UNIVERSIDADE FEDDERAL DE PERNAMBUCO
DEPARTAMENTO DE FÍSICA
PÓS GRADUAÇÃO EM FÍSICA

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DICKE STATES: PRODUCTION FROM EPR PAIRS AND ENTANGLEMENT ANALYSIS

Recife
2017

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Thesis presented to the postgraduate program of the Physics Department of the Universidade Federal de Pernambuco, as partial requirement to obtain the degreed of Doctor in Physics.

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Recife
2017

Catalogação na fonte
Bibliotecária Joana D'Arc Leão Salvador CRB 4-572

M843d Moreno Filho, Marcos George M.
Dicke states: production from EPR pairs and entanglement analysis / Marcos George M. Moreno Filho. – 2017.
83 f.: fig., tab.

Orientador: Fernando Roberto de Luna Parisio Filho.
Tese (Doutorado) – Universidade Federal de Pernambuco. CCEN. Física, Recife, 2017.
Inclui referências.

1. Física quântica. 2. Emaranhamento. 3. Fenômenos críticos. 4. Testemunha de emaranhamento. I. Parisio Filho, Fernando Roberto de Luna (Orientador). II. Titulo.

530.12 CDD (22. ed.) UFPE-FQ 2017-45

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DICKE STATES: PRODUCTION FROM EPR PAIRS AND ENTANGLEMENT ANALYSIS

Tese apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutor Doutora em Física.

Aprovada em: 11/08/2017.

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Acknowledgements

Agradeço ao Conselho Nacional de Pesquisa (CNPq) e ao meu orientador Prof. Fernando Roberto de Luna Parisio Filho.

Abstract

In this work we analyze the problem of creation of Dicke states through the process of entanglement swapping. First we deal with the tripartite case, presenting a deterministic protocol based on a source of EPR pairs. Still in this topic we show that it is possible to create perfect W states (tripartite Dicke states), even when the EPR source does not yield maximally entangled pairs, with a finite success rate. Subsequently we cope with the n-partite case, for which we present a probabilistic procedure, also employing EPR-pair sources, where the output, whenever successful, is perfect. The analysis of this protocol reveals a critical transition for the success probability, which is characterized in the framework of Landau theory of second order phase-transitions. Finally we proceed with a study of the entanglement between bipartitions of systems described by Dicke states, which allows for the implementation of entanglement witnesses. For each Dicke state we present a highly sensitive witness.

Keywords: Quantum entanglement. Dicke states. Critical phenomena. Entanglement swapping. Entanglement witness.

Resumo

Neste trabalho analisamos o problema de criação de estados de Dicke através de um processo de *entanglement swapping*. Inicialmente tratamos o caso tripartite, para o qual apresentamos um protocolo determinístico empregando uma fonte de estados EPR. Ainda nesse tópico, mostramos que, com uma taxa de sucesso finita, é possível produzir estados W (estados de Dicke tripartite) perfeitos mesmo quando utilizamos uma fonte de estados EPR não maximamente emaranhados. Em seguida consideramos o caso de n partes, para o qual apresentamos um procedimento probabilístico, também fazendo uso de uma fonte de estados EPR, em que o resultado, sempre que bem-sucedido, é perfeito. A análise deste protocolo revela uma transição crítica, que é caracterizada no contexto da teoria de transição de fase de segunda ordem de Landau. Por último fazemos um estudo do emaranhamento entre bipartições de sistemas descritos por estados de Dicke, que nos permite a implementação de testemunhas de emaranhamento. Para cada estado de Dicke, apresentamos uma testemunha com alta sensibilidade.

Palavras-chave: Emaranhamento quântico. Estados de Dicke. Fenômenos críticos. Troca de emaranhamento. Testemunhas de emaranhamento.

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Introduction

The term entanglement was formally introduced by E. Schrödinger (1935) in a work entitled "Discussion of Probability Relations Between Separated Systems" [Sch35]. In this work Schrödinger uses this term to describe what he claims to be the phenomenon of quantum mechanics that departs the most from the classical world. In his own words:

> I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.

Earlier, in the same year, entangled states had been the main structure of an argument presented by Einstein, Podolsky and Rosen [EPR35] aiming at proving that quantum mechanics was an incomplete theory, meaning that it would not completely describe all elements of physical reality. According to them a satisfactory theory must not only be correct, agreeing with experiments, it must also be complete, i. e., "...every element of the physical reality must have a counterpart in the physical theory". The argument relied on the assumption of local realism, i.e., elements of reality, the spin of a particle for example, cannot be affected by events that happen at arbitrary distances within arbitrary time intervals. This argument was named Einstein-Podolsky-Rosen paradox, or yet EPR paradox.

The Einstein-Podolsky-Rosen paradox brought up an apparent problem that led to the search for a hypothetical theory capable of describing reality in a more trustworthy way than quantum theory. This search had a huge turnover in 1964 with the publication of the paper "On the Einstein-Podolsky-Rosen paradox" by John S. Bell [Bel64] in which entanglement was used to demonstrate, through the violation of an inequality, (later named Bell's inequality) that no theory restricted to local realism could possibly agree with the predictions of quantum theory. Further works tested this violation on experiments and the violation predicted by the quantum theory was indeed observed [CHSH69, FC72, ADR82, Asp99, RKM$^+$01], changing the way many scientists understood reality.

As is illustrated above, a better understanding of entangled states even in very simple systems has been pushing forward our knowledge. Since the advent of the EPR paradox, a certain attention was directed to this kind of state. Nevertheless it took time for scientists to realize the potential behind this phenomenon. It was, however, a few years after the publication of Bell's paper, that we had the first experimental realization of entangled states. In 1967 Kocher and Commins [KC67] reported the execution of an experiment which consisted in observing correlations in the linear polarization of photons which were emitted in a cascade decay process in calcium atoms. R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki [HHHH09]

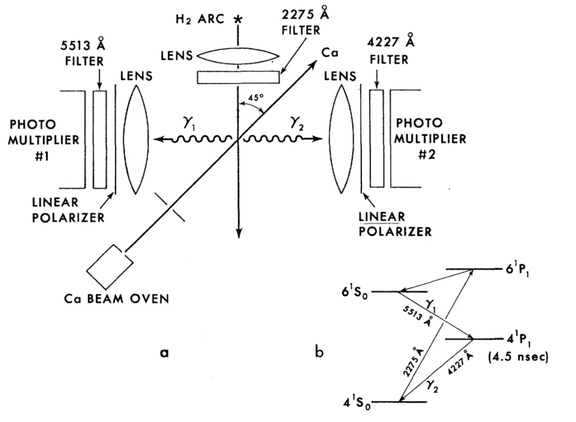

FIG. 1. (a) Schematic diagram of apparatus. (b) Level scheme for calcium.

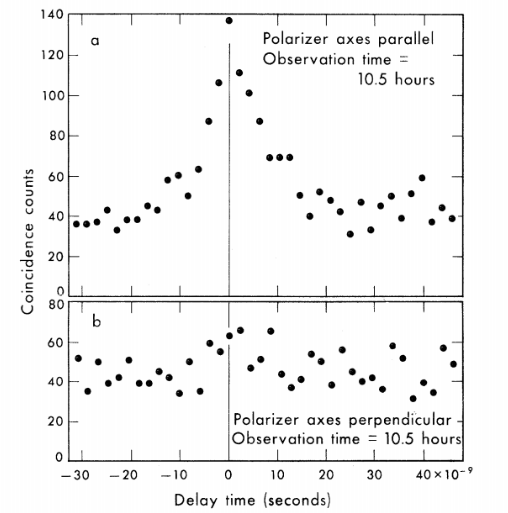

FIG. 2. Coincidence counts, as a function of relative time delay, showing polarization correlation. Each point represents a sum over three analyzer channels. Very slight peak in (b) is probably due to 6% transmission of crossed polarizers.

Figure 1.1 In the left, the setup used by Kocher and E. D. Commins and the scheme of levels for the atom of calcium [KC67]. In the right, the plot of the data from the realization of the experiment [KC67]. Both pictures are from the original paper.

regard the work of C. A. Kocher and E. D. Commins as possibly the first experimental demostration of entangled states. Considering the remarkable historical value of this work, we present in figure the setup and the data plot of what was perhaps the first time entanglement was intentionally created in the controlled environment of a laboratory. In the left part of figure it is displayed the experimental setting employed, the setting was based on the decay of an atom of calcium (whose detailed transition can be found in the same figure), which generates two photons in the singlet state of polarization, represented by γ_1 and γ_2. The photons are directed to distinct polarizers and, after that to the detectors. In the right side a plot of the coincidence of clicks in both detectors is depicted for two cases: down we have the axes of the polarizers seted perpendicular to each other leading to a constant signal, up we have a parallel setting of the polarizers, revealing a peak when the hits happens simultaneously as expected for the singlet state.

In a first moment the efforts towards the experimental realization of entanglement were mainly directed to observe the correlations predicted by the EPR paradox. Naturally, after 1964, with the publication of Bell's work, the interest in testing Bell's inequality (which would only be tested with more rigorously in 1981 [AGR81]) took the lead of this struggle. It was only in the beginning of the decade of 1990, as some theoretical proposals of using entanglement as a tool to perform fundamental tasks of several areas of applied sciences were presented to the scientific community, that the interest of experimentally implementing such states grew with a richer structure.

Amid these groundbreaking theoretical proposals, reference [HHH96] highlights five of them: quantum key distribution [Eke91], quantum dense coding [BW92], quantum teleportation [BBC$^+$93], entanglement swapping [ZZHE93, YS92] (which will receive further attention in the next chapter), and the capacity of overcome the limitations of classical communication necessary to solve problems distributed among separated parts [BCVD01]. The first of these advances was made in Ekert's work [Eke91], in 1991, entitled “Quantum cryptography based on Bell's theorem”, which showed that entangled states could be used to establish a cryptographic key shared between two separated parts in a way that no eavesdroper could ever spy (to obtain information about the key without being detected). After that, Bennett and Wiesner [BW92] showed that it is possible, by using entangled states, to store more information in one qubit than what was predicted by the Holevo bound [NC02]. Following the chronological line, in 1993, Bennett *et al.* [BBC$^+$93] published the seminal work “Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels”. In 1992-1993 two papers [YS92, ZZHE93] were published showing that it was possible to implement quantum correlations between separated parts that never met or interacted, a process that was called entanglement swapping and to which we devote special attention in what follows. The last of these outstanding works was published in 2001 by Buhrman *et al* [BCVD01], showing the advantages of employing entanglement to assist distributed computation.

It was also in the beginning of the 90's that a new interest in a different kind of entangled states emerged: multipartite entangled states. It is important to make it clear, however, that this kind of state appeared in the literature quite earlier. In fact the work of R. H. Dicke [Dic54], which reports on the kind of multipartite entangled states addressed in the present work, was published in 1954. By new interest we mean the interest in the states themselves as tools that could facilitate computational or informational tasks.

The work which seems to be the starting point in the use of multipartite states as an investigative tool dates from 1990. In that year Daniel M. Greenberger, Michael A. Horne, Abner Shimony, and Anton Zeilinger, [GHSZ90] published the manuscript “Bell's theorem without inequalities” where they used the entangled qubit state $|GHZ\rangle$ and for the case of four qubits can be defined as follows:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right). \tag{1.1}$$

Their work, as the title suggests, demonstrated that correlations beyond classical limits could be observed between separated entangled systems. Reference [Bar09] presents a very didactic formulation of this work, however this topic is outside the scope of this thesis.

From that point, in the beginning of the 90's, with all these mentioned publications, it would be reasonable to say that a new area was established in theoretical physics, devoted to study the mechanisms, applicability and nature of entangled states. Entanglement started to be treated as a resource, as it is fundamental to the execution of several new procedures. In 2001, with the publication of “A one-way quantum computer” by Robert Raussendorf, and Hans J. Briegel [RB01] raising the possibility of universal computation with single qubit operations plus multipartite entanglement this approach received even more attention. In fact, modern books as reference [NC02] start the chapters dedicated to this subject already introducing concepts of resource theory and in the review “Quantum entanglement” [HHHH09] it is presented as the “feature of quantum machinery which lies at the center of interest of physics of the 21st century”.

One interesting fact about multipartite entanglement is that it has a far richer structure than bipartite entanglement. When we consider two entangled qubits, any realization of their entanglement is, in a way, equivalent to each other. This is not the case when we deal with multipartite entanglement. In this case there are completely inequivalent ways of having entanglement among the parts that compose the system. As an illustration, we recall the work of Wolfgang Dür, Guifre Vidal, and J Ignacio Cirac [DVC00], entitled “Three qubits can be entangled in two inequivalent ways”, published in 2000, which shows that for three partite qubits systems there is a kind of entanglement inequivalent to that of a GHZ state (the context of this nonequivalence is addressed forward in this thesis), a state they called W state, which can be represented in the following way:

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right). \tag{1.2}$$

Interestingly, this state, is just a special case of an already known class of states called Dicke states [Dic54]. Despite the long time since its discovery, the properties of Dicke states are modestly known, when compared with that of GHZ states or cluster states [BR01]. While we have many generalizations of procedures and structures used in bipartite systems to multpartite systems in GHZ states, almost none of them has a similar form for Dicke states. However, it is already known that Dicke states are more resistant to environment effects than GHZ states and in the specific scenario of an even number of parts, say $2k$-parts, the singlet state is a particular case of Dicke state, and thus we can add to the applicability of these states all the machinery of singlet states reported by Adán Cabello in the manuscript “Supersinglets”[Cab03], as for instance the solution of the N stranges problem, the secret. sharing problem and the liar detection problem. Also, Dicke states have been used to detect multipartite entanglement [LPV$^+$14] and several reports highlights its optimal metrological properties [KSW$^+$11] .

The next chapter, is a collection of tools that we consider to be important for the reminder of this thesis. First we present a formal definition of entanglement. Although in this work we are mostly concerned with pure states, the definition will be presented in the general case for the sake of completeness. After that, we discuss methods of quantifying entanglement, in the perspective of local operations and classical communication, by introducing the concepts of entropy of entanglement and entanglement of formation. Then, we present a section about entanglement witnesses, where we address this subject in a direct and simple way. In the fourth section the concept of entanglement swapping is discussed in a didactic way. The last section presents the Dicke state, a family of states whose features are the core of this thesis.

The introduction of Dicke states is provided through the formalism of creation and annihilation operators.

In chapter 3 we start by showing that a process similar to entanglement swapping can be used to remotely create W states (or yet three partite Dicke states) in a deterministic way. The process is based on a joint entangled measurement in one part of each qubit of three Einstein-Podolsky-Rosen entangled pairs [MCP16]. The joint measurement is performed in a complete basis build with states equivalent to W states under single-qubits operations. After that we show that if the we relax the determinism condition it is possible to have perfect outputs whenever the protocol succeeds. We argue that this phenomenon amounts to a process of entanglement concentration.

In the fourth chapter we provide a generalization of the protocol of chapter two for the case of n qubits. The process here is no longer deterministic. A sketch of an experimental realization is provided which would in principle be performed using only components of linear optics and sources of EPR entangled pairs. Curiously by tracking the probability of each output as a function of the source-state entanglement, we observe a phenomenon analogous to a second order phase transition in the context of the classical theory of Landau, which allows us to adjust which kind of Dicke state we want to produce with higher probability. The phenomenon of critical transition is characterized, and through some relations present in the system we can effectively establish an upper bound to the amount of entanglement between partitions of Dicke states based on the amount of entanglement present in the EPR pair used in the protocol.

Chapter five introduces a different way of expressing a general Dicke state. With this we are able to derive a general form for the Schmidt decomposition of any possible bipartition of the system. This allows us to quantify the amount of entanglement between these bipartitions and compare it with the GHZ case. That also allows us to devise whether or not different Dick states are equivalent to each other. Furthermore we use these results to provide optimal entanglement witnesses for each possible Dicke state, specifying its tolerance against noise admixture and imperfections in entanglement.

Chapter six is entitled conclusion and reiterate in few words the results of this theses.

Entanglement

Let us start this chapter by introducing the concept of entanglement in the simplest scenario, a bipartite system, and then generalize it to the more complex case of multipartite systems. Consider two systems A and B whose states lie inside the Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively. The dimension of the first Hilbert space, $\mathscr{H}_A$, is denoted by d_A and $\{|a\rangle\}_{a=1...d_A}$ form an orthogonal basis of $\mathscr{H}_A$. Likewise, the dimension of $\mathscr{H}_B$ is denoted by d_B and it is spanned by the basis $\{|b\rangle\}_{b=1..d_B}$. Hence an arbitrary pure state can be represented by a ket $|\Psi\rangle$ of the joint system AB, in the total Hilbert space $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$, where $\otimes$ stands for tensor product. The global state can be written as:

$$|\Psi\rangle = \sum_{a=1}^{d_A} \sum_{b=1}^{d_B} \gamma_{ab} |a\rangle \otimes |b\rangle, \tag{2.3}$$

where $\sum_{a=1}^{d_A} \sum_{b=1}^{d_B} |\gamma_{ab}|^2 = 1$.

Whenever $|\Psi\rangle$ cannot be written as the tensor product of a state in $\mathscr{H}_A$ and a state in $\mathscr{H}_B$, it is said to be entangled. Otherwise we call it separable. More explicitly, when there are no $|\psi_A\rangle \in \mathscr{H}_A$ and $|\psi_B\rangle \in \mathscr{H}_B$ such that:

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle, \tag{2.4}$$

then $|\Psi\rangle$ is entangled. At this point we can see that it is characteristic of entanglement the connection to correlated states, because the absence of the form displayed in equation (2.4) indicates some superposition like $|\psi\rangle \otimes |\phi\rangle + |\psi^\perp\rangle \otimes |\phi^\perp\rangle$ (assuming $\langle \psi^\perp|\psi\rangle = 0$ and $\langle \phi^\perp|\phi\rangle = 0$) which has outputs correlated under a measurement defined by the detection operators $|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$ and $|\psi^\perp\rangle\langle\psi^\perp| \otimes |\phi^\perp\rangle\langle\phi^\perp|$. The kind of correlation that emerges from these states cannot be reproduced classically, as will become clear when we consider mixed states.

The generalization of this concept from two parts to n parts compounding a system is straightforward. If we have a set of systems A_1, A_2, ..., A_n whose Hilbert spaces are $\mathscr{H}_1$, $\mathscr{H}_2$, ..., $\mathscr{H}_n$ respectively, a pure state $|\Psi\rangle$ in $\mathscr{H}_{tot} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes ... \otimes \mathscr{H}_n$ is said to be separable if there is a set of states $\{|\psi_l\rangle \in \mathscr{H}_l\}_{l=1,2,...,n}$ such that:

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle. \tag{2.5}$$

Otherwise it is said to be entangled.

Despite the simplicity of this generalization it is important to notice that as the number of parts compounding the system grows, new inequivalent ways to entangle subsystems may emerge in the context of resource theory. It is also possible to have entanglement between specific parts of the system while others remain completely separable, as pointed out in reference [HHHH09].

The appearance of new structures of entanglement is observed already in three-partite systems of qubits. In this case two parts can be entangled and one separable, representing a bipartite-like entanglement. But we also may have all parts entangled, and this can happen in two different ways: GHZ-like or W-like [DVC00]. We will come back to this subject latter.

Having in mind the definition of entanglement for pure states, we can extend it to a more complex scenario where we may have a source randomly generating states $|\Psi^{(i)}\rangle$ each of them with probability p_i (imposing the natural restriction $\sum_i p_i = 1$), or more precisely a scenario where we have an ensemble composed of different quantum states. In this case the system can no longer be completely described by a ket in $\mathscr{H}_{tot}$. Instead we will need a new tool to describe this ensemble of states. This tool is a Hermitian operator named density operator (or yet state operator) which is usually represented by:

$$\rho = \sum_i p_i |\Psi^{(i)}\rangle\langle\Psi^{(i)}|. \tag{2.6}$$

Given a system with Hilbert space $\mathscr{H}$, the density operator is an element of the set $\mathscr{L}(\mathscr{H})$ of operators acting in $\mathscr{H}$. The formalism behind the density operator is of common knowledge and, among all of its special features, the most relevant, for the definition of entanglement, to be highlighted here is that the representation of ρ is not unique. In fact, unless the state is pure, there are infinite different representations of ρ using a convex combination (a combination in which the sum of the coefficients, which are always positive, is equal to 1) of other density operators. In this case the definition of entanglement acquires a more subtle form. An state ρ of n parts is said to be separable if, and only if, there is at least one possible representation for ρ with the following structure:

$$\rho = \sum_q \tilde{p}_q \bigotimes_{r=1}^{n} \tilde{\rho}_r^{(q)} \tag{2.7}$$

In equation (2.7), ρ is described in therms of the operators $\tilde{\rho}_r^{(q)}$ which are density operators in $\mathscr{L}(\mathscr{H}_r)$ and $\sum_q \tilde{p}_q = 1$.

The formulation of separability as presented above, enlightens what the realization of an entangled state is. It is well known that any classical correlation between systems has to have the form of equation (2.7), while entangled states are literally defined as something else. Entangled states go beyond by presenting correlations that can not be reproduced in a classical system.

2.1 Quantifying entanglement

Entanglement allowed for a plethora of new protocols, most of them on processing and transporting information. All these advances are related to the possibility of having systems more correlated than one would have classically. Hence there was an increasing interest into this extra capacity of correlation which motivated the perspective of entanglement as a resource, and to build up a theory to establish whether this resource is available and the amount of it needed to perform some tasks [BP08, BHO$^+$13, BHN$^+$15]. This kind of approach is called resource theory. To build this kind of theory it is necessary to define in what context the object under study represents indeed a resource. In this case the context was pretty clear: entanglement represents a resource over local operations and classical communication (a formal introduction of this concept will be provided in what follows).

The demand of the capacity to quantify the amount of entanglement available in one system, inherent of this formulation, however is a problem of high complexity and has been the core of several discussions in the field. V. Vedral *et al.* presented a work in 1997 [VPRK97], of which this section borrows its title, listing all necessary features that a good measure of entanglement should have. In the mentioned reference the authors show an interesting measure based on distance. However it is important to stress that all of their arguments concern only bipartite systems of arbitrary dimensions. The desired features of a measure of entanglement $E(\rho)$ for a state ρ are listed bellow.

