UNIVERSIDADE FEDERAL DE PERNAMBUCO
DEPARTAMENTO DE FÍSICA – CCEN
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA

FILIPE RUDRIGUES DE ALMEIDA LIRA

LARGE GRAVITONS AND NEAR HORIZON DIFFEOMORPHISMS

Recife
2017

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Dissertação apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Física.

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Recife
2017

Catalogação na fonte
Bibliotecário Jefferson Luiz Alves Nazareno CRB 4-1758

L768l Lira, Filipe Rudrigues de Almeida.
Large gravitons and near horizon diffeomorphisms / Filipe Rudrigues de Almeida Lira. – 2017.
59 f.: fig.

Orientador: Bruno Geraldo Carneiro da Cunha.
Dissertação (Mestrado) – Universidade Federal de Pernambuco. CCEN. Física, Recife, 2017.
Inclui referências.

1. Teoria de campos. 2. Gravitação I. Cunha, Bruno Geraldo Carneiro da. (Orientador). II. Titulo.

530.142 CDD (22. ed.) UFPE-FQ 2017-39

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Aprovada em: 21/07/2017.

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Resumo

Propostas recentes foram feitas para associar os graus de liberdade (cabelos) de buracos negros com as isometrias do horizonte de eventos. Essas abordagens, porém, sofrem da falta de invariância de calibre de forma explícita, normalmente dependendo de sistemas de coordenadas específicos. Neste trabalho, propomos um método para associar uma isometria assintótica correspondente para cada isometria do horizonte por meio de procedimentos usuais de fixação de calibre em Relatividade Geral clássica. As simetrias assintóticas podem ser definidas de uma forma puramente geométrica, independente de escolha de coordenadas, por meio da construção do infinito conforme e, então, nosso procedimento resulta em uma definição mais precisa para os cabelos dos buracos negros. Primeiramente revisamos algumas noções gerais sobre infinito conforme e definimos os difeomorfismos grandes que mapeam isometrias assintóticas para transformações no interior do espaço-tempo. Nós, então, calculamos as soluções para os buracos negros de BTZ e Kerr. No primeiro caso, as simetrias de AdS_3 nos permitem encontrar soluções fechadas.

Palavras-chave: BTZ. Kerr. Cabelo de buraco negro. Isometria do horizonte. Difeomorfismo grande.

Abstract

Recent proposals to associate degrees of freedom ("hair") of black hole backgrounds with isometries of the event horizon have been put forward. These approaches suffer, however, from lack of explicit gauge-invariance in their construction, usually relying on some preferred coordinate system. In this work, we propose a way of assigning a corresponding asymptotic isometry to each horizon isometry by means of the usual gauge fixing procedures of classical General Relativity. The asymptotic symmetries can be defined in a gauge-invariant way through the framework of conformal infinity and thus our recipe makes the definitions of black hole hair more precise. We first review some general notions of conformal infinity and define the large diffeomorphisms that map asymptotic isometries to bulk transformations. We then calculate the solutions for the BTZ and Kerr black holes. In the first case, the symmetries of AdS_3 allow us to find closed solutions.

Keywords: BTZ. Kerr. Black hole hair. Horizon isometry. Large diffeomorphism.

Contents

1 Introduction

In some ways, black hole solutions may be thought of as point particle solutions of the Einstein's equations. Drawing a parallel with Maxwell's equations, the Schwarzschild black hole resembles the electrostatic Coulomb's law, both being the vacuum solutions with a singularity at the origin. However, the intrinsic geometrical character of General Relativity makes these solutions divert from the structureless point particle picture. In particular, the existence of an event horizon endows black holes with not only a geometrical structure but also thermodynamic properties, which have been essential guidelines in the search of a quantum theory of gravity.

The black hole horizon can be defined in more than one way. One definition involves the notion of trapped surfaces, which formalizes the idea that nearby trajectories tend to converge to it, being a local description of the horizon. From a global point of view, the horizon gives a non-trivial causal structure to spacetime; an infalling object will experience an infinite redshift while crossing the horizon and hence it will lose causal contact with an observer outside. More precisely, the event horizon is the boundary of the causal future for ingoing observers, and of the causal past for outgoing observers, in the case of eternal black holes.

The observable charges of black holes are subject to a first law of thermodynamics, with the area of the event horizon playing the role of entropy. The Bekenstein-Hawking entropy,

$$S = \frac{A}{4}\,, \tag{1.1}$$

in natural units, was found in the 1970's and it was shown to satisfy the second law of thermodynamics. From a statistical point of view, there must be a microscopic description of the black hole degrees of freedom that give rise to these effective, thermodynamic properties. What is this microscopic description? The correct answer to this question should be an invaluable guide to consistently unify gravity and quantum mechanics.

Some answers have been found in different frameworks but with limited validity. In string theory, the degeneracy of BPS bound states correctly counts the degrees of freedom of higher dimensional supersymmetric black holes [1]. The counting also matches in Loop Quantum Gravity, after fixing a free parameter of the theory [2]. General Relativity was also able to predict the correct value of the entropy in some cases [3]. The latter case generally relies on the idea of Holography: the degrees of freedom are encoded in a lower dimensional surface. The Bekenstein-Hawking entropy formula corroborates this viewpoint, since it is proportional to the surface area of the event horizon. Intimately tied to this description is the notion of asymptotic symmetries; the holographic surface should inherit its symmetries from the spacetime and a great deal of information can be attained by the symmetries alone. This program is particularly successful in asymptotically AdS_3 black holes by studying the asymptotic symmetries at infinity [4].

The idea of asymptotic infinity is necessary for a proper definition of spacetime observables, like energy and angular momentum, which are dependent on a notion of Poincaré invariance. This symmetry is expected to hold only infinitely far from sources and one is thus led to the notion of asymptotic isometries. The most famous examples are the Bondi-Metzner-Sachs (BMS) group in four dimensions with zero cosmological constant [5, 6] and the Brown-Henneaux group in three dimensions with negative cosmological constant [7]. In each case, the asymptotic isometries include the expected symmetries of Poincaré and AdS_3, but come with infinitely many non-trivial transformations other than those. In the Brown-Hennaux case, the classical charges associated with the asymptotic isometries generates a Virasoro algebra and result in a consistent way of counting the black hole degrees of freedom. This has not been achieved for the BMS case in four dimensions yet.

Proposals for local, geometric diffeomorphisms that count the black hole degrees of freedom – black hole hair – have been put forward by a number of authors over the years, see, e.g., [3, 8]. Recently, Hawking and collaborators proposed the idea of associating the hair with isometries of the black hole horizon [9, 10] – see also Donnay *et al.* [11, 12]. This could in principle solve the problems of counting the Bekenstein-Hawking entropy. Like others before, their construction suffer from an unclear gauge invariance.

In this work , we propose a way of assigning a corresponding asymptotic isometry to each horizon isometry by means of the usual gauge fixing procedures of classical General Relativity [13]. The recipe given allows the extension of the asymptotic isometries to bulk transformations, which will induce transformations at the event horizon, by means of a large diffeomorphism acting on the entire space. Since asymptotic symmetries can be defined in a gauge-invariant way, this identification highlights the gauge invariance on the definition of black hole hair.

This dissertation is organized as follows. In chapter 2 we review the notions of asymptotic infinity and give the procedure of gauge fixing. We use the notion of conformal infinity to fix the boundary conditions we will consider asymptotically. In the following chapters, 3 and 4, we work out the explicit calculations for the BTZ and Kerr black holes, respectively. In BTZ we were able to find closed solutions for the large diffeomorphism matching the Brown-Henneaux generators at the AdS boundary. In Kerr we present preliminary results along with some specific solutions matching some generators of the Poincaré group at conformal infinity. We conclude with a summary and prospects.

2 Asymptotics and Symmetries

The study of the symmetries of a given theory is indispensable if one wants to fully understand it. Sometimes the knowledge about the symmetries precedes the theory; Newtonian mechanics arose as a theory that accommodated the symmetries of Galileo: translations, rotations and Galilean boost – change of inertial frames. But it is often found *a posteriori*, like the discovery of another symmetry group, with Maxwell's electromagnetism, which replaced the Galilean with the Lorentz boosts and put time and space on equal footing. These compose the Poincaré group that is understood today as the fundamental symmetry group of nature, being the most basic ingredient in the Standard Model of particles.

In General Relativity, Lorentz symmetry is only locally valid; it does not have the status of a global symmetry anymore. It was usually assumed that when going infinitely far from gravitational sources, one would recover the symmetries of Special Relativity. However, in 1962 Sachs found a much larger symmetry group while studying gravitational radiation in 4d asymptotically flat spacetimes [5]. Bondi, van de Burg and Metzner also contributed in the study of gravitational radiation [6, 14]. The discovery of the so called BMS group gave rise to the study of asymptotic symmetries in Einstein's gravity.

In this chapter we will study the notions of conformal infinity, a geometric framework developed to study the asymptotics of spacetimes, and derive its symmetries. We then define large diffeomorphisms as the coordinate transformations in spacetime that act non trivially at infinity, modifying the conserved charges, and give a way of extending the action of the asymptotic symmetries to the bulk.

2.1 Conformal Infinity

One of the notions that is familiar in prerelativistic theories is that of an object located infinitely far from an observer. One usually uses such construction in order to define conserved charges - be it mass, in Newtonian gravity, or electric charge, in classical electromagnetism - through an integral evaluated on a surface at infinity. In general relativity, however, limits like $r \to \infty$

become less precise, since the metric tensor, what is used to measure distance, is dynamical. Thus a more geometrical, coordinate independent definition of infinity is needed. Such a construction was originally made by Penrose [15] and further studied by many others in different settings [16]. In this section we will review the idea of conformal infinity and its symmetries, first from a general point of view, in a d-dimensional spacetime satisfying Einstein's equations possibly with a cosmological constant, and then specializing for (anti)-de Sitter and flat space asymptotics.

2.1.1 Asymptotically Simple Spacetimes

Let $(\mathcal{M}, g_{ab})$ be a d-dimensional spacetime of Lorentzian signature $(- + \cdots +)$. In order to work with infinity as a true boundary of spacetime, the points at "$r = \infty$" must somehow be added to $\mathcal{M}$. The idea is to bring them to a "finite distance" by a suitable coordinate transformation. However, what happens in general is that the metric will be ill defined at those points and an auxiliary, unphysical spacetime $\hat{\mathcal{M}}$, conformally related to $\mathcal{M}$, will be defined to include them.

Let us illustrate this with 4d Minkowski space. The metric, in spherical coordinates, is given by

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2.1}$$

Define null coordinates $u = t - r$ and $v = t + r$. Light rays propagating in the radial direction move in trajectories given by constant u, if moving outwards, or constant v, if ingoing. The endpoints of these trajectories are given by $u = -\infty$ (past) and $v = \infty$ (future). Note that after crossing $r = 0$ an ingoing ray becomes outgoing, thus $u = \infty$ must be identified with $v = \infty$ and the same applies for $u, v = -\infty$. These endpoints may be brought to a "finite distance" by the following coordinate transformations:

$$u = \tan p\,, \quad v = \tan q\,, \tag{2.2}$$

which takes the metric to

$$ds^2 = \frac{1}{\cos^2 p \cos^2 q}\left[-dpdq + \frac{\sin^2(p-q)}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]\,. \tag{2.3}$$

As advertised, the metric diverges at infinity, now labeled by the points at $p, q = \pm\pi/2$. If we define an unphysical spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$ with metric related to that of the physical space-

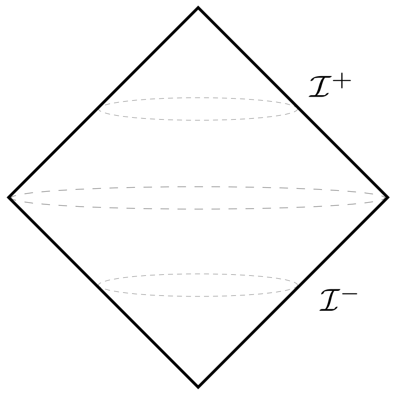

Figure 2.1: Penrose diagram of Minkowski space.

time by a conformal transformation

$$d\hat{s}^2 = \Omega^2 \, ds^2 \,, \tag{2.4}$$

with

$$\Omega = \cos p \cos q \,, \tag{2.5}$$

the unphysical metric (2.4) is now regular at infinity and these points may be added to the unphysical spacetime. In a general spacetime, the metric is not that of Minkowski space. However, if one assumes that the fields in it satisfy appropriate asymptotic conditions, it is expected that the space will be asymptotically flat (if the cosmological constant is nonzero, the asymptotic behavior should be that of (anti-)de Sitter space).

Using the example above, we may extract some general information. First note that in order to add the points at infinity we had to define an unphysical spacetime whose metric is conformally related to the physical metric. The conformal factor (2.5) vanishes at infinity. That can be understood as a means of renormalizing the infinite distances at the boundary. The boundary (infinity) is then labeled by $\Omega = 0$ and has the feature that $\nabla_a \Omega$ does not vanish there. Using these as general properties, we may extend to other spacetimes.

