $n \geq 3$, a solução fundamental é dada por

$$\phi(\boldsymbol{x}) = \frac{|\boldsymbol{x}|^{2-n}}{n(n-2)\alpha(n)}, \quad \boldsymbol{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\},$$

onde $\alpha(n)$ é o volume da bola unitária em $\mathbb{R}^n$ (veja [43]). Assim,

$$P(\boldsymbol{x}, t) = \frac{1}{4\pi} \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \frac{\partial^2}{\partial y_i \partial y_j} (v_i u_j - \overline{v}_i b_j)(\boldsymbol{y}, t)\, d\boldsymbol{y}.$$

Logo, pelo teorema de Calderón-Zygmund (veja o Lema 2.38), obtemos, para $1 < q < \infty$, que

$$\int_{\mathbb{R}^3} |P|^q \, d\boldsymbol{x} \leq C_q \int_{\mathbb{R}^3} [(|\boldsymbol{v}||\boldsymbol{u}|)^q + (|\overline{\boldsymbol{v}}||\boldsymbol{b}|)^q] \, d\boldsymbol{x}.$$

Em particular, para $\boldsymbol{u}$ e $\boldsymbol{b}$ como nos Lemas 5.7 e 5.11, temos

$$\int_0^T \int_{\mathbb{R}^3} |P|^{5/3} \, d\boldsymbol{x}\, dt \leq C \int_0^T \int_{\mathbb{R}^3} \left(|\boldsymbol{u}|^{10/3} + |\boldsymbol{b}|^{10/3} \right) d\boldsymbol{x}\, dt.$$

Usando desigualdades de Interpolação e Sobolev, obtemos

$$\int_{\mathbb{R}^3} |\boldsymbol{u}|^{10/3} \, d\boldsymbol{x} \leq C \left(\int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2 \, d\boldsymbol{x} \right) \left(\int_{\mathbb{R}^3} |\boldsymbol{u}|^2 \, d\boldsymbol{x} \right)^{2/3}$$

e

$$\int_{\mathbb{R}^3} |\boldsymbol{b}|^{10/3} \, d\boldsymbol{x} \leq C \left(\int_{\mathbb{R}^3} |\nabla \boldsymbol{b}|^2 \, d\boldsymbol{x} \right) \left(\int_{\mathbb{R}^3} |\boldsymbol{b}|^2 \, d\boldsymbol{x} \right)^{2/3}.$$

Portanto,

$$\int_0^T \int_{\mathbb{R}^3} |P|^{5/3} \, d\boldsymbol{x}\, dt < \infty,$$

isto é, $P \in L^{5/3}(\mathcal{Q}_T)$, para todo $T > 0$, concluindo, assim, a demonstração do lema. □

Lema 5.15. *Suponha que* $\boldsymbol{f}, \boldsymbol{g} \in L^2(0, T; \boldsymbol{H}^{-1}(\mathbb{R}^3))$, $\operatorname{div} \boldsymbol{f} = 0$, $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \boldsymbol{H}$ *e* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$. *Seja* $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N), N \in \mathbb{N}^*$, *a única solução do problema* (5.3). *Então*

$$\begin{aligned}
&\left(\boldsymbol{u}^N, \boldsymbol{w}^N, \boldsymbol{b}^N \right) \to (\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b}) \quad \textit{fortemente em } L^2(\mathcal{Q}_T), \\
&P^N \to P \quad \textit{fracamente em } L^{5/3}(\mathcal{Q}_T), \\
&\left(\Psi_\delta(\boldsymbol{u}^N), \Psi_\delta(\boldsymbol{b}^N) \right) \to (\boldsymbol{u}, \boldsymbol{b}) \quad \textit{fortemente em } L^2(\mathcal{Q}_T), \\
&(\boldsymbol{u}^N(0), \boldsymbol{w}^N(0), \boldsymbol{b}^N(0)) \xrightarrow[N \to \infty]{} (\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0),
\end{aligned}$$

e $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ *é uma solução fraca das equações magneto-micropolares* (5.1) *com forças externas* $\boldsymbol{f}$ *e* $\boldsymbol{g}$, *pressão total* P *e dados iniciais* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$.

Demonstração. A existência da solução $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$ do problema (5.3) segue-se aplicando o Teorema 5.6 indutivamente sobre cada intervalo de tempo $(k\delta, (k+1)\delta)$, com $0 \leq k \leq N-1$. Basta, portanto, mostrar as convergências acima. Repetindo os argumentos utilizados no Lema 4.8 e fazendo uso do Lema 5.13, obtemos a seguinte estimativa de energia:

$$\begin{aligned}\frac{d}{dt}\left(\|\boldsymbol{u}^N\|_2^2+\|\boldsymbol{w}^N\|_2^2+\|\boldsymbol{b}^N\|_2^2\right) &+ \frac{1}{2}\left(\|\nabla\boldsymbol{u}^N\|_2^2+\|\nabla\boldsymbol{w}^N\|_2^2+\|\nabla\boldsymbol{b}^N\|_2^2\right)\\ &\le 2(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}^N\|_{\boldsymbol{V}}+\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}^N\|_{\boldsymbol{H}^1}), \qquad (5.49)\end{aligned}$$

para $t\in(0,T)$. Integrando a desigualdade acima no intervalo de tempo $[0,s]$, com $s>0$, obtemos

$$\begin{aligned}\|\boldsymbol{u}^N(s)\|_2^2+\|\boldsymbol{w}^N(s)\|_2^2+\|\boldsymbol{b}^N(s)\|_2^2 &\le \|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{w}_0\|_2^2+\|\boldsymbol{b}_0\|_2^2\\ &\quad +2\int_0^s(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}^N\|_{\boldsymbol{V}}+\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}^N\|_{\boldsymbol{H}^1})\,dt\\ &\le \|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{w}_0\|_2^2+\|\boldsymbol{b}_0\|_2^2\\ &\quad +2\int_0^T(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}^N\|_{\boldsymbol{V}}+\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}^N\|_{\boldsymbol{H}^1})\,dt.\end{aligned}$$

Por outro lado, integrando a estimativa (5.49) de 0 até T, encontramos

$$\begin{aligned}\|\boldsymbol{u}^N(T)\|_2^2+\|\boldsymbol{w}^N(T)\|_2^2+\|\boldsymbol{b}^N(T)\|_2^2+\frac{1}{2}\int_0^T\left(\|\nabla\boldsymbol{u}^N(t)\|_2^2+\|\nabla\boldsymbol{w}^N(t)\|_2^2+\|\nabla\boldsymbol{b}^N(t)\|_2^2\right)dt\\ \le \|\boldsymbol{u}_0\|_2^2+\|\boldsymbol{w}_0\|_2^2+\|\boldsymbol{b}_0\|_2^2+2\int_0^T(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}^N\|_{\boldsymbol{V}}+\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}^N\|_{\boldsymbol{H}^1})\,dt.\end{aligned}$$

Desta forma, concluímos que

$$\{\boldsymbol{u}^N\},\{\boldsymbol{b}^N\} \text{ são limitadas em } L^\infty(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{V}) \qquad (5.50)$$

e

$$\{\boldsymbol{w}^N\} \text{ é limitada em } L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3))\cap L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)). \qquad (5.51)$$