- First, an entanglement measure must be zero only when the state is completely separable. That is expected, taking into account that from n copies of non separable systems is always possible to extract asymptomatically an amount m of maximally entangled states, a process usually named as entanglement distillation [HW97, BBP$^+$96].

- The second requirement is that $E(\rho)$ must be invariant under local unitary operations. The idea behind this requirement is the fact that local unitary operations represent only local change of basis and thus it should not increase nor decrease any correlation among separate parts.

- Finally, the third feature refers to local operations and classical communications (LOCC) and can be thought as a generalization of the second requirement. At this point it is important to define what we consider as local operation: by LOCC we mean any deterministic evolution, whether achieved by unitary transformation or by positive operator valued measurements (POVM) classically correlated. This kind of process, for a bipartite system, was generalized in in reference [VPRK97]. Here we present a generalization for the n-partite case:

$$\rho \to \sum_j A_1^{(j)} \otimes A_2^{(j)} \otimes \ldots \otimes A_n^{(j)} \rho A_1^{(j)\dagger} \otimes A_2^{(j)\dagger} \otimes \ldots \otimes A_n^{(j)\dagger}, \tag{2.8}$$

where we must have:

$$\sum_j A_k^{(j)} A_k^{(j)\dagger} = 1, \tag{2.9}$$

for all k.

The third requirement is based in equation (2.8) itself, from which it is evident that this kind of operation can only create classical correlations.Thus we state it in the following way: local operations and classical communication can only decrease the amount of entanglement observed in a system.

In this section we are going to explore two of those concepts of measures: entropy of entanglement and entanglement of formation.

2.1.1 Entropy of entanglement

Entropy of entanglement is a measure for pure states of bipartite systems, for which we always have a Schmidt decomposition. Any pure bipartite state $|\Psi\rangle$ in $\mathscr{H}_{tot} = \mathscr{H}_A \otimes \mathscr{H}_B$, composed by systems A and B, are described by:

$$|\Psi\rangle = \sum_j \sum_k \lambda_{jk} |a_j\rangle_A \otimes |b_k\rangle_B, \tag{2.10}$$

where the labels A and B indicates the ket belongs to $\mathscr{H}_A$ and $\mathscr{H}_B$, respectively. This state can also be written in the following way:

$$|\Psi\rangle = \sum_j \lambda'_j |a'_j\rangle_A \otimes |b'_j\rangle_B, \tag{2.11}$$

where $|a_j\rangle_A$ and $|a'_j\rangle_A$ are related trough local unitary operations as well as $|b_j\rangle_B$ and $|b'_j\rangle_B$. This is what we call the Schmidt decomposition.

Not only the Schmidt decomposition exists for every pure bipartite state, but also it is unique [NC02]. It is characterised by the values of λ'_j and the amount of them that are non-vanishing , which is usually referred to as the Schmidt number (which is itself a good measure of entanglement). This decomposition gives us a direct notion of entanglement: whenever the Schmidt number is one, the state is separable, otherwise it is entangled.

Considering the density operator representing the same system, we can write:

$$\rho_{tot} = |\Psi\rangle\langle\Psi| = \sum_j \sum_k \lambda'_j \lambda'^*_k |a'_j\rangle\langle a'_k|_A \otimes |b'_j\rangle\langle b'_k|_B. \tag{2.12}$$

So if we trace one of the parts we have either ρ_A or ρ_B, representing the reduced matrix of systems A and B:

$$\begin{cases} \rho_A = \sum_j |\lambda_j|^2 |a'_j\rangle\langle a'_j|_A \\ \rho_B = \sum_j |\lambda_j|^2 |b'_j\rangle\langle b'_j|_B \end{cases}. \tag{2.13}$$

As equation (2.13) shows, both density operators have the same eigenvalues. Only for separable states, or yet whenever we have $\lambda'_j = 1$ for a specific j, the partial density operators represent a pure state. Hence the purity of ρ_A (or ρ_B) agrees with the first requirement of a measure of entanglement. The closer of $\dim(A)^{-1}$ the value of $|\lambda'_j|^2$ gets, for all j, and consequently ρ_A and ρ_B approach a maximally mixed state, the closer to the maximally entangled system is $|\Psi\rangle$. It is also a fact that the eigenvalues of ρ_A (or ρ_B) do not depend on local changes of basis, thus they are invariant under local unitary operations, satisfying the second requirement for an entanglement measure. Further it is possible to see, from equation (2.8), that POVMs can only take farther away the system from the maximally entangled state.

These arguments show that a function of $\{|\lambda'_j|^2\}_{j=1,\dots}$ which is capable of detecting whether the partial systems are pure states or statistical mixtures, and if the last is true, how close the partial density operator of each system is from a maximally mixed state would be a good measure of entanglement.

This leads us to the von Neumann entropy defined as:

$$S(\rho) = -\operatorname{Tr}(\rho \log \rho). \tag{2.14}$$

In equation (2.14), Tr(...) is the trace of the argument, and we take log in base 2 for convenience.

The von Neumann entropy is a positive function of ρ and consequently of its eigenvalues. The entropy is zero if and only if the state is pure. Also the maximum value of $S(\rho)$ happens when ρ is a maximally mixed state.

Regarding this scenario it is quite natural to define the entanglement amount S_{ent} for a bipartition of a pure state as the von Neumann entropy of the partial trace of its density operator:

$$S_{ent} = S(\rho_A) = S(\rho_B). \tag{2.15}$$

This quantity is called entropy of entanglement.

A very interesting property of entropy of entanglement for an arbitrary state $|\Psi\rangle$ of qubits, which by the Schmidt decomposition can always be written as $\lambda_1|00\rangle + \lambda_2|11\rangle$, is that it quantifies the amount of asymptotically perfect entangled states $\frac{1}{\sqrt{2}}(|00\rangle + |11)$ that can be produced from a infinite set of copies of $|\Psi\rangle$ using only classical communications and local procedures. More precisely, assume that we have a set of n copies of $|\Psi\rangle$, then it is possible to extract $m < n$ copies of $|\Psi'\rangle = a|00\rangle + b|11\rangle$, and defining $\delta = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)|\Psi'\rangle$, we must have:

$$\begin{cases} \lim\limits_{n\to\infty} \delta = 1 \\ \lim\limits_{n\to\infty} \left(\frac{m}{n}\right) = S_{ent}(|\Psi\rangle) \end{cases} \tag{2.16}$$

This process is called entanglement distillation. It was introduced by Bennett, Popescu and Schumacher in reference [BBPS96], and in this context $\frac{m}{n}$ is also a good way for quantifying entanglement in more general scenarios. However when we go beyond pure bipartite states of two qubits, the evaluation of the fraction $\frac{m}{n}$ becomes a challenging task.

2.1.2 Entanglement of formation

When dealing with more complex states, in general, we will employ a measure called entanglement of formation. Particularly, this concept seems to be the most well succeeded among many other potential entanglement measures. For instance, by employing the measure of distillable entanglement (the proportion of maximally entangled states that can be extracted from some ensemble ρ [BDSW96]), as pointed out by Wootters in reference [Woo01], one can find systems presenting entanglement for which this measure gives zero, a kind of entanglement classified as bound entanglement [HHH98]. In the same reference Wootters also points out some unsolved problems with the entropy of formation, an important example being the problem of additivity: it is not known whether the entropy or formation of a set of n copies of a quantum state is n times the entropy of formation for a single copy of the same system.

The idea behind this new quantity is to describe the amount of maximally entangled states necessary to produce some mixed state ρ. This quantity is defined in reference [BDSW96], and has the following form:

$$E \equiv \min(\sum_j p_j S_{ent}(|\phi_j\rangle)), \tag{2.17}$$

where S_{ent} is defined in equation 2.15.

In this equation, p_j and $|\phi_j\rangle$ are such that we can write:

$$\rho = \sum_j p_j |\phi_j\rangle\langle\phi_j|, \tag{2.18}$$

and the minimization is over all possible representations of ρ.

The quantity E is called entanglement of formation and, from equation 2.17, it becomes clear that it is nothing but the minimal mean value of the entropy of entanglement, which is a intuitive way of generalizing this measure. However, it must be considered that mixtures of entanglement can generate classical correlations that may destroy the quantum ones, as in the case of state:

$$\rho_{wn} = \frac{1}{4}\left(|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|\right), \tag{2.19}$$

where:

$$\begin{cases} |\Phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \\ |\Psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right) \end{cases} \tag{2.20}$$

form a basis of a two-qubit system (from now on we will refer to these states as EPR states, a reference to the work presented by Einstein, Podolsky and Rosen [EPR35]). It is easy to see that in this case ρ_{wn} can also be written in the following way:

$$\begin{aligned} \rho_{wn} &= \frac{1}{4}\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\langle 01| + | + |10\rangle\langle 10|\right) \\ &= \frac{1}{4}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \otimes \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right), \end{aligned} \tag{2.21}$$

which represents a system containing only classical correlations, whose implementation would not require any level of entanglement. Thus justifying the need of taking the minimum value over all the representations of a state.

2.2 Entanglement witness

In the previous sections we presented concepts with which it is possible to quantify the amount of entanglement present in a system. However those concepts have a very limited structure, they can only evaluate entanglement between bipartitions of the entire system. Even when dealing with bipartitions, if the state of the system of interest is not pure the quantity E, defined in equation (2.17), and many others, is a convex roof construction being very hard to be evaluated (though recently a great progress was done in this area and reported in reference [TMG15]).

Fortunately in 1996 M. Horodecki, P. Horodecki and R. Horodecki published a work analyzing sufficient and necessary conditions for biseparability of mixed states [HHH96]. This section aims to provide a detailed formulation of a concept build up in one of the results reported by the authors: the concept of entanglement witness. Hence many steps presented here have the same structure of those present in the referred paper. Given the nature of the results presented here, this section will have a slightly different approach from the rest of this work, being more mathematical in nature.

First we go back to equation (2.7), where separable states are defined. Accordingly, a biseparable density operator ρ, defined in the space $\mathscr{L}(\mathscr{H}_{tot})$ of linear operators acting in $\mathscr{H}_{tot} = \mathscr{H}_A \otimes \mathscr{H}_B$ would be characterized by the following structure:

$$\rho = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)}. \tag{2.22}$$

Where we must have p_k real and positive, and $\sum_k p_k = 1$. In addition $\rho_k^{(A)}$ and $\rho_k^{(B)}$ are density operators inside the set o linear operators acting on $\mathscr{H}_A$ and $\mathscr{H}_B$, respectively.

Now some properties of the set of separable states must be highlighted. First, it is a closed set, for entangled states can present arbitrarily low quantum correlation (for example, the entropy of entanglement as well as the entropy of formation are continuous and they go to zero if and only if the state is separable) implying that there are entangled states lying in the neighborhood of separable ones.

The second feature of the set of separable states that must be emphasized here is that, by definition, it is a convex set. This means that any convex combination of a separable state is also a separable state.

The last important characteristic of this set is that it is a subset of a Banach space. This comes from the fact that they are a subset of the Hilbert-Schmidt space, which is a complete normed space and thus a Banach space.

Those features are important because they allow us to use the Hahn-Banach Theorem, which, quoting reference [HHH96], states that "If W_1 and W_2 are convex closed sets in a real Banach space and one of them is compact, then there exists a continuous functional f and $\alpha \in \mathbb{R}$ such that for all pairs $w_1 \in W_1$, $w_2 \in W_2$ we have: $f(w_1) < \alpha \leq f(w_2)$". Thus if we choose the set of all separable states A to play the role of W_2, and take a set composed by a single arbitrary entangled state ρ, which form a convex complete compact set, to be W_1, for $\sigma \in A$ and some $\alpha \in \mathbb{R}$ we can find a functional f so we can write:

$$f(\rho) < \alpha \leq f(\sigma). \tag{2.23}$$

Now recalling that the set of functionals on a vector space is isomorphic to the vector space itself, that our functional is real and that the scalar product of the Hilbert-Schmidt space is defined as the trace $\langle A, B\rangle = \mathrm{Tr}(B^\dagger A)$, we can represent any functional of one element in the set of operators as the trace between this element and another operator $W = W^\dagger$:

$$\mathrm{Tr}(W\rho) < \alpha \leq \mathrm{Tr}(W\sigma). \tag{2.24}$$

Once the trace of any density operator is one, we can define the operator $\mathbf{W} = W - \alpha \mathbb{1}$, so that we still have $\mathbf{W} = \mathbf{W}^\dagger$ and:

$$\begin{cases} \mathrm{Tr}(\mathbf{W}\rho) & < & 0 \\ \mathrm{Tr}(\mathbf{W}\sigma) & \geq & 0 \end{cases}. \tag{2.25}$$

This means that given an entangled state ρ it is always possible to find an observable whose mean value is negative if and only if the state is entangled. We will call these observables entanglement witnesses.

In the same reference [HHH96] mentioned before, there is also a theorem that provides a necessary and sufficient condition for separability. The theorem is formulated in the following way:

Theorem 1. A state $\rho \in \mathscr{A}_1 \otimes \mathscr{A}_2$ is separable iff

$$\mathrm{Tr}(A\rho) \geq 0$$

for any Hermitian operator A satisfying $\mathrm{Tr}(AP \otimes Q) \geq 0$, where P and Q are projections acting on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

In the above quote, $\mathscr{A}_1$ and $\mathscr{A}_2$ represent the space of linear operators acting on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

To prove the above theorem first we consider the case where ρ is separable. In this case the theorem is satisfied by the definition of A itself. Assuming now that ρ is inseparable and nonetheless there is an operator $\mathbf{W}$ such that $\mathrm{Tr}(\mathbf{W}P \otimes Q) \geq 0$, for P and Q representing projections in separated Hilbert spaces, and $\mathrm{Tr}(\mathbf{W}\rho) \geq 0$, we have a contradiction: by the previous statement, if ρ is inseparable, there must exist a hermitian operator $\mathbf{W}'$ such that $\mathrm{Tr}(\mathbf{W}'\rho) < 0$ and $\mathrm{Tr}(\mathbf{W}'\sigma) \geq 0$ for all separable states σ, however a separable state happens to be the tensor product of projections, hence the features of $\mathbf{W}'$ agree with the definition of $\mathbf{W}$ itself, and lead us to the relation $\mathrm{Tr}(\mathbf{W}\rho) < 0$.

Entanglement witnesses not only detect entanglement but specifically they detect genuine multipartite entanglement, because in a genuine multipartite entangled system there are no bipartitions in which it is separable, all parts must be entangled with each other.

As an example of these observables we mention the work of M. Bourennane *et al* [BEK$^+$04], where a general form to build entanglement witnesses is provided. The proposed observable W reads:

$$W = \beta \mathbb{1} - |\psi\rangle\langle\psi|. \tag{2.26}$$

The form above depends on the previous knowledge of a genuinely entangled state $|\psi\rangle$. Also, the scalar β is defined as the maximum value of the modulus squared of the inner product of $|\psi\rangle$ and a biseparable state:

$$\beta = \max_{|\phi\rangle \in B} \left(|\langle\phi|\psi\rangle|^2 \right). \tag{2.27}$$

To show that W is indeed a witness of entanglement all we have to do is to show that for any projectors P and Q of any bipartition $\mathscr{H}_A \otimes \mathscr{H}_B$ of $\mathscr{H}_{tot}$, we have $\mathrm{Tr}(WP \otimes Q) \geq 0$. This property follows immediately from the definition of β. In order to provide an efficient way to evaluate β, the authors prove that it is "given by the square of the Schmidt coefficient, which is maximal over all possible bipartite partitions of $|\psi\rangle$".

2.3 Entanglement swapping

As entanglement became a powerful resource, the problem of its distribution among spatially separated parts turned out to be a relevant issue in the field of quantum information.

In this context, in 1992 Bernard Yurke and David Stoler showed, in reference [YS92], the possibility of observing entanglement among two particles coming from different, completely uncorrelated, and far apart sources. A simplified description of their work is presented in this section.

Consider two different sources of the EPR states $|\Phi^+\rangle$, S_1 and S_2, working independently. S_1 sends one of its particles to Alice and the other to Charlie, while the second source S_2 sends one particle to Charlie and one to Bob, the situation is depicted in figure 2.2. The total state of the system is given by the tensor product of the state of each source:

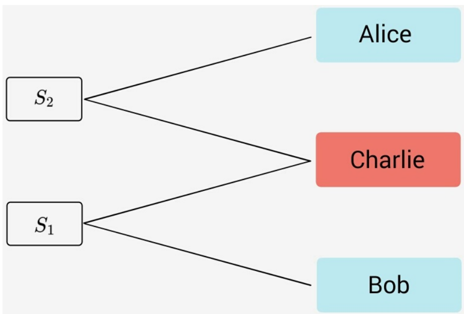

Figure 2.2 Two sources of EPR pairs S_1 and S_2 are represented. S_1 sends one of the particles to Alice and the other to Charlie, S_2 sends one to Bob and one to Charlie

$$|\Psi\rangle = \frac{1}{2}\left(|0_A 0_C\rangle + |1_A 1_C\rangle\right) \otimes \left(|0_C 0_B\rangle + |1_C 1_B\rangle\right), \tag{2.28}$$

here the index indicates who is holding each particle.

It is possible to rewrite $|\Psi\rangle$ in the following way:

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{2}\left(|0_C 0_C 0_A 0_B\rangle + |0_C 1_C 0_A 1_B\rangle + |1_C 0_C 1_A 0_B\rangle + |1_C 1_C 1_A 1_B\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\big(|\Phi_C^+\rangle|0_A 0_B\rangle + |\Phi_C^-\rangle|0_A 0_B\rangle + |\Psi_C^+\rangle|0_A 1_B\rangle + |\Psi_C^-\rangle|0_A 1_B\rangle \\
&\quad + |\Psi_C^+\rangle|1_A 0_B\rangle - |\Psi_C^-\rangle|1_A 0_B\rangle + |\Phi_C^+\rangle|1_A 1_B\rangle - |\Phi_C^-\rangle|1_A 1_B\rangle\big) \qquad (2.29) \\
&= \frac{1}{2}\big(|\Phi_C^+\rangle|\Phi_{AB}^+\rangle + |\Phi_C^-\rangle|\Phi_{AB}^-\rangle + |\Psi_C^+\rangle|\Psi_{AB}^+\rangle + |\Psi_C^-\rangle|\Psi_{AB}^-\rangle\big). \qquad (2.30)
\end{aligned}$$

The collective index AB indicates that the particles in the labeled EPR state are those of Alice and Bob. This equation enlightens that whenever Charlie makes a measurement in the EPR basis, Alice's particle gets maximally entangled with Bob's one despite the fact that no interaction ever happened between them.

A generalization of this process for multipartite systems was proposed by Sougato Bose, Vlatko Vedral, and Peter L. Knight in reference [BVK98]. In their work, the process of swapping occurs between systems of arbitrary size in the GHZ state, which in the mentioned reference is called “cat states”. They show that:

> ...the entanglement swapping scheme of Zukowski et al. can be generalized to the case of starting with cat states involving any number of particles, doing local measurements by selecting any number of particles from the different cat states and also ending up with cat states involving any number of particles.

To show this mechanism, we are going to use a notation slightly different from the one used by the authors.

First, let us define the n partite GHZ state:

$$|GHZ_n^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n} \pm |1\rangle^{\otimes n}\right). \tag{2.31}$$

The power $\otimes n$ in equation (2.31) indicates the tensor product n times, for instance $|0\rangle^{\otimes 3} = |000\rangle$. Also in our demonstration we take into account that proving the relation for one element of a class of states defined by local operations is enough for prove it to all elements of the same class.

For some natural $l < n$ the state $|GHZ_n^{\pm}\rangle$ has the following property:

$$\begin{aligned}
|GHZ_n^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes l}\otimes|0\rangle^{\otimes(n-l)} \pm |1\rangle^{\otimes l}\otimes|1\rangle^{\otimes(n-l)}\right)\\
&= \frac{1}{2}\left[\left(|GHZ_l^{+}\rangle + |GHZ_l^{-}\rangle\right)\otimes|0\rangle^{\otimes(n-l)} \pm \left(|GHZ_l^{+}\rangle - |GHZ_l^{-}\rangle\right)\otimes|1\rangle^{\otimes(n-l)}\right]\\
&= \frac{1}{\sqrt{2}}\left(|GHZ_l^{+}\rangle\otimes|GHZ_{n-l}^{\pm}\rangle + |GHZ_l^{-}\rangle\otimes|GHZ_{n-l}^{\mp}\rangle\right)
\end{aligned} \tag{2.32}$$

Implying that if we perform a measurement in a complete GHZ basis in some elements of a GHZ state, the remaining system will be in a GHZ state too.