Let $\mathcal{I}$ be the set of all points at infinity. Penrose defined an *asymptotically simple spacetime* $\mathcal{M}$ as that satisfying [15]

1. There exists a conformal factor Ω such that $d\hat{s} = \Omega \, ds$ is smooth on $\hat{\mathcal{M}} \simeq \mathcal{M} \cup \mathcal{I}$;

2. $\Omega \big|_{\mathcal{I}} = 0$ and nowhere else vanishing; and

3. $\nabla_a \Omega \big|_{\mathcal{I}} \neq 0$.

Here, $A \simeq B$ means that A and B are homeomorphic. There is one extra property in the original definition for asymptotically flat spaces: every null geodesic must have two endpoints on $\mathcal{I}$. This will not hold in the presence of a black hole due to its event horizon, hence we will not require such property.

We will now show that the local properties of $\mathcal{I}$ depend on the value of the cosmological constant given that matter fields satisfy appropriate boundary conditions. To do so, we shall work on the unphysical spacetime $\hat{\mathcal{M}}$, where infinity is included as its boundary. The unphysical metric $\hat{g}_{ab} = \Omega^2 g_{ab}$ has its compatible covariant derivative $\hat{\nabla}_a$ related to physical ∇_a by the connection

$$C^c{}_{ab} = \frac{1}{2}\hat{g}^{cd}\left(\nabla_a \hat{g}_{bd} + \nabla_b \hat{g}_{ad} - \nabla_d \hat{g}_{ab}\right) , \tag{2.6}$$

where

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b - C^c{}_{ab}\omega_c , \tag{2.7}$$

for any ω_c. Using this last equation, one can relate the Riemann tensor in $\hat{\mathcal{M}}$ with that of $\mathcal{M}$. In particular, the Ricci tensor and scalar transform as [17]

$$\begin{aligned} \hat{R}_{ab} &= R_{ab} - g_{ab}\nabla^2 \log\Omega - (d-2)\left(\nabla_a\nabla_b \log\Omega - \nabla_a \log\Omega \nabla_b \log\Omega + g_{ab}\nabla_c \log\Omega\nabla^c \log\Omega\right) , \\ \hat{R} &= \Omega^{-2}\left[R - (d-1)\left(2\nabla^2 \log\Omega + (d-2)\nabla_c \log\Omega \nabla^c \log\Omega\right)\right] . \end{aligned} \tag{2.8}$$

Instead of using the Ricci tensor, let us reformulate the transformation in terms of[1]

$$P_{ab} = \frac{1}{d-2}\left(R_{ab} - \frac{R}{2(d-1)}\, g_{ab}\right) . \tag{2.9}$$

If Einstein's equations hold on $\mathcal{M}$,

$$R_{ab} - \frac{R}{2}\, g_{ab} + \Lambda g_{ab} = 8\pi G^{(d)} T_{ab} , \tag{2.10}$$

then

$$P_{ab} = \frac{\Lambda}{(d-1)(d-2)}\, g_{ab} + \frac{8\pi G^{(d)}}{d-2}\left(T_{ab} - \frac{T}{d-1}\, g_{ab}\right) . \tag{2.11}$$

[1]Note that in terms of P_{ab}, the Riemann tensor decomposition is simply

$$R_{abcd} = C_{abcd} + 2\left(g_{a[c}P_{d]b} - g_{b[c}P_{d]b}\right) .$$

Using (2.8), it can be shown that

$$P_{ab} = \hat{P}_{ab} + \Omega^{-1}\hat{\nabla}_a\hat{\nabla}_b\Omega - \frac{1}{2}\Omega^{-2}\hat{g}_{ab}\hat{g}^{cd}\hat{\nabla}_c\Omega\hat{\nabla}_d\Omega\,. \tag{2.12}$$

By construction, the curvature tensors on the unphysical spacetime are all regular at $\mathcal{I}$. Thus, multiplying (2.12) by Ω^2 and evaluating the result at $\mathcal{I}$, where Ω vanishes, leaves only the last term on the RHS finite. Defining $\hat{n}_a = \hat{\nabla}_a\Omega$, this procedure gives

$$\hat{g}^{ab}\,\hat{n}_a\,\hat{n}_b\big|_{\mathcal{I}} = -\frac{2\Lambda}{(d-1)(d-2)} + \frac{16\pi G^{(d)}}{(d-1)(d-2)}\left(\Omega^2\,T\right)\big|_{\mathcal{I}}\,. \tag{2.13}$$

Here, and in what follows, indices of hatted tensors are raised and lowered using the unphysical metric $\hat{g}$. If we require the asymptotic condition $(\Omega^2\,T)\big|_{\mathcal{I}} = 0$, which is not very restrictive, since one expects that the energy-momentum tensor itself vanish at infinity, then

$$\hat{g}^{ab}\,\hat{n}_a\,\hat{n}_b\big|_{\mathcal{I}} = -\frac{2\Lambda}{(d-1)(d-2)} = -\frac{\epsilon}{l^2}\,, \tag{2.14}$$

where l is the (A)dS radius and ($\epsilon = -1$) $\epsilon = 1$. The asymptotically flat case can be recovered either by putting $\epsilon = 0$ or in the limit $l \to \infty$. Note that $\hat{n}_a$ is normal to $\mathcal{I}$ and hence (2.14) is a statement about the geometry of infinity. Three distinct cases arise:

(i) Asymptotically AdS ($\Lambda < 0$): $\mathcal{I}$ is a timelike hypersurface;

(ii) Asymptotically flat ($\Lambda = 0$): $\mathcal{I}$ is a null hypersurface; and

(iii) Asymptotically dS ($\Lambda > 0$): $\mathcal{I}$ is a spacelike hypersurface.

In order to define quantities on $\mathcal{I}$, we need to project the tensors on $\hat{\mathcal{M}}$ at $\Omega = 0$. To do so, define a projection operator $\Pi^a{}_b$ that can be viewed as a map between both the tangent or cotangent spaces of the unphysical spacetime and the conformal infinity at corresponding points. This operator must satisfy

$$\begin{aligned} \Pi^a{}_c\,\Pi^c{}_b &= \Pi^a{}_b\,, \\ \hat{n}_a\,\Pi^a{}_b &= 0\,. \end{aligned} \tag{2.15}$$

The first identity is the usual property of projection operators, while the second states that $\hat{n}_a$ is normal to $\mathcal{I}$. Define a vector $\hat{l}^a$ such that

$$\hat{l}^a\,\hat{n}_a = 1\,; \tag{2.16}$$

the projection operator can then be written as

$$\Pi^a{}_b = \delta^a{}_b - \hat{l}^a\,\hat{n}_b\,. \tag{2.17}$$

The vector $\hat{l}^a$ is (almost) determined by noting that

$$\Pi^a{}_b\,\hat{l}^b = 0\,, \tag{2.18}$$

i.e., $\hat{l}^a$ is also normal to $\mathcal{I}$. One may then project the metric tensor $\hat{g}_{ab}$ to define the induced metric at $\mathcal{I}$

$$\gamma_{ab} = \hat{g}_{cd}\,\Pi^c{}_a\,\Pi^d{}_b\,. \tag{2.19}$$

Depending on the value of ϵ – the sign of the cosmological constant – the $(d-1)$-dimensional induced metric has different signatures. In the AdS case (i), its signature is $(-+\cdots+)$; for asymptically flat (ii), the signature is $(0+\cdots+)$; and in de Sitter asymptotics (iii) is $(++\cdots+)$.

We may then decompose the unphysical metric near $\mathcal{I}$ as

$$\hat{g}_{ab} = -L\,\hat{n}_a\,\hat{n}_b + 2\hat{l}_{(a}\,\hat{n}_{b)} + \gamma_{ab}\,, \tag{2.20}$$

with the norm of $\hat{l}^a$,

$$L = \hat{g}_{ab}\,\hat{l}^a\,\hat{l}^b\,, \tag{2.21}$$

undetermined. Some other properties arise from this conformal compactification of spacetime. In particular, take the trace free part of (2.12) multiplied by Ω and evaluate it at infinity. If the energy momentum tensor satisfies

$$\left(T_{ab} - \frac{T}{d}g_{ab}\right)\Omega\,\Big|_{\mathcal{I}} = 0\,, \tag{2.22}$$

then it follows that

$$\hat{\nabla}_a\hat{n}_b = \frac{1}{d}(\hat{\nabla}^2\Omega)\,\hat{g}_{ab} + \mathcal{O}\left(\Omega\right)\,. \tag{2.23}$$

Note that, anywhere on $\hat{\mathcal{M}}$, $\hat{\nabla}_a\,\hat{n}_b = \hat{\nabla}_{(a}\,\hat{n}_{b)}$, assuming there is no torsion. Hence, using (2.16), one can see that

$$\begin{aligned}\mathcal{L}_{\hat{l}}\,\hat{n}_a &= \hat{l}^b\,\hat{\nabla}_b\,\hat{n}_a + \hat{n}_b\,\hat{\nabla}_a\,\hat{l}^b = 2\hat{l}^b\,\hat{\nabla}_{[b}\,\hat{n}_{a]}\\ &= 0\,.\end{aligned} \tag{2.24}$$

A direct consequence of this is that the projection operator (2.17) is also constant along the integral lines of $\hat{l}^a$, and thus we can use it to project tensors at any point of the unphysical spacetime into hypersurfaces normal to $\hat{n}_a$, i.e., we can foliate $\hat{\mathcal{M}}$ with hypersurfaces Σ_Ω labeled by constant Ω and use $\Pi^a{}_b$ to project tensors on $\hat{\mathcal{M}}$ onto Σ_Ω. The unphysical metric (2.20) may then be seen as defined throughout $\hat{\mathcal{M}}$ with L and γ_{ab} – the norm of $\hat{l}$ and the induced metric at Σ_Ω – now functions of Ω.

2.1.2 Residual Conformal Transformation and Asymptotic Symmetry

The definition of the conformal infinity relied on the existence of a function Ω on $\hat{\mathcal{M}}$ that vanishes asymptotically and satisfies the conditions stated in the previous subsection. However, this function is not uniquely determined; one may consider instead Ω' given by

$$\Omega' = \mathrm{e}^{\omega}\,\Omega\,, \tag{2.25}$$

where ω is a function on $\hat{\mathcal{M}}$. The new conformal factor still vanishes at infinity and has nonzero gradient. Hence all properties discussed in last section should hold if we "prime" all quantities. The transformed metric tensor is

$$\hat{g}'_{ab} = \mathrm{e}^{2\omega}\,\hat{g}_{ab}\,, \tag{2.26}$$

while the new normal at $\mathcal{I}$ is

$$\hat{n}'_a = \hat{\nabla}_a(\mathrm{e}^{\omega}\,\Omega)\,. \tag{2.27}$$

The new normal vector can be defined as before by $\hat{l}'^a\,\hat{n}'_a = 1$. In general it is not possible to write $\hat{l}'^a$ exactly in terms of $\hat{l}^a$; there is no unique way of choosing a vector satisfying this relation. This is not a problem here for we do not need an explicit expression for this vector.

Further conformal transformations can now be dubbed as $\omega \to \omega'$. This is a residual transformation, for it does not hinder the properties of conformal infinity as long as ω' is non singular. Thus the group of asymptotic symmetries must be isomorphic to the residual conformal transformations. It may seem unnatural that these symmetries should be conformal; the first guess would be that they are isometries. But remember that infinity is well defined in $\hat{\mathcal{M}}$, and not in the physical spacetime $\mathcal{M}$, so when viewed as symmetries on the unphysical space, this should not be very counterintuitive. In particular, isometries in $\mathcal{M}$ correspond to conformal transformations in $\hat{\mathcal{M}}$. To see this, suppose that η^a is an isometry of the physical spacetime:

$$\mathcal{L}_\eta\, g_{ab} = 0\,. \tag{2.28}$$

This vector can be naturally extended to $\hat{\mathcal{M}} \setminus \mathcal{I}$ and thus

$$\mathcal{L}_\eta \, \hat{g}_{ab} = 2 \left(\Omega^{-1} \, \mathcal{L}_\eta \, \Omega \right) \hat{g}_{ab} \, . \tag{2.29}$$

If η^a has no component along $\hat{l}^a$, or it is at least of order Ω in that direction, near infinity, then it can be defined at $\mathcal{I}$ as well. Hence, Killing vectors on the physical spacetime are conformal Killing vectors on the unphysical.

We shall then define the (infinitesimal) Asymptotic Symmetry Group (ASG) to be the (infinitesimal) diffeomorphisms generated by vectors ξ^a_α that induce a residual conformal transformation $\omega \to \omega + \alpha$ near infinity. Writing δ_α instead of $\mathcal{L}_{\xi_\alpha}$, the ASG acts on $\hat{\mathcal{M}}$ as

$$\begin{aligned} \delta_\alpha \, \hat{g}_{ab} &= 2\alpha \, \hat{g}_{ab} \, , \\ \delta_\alpha \, \hat{n}_a &= \hat{\nabla}_a(\alpha\Omega) \, . \end{aligned} \tag{2.30}$$

We will show that the vectors ξ^a_α form a group under the Lie algebra, given that they satisfy some condition stated below.