A seguir, denotaremos por $\boldsymbol{V}_2$ o fecho de $\boldsymbol{\mathcal{V}}$ em $\boldsymbol{H}^2(\mathbb{R}^3)$ e por $\boldsymbol{V}_2'$ seu espaço dual. Multiplicando a equação $(5.3)_1$ por $\boldsymbol{\nu}\in\boldsymbol{\mathcal{V}}$, obtemos

$$\langle\boldsymbol{u}_t^N,\boldsymbol{\nu}\rangle+\langle(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N,\boldsymbol{\nu}\rangle-\langle\Delta\boldsymbol{u}^N,\boldsymbol{\nu}\rangle=\frac{1}{2}\langle\operatorname{rot}\boldsymbol{w}^N,\boldsymbol{\nu}\rangle+\langle(\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\boldsymbol{b}^N,\boldsymbol{\nu}\rangle+\langle\boldsymbol{f},\boldsymbol{\nu}\rangle. \qquad (5.52)$$

Usando a imersão $\boldsymbol{H}_0^1(\mathbb{R}^3)\subset\boldsymbol{L}^6(\mathbb{R}^3)$, obtemos, pelo Lema 2.30, que

$$\begin{aligned}\left|\langle(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N,\boldsymbol{\nu}\rangle\right| &= \left|\int_0^T\int_{\mathbb{R}^3}(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N\,\boldsymbol{\nu}\,d\boldsymbol{x}\,dt\right|\\ &= \left|-\int_0^T\int_{\mathbb{R}^3}(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{\nu}\,\boldsymbol{u}^N\,d\boldsymbol{x}\,dt\right|\\ &\le \int_0^T\|\Psi_\delta(\boldsymbol{u}^N)\|_3\,\|\boldsymbol{u}^N\|_2\,\|\nabla\boldsymbol{\nu}\|_6\,dt\\ &\le C\,\|\Psi_\delta(\boldsymbol{u}^N)\|_{L^4(0,T;\boldsymbol{L}^3)}\|\boldsymbol{u}^N\|_{L^4(0,T;\boldsymbol{L}^2)}\,\|\boldsymbol{\nu}\|_{L^2(0,T;\boldsymbol{V}_2)}\,. \qquad (5.53)\end{aligned}$$

Assim, pelas propriedades da função regularizante Ψ_δ, segue, de (5.50) e do Lema 2.30, que

$$\|(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\,\boldsymbol{u}^N\|_{L^2(0,T;\boldsymbol{V}_2')}\le C,$$

onde $C > 0$ é uma constante independente de N. Analogamente, temos que

$$\begin{gathered}
\|(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\,\boldsymbol{w}^N\|_{L^2(0,T;\boldsymbol{H}^{-2}(\mathbb{R}^3))} \leq C,\\
\|(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\,\boldsymbol{b}^N\|_{L^2(0,T;\boldsymbol{V}_2')} \leq C,\\
\|(\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\,\boldsymbol{u}^N\|_{L^2(0,T;\boldsymbol{V}_2')} \leq C,\\
\|(\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\,\boldsymbol{b}^N\|_{L^2(0,T;\boldsymbol{V}_2')} \leq C,
\end{gathered}$$

onde $C \in \mathbb{R}^+$ não depende de N. Logo,

$$\{\boldsymbol{u}_t^N\}, \{\boldsymbol{b}_t^N\} \text{ são limitadas em } L^2(0,T;\boldsymbol{V}_2') \tag{5.54}$$

e

$$\{\boldsymbol{w}_t^N\} \text{ é limitada em } L^2(0,T;\boldsymbol{H}^{-2}(\mathbb{R}^3)). \tag{5.55}$$

Portanto, pelo Lema de Aubin-Lions (veja o Lema 2.24), temos que

$$\{\boldsymbol{u}^N\}, \{\boldsymbol{w}^N\} \text{ e } \{\boldsymbol{b}^N\} \text{ estão em um subcontjunto compacto de } L^2(\mathcal{Q}_T). \tag{5.56}$$

Por outro lado, usando os Lemas 5.13 e 5.14, temos que

$$\{P^N\} \text{ é limitada em } L^{5/3}(\mathcal{Q}_T). \tag{5.57}$$

Logo, por (5.50)–(5.51) e (5.54)–(5.57), concluímos que existe uma subsequência, novamente denotada por $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$, convergindo para o limite $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ no seguinte sentido:

$$\big(\boldsymbol{u}^N, \boldsymbol{w}^N, \boldsymbol{b}^N\big) \to (\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b}) \text{ forte em } L^2(\mathcal{Q}_T), \tag{5.58}$$

$$P^N \to P \text{ fraco em } L^{5/3}(\mathcal{Q}_T), \tag{5.59}$$

$$\big(\boldsymbol{u}^N, \boldsymbol{b}^N\big) \to (\boldsymbol{u}, \boldsymbol{b}) \text{ fraco em } L^2(0,T;\boldsymbol{V}), \tag{5.60}$$

$$\boldsymbol{w}^N \to \boldsymbol{w} \text{ fraco em } L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)), \tag{5.61}$$

$$\big(\boldsymbol{u}^N, \boldsymbol{b}^N\big) \to (\boldsymbol{u}, \boldsymbol{b}) \text{ fraco estrela em } L^\infty(0,T;\boldsymbol{H}), \tag{5.62}$$

$$\boldsymbol{w}^N \to \boldsymbol{w} \text{ fraco estrela em } L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)). \tag{5.63}$$

Da definição de Ψ_δ e de (5.58)–(5.63), obtemos

$$\big(\Psi_\delta(\boldsymbol{u}^N), \Psi_\delta(\boldsymbol{b}^N)\big) \to (\boldsymbol{u}, \boldsymbol{b}) \text{ fortemente em } L^2(\mathcal{Q}_T).$$

Finalmente, por (5.54), concluímos que $\{\boldsymbol{u}^N\}$ e $\{\boldsymbol{b}^N\}$ são uniformemente contínuas de $(0,T)$ em $\boldsymbol{V}_2'$ e, por (5.55), temos que $\{\boldsymbol{w}^N\}$ é uniformemente contínua de $(0,T)$ em $\boldsymbol{H}^{-2}(\mathbb{R}^3)$. Desta forma,

$$(\boldsymbol{u}^N(0), \boldsymbol{w}^N(0), \boldsymbol{b}^N(0)) \xrightarrow[N\to\infty]{} (\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0), \tag{5.64}$$

completando, assim, a demonstração do Lema 5.15. □

5.2 Resultados de decaimento

5.2.1 Forças externas nulas

Primeiro consideraremos o caso em que $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$.

Teorema 5.16. *Sejam* $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \boldsymbol{H} \cap \boldsymbol{L}^1(\mathbb{R}^3)$ *e* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$*. Então existe uma solução fraca* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *do problema* (5.1) *com* $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$ *tal que*

$$\|\boldsymbol{u}(\cdot,t)\|_2 + (t+1)^{\frac{1}{2}}\|\boldsymbol{w}(\cdot,t)\|_2 + \|\boldsymbol{b}(\cdot,t)\|_2 \leq C(t+1)^{-\frac{3}{4}}, \quad \forall\, t \geq 0,$$

onde a constante $C \in \mathbb{R}^+$ *depende somente das normas de* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$ *em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$.

Para provar o Teorema 5.16, mostraremos que os argumentos formais apresentados na Seção 5.1 são válidos para as soluções aproximadas $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$.