On the other hand, for natural numbers l, m, p, and q the following property also holds:

$$|GHZ_l^{+}\rangle\otimes|GHZ_m^{+}\rangle = \frac{1}{2}(|0\rangle^{\otimes p}|0\rangle^{\otimes(l-p)} + |1\rangle^{\otimes p}|1\rangle^{\otimes(l-p)})\otimes(|0\rangle^{\otimes q}|0\rangle^{\otimes(m-q)} + |1\rangle^{\otimes q}|1\rangle^{\otimes(m-q)})$$

Assembling p parts of the first system and q parts of the second together:

$$
\begin{aligned}
|GHZ_l^+\rangle \otimes |GHZ_m^+\rangle &= \frac{1}{2}\Big[|0\rangle^{\otimes(p+q)} \otimes |0\rangle^{\otimes(l+m-p-q)} + \\
&\Big(\mathbb{1}^{\otimes p} \otimes \sigma_x^{\otimes q}|0\rangle^{\otimes(p+q)}\Big) \otimes \Big(\mathbb{1}^{\otimes(l-p)} \otimes \sigma_x^{\otimes(m-q)}|0\rangle^{\otimes(l+m-p-q)}\Big) + \\
&\Big(\mathbb{1}^{\otimes p} \otimes \sigma^{\otimes q}|1\rangle^{\otimes(p+q)}\Big) \otimes \Big(\mathbb{1}^{\otimes(l-p)} \otimes \sigma_x^{\otimes(m-q)}|1\rangle^{\otimes(l+m-p-q)}\Big) + \\
&|1\rangle^{\otimes(p+q)} \otimes |1\rangle^{\otimes(l+m-p-q)}\Big]
\end{aligned}
$$

Writing the first $p+q$ qubits in terms of GHZ states:

$$
\begin{aligned}
|GHZ_l^+\rangle \otimes |GHZ_m^+\rangle &= \frac{1}{2\sqrt{2}}\Big\{\big(|GHZ_{p+q}^+\rangle + |GHZ_{p+q}^-\rangle\big) \otimes |0\rangle^{\otimes(l+m-p-q)} + \\
&\big[\mathbb{1}^{\otimes p} \otimes \sigma_x^{\otimes q}\big(|GHZ_{p+q}^+\rangle + \\
&|GHZ_{p+q}^-\rangle\big)\big] \otimes \Big(\mathbb{1}^{\otimes(l-p)} \otimes \sigma_x^{\otimes(m-q)}|0\rangle^{\otimes(l+m-p-q)}\Big) + \\
&\big[\mathbb{1}^{\otimes p} \otimes \sigma^{\otimes q}\big(|GHZ_{p+q}^+\rangle - \\
&|GHZ_{p+q}^-\rangle\big)\big] \otimes \Big(\mathbb{1}^{\otimes(l-p)} \otimes \sigma_x^{\otimes(m-q)}|1\rangle^{\otimes(l+m-p-q)}\Big) \\
&\big(|GHZ_{p+q}^+\rangle - |GHZ_{p+q}^-\rangle\big) \otimes |1\rangle^{\otimes(l+m-p-q)}\Big\}.
\end{aligned}
$$

Rearranging the terms of the above equation:

$$
\begin{aligned}
|GHZ_l^+\rangle \otimes |GHZ_m^+\rangle &= \frac{1}{\sqrt{2}}\Big[|GHZ_{p+q}^+\rangle \otimes |GHZ_{l+m-p-q}^+\rangle + |GHZ_{p+q}^-\rangle \otimes |GHZ_{l+m-p-q}^-\rangle + \\
&\big(\mathbb{1}^{\otimes p} \otimes \sigma_x^{\otimes q}|GHZ_{p+q}^+\rangle\big) \otimes \Big(\mathbb{1}^{\otimes(l-p)} \otimes \sigma_x^{\otimes(m-q)}|GHZ_{l+m-p-q}^+\rangle\Big) + \\
&\big(\mathbb{1}^{\otimes p} \otimes \sigma_x^{\otimes q}|GHZ_{p+q}^-\rangle\big) \otimes \Big(\mathbb{1}^{\otimes(l-p)} \otimes \sigma_x^{\otimes(m-q)}|GHZ_{l+m-p-q}^-\rangle\Big)\Big]
\end{aligned} \tag{2.33}
$$

Thus proving that whenever a joint measurement in the GHZ basis is performed among parts of two distinct systems in the GHZ state, the remainder of the system can be deterministically set in the GHZ state.

With those properties the general entanglement procedure, as described in the manuscript in question, is assured. This idea of entanglement swapping is very general and can be used to remotely manage entanglement among separate parts. The authors of reference [BVK98] presented some diagrams (reproduced here in figure 2.3) to illustrate possible structural changes in the entanglement of systems due this kind of entanglement swapping.

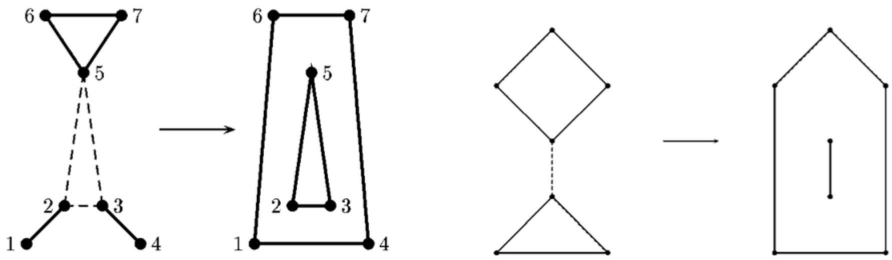

Figure 2.3 Figures from reference [BVK98]. Dots are particles and lines represent GHZ entanglement between the parts they connect. The dashed line represent measurements to be performed in a complete GHZ basis. In the left of each figure we have the states before measurement and in the right the state after measurement

2.4 Dicke states

Dicke states represent a class of entangled states defined for a system of n qubits. They where first reported in 1954 by R. H. Dicke [Dic54] in a study on the spontaneous emission of light from a system of two level particles in a small cavity in comparison with the wave length of the emitted radiation. Whenever a photon was emitted by the cavity, it was not possible to distinguish from which particle it came.

With this in mind it is expected that the idea of Dicke states must have a close relation with angular momentum theory. This must be true since the structure of a two level system is, in all aspects, similar to a particle with spin half; in a way they are pseudo spin-$\frac{1}{2}$ systems. In this section we review the phenomena studied by R. Dicke on the view of angular momentum theory, using creation and annihilation operators to provide a more generic introduction to Dicke states, considering only the points more relevant from a fundamental perspective. The details of theory of angular momenta that where not covered here can be found on reference [STC95].

First, we use the total angular momentum operators $\hat{J}_z$, $\hat{J}_x$, and $\hat{J}_y$ to define the creation and annihilation operators ($\hat{J}_+$ and $\hat{J}_-$):

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y; \tag{2.34}$$

The effect of the operator $\hat{J}_\pm$ acting on an eigenstate of $\hat{J}_z$, is given by:

$$\hat{J}_\pm|J,J_z\rangle = c^{\pm}_{J_z}|J,J_z+1\rangle, \tag{2.35}$$

where:

$$|c^{\pm}_{J_z}|^2 = \langle J,J_z|\hat{J}_\mp\hat{J}_\pm|J,J_z\rangle = J_z^{\max}\left(J_z^{max}+1\right) - J_z\left(J_z \mp 1\right).$$

So, except by a global phase:

$$c_{J_z}^{\pm} = \sqrt{\left(J_z^{max} \mp J_z\right)\left(J_z^{max} \pm J_z + 1\right)} \tag{2.36}$$

Now let us consider kets written in the computational basis $|j_1 j_2 ... j_n\rangle$, $j_i = 0, 1 \forall i$, and define the number of excitations k in the following way:

$$k \equiv \sum_{l=1}^{n} j_l. \tag{2.37}$$

It is possible to rewrite $c_{J_z}^{\pm}$ in therms of k and n:

$$c_{J_z}^{\pm} = c_k^{\pm} = \sqrt{\left(\frac{n}{2} \mp k \pm \frac{n}{2}\right)\left(\frac{n}{2} \pm k \mp \frac{n}{2} + 1\right)} \tag{2.38}$$

If the Hamiltonian is proportional to $\hat{J}_z$, the action of $\hat{J}_{\pm}$ is equivalent to an absorption or emission of a energy quantum. Hence, starting from the non excited state $|00...0\rangle$ the absorption of k quanta leads to an eigenstate of $\hat{J}_z$, which for convenience we will call $|D_n^{(k)}\rangle$, where $\hat{J}_z|D_n^{(k)}\rangle = \left(-\frac{n}{2}\right)|D^{(k)}\rangle$, so we can write:

$$|D_n^{(k)}\rangle = \frac{1}{C_k^{+}} \hat{J}_{+}^{k} |00...0\rangle. \tag{2.39}$$

The value of C_k^{+} is given by:

$$\begin{aligned} C_k^{+} &= \prod_{l=0}^{k-1} c_l^{+} \\ &= \sqrt{(n)(1)} \cdot \sqrt{(n-1)(2)} \cdot \ldots \cdot \sqrt{(n-k+1)(k)} \\ &= \left(\frac{n!}{(n-k)!} k!\right)^{\frac{1}{2}} \end{aligned} \tag{2.40}$$

Another feature of $\hat{J}_{\pm}$ is that, by its definition, we can write:

$$\hat{J}_{\pm} = \sigma_{\pm} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \quad + \quad \mathbb{1} \otimes \sigma_{\pm} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \quad + \quad \ldots \quad + \quad \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \sigma_{\pm}, \tag{2.41}$$

where $\sigma_{+}|0\rangle = |1\rangle$, $\sigma_{+}|1\rangle = 0$, $\sigma_{-}|0\rangle = 0$, and $\sigma_{-}|1\rangle = |0\rangle$.

Thus k applications of $\hat{J}_{+}$ can be described as follows:

$$\hat{J}_+^k = (\sigma_+ \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_+ \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \quad \ldots \quad + \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \sigma_+)^k. \qquad (2.42)$$

However $\sigma_+^l|\psi\rangle = 0$, $\forall|\psi\rangle$ whenever $l > 1$. Hence we will make $\sigma_+^l = 0$ for $l > 1$. This leads to:

$$\hat{J}_+^k = k! \sum_{l=1}^{\binom{n}{k}} \mathscr{P}_l^{(k)} \overbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}^{n-k} \otimes \underbrace{\sigma_+ \otimes \ldots \otimes \sigma_+}_{k}. \qquad (2.43)$$

Combining equations (2.39), (2.40), and (2.43), we have:

$$|D_n^{(k)}\rangle = \frac{1}{\binom{n}{k}^{\frac{1}{2}}} \sum_{l=1}^{\binom{n}{k}} P_l^{(k)} |\overbrace{0\ldots0}^{n-k}\underbrace{1\ldots1}_{k}\rangle. \qquad (2.44)$$

The same relation can be derived from successive applications of $\hat{J}_-$ on state $|11\ldots1\rangle$.

The state $|D_n^{(k)}\rangle$ represents a Dicke state with k excitations. As became clear by the above demonstration, these states are the result of a sequence of absorptions (emissions) of k $(n-k)$ energy quanta by a system of n pseudo spin-$\frac{1}{2}$ parties or, as we will refer from now on, n qubits.

To better illustrate, the two-qubit Dicke state with zero, one and two excitations reads:

$$\begin{cases} |D_2^{(0)}\rangle &= |00\rangle \\ |D_2^{(1)}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |D_2^{(2)}\rangle &= |11\rangle \end{cases} \qquad (2.45)$$

For a three-qubit Dicke states we have:

$$\begin{cases} |D_3^{(0)}\rangle &= |000\rangle \\ |D_3^{(1)}\rangle &= \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \\ |D_3^{(2)}\rangle &= \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle) \\ |D_3^{(3)}\rangle &= |111\rangle \end{cases} \qquad (2.46)$$

Notice that the W state is defined as $|D_3^{(1)}\rangle$.

Remote preparation of W states from imperfect bipartite sources

The inventory of potential achievements that quantum entanglement may bring about has steadily grown for decades. It is the concept behind most of the non-trivial, classically prohibitive tasks in information science [NC02]. However, for entanglement to become a useful resource in general, much work is yet to be done. The efficient creation of entanglement, often involving many degrees of freedom, is one of the first challenges to be coped with, whose simplest instance are the sources of correlated pairs of two-level systems. These sources have been used to demonstrate the possibility of teleporting an unknown qubit [BBC$^+$93] and to disclose the nonlocality of quantum mechanics [BCP$^+$14]. Some other important tasks require more involved kinds of entanglement, e. g., a prominent framework for measurement-based quantum computation is possible only if cluster sates are available [RB01].

In the last two decades the increasing interest in complex entanglement motivated a fair amount of works aiming at the creation of larger than Einstein-Podolsky-Rosen (EPR) states. Most of these efforts have produced Greenberger-Horne-Zeilinger (GHZ) states [GHSZ90] involving three [BPD$^+$99], four [SKK$^+$00], five [ZCZ$^+$04], six [LZG$^+$07], and eight [YWX$^+$12] qubits. Other multiqubit states have been built though more sparsely, for instance, Dicke states [WKK$^+$09] and graph states [LZG$^+$07]. Intermediate kinds of entanglement have been reported in Ref. [WY08].

In this chapter of the thesis we are mainly interested in the creation of tripartite W states, or yet tripartite Dicke states with one excitation. In the first experimental realization of such a state [EKB$^+$04], four photons originated from second order parametric down conversion (PDC) are sent to distinct spatial modes and through linear optical elements. The conditional detection of one of the photons leaves the remaining three photons in the desired W state with a probability of $1/32$ for each second order PDC event. Since these occurrences are by themselves rare, the whole process lacks efficiency. A sufficiently robust experiment to enable state tomography is described in Ref. [MLFK05], where three-fold coincidences were observed with a rate about 40 times higher than that of [EKB$^+$04]. However, the creation of W states remained probabilistic, thus, requiring postselection. In Ref. [MLK04], an experiment based on two independent first order PDC events is described, resulting in W states of four and three photons. The latter being achieved, again, only probabilistically, through a measurement on one of the parties. Projective measurements, that present intrinsic stochasticity, can also be used to produce W-type entanglement [WKSW09]. More in the spirit of the present work, bipartite entanglement can be considered as an available resource, as is the case of Ref.

[TWÖ+09], that takes two pairs of photons in EPR states as the building blocks to probabilistically produce W states of three parties. Very recently general tripartite entangled states were encoded in the nuclei of the fluorine atoms of trifluoroiodoethylene molecules, via nuclear magnetic resonance [DD+15]. This technique, however, is unsuited for remote preparation because the state is encoded in a spatially localized structure.

In the theoretical front, protocols based on PDC [YTKI02], linear optics [YBFGHY08, TÖY+08, TWÖ+09, TKÖ+10], sources of EPR states [XZL08], atomic systems [YZ11], and nitrogen-vacancy centers [TWC+14] can be found, all being probabilistic. In contrast, the authors of Ref. [ÖYB+15] propose a deterministic scheme to create four-partite W states, which can be extended to generate W states of arbitrary dimension [YBO+16]. More recently, a procedure was suggested to create entangled states of several degrees of freedom, which is particularly suited for W states. This fusion operation [ÖMT+11] corresponds to a swapping procedure and employs two entangled states with dimension d to produce an entangled state of dimension $D = 2d - 2$.

3.1 Deterministic production of W states

We assume that there is a source of bipartite entanglement, ideally delivering identical EPR pairs. Three such doubles are needed in each run, with one particle of each pair being sent to three detectors (D_1, D_2 e D_3). At this point let us simply consider that the state of three pairs coming from the source is $|\phi^+\rangle^{\otimes 3}$:

$$\begin{aligned}
|\phi^+\rangle^{\otimes 3} &= \frac{1}{2\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12}) \otimes (|00\rangle_{34} + |11\rangle_{34}) \otimes (|00\rangle_{56} + |11\rangle_{56}) \\
&= \frac{1}{2\sqrt{2}}(|00\rangle_{12} \otimes |00\rangle_{34} \otimes |00\rangle_{56} + |00\rangle_{12} \otimes |00\rangle_{34} \otimes |11\rangle_{56} \\
&\quad |00\rangle_{12} \otimes |11\rangle_{34} \otimes |00\rangle_{56} + |00\rangle_{12} \otimes |11\rangle_{34} \otimes |11\rangle_{56} + \\
&\quad |11\rangle_{12} \otimes |00\rangle_{34} \otimes |00\rangle_{56} + |11\rangle_{12} \otimes |00\rangle_{34} \otimes |11\rangle_{56} + \\
&\quad |11\rangle_{12} \otimes |11\rangle_{34} \otimes |00\rangle_{56} + |11\rangle_{12} \otimes |11\rangle_{34} \otimes |11\rangle_{56})
\end{aligned}$$

Of course, it does not matter which ERP state one choses, provided that it is known. We intend to make a triple measurement on the odd labelled particles (subsystem A) which are sent to the detectors, see Fig. 3.4. We note from the outset that the measurement on subsystem A may be destructive and will be assumed to be so, although this is not a logical necessity. The key point of our protocol, which enables the deterministic creation of W states, is the the following set of kets:

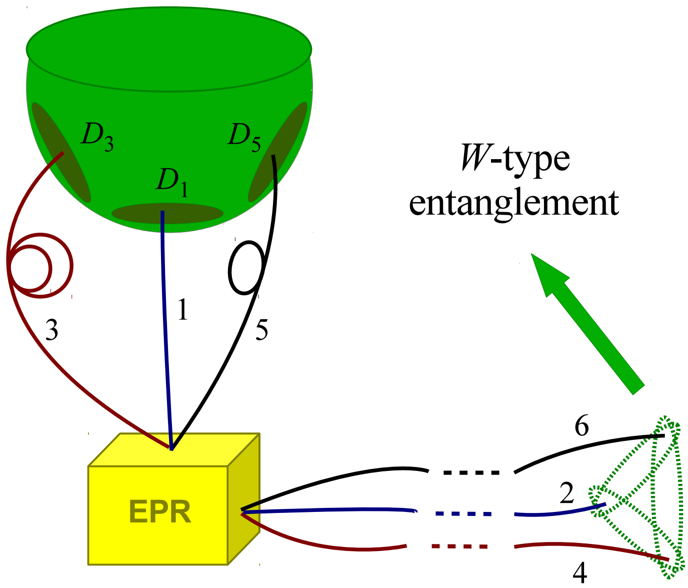

Figure 3.4 EPR entangled pairs (1-2, 3-4, and 5-6) produced in sequence in a single source are spatially separated. Odd-labeled particles are appropriately measured, after which, the remaining particles are left in a genuine tripartite entangled state.

$$\begin{aligned}
|W_1\rangle &= \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)\,, \\
|W_2\rangle &= \frac{1}{\sqrt{3}}(|000\rangle - |011\rangle + |101\rangle)\,, \\
|W_3\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |011\rangle - |110\rangle)\,, \\
|W_4\rangle &= \frac{1}{\sqrt{3}}(|000\rangle - |101\rangle + |110\rangle)\,, \\
|W_5\rangle &= \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle)\,, \\
|W_6\rangle &= \frac{1}{\sqrt{3}}(|111\rangle - |100\rangle + |010\rangle)\,, \\
|W_7\rangle &= \frac{1}{\sqrt{3}}(|111\rangle + |100\rangle - |001\rangle)\,, \\
|W_8\rangle &= \frac{1}{\sqrt{3}}(|111\rangle - |010\rangle + |001\rangle)\,.
\end{aligned} \tag{3.47}$$

The state $|W_1\rangle (\equiv |W\rangle)$ is the tripartite W state as we are used to it and $|W_5\rangle$ its negation. The other six states may not seem to be of the same category, since they are not combinations of kets with the same number of "excitations". However, it is easy to see that they also represent perfect W states, for, $|W_1\rangle$ is obtainable from any of them via local, unitary operations, e. g., $|W_1\rangle = (\mathbb{I}\otimes\sigma_z\otimes\sigma_x)|W_2\rangle = (\sigma_z\otimes\sigma_x\otimes\mathbb{I})|W_3\rangle$, etc. Note also that $|W_{j+4}\rangle = \hat{\sigma}_x^{\otimes 3}|W_j\rangle$ and $\langle W_j|W_k\rangle = \delta_{jk}$. Therefore, we have at our disposal an orthonormal W-state basis, that, in principle, corresponds to the eigenvectors of some observable that can be measured. Given the ideal source state $|\phi^+\rangle^{\otimes 3}$, after conveniently reordering the kets, it can be written as

$$\begin{aligned}
|\phi\rangle^{\otimes 3} &= \frac{1}{2\sqrt{2}}[|000\rangle_A|000\rangle_B + |001\rangle_A|001\rangle_B + |010\rangle_A|010\rangle_B \\
&+|011\rangle_A|011\rangle_B + |100\rangle_A|100\rangle_B + |101\rangle_A|101\rangle_B \\
&+|110\rangle_A|110\rangle_B + |111\rangle_A|111\rangle_B]\,,
\end{aligned}$$

where particles 1, 3, and 5 belong to subsystem A and the remaining particles to the subsystem B.

Using the inverse of relations 3.47:

$$\begin{aligned}
|000\rangle &= \frac{1}{\sqrt{3}}(|W_2\rangle + |W_3\rangle + |W_4\rangle)\,, \\
|001\rangle &= \frac{1}{\sqrt{3}}(|W_1\rangle - |W_6\rangle + |W_7\rangle)\,, \\
|010\rangle &= \frac{1}{\sqrt{3}}(|W_1\rangle + |W_6\rangle + |W_8\rangle)\,, \\
|011\rangle &= \frac{1}{\sqrt{3}}(-|W_2\rangle + |W_3\rangle + |W_5\rangle)\,, \\
|100\rangle &= \frac{1}{\sqrt{3}}(|W_1\rangle - |W_6\rangle + |W_7\rangle)\,, \\
|101\rangle &= \frac{1}{\sqrt{3}}(|W_2\rangle - |W_4\rangle + |W_5\rangle)\,, \\
|110\rangle &= \frac{1}{\sqrt{3}}(-|W_3\rangle + |W_4\rangle + |W_5\rangle)\,, \\
|111\rangle &= \frac{1}{\sqrt{3}}(|W_6\rangle + |W_7\rangle + |W_8\rangle)\,. \qquad (3.48)
\end{aligned}$$

We have:

$$
\begin{aligned}
|\phi\rangle^{\otimes 3} = & \frac{1}{6\sqrt{2}}[(|W_2\rangle_A + |W_3\rangle_A + |W_4\rangle_A) \otimes (|W_2\rangle_B + |W_3\rangle_B + |W_4\rangle_B) + \\
& (|W_1\rangle_A - |W_6\rangle_A + |W_7\rangle_A) \otimes (|W_1\rangle_B - |W_6\rangle_B + |W_7\rangle_B) + \\
& (|W_1\rangle_A + |W_6\rangle_A + |W_8\rangle_A) \otimes (|W_1\rangle_B + |W_6\rangle_B + |W_8\rangle_B) + \\
& (-|W_2\rangle_A + |W_3\rangle_A + |W_5\rangle_A) \otimes (-|W_2\rangle_B + |W_3\rangle_B + |W_5\rangle_B) + \\
& (|W_1\rangle_A - |W_6\rangle_A + |W_7\rangle_A) \otimes (|W_1\rangle_B - |W_6\rangle_B + |W_7\rangle_B) + \\
& (|W_2\rangle_A - |W_4\rangle_A + |W_5\rangle_A) \otimes (|W_2\rangle_B - |W_4\rangle_B + |W_5\rangle_B) + \\
& (-|W_3\rangle_A + |W_4\rangle_A + |W_5\rangle_A) \otimes (-|W_3\rangle_B + |W_4\rangle_B + |W_5\rangle_B) + \\
& (|W_6\rangle_A + |W_7\rangle_A + |W_8\rangle_A) \otimes (|W_6\rangle_B + |W_7\rangle_B + |W_8\rangle_B)]
\end{aligned}
$$

Which has the totally correlated form:

$$
\frac{1}{2\sqrt{2}} \sum_{j=1}^{8} |W_j\rangle_A \otimes |W_j\rangle_B \,. \tag{3.49}
$$

By executing a projective measurement in this basis for particles in A we *deterministically* obtain, after local operations and classical communication (LOCC), a perfect set of pure states in the standard form $|W_1\rangle$ in subsystem B. Of course, the measurements in the entangled basis (3.47) constitute a technical difficulty in practice, but there is nothing that prevents their realization, in principle (more on this point in the next sections). The required local unitary operations are shown in table 3.1. Note, in addition, that it does not matter how far apart are the particles of system B, thus, the preparation may be remote.