Define

$$[\alpha, \beta] = \delta_\alpha \, \beta - \delta_\beta \, \alpha \, . \tag{2.31}$$

It is straightforward to show that

$$\delta_{[\alpha} \, \delta_{\beta]} \, \hat{g}_{ab} = 2[\alpha, \beta] \, \hat{g}_{ab} \, , \tag{2.32}$$

where $\delta_{[\alpha} \, \delta_{\beta]} = \delta_\alpha \, \delta_\beta - \delta_\beta \, \delta_\alpha = \mathcal{L}_{[\xi_\alpha, \xi_\beta]}$. It then seems that the algebra of ASG is isomorphic to the Lie algebra of vectors. However, the situation is not so simple with variation of $\hat{n}_a$ in (2.30). Namely,

$$\delta_{[\alpha} \, \delta_{\beta]} \, \hat{n}_a = \hat{\nabla}_a([\alpha, \beta] \, \Omega) - \hat{\nabla}_a \left(\alpha \, \delta_\beta \, \Omega - \beta \, \delta_\alpha \, \Omega \right) \tag{2.33}$$

There are two approaches for this issue. First note that this does not correspond to a central extension of the Lie algebra of the generating vectors, since the extra term is not present at the variation of the metric. Hence we can either restrict our attention to vectors satisfying

$$\alpha \, \delta_\beta \, \Omega = \beta \, \delta_\alpha \, \Omega \, , \tag{2.34}$$

or modify the algebra of the ASG, as is common when dealing with field-dependent transformations – see [18]. The modified algebra that accommodates the changes above is

$$\delta_{[\alpha,\beta]} = \delta_{[\alpha}\,\delta_{\beta]} - \delta^{\Omega}_{\alpha}\,\delta_{\beta} + \delta^{\Omega}_{\beta}\,\delta_{\alpha}\,, \tag{2.35}$$

where $\delta^{\Omega}_{\alpha}\,X$ denotes the variation in X – an arbitrary tensor – due to the variation of the conformal factor induced by ξ_{α}, i.e., $\delta^{\Omega}_{\alpha}\,X = X(\Omega + \delta_{\alpha}\,\Omega) - X(\Omega)$.

We shall initially take the conservative approach and consider those symmetries satisfying (2.34). Any vector can be decomposed as a vector along $\hat{l}^a$ and another perpendicular to it, using $\delta^a{}_b = \Pi^a{}_b + \hat{l}^a\,\hat{n}_b$. One can see that $\delta_{\alpha}\,\Omega = \xi^a_{\alpha}\,\hat{n}_a$, and thus (2.34) is a condition imposed on the $\hat{l}^a$ component of the ASG generators. It can be fulfilled if

$$\xi^a_{\alpha} = \alpha\,\Omega\,\hat{l}^a + \xi^a_{\|\alpha}\,, \tag{2.36}$$

where $\xi^a_{\|\alpha} = \Pi^a{}_b\,\xi^b_{\alpha}$. We must then plug this into (2.30) to find the conditions imposed on α and the component parallel to $\mathcal{I}$. The second equation turns out to be automatically satisfied:

$$\begin{aligned}
\delta_{\alpha}\,\hat{n}_a &= \mathcal{L}_{\alpha\Omega\hat{l}}\,\hat{n}_a + \mathcal{L}_{\xi_{\|\alpha}}\,\hat{n}_a \\
&= \hat{\nabla}_a(\alpha\Omega) + \alpha\Omega\,\mathcal{L}_{\hat{l}}\,\hat{n}_a + \mathcal{L}_{\xi_{\|\alpha}}\,\hat{n}_a \\
&= \hat{\nabla}_a(\alpha\Omega) + 2\xi^a_{\alpha}\,\hat{\nabla}_{[b}\,\hat{n}_{a]} \\
&= \hat{\nabla}_a(\alpha\Omega)\,.
\end{aligned} \tag{2.37}$$

From the second to third line we used $\hat{l}^a\,\hat{n}_a = 1$ and $\xi^a_{\|\alpha}\,\hat{n}_a = 0$ to organize the terms in a similar way as in (2.24). The last line results from the fact that $\hat{n}_a$ is a gradient, hence its derivative is symmetric. Note that there is really no need for the modified algebra (2.35), unless we require that the variation of $\hat{n}_a$ induced by the ASG agrees with (2.30) only to order zero on Ω. This means including terms of order Ω^2 in ξ^a_{α}, which are really small diffeomorphisms on spacetime and thus have no physical consequences. Hence vectors of the form (2.36) exhaust all possibilities.

It is desired that after the transformation, the projection operator (2.17) does not change to order zero in order to allow quantities on $\mathcal{I}$ to remain there. Let us first calculate the change in $\hat{l}^a$:

$$\begin{aligned}
\delta_{\alpha}\,\hat{l}^a &= \mathcal{L}_{\alpha\Omega\hat{l}}\,\hat{l}^a + \mathcal{L}_{\xi_{\|\alpha}}\,\hat{l}^a \\
&= -\mathcal{L}_{\hat{l}}(\alpha\Omega)\hat{l}^a - \partial_{\Omega}\,\xi^a_{\|\alpha}\,,
\end{aligned} \tag{2.38}$$

where the second term on the RHS of the last line comes from using the ordinary derivative

∂_a for evaluation of the Lie derivative along $\xi^a_{\|\alpha}$ and noting the $\hat{l}^a \partial_a = \partial_\Omega$. The change in the projection operator is then

$$\begin{aligned}\delta_\alpha \, \Pi^a{}_b &= -\delta_\alpha \, \hat{l}^a \, \hat{n}_b - \hat{l}^a \, \delta_\alpha \, \hat{n}_b \\ &= -\left(\partial_\Omega \, \xi^a_{\|\alpha}\right) \hat{n}_b - \Omega \, \hat{l}^a \, \Pi^c{}_b \, \hat{\nabla}_c \, \alpha + \mathcal{O}\left(\Omega^2\right) \, .\end{aligned} \tag{2.39}$$

We shall require that $\xi^a_{\|\alpha}$ deviates from its value at $\mathcal{I}$ only to second order in Ω. Thus (2.36) corresponds to all diffeomorphisms on spacetime modulo the small diffeos – those resulting in naive coordinate transformations.

In order to determine $\xi^a_{\|\alpha}$, let us project the first equation in (2.30) into Σ_Ω:

$$\begin{aligned}\Pi^c{}_a \, \Pi^d{}_b \, \delta_\alpha \, \hat{g}_{cd} &= \delta_\alpha \, \gamma_{ab} - 2\hat{g}_{cd} \, \Pi^c{}_{(a} \, \delta_\alpha \, \Pi^d{}_{b)} \\ &= \delta_\alpha \, \gamma_{ab} + 2\Omega \, \hat{g}_{cd} \, \hat{l}^c \, \Pi^d{}_{(a} \, \Pi^e{}_{b)} \, \hat{\nabla}_e \, \alpha + \mathcal{O}\left(\Omega^2\right) \, .\end{aligned} \tag{2.40}$$

The variation of the induced metric on Σ_Ω can be separated from contributions normal and parallel to the hypersurface; namely

$$\delta_\alpha \, \gamma_{ab} = \alpha \, \Omega \, \mathcal{L}_{\hat{l}} \, \gamma_{ab} + \mathcal{L}_{\xi_{\|\alpha}} \, \gamma_{ab} \, . \tag{2.41}$$

Plugging this all in the first equation of (2.30) results in

$$\mathcal{L}_{\xi_{\|\alpha}} \, \gamma_{ab} = 2\alpha \, \gamma_{ab} - \Omega \left[\alpha \, \mathcal{L}_{\hat{l}} \, \gamma_{ab} + 2\hat{g}_{cd} \, \hat{l}^c \, \Pi^d{}_{(a} \, \Pi^d{}_{b)} \, \hat{\nabla}_e \, \alpha\right] + \mathcal{O}\left(\Omega^2\right) \, . \tag{2.42}$$

Evaluation of the equation above at $\Omega = 0$ shows that the projection of the ASG generating vector is a conformal Killing vector of $\mathcal{I}$. This can be written in a better way if we define a map $\pi : \hat{\mathcal{M}} \to \mathcal{I}$. We can then define the pushforward of a general tensor on $\hat{\mathcal{M}}$ to tensors on $\mathcal{I}$ by the operation

$$\pi^* \, \hat{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} = \Pi^{a_1}{}_{c_1} \cdots \Pi^{a_n}{}_{c_n} \hat{T}^{c_1 \cdots c_n}{}_{d_1 \cdots d_m} \Pi^{d_1}{}_{b_1} \cdots \Pi^{d_m}{}_{b_m} \Big|_{\Omega=0} \tag{2.43}$$

We shall denote the pushforward by $T^{A_1 \cdots A_n}{}_{B_1 \cdots B_m}$; capital letters will be used to denote tensors on $\mathcal{I}$. Note that this operation is not hindered by ASG transformations, for the projection operator is unchanged at $\mathcal{I}$. Then, let $\xi^A_\alpha = \pi^* \xi^a_\alpha = \pi^* \xi^a_{\|\alpha}$ and $\gamma_{AB} = \pi^* \gamma_{ab}$, we have

$$\mathcal{L}_{\xi_\alpha} \, \gamma_{AB} = 2\alpha \, \gamma_{AB} \, . \tag{2.44}$$

The discussion up to this point was completely general in the sense that it is equally appli-

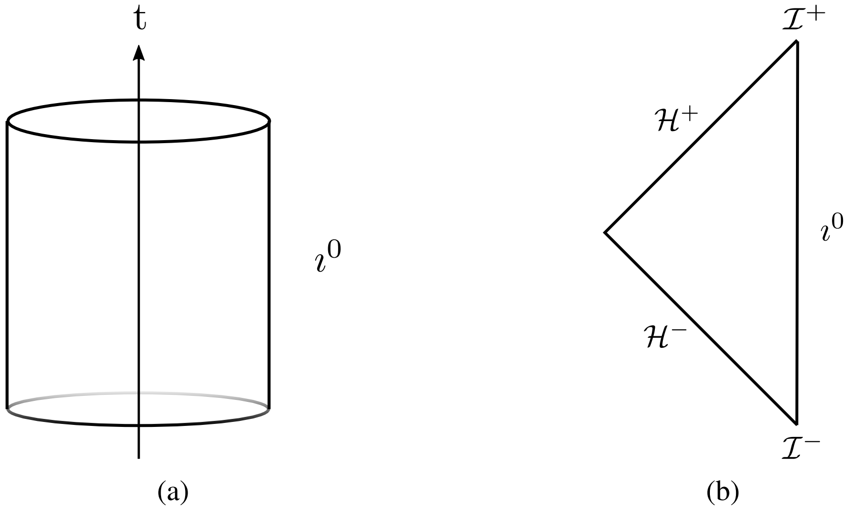

Figure 2.2: (a) Topological structure of AdS spaces. (b) Penrose diagram of asymptotically AdS space with an event horizon $\mathcal{H}$.

cable to any of the three cases of asymptotics discussed in the last subsection: de Sitter, anti-de Sitter or flat space. In the next two subsections we shall discuss specific properties of asymptotically (A)dS and flat spacetimes separately; the (non)degeneracy of the induced metric at $\mathcal{I}$ poses some constraints on the vector $\hat{l}^a$ that was not completely determined by the conditions of asymptotic simplicity.

2.1.3 (Anti)-de Sitter Asymptotics

While the treatment of conformal infinity in de Sitter and (Anti)-de Sitter spaces are similar – only differing on the sign of the cosmological constant – the structure of infinity is different. In de Sitter, the positive cosmological constant acts accelerating the expansion of spacetime and hence no observer, or light rays, manage to reach spatial infinity; the infinity is located in the absolute future and/or absolute past [19]. We shall consider only the absolute future infinity $\imath^+$ (treatment of past infinity $\imath^-$ is equivalent). The boundary of $\imath^+$ can only be reached by light rays and corresponds to a codimension 2 surface $\mathcal{I}^+$, future null infinity. In anti-de Sitter, on the other hand, the negative cosmological constant renders spatial infinity $\imath^0$ as a possible destination for observers. The boundaries of $\imath^0$ are two disjoint codimension 2 surfaces corresponding to temporal infinities $\imath^\pm$, if no black hole is present, or future and past null infinities $\mathcal{I}^\pm$, otherwise. In what follows we will refer to the codimension 1 part of conformal infinity – $\imath^+$ for dS and $\imath^0$ for AdS asymptotics – simply as infinity, still being referred to by $\mathcal{I}$, and treat them on equal footing unless some specification is necessary.