Teorema 5.17. *Suponha que* $\boldsymbol{u}_0$, $\boldsymbol{b}_0 \in \boldsymbol{H} \cap \boldsymbol{L}^1(\mathbb{R}^3)$ *e que* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$*. Seja* $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$*,* $N \in \mathbb{N}^*$*, uma solução do problema* (5.3)*. Então*

$$\|\boldsymbol{u}^N(\cdot,t)\|_2 + (t+1)^{\frac{1}{2}}\|\boldsymbol{w}^N(\cdot,t)\|_2 + \|\boldsymbol{b}^N(\cdot,t)\|_2 \leq C(t+1)^{-\frac{3}{4}}, \quad \forall\, t \geq 0,$$

onde a constante $C > 0$ *depende apenas das normas em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$ *de* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$.

Demostraremos primeiro o Teorema 5.17, pois o Teorema 5.16 é uma consequência do Teorema 5.17.

Demonstração do Teorema 5.17

Por simplicidade, escreveremos $\boldsymbol{u}^N = \boldsymbol{u}$, $P^N = P$, $\boldsymbol{w}^N = \boldsymbol{w}$ e $\boldsymbol{b}^N = \boldsymbol{b}$. O primeiro passo para provar o Teorema 5.17 é mostrar que, para $\boldsymbol{\xi} \in S(t)$, vale

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1}. \tag{5.65}$$

Desta forma, basta seguirmos os mesmo passos da demonstração do Teorema 4.3. Devido aos Lemas 5.5, 5.7, 5.9 e 5.11 e aos Corolários 5.8, 5.10 e 5.12, os termos

$$J(t) := \|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 + \|\boldsymbol{b}(\cdot,t)\|_2^2, \qquad Q(t) := (\boldsymbol{u},\boldsymbol{u}_t) + (\boldsymbol{w},\boldsymbol{w}_t) + (\boldsymbol{b},\boldsymbol{b}_t), \tag{5.66}$$

satisfazem $J, Q \in L^2_{loc}(\mathbb{R})$ e

$$\frac{1}{2}\frac{d}{dt}J(t) = Q(t) \tag{5.67}$$

como distribuições. Além disso, J é absolutamente contínua. Assim, a identidade (5.67) é válida no sentido clássico. A estimativa (5.65) será estabelecida pela seguinte

Proposição 5.18. *Sejam* $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \boldsymbol{H} \cap \boldsymbol{L}^1(\mathbb{R}^3)$ *e* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$*. Sejam* $\boldsymbol{v}(\cdot,t), \overline{\boldsymbol{v}}(\cdot,t) \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{C}^\infty(\mathbb{R}^3)$ *com* $\operatorname{div} \boldsymbol{v}(\cdot,t) = \operatorname{div} \overline{\boldsymbol{v}}(\cdot,t) = 0$*. Se* $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ *satisfaz*

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla P = \dfrac{1}{2}\operatorname{rot}\boldsymbol{w} + (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b}, \\ \boldsymbol{w}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} - \Delta\boldsymbol{w} - \nabla(\operatorname{div}\boldsymbol{w}) + \boldsymbol{w} = \dfrac{1}{2}\operatorname{rot}\boldsymbol{u}, \\ \boldsymbol{b}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - \Delta\boldsymbol{b} = (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u}, \\ \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{b} = 0, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{w}(\boldsymbol{x},0) = \boldsymbol{w}_0(\boldsymbol{x}), \quad \boldsymbol{b}(\boldsymbol{x},0) = \boldsymbol{b}_0(\boldsymbol{x}), \end{cases} \tag{5.68}$$

então, para $\boldsymbol{\xi} \in \mathcal{K}$, $\mathcal{K} \subset \mathbb{R}^3$ um conjunto compacto, temos

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{b}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1}, \ \text{q.s. em } t, \tag{5.69}$$

onde a constante $C \in \mathbb{R}^+$ depende somente do compacto $\mathcal{K}$, das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ e das normas de $\boldsymbol{v}$ e $\overline{\boldsymbol{v}}$ em $\boldsymbol{L}^2$.

Demonstração. Seguindo as mesmas etapas da demonstração da Proposição 4.2, escrevemos (5.68) como

$$\widehat{\boldsymbol{z}}_t + A(\boldsymbol{\xi})\widehat{\boldsymbol{z}} = \boldsymbol{G}(\boldsymbol{\xi},t),$$

no sentido das distribuições, onde $\boldsymbol{z} := (\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})^T$ e $\boldsymbol{G} := (\boldsymbol{G}_1,\boldsymbol{G}_2,\boldsymbol{G}_3)^T$, com

$$\begin{cases} \boldsymbol{G}_1(\boldsymbol{\xi},t) := -\mathcal{F}\{\nabla P + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t), \\ \boldsymbol{G}_2(t,\boldsymbol{\xi}) := -\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t), \\ \boldsymbol{G}_3(t,\boldsymbol{\xi}) := -\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t), \end{cases}$$

e $A(\boldsymbol{\xi})$ é a matriz Hermitiana definida em (4.16). Desta forma, para provar a proposição, precisamos apenas demonstrar que

$$|\boldsymbol{G}_1(\boldsymbol{\xi},t)| + |\boldsymbol{G}_2(\boldsymbol{\xi},t)| + |\boldsymbol{G}_3(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}| \tag{5.70}$$

e repetir a análise na demonstração do Lema 4.1 e da Proposição 4.2. Primeiro mostraremos que $\boldsymbol{G}_1, \boldsymbol{G}_2$ e $\boldsymbol{G}_3$ estão bem definidos. Pelo Lema 5.14, temos

$$\int_0^t \int_{\mathbb{R}^3} |P(\boldsymbol{x},s)|^{5/3}\, d\boldsymbol{x}\, ds \leq \text{constante},$$

de modo que

$$\int_{\mathbb{R}^3} |P(\boldsymbol{x},t)|^{5/3}\, d\boldsymbol{x} \leq \text{constante q.s. em } t.$$

Logo, pela desigualdade de Hausdorff-Young (veja o Lema 2.34), temos

$$\|\widehat{P}(\cdot,t)\|_{5/2} \leq C\|P(\cdot,t)\|_{5/3},$$

o que implica que $\widehat{P}(\cdot,t) \in L^{5/2}(\mathbb{R}^3)$ q.s. em t. Ademais, pelo Lema 5.14, temos

$$\Delta P = -\text{div}\,[(\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b}] = -\sum_{j,k=1}^{3} \frac{\partial^2}{\partial x_j \partial x_k}(v_j u_k - \overline{v}_j, b_k). \tag{5.71}$$

Tomando a tranformada de Fourier de (5.71), como no Lema 4.1, obtemos

$$-|\boldsymbol{\xi}|^2\widehat{P}(\boldsymbol{\xi},t) = -\sum_{j,k=1}^{3} \xi_j\xi_k\mathcal{F}\{b_j\overline{v}_k\}(\boldsymbol{\xi},t) + \sum_{j,k=1}^{3} \xi_j\xi_k\mathcal{F}\{u_jv_k\}(\boldsymbol{\xi},t).$$