Result in A	Operation to be done on B
$\|W_1\rangle$	$\hat{O}_1 = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}$
$\|W_2\rangle$	$\hat{O}_2 = \mathbb{I} \otimes \sigma_z \otimes \sigma_x$
$\|W_3\rangle$	$\hat{O}_3 = \sigma_z \otimes \sigma_x \otimes \mathbb{I}$
$\|W_4\rangle$	$\hat{O}_4 = \sigma_x \otimes \mathbb{I} \otimes \sigma_z$
$\|W_5\rangle$	$\hat{O}_5 = \sigma_x \otimes \sigma_x \otimes \sigma_x$
$\|W_6\rangle$	$\hat{O}_6 = \sigma_x \otimes i\sigma_y \otimes \mathbb{I}$
$\|W_7\rangle$	$\hat{O}_7 = i\sigma_y \otimes \mathbb{I} \otimes \sigma_x$
$\|W_8\rangle$	$\hat{O}_8 = \mathbb{I} \otimes \sigma_x \otimes i\sigma_y$

Table 3.1 Operations to be done on system B after the measurements on A to deterministically produce $|W_1\rangle$.

A more evident, though analogous procedure is possible for GHZ states. The corresponding measurement involves the basis $|\mathrm{GHZ}^{\pm}_{ijk}\rangle = (|ijk\rangle \pm |\overline{ijk}\rangle)/\sqrt{2}$, where the overbar denotes negation and $i, j, k = 0, 1$. For the ideal source of states (3.47) one gets the result

$$
\sim |\mathrm{GHZ}^{+}_{000}\rangle \otimes |\mathrm{GHZ}^{+}_{000}\rangle + \cdots + |\mathrm{GHZ}^{-}_{111}\rangle \otimes |\mathrm{GHZ}^{-}_{111}\rangle \,,
$$

also enabling deterministic creation of this kind of tripartite state. However, when the source is imperfect, the process for the two classes of states lead to quite distinct results, as we discuss below.

3.2 Entanglement concentration and lifting

We are now in position to address a more realistic entanglement source containing a systematic error. The consideration of source flaws is relevant in the promotion of any theoretical proposal into a feasible process. This has been considered, for instance, in an experiment for secure quantum key distribution [XWS+15] (see also [MIT15, YBO+16]) and also in state preparation [ÖMKI02, SHST09]. The source is now described by $|\Phi_0\rangle = (a|00\rangle_{12} + b|11\rangle_{12}) \otimes (a|00\rangle_{34} + b|11\rangle_{34}) \otimes (a|00\rangle_{56} + b|11\rangle_{56})$, where, hereafter, a is assumed to be real and positive, and $a \neq |b|$. This corresponds to

$$\begin{aligned} |\Phi_0\rangle \quad = \quad & a^3|000\rangle_A|000\rangle_B + a^2b|001\rangle_A|001\rangle_B + \\ & a^2b|010\rangle_A|010\rangle_B + ab^2|011\rangle_A|011\rangle_B + \\ & a^2b|100\rangle_A|100\rangle_B + ab^2|101\rangle_A|101\rangle_B + \\ & ab^2|110\rangle_A|110\rangle_B + b^3|111\rangle_A|111\rangle_B \,, \end{aligned} \tag{3.50}$$

where we reordered the state, which, in this case, is expressed in terms of basis (3.47) as

$$\begin{aligned} |\Phi_0\rangle = a^2b|W_1\rangle \otimes |W_1\rangle + ab^2|W_5\rangle \otimes |W_5\rangle \\ + \sum_{k=2,3,4} |W_k\rangle \otimes \left[ab^2|W_k\rangle + \frac{1}{\sqrt{3}}(a^3 - ab^2)|000\rangle \right] \\ + \sum_{k=6,7,8} |W_k\rangle \otimes \left[a^2b|W_k\rangle + \frac{1}{\sqrt{3}}(b^3 - a^2b)|111\rangle \right] \,. \end{aligned} \tag{3.51}$$

Therefore, it remains possible to get an ideal set of W states after postselection, because whenever the result of the measurement is $|W_1\rangle$ ($|W_5\rangle$) the produced state on B is $|W_1\rangle$ ($|W_5\rangle$) *no matter* the values of a and b. This occurs with probability $|a^2b|^2$ ($|ab^2|^2$). Thus, by postselecting the outcomes $|W_1\rangle$ and $|W_5\rangle$, proceeding the appropriate LOCC for the latter, one gets a pure ensemble of standard W states. The success probability is:

$$P = |a^2b|^2 + |ab^2|^2 = a^2 - a^4. \tag{3.52}$$

Note that it is bounded from above by $P = 1/4$, since we are eliminating the other six outcomes even if they lead to states with high fidelity.

In a more realistic scenario, one may not be able to carry out a full measurement in the basis (3.47). This is indeed the case of our present technical development. Note however that if we can unambiguously distinguish $|W_1\rangle$ from $|W_5\rangle$, and these from the other states, then we are

Result in B after measurement in A	State left in B after LOCC (Table 3.1)
$\|W_1\rangle$	$\|W_1\rangle$
$\sim (a^2+2b^2)\|W_2\rangle + (a^2-b^2)(\|W_3\rangle + \|W_4\rangle)$	$\sim (a^2+2b^2)\|W_1\rangle + (a^2-b^2)(\|W_8\rangle - \|W_7\rangle)$
$\sim (a^2+2b^2)\|W_3\rangle + (a^2-b^2)(\|W_2\rangle + \|W_4\rangle)$	$\sim (a^2+2b^2)\|W_1\rangle + (a^2-b^2)(-\|W_8\rangle + \|W_6\rangle)$
$\sim (a^2+2b^2)\|W_4\rangle + (a^2-b^2)(\|W_2\rangle + \|W_3\rangle)$	$\sim (a^2+2b^2)\|W_1\rangle + (a^2-b^2)(\|W_7\rangle - \|W_6\rangle)$
$\|W_5\rangle$	$\|W_1\rangle$
$\sim (b^2+2a^2)\|W_6\rangle + (b^2-a^2)(\|W_7\rangle + \|W_8\rangle)$	$\sim (b^2+2a^2)\|W_1\rangle + (b^2-a^2)(\|W_8\rangle - \|W_7\rangle)$
$\sim (b^2+2a^2)\|W_7\rangle + (b^2-a^2)(\|W_6\rangle + \|W_8\rangle)$	$\sim (b^2+2a^2)\|W_1\rangle + (b^2-a^2)(-\|W_8\rangle + \|W_6\rangle)$
$\sim (b^2+2a^2)\|W_8\rangle + (b^2-a^2)(\|W_6\rangle + \|W_7\rangle)$	$\sim (b^2+2a^2)\|W_1\rangle + (b^2-a^2)(\|W_7\rangle - \|W_6\rangle)$

Table 3.2 States left in B after the measurement in A (first column), and after the LOCC prescribed in Table 3.1 (second column). Note that the states $|W_2\rangle$, $|W_3\rangle$, $|W_4\rangle$, and $|W_5\rangle$ do not appear after the unitary operations. For the sake of clarity the states are not normalized.

Result in B after measurement in A	Probability
$\|W_1\rangle$	$a^4\|b\|^2$
$\sim (a^2+2b^2)\|W_2\rangle + (a^2-b^2)(\|W_3\rangle + \|W_4\rangle)$	$a^2(a^4+2\|b\|^4)/3$
$\sim (a^2+2b^2)\|W_3\rangle + (a^2-b^2)(\|W_2\rangle + \|W_4\rangle)$	$a^2(a^4+2\|b\|^4)/3$
$\sim (a^2+2b^2)\|W_4\rangle + (a^2-b^2)(\|W_2\rangle + \|W_3\rangle)$	$a^2(a^4+2\|b\|^4)/3$
$\|W_5\rangle$	$a^2\|b\|^4$
$\sim (b^2+2a^2)\|W_6\rangle + (b^2-a^2)(\|W_7\rangle + \|W_8\rangle)$	$\|b\|^2(\|b\|^4+2a^4)/3$
$\sim (b^2+2a^2)\|W_7\rangle + (b^2-a^2)(\|W_6\rangle + \|W_8\rangle)$	$\|b\|^2(\|b\|^4+2a^4)/3$
$\sim (b^2+2a^2)\|W_8\rangle + (b^2-a^2)(\|W_6\rangle + \|W_7\rangle)$	$\|b\|^2(\|b\|^4+2a^4)/3$

Table 3.3 States left in B after the measurement in A (first column). The second column shows the probabilities for each outcome. For the sake of clarity the states are not normalized.

able to remotely produce perfect W states from an imperfect bipartite source with probability $P = a^2 - a^4$. It is possible to tackle an equivalent task with linear-optics elements in the case of a GHZ basis [PZ98].

Although we cannot directly compare the entanglement of systems with two and three parties, we argue that the previous procedure leads to entanglement concentration [BBPS96]. The source of bipartite states may have arbitrarily low entanglement and the result is always a highly entangled tripartite state, of course at the cost of a proportional reduction in the number of elements in the final ensemble. But this is exactly what happens for entanglement concentration between ensembles of states with the same dimensionality [BVK99]. If the source is composed by N quasi-separable pairs with state:

$$|\psi_\varepsilon\rangle = \varepsilon|00\rangle + \sqrt{1-\varepsilon^2}|11\rangle, \tag{3.53}$$

then we get $\varepsilon^2 N/2 + O(\varepsilon^4)$ perfect W states, asymptotically. It is evident that any reasonable measure of multipartite entanglement E, would give $E(|W\rangle) > E(|\psi_\varepsilon\rangle^{\otimes 3})$, for sufficiently small ε. Also, because the entanglement delivered by the source is bipartite while the product presents genuine tripartite entanglement, we say that the entanglement has been *lifted* (from a "small" Hilbert space to a larger one). This is a remarkable property of W states, and an analogous situation does not exist for GHZ states. If the source has $a \neq |b|$, by any small extent, then the probability to get an exact GHZ state in any run is zero, and no postselection could help in getting an ideal GHZ ensemble. This adds to the reasoning that W states are less entangled than GHZ states, in the sense that from three non-maximally entangled states one can, via stochastic LOCC (SLOCC) [DVC00] get a perfect W but not a perfect GHZ.

3.3 High-fidelity W states without postselection

The referred postselection may be a waste of resources if some amount of error can be tolerated in a scenario where full measurements can be carried out. For a good, but imperfect source, the outputs are either perfect or so close to the ideal states that the chance that they can be distinguished in practice is very small. Note from the second line of Eq. (3.51), that if, e. g., we obtain $|W_2\rangle$ in subsystem A, then the state in B is proportional to $\sqrt{3}|W_2\rangle + (a^2/b^2 - 1)|000\rangle$. So, for $a^2 = 1/2 + \varepsilon$ ($|b|^2 = 1/2 - \varepsilon$), with $\varepsilon << 1$, we get $\sqrt{3}|W_2\rangle + 4\varepsilon e^{-2i\theta}|000\rangle + O(\varepsilon^2)$, where $b = |b|e^{i\theta}$.

We proceed to show that even if all measurement outcomes are utilized, the fidelity of the resulting state with respect to $|W_1\rangle$ is slightly improved in comparison to the fidelity of (3.50) with respect to the ideal source state $|\phi^+\rangle^{\otimes 3}$. Without postselection, the possible states left in B after the measurement in A are listed in the first column of Table 3.2. The next step is to proceed with the operations in Table 3.1. That leaves the first state in Table 3.2 unchanged, while the second (not normalized) state undergoes the transformation

$$(a^2+2b^2)|W_2\rangle + (a^2-b^2)(|W_3\rangle + |W_4\rangle) \longrightarrow$$
$$(a^2+2b^2)|W_1\rangle + (a^2-b^2)(|W_8\rangle - |W_7\rangle)\ ,$$

and so on, see the second column of Table 3.2. Gathering these results together and considering their probabilities (second column of Table 3.3), the resulting density operator for the mixed state reads, in the W basis [Eq. (3.47)],

$$\begin{aligned}\rho &= F|W_1\rangle\langle W_1|_B + \frac{1-F}{3}\left(|W_6\rangle\langle W_6|_B + |W_7\rangle\langle W_7|_B + |W_8\rangle\langle W_8|_B\right) - \\ &\quad K\left(|W_6\rangle\langle W_7|_B + |W_6\rangle\langle W_8|_B + |W_7\rangle\langle W_6|_B + \right. \\ &\quad \left. |W_7\rangle\langle W_8|_B + |W_8\rangle\langle W_6|_B + |W_8\rangle\langle W_7|_B\right)\end{aligned} \tag{3.54}$$

Or in matrix notation:

$$\rho = \begin{pmatrix} F & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-F}{3} & -K & -K \\ 0 & 0 & 0 & 0 & 0 & -K & \frac{1-F}{3} & -K \\ 0 & 0 & 0 & 0 & 0 & -K & -K & \frac{1-F}{3} \end{pmatrix}, \tag{3.55}$$

where the absolute value of the coherences reads $K = [4a^2(1-a^2)(1+\sin^2\theta)+1]/9$ and

$$F = \frac{1}{3}(8\Gamma^2+1)\,, \tag{3.56}$$

with $\Gamma = a\sqrt{1-a^2}\cos\theta$ and $b = \sqrt{1-a^2}e^{i\theta}$. The quantity in Eq. (3.56) is the final fidelity, $F = \mathrm{Tr}(\rho|W_1\rangle\langle W_1|)$. Since the described protocol amounts to lifting entanglement from a Hilbert space with dimension 4 to a Hilbert space with dimension 8, it could be considered useful even if the final fidelity were smaller than the fidelity of the source, to an acceptable extent. However, we note that the fidelity of $|\Phi_0\rangle$, Eq. (3.50), with respect to an ideal source is $F_0 = |\langle\Phi_0|(|\phi^+\rangle)^{\otimes 3}|^2$, or $F_0 = (\Gamma+1/2)^3$, with $-\pi/2 \le \theta \le \pi/2$. For phase arguments outside this range the source would better described by the Bell state $|\phi^-\rangle \sim |00\rangle - |11\rangle$. This leads to

$$F = \frac{2}{3}\left(1-2F_0^{1/3}\right)^2 + \frac{1}{3} \ge F_0\,. \tag{3.57}$$

So that if the source produces single pairs with states $|00\rangle + e^{i\pi/5}|11\rangle$, whose fidelity is $F_0 = (0.905)^3 = 0.74$, then we obtain an ensemble whose fidelity with respect to an ideal W state is $F = 0.77$ (see Fig. 3.5). In particular, a borderline initial fidelity of 0.5 leads to $F = 0.56$.

It is important to recall that in this whole process, without postselection, there are no losses but those due to the measurement in A. Whenever the EPR source has N particles the resulting W-state ensemble has $N/2$ particles, which renders statistical efficiency to the protocol. In addition, if the physical realization of the bipartite source is provided by entangled photons, we only demand first order PDC, that is, three pairs generated in sequence, provided that the

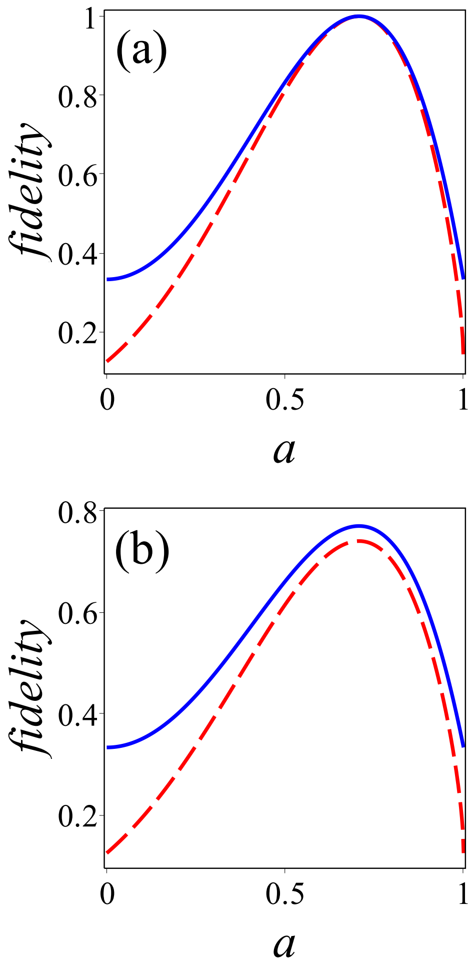

Figure 3.5 Fidelities F (solid) and F_0 (dashed) as functions of a for (a) $\theta = 0$ and (b) $\theta = \pi/5$

setup is capable of keeping the coherence of the pairs in the typical window of time for the occurrence of three PDC events. Let us finally comment on the scenario in which the source, in addition to the systematic error, presents a fraction f of white noise. A conservative hypothesis for the state of the triples in the source is:

$$\rho_0 = (1-f)|\Phi_0\rangle\langle\Phi_0| + f\,\mathbb{I}_{64}/64, \tag{3.58}$$

where $\mathbb{I}_D$ is the $D \times D$ identity operator. In this case it is immediate to see that the noise fraction remains unchanged throughout the protocol, and the obtained density operator reads $\rho = (1-f)\rho + f\,\mathbb{I}_8/8$, where ρ is given by Eq. (3.55). That is to say, our lifting procedure leads to entanglement concentration but probably not to entanglement distillation upon depolarizing noise.

3.4 Discussion

It is arguably easier, both conceptually and experimentally, to create a four-partite W state than to produce a tripartite $|W\rangle$. This is partly due to the fact that most of the measurements involved in the previous protocols are of Bell type. In this work we provide a full basis of W states which induces more appropriate measurements to produce the desired type of entanglement. Of course, we are aware of the technical difficulties associated with full measurements involving entangled bases, either without interactions between the subsystems [VY99] or with passive linear elements only [LCS99]. In the last fifteen years, many ways to circumvent these difficulties have been proposed, employing ancillary systems (hyperentanglement) [SHKW06, WBK07], nonlinear optics [KKS01], or even active linear optical elements [ZvL13]. In addition, we note that in the case of a perfect source, if one can distinguish n basis elements out of 8, then we have a statistical efficiency of $n/8$. Even in the case of imperfect sources, to produce ideal W states only requires the ability to distinguish $|W_1\rangle$ from $|W_5\rangle$, and these states from the rest of the basis elements. This is a much more modest task. Finally, we note that the measurement on subsystem A can be made in a very localized region of space (without affecting the remote character of the state left in B), enabling the execution of CNOT gates involving particles 1, 3, and 5.

Our results can be summarized as follows: When the EPR source is ideal, a pure ensemble of W states can be deterministically produced. In addition, if the bipartite source is non-maximally entangled, one can still obtain a set of pure W states, this time at the cost of postselection. This is a peculiarity of the states $|W\rangle$ that, for instance, does not hold for GHZ states. In fact one can produce perfect W states from arbitrarily poor entanglement, but, the poorest the entanglement of the source the smaller the number of states $|W\rangle$ in the ensemble. Alternatively, it is possible to profit from every individual run, for, even without post selection one gets a mixed tripartite state whose fidelity (with respect to $|W\rangle$) is higher than that of the triple of pairs coming from the source (in comparison to a perfect triple of EPR pairs). Finally we emphasize the possibility of creating remote entangled systems, allowing, e. g., distribution of keys.

Critical behaviour in the optimal generation of multipartite entanglement

Quantum entanglement involving two parties has led to groundbreaking advances, of which preeminent examples are Bell inequalities and nonlocality [Bel04], teleportation [BBC$^+$93], and dense coding [BW92, MWKZ96]. It would be natural to think, at a first sight, that multipartite entangled systems would present the same features in a larger scale. However, since the seminal study on the nonlocality of certain tripartite states in the early 90's [GHSZ90], it became gradually clear that completely new phenomena and potential applications could arise. It is now known that n particles can be highly entangled without any pairwise correlation: entanglement appears in many different, inequivalent forms, of which the Einstein-Podolsky-Rosen (EPR) type is the simplest instance [LL12, DVC00]. As for applications, there are, e. g., paradigms for quantum computation which rely on the feasibility of cluster states of several qubits [RB01].

It is, then, clear that conceiving means to produce multipartite entangled states is of relevance. To date, there are three ways to meet this goal: controlled interactions between qubits, measurements in entangled basis, and indistinguishability of identical particles. The first way relies on the fact that initially uncorrelated interacting subsystems may become entangled. Of course, the interactions must be finely tuned and the qubits protected from noise. The second possibility is related to the fact that, to any orthogonal entangled basis corresponds a physical observable that can be measured, in principle, leaving the system entangled. There are, however, serious provisos regarding this simplistic picture. Firstly, measurements are commonly destructive, photo-detection for instance, so, after their realisation there remains no system whatsoever. This problem can be circumvented if bipartite entanglement is an available resource. So, instead of using n qubits, one employs n EPR pairs and proceed with the n-partite measurement on one particle of each pair. The remaining qubits will end up, with some non-zero probability, in a n-partite entangled state, see, e. g., [MCP16]. This brings the second difficulty. In practice, it is very hard to make full measurements in entangled bases, even in the bipartite case [VY99, LCS99]. The third way is more related to a fundamental principle than to deliberate procedures, see however [BH02]. It simply amounts to the fact that two indistinguishable electrons, e. g., can only exist in an entangled state.

In this chapter we present a procedure for creating arbitrary Dicke states, $|D_n^{(k)}\rangle$, of n particles and $1 \leq k \leq n-1$ excitations. It employs a combination of induced indistinguishability and, differently from the previous chapter, unentangled Fock measurements. The protocol has the property that, irrespective of how poor is the source of entanglement, every time a Dicke state is created, it is ideal. In addition, the probability of creating $|D_n^{(k)}\rangle$ behaves similarly to a thermodynamic potential during a second-order phase transition, as n grows, where the entanglement of the optimal source undergoes a qualitative change at a critical n. Finally, we establish exact results on the asymptotic entanglement between any qubit belonging to $|D_n^{(k)}\rangle$ and the rest of the system.