The induced metric at infinity is nondegenerate and thus there exists a well defined inverse. Instead of inverting the induced metric itself, we may calculate the induced inverse metric using the projection operator defined in last subsection, i.e.,

$$\begin{aligned}\gamma^{ab} &= \Pi^a{}_c\,\Pi^b{}_d\,\hat{g}^{cd}\\ &= \hat{g}^{ab} - 2\hat{l}^{(a}\,\hat{n}^{b)} + N\hat{l}^a\,\hat{l}^b\,,\end{aligned} \tag{2.45}$$

where $N = \hat{g}^{ab}\,\hat{n}_a\,\hat{n}_b$. Nondegeneracy implies that γ^{ab} is the inverse of γ_{ab}, and thus

$$\gamma^{ac}\,\gamma_{cb} = \Pi^a{}_b\,, \tag{2.46}$$

for the projection operator acts as the $(d-1)$ identity operator when restricted to tensors on $\mathcal{I}$. This implies that

$$\hat{l}^a = L\,\hat{n}^a\,, \tag{2.47}$$

so that $\hat{n}^a$ has no component tangent to $\mathcal{I}$. The metric then assumes the simpler form

$$\hat{g}_{ab} = \frac{1}{N}\hat{n}_a\,\hat{n}_b + \gamma_{ab}\,, \tag{2.48}$$

with N given by (2.14)

Equation (2.44) simply tells us that ξ^A_α is a conformal Killing vector on infinity; let D_A be the derivative compatible with γ_{AB}[2], then

$$D_{(A}(\xi_\alpha)_{B)} = \alpha\,\gamma_{AB}\,, \tag{2.49}$$

where $(\xi_\alpha)_A = \gamma_{AB}\,\xi^B_\alpha$. The ASG is then isomorphic to the conformal group of $\mathcal{I}$.

2.1.4 Flat Space Asymptotics

The structure of infinity when the cosmological constant is zero (ii) is different. The codimension 1 part of the conformal infinity is now the destination of outgoing light rays, future null infinity $\mathcal{I}^+$. Its boundaries correspond to future infinity $\imath^+$ and spatial infinity $\imath^0$. By time reversion, conformal infinity actually is composed by another, disjoint part: past null infinity $\mathcal{I}^-$,

[2]In [17] it is shown that this derivative is unique. In our notation, it given by

$$D_C\,T^{A_1\cdots A_n}{}_{B_1\cdots B_m} = \pi^*\hat{\nabla}_c\left(\Pi^{a_1}{}_{c_1}\cdots\Pi^{a_n}{}_{c_n}\hat{T}^{c_1\cdots c_n}{}_{d_1\cdots d_m}\Pi^{d_1}{}_{b_1}\cdots\Pi^{d_m}{}_{b_m}\right)\,.$$

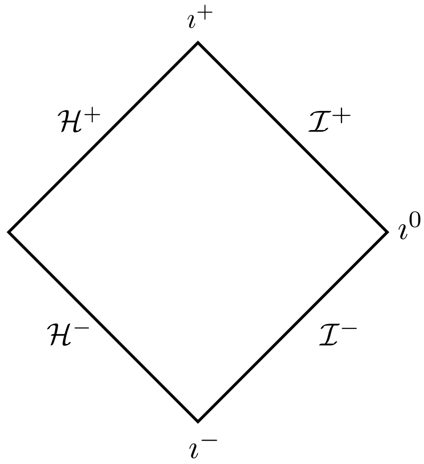

Figure 2.3: Penrose diagram of an asymptotically flat spacetime in the the presence of a black hole with event horizon $\mathcal{H}$.

which is the origin of ingoing null rays and has $\imath^-$ and $\imath^0$ and its boundaries. We will primarily be concerned in describing the symmetries of $\mathcal{I}^+$ and shall mention the corresponding results for $\mathcal{I}^-$ whenever relevant.

Note that

$$\Pi^a{}_b \, \hat{n}^b = \hat{n}^a \, , \tag{2.50}$$

for $\hat{n}^a \, \hat{n}_a = 0$, cf. (2.14). Thus $\hat{n}^a$ is parallel to null infinity and we may write n^A as its pushforward $\pi^* \hat{n}^a$ on $\mathcal{I}$. From the definition of the induced metric, one may check that it is degenerate and its null direction corresponds to that of n^A:

$$\gamma_{AB} \, n^B = 0 \, . \tag{2.51}$$

In the 4d case, Geroch proved that the topology of null infinity is $\mathbb{R} \times S^2$, the $\mathbb{R}$ part corresponding to the null direction generated by n^A [20]. We shall consider the more general situation in which the topology of $\mathcal{I}^\pm$ is $\mathbb{R} \times \mathcal{B}$, where $\mathcal{B}$ is a $(d-2)$ dimensional compact space. It is then natural to assume that γ_{AB} is associated to a positive definite metric σ_{AB} over $\mathcal{B}$; we shall soon make that more precise.

We also note that $\hat{l}_a - L\hat{n}_a$ is parallel to null infinity and satisfies $\gamma^{ab}(\hat{l}_b - L\hat{n}_b) = -L\hat{n}^a$, where γ^{ab} is the induced inverse metric defined in (2.45). We would like to think of the induced inverse metric as being related to the inverse of σ_{AB}, hence we shall take $L = 0$. This implies that $l_A = \pi^* \hat{l}_a$ is also related to the null direction on $\mathcal{I}^+$. The metric on $\hat{\mathcal{M}}$ near null infinity is thus given by

$$\hat{g}_{ab} = 2\hat{l}_{(a}\hat{n}_{b)} + \gamma_{ab} \, . \tag{2.52}$$

It is now possible to define an operator that projects any tensor to components perpendicular to the null direction. Let $\mathcal{P}^A{}_B$ be such operator. It must annihilate both n^A and l_A, thus

$$\mathcal{P}^A{}_B = \delta^A{}_B - n^A l_B \,, \tag{2.53}$$

where $\delta^A{}_B = \pi^* \Pi^a{}_b$ is the identity on $\mathcal{I}^+$. We may then decompose any vector in a uniquely defined way. In particular, the restriction ξ^A_α of the ASG generator on null infinity may be written as

$$\xi^A_\alpha = \kappa n^A + \mathcal{P}\xi^A_\alpha \,, \tag{2.54}$$

where $\mathcal{P}\xi^A = \mathcal{P}^A{}_B \xi^B$. We have all the tools to analyze the ASG on null infinity. When restricted to $\mathcal{I}^+$, the residual conformal transformations act as

$$\begin{aligned} \delta_\alpha \, \gamma_{AB} &= 2\alpha \gamma_{AB} \,. \\ \delta_\alpha \, n^A &= -\alpha n^A \,. \end{aligned} \tag{2.55}$$

These two equations lead to

$$\kappa \mathcal{L}_n \, \gamma_{AB} + \mathcal{L}_{\mathcal{P}\xi_\alpha} \, \gamma_{AB} = 2\alpha \gamma_{AB} \tag{2.56}$$

and

$$-\left(\mathcal{L}_n \kappa\right) n^A - \mathcal{L}_n \, \mathcal{P}\xi^A_\alpha = -\alpha n^A \,. \tag{2.57}$$

We now pause to give some physical arguments leading to constraints on the geometry of conformal infinity that will allow us to solve these two equations.

There are no conditions imposed on the dynamics of the physical spacetime for the construction of conformal infinity. In particular, $\mathcal{M}$ need not even be stationary. We shall then assume that time evolution does not hinder the conditions of asymptotic simplicity. Apart from the de Sitter case, this poses some interesting physical consequences on the structure of infinity. Basically, since one can think of time evolution as changing the physical spacetime to different configurations that are still asymptotically simple, the geometrical features on $\mathcal{I}$ should remain unchanged. In the asymptotically flat case, it means that the induced metric on $\mathcal{I}^{+(-)}$ is unchanged under variations of the retarded (advanced) time:

$$\mathcal{L}_n \, \gamma_{AB} = 0 \,. \tag{2.58}$$

Since this direction is unique, i.e., invariant under residual conformal transformations, we must restrict the ASG to those ξ^a_α with

$$\mathcal{L}_n\,\alpha = 0\,. \tag{2.59}$$

In the AdS case, there should be a timelike Killing vector η^A on spatial infinity. In that case, however, there is no preferred time direction and α may be time dependent. This is not a problem since $\eta^A + \delta_\alpha\eta^A$ is a (timelike) Killing vector of $\gamma_{AB} + \delta_\alpha\gamma_{AB}$, and so the existence of such symmetry is gauge independent. This suggests that the topology of spatial infinity in AdS asymptotics should also be $\mathbb{R}\times\mathcal{B}$.

As done by Geroch in [20], we can define a map from null infinity to $\mathcal{B}$, its base space, $\zeta : \mathcal{I}^+ \to \mathcal{B}$, and view $\mathcal{I}$ as a fiber bundle over the base space. A cross section of $\mathcal{I}^+$ is a $(d-2)$ surface homeomorphic to $\mathcal{B}$ that intercepts the integral curves n^A only once. Equation (2.58) implies that the metric is the same in any cross section of $\mathcal{I}^+$, hence it can be identified with a metric σ_{AB} over the base space $\mathcal{B}$: $\gamma_{AB} = \zeta^*\sigma_{AB}$. The conclusion also holds for the parameter α of the ASG generators; it is the pushforward of a field over the base space.

These considerations lead us to view $\mathcal{P}\xi^A_\alpha$ in (2.56) as a vector field on the base space and that equation can be written as

$$\mathcal{L}_{\mathcal{P}\xi_\alpha}\,\sigma_{AB} = 2\alpha\sigma_{AB}\,, \tag{2.60}$$

where all fields are thought of as defined over $\mathcal{B}$. In doing so, we have assumed that

$$\mathcal{L}_n\,\mathcal{P}\xi^A_\alpha = 0\,, \tag{2.61}$$

hence (2.57) implies that

$$\kappa = \alpha u + \alpha_0\,. \tag{2.62}$$

We defined the null coordinate u, such that $n^A D_A = \partial_u$. The term α_0 is u-independent and thus viewed as a field over $\mathcal{B}$. The whole generator on $\hat{\mathcal{M}}$ is

$$\xi^a_{\alpha,\alpha_0} = \alpha\Omega\hat{l}^a + (\alpha u + \alpha_0)\hat{n}^a + \mathcal{P}\xi^a_\alpha\,, \tag{2.63}$$

where $\mathcal{P}\xi_\alpha$ is a conformal Killing vector on the base space $\mathcal{B}$. It has now a dependence on two parameters α and α_0. It is suggestive to write $\xi^a_{\alpha,\alpha_0} = \xi^a_\alpha + \alpha_0\,\hat{n}^a$ and analyze the algebras separately. One can see that

$$\begin{aligned} [\xi_\alpha, \xi_\beta] &= \xi_{[\alpha,\beta]}\,, \\ [\xi_\alpha, \beta_0\,\hat{n}] &= (\mathcal{L}_{\mathcal{P}\xi_\alpha}\,\beta_0 - \alpha\beta_0)\,\hat{n}\,, \\ [\alpha_0\,\hat{n}, \beta_0\,\hat{n}] &= 0\,. \end{aligned} \tag{2.64}$$

Thus, the ASG is the semidirect sum of two algebras that are commonly called superrotations, generated by ξ^a_α, and supertranslations, generated by $\alpha_0\,\hat{n}^a$. The former is a subalgebra of the ASG isomorphic to the conformal algebra of $\mathcal{B}$ and the latter is an abelian subalgebra that is an ideal of the whole group.

2.2 Perturbations and Large Diffeomorphisms

Before turning to specific cases, let us digress over metric perturbations. Suppose we have a solution to Einstein's equations in vacuum and possibly with a cosmological constant in d dimensions, like the black holes we will study in chapters 3 and 4,

$$R_{ab} = \frac{\epsilon(d-1)}{l^2} g_{ab}\,. \tag{2.65}$$

(See eq. (2.14) for the relation between l and the cosmological constant.) If we vary the metric by a "small amount" $\delta g_{ab} = h_{ab}$ around the solution, the change in the Ricci tensor is, to first order, [17]

$$\delta R_{ab} = -\frac{1}{2}\nabla_a\nabla_b h_c{}^c - \frac{1}{2}\nabla_c\nabla^c h_{ab} + \nabla_{(a}\nabla^c h_{b)c} + R_{cabd}h^{cd} + R_{(a}{}^c h_{b)c}\,. \tag{2.66}$$

Here we raise and lower indices with the background metric g_{ab}.

The linearized Einstein's equations

$$\delta R_{ab} = \frac{\epsilon(d-1)}{l^2}\delta g_{ab} \tag{2.67}$$

thus puts $d(d+1)/2$ dynamical constraints in the metric perturbation, assuming it is on-shell. However, diffeomorphism invariance of the background metric can be translated into the perturbation by saying that h_{ab} and $h_{ab} + \nabla_a\xi_b + \nabla_b\xi_a$ are physically equivalent for well behaved vector fields ξ^a. This allows us to choose a gauge in which the perturbation is traceless, $g^{ab}h_{ab} = 0$, and transverse $\nabla^a h_{ab} = 0$. To see this, let $h'_{ab} = h_{ab} + 2\nabla_{(a}\xi_{b)}$ be the gauge transformed perturbation. One can see that

$$g^{ab}h'_{ab} = g^{ab}h_{ab} + 2\nabla^a\xi_a\,, \quad \nabla^a h'_{ab} = \nabla^a h_{ab} + \nabla^2\xi_b + \nabla^c\nabla_b\xi_c\,. \tag{2.68}$$

The vanishing of the left hand side of both equations yields differential equations for ξ^a that can be solved generically.[3]

We still have the freedom of doing residual gauge transformations, those that do not hinder the traceless transverse gauge. The generating vector must satisfy

$$\nabla_a \xi^a = 0 \,, \quad \nabla^2 \xi^a + R^a{}_b \xi^b = 0 \,. \tag{2.69}$$

Note that each of the above transformations eliminates d degrees of freedom from the metric perturbation, thus leaving only $d(d-3)/2$ independent components. This amounts to the 2 polarizations of the graviton in four dimensions and zero in the three dimensional case. The latter is a statement that Einstein's gravity in three dimensions is a topological theory; it has no dynamics in the bulk. In solving the gauge imposing equations, one usually assume fast enough fall-offs at infinity – the small gauge transformations, or small diffeos.