Desta forma,

$$\begin{aligned} |\boldsymbol{\xi}|^2|\widehat{P}(\boldsymbol{\xi},t)| &= \left| -\sum_{j,k=1}^{3} \xi_j\xi_k\mathcal{F}\{b_j\overline{v}_k\}(\boldsymbol{\xi},t) + \sum_{j,k=1}^{3} \xi_j\xi_k\mathcal{F}\{u_jv_k\}(\boldsymbol{\xi},t) \right| \\ &\leq \sum_{j,k=1}^{3} |\xi_j\xi_k|\|b_j\overline{v}_k(\cdot,t)\|_1 + \sum_{j,k=1}^{3} |\xi_j\xi_k|\|u_jv_k(\cdot,t)\|_1. \end{aligned}$$

Como u_j, b_j, v_k e $\overline{v}_k$ pertencem a $L^2(\mathbb{R}^3)$, segue que

$$|\widehat{P}(\boldsymbol{\xi},t)| \leq C.$$

Por outro lado, uma vez que

$$\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t) = i\boldsymbol{\xi}\widehat{P}(\boldsymbol{\xi},t),$$

concluímos que

$$|\mathcal{F}\{\nabla P\}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|, \tag{5.72}$$

onde $C > 0$ é uma constante dependendo somente das normas de $\boldsymbol{u}_0$, $\boldsymbol{w}_0$, $\boldsymbol{b}_0$, $\boldsymbol{v}$ e $\overline{\boldsymbol{v}}$ em $\boldsymbol{L}^2$.

A fim de obtermos (5.70) para os termos $\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\}$, $\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\}$, $\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{b}\}$, $\mathcal{F}\{(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u}\}$ e $\mathcal{F}\{(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b}\}$, devemos mostrar que isto faz sentido em $L^2(\mathcal{Q}_T)$, isto é, $\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\}$, $\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\}$, $\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{b}\}$, $\mathcal{F}\{(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u}\}$ e $\mathcal{F}\{(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b}\} \in L^2(\mathcal{Q}_T)$. Mas, isto segue dos Lemas 5.7, 5.9 e 5.11, pois

$$\int_0^T\int_{\mathbb{R}^3} |(\boldsymbol{v}\cdot\nabla)u_j|^2\,d\boldsymbol{x}\,dt \leq \sum_{k=1}^3 \int_0^T\int_{\mathbb{R}^3} \left|v_k\frac{\partial u_j}{\partial x_k}\right|^2 d\boldsymbol{x}\,dt \leq C\int_0^T\int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2\,d\boldsymbol{x}\,dt \leq C,$$

$$\int_0^T\int_{\mathbb{R}^3} |(\boldsymbol{v}\cdot\nabla)w_j|^2\,d\boldsymbol{x}\,dt \leq \sum_{k=1}^3 \int_0^T\int_{\mathbb{R}^3} \left|v_k\frac{\partial w_j}{\partial x_k}\right|^2 d\boldsymbol{x}\,dt \leq C\int_0^T\int_{\mathbb{R}^3} |\nabla \boldsymbol{w}|^2\,d\boldsymbol{x}\,dt \leq C,$$

$$\int_0^T\int_{\mathbb{R}^3} |(\boldsymbol{v}\cdot\nabla)b_j|^2\,d\boldsymbol{x}\,dt \leq \sum_{k=1}^3 \int_0^T\int_{\mathbb{R}^3} \left|v_k\frac{\partial b_j}{\partial x_k}\right|^2 d\boldsymbol{x}\,dt \leq C\int_0^T\int_{\mathbb{R}^3} |\nabla \boldsymbol{b}|^2\,d\boldsymbol{x}\,dt \leq C,$$

$$\int_0^T\int_{\mathbb{R}^3} |(\overline{\boldsymbol{v}}\cdot\nabla)u_j|^2\,d\boldsymbol{x}\,dt \leq \sum_{k=1}^3 \int_0^T\int_{\mathbb{R}^3} \left|\overline{v}_k\frac{\partial u_j}{\partial x_k}\right|^2 d\boldsymbol{x}\,dt \leq C\int_0^T\int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2\,d\boldsymbol{x}\,dt \leq C,$$

$$\int_0^T\int_{\mathbb{R}^3} |(\overline{\boldsymbol{v}}\cdot\nabla)b_j|^2\,d\boldsymbol{x}\,dt \leq \sum_{k=1}^3 \int_0^T\int_{\mathbb{R}^3} \left|\overline{v}_k\frac{\partial b_j}{\partial x_k}\right|^2 d\boldsymbol{x}\,dt \leq C\int_0^T\int_{\mathbb{R}^3} |\nabla \boldsymbol{b}|^2\,d\boldsymbol{x}\,dt \leq C.$$

Uma vez que $\operatorname{div}\boldsymbol{v} = \operatorname{div}\overline{\boldsymbol{v}} = 0$ e $v_k, \overline{v}_k, u_j, w_j, b_j \in L^2(\mathbb{R}^3)$, temos

$$\left|[\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)]_j\right| = \left|i\sum_{k=1}^3 \xi_k\,\mathcal{F}\{u_jv_k\}(\boldsymbol{\xi},t)\right| \leq |\boldsymbol{\xi}|\sum_{k=1}^3 |\mathcal{F}\{u_jv_k\}(\boldsymbol{\xi},t)| \leq C\,|\boldsymbol{\xi}|, \tag{5.73}$$

$$\left|[\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi},t)]_j\right| = \left|i\sum_{k=1}^3 \xi_k\,\mathcal{F}\{w_jv_k\}(\boldsymbol{\xi},t)\right| \leq |\boldsymbol{\xi}|\sum_{k=1}^3 |\mathcal{F}\{w_jv_k\}(\boldsymbol{\xi},t)| \leq C\,|\boldsymbol{\xi}|, \tag{5.74}$$

$$\left|[\mathcal{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)]_j\right| = \left|i\sum_{k=1}^3 \xi_k\,\mathcal{F}\{b_jv_k\}(\boldsymbol{\xi},t)\right| \leq |\boldsymbol{\xi}|\sum_{k=1}^3 |\mathcal{F}\{b_jv_k\}(\boldsymbol{\xi},t)| \leq C\,|\boldsymbol{\xi}|, \tag{5.75}$$

$$\left|[\mathcal{F}\{(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi},t)]_j\right| = \left|i\sum_{k=1}^3 \xi_k\,\mathcal{F}\{u_j\overline{v}_k\}(\boldsymbol{\xi},t)\right| \leq |\boldsymbol{\xi}|\sum_{k=1}^3 |\mathcal{F}\{u_j\overline{v}_k\}(\boldsymbol{\xi},t)| \leq C\,|\boldsymbol{\xi}|, \tag{5.76}$$

$$\left|[\mathcal{F}\{(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{b}\}(\boldsymbol{\xi},t)]_j\right| = \left|i\sum_{k=1}^3 \xi_k\,\mathcal{F}\{b_j\overline{v}_k\}(\boldsymbol{\xi},t)\right| \leq |\boldsymbol{\xi}|\sum_{k=1}^3 |\mathcal{F}\{b_j\overline{v}_k\}(\boldsymbol{\xi},t)| \leq C\,|\boldsymbol{\xi}|, \tag{5.77}$$

onde $C \in \mathbb{R}^+$ é uma constante dependendo somente das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \boldsymbol{v}$ e $\overline{\boldsymbol{v}}$ em $\boldsymbol{L}^2$. Das estimativas (5.72)–(5.77), obtemos (5.70). Agora, usamos a desigualdade (5.70) para mostrar (5.69). Defina

$$\varphi(\boldsymbol{\xi}, t) := e^{tA(\boldsymbol{\xi})}\widehat{\boldsymbol{z}}(\boldsymbol{\xi}, t).$$