4.1 Entanglement lifting

We begin by sketching an optical realisation of the scheme in the simple case of three photon pairs. It illustrates what we refer to as entanglement *lifting*. Differently from usual entanglement concentration, we start with M bipartite entangled pairs and, in the end, we obtain j n-partite entangled systems. More precisely, in the pure case, we will consider the process $|\phi\rangle^{\otimes M} \longrightarrow |\Psi\rangle^{\otimes j}$ with $|\phi\rangle \in \mathscr{H}_i$ and $|\Psi\rangle \in \mathscr{H}_f$, where not only $j < M$, but also $D > d$, with $d = \dim\mathscr{H}_i$, and $D = \dim\mathscr{H}_f$. It will become clear that lifting also entails concentration.

Consider three identical sources of pairs of polarisation-entangled photons (alternatively one can consider that there is a single source producing the pairs which are posteriorly distributed). We initially assume that each pair is maximally entangled, with state:

$$|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}, \tag{4.59}$$

with 0 (1) denoting horizontal (vertical) polarisation.

One photon of each pair is delivered to Alice, while the others are sent to Bob, Brian and Brandon. After this, the total quantum state reads:

$$|\Psi_0\rangle = |\phi^+\rangle^{\otimes 3} = 2^{-3/2} \sum_{i,j,k} |ijk\rangle_A \otimes |ijk\rangle_B, \tag{4.60}$$

where A stands for Alice and B for the spatially separated system of Brandon, Brian and Bob (fig. 4.6).

Alice synchronizes her photons, e.g., using quantum memories [FCL$^+$06, YCC$^+$07, MHY$^+$16] (process denoted by S) before sending them to a polarization beam splitter (PBS). It is important that the three wave packets overlap so that the photons are undistinguishable when they arrive at the PBS. With this, Alice intentionally discards the information on the former holder of each photon.

After passing the PBS, the photons proceed to detectors capable of discriminating the Fock state in each spatial mode of Alice's system [DMB$^+$08, MZG$^+$15]. The indistinguishability induces a natural partition of Alice system's Hilbert space into subspaces generated by kets with a fixed number of photons with a given polarisation, so that, states like $|001\rangle$ and $|010\rangle$ coalesce into $|2\text{ horizontal}; 1\text{ vertical}\rangle$. In this way, we have four detection possibilities:

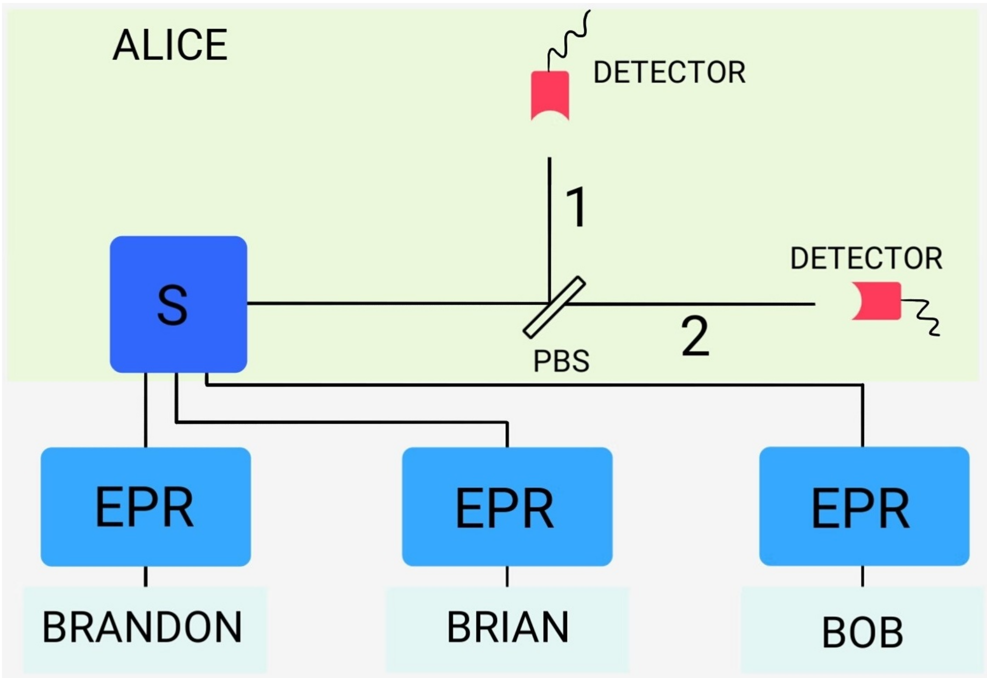

Figure 4.6 Three photon pairs are produced in an EPR polarisation state. One photon of each pair is sent to Alice, while the B-group members receive one photon each. Alice send her synchronised photons to a PBS, after which they are detected, leaving the B-group with an ideal Dick state with probability $p = 0.75$.

$$\begin{aligned} |000\rangle &\rightarrow |3;0\rangle, \\ \{|001\rangle, |010\rangle, |100\rangle\} &\rightarrow |2;1\rangle, \\ \{|011\rangle, |101\rangle, |110\rangle\} &\rightarrow |1;2\rangle, \\ |111\rangle &\rightarrow |0;3\rangle. \end{aligned} \tag{4.61}$$

The first entry in the kets on the right-hand side gives the photon number of path 2, and the second entry the photon number of path 1, see fig. 4.6. It is clear that this process is non unitary since it can map orthogonal states into the same final state. This is due to the fact that synchronisation can only be achieved through the interaction between the photons and some ancillary systems which, generally speaking, store the information on the initial delay between the photons. In [YCC+07], where the possibility of using this technique in the scalable generation of photonic entanglement is already mentioned, the auxiliary systems consist of Rubidium atom ensembles. The global unitary evolution of the larger system leads to the non unitary evolution of the reduced system of the photons to be synchronised.

In the full indistinguishability situation, just before the detection, the state is proportional to:

$$|3;0\rangle|000\rangle + \sqrt{3}|2;1\rangle\left(\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle\right)$$
$$+\sqrt{3}|1;2\rangle\left(\frac{1}{\sqrt{3}}|011\rangle + \frac{1}{\sqrt{3}}|101\rangle + \frac{1}{\sqrt{3}}|110\rangle\right) + |0;3\rangle|111\rangle.$$

Therefore, after the detection of Alice's photons there are four possible states left to the B-group. If all three photons are either detected in path 1 or in path 2 (probability 1/8 for each event), then, B-group's state is separable and, after appropriate classical communication, it is disposed. With probability $3/8$ two photons are detected in path 2 and one in path 1 and, then, B-group's state is already a W state, $|D_3^{(1)}\rangle \equiv |W_3\rangle \propto |001\rangle + |010\rangle + |100\rangle$. Lastly, also with probability 3/8, two photons are detected in path 1 and one in path 2. In this case, Alice communicates the B-group members that each of them have to perform a bit-flip operation in his qubit, and, in the end, ideal W states are prepared with probability $3/4$.

4.2 Dicke states of arbitrary dimension

We now generalise the lifting procedure to the case where n EPR pairs are used per run. We will see that this leads to the production of arbitrary, unambiguously heralded Dicke states. The initial state is given by $|\Psi_0\rangle = |\phi^+\rangle^{\otimes n}$, and it is easy (but crucial) to see that it can be written as:

$$|\Psi_0\rangle = \left(\frac{1}{2}\right)^{n/2} \sum |j_1 j_2 \ldots j_n\rangle_A \otimes |j_1 j_2 \ldots j_n\rangle_B, \tag{4.62}$$

where $j_i = 0,1$.

Again, one photon of each pair is sent to Alice and the others to each of the elements of the B-group, now composed by n parties. Alice's photons are synchronised and sent to the PBS. Given the indistinguishability:

$$\begin{array}{rcl} |00...000\rangle_A & \rightarrow & |n;0\rangle, \\ \{\hat{\mathscr{P}}_j^{(1)}|00...001\rangle_A\}_j & \rightarrow & |n-1;1\rangle, \\ & . & \\ & . & \\ & . & \\ \{\hat{\mathscr{P}}_j^{(n-1)}|01...111\rangle_A\}_j & \rightarrow & |1;n-1\rangle \\ |11...111\rangle_A & \rightarrow & |0;n\rangle, \end{array} \tag{4.63}$$

where $\hat{\mathscr{P}}_j^{(k)}$ represents the jth non-trivial permutation of the ket entries.

Thus, for each k, we have $j = 1,2,..., \binom{n}{k}$. It is simple to show that the global state of the system immediately before the detection of Alice's photons is

$$|\Psi\rangle = \left(\frac{1}{2}\right)^{\frac{n}{2}} \Bigg[|n;0\rangle_A|00...00\rangle_B + |0;n\rangle_A|1...111\rangle_B + \sum_{k=1}^{n-1} \binom{n}{k}^{\frac{1}{2}} |n-k;k\rangle_A|D_n^{(k)}\rangle_B\Bigg], \tag{4.64}$$

where

$$|D_n^{(k)}\rangle \equiv \binom{n}{k}^{-\frac{1}{2}} \sum_{j=1}^{\binom{n}{k}} \hat{\mathscr{P}}_j^{(k)} |\underbrace{0...0}_{n-k}\overbrace{1...11}^{k}\rangle \tag{4.65}$$

is a n-partite Dicke state with $1 \leq k \leq n-1$ excitations. Note in Eq. (4.64), as in eq. (3.3), the perfect correlation between Alice's output of an unentangled Fock measurement and the production of a specific ideal Dicke state shared by the elements of the B-group. The probability of having output $|n-k;k\rangle$, or yet, to remotely produce $|D_n^{(k)}\rangle$, is $\binom{n}{k} 2^{-n}$. Dicke states with different number of excitations are generally inequivalent under local operations and classical communications (LOCC). The obvious exception occurs for k and $n-k$ excitations, since $|D_n^{(k)}\rangle = \hat{\sigma}_x^{\otimes n}|D_n^{(n-k)}\rangle$. Therefore, after Alice communicates her outcome, the B-group may transform all the states with $2k > n+1$ ($2k > n$), into states with $2k \leq n-1$ ($2k < n$), for n odd (even). We will assume that this procedure is always adopted in the remainder of this chapter. For n even and $k = n/2$, we already have $|D_n^{(k)}\rangle = |D_n^{(n-k)}\rangle$.

We are in a position to further extend the scheme to the case of tunable sources producing pairs with arbitrary entanglement [WJEK99, XWS+15, MIT15, MFC+15]: $|\phi\rangle = a|0_A 0_B\rangle +$

$b|1_A 1_B\rangle$. In this case the total initial state $|\Psi_0\rangle$ can be written as:

$$\sum_{j_1,j_2,\ldots,j_n=0,1} a^n \left(\frac{b}{a}\right)^{\sum_{\ell=1}^{n} j_\ell} |j_1 j_2 \ldots j_n\rangle_A \otimes |j_1 j_2 \ldots j_n\rangle_B,$$

for $a \neq 0$.

Under full indistinguishability, the surprising result is that after the PBS, and before detection, the state reads:

$$|\Psi\rangle = a^n |n;0\rangle_A |00\ldots00\rangle_B + b^n |0;n\rangle_A |1\ldots111\rangle_B + \sum_{k=1}^{n-1} a^{n-k} b^k \binom{n}{k}^{\frac{1}{2}} |n-k;k\rangle_A |D_n^{(k)}\rangle_B, \quad (4.66)$$

that is, despite the asymmetry of the source states, whenever an entangled state is produced it is still a perfect $|D_n^{(k)}\rangle$. It is of great importance to note that, had we kept all initial information, namely, the distinguishability of Alice's photons, then entangled measurements would be required to leave the B-group with Dicke states. The unbalanced character of the bipartite source, rather than affecting the ideality of the outputs, only changes their probabilities of occurrence, which are given by

$$P_n^{(k)} = \binom{n}{k} \left(\left|a^{n-k} b^k\right|^2 + \left|a^k b^{n-k}\right|^2 \right) \mathscr{G}_{n,k}, \quad (4.67)$$

with $\mathscr{G}_{n,k} = 1$ for n odd and $\mathscr{G}_{n,k} = (1+\delta_{k,n/2})^{-1}$ for n even, and, with the LOCC-equivalence between k and $n-k$ excitations already considered.

In fig. 4.7 we display (a) $P_n^{(1)}$, (b) $P_n^{(2)}$, (c) $P_n^{(3)}$, (d) $P_n^{(4)}$, (e) $P_n^{(25)}$, and (f) $P_n^{(50)}$, as functions of $|a|^2$, for selected values of n. The left panel shows that for $n = 3$, the optimal source corresponds to maximally entangled states ($|a|^2 = 1/2$). This is also true for $n = 4$, but, in this case there is a broad plateau around $|a|^2 = 1/2$. For $n \geq 5$, $P_n^{(1)}$ is maximal for non-maximally entangled sources and, two symmetric optimal values of $|a|^2$ appear. By using the general expression (4.67) one can analytically determine the bifurcation condition:

$$n > 2k + \frac{1}{2} + \sqrt{2k + \frac{1}{4}} \equiv \eta_c. \quad (4.68)$$

Since the right-hand side of the above equation is not necessarily integer, we define $n_c = \lceil \eta_c \rceil$, which stands for the ceiling function (smallest following integer). For $k = 1$ and $k = 3$, $\eta_c = n_c = 4$ and $\eta_c = n_c = 9$, respectively. In these cases the critical value is indeed integer and one can see the typical behaviour shown in fig. 4.7(a) and 4.7(c) for $n = 4$ and $n = 9$, respectively. In contrast, for $k = 2$, $\eta_c \approx 6.56$ ($\Rightarrow n_c = 7$), see fig. 4.7(b), and the exact critical point is passed by, for, there is no n for which the plateau appears: for $n = 6$ there is a single maximum in $P_n^{(2)}$, whereas for $n = 7$ there are two maxima. The same happens in figures 4.7(d), 4.7(e) and 4.7(f). However, the properties hereafter derived are the same no matter if η_c is an integer or not. In fig. 4.8(a) we plot the optimal source parameter $|a_{opt}|^2 \equiv |\tilde{a}|^2$ against n for $k = 1$

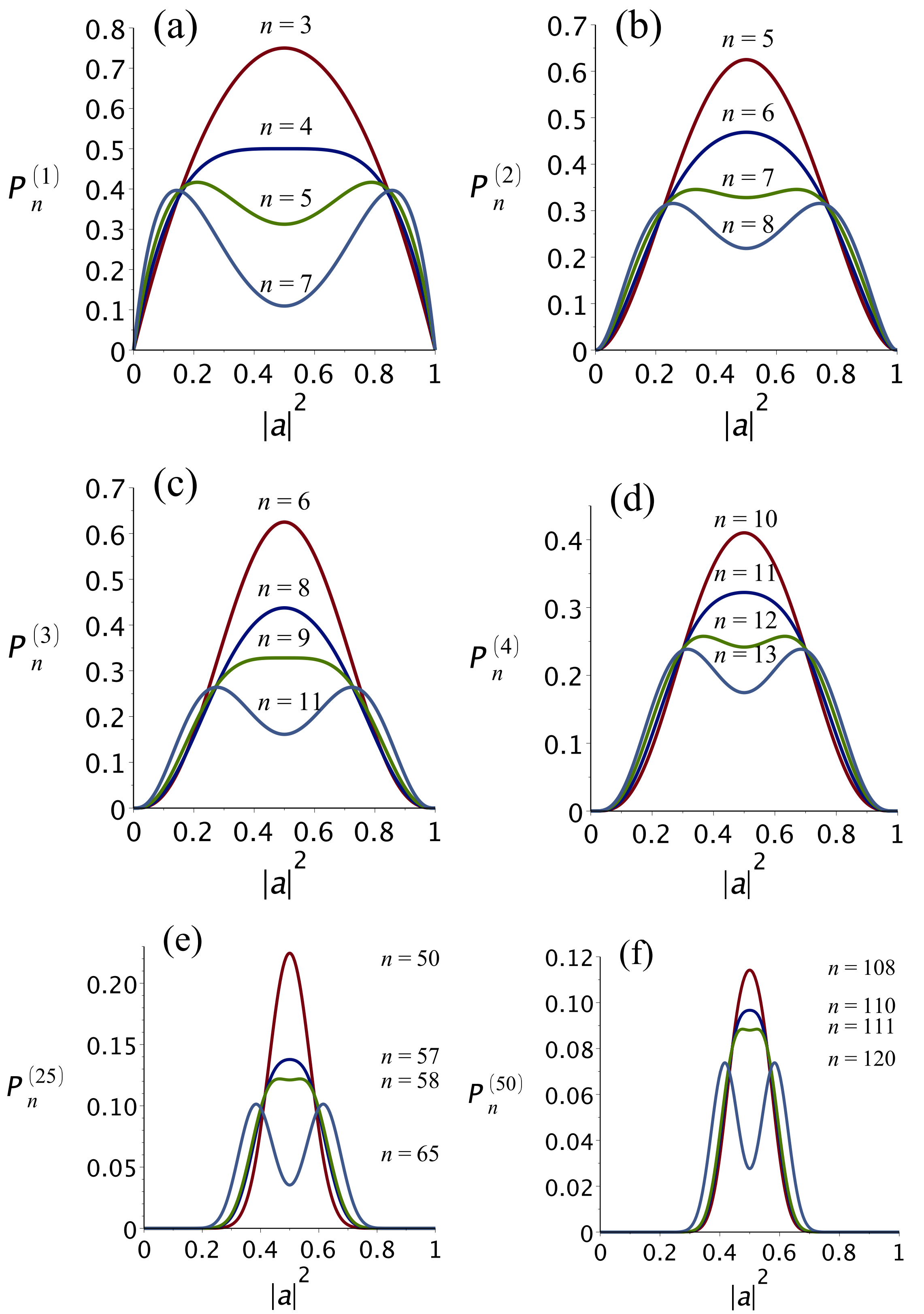

Figure 4.7 Probability of producing Dicke states with (a) one excitation and (b) two excitations (c) three excitations (d) four excitations (e) twenty five excitations (f) fifty excitations for selected values of n. As n grows there is a transition from functions with a single maximum at $|a|^2 = 1/2$ to curves with two maxima.

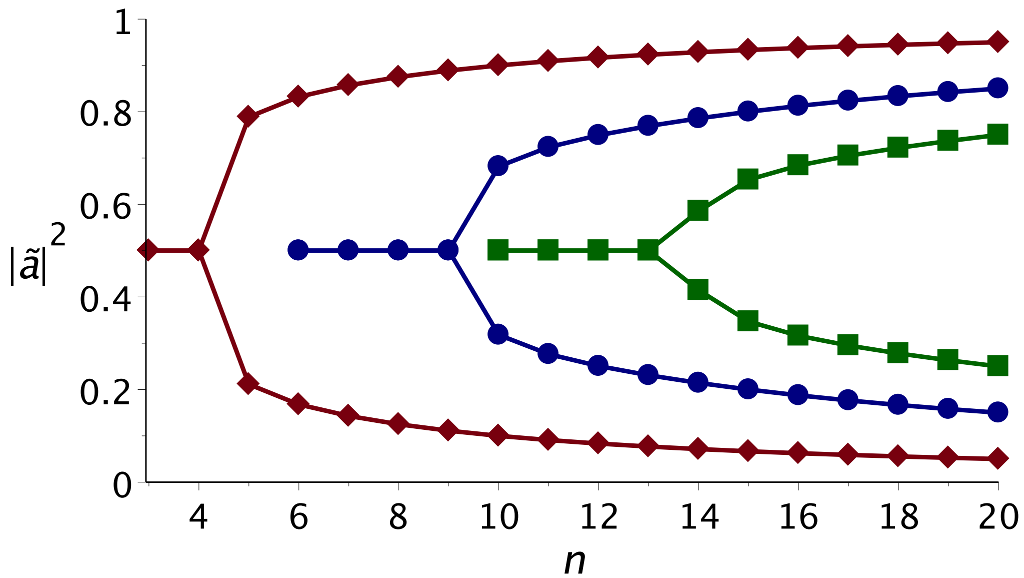

Figure 4.8 "Pitchfork" bifurcation diagram of $|a_{opt}|^2 \equiv |\tilde{a}|^2$ versus n for $k = 1$ (diamonds), $k = 3$ (bullets), and $k = 5$ squares.

($n_c = 4$), $k = 3$ ($n_c = 9$) and $k = 5$ ($n_c = 14$). The optimal source state remains maximally entangled up to $n = n_c$, after which a bifurcation develops.