Our idea is to relax this last condition. Instead, we will seek solutions of the residual gauge transformations with the condition that the vector will tend to an asymptotic symmetry at conformal infinity, as described in section 2.1. These are the so called large diffeos, would-be gauge transformations that leads to a different physical situation. For instance, by such transformation, one can turn a stationary black hole to a moving black hole by means of the action of the Poincaré group at infinity.

Said in a different way, the solutions of (2.69) which tends to asymptotic symmetries at conformal infinity will induce an "active" transformation in the interior. Following [21], one can associate a conserved charge to an infinitesimal coordinate transformation ξ^a. In Einstein's gravity, this is

$$Q[\xi] = -\frac{1}{16\pi G} \int_{\Sigma} \nabla^b \xi^c \epsilon_{bca_1 \cdots a_{d-2}} \,. \tag{2.70}$$

One can calculate the total mass M, associated with time translation, and angular momentum J, related to azimuthal rotation. The charge associated with a general asymptotic symmetry is less clear. But regardless of their interpretation, asymptotic symmetries that fail to commute with time translation or azimuthal rotation, for instance, cannot be pure gauge transformations because they will change the total mass or angular momentum of spacetime. We will save further discussion for when they are due. In chapter 3 we will apply these ideas to the BTZ black hole, where the lack of dynamics in 3d gravity will allow us to find analytic solutions. Finally, we will tackle the same problem for the Kerr black hole in chapter 4.

[3] One can always make $\bar{h}_{ab} = h_{ab} - \frac{1}{2} h g_{ab}$ transverse [17]. So even in case where the traceless transverse gauge cannot be achieved, one can think of the residual gauge transformations (2.69) as preserving the transverse condition on $\bar{h}_{ab}$, though the condition $\nabla_a \xi^a = 0$ is unnecessary in this case.

3 BTZ Black Hole

It has been known that in flat three dimensional Einstein's gravity there is no solution with a black hole event horizon. So in 1992 it came as a surprise when Bañados, Teitelboim and Zanelli found a black hole solution for the case of negative cosmological constant [22]. It is very similar to the four-dimensional flat black hole, although much simpler to treat. Hence the BTZ solution has been used extensively in the last two decades as a toy model for the more realistic Kerr black hole. We shall do so as well; this chapter covers the action of large diffeos in BTZ black hole.

The metric, in BTZ coordinates, is given by

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2} dt^2 + \frac{l^2 r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2 \left(d\phi - \frac{r_+ r_-}{l r^2} dt \right)^2 . \tag{3.1}$$

Here l is the AdS radius. The two parameters $r_\pm$ are the radial locations of the event and Cauchy, or outer and inner, horizons. They are also related to the mass and angular momentum

$$M = \frac{r_+^2 + r_-^2}{l^2} , \quad J = \frac{2 r_+ r_-}{l} . \tag{3.2}$$

The global AdS solution is separated by a mass gap from the black hole spectrum. It is obtained by setting $M = -1$ and $J = 0$, or $r_+ = il$ and $r_- = 0$. The Killing vectors whose norm vanish at the horizons are

$$\chi_\pm = \partial_t + \Omega_\pm \partial_\phi , \tag{3.3}$$

where

$$\Omega_\pm = \frac{r_\mp}{l r_\pm} \tag{3.4}$$

are the angular velocities of the horizons. The surface gravities are

$$\kappa_\pm = \frac{r_+^2 - r_-^2}{l^2 r_\pm} . \tag{3.5}$$

See [23] for more details.

3.1 AdS$_3$ Kinematics

AdS$_3$ can be viewed as the universal covering of the hyperboloid $-T_1^2 - T_2^2 + X_1^2 + X_2^2 = -l^2$ in $\mathbb{R}^{(2,2)}$ with flat metric metric $ds^2 = -dT_1^2 - dT_2^2 + dX_1^2 + dX_2^2$. For instance, if

$$\begin{aligned} X_1 &= l\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}\sinh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right)\,, \quad X_2 = l\sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}}\cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)\,, \\ T_1 &= l\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}\cosh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right)\,, \quad T_2 = l\sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}}\sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)\,, \end{aligned} \tag{3.6}$$

one obtains the region $r > r_+$ of the BTZ black hole. Other parametrizations give the remaining regions [23]. Let us explore this construction in a general setting.

Points in the hyperboloid can be viewed as elements of SL$(2,\mathbb{R})$, the group of 2x2 matrices with unit determinant:

$$g = \frac{1}{l}\begin{pmatrix} T_1 + X_1 & T_2 + X_2 \\ -T_2 + X_2 & T_1 - X_1 \end{pmatrix} = \begin{pmatrix} U_1 & U_2 \\ -V_2 & V_1 \end{pmatrix}\,, \tag{3.7}$$

where we defined lightcone coordinates $U_i = (T_i + X_i)/l$ and $V_i = (T_i - X_i)/l$. In this context, the points are defined up to the action of elements of the group, i.e., g and $h_L\, g\, h_R$ are equivalent, as long as $h_{L,R}$ are elements of SL$(2,\mathbb{R})$. Thus the isometry group is recognized as two copies of SL$(2,\mathbb{R})$acting on the left and right: SL$(2,\mathbb{R})_L \times$SL$(2,\mathbb{R})_R$. By introducing the Pauli matrices σ^i as the Lie generators of the group, one can endow it with a metric structure by defining the Killing-Cartan form

$$\eta^{ij} = \frac{1}{2}\text{Tr}(\sigma^i\sigma^j) \tag{3.8}$$

and its inverse η_{ij}. Let us now transfer these group properties into spacetime.

We first need to define the spacetime metric, which takes the form $ds^2 = -dU_1 dV_1 - dU_2 dV_2$ in terms of the lightcone coordinates. To respect the symmetries above, the metric should be obtained from a symmetric bilinear form that is invariant under $g \to h_L\, g\, h_R$. The only such structure available is the SL$(2,\mathbb{R})$ bi-invariant Tr$(g^{-1}dg)^2$. Explicit evaluation shows that the metric is

$$ds^2 = \frac{1}{2}\text{Tr}(g^{-1}dg)^2\,. \tag{3.9}$$

Using spacetime tensor notation, the metric is $g_{ab} = \text{Tr}(g^{-1}(dg)_a\, g^{-1}(dg)_b)/2$, where $(dg)_a =$

$\partial_a g$.

The group isometries induce a set of Killing vectors on spacetime. To see this, let us consider infinitesimal SL$(2,\mathbb{R})_L$ transformations $h_L \approx 1 + \alpha_i^L \sigma^i$, where α_i^L are infinitesimal parameters. The infinitesimal change due to the left action $\delta g = \alpha_i^L \sigma^i g$, induces a change on scalar functions that are generated by the vector fields. On g itself, we have

$$\sigma^i g = J^{i,a}(dg)_a \,. \tag{3.10}$$

The right hand side is the Lie derivative of g along J^i, the generators of left transformations on spacetime. The equation above can be solved for the vector field by multiplying both sides with $g^{-1}(dg)_b\, g^{-1}$ on the right and taking the trace. The right hand side will result in $2J^{i,a}g_{ab} = 2J^i_b$. Therefore,

$$J^i_a = \frac{1}{2}\mathrm{Tr}((dg)_a\, g^{-1}\sigma^i)\,. \tag{3.11}$$

Note that J^i is the σ_i component of the SL$(2,\mathbb{R})$ element current

$$J_a = (dg)_a\, g^{-1} = J^i_a \sigma_i \,. \tag{3.12}$$

This is also called the right invariant current, since it is unchanged under global transformations $g \to g h_R$. A few properties follow from the definitions above, namely

$$g_{ab} = J^i_a J_{i,b}\,, \quad \eta^{ij} = J^i_a J^{j,a}\,. \tag{3.13}$$

Also, from (3.10), one can derive the commutation relations among the J^i. All one has to do is take the Lie derivative of both sides with respect to J^j and then subtract the $i \leftrightarrow j$ equation. Using the identity $\mathcal{L}_{[J^i,J^j]} = \mathcal{L}_{J^i}\mathcal{L}_{J^j} - \mathcal{L}_{J^j}\mathcal{L}_{J^i}$, we see that if $[\sigma^i,\sigma^j] = -C^{ij}{}_k\sigma^k$, then

$$[J^i, J^j] = C^{ij}{}_k J^k\,; \tag{3.14}$$

i.e, the Lie algebra of spacetime generators is isomorphic to the algebra of the group generators.

The same can be done for the SL$(2,\mathbb{R})_R$ symmetry. Now, the infinitesimal transformation generated by $h_R = 1 + \alpha_i^R \sigma^i$ is

$$g\sigma^i = \bar{J}^{i,a}(dg)_a\,, \tag{3.15}$$

where $\bar{J}^i$ are the generators of right transformations on spacetime. Inversion of (3.15) is done by multiplying both sides by $g^{-1}(dg)_b\, g^{-1}$ on the left and taking the trace. This yields

$$\bar{J}^i_a = \frac{1}{2}\mathrm{Tr}(g^{-1}(dg)_a\,\sigma^i)\,. \tag{3.16}$$

The same identities (3.13) hold for the barred, left invariant currents, $\bar{J}_a = g^{-1}(dg)_a$. And it follows from (3.15), through the same calculation performed for the left generators, that

$$[\bar{J}^i, \bar{J}^j] = -C^{ij}{}_k \bar{J}^k\,, \tag{3.17}$$

with the same structure constants, apart from the minus sign. It also follows from (3.10) and (3.15) that

$$[J^i, \bar{J}^j] = 0\,. \tag{3.18}$$

Thus the isometry algebra on spacetime is exactly $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$.

Now, introduce the Gauss parametrization for $\mathrm{SL}(2,\mathbb{R})$,

$$g = \mathrm{e}^{u\sigma^3}\mathrm{e}^{\varrho\sigma^1}\mathrm{e}^{v\sigma^3} = \begin{pmatrix} \mathrm{e}^{u+v}\cosh\varrho & \mathrm{e}^{u-v}\sinh\varrho \\ \mathrm{e}^{-u+v}\sinh\varrho & \mathrm{e}^{-u-v}\cosh\varrho \end{pmatrix}\,. \tag{3.19}$$

The left and right currents are

$$\begin{aligned} J &= (du + \cosh 2\varrho\, dv)\sigma^3 + \mathrm{e}^{2u}(d\varrho - \sinh 2\varrho\, dv)\sigma^+ + \mathrm{e}^{-2u}(d\varrho + \sinh 2\varrho\, dv)\sigma^-\,, \\ \bar{J} &= (dv + \cosh 2\varrho\, du)\sigma^3 + \mathrm{e}^{-2v}(d\varrho + \sinh 2\varrho\, du)\sigma^+ + \mathrm{e}^{2v}(d\varrho - \sinh 2\varrho\, du)\sigma^-\,, \end{aligned} \tag{3.20}$$

where $\sigma^\pm = (\sigma^1 \pm i\sigma^2)/2$. From the expressions above we may read the components of the one-forms in the basis $(\sigma^3, \sigma^+, \sigma^-)$. In this basis, the Killing-Cartan form (3.8) and its inverse are

$$\eta^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}\,, \quad \eta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}\,. \tag{3.21}$$

It then follow from (3.13) that the spacetime metric is

$$ds^2 = d\varrho^2 + du^2 + dv^2 + 2\cosh 2\varrho\, du dv\,. \tag{3.22}$$

Using the inverse metric, we find the isometry generating vectors

$$J_3 = \partial_u\,, \quad J_\pm = \mathrm{e}^{\pm 2u}\left(\partial_\varrho \mp \frac{\cosh 2\varrho}{\sinh 2\varrho}\partial_u \pm \frac{1}{\sinh 2\varrho}\partial_v\right)\,, \tag{3.23}$$

$$\bar{J}_3 = \partial_v\,, \quad \bar{J}_\pm = \mathrm{e}^{\mp 2v}\left(\partial_\rho \pm \frac{\cosh 2\varrho}{\sinh 2\varrho}\partial_v \mp \frac{1}{\sinh 2\varrho}\partial_u\right)\,. \tag{3.24}$$

The Lie algebra is

$$[J_3, J_\pm] = \pm 2J_\pm\,, \quad [J_+, J_-] = 4J_3\,, \tag{3.25}$$

$$[\bar{J}_3, \bar{J}_\pm] = \mp 2\bar{J}_\pm\,, \quad [\bar{J}_+, \bar{J}_-] = -4\bar{J}_3\,. \tag{3.26}$$

The structure constants are completely antisymmetric with $C_{3+-} = 4$ and $C^{3+-} = -1$. One can show that

$$C^{ijk}C_{ijk} = 3!\det(\eta^{ij}) = -24\,. \tag{3.27}$$

One last result, obtained after a little algebra from the commutation relations and with the aid of (3.13), is

$$\nabla_a(J_i)_b = \epsilon_{abc}J_i^c\,, \quad \nabla_a(\bar{J}_i)_b = -\epsilon_{abc}J_i^c\,. \tag{3.28}$$

Here, ∇_a is the derivative compatible with the metric and

$$\epsilon_{abc} = -\frac{1}{2}C_{ijk}J_a^iJ_b^jJ_c^k = \frac{1}{2}C_{ijk}\bar{J}_a^i\bar{J}_b^j\bar{J}_c^k \tag{3.29}$$

is the volume form obtained from (3.22).