Então

$$\varphi_t(\boldsymbol{\xi}, t) = e^{tA(\boldsymbol{\xi})}\boldsymbol{G}(\boldsymbol{\xi}, t) \tag{5.78}$$

no sentido das distribuições. Da estimativa (5.70), segue que $e^{-tA(\boldsymbol{\xi})}G(\boldsymbol{\xi}, \cdot) \in \boldsymbol{L}^{\infty}_{loc}(\mathbb{R})$. Assim, a identidade (5.78) vale no sentido clássico q.s. em t. Desta forma,

$$\varphi(\boldsymbol{\xi}, t) = \widehat{\boldsymbol{z}}(\boldsymbol{\xi}, 0) + \int_0^t e^{sA(\boldsymbol{\xi})}\boldsymbol{G}(\boldsymbol{\xi}, s)\, ds$$

e, assim, obtemos a estimativa (5.69). Isto completa a demonstração da Proposição 5.18. □

Para completar a demonstração do Teorema 5.17, considere $\boldsymbol{v} = \Psi_\delta(\boldsymbol{u})$ e $\overline{\boldsymbol{v}} = \Psi_\delta(\boldsymbol{b})$ na Proposição 5.18, repita os argumentos da demonstração do Teorema 4.3 e aplique o Lema 5.13 para mostrar que

$$\|\boldsymbol{u}(\cdot, t)\|_2^2 + \|\boldsymbol{w}(\cdot, t)\|_2^2 + \|\boldsymbol{b}(\cdot, t)\|_2^2 \leq C(t+1)^{-3/2}, \quad \forall\, t \geq 0,$$

onde a constante $C > 0$ depende somente das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Em seguida, procedendo como no Lema 4.4, na Proposição 4.6 e no Teorema 4.7, obtém-se que

$$\|\boldsymbol{w}(\cdot, t)\|_2^2 \leq \widetilde{C}(t+1)^{-5/2}$$

para todo $t \geq 0$ e para alguma cosntante $\widetilde{C} \in \mathbb{R}^+$ dependendo apenas das normas em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$. Isto completa a demonstração do teorema.

Demonstração do Teorema 5.16

Seja $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ a solução fraca obtida como limite forte de $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$ em $L^2(\mathcal{Q}_T)$ (veja o Lema 5.15). Então,

$$\begin{aligned}\lim_{N\to\infty}\int_{\mathbb{R}^3}|\boldsymbol{u}^N(\boldsymbol{x}, t) - \boldsymbol{u}(\boldsymbol{x}, t)|^2 d\boldsymbol{x} &= \lim_{N\to\infty}\int_{\mathbb{R}^3}|\boldsymbol{w}^N(\boldsymbol{x}, t) - \boldsymbol{w}(\boldsymbol{x}, t)|^2 d\boldsymbol{x}\\ &= \lim_{N\to\infty}\int_{\mathbb{R}^3}|\boldsymbol{b}^N(\boldsymbol{x}, t) - \boldsymbol{b}(\boldsymbol{x}, t)|^2 d\boldsymbol{x} = 0.\end{aligned}$$

Pelo Teorema 5.17, temos que

$$\begin{aligned}&\|\boldsymbol{u}(\cdot, t)\|_2 \leq \|\boldsymbol{u}^N(\cdot, t)\|_2 + \|(\boldsymbol{u} - \boldsymbol{u}^N)(\cdot, t)\|_2 \leq C(t+1)^{-3/4} + \|(\boldsymbol{u} - \boldsymbol{u}^N)(\cdot, t)\|_2,\\ &\|\boldsymbol{w}(\cdot, t)\|_2 \leq \|\boldsymbol{w}^N(\cdot, t)\|_2 + \|(\boldsymbol{w} - \boldsymbol{w}^N)(\cdot, t)\|_2 \leq C(t+1)^{-5/4} + \|(\boldsymbol{w} - \boldsymbol{w}^N)(\cdot, t)\|_2,\\ &\|\boldsymbol{b}(\cdot, t)\|_2 \leq \|\boldsymbol{b}^N(\cdot, t)\|_2 + \|(\boldsymbol{b} - \boldsymbol{b}^N)(\cdot, t)\|_2 \leq C(t+1)^{-3/4} + \|(\boldsymbol{b} - \boldsymbol{b}^N)(\cdot, t)\|_2,\end{aligned}$$

para alguma constante $C > 0$ dependendo somente das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ (mas não depende de N). Fazendo $N \to \infty$, obtemos

$$\|\boldsymbol{u}(\cdot, t)\|_2^2 + (t+1)\|\boldsymbol{w}(\cdot, t)\|_2^2 + \|\boldsymbol{b}(\cdot, t)\|_2^2 \leq C(t+1)^{-3/2}, \quad \forall\, t \geq 0,$$

onde a constante $C \in \mathbb{R}^+$ depende apenas das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$, concluindo, assim, a demonstração do Teorema 5.16.

5.2.2 Forças externas não nulas

Quando as forças externas $\boldsymbol{f}$ e $\boldsymbol{g}$ estão presentes, temos o seguinte

Teorema 5.19. *Suponha que* $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \boldsymbol{H} \cap \boldsymbol{L}^1(\mathbb{R}^3)$, $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$, $\boldsymbol{f} \in L^2(0, \infty; \boldsymbol{V}')$, $\boldsymbol{g} \in L^2(0, \infty; \boldsymbol{H}^{-1}(\mathbb{R}^3))$, $\operatorname{div} \boldsymbol{f} = 0$ *e*

$$(t+1)^{1/2}\|\boldsymbol{f}(\cdot,t)\|_2 + \|\boldsymbol{g}(\cdot,t)\|_2 \leq K_1(t+1)^{-5/4}$$

e

$$|\widehat{\boldsymbol{f}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{g}}(\boldsymbol{\xi},t)| \leq K_2(t+1)^{-1/2}|\boldsymbol{\xi}|,$$

para todo $t \geq 0$ *e* $|\boldsymbol{\xi}| \leq 1$. *Então, a única solução* $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$, $N \in \mathbb{N}^*$, *do problema* (5.3) *satisfaz*

$$\|\boldsymbol{u}^N(\cdot,t)\|_2 + \|\boldsymbol{w}^N(\cdot,t)\|_2 + \|\boldsymbol{b}^N(\cdot,t)\|_2 \leq C(t+1)^{-\frac{3}{4}}, \quad \forall\, t \geq 0,$$

onde a constante $C > 0$ *depende somente de* K_1, K_2 *e das normas de* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$ *em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$. *Além disso, se* $\boldsymbol{f}$ *e* $\boldsymbol{g}$ *também satisfazem* (4.74)*, então*

$$\|\boldsymbol{w}^N(\cdot,t)\|_2 \leq \widetilde{C}(t+1)^{-\frac{5}{4}}, \quad \forall\, t \geq 0,$$

onde a constante $\widetilde{C} \in \mathbb{R}^+$ *depende apenas de* $\tilde{t}_1, K_1, K_2, K_3$ *e das normas em* $\boldsymbol{L}^1$ *e* $\boldsymbol{L}^2$ *de* $\boldsymbol{u}_0, \boldsymbol{w}_0$ *e* $\boldsymbol{b}_0$.