These features remind us of the classical Landau theory [Cal98, PP72] of second order phase transitions, with the role of the thermodynamic potential being played by $P_n^{(k)}$, $|a|^2$ as the order parameter, and n inversely related to the temperature. Given this similarity, we set to find further evidence to support our analogy. Figure 4.9 displays the probability $P_n^{(3)}$ as n varies. The filled bullets represent $P_n^{(3)}$ if one employs a maximally entangled source, $|a|^2 = 1/2 \; \forall \; n$, leading to a steep decay toward zero. As soon as $n > n_c$, if, instead, one uses the optimal, but less entangled source determined by the lateral maxima, the decay follows a power law with a *finite*, non-vanishing asymptotic value. This regime is represented by the stars in fig. 4.9. We found that the state of the optimal bipartite source for the production of $|D_n^{(k)}\rangle$, is asymptotically given by:

$$|\tilde{\phi}_n^{(k)}\rangle \to \sqrt{\frac{k}{n}}|00\rangle + \sqrt{1 - \frac{k}{n}}|11\rangle, \; n \to \infty, \tag{4.69}$$

with k finite. The other optimal state is obtained via $|a| \rightleftarrows |b|$ [Lower branches in fig. 4.8 tend to Eq. (4.69)]. Thus, the optimal source tends to a collection of quasi-separable states. Yet, by using it we end up with a non-zero probability of obtaining n-partite entangled states. In this limit, the behaviour of the optimal probability, $\tilde{P}_n^{(k)}$, is given by:

$$\tilde{P}_n^{(k)} = \tilde{P}_\infty^{(k)} \left[1 + k\,(2n)^{-1}\right] + O\left(n^{-2}\right), \tag{4.70}$$

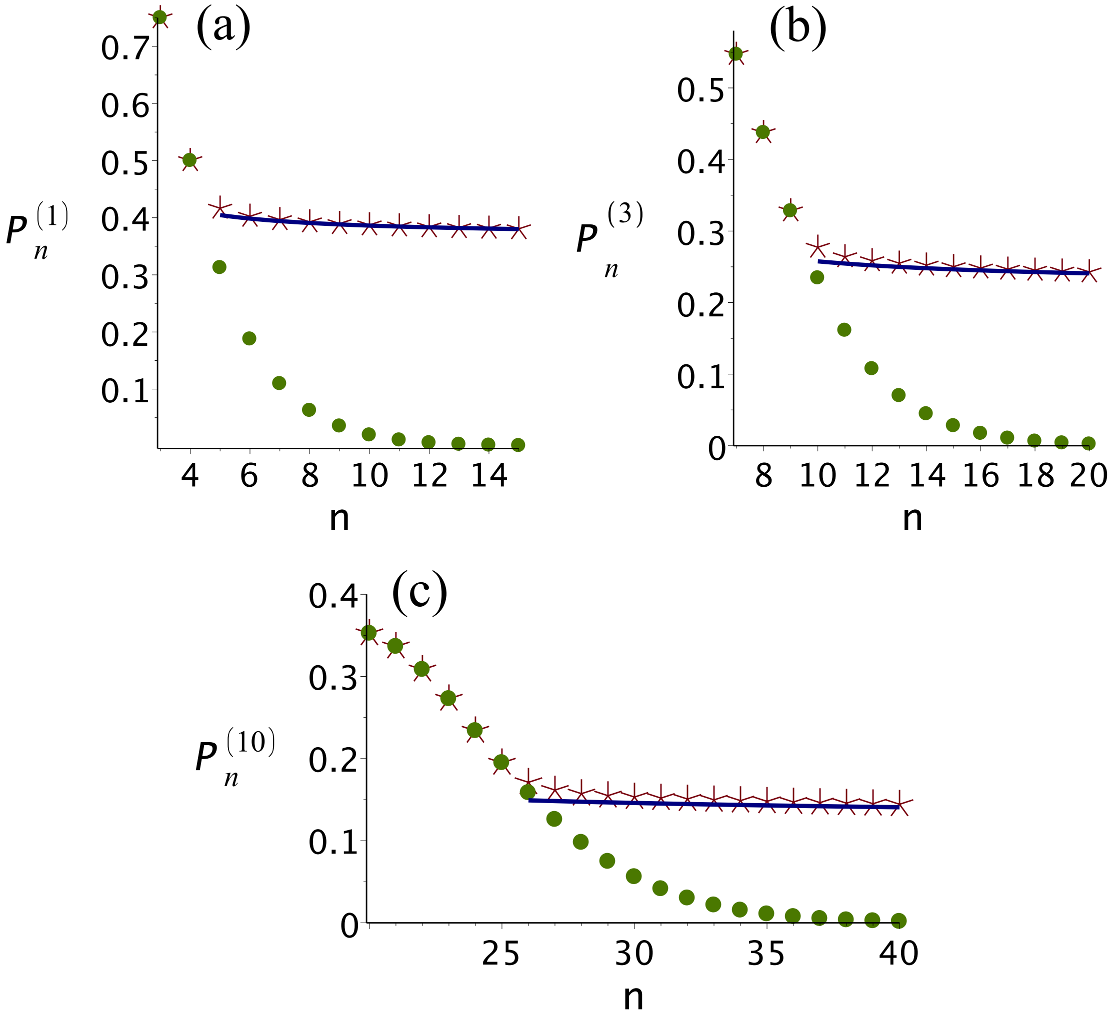

Figure 4.9 Probabilities of creating a Dicke state with (a) 3 excitations, (b) 4 excitations, (c) 10 excitations versus n. If EPR pairs are used $P_n^{(k)}$ drops exponentially (bullets). If the optimal source is used, it undergoes a power-law decay (stars).

where the constant asymptotic probability reads

$$\lim_{n\to\infty} \tilde{P}_n^{(k)} \equiv \tilde{P}_\infty^{(k)} = \frac{k^k\, e^{-k}}{k!}. \tag{4.71}$$

Although these are asymptotic results, the convergence is fast for small k's . The stars in fig. 4.8 are close to the continuous curve $[0.224(1+3/2n)]$ already for $n > 15$, see the lower-order non-constant term in the asymptotic expression for $\tilde{P}_n^{(k)}$, derived further.

Here we give the main steps to demonstrate equations (4.69) to (4.71). We intend to find the optimal probability of obtaining the state $|D_n^{(k)}\rangle$ in the situation $n > n_c$. For $n \neq k/2$ the probability reads

$$P_n^{(k)} = \binom{n}{k} \left[A^{(n-k)}\,(1-A)^k + A^k\,(1-A)^{n-k}\right], \tag{4.72}$$

where $A = |a|^2$. Initially we had numeric evidence that the optimized probability would follow a power law to reach a finite value as $n \to \infty$. We, therefore, set to look for a source state that would support this behaviour. Looking at Eq. (4.72), since $0 \leq A \leq 1$, we see that as $n \to \infty$, we must have either $A \to 0$ [lower branches in fig. 3 (a)] or $A \to 1$ [uper branches in fig. 3 (a)] in order to observe a $P_n^{(k)}$ finite. These two choices are equivalent due to the symmetry of equation (4.72), so we address the lower branch $A \to 0$ as $n \to \infty$. In this case, the first term in Eq. (4.72) vanishes, leading to the following result:

$$\lim_{\substack{n\to\infty \\ A\to 0}} P_n^{(k)} = \frac{n\,(n-1)\ldots(n-k+1)}{k!} A^k \frac{(1-A)^n}{(1-A)^k}$$

The product $n\,(n-1)\cdot\ldots\cdot(n-k+1)$ gives rise to a term of order n^k plus lower-order terms. So the only way to keep $P_n^{(k)}$ finite is to assume that $A \sim n^{-1}$, say, $A = \frac{\alpha}{n}$. With this we indeed obtain the finite asymptotic value:

$$\lim_{\substack{n\to\infty \\ A\to 0}} P_n^{(k)} = \lim_{n\to\infty} \frac{\alpha^k}{k!}\left(1-\frac{\alpha}{n}\right)^n = \frac{\alpha^k}{k!} e^{-\alpha} \equiv P_\infty^{(k)},$$

with α to be determined. Since we are seeking the optimal source, we simply maximize $P_\infty^{(k)}$:

$$\frac{d}{d\alpha} P_\infty^{(k)} = 0 \Rightarrow \alpha = k,\; A_{opt} \equiv \tilde{A} = \frac{k}{n},$$

which justifies Eqs. (4.69) to (4.71). Equation (4.70) is obtained by collecting the following lower order term, leading to

$$\tilde{P}_n^{(k)} \sim \frac{k^k}{k!} e^{-k}\left(1+\frac{k}{2n}\right).$$

In general, for small k this regime is quickly reached and, as the number of excitations increases (remaining finite), we observe a longer transient.

The properties derived so far provide information on the entanglement of Dicke states. Result (4.71) leads to the conclusion that, if entangled pairs are an available resource, the probability to create Dicke states as $n \to \infty$, with a finite number of excitations k is *non-vanishing even if the source has an arbitrarily low entanglement per pair*. For $k = 1$, e. g., one obtains $\tilde{P}_\infty^{(1)} \approx 0.368$, with $|\tilde{a}|^2 \to 1/n$. This is the signature of entanglement concentration: the exchange of a large number of copies with low entanglement by a small quantity of more entangled systems [BBPS96]. Here, in addition to concentration, the initial bipartite entanglement is lifted to a larger Hilbert space.

Finally, we derive asymptotic results on the amount of entanglement between any single qubit which is part of a system in a Dicke state and the rest of the system. Therefore we address the partition (Alice, $B_1, \cdots, B_{j-1}, B_{j+1}, \cdots, B_N | B_j) \equiv (I|II)$, where B_j represents any of the elements in the B group (due to the symmetry of the Dicke state the analysis does not depend on which qubit is being singled out). Note that, after Alice's local measurements this same partition reads $(B_1, \cdots, B_{j-1}, B_{j+1}, \cdots, B_N | B_j)$, which indeed refers to the entanglement between an arbitrary B-group member and the rest of the group. Suppose one intends to produce $|D_n^{(k)}\rangle$ with k finite and n arbitrarily large from a reservoir of optimal bipartite states, given by Eq. (4.69). The initial entanglement between I and II is precisely the entanglement between B_j and the rest of the system, i. e., $E(|\tilde{\phi}_n^{(k)}\rangle)$, once local operations and classical communications should not increase the entanglement between any bipartition, the final amount of entanglement between theses parts are upperbounded by $E(|\tilde{\phi}_n^{(k)}\rangle)$, where E is an arbitrary bipartite measure or monotone. Nevertheless it must also be lowerbounded by the amount of entanglement $\tilde{P}_n^{(k)} \times E_{(I|II)}(|D_n^{(k)}\rangle)$ between B_j and the rest of the B group –at this point Alice's part is completely factorable and plays no role. This holds because we are disregarding the contributions coming from the other Dick states that may be created. So, we must have,

$$E(|\tilde{\phi}_n^{(k)}\rangle) > \tilde{P}_n^{(k)} E_{(I|II)}(|D_n^{(k)}\rangle). \tag{4.73}$$

As $n \to \infty$ the left-hand side goes to zero from above. The same must be true for the positive-definite quantity in the right-hand side. However, in this limit the asymptotic probability is non-vanishing, which demands

$$E_{(I|II)}(|D_n^{(k)}\rangle) \to 0, \;\; n \to \infty. \tag{4.74}$$

This result is to be contrasted with the entanglement referring to the same partition of a system in a GHZ state of n qubits: $|GHZ_n\rangle = 2^{-1/2}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$, which gives $E_{(I|II)}(|GHZ_n\rangle) = 1$ ebit, for arbitrary n. As a specific example let us consider the 2-tangle as the bipartite measure. In this case, equation (4.73) reads, $\tau_{2(I|II)}(|D_n^{(k)}\rangle) < (4k/\tilde{P}_n^{(k)})n^{-1} + O(n^{-2})$. Therefore, the 2-tangle between an arbitrary qubit in a Dicke state and the rest of the system must go to zero, at least, as fast as n^{-1}, as $n \to \infty$ with k fixed.

4.3 A simplified experimental approach for alignment

In this section we intend to demonstrate one possible execution of the alignment part of our protocol. The main problem addressed is alignment, which in figure 4.6 is performed by the box S. The process uses only beam-splitters and polarization-beam-splitters, both objects of linear optics. Also, our proposal considers only the tripartite case, however a generalization of the protocol is straightforward. The sketch is depicted in figure 4.10.

First we have the three sources creating the state $|\Psi\rangle$:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{2\sqrt{2}}\left(a^\dagger_{B1}(H)a^\dagger_{E1}(H)+a^\dagger_{B1}(V)a^\dagger_{E1}(V)\right)\left(a^\dagger_{B2}(H)a^\dagger_{E2}(H)+a^\dagger_{B2}(V)a^\dagger_{E2}(V)\right)\\ &\left(a^\dagger_{B3}(H)a^\dagger_{E3}(H)+a^\dagger_{B3}(V)a^\dagger_{E3}(V)\right)|0\rangle \qquad (4.75)\end{aligned}$$

Here we have changed our notation. H (V) stand for horizontal (vertical) polarization. The state $|0\rangle$ represents vacuum. The operator $a^\dagger_X(Y)$ creates a photon in path X with polarization Y. The reference to the paths are made according to figure 4.10.

Using this notation the effect of a beam-splitter in the evolution of a state is the following:

$$\begin{aligned}a^\dagger_{E1}(X) &\rightarrow \frac{1}{\sqrt{2}}\left(a^\dagger_B(X)-a^\dagger_A(X)\right),\\ a^\dagger_{E2}(X) &\rightarrow \frac{1}{\sqrt{2}}\left(a^\dagger_{C2}(X)-a^\dagger_{C1}(X)\right),\\ a^\dagger_{E3}(X) &\rightarrow \frac{1}{\sqrt{2}}\left(a^\dagger_{C2}(X)+a^\dagger_{C1}(X)\right),\\ a^\dagger_{E1}(X) &\rightarrow \frac{1}{\sqrt{2}}\left(a^\dagger_B(X)+a^\dagger_A(X)\right).\end{aligned}$$

Thus, after the photons in path $E2$ and $E3$ hit the beam-splitter, we employ the relations above to have:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{4}\left(a^\dagger_{B1}(H)a^\dagger_{E1}(H)+a^\dagger_{B1}(V)a^\dagger_{E1}(V)\right)\\ &\Big[-a^\dagger_{B3}(V)a^\dagger_{B2}(V)a^{\dagger 2}_{C1}(V)-a^\dagger_{B3}(H)a^\dagger_{B2}(H)a^{\dagger 2}_{C1}(H)-\\ &\left(a^\dagger_{B3}(H)a^\dagger_{B2}(V)+a^\dagger_{B3}(V)a^\dagger_{B2}(H)\right)a^\dagger_{C1}(H)a^\dagger_{C1}(V)+\\ &C\Big]|0\rangle. \qquad (4.76)\end{aligned}$$

Here C represents all the terms with at least one photon in path $C2$, which represents evolutions out of our interest.

After the second beam-splitter, we have:

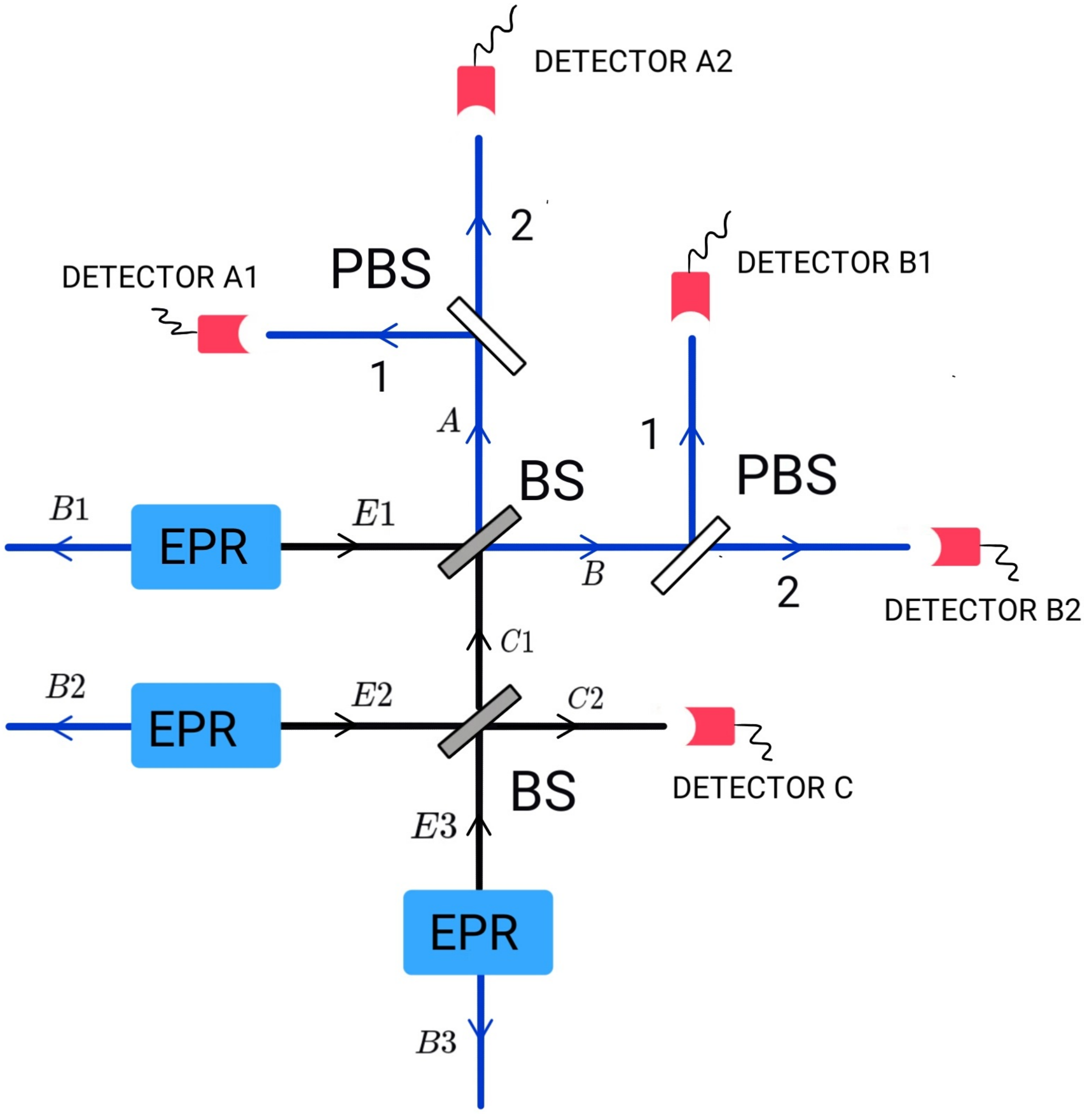

Figure 4.10 It is presented a protocol to align the photons emitted by the three EPR sources. Whenever some photon is detected in by detector C we have a failure. The same happens if we have $A1B1$, $A1B2$, $A2B1$, and $A2B2$.

$$
\begin{aligned}
|\Psi\rangle \;=\; & \frac{1}{8\sqrt{2}}\Big[a^{\dagger}_{B1}(H)a^{\dagger}_{B2}(H)a^{\dagger}_{B3}(H)a^{\dagger 3}_{B}(H)-a^{\dagger}_{B1}(H)a^{\dagger}_{B2}(H)a^{\dagger}_{B3}(H)a^{\dagger 3}_{A}(H)+\\
& a^{\dagger}_{B1}(V)a^{\dagger}_{B2}(V)a^{\dagger}_{B3}(V)a^{\dagger 3}_{V}(V)-a^{\dagger}_{B1}(V)a^{\dagger}_{B2}(V)a^{\dagger}_{B3}(V)a^{\dagger 3}_{A}(V)+\\
& a^{\dagger}_{B1}(H)a^{\dagger}_{B2}(V)a^{\dagger}_{B3}(V)a^{\dagger}_{B}(H)a^{\dagger 2}_{B}(V)-a^{\dagger}_{B1}(H)a^{\dagger}_{B2}(V)a^{\dagger}_{B3}(V)a^{\dagger}_{A}(H)a^{\dagger 2}_{A}(V)+\\
& a^{\dagger}_{B1}(V)a^{\dagger}_{B2}(H)a^{\dagger}_{B3}(H)a^{\dagger}_{B}(V)a^{\dagger 2}_{B}(H)-a^{\dagger}_{B1}(V)a^{\dagger}_{B2}(H)a^{\dagger}_{B3}(H)a^{\dagger}_{A}(V)a^{\dagger 2}_{A}(H)+\\
& a^{\dagger}_{B1}(H)a^{\dagger}_{B2}(V)a^{\dagger}_{B3}(H)a^{\dagger}_{B}(V)a^{\dagger 2}_{B}(H)-a^{\dagger}_{B1}(H)a^{\dagger}_{B2}(V)a^{\dagger}_{B3}(H)a^{\dagger}_{A}(V)a^{\dagger 2}_{A}(H)+\\
& a^{\dagger}_{B1}(H)a^{\dagger}_{B2}(H)a^{\dagger}_{B3}(V)a^{\dagger}_{B}(V)a^{\dagger 2}_{B}(H)-a^{\dagger}_{B1}(H)a^{\dagger}_{B2}(H)a^{\dagger}_{B3}(V)a^{\dagger}_{A}(V)a^{\dagger 2}_{A}(H)+\\
& a^{\dagger}_{B1}(V)a^{\dagger}_{B2}(V)a^{\dagger}_{B3}(H)a^{\dagger}_{B}(H)a^{\dagger 2}_{B}(V)-a^{\dagger}_{B1}(V)a^{\dagger}_{B2}(V)a^{\dagger}_{B3}(H)a^{\dagger}_{A}(H)a^{\dagger 2}_{A}(V)+\\
& a^{\dagger}_{B1}(V)a^{\dagger}_{B2}(H)a^{\dagger}_{B3}(V)a^{\dagger}_{B}(H)a^{\dagger 2}_{B}(V)-a^{\dagger}_{B1}(V)a^{\dagger}_{B2}(H)a^{\dagger}_{B3}(V)a^{\dagger}_{A}(H)a^{\dagger 2}_{A}(V)+\\
& AB+\tilde{C}\Big]|0\rangle
\end{aligned}
\tag{4.77}
$$

In equation (4.77) $\tilde{C}$ represents the evolution of C, which is not of our interest and AB is the term with all evolutions leading to at least one photon in A and one in B simultaneously, whose realization represents a failure of our protocol.

Finally, after passing the polarization Beam-splitter, we have:

$$
\begin{aligned}
|\Psi_F\rangle \;=\; & \frac{1}{8\sqrt{2}}\Big[|HHH\rangle\otimes|B2:3\rangle+|HHH\rangle\otimes|A2:3\rangle+\\
& |VVV\rangle\otimes|B1:3\rangle+|VVV\rangle\otimes|A1:3\rangle+\\
& \sqrt{3}\left(\frac{1}{\sqrt{3}}|HHV\rangle+\frac{1}{\sqrt{3}}|HVH\rangle+\frac{1}{\sqrt{3}}|VHH\rangle\right)\otimes|B1:1,B2:2\rangle+\\
& \sqrt{3}\left(\frac{1}{\sqrt{3}}|VVH\rangle+\frac{1}{\sqrt{3}}|VHV\rangle+\frac{1}{\sqrt{3}}|HVV\rangle\right)\otimes|B1:2,B2:1\rangle-\\
& \sqrt{3}\left(\frac{1}{\sqrt{3}}|HHV\rangle+\frac{1}{\sqrt{3}}|HVH\rangle+\frac{1}{\sqrt{3}}|VHH\rangle\right)\otimes|A1:1,A2:2\rangle-\\
& \sqrt{3}\left(\frac{1}{\sqrt{3}}|VVH\rangle+\frac{1}{\sqrt{3}}|VHV\rangle+\frac{1}{\sqrt{3}}|HVV\rangle\right)\otimes|A1:2,A2:1\rangle+\\
& |\psi\rangle\Big]
\end{aligned}
\tag{4.78}
$$

Therefore whenever we have detections on $A1$ and $A2$ or $B1$ and $B2$, which will occur with probability $\frac{3}{64}$, a Dick state with one excitation, or equivalent under LOCC, is created, otherwise the output is rejected.