The general BTZ metric (3.1) can be obtained from the $\mathrm{SL}(2,\mathbb{R})$metric (3.22) by the change of variables

$$u = \frac{r_+ - r_-}{2}(\phi + t)\,, \quad \cosh^2\varrho = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}\,, \quad v = \frac{r_+ + r_-}{2}(\phi - t)\,, \tag{3.30}$$

where we set $l = 1$. Note that this is much in the spirit of what we will be studying here: coordinate transformations that change the value of charges, like mass and angular momentum. We can foresee that the situation is indeed simple in the BTZ case, for we can obtain any solution from the general $\mathrm{SL}(2,\mathbb{R})$metric through the coordinate choice above.

A final remark concerns the eigenvalues of J_3 and $\bar{J}_3$. One can see that the Killing vectors J_3 and $\bar{J}_3$ are linear combinations of the time translation and azimuthal rotation vectors. Thus the eigenvalues $2p$ and $2q$ of J_3 and $\bar{J}_3$, respectively, are related to those of ∂_t, $-i\omega$, and ∂_ϕ,

im, by

$$p = \frac{i}{2}\frac{m-\omega}{r_+ - r_-}\,, \quad q = \frac{i}{2}\frac{m+\omega}{r_+ + r_-}\,. \tag{3.31}$$

3.2 Brown-Henneaux group

The asymptotic symmetry group of anti-de Sitter space in three dimensions was studied by Brown and Henneaux in 1986 when they derived it by assuming appropriate boundary conditions for the falloff of the metric at spatial infinity [7]; they obtained two copies of the Witt algebra. We shall derive it from the framework built in chapter 2, where we showed that the generators of the ASG for asymptotically AdS spaces are isomorphic to the conformal Killing vectors of the boundary, spatial infinity.

From the relations to BTZ coordinates, (3.30), we can see that spatial infinity is approached as $\varrho \to \infty$. In this limit, the metric (3.22) tends to $ds^2 = \mathrm{e}^{2\varrho}dudv$. Thus a possible choice for the conformal factor is $\Omega = \mathrm{e}^{-\varrho}$. The induced metric at infinity is

$$d\hat{s}^2 = dudv \tag{3.32}$$

This form of the metric allows us to easily visualize the types of diffeos, or coordinate changes, that give conformally related metrics, namely $u \to f(u)$ and $v \to g(v)$ for arbitrary monotonically increasing functions. Each of these independent transformations have infinite degrees of freedom. One way to label them as infinitesimal symmetries are

$$u \to u + \frac{1}{2}\mathrm{e}^{2nu}\,, \quad v \to v + \frac{1}{2}\mathrm{e}^{2nv}\,. \tag{3.33}$$

These are generated by the vectors $l_n = \frac{1}{2}\mathrm{e}^{2nu}\partial_u$ and $\bar{l}_n = \frac{1}{2}\mathrm{e}^{2nv}\partial_v$ at infinity, which correspond to the parallel components $\xi_{\|}$ of the ASG generators in (2.36).

In order to find the full generators, we need the metric rescaling factors α and $\bar{\alpha}$ in (2.44). Those are

$$\alpha = \frac{1}{2}n\mathrm{e}^{2nu}\,, \quad \bar{\alpha} = \frac{1}{2}n\mathrm{e}^{2nv}\,. \tag{3.34}$$

The vector perpendicular to spatial infinity is $\hat{l} = -\mathrm{e}^{\varrho}\partial_\varrho$, for $\hat{l}^a\partial_a\Omega = 1$. Plugging all in (2.36) yields

$$l_n = \frac{1}{2}\mathrm{e}^{2nu}\left(-n\partial_\varrho + \partial_u\right) \tag{3.35}$$

and

$$\bar{l}_n = \frac{1}{2}\mathrm{e}^{2nv}\left(-n\partial_\varrho + \partial_v\right) . \tag{3.36}$$

These give exactly the algebra of conformal symmetries in 2d, two copies of the Witt algebra:

$$[l_n, l_m] = (m-n)l_{n+m} , \quad [\bar{l}_n, \bar{l}_m] = (m-n)\bar{l}_{n+m} , \quad [l_n, \bar{l}_m] = 0 . \tag{3.37}$$

The global part of the algebra is generated by l_{-1}, l_0, l_1 and $\bar{l}_{-1}, \bar{l}_0, \bar{l}_1$, which we recognize as the limits of (3.25) and (3.26) when $\varrho \to \infty$, apart from $1/2$ factors. Thus they realize the $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ three dimensional anti-de Sitter algebra, while the remaining vectors induce local conformal transformations at spatial infinity.

3.3 Large Diffeomorphisms

As discussed in the end of chapter 2, transformations at conformal infinity related to the asymptotic symmetries will induce a non trivial transformation in the bulk. In this section we will solve the traceless transverse gauge preserving residual transformations (2.69) imposing the boundary condition that ξ^a tends to the asymptotic symmetries of last section. Let us first find solutions for which $\xi^a \to l_n^a$. The derivation of the barred counterpart will be equivalent and we shall not delve into the details.

In order to solve the differential equations, we will make use of the properties of spacetime inherited by the $\mathrm{SL}(2,\mathbb{R})$symmetry derived in section 3.1 . Let $\xi_i = J_i^a \xi_a$ and $\nabla_i = J_i^a \nabla_a$. Using equations (3.13) and (3.29), one can show that

$$\nabla^a \xi_a = \nabla^i \xi_i , \quad J_i^a \nabla^2 \xi_a = \nabla^2 \xi_i + C_i^{\ jk} \nabla_j \xi_k + 2\xi_i . \tag{3.38}$$

Define

$$\lambda_i = C_i^{\ jk} \nabla_j \xi_k . \tag{3.39}$$

The idea here is to transform the second order differential equation in a set of two linear equations on ξ_i and λ_i. First notice that the second equation in (2.69) gives the relation

$$\nabla^2 \xi_i + \lambda_i = 0 . \tag{3.40}$$

Now take the curl of (3.39)

$$\begin{aligned} C_i^{\ jk}\nabla_j\lambda_k &= 4\nabla^2\xi_i - 4C_i^{\ jk}\nabla_j\xi_k - 4\nabla_i(\nabla^j\xi_j) \\ &= -8\lambda_i - 4\nabla_i(\nabla^j\xi_j)\,. \end{aligned} \tag{3.41}$$

When we impose the condition that ξ^a is divergenceless, the equation above for λ_i decouples. However, the solution fails to satisfy the required boundary condition. Therefore we shall set $\lambda = 0$.[1]

The components of the vector field thus satisfy Laplace's equation, though not quite independently – they are subjected to the constraint $C_i^{\ jk}\nabla_j\xi_k = 0$. In terms of

$$z = \cosh^2\varrho = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}\,, \tag{3.42}$$

the Laplace's equation is given by

$$\nabla^2\xi_i = \left[\frac{\partial}{\partial z}\left(z(z-1)\frac{\partial}{\partial z}\right) + \frac{1}{16z(z-1)}\left((2z-1)\frac{\partial^2}{\partial u\partial v} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right)\right]\xi_i = 0\,. \tag{3.43}$$

In general, a solution to this equation can be expanded in simultaneous eigenvalues of J_3 and $\bar{J}_3$, namely $\mathrm{e}^{2pu+2qv}$ as discussed in the end of section 3.1. However, in order to have our solution matching one of the asymptotic vectors l_n, we must set $q = 0$. Hence ξ^a is only a function of z and u. This simplifies the Laplace's equation to the point that its solutions can be written in terms of elementary functions. For instance, the solution for ξ_3 is

$$\xi_3 = -\frac{p^2-1}{2}\left[c_+\left(\frac{z-1}{z}\right)^{p/2} + c_-\left(\frac{z}{z-1}\right)^{p/2}\right]\mathrm{e}^{2pu}\,. \tag{3.44}$$

The reason for the choice of normalization will become apparent when we take the limit $z \to \infty$ and compare with the generators of the ASG. The other components can be obtained from $C_i^{\ jk}\nabla_j\xi_k = 0$, yielding

$$\xi_+ = \frac{p(p-1)}{2}\left[c_+\left(\frac{z-1}{z}\right)^{(p+1)/2} + c_-\left(\frac{z}{z-1}\right)^{(p+1)/2}\right]\mathrm{e}^{2(p+1)u} \tag{3.45}$$

[1]In case the divergence of ξ^a is nonzero, as discussed in the footnote 3 of chapter 2, we still have that the solutions of the homogeneous equation does not satisfy the boundary condition. However, one cannot set $\lambda_i = 0$. It turns out that the solution of (3.41) is simple: $\lambda_i = -\nabla_i(\nabla^j\xi_j)$.

and

$$\xi_- = -\frac{p(p+1)}{2}\left[c_+\left(\frac{z-1}{z}\right)^{(p-1)/2} + c_-\left(\frac{z}{z-1}\right)^{(p-1)/2}\right]\mathrm{e}^{2(p-1)u}\,. \tag{3.46}$$

When we relate to BTZ coordinates (3.30), the vanishing of the $\bar{J}_3$ eigenvalue q means that the frequency and angular momentum are related by $\omega = -m$ (see (3.31)). We shall then refer to these solutions as the left moving large diffeos or simply as left diffeos.

Going back to spacetime vector notation, we can write $\xi^a = \eta^{ij} J_i^a \xi_j$. Using coordinates (z, u, v), we have

$$\begin{aligned}\xi_p = &\frac{c_+}{2}\mathrm{e}^{2pu_+}\left[-p(p+2z-1)\partial_z + \left(\frac{p(p+2z-1)}{4z(z-1)}+1\right)\partial_u - \frac{p(p(2z-1)+1)}{4z(z-1)}\partial_v\right] \\ &+ \frac{c_-}{2}\mathrm{e}^{2pu_-}\left[p(p-2z+1)\partial_z + \left(\frac{p(p-2z+1)}{4z(z-1)}+1\right)\partial_u - \frac{p(p(2z-1)-1)}{4z(z-1)}\partial_v\right]\,. \end{aligned} \tag{3.47}$$

where we defined null coordinates on the $z-u$ plane

$$u_\pm = u \pm \frac{1}{4}\log\left(\frac{z-1}{z}\right)\,, \tag{3.48}$$

which are respectively ingoing and outgoing left moving coordinates. Notice that ξ_p reduces to the $\mathrm{SL}(2,\mathbb{R})_L$ current vectors when $p = 0, \pm 1$. At spatial infinity both $u_\pm$ tends to u and the vector

$$\xi_p \to (c_+ + c_-)\,\mathrm{e}^{2pu}\left(-nz\partial_z + \frac{1}{2}\partial_u\right)\,, \tag{3.49}$$

which is the expression for l_p in terms of the z coordinate. Thus, the only restriction that must be imposed on the coefficients $c_\pm$ is that their sum must not vanish, otherwise we end up with a small diffeomorphism. The actual change in the metric can be calculated with the aid of eq. (3.29). The result is

$$\mathcal{L}_{\xi_p} g_{ab} = 2p(1-p^2)\left[c_+\mathrm{e}^{2pu_+}(du_+)_a(du_+)_b + c_-\mathrm{e}^{2pu_-}(du_-)_a(du_-)_b\right]\,. \tag{3.50}$$

As expected, this vanishes for $p = 0, \pm 1$, for these are Killing vectors. It should be noted that if we loosen the constraints that the large diffeos must preserve the trace and divergence of metric perturbations, we can readily obtain infinitely many solutions from (3.47). Namely $[l_{p-n}, \xi_n]$ will tend to l_p as one approaches spatial infinity, due to the Witt algebra of the ASG.

Before analyzing the effects of the large diffeos at the horizons, let us discuss the right diffeos – the solutions to (2.69) that approach $\bar{l}_n$ at spatial infinity. We can consider, in the same way, the components of tensors in the basis of the $\mathrm{SL}(2,\mathbb{R})_R$ current vectors. For instance, let $\bar{\xi}_i = \bar{J}_i^a \xi_a$. All the calculations that follow are analogous to what was done above, the only difference being the change of sign in (3.29), which will effectively make $C_{ijk} \to -C_{ijk}$ in all equations. The solutions now will have zero eigenvalue of J_3 and, in relation to BTZ coordinates, will have $\omega = m$.