Demonstração. Sejam

$$\begin{aligned}
\boldsymbol{G}_1(\boldsymbol{\xi},t) &:= \mathcal{F}\{\boldsymbol{f} + (\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\boldsymbol{b}^N - (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N - \nabla P^N\}(\boldsymbol{\xi},t),\\
\boldsymbol{G}_2(\boldsymbol{\xi},t) &:= \mathcal{F}\{\boldsymbol{g} - (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{w}^N\}(\boldsymbol{\xi},t),\\
\boldsymbol{G}_3(\boldsymbol{\xi},t) &:= \mathcal{F}\{(\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\boldsymbol{u}^N - (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{b}^N\}(\boldsymbol{\xi},t).
\end{aligned}$$

O primeiro passo é mostrar que

$$|\boldsymbol{G}_1(\boldsymbol{\xi},t)| + |\boldsymbol{G}_2(\boldsymbol{\xi},t)| + |\boldsymbol{G}_3(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|, \tag{5.79}$$

para $|\boldsymbol{\xi}| \leq 1$. Usando o mesmo argumento do Teorema 5.17, temos que

$$\begin{aligned}
&|\mathcal{F}\{\nabla P^N\}(\boldsymbol{\xi},t)| \leq \left(\|\boldsymbol{u}^N(\cdot,t)\|_2^2 + \|\boldsymbol{b}^N(\cdot,t)\|_2^2\right)|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{u}^N(\cdot,t)\|_2^2\,|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{b}^N\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{u}^N(\cdot,t)\|_2\|\boldsymbol{b}^N(\cdot,t)\|_2\,|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{w}^N\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{u}^N(\cdot,t)\|_2\|\boldsymbol{w}^N(\cdot,t)\|_2\,|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\boldsymbol{u}^N\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{b}^N(\cdot,t)\|_2\|\boldsymbol{u}^N(\cdot,t)\|_2\,|\boldsymbol{\xi}|,\\
&|\mathcal{F}\{(\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\boldsymbol{b}^N\}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{b}^N(\cdot,t)\|_2^2\,|\boldsymbol{\xi}|.
\end{aligned}$$

Em particular,

$$\begin{aligned}
&|\mathcal{F}\{\nabla P^N\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\,\boldsymbol{u}^N\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\,\boldsymbol{b}^N\}(\boldsymbol{\xi},t)|\\
&+ |\mathcal{F}\{(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\,\boldsymbol{w}^N\}(\boldsymbol{\xi},t)| + |\mathcal{F}\{(\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\,\boldsymbol{u}^N\}(\boldsymbol{\xi},t)|\\
&+ |\mathcal{F}\{(\Psi_\delta(\boldsymbol{b}^N)\cdot\nabla)\,\boldsymbol{b}^N\}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|,
\end{aligned}$$

para todo $\boldsymbol{\xi} \in \mathbb{R}^3$. Como

$$|\widehat{\boldsymbol{f}}(\boldsymbol{\xi}, t)| + |\widehat{\boldsymbol{g}}(\boldsymbol{\xi}, t)| \le K_2(t+1)^{-1/2}|\boldsymbol{\xi}|, \quad \text{para todo} \quad t \ge 0 \quad \text{e} \quad |\boldsymbol{\xi}| \le 1,$$

obtemos a estimativa (5.79). Isto implica, como no Teorema 4.10, que

$$|\mathcal{F}\{\boldsymbol{u}^N\}(\boldsymbol{\xi}, t)| + |\mathcal{F}\{\boldsymbol{w}^N\}(\boldsymbol{\xi}, t)| + |\mathcal{F}\{\boldsymbol{b}^N\}(\boldsymbol{\xi}, t)| \le C\,|\boldsymbol{\xi}|^{-1},$$

para todo $0 < |\boldsymbol{\xi}| \le 1$. Continuando como na demonstração do Teorema 4.10, obtemos

$$\|\boldsymbol{u}^N(\cdot, t)\|_2^2 + \|\boldsymbol{w}^N(\cdot, t)\|_2^2 + \|\boldsymbol{b}^N(\cdot, t)\|_2^2 \le C(t+1)^{-1/2}, \tag{5.80}$$

para todo $t \ge 0$ e alguma constante $C > 0$ dependendo somente de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ (independente de N). Agora, usamos isso para obtermos a estimativa mais precisa

$$|\mathcal{F}\{\boldsymbol{u}^N\}(\boldsymbol{\xi}, t)| + |\mathcal{F}\{\boldsymbol{w}^N\}(\boldsymbol{\xi}, t)| + |\mathcal{F}\{\boldsymbol{b}^N\}(\boldsymbol{\xi}, t)| \le C, \tag{5.81}$$

para todo $t \ge 0$ e para alguma constante $C \in \mathbb{R}^+$ dependendo apenas de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ (em particular independente de N). Isto permite aplicar o procedimento descrito no final do Teorema 4.10 para $(\boldsymbol{u}^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$ melhorando a estimativa (5.80) para

$$\|\boldsymbol{u}^N(\cdot, t)\|_2^2 + \|\boldsymbol{w}^N(\cdot, t)\|_2^2 + \|\boldsymbol{b}^N(\cdot, t)\|_2^2 \le C(t+1)^{-3/2}, \tag{5.82}$$

para todo $t \ge 0$, com a constante $C > 0$ dependendo somente de K_1 e K_2 (ver (4.44) e (4.45)) e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ e independente de N. Isto prova a primeira parte do Teorema 5.19. Agora, assumindo que as forças externas $\boldsymbol{f}$ e $\boldsymbol{g}$ satisfazem (4.74) para algum dado $\tilde{t}_1 \ge 0$, usando (5.82), repetindo os argumentos nas demonstrações da Proposição 4.12 e do Teorema 4.13, obtemos a estimativa melhorada

$$\|\boldsymbol{w}^N(\cdot, t)\|_2 \le \widetilde{C}(t+1)^{-5/4}, \tag{5.83}$$

para todo $t \ge 0$, onde a constante $\widetilde{C} \in \mathbb{R}^+$ depende apenas de $\tilde{t}_1, K_1, K_2, K_3$ e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ (e não depende de N). Isto completa a demonstração do teorema. □

Teorema 5.20. *Assuma $\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \boldsymbol{f}$ e $\boldsymbol{g}$ como no Teorema 5.19. Seja $(\boldsymbol{u}, P, \boldsymbol{w}, \boldsymbol{b})$ a solução fraca de* (5.1) *obtida como limite em $\boldsymbol{L}^2$ da solução $(\boldsymbol{u}^N, P^N, \boldsymbol{w}^N, \boldsymbol{b}^N)$ de* (5.3)*. Então*

$$\|\boldsymbol{u}(\cdot, t)\|_2 + \|\boldsymbol{b}(\cdot, t)\|_2 + \|\boldsymbol{w}(\cdot, t)\|_2 \le C(t+1)^{-\frac{3}{4}}$$

para todo $t \ge 0$, onde a constante $C > 0$ depende somente de K_1, K_2 e das normas de $\boldsymbol{u}_0, \boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$. Ademais, se $\boldsymbol{f}$ e $\boldsymbol{g}$ também satisfazem (4.74)*, então*

$$\|\boldsymbol{w}(\cdot, t)\|_2 \le \widetilde{C}(t+1)^{-\frac{5}{4}}, \quad \forall\, t \ge 0,$$

onde a constante $\widetilde{C} \in \mathbb{R}^+$ depende apenas de $\tilde{t}_1, K_1, K_2, K_3$ e das normas dos dados iniciais em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$.