It is worth stressing that this sketch does not represent an optimal protocol. Its presentation aims at providing some insight on the nature of the process of alignment. The synchronism is not mentioned above for, in principle, it would be possible to select only detections within some a priori established time window.

4.4 Discussion

We presented an efficient protocol to create Dicke states without the need of entangled measurements, a major technical difficulty. The efficiency is due to a high success probability and to the fact that whenever a Dicke state is produced it is ideal. In optimally executing this protocol the tunable source states present a behaviour that is analogous to a second-order phase transition in thermodynamics. In particular, the optimal source is not necessarily made of maximally entangled pairs.

Although our conclusions rely on the fact that our scheme is possible. In principle, an experiment seems to be already feasible for a relatively small number of parties. The most important ingredients, namely, tunable bipartite sources [WJEK99, XWS$^+$15, MIT15, MFC$^+$15], photon-number-resolving detectors [DMB$^+$08, MZG$^+$15] and synchronisation techniques [FCL$^+$06, YCC$^+$07, MHY$^+$16] are presently available, although, the conjunction of these elements may pose technical difficulties. The result of such an implementation would be the production of highly entangled multipartite states with a strong robustness against poor sources and with an increasing success rate for large n's. We finally call attention to the fundamental relation between the information loss induced by synchronisation and the simpler nature of the required measurements. This counterintuitive fact has been first reported, in a quite distinct context, in [FL07] and it deserves further investigation.

All bipartitions of arbitrary Dicke states

The growing interest in multipartite entangled states is partly a consequence of the rush to find, classify, and quantify useful resources to a great variety of tasks in the field of quantum information. For this reason the history of these states is relatively recent. To give a few outstanding examples, Greenberger-Horne-Zeilinger (GHZ) states appeared in 1990 in the context of Bell nonlocality, while cluster states were defined in 2001 as a fundamental resource to the realization of one-way quantum computing. These states have been formally introduced before any actual counterpart could be produced in the laboratory.

There is, however, one noticeable exception: Dicke states. They were introduced back in 1954 in an acknowledgedly important study on the spontaneous radiation emitted by a molecular gas. In spite of this specific, but important original goal, these states are becoming the core of several researches. In this chapter we analyze the amount of entanglement between bipartitions of systems of qubits in an arbitrary pure Dicke state. The evaluation of the amount of entanglement is possible through a relation which explicits the Schmidt decomposition of an arbitrary bipartition. This relation is proved through the principle of finite induction. Regarding this decomposition we are able to derive a property of Dicke states, which so far remained unclear in the literature: whether Dicke states are equivalent under LOCC. In this regard, we show that two of them are only equivalent when their number of excitations can be written as k and $n-k$. Also, using the decomposition, we derive a form for highly sensitive entanglement witnesses with respect to the fidelity of an arbitrary state ρ and the Dicke state associated with the witness.

5.1 Setting one qubit apart from a Dicke state

As mentioned in the introduction, Dicke states have been introduced as describing the internal degrees of freedom, ground (0) or excited (1), of a gas composed by n molecules. The system is admitted to be in a container whose dimensions are small in comparison to the wave length of the radiation (corresponding to the transitions $0 \leftrightarrow 1$), but large enough for the overlap between the individual molecular wave functions to be negligible, thus, avoiding the need of symmetrization. In this regime, although the molecules are distinguishable, one cannot identify which molecule emitted or absorbed a photon. Under these conditions the state of the

n-molecule gas with k excited molecules is

$$|D_n^{(k)}\rangle = \binom{n}{k}^{-\frac{1}{2}} \sum_{q=1}^{\binom{n}{k}} \hat{P}_q^{(n)} |\underbrace{0...0}_{n-k}\underbrace{1...1}_{k}\rangle, \tag{5.79}$$

which is the general form of a Dicke state. The operator $\hat{P}_q^{(n)}$ performs the qth permutation on the n entries of the ket.

The starting point of our study is the observation that definition (5.79) can be rewritten as:

$$|D_n^{(k)}\rangle = \binom{n}{k}^{-\frac{1}{2}} |0\rangle \otimes \sum_{q=0}^{\binom{n-1}{k}} \hat{P}_q^{(n-1)} |\underbrace{0...0}_{n-k-1}\underbrace{1...1}_{k}\rangle + \binom{n}{k}^{-\frac{1}{2}} |1\rangle \otimes \sum_{q=0}^{\binom{n-1}{k-1}} \hat{P}_q^{(n-1)} |\underbrace{0...0}_{n-k}\underbrace{1...1}_{k-1}\rangle,$$

or simply

$$|D_n^{(k)}\rangle = \left(\frac{n-k}{n}\right)^{\frac{1}{2}} |0\rangle|D_{n-1}^{(k)}\rangle + \left(\frac{k}{n}\right)^{\frac{1}{2}} |1\rangle|D_{n-1}^{(k-1)}\rangle, \tag{5.80}$$

where we dropped the tensor product symbol $\otimes$ (we will do so hereafter). Some comments are in order. First note that, due to the symmetry of the Dicke state, there is no need to say which qubit is being singled out from the other $(n-1)$ qubits. So that, all partitions of the system into 1 qubit and $(n-1)$ qubits, which we denote by $(1|n-1)$ are equivalently described by the previous state. Note, in addition, that Eq. (5.80) is a Schmidt decomposition for a bipartition with one qubit in one side, say with Alice and $(n-1)$ qubits in the other with Bob. This allows us to use the well-settled theoretical apparatus concerning the entanglement of pure states of bipartite systems (the fact that the two Hilbert spaces have different dimensions is immaterial). One can immediately write the entropy of entanglement, $S = \mathrm{Tr}_A[-\mathrm{Tr}_B(\rho_n^{(k)}) \log \mathrm{Tr}_B(\rho_n^{(k)})]$, bearing the partition $(1|n-1)$, with $\rho_n^{(k)} = |D_n^{(k)}\rangle\langle D_n^{(k)}|$. The result is

$$S_{ent}(n,k) = -\left(\frac{n-k}{n}\right) \log\left(\frac{n-k}{n}\right) - \left(\frac{k}{n}\right) \log\left(\frac{k}{n}\right),$$

where we are going to use log in basis 2. For $k = n/2$, which can only happen for n even, this function has an extreme value $S(n,n/2) = 1$ which means the qubit held by Alice is maximally entangled with Bob's part, that is, the entanglement between the qubit and the rest of the system is equivalent to that between the qubits of an EPR-Bell state. However as the number of parts n grows and the number of excitations is fixed, the amount of entanglement shared between Alice and Bob decreases and approaches zero as $n \to \infty$. This loss of entanglement present as n grows and k is fixed was already derived in reference [MP16]. A graphic representation of the entropy of entanglement behaviour as a function of n for different values of k is displayed in figure 5.11.

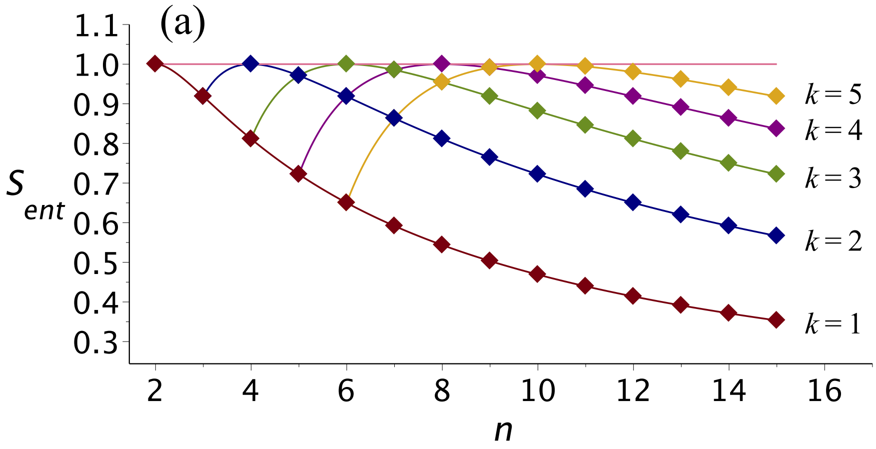

Figure 5.11 The entropy of entanglement of the partition $(1|n-1)$ is plotted for different values of k. In the graphic we can see the relation between $|D_n^{(k)}\rangle$ and $|D_n^{(n-k)}\rangle$ in the intersections of the curves. The purple line indicates the maximum amount of entanglement that tho qubits can have.

In graphic 5.11 it is possible to notice that some points superimpose others referring to different number of excitations. This is expected as a consequence of the fact that Dicke states $|D_n^{(k)}\rangle$ and Dicke states $|D_n^{(n-k)}\rangle$ are related trough local operations and classical communication (LOCC).

5.2 All Bipartitions

Note that we can use Eq. (5.80) into itself to expand the states $|D_{n-1}^{(k)}\rangle$ and $|D_{n-1}^{(k-1)}\rangle$ in terms of Dicke states with k, $k-1$, and $k-2$ excitations and so forth. By repeating this procedure self-consistently j times we obtain:

$$|D_n^{(k)}\rangle = \sum_{q=\max(0,j-n+k)}^{\min(j,k)} \left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right]^{\frac{1}{2}} |D_j^{(q)}\rangle \otimes |D_{n-j}^{(k-q)}\rangle. \quad (5.81)$$

To prove relation (5.81) we use the principle of induction. Thus let us show that it is valid for $j = 1$:

$$
\begin{aligned}
|D_n^{(k)}\rangle &= \sum_{q=0}^{1} \left[\frac{(n-1)!k!(n-k)!}{n!(k-q)!(n-k-1+q)!} \binom{j}{q} \right]^{\frac{1}{2}} |D_1^{(q)}\rangle |D_{n-1}^{(k-q)}\rangle \\
&= \left[\frac{(n-1)!k!(n-k)!}{n!k!(n-k-1)!} \right]^{\frac{1}{2}} |D_1^{(0)}\rangle |D_{n-1}^{(k)}\rangle + \\
& \left[\frac{(n-1)!k!(n-k)!}{n!(k-1)!(n-k)!} \right]^{\frac{1}{2}} |D_1^{(1)}\rangle |D_{n-1}^{(k-1)}\rangle \\
&= \left(\frac{n-k}{n} \right)^{\frac{1}{2}} |0\rangle |D_{n-1}^{(k)}\rangle + \left(\frac{k}{n} \right)^{\frac{1}{2}} |1\rangle |D_{n-1}^{(k-1)}\rangle,
\end{aligned}
$$

which is relation (5.80).

Now we are going to test if assuming its validity for some arbitrary j it is also valid for $j+1$. First we consider $j \leq k$ and $0 > j-n+k$, and use relation (5.80) into (5.81) obtaining:

$$
\begin{aligned}
|D_n^{(k)}\rangle &= \sum_{q=0}^{j} \left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!} \binom{j}{q} \right]^{\frac{1}{2}} \left(\frac{n-j-k+q}{n-j} \right)^{\frac{1}{2}} |D_j^{(q)}\rangle |0\rangle |D_{n-j-1}^{(k-q)}\rangle + \\
& \sum_{q'=0}^{j} \left[\frac{(n-j)!k!(n-k)!}{n!(k-q')!(n-k-j+q')!} \binom{j}{q'} \right]^{\frac{1}{2}} \left(\frac{k-q'}{n-j} \right)^{\frac{1}{2}} |D_j^{(q')}\rangle |1\rangle |D_{n-j-1}^{(k-q'-1)}\rangle.
\end{aligned}
$$

Leading to:

$$
\begin{aligned}
|D_n^{(k)}\rangle &= \sum_{q=0}^{j} \left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q)!(n-k-j+q-1)!} \binom{j}{q} \right]^{\frac{1}{2}} |D_j^{(q)}\rangle |0\rangle |D_{n-j-1}^{(k-q)}\rangle + \\
& \sum_{q'=0}^{j} \left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q'-1)!(n-k-j+q')!} \binom{j}{q'} \right]^{\frac{1}{2}} |D_j^{(q')}\rangle |1\rangle |D_{n-j-1}^{(k-q'-1)}\rangle.
\end{aligned}
$$

Now define q' such that $q = q'+1$:

$$
\begin{aligned}
|D_n^{(k)}\rangle &= \sum_{q=0}^{j} \left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q)!(n-k-j-1+q)!} \binom{j+1}{q} \right]^{\frac{1}{2}} \left(\frac{j+1-q}{j+1} \right)^{\frac{1}{2}} |D_j^{(q)}\rangle |0\rangle |D_{n-j-1}^{(k-q)}\rangle \\
& \sum_{q=1}^{j+1} \left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q)!(n-k-j-1+q)!} \binom{j+1}{q} \right]^{\frac{1}{2}} \left(\frac{q}{j+1} \right)^{\frac{1}{2}} |D_j^{(q-1)}\rangle |1\rangle |D_{n-j-1}^{(k-q)}\rangle.
\end{aligned}
$$

Let us also define $j' = j+1$:

$$
\begin{aligned}
|D_n^{(k)}\rangle \;=\; & \sum_{q=0}^{j'-1}\left[\frac{(n-j')!k!(n-k)!}{n!(k-q)!(n-k-j'+q)!}\binom{j'}{q}\right]^{\frac{1}{2}}\left(\frac{j'-q}{j'}\right)^{\frac{1}{2}}|D_{j'-1}^{(q)}\rangle|0\rangle|D_{n-j'}^{(k-q)}\rangle+\\
& \sum_{q=1}^{j'}\left[\frac{(n-j')!k!(n-k)!}{n!(k-q)!(n-k-j'+q)!}\binom{j'}{q}\right]^{\frac{1}{2}}\left(\frac{q}{j'}\right)^{\frac{1}{2}}|D_{j'-1}^{(q-1)}\rangle|1\rangle|D_{n-j'}^{(k-q)}\rangle.
\end{aligned}
$$

Using relation (5.80):

$$
|D_n^{(k)}\rangle=\sum_{q=0}^{j'}\left[\frac{(n-j')!k!(n-k)!}{n!(k-q)!(n-k-j'+q)!}\binom{j'}{q}\right]^{\frac{1}{2}}|D_{j'}^{(q)}\rangle|D_{n-j'}^{(k-q)}\rangle.
$$

Recovering the original form.
Now we consider $j>k$ and $0>j-n+k$. Again we start using relation (5.80) into (5.81):

$$
\begin{aligned}
D_n^{(k)}\rangle \;=\; & \sum_{q=0}^{k-1}\left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right]^{\frac{1}{2}}\left(\frac{n-j-k+q}{n-j}\right)^{\frac{1}{2}}|D_j^{(q)}\rangle|0\rangle|D_{n-j-1}^{(k-q)}\rangle+\\
& \sum_{q'=0}^{k-1}\left[\frac{(n-j)!k!(n-k)!}{n!(k-q')!(n-k-j+q')!}\binom{j}{q'}\right]^{\frac{1}{2}}\left(\frac{k-q'}{n-j}\right)^{\frac{1}{2}}|D_j^{(q')}\rangle|1\rangle|D_{n-j-1}^{(k-q'-1)}\rangle+\\
& \left[\frac{k!(n-k)!}{n!}\binom{j}{k}\right]^{\frac{1}{2}}|D_j^{(k)}\rangle|0\rangle|D_{n-j-1}^{(0)}\rangle.
\end{aligned}
$$

Making $q=q'+1$, we have:

$$
\begin{aligned}
D_n^{(k)}\rangle \;=\; & \sum_{q=0}^{k-1}\left[\frac{(n-j-1)!k!(n-k!)}{n!(k-q)!(n-k-j-1+q)!}\binom{j+1}{q}\right]^{\frac{1}{2}}\left(\frac{j+1-q}{j+1}\right)^{\frac{1}{2}}|D_j^{(q)}\rangle|0\rangle|D_{n-j-1^{(k-q)}}\rangle+\\
& \sum_{q=1}^{k-1}\left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q)!(n-k-j-1+q)}!\binom{j+1}{q}\right]^{\frac{1}{2}}\left(\frac{q}{j+1}\right)|D_j^{(q-1)}\rangle|1\rangle|D_{n-j-1}^{(k-q)}\rangle \qquad (5.82)\\
& \left[\frac{k!(n-k)!}{n!}\binom{j+1}{k}\right]^{\frac{1}{2}}\left(\frac{k}{j+1}\right)^{\frac{1}{2}}|D_j^{(k-1)}\rangle|1\rangle|D_{n-j-1}^{(0)}\rangle+\\
& \left[\frac{k!(n-k)!}{n!}\binom{j+1}{k}\right]^{\frac{1}{2}}\left(\frac{j+1-k}{j+1}\right)^{\frac{1}{2}}|D_j^{(k)}\rangle|0\rangle|D_{n-j-1}^{(0)}\rangle.
\end{aligned}
$$

Defining $j'\equiv j+1$:

$$
\begin{aligned}
|D_n^{(k)}\rangle \;=\; & \sum_{q=0}^{k-1}\left[\frac{(n-j')!k!(n-k!)}{n!(k-q)!(n-k-j'+q)!}\binom{j'}{q}\right]^{\frac{1}{2}}\left(\frac{j'-q}{j'}\right)^{\frac{1}{2}}|D_{j'-1}^{(q)}\rangle|0\rangle|D_{n-j'^{(k-q)}}\rangle+\\
& \sum_{q=1}^{k-1}\left[\frac{(n-j')!k!(n-k)!}{n!(k-q)!(n-k-j'+q)}!\binom{j'}{q}\right]^{\frac{1}{2}}\left(\frac{q}{j'}\right)|D_{j'-1}^{(q-1)}\rangle|1\rangle|D_{n-j'}^{(k-q)}\rangle \qquad (5.83)\\
& \left[\frac{k!(n-k)!}{n!}\binom{j'}{k}\right]^{\frac{1}{2}}\left(\frac{k}{j'}\right)^{\frac{1}{2}}|D_{j'-1}^{(k-1)}\rangle|1\rangle|D_{n-j'}^{(0)}\rangle+\\
& \left[\frac{k!(n-k)!}{n!}\binom{j'}{k}\right]^{\frac{1}{2}}\left(\frac{j'-k}{j'}\right)^{\frac{1}{2}}|D_{j'-1}^{(k)}\rangle|0\rangle|D_{n-j'}^{(0)}\rangle.
\end{aligned}
$$

Using relation (5.80) we recover the original form:

$$
|D_n^{(k)}\rangle \;=\; \sum_{q=0}^{k}\left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right]^{\frac{1}{2}}|D_j^{(q)}\rangle\otimes|D_{n-j}^{(k-q)}\rangle.
$$

We follow the same steps for the case $j<k$ and $0\leq j-n+k$:

$$
\begin{aligned}
|D_n^{(k)}\rangle \;=\; & \sum_{q=j-n+k+1}^{j}\left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right]^{\frac{1}{2}}|D_j^{(q)}\rangle|D_{n-j}^{(k-q)}\rangle\\
=\; & \sum_{q=j-n+k+1}^{j}\left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right]^{\frac{1}{2}}\left(\frac{n-j-k+q}{n-j}\right)^{\frac{1}{2}}|D_j^{(q)}\rangle|0\rangle|D_{n-j-1}^{(k-q)}\rangle+\\
& \sum_{q'=j-n+k+1}^{j}\left[\frac{(n-j)!k!(n-k)!}{n!(k-q')!(n-k-j+q')!}\binom{j}{q'}\right]^{\frac{1}{2}}\left(\frac{k-q'}{n-j}\right)^{\frac{1}{2}}|D_j^{(q')}\rangle|1\rangle|D_{n-j-1}^{k-q'-1}+\\
& \left[\frac{k!(n-k)!}{n!}\binom{j}{j+k-n}\right]^{\frac{1}{2}}|D_j^{(j+k-n)}\rangle|1\rangle|D_{n-j-1}^{(n-j-1)}\rangle.
\end{aligned}
$$

Choosing $q=q'+1$, we get to:

$$
\begin{aligned}
|D_n^{(k)}\rangle \;=\; & \sum_{q=j-n+k+1}^{j}\left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q)!(n-k-j-1+q)!}\binom{j+1}{q}\right]^{\frac{1}{2}}\left(\frac{j+1-q}{j+1}\right)^{\frac{1}{2}}|D_j^{(q)}\rangle|0\rangle|D_{n-j-1}^{(k-q)}\rangle+\\
& \sum_{q=j-n+k+2}^{j+1}\left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q)!(n-k-j-1-q)!}\binom{j+1}{q}\right]^{\frac{1}{2}}\left(\frac{q}{j+1}\right)^{\frac{1}{2}}|D_j^{(q-1)}\rangle|1\rangle|D_{n-j-1}^{(k-q)}\rangle+\\
& \left[\frac{k!(n-k)!}{n!}\binom{j+1}{j+1+k-n}\right]^{\frac{1}{2}}\left(\frac{j+1+k-n}{j+1}\right)^{\frac{1}{2}}|D_j^{(j+k-n)}\rangle|1\rangle|D_{n-j-1}^{(n-j-1)}\rangle.
\end{aligned}
$$

Setting $j' = j+1$:

$$
\begin{aligned}
|D_n^{(k)}\rangle \;=\; & \sum_{q=j'-n+k+1}^{j'-1} \left[\frac{(n-j')!k!(n-k)!}{n!(k-q)!(n-k-j'+q)!}\binom{j'}{q}\right]^{\frac{1}{2}} \left(\frac{j'-q}{j'}\right)^{\frac{1}{2}} |D_{j'-1}^{(q)}\rangle|0\rangle|D_{n-j'}^{k-q}\rangle + \\
& \sum_{q=j'-n+k+1}^{j'} \left[\frac{(n-j')!k!(n-k)!}{n!(k-q)!(n-k-j'+q)!}\binom{j'}{q}\right]^{\frac{1}{2}} \left(\frac{q}{j'}\right)^{\frac{1}{2}} |D_{j'-1}^{(q-1)}\rangle|1\rangle|D_{n-j'}^{(k-q)}\rangle + \\
& \left[\frac{k!(n-k)!}{n!}\binom{j'}{j'-n+k}\right]^{\frac{1}{2}} \left(\frac{n-k}{j'}\right)^{\frac{1}{2}} |D_{j'-1}^{(j'-n+k)}\rangle|0\rangle|D_{n-j'}^{(k-q)}\rangle \\
& \left[\frac{k!(n-k)!}{n!}\binom{j'}{j'+k-n}\right]^{\frac{1}{2}} \left(\frac{j'+k-n}{j'}\right)^{\frac{1}{2}} |D_{j'-1}^{(j'-1+k-n)}\rangle|1\rangle|D_{n-j'}^{(n-j')}\rangle.
\end{aligned}
$$