The fact that when changing from left to right currents, we make $u \to v$ and $J_\pm \to \bar{J}_\mp$, implies that the components $\bar{\xi}_i$ are not obtained by simply making $u \to v$. However, since the asymptotic vectors have this symmetry, the full vector $\bar{\xi}_q^a$ turns out to be related to ξ_p^a by $u \leftrightarrow v$. Thus

$$\begin{aligned}\bar{\xi}_q = \frac{\bar{c}_+}{2}\mathrm{e}^{2qv_+}&\left[-q(q+2z-1)\partial_z + \left(\frac{q(q+2z-1)}{4z(z-1)}+1\right)\partial_v - \frac{q(q(2z-1)+1)}{4z(z-1)}\partial_u\right]\\ + \frac{\bar{c}_-}{2}\mathrm{e}^{2qv_-}&\left[q(q-2z+1)\partial_z + \left(\frac{q(q-2z+1)}{4z(z-1)}+1\right)\partial_v - \frac{q(q(2z-1)-1)}{4z(z-1)}\partial_u\right]. \end{aligned} \quad (3.51)$$

Now the null coordinates

$$v_\pm = v \pm \frac{1}{4}\log\left(\frac{z-1}{z}\right) \quad (3.52)$$

represent outgoing and ingoing right moving coordinates, respectively. All the discussion following equation (3.47) has a straightforward parallel for the right diffeos.

3.4 Near horizon

In terms of the radial z coordinate, the inner and outer horizons are located at $z = 0$ and $z = 1$, respectively. One can see that the vector fields (3.47) and (3.51) seem to be ill defined at the horizons, but this is not true. The divergence of the expressions is just a reflection of the fact that the coordinates u and v, or rather ϕ and t (see (3.30)), fail to cover the horizons; this is simply an apparent singularity.

Consider the following. Particles travelling at constant values of u_+, for instance, are ingoing and will eventually reach the event horizon and dive into the black hole. The situation for u_- is the opposite; particles in level curves of this coordinate will travel towards spatial infinity. However, when we look at its time reversed trajectory, it seems it was originated at the event horizon as well. This reasoning shows us that the event horizon has two disjoint surfaces: future and past event horizon $\mathcal{H}^\pm$. This gives a reason for the failure of the time coordinate, as measured by an observer at infinity; the infinite redshift caused by the event horizon implies that the future event horizon is located at $t = \infty$ and the past horizon at $t = -\infty$. The bad behavior with the angular coordinate ϕ only happens when the black hole has a nonzero angular momentum, as it causes an indefinite amount of winding from the point of view of the observer at infinity. Therefore, in order to describe the effect of the large diffeos near horizon, we must find appropriate coordinates.

Concerning the left diffeos, from the near horizon perspective, the two solutions in (3.47) controlled by the constants $c_\pm$ act in different parts of the event horizon: future or past, respectively. From a physical point of view, black holes can arise by the collapse of matter and the past event horizon is not a real location, unless one consider an unrealistic eternal black hole. In any case, the idea here is to infer what changes are induced on the black hole through some measurement made by an observer at spatial infinity. Causality thus restricts our attention to the future event horizon $\mathcal{H}^+$. We shall then set $c_- = 0$. As for the right diffeos, the ingoing coordinate is v_- and thus we set $\bar{c}_+ = 0$ in (3.51).

Let us get an intuition in to what the regular coordinates at $\mathcal{H}^+$ are by working with BTZ coordinates, in which the metric is given by (3.1). We shall make a change of variables similar to the ingoing Eddington-Finkelstein coordinates in 4d black holes. First define the tortoise coordinate

$$r_* = \int \frac{r^2 dr}{(r^2 - r_+^2)(r^2 - r_-^2)} = \frac{r_+}{2(r_+^2 - r_-^2)} \log\left(\frac{r - r_+}{r + r_+}\right) - \frac{r_-}{2(r_+^2 - r_-^2)} \log\left(\frac{r - r_-}{r + r_-}\right) \tag{3.53}$$

and the retarded time $\tilde{t} = t + r_*$. An angular comoving coordinate can then be defined by $\tilde{\phi} = \phi + \hat{r}$, where

$$\hat{r} = \int \frac{r_+ r_- dr}{(r^2 - r_+^2)(r^2 - r_-^2)} = \frac{r_-}{2(r_+^2 - r_-^2)} \log\left(\frac{r - r_+}{r + r_+}\right) - \frac{r_+}{2(r_+^2 - r_-^2)} \log\left(\frac{r - r_-}{r + r_-}\right) . \tag{3.54}$$

The ingoing BTZ coordinates are $(\tilde{t}, r, \tilde{\phi})$, in terms of which the metric is

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} d\tilde{t}^2 + 2d\tilde{t}dr + r^2 \left(d\tilde{\phi} - \frac{r_+ r_-}{r^2} d\tilde{t}\right)^2 . \tag{3.55}$$

It is now regular at both horizons. Note that the radial vector ∂_r, in this coordinate system, describes the path of ingoing null rays. The Killing vector that vanishes at the event horizon (3.3) is given by $\chi_+ = \partial_{\tilde{t}} + \Omega_+ \partial_{\tilde{\phi}}$. It picks out a preferred corotating angular coordinate,

$$\phi_H = \tilde{\phi} - \Omega_+ \tilde{t} \, , \tag{3.56}$$

whose level surfaces are normal to χ_+. We then define the event horizon coordinates $(\tilde{t}, r, \phi_H)$. The metric now reads

$$ds^2 = 2d\tilde{t}dr + r^2 d\phi_H^2 - \frac{r^2 - r_+^2}{r_+} d\tilde{t} \left(\kappa_+ d\tilde{t} - 2\Omega_+ d\phi_H\right) . \tag{3.57}$$

At $r = r_+$, the retarded time is the null coordinate corresponding to the null part of the future event horizon and the vector field $\partial_{\tilde{t}}$ coincides with the Killing vector that vanishes at $\mathcal{H}^+$. Cross sections of the horizon yield a 1d compact space which is parametrized by the corotating angular coordinate ϕ_H.

The description we will give of the large diffeos near the event horizon is not exactly obtained from the coordinates above. One reason for that can be seen from the definition of u_+ in (3.48); the z dependent part is not the same as the tortoise coordinate, it is, however, the coordinate that puts the Laplacian (3.43) in the normal form. Yet, there is a close resemblance between the event horizon coordinate system showed above and the ones we will use to describe the left and right diffeos. Note, for instance, that $\tilde{\phi} + \tilde{t}$ differs from u_+ by an analytic function at both horizons:

$$u_+ = \frac{r_+ - r_-}{2}(\tilde{\phi} + \tilde{t}) + \frac{1}{2}\log\left(\frac{r + r_+}{r + r_-}\right) . \tag{3.58}$$

Thus we would like that our coordinate system have some similarities with the obtained above. However, in order to keep explicit the chiral properties of our solution, we will have u_+ as the retarded time. Note that in (z, u, v) coordinates, the Killing vector that vanishes at the event horizon is proportional to $\partial_u - \partial_v$. In order to mirror the properties of $(\tilde{t}, r, \phi_H)$, we want a coordinate system in which the vector along u_+ coincides with the Killing vector that vanishes at the event horizon. After a little algebra, one can show that (u_+, z, ψ_H), where

$$\psi_H = u + v - \frac{1}{2}\log z\,, \tag{3.59}$$

accomplishes that. The metric,

$$ds^2 = 2du_+ dz + d\psi_H^2 - 4(z-1)du_+(du_+ - d\psi_H)\,, \tag{3.60}$$

is regular at the event horizon and ψ_H functions as the angular coordinate there. In fact, since $u+v = r_+(\phi - \Omega_+ t)$, one can see that ψ_H is closely related to the corotating angular coordinate ϕ_H.

Now, in order to write the left diffeo in this coordinate system, one only needs to calculate the components $\xi_p^a(du_+)_a, \xi_p^a(dz)_a$ and $\xi_p^a(d\psi_H)_a$. This leads us to the expression

$$\xi_p = \frac{1}{2}\mathrm{e}^{2pu_+}\left[\partial_+ + (p+1)\partial_H - p(p+2z-1)\partial_z\right] , \tag{3.61}$$

where we set $c_+ = 1$. In this coordinate system it is easy to verify that the ingoing part of the left diffeos also satisfy the Witt algebra

$$[\xi_n, \xi_m] = (m-n)\xi_{n+m} , \tag{3.62}$$

thus transposing the whole structure of the left ASG at spatial infinity to the bulk and, in particular, the event horizon. The transformations induced at the horizon are time dependent; each cross section will be affected differently due to the e^{2pu_+} factor. It is a composition of a time dependent supertranslation, a rotation and a radial displacement.

Similarly for the right diffeos, we define the coordinate system $(v_-, z, \bar{\phi}_H)$, where

$$\bar{\psi}_H = u + v + \frac{1}{2}\log z\,. \tag{3.63}$$

The metric assumes a similar form,

$$ds^2 = -2dv_-dz + d\bar{\psi}_H^2 - 4(z-1)dv_-(dv_- - d\bar{\psi}_H), \tag{3.64}$$

while the right diffeos with $\bar{c}_- = 1$ are

$$\bar{\xi}_q = \frac{1}{2}\mathrm{e}^{2qv_-}\left[\bar{\partial}_- - (q-1)\bar{\partial}_H + q(q-2z+1)\bar{\partial}_z\right] , \tag{3.65}$$

also satisfying the Witt algebra $[\bar{\xi}_n, \bar{\xi}_m] = (m-n)\bar{\xi}_{n+m}$. However, left and right diffeos do not commute, as the ASG generators do. In fact, we have, in left ingoing coordinates,

$$[\xi_p, \bar{\xi}_q] = \frac{pq(p-1)(q+1)}{4z^2}\mathrm{e}^{2pu_+ + 2qv_-}\left[\frac{q-1}{2}\partial_+ + (q-1)\partial_H + (q+p)z\partial_z\right] . \tag{3.66}$$

Note that whenever p and q assume the values -1,0 or 1, the commutator vanishes, as expected since one returns to the global SL$(2,\mathbb{R})_L\times$SL$(2,\mathbb{R})_R$. It also vanishes as $z \to \infty$.

4 Kerr Black Hole

Shortly after General Relativity was proposed in 1915, a static black hole solution was found by Schwarzschild [24]. A few years later another solution was found; a charged, yet still static, black hole, discovered by Reissner [25] and Nordström [26]. It was only in 1963 that a rotating solution was found by Kerr when the algebraic properties of spacetimes were better understood [27]. Since the universe is, in general, electrically neutral, this is the solution that most accurately describes the black holes found in our universe. In particular, gravitational waves detected by LIGO in 2016 [28] originated from two Kerr black holes merging, with the end state also being described by this solution. In this chapter we will study the action of large diffeos in Kerr black holes.

In Boyer-Lindquist coordinates, the Kerr metric, with mass M and angular momentum $J = Ma$, is

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)d\phi - adt\right)^2 + \Sigma d\theta^2\,, \tag{4.1}$$

where

$$\begin{aligned} \Delta &= r^2 + a^2 - 2Mr = (r - r_+)(r - r_-)\,, \\ \Sigma &= r^2 + a^2\cos^2\theta\,. \end{aligned} \tag{4.2}$$

The constant a is called rotation parameter; when it is set to zero, the metric becomes that of the Schwarzschild black hole. Like BTZ, the Kerr black hole has two horizons, located at

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{4.3}$$

Both are Killing horizons with vanishing Killing vectors given by

$$\chi_\pm = \partial_t + \Omega_\pm \partial_\phi\,, \tag{4.4}$$

where the angular velocities of the horizons and their surface gravities are

$$\Omega_\pm = \frac{a}{2Mr_\pm} \,. \quad \kappa_\pm = \frac{r_+ - r_-}{4Mr_\pm} \,. \tag{4.5}$$

For a recent review of the properties and history of the Kerr black hole, see [29].

4.1 Newman-Penrose formalism

Unlike anti-de Sitter space in three dimensions, flat 4d spaces have not enough symmetries that allow for complete integrability. As stated earlier, the metric is dynamical and the graviton – metric perturbations around a background solution – has two independent polarizations. There are, however, some solutions of the Einstein's equations that possess some algebraic properties which provide great computational power. Those are properties of the null geodesics on the spacetime. It was only after a systematic study of them that the Kerr black hole was found. In 1962, Newman and Penrose (NP) introduced a formalism based on null tetrads that make such properties more explicit [30]. In this section we will review their formalism and apply to the Kerr black hole.