Demonstração. Do Lema 5.15, segue que

$$\begin{aligned}\lim_{N\to\infty}\int_{\mathbb{R}^3}|\boldsymbol{u}^N(\boldsymbol{x},t)-\boldsymbol{u}(\boldsymbol{x},t)|^2\,d\boldsymbol{x} &= \lim_{N\to\infty}\int_{\mathbb{R}^3}|\boldsymbol{w}^N(\boldsymbol{x},t)-\boldsymbol{w}(\boldsymbol{x},t)|^2\,d\boldsymbol{x}\\ &= \lim_{N\to\infty}\int_{\mathbb{R}^3}|\boldsymbol{b}^N(\boldsymbol{x},t)-\boldsymbol{b}(\boldsymbol{x},t)|^2\,d\boldsymbol{x}=0.\end{aligned}$$

Desta forma, pela estimativa (5.82), temos

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|_2 \le \|\boldsymbol{u}^N(\cdot,t)\|_2+\|(\boldsymbol{u}-\boldsymbol{u}^N)(\cdot,t)\|_2 \le C(t+1)^{-3/4}+\|(\boldsymbol{u}-\boldsymbol{u}^N)(\cdot,t)\|_2,\\ \|\boldsymbol{w}(\cdot,t)\|_2 \le \|\boldsymbol{w}^N(\cdot,t)\|_2+\|(\boldsymbol{w}-\boldsymbol{w}^N)(\cdot,t)\|_2 \le C(t+1)^{-3/4}+\|(\boldsymbol{w}-\boldsymbol{w}^N)(\cdot,t)\|_2,\\ \|\boldsymbol{b}(\cdot,t)\|_2 \le \|\boldsymbol{b}^N(\cdot,t)\|_2+\|(\boldsymbol{b}-\boldsymbol{b}^N)(\cdot,t)\|_2 \le C(t+1)^{-3/4}+\|(\boldsymbol{b}-\boldsymbol{b}^N)(\cdot,t)\|_2,\end{aligned}$$

para alguma constante $C>0$ dependendo somente de K_1 e K_2 (ver (4.44) e (4.45)) e das normas de $\boldsymbol{u}_0,\boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ e não depende de N. Fazendo $N\to\infty$, obtemos

$$\|\boldsymbol{u}(\cdot,t)\|_2+\|\boldsymbol{w}(\cdot,t)\|_2+\|\boldsymbol{b}(\cdot,t)\|_2 \le C(t+1)^{-3/4},$$

para todo $t\ge 0$, o que prova a primeira parte do Teorema 5.20. No caso em que a condição (4.74) também é satisfeita, temos, por (5.83), que

$$\|\boldsymbol{w}(\cdot,t)\|_2 \le \|\boldsymbol{w}^N(\cdot,t)\|_2+\|(\boldsymbol{w}-\boldsymbol{w}^N)(\cdot,t)\|_2 \le \widetilde{C}(t+1)^{-5/4}+\|(\boldsymbol{w}-\boldsymbol{w}^N)(\cdot,t)\|_2$$

para todo N e para todo $t\ge 0$ onde a constante $\widetilde{C}>0$ depende somente de $\tilde{t}_1,K_1,K_2,K_3$ e das normas de $\boldsymbol{u}_0,\boldsymbol{w}_0$ e $\boldsymbol{b}_0$ em $\boldsymbol{L}^1$ e $\boldsymbol{L}^2$ e não de N. Fazendo $N\to\infty$, obtemos

$$\|\boldsymbol{w}(\cdot,t)\|_2 \le \widetilde{C}(t+1)^{-5/4},$$

para todo $t\ge 0$, concluindo, assim, a demonstração do Teorema 5.20. $\square$

REFERÊNCIAS

1 ERINGEN, A. C. *Theory of Micropolar Fluids*. J. Math. Mech., 16(1):1–18, 1966. 9, 10

2 BRAZ E SILVA,P.; CRUZ, F. W.; ROJAS-MEDAR, M. A. *Semi-strong and strong solutions for variable density asymmetric fluids in unbounded domains*. Math. Meth. Appl. Sci. 40:757–774, 2017. 10

3 BRAZ E SILVA,P.; CRUZ, F. W.; ROJAS-MEDAR, M. A. *Vanishing viscosity for non-homogeneous asymmetric fluids in* $\mathbb{R}^3$*: The* L^2 *case*. J. Math. Anal. Appl. 420:207–221, 2014. 10

4 BRAZ E SILVA,P.; CRUZ, F. W.; ROJAS-MEDAR, M. A.; SANTOS, E. G. *Weak solutions with improved regularity for the nonhomogeneous asymmetric fluids equations with vacuum*. J. Math. Anal. Appl. 473:567–586, 2019. 10

5 BRAZ E SILVA,P.; FERNÁNDEZ-CARA, E.; ROJAS-MEDAR, M. A. *Vanishing viscosity for non-homogeneous asymmetric fluids in* $\mathbb{R}^3$. J. Math. Analysis and Appl., 332:833–845, 2007. 10

6 BRAZ E SILVA,P.; SANTOS, E. G. *Global weak solutions for variable density asymmetric incompressible fluids*. J. Math. Anal. Appl., 387:953–969, 2012. 10

7 CONCA, C; GORMAZ, R.; ORTEGA-TORRES , E.; ROJAS-MEDAR, M. *Existence and uniqueness of a strong solution for nonhomogeneous micropolar fluids*. In Nonlinear partial differential equations and their applications. Collège de France Seminar, VOL. XIV (Paris, 1997/1998), volume 31 of Stud. Math. Appl., pages 213–241, Amsterdam, 2002. North-Holland. 10

8 CONCA, C; GORMAZ, R.; ORTEGA-TORRES , E.; ROJAS-MEDAR, M. *The equations of nonhomogeneous asymmetric fluids: an interative approach*. Math. Meth. Appl. Sci., 25:1251–1280, 2002. 10

9 CRUZ, F. W.; BRAZ E SILVA, P. *Error estimates for spectral semi-Galerkin approximations of incompressible asymmetric fluids with variable density*. J. Math. Fluid Mech., 27 pages, 21:2, 2019. 10

10 GALDI, G. P.; RIONERO, S. *A note on the existence and uniqueness of solutions of the micropolar fuid equations*. Int. J. Eng. Sci. 15, 105–108, 1977. 10

11 ŁUKASZEWICZ, G. *On non-stationary flows of incompressible asymmetric fluids*. Math. Methods Appl. Sci., 13(3):219–232, 1990. 10

12 ORTEGA-TORRES, E. E.; ROJAS-MEDAR, M. A. *On the regularity for solutions of the micropolar fluid equations*. Rend. Sem. Mat. Univ. Padova 122, 27–37, 2009. 10

13 ORTEGA-TORRES, E. E.; VILLAMIZAR-ROA, E. J.; ROJAS-MEDAR, M. A. *Micropolar fluids with vanishing viscosity*. Abstract and Applied Analysis, pp. 1–18, 2010. 10

14 CHEN, Q.; MIAO, C. *Global well-posedness for the micropolar fuid system in critical Besov spaces*. J. Difer. Equ. 252(3), 2698–2724, 2012. 10