Through relation (5.80) we have:

$$
|D_n^{(k)}\rangle \;=\; \sum_{q=j-n+k}^{j} \left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right]^{\frac{1}{2}} |D_j^{(q)}\rangle \otimes |D_{n-j}^{(k-q)}\rangle.
$$

Finally we consider the case where $j > k$ and $0 \leq j-n+k$. As in the previous times, we start by applying equation (5.80) into equation (5.81):

$$
\begin{aligned}
|D_n^{(k)}\rangle \;=\; & \sum_{q=j-n+k+1}^{k} \left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right]^{\frac{1}{2}} \left(\frac{n-j-k+q}{n-j}\right)^{\frac{1}{2}} |D_j^{(q)}\rangle|0\rangle|D_{n-j-1}^{(k-q)}\rangle + \\
& \sum_{q'=j-n+k+1}^{k} \left[\frac{(n-j)!k!(n-k)!}{n!(k-q')!(n-k-j+q')!}\binom{j}{q'}\right]^{\frac{1}{2}} \left(\frac{k-q'}{n-j}\right)^{\frac{1}{2}} |D_j^{(q')}\rangle|1\rangle|D_{n-j-1}^{(k-q'-1)}\rangle + \\
& \left[\frac{k!(n-k)!}{n!}\binom{j}{j-n+k}\right]^{\frac{1}{2}} |D_j^{(j-n+k)}\rangle|1\rangle|D_{n-j-1}^{(k-q-1)}\rangle.
\end{aligned}
$$

Choosing $q = q'+1$:

$$
\begin{aligned}
|D_n^{(k)}\rangle \;=\; & \sum_{q=j-n+k+1}^{k} \left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q)!(n-k-j+q-1)!}\binom{j+1}{q}\right]^{\frac{1}{2}} \left(\frac{j+1-q}{j+1}\right)^{\frac{1}{2}} |D_j^{(q)}\rangle|0\rangle|D_{n-j-1}^{(k-q)}\rangle + \\
& \sum_{q=j-n+k+2}^{k+1} \left[\frac{(n-j-1)!k!(n-k)!}{n!(k-q)!(n-k-j-1+q)!}\binom{j+1}{q}\right]^{\frac{1}{2}} \left(\frac{q}{j+1}\right)^{\frac{1}{2}} |D_j^{(q-1)}\rangle|1\rangle|D_{n-j-1}^{(k-q)}\rangle + \\
& \left[\frac{k!(n-k)!}{n!}\binom{j+1}{j+1-n+k}\right]^{\frac{1}{2}} \left(\frac{j+1-n+k}{j+1}\right)^{\frac{1}{2}} |D_j^{(j-n+k)}\rangle|1\rangle|D_{n-j-1}^{(k-q-1)}\rangle.
\end{aligned}
$$

Setting again $j' = j+1$, we have:

$$
\begin{aligned}
|D_n^{(k)}\rangle &= \sum_{q=j'-n+k+1}^{k} \left[\frac{(n-j')!k!(n-k)!}{n!(k-q)!(n-k-j'+q)!}\binom{j'}{q}\right]^{\frac{1}{2}} \left(\frac{j'-q}{j'}\right)^{\frac{1}{2}} |D_{j'-1}^{(q)}\rangle|0\rangle|D_{n-j'}^{(k-q)}\rangle + \\
&\quad \sum_{q=j'-n+k+1}^{k+1} \left[\frac{(n-j')!k!(n-k)!}{n!(k-q)!(n-k-j'+q)!}\binom{j'}{q}\right]^{\frac{1}{2}} \left(\frac{q}{j'}\right)^{\frac{1}{2}} |D_{j'-1}^{(q-1)}\rangle|1\rangle|D_{n-j'}^{(k-q)}\rangle + \\
&= \left[\frac{k!(n-k)!}{n!}\binom{j'}{j'-n+k}\right]^{\frac{1}{2}} \left(\frac{n-k}{j'}\right)^{\frac{1}{2}} |D_{j'-1}^{(j'-n+k)}\rangle|0\rangle|D_{n-j'}^{(n-j')}\rangle \\
&\quad \left[\frac{k!(n-k)!}{n!}\binom{j'}{j'-n+k}\right]^{\frac{1}{2}} \left(\frac{j'-n+k}{j'}\right)^{\frac{1}{2}} |D_{j'-1}^{(j'-1-n+k)}\rangle|1\rangle|D_{n-j'}^{(n-j')}\rangle.
\end{aligned}
$$

Which, by using (5.80), leads us to:

$$
|D_n^{(k)}\rangle = \sum_{q=j-n+k}^{k} \left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right]^{\frac{1}{2}} |D_j^{(q)}\rangle \otimes |D_{n-j}^{(k-q)}\rangle,
$$

completing our demonstration.

Note that Eq. (5.81) is a Schmidt decomposition for all bipartitions $(j|n-j)$. We, therefore, can immediately write the entropy of entanglement corresponding to any bipartition. It reads:

$$
\begin{aligned}
S = -\sum_{q=\max(0,j-n+k)}^{\min(j,k)} \quad & \frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q} \\
& \log\left[\frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}\right],
\end{aligned}
\tag{5.84}
$$

Graphics of S_{ent} for $j=2$ and $j=3$ are framed in figure 5.12. In this graphics we can see that for $k=1$ the entropy is limited to 1 in both cases, suggesting that the partition entangles like a single qubit. However for $k=2$ the system may present a stronger entanglement, in fact in both graphics there is a gap between the curve $k=1$ and $k=2$.

Regarding the relation $|D_n^{(k)}\rangle \overset{LOCC}{\longleftrightarrow} |D_n^{(n-k)}\rangle$ we expect that the function $S(n,k,j)$ will present an extreme value for $k=n/2$, whenever n is even. Setting this value of k in equation 5.84 in the limit $n\to\infty$, for $j \le n/2$ we have:

$$
\lim_{n\to\infty} S(n,n/2,j) = j - \frac{1}{2^j}\sum_{q=0}^{j}\binom{j}{q}\log\left[\binom{j}{q}\right]. \tag{5.85}
$$

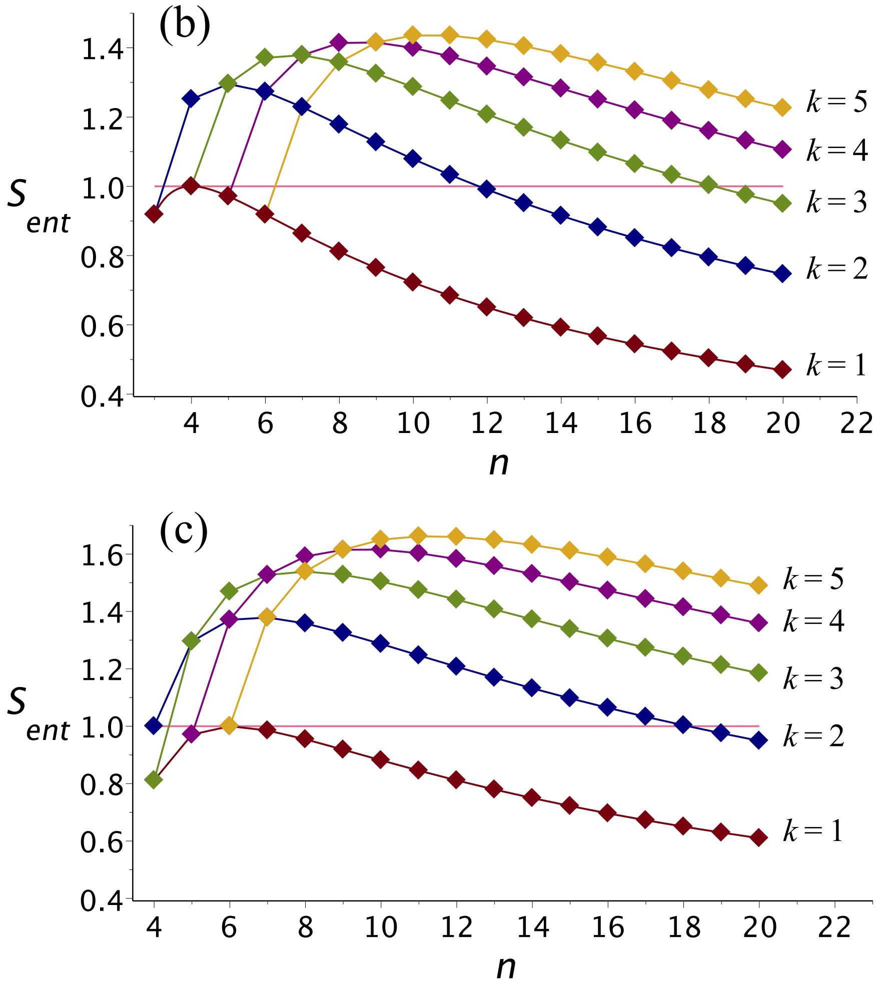

Figure 5.12 Entropy of entanglement of the partitions (a)$(2|n-2)$ (b)$(3|n-3)$ are for different values of k. In the graphic we can see the relation between $|D_n^{(k)}\rangle$ and $|D_n^{(n-k)}\rangle$ in the intersections of the curves. The purple line indicates the maximum amount of entanglement that tho qubits can have.

In equation 5.85 we see that for a fixed j, the entropy of entanglement grows with n but it is far from the maximum value of entropy by an amount $\frac{1}{2^j}\sum_{q=0}^{j}\binom{j}{q}\log\left[\binom{j}{q}\right]$, which also grows with n. Figure 5.13 represents the behaviour of $S(n.k,j)$ for different values of n and different values of k. We notice the expected behavior: states with more parts can present higher values for the entropy of entanglement. Nevertheless it is worth noticing that for small partitions and a fixed value of excitations systems with smaller number of parties are more entangled than those with bigger number of parts, and as we approach $j=\frac{n}{2}$, this situation inverts. This is coherent with the idea presented in reference [MP16] of a dilution of entanglement.

Also, we highlight that in general $S(n,k,j)$ is different from $S(n,k',j)$, but the mentioned situation where $k'=n-k$. *This allow us to ensure that Dicke states with different excitations are not LOCC equivalent.*

In a similar argument we must have $S(n.k,j)=S(n.k,n-j)$, for $(j|n-j)$ and $(n-j|j)$ represent the same bipartition, thus we expect to have an extreme value for $j=n/2$. Once j and k are symmetric in the equation for $n\to\infty$ we must have:

$$\lim_{n\to\infty} S(n,k,n/2) = k-\frac{1}{2^k}\sum_{q=0}^{k}\binom{k}{q}\log\left[\binom{k}{q}\right]. \tag{5.86}$$

We notice that when $k=n/2$ and $j=n/2$ the limits of the sum in equation 5.84 show the highest Schmidt measure [EB01, BR01]. Hence it shall represent the configuration with stronger quantum binding for this system.

It is possible to see in equation (5.81) that both j and k play symmetric roles. This implies in the following relation: a bipartition $(j|n-j)$ of a system in a Dicke state with k excitations have the same entropy of entanglement than a a partition $(k|n-k)$ of the same system in a Dicke state with j excitations.

Further more, from equation 5.84 one can easily see that for a fixed value of k whenever j is small in comparison with n, the entropy of entanglement decreases with n. This indicates that small partitions of a Dicke state with a fixed value of excitations become nearly separable as the system grows.

Another interesting relation is that the reduced density operator $\rho_n'^{(k)}$ of a part of the bipartition, which can be obtained from equation 5.81 has the form:

$$\sum_{q=q'}^{q''} \frac{(n-j)!k!(n-k)!}{n!(k-q)!(n-k-j+q)!}\binom{j}{q}|D_j^{(q)}\rangle\otimes\langle D_j^{(q)}|,$$

which is a mixture of Dicke states and thus is likely to present quantum correlations itself. For the case of $j=2$ this property reveals that as n increases two qubits, elements of a Dicke state of n qubits and k excitations, behave nearly like classically correlated qubits.

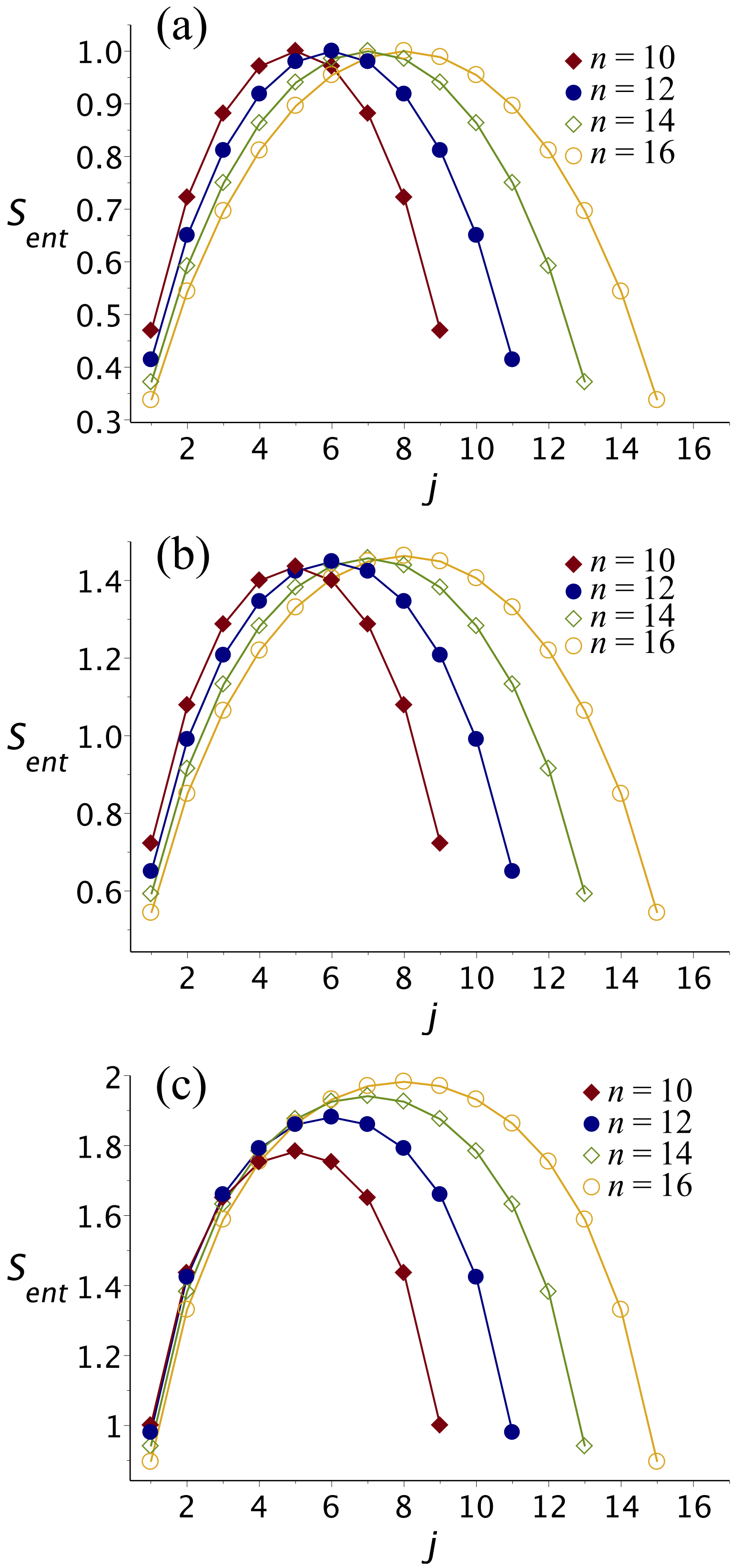

Figure 5.13 Entropy of entanglement as a function of the number of elements in the bipartition j for selected values of n for (a) $k = 1$ (b)$k = 2$ and (c)$k = 5$.

5.3 Tight multipartite entanglement witness

M. Bourennane *et al.* [BEK$^+$04] provided a scheme for the construction of multipartite entanglement witnesses [HHH96, HHHH09]. Given an entangled state $|\Psi\rangle$ the witness they build would be able to detect entanglement of $|\Psi\rangle$ and states close to it. The witness $\mathscr{W}$ reads:

$$\mathscr{W} = \alpha \mathbb{1} - |\Psi\rangle\langle\Psi|, \tag{5.87}$$

where α is the squared modulus of the highest Schmidt coefficient over all possible bipartitions.

Since we have an explicit Schmidt representation for all possible bipartitions of a n-partite system, the implementation of such witness becomes simple. Considering that the partition that maximizes the Schmidt coefficient must be the same that minimizes the entropy, to find α we must choose $j = 1$. A simple calculation shows that the maximization choice for q is $q = 0$ if $1 \le k \le n/2$ and $q = 1$ for $n/2 < k < n$. Those results define a set of $n-2$ witnesses $\mathscr{W}_k$, for $1 \le k < n$, each of them associated with a specific Dicke state $|D_n^{(k)}\rangle$.

A direct application of those witnesses is to test the entanglement resistance of Dicke states against noise. Consider the state $\rho_k = p_k \frac{1}{n}\mathbb{1} + (1-p_k)|D_n^{(k)}\rangle\langle D_n^{(k)}|$, the question is how large can p_k be so that we can still witness entanglement with $\mathscr{W}_k$. This limit can be appreciated through the following relation:

$$\mathrm{Tr}(|D_n^{(k)}\rangle\langle D_n^{(k)}|\rho_k) \le \begin{cases} \frac{n-k}{n}, & 1 \le k \le \frac{n}{2} \\ \frac{k}{n}, & \frac{n}{2} < k < n \end{cases}, \tag{5.88}$$

Which leads to the following condition:

$$p_k^{max} = \begin{cases} \frac{k}{n(1-2^{-n})}, & 1 \le k \le \frac{n}{2} \\ \frac{n-k}{n(1-2^{-n})}, & \frac{n}{2} < k < n \end{cases}, \tag{5.89}$$

Géza Tóth [Tót07] have already considered the case for $k = n/2$, which derived the values:

$$p_k^{max} = \frac{1}{2}\left[\frac{n-2}{(n-1)(1-2^{-n})}\right]. \tag{5.90}$$

However, here we present a tighter relation and extend it to all possible values of k.

5.4 Discussion

We provide a closed form for a Schmidt decomposition of any bipartition of a n-qubit system in an arbitrary Dicke state. Using this result we characterize these bipartitions through the entanglement entropy and derived some properties so far unknown.

In general, the behavior of the amount of entanglement encompassed by bipartitions in Dicke states has a rich structure. Generalizations of n-partite Greenberger-Horne-Zeilinger (GHZ) states [GHZ89, GHSZ90], for example, have a fixed value $S = 1$ for any possible bipartition no matter the size of the state displaying a less interesting structure.

Based on our analysis, it was possible to ensure that Dicke states with the same number of parts and different number of excitations are not LOCC equivalent, unless the excitations k and k' of each of them are related as $k' = n - k$.

Another interesting feature derived was the decrease of quantum correlations between bipartitions $(j|n-j)$ with small j compared to n for a fixed k.

Conclusion

In this thesis we presented a study on the production and entanglement features of Dicke states. The process of production here is always based on entanglement-swapping like processes starting with entangled pairs of qubits. The features of entanglement are considered under bipartitions of systems with arbitrary number of qubits.

In chapter three we initiate our study on the production of Dicke states. We have considered the specific case of three-partite Dicke states. We have show that there is a basis whose elements are equivalent under local operations and classical communications to the one-excitation Dicke state, also known as W state. This fact allows us to execute the process deterministically. Also in that chapter we care for another procedure of creation of W states, which considered sources of unbalanced EPR states, in this case we present a probabilistic scheme for which the output, when successful, happens to be always a perfect W state. This last procedure led to an interpretation of the phenomenon as a process of entanglement concentration and yielded a final state with high fidelity relative to the W state. With these results in mind and considering the practical difficulties involved in full entangled measurements, we set to find a way to produce similar results with unentangled measurements. Our findings were presented in chapter four.

In chapter 4, we work on the n-partite case. We first introduced a probabilistic way of creating photons in W states using only EPR pairs and elements of linear optics. The process is very similar to the probabilistic scheme showed in the third chapter, also having a perfect output, when successful. Then we generalized this protocol to the case of n qubits. In this scenario we demonstrated that the protocol can create any kind of Dicke state using unentangled measurements of photons Fock states in paths of an interferometer. This procedure was inspired in a controlled loss of information of the photons to be measured, which during the process become indistinguishable. Tracking the success probability as a function of the coefficient a of the unbalanced EPR sources for each case we found a critical transition similar to the Landau's model of phase transition. After the critical phenomena the maximum value of the success probability happens for a value of a different of $\frac{1}{\sqrt{2}}$. We characterized the critical phenomenon. We also showed that by tracking the a that yields the maximum value, it is possible to create arbitrarily large Dicke states, with a fixed number of excitations, with finite probability. That last step provided an opportunity to establish an upperbound to the amount of entanglement between one part of a system in a Dicke state and the rest of it. Inspired by these results, we systematically investigated the bipartitions of systems described by Dicke states. The results of this investigation are displayed in chapter five.

The fifth chapter is dedicated to an analysis of the amount of entanglement between bipartitions of systems in a Dicke state. First we have considered the case where a single qubit of the system is isolated, deriving a formula for the Schmidt decomposition in this case. Using this result, we calculated the entropy of this specific bipartition for the case of an arbitrary Dicke state. Afterwards we introduces a representation of the Schmidt decomposition of an arbitrary bipartition. We proved by induction the relation to be valid and employed it to evaluate the entropy of entanglement of an arbitrary bipartition of a system in an arbitrary Dicke state. We also used this Schmidt decomposition to build entanglement witnesses which we showed to be sensitive to Dicke states even in the presence of noise and imperfections.

In a summary, this thesis presentes protocols for creation and detection of Dicke states with linear optics devices. The process employed in the creation of these states led to the theoretical prediction of a new critical phenomenon. We would like to raise the perspective of experimental verification of this phenomenon. Although we have not reached the point of presenting an experimental proposal, we believe that the results of chapter 4 and 5, when assembled together represent a starting point for this quest. Also as a perspective, we intent to conduct a deeper investigation on the results of chapter five concerning the amount of entanglement between bipartitions of systems in Dicke states, testing limits and possible connections with Thermodynamics.

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