15 BRAZ E SILVA,P.; CRUZ, F. W.; FREITAS, L. B. S.; ZINGANO, P. R. *On the L^2 decay of weak solutions for the 3D asymmetric fluids equations*. J. Differential Equations, v. 267, 3578–3609, 2019. 10, 11

16 CRUZ, F. W. *Global strong solutions for the incompressible micropolar fluids equations*. Arch. Math., 113, 201–212, 2019. 10, 11

17 FERREIRA, L. C. F.; VILLAMIZAR-ROA, E. J. *Micropolar fluid system in a space of distributions and large time behavior*. J. Math. Anal. Appl., v. 332, n. 2, p. 1425–1445, 2007 . 10, 20

18 ŁUKASZEWICZ, G. *Micropolar Fluids: Theory and Applications*. Modelling and Simulation in Science, Engineering & Technology. Birkhäuser Boston, Inc., Boston, MA, 1999. 10, 84, 85

19 BOLDRINI, J. L.; ROJAS-MEDAR, M. A.; FERNÁNDEZ-CARA, E. *Semi-Galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids*. J. Math. Pures Appl., v. 82, n. 11, p. 1499–1525, 2003. 10, 86, 87

20 GUTERRES, R. H.; NUNES, J. R.; PERUSATO, C. F. *On the large time decay of global solutions for the micropolar dynamics in $L^2(\mathbb{R}^n)$*. Nonlinear Analysis: Real World Applications 45:789–798, 2019. 10

21 WU, J. *Bounds and new approaches for the 3D MHD equations*. J. Nonlinear Sci. vol. 12, 395–413, 2002. 10

22 AHMADI, G.; SHAHINPOOR, M. *Universal stability of magneto-micropolar fluid motions*. Int. J. Engng Sci. 12 , 657–663, 1974. 10

23 ROJAS-MEDAR, M. A.; BOLDRINI, J. L. *Magneto-micropolar fluid motion: existence of weak solution*. Rev. Mat. Complut. (Madrid) 11(2), 443–460, 1998. 10

24 GUTERRES, R. H.; NUNES, J. R.; PERUSATO, C. F. *Decay rates for the magneto-micropolar system in $L^2(\mathbb{R}^n)$*. Arch. Math. 111:431–442, 2018. 10

25 BRAZ E SILVA P.; FRIZ, L.; ROJAS-MEDAR, M. A. *Exponential stability for magneto-micropolar fluids*. Nonlinear Analysis 143, 211–223, 2016. 11

26 ORTEGA-TORRES, E. E.; ROJAS-MEDAR, M. A. *On the uniqueness and regularity of the weak solutions for magneto-micropolar fluid equations*. Rev. Mat. Apl. 17, 75–90, 1996. 11

27 ROJAS-MEDAR, M. A. *Magneto-micropolar fluid motion: Existence and uniqueness of strong solution*. Math. Nachr. 188, 301–319, 1997. 11, 31

28 ORTEGA-TORRES, E. E.; ROJAS-MEDAR, M. A. *Magneto-micropolar fluid motion: global existence of strong solutions*. Abstr. Appl. Anal. 4, 109–125, 1999. 11

29 ORTEGA-TORRES, E. E.; ROJAS-MEDAR, M. A.; CABRALES, R. C. *A uniform error estimate in time for spectral Galerkin approximations of the magneto-micropolar fluid equations*. Numer. Methods Partial Differ. Equ. 28(2), 689–706, 2012. 11

30 ZHONG, X. *Global well-posedness to the incompressible Navier-Stokes equations with damping*. Electron. J. Qual. Theory Differ. Equ. 62, 1–9, 2017. 11

31 LERAY, J. *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. Volume 63, 193–248, 1934. 11

32 KATO, T., *Strong L^p-solutions of the Navier-Stokes equations in R^m, with applications to weak solutions*, Math. Z. 187, 471–480, 1984. 11

33 WIEGNER, M. *Decay results for weak solutions of the Navier-Stokes equations on* $\mathbb{R}^n$ n J. London Math. Soc. 35, 303-313, 1987. 11

34 MASUDA, K. *Weak solutions of the Navier-Stokes equations, Tôhoku* Math. Journal 36, 623–646, 1984. 11

35 SCHONBEK, M. E. *L^2 decay for weak solutions of the Navier-Stokes equations*. Arch. Ration. Mech. Anal., v. 88, n. 3, p. 209–222, 1985. 11

36 SCHONBEK, M. E. *Uniform decay rates for parabolic conservation laws*. Nonlinear Analysis: Theory, Methods e Applications, v. 10, n. 9, p. 943–956, 1986. 11

37 LI, M.; SHANG, H. *Large time decay of solutions for the 3D magneto-micropolar equations*. Nonlinear Analysis: Real World Applications 44, 479–496, 2018. 11

38 BREZIS, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext, Springer, New York, 2011. 13

39 IORIO, R.; IORIO, V. M. *Fourier Analysis and Partial Differential Equations*. Cambridge Stud. Adv. Math., 2001. 13

40 LIONS, J.-L. *Quelques méthodes de résolution des problemes aux limites non linéaires*. Dunod, 1969. 13, 17

41 STEIN, E. M. *Singular Integrals and Differentiability Properties of Functions* (PMS-30). Princeton Univ. Press, 1970. 13, 19, 20

42 TEMAM, R. *Navier-Stokes Equations: Theory and Numerical Analysis*, vol. 2 of Studies in mathematics and its applications. North-Holland Amsterdam, 1984. 13, 16, 90

43 EVANS, L. *Partial Differential Equations*, Vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 1998. 13, 14, 92

44 ROBSON, James C.;Rodrigo, José L.;Sadowski,Witold *The three-Dimensional Navier-Stokes Equations* Cambridge Studies in advanced mathematics,2016. 14

45 RIVERA, M.; EDILBERTO, J. *Teoria das distribuições e Equações Diferenciais Parciais* Laboratório Nacional de Computação Científica, 242.p, 2004. 15

46 SCHÜTZ, L. *Equações de Advecção-Difusão com Aplicações às Equações de Navier-Stokes*. Tese de doutorado, Universidade Federal do Rio Grande do Sul, Porto Alegre, 2008. 19

47 KREISS, H.-O.; HAGSTROM, T.; LORENZ, J.; ZINGANO, P. *Decay in time of incompressible flows*. J. Math. Fluid Mech. 5, 231–244, 2003. 19

48 ZINGANO, P. R. *Dois Problemas em Equações Diferenciais Parciais*. Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, 2015 (disponível em http://lume.ufrgs.br). 19

49 GUADAGNIN, M. A. *Alguns resultados para a Equação do Calor e Equações de Advecção-Difusão*. Dissertação de mestrado, Universidade Federal do Rio Grande do Sul, Porto Alegre, 2005. 21

50 SIMON, J. *Nonhomogeneous viscous incompressible fluids: Existence of velocity, density, and pressure*. SIAM J. Math. Anal. 21, 1093–1117, 1990. 28

51 CAFFARELLI, L.; KOHN, R.; NIRENBERG, L. *Partial regularity of suitable weak solutions of the Navier-Stokes equations*. Communications on Pure and Applied Mathematics, Wiley Online Library, v. 35, n. 6, p. 771–831, 1982. 84, 85

52 MOHGAONKAR, S. D.; SARAYKAR, R. V. *L^2-decay for solutions of the magnetohydrodynamic equations*. J. Math. Phys. Sci. 23, no. 1, 35–55, 1989. 85