In general, one may write the metric and its inverse in terms of a tetrad e^a_μ:

$$g^{ab} = \eta^{\mu\nu} e^a_\mu e^b_\nu \,, \tag{4.6}$$

where Greek indices label the tetrads and, as before, Latin indices are for spacetime tensors. This relation can be inverted to

$$\eta_{\mu\nu} = g_{ab} e^a_\mu e^b_\nu \,. \tag{4.7}$$

Like the Killing-Cartan form of section 3.1, the constant invertible matrix $\eta_{\mu\nu}$ may be used to raise and lower tetrad indices. Now, instead of dealing with Christoffel symbols, one defines the Ricci rotation coefficients,

$$\lambda_{\rho\mu\nu} = e^a_\rho e^b_\mu \nabla_a (e_\nu)_b \,, \tag{4.8}$$

which are antisymmetric in the last two indices due to the identity (4.7). One has therefore 24 independent components, instead of the 40 Christoffel symbols. This equation can be inverted to give $\nabla_a (e_\nu)_b = (e^\mu)_b \lambda_{a\mu\nu}$, where $\lambda_{a\mu\nu} = \lambda_{\rho\mu\nu} e^\rho_a$. Using the spacetime differential form notation $\boldsymbol{e}_\mu = (e_\mu)_a$ and $\boldsymbol{\lambda}_{\mu\nu} = \lambda_{a\mu\nu}$, one can rewrite (4.8) as

$$\mathrm{d}\boldsymbol{e}_\mu = \boldsymbol{e}^\nu \wedge \boldsymbol{\lambda}_{\mu\nu} \,, \tag{4.9}$$

where d is the exterior derivative. The same can be done for the Riemann tensor. Its components on the tetrad basis may also be put in a differential form notation.

$$R_{ab\mu\nu} = R_{abcd}e^c_\mu e^d_\nu = e^c_\mu(\nabla_a\nabla_b - \nabla_b\nabla_a)(e_\nu)_c \tag{4.10}$$

may be written as a 2-form $\boldsymbol{R}_{\mu\nu}$. Explicit evaluation of the derivatives above yields

$$\boldsymbol{R}_{\mu\nu} = \mathrm{d}\boldsymbol{\lambda}_{\mu\nu} + \boldsymbol{\lambda}_\mu{}^\rho \wedge \boldsymbol{\lambda}_{\rho\nu}\,. \tag{4.11}$$

The two equations (4.9) and (4.11) are the first and second Cartan equations; complemented by the Bianchi identities, they provide the full description of the space in the tetrad formalism.

Newman and Penrose considered the case in which the tetrad metric

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \eta^{\mu\nu}\,; \tag{4.12}$$

all vectors in the tetrad are null. Using the notation

$$e^a_\mu = (l^a, n^a, m^a, \overline{m}^a)\,, \tag{4.13}$$

with tetrad indices assuming the values 1,2,3 and 4, we have

$$l^a n_a = -m^a \overline{m}_a = -1\,. \tag{4.14}$$

and any other product vanishing. While there are many possible choices for these vectors, in order to take full advantage of the formalism, it is more suitable to choose l^a and n^a to be the generators of null geodesics on spacetime. With the freedom of rescaling these two vectors, while maintaining their inner product equal to -1, l^a can be made affinely parametrized. The remaining vectors are then chosen to satisfy the relations above. The resulting vectors, known as the Kinnersley tetrad [31], are given by

$$
\begin{aligned}
l &= \frac{r^2+a^2}{\Delta}\partial_t + \frac{a}{\Delta}\partial_\phi + \partial_r \,, \\
n &= \frac{\Delta}{2\Sigma}\left(\frac{r^2+a^2}{\Delta}\partial_t + \frac{a}{\Delta}\partial_\phi - \partial_r\right) \,, \\
m &= \frac{1}{\sigma\sqrt{2}}\left(ia\sin\theta\,\partial_t + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \,, \\
\overline{m} &= (m)^* \,.
\end{aligned}
\tag{4.15}
$$

Here $\sigma = r + ia\cos\theta$, so that $\Sigma = \sigma\sigma^*$. Note that l^a describes outgoing trajectories and n^a, ingoing. From an outgoing perspective, m^a and $\overline{m}^a$ correspond to right and left circular polarizations, respectively. From an ingoing perspective, the roles are reversed. For a nice derivation of the null geodesics in Kerr using the Hamilton-Jacobi equation, see [32].

The Ricci rotation coefficients may be calculated with the aid of the structure constants in the commutation relations $[e_\mu, e_\nu] = C^\rho{}_{\mu\nu} e_\rho$. The left hand side can be written as $(\lambda_{\mu\rho\nu} - \lambda_{\nu\rho\mu})e^\rho$, expressing C in terms of λ. Inverting this relation yields

$$
\lambda_{\rho\mu\nu} = \frac{1}{2}(C_{\rho\mu\nu} - C_{\mu\rho\nu} - C_{\nu\rho\mu}) \,. \tag{4.16}
$$

The commutation relations are easier to calculate because the ordinary derivative operator can be used. The 24 independent coefficients may be written in terms of 12 complex quantities known as the spin coefficients. Those come from a spinor formalism which is naturally related to the null tetrad framework in the NP formalism. Please refer to their original paper [30] for more details on the spinorial viewpoint. The coefficients are

$$
\begin{aligned}
&\bar{\kappa} = -\lambda_{131} = 0\,, \qquad \bar{\sigma} = -\lambda_{331} = 0\,, \qquad \bar{\rho} = -\lambda_{431} = -\frac{1}{\sigma^*}\,, \\
&\bar{\lambda} = -\lambda_{424} = 0\,, \qquad \bar{\nu} = -\lambda_{224} = 0\,, \qquad \bar{\mu} = -\lambda_{324} = -\frac{\Delta}{2|\sigma|^2\sigma^*}\,, \\
&\bar{\tau} = -\lambda_{231} = -\frac{ia\sin\theta}{|\sigma|^2\sqrt{2}}\,, \qquad \bar{\pi} = -\lambda_{124} = \frac{ia\sin\theta}{(\sigma^*)^2\sqrt{2}}\,, \\
&\bar{\varepsilon} = -\frac{1}{2}(\lambda_{121} + \lambda_{134}) = 0\,, \qquad \bar{\gamma} = -\frac{1}{2}(\lambda_{221} + \lambda_{234}) = -\frac{\Delta}{2|\sigma|^2\sigma^*} + \frac{\partial_r\Delta}{4|\sigma|^2}\,, \\
&\bar{\alpha} = -\frac{1}{2}(\lambda_{434} + \lambda_{421}) = \frac{ia\sin\theta}{(\sigma^*)^2\sqrt{2}} - \frac{\cot\theta}{2\sigma^*\sqrt{2}}, \\
&\bar{\beta} = -\frac{1}{2}(\lambda_{321} + \lambda_{334}) = \frac{\cot\theta}{2\sigma\sqrt{2}}\,.
\end{aligned}
\tag{4.17}
$$

See [32] for the explicit evaluation. The coefficients were barred to avoid confusion with symbols used throughout this work.

The algebraic properties we referred to in the beginning of this section are those of the Weyl tensor. Since $C_{abcd} = C_{[ab][cd]} = C_{[cd][ab]}$, it can be regarded as a quadratic form on the space of bivectors $X^{ab} = X^{[ab]}$. At any given event on spacetime there are four eigenbivectors, which can be associated to null vectors, giving rise to the Principal Null Directions (PNDs). This leads to the Petrov classification [33]; one can organize solutions of the Einstein's equations according to the degeneracies of the PNDs. Black holes are generally of type D, meaning they have two double PNDs, corresponding to the ingoing and outgoing null directions. A double PND has the property of being a shear-free null geodesic and this is directly encoded on the spin coefficients; the vanishing of $\bar{\kappa}$ and $\bar{\sigma}$ is a reflection of the fact that l^a lies along a PND, while $\bar{\lambda} = \bar{\nu} = 0$ is the corresponding result for n^a. The other vanishing coefficient, $\bar{\varepsilon}$, is so because l^a was chosen to be affinely parametrized. The corresponding coefficient for n^a is $\bar{\gamma}$ and it cannot be made zero simultaneously due to the normalization $l^a n_a = -1$.

In the absence of cosmological constant, Einstein's equations imply that the Ricci tensor vanishes. Thus the Weyl tensor completely characterizes the curvature of spacetime. Due to its symmetries and the fact that it is completely traceless, the Weyl tensor has 10 independent components. In the NP formalism, the components $C_{\mu\nu\rho\sigma} = C_{abcd} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma$ are organized into 5 complex scalars given by

$$\Psi_0 = C_{1313}\,, \quad \Psi_1 = C_{1213}\,, \quad \Psi_2 = C_{1342}\,, \quad \Psi_3 = C_{1242}\,, \quad \Psi_4 = C_{2424}\,. \tag{4.18}$$

Now, the Goldberg-Sachs theorem states that, in type D spacetimes, the only nonzero component is Ψ_2. It can be calculated by using the equations (4.11). Those are listed in [30]. In particular, using the equation (4.2f) in that reference,

$$D\bar{\gamma} - \overline{D}\bar{\varepsilon} = (\bar{\tau} + \bar{\pi}^*)\bar{\alpha} + (\bar{\tau}^* + \bar{\pi})\overline{\beta} - (\bar{\varepsilon} + \bar{\varepsilon}^*)\bar{\gamma} - (\bar{\gamma} + \bar{\gamma}^*)\bar{\varepsilon} + \bar{\tau}\bar{\pi} - \bar{\nu}\bar{\kappa} + \Psi_2 - \Lambda + \Phi_{11}\,, \tag{4.19}$$

where $\Lambda = \Psi_{11} = 0$ are components of the Ricci tensor, $D = l^a \partial_a$ and $\overline{D} = n^a \partial_a$. One finds that

$$\Psi_2 = -\frac{M}{(\sigma^*)^3}\,. \tag{4.20}$$

This completes the description of the Kerr black hole in the NP formalism. Let us now discuss the ASG of 4d asymptotic flat spaces.

4.2 BMS group

As discussed in subsection 2.1.1, conformal infinity in asymptotically flat spacetimes is separated in two null hypersurfaces, one in the future and other in the past. The Boyer-Lindquist coordinates in which we described the Kerr black hole earlier is not suitable for future null infinity $\mathcal{I}^+$ or past null infinity $\mathcal{I}^-$. For each case we will need appropriate coordinate systems which are regular at the respective null infinity. Those are ingoing and outgoing Eddington-Finkelstein-type coordinates [34].

For the outgoing coordinate system, we need a retarded time $u = t - r_*$, where the tortoise coordinate

$$r_* = \int \frac{r^2 + a^2}{\Delta} dr = r + \frac{2M}{r_+ - r_-} \left(r_+ \log \left(\frac{r}{r_+} - 1 \right) - r_- \log \left(\frac{r}{r_-} - 1 \right) \right) , \tag{4.21}$$

and a comoving angular coordinate $\phi_u = \phi - \hat{r}$, with

$$\hat{r} = \int \frac{a}{\Delta} dr = \frac{a}{r_+ - r_-} \log \left(\frac{r - r_+}{r - r_-} \right) . \tag{4.22}$$

Notice the similarities with the construction made in section 3.4 for the BTZ black hole, when we considered coordinates that are regular at the future event horizon. It turns out that the coordinates we will find here are also well defined in the future or past event horizon of the Kerr black hole. The metric in outgoing Eddington-Finkelstein coordinates (u, r, θ, ϕ_u) is

$$ds^2 = -2dr(du - a \sin^2\theta d\phi_u) - \frac{\Delta}{\Sigma}(du - a\sin^2\theta d\phi_u)^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2 + a^2)d\phi_u - a du\right)^2 + \Sigma d\theta^2 . \tag{4.23}$$

In the large r limit, the unphysical metric $d\hat{s}^2 = \Omega^2 ds^2$ with $\Omega = 1/r$ is given by

$$d\hat{s}^2 = 2d\Omega(du - a\sin^2\theta d\phi_u) + d\theta^2 + \sin^2\theta d\phi_u^2 + \mathcal{O}\left(\Omega^2\right) , \tag{4.24}$$

with inverse metric

$$\partial\hat{s}^2 = 2\partial_u\partial_\Omega + a\partial_u\left(a\sin^2\theta\partial_u + 2\partial_{\phi_u}\right) + \partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_{\phi_u}^2 + \mathcal{O}\left(\Omega^2\right) . \tag{4.25}$$

In this coordinate system, future null infinity is located at $\Omega = 0$ and the null vector parallel to $\mathcal{I}^+$ is $\hat{n}^a = \partial_u^a$, according to the notation used in subsection 2.1.4. The base space of future null infinity is the sphere S^2 parametrized by the coordinates (θ, ϕ_u).

The ASG at $\mathcal{I}^+$ is known as the BMS group, the semidirect sum of supertranslations and superrotations. The former corresponds to angle dependent null translations and the latter are isomorphic to the conformal group of the base space. Supertranslations are generated by vectors of the form $\alpha(\theta, \phi_u)\partial_u^a$. The $l = 0$ and $l = 1$ modes of the decomposition in spherical harmonics are the time and spatial translations, respectively. The conformal group of the sphere is best visualized if we change to stereographic coordinates $(\zeta_u, \bar{\zeta}_u)$, where

$$\zeta_u = \mathrm{e}^{i\phi_u} \cot\frac{\theta}{2}\,, \quad \bar{\zeta}_u = (\zeta_u)^*\,. \tag{4.26}$$

The metric of the sphere becomes

$$d\sigma^2 = \frac{d\zeta_u d\bar{\zeta}_u}{4(1+\zeta_u\bar{\zeta}_u)^2}\,, \tag{4.27}$$

which is conformally related to the metric of the plane. Thus, just as in spatial infinity in AdS_3, conformal transformations are simply $\zeta_u \to \zeta_u'(\zeta_u)$. The infinitesimal transformation has an infinite number of generators given by

$$l_n = \zeta_u^n\,\partial_{\zeta_u}\,, \quad \bar{l}_n = \bar{\zeta}_u^n\,\partial_{\bar{\zeta}_u}\,. \tag{4.28}$$

Of those, only the $n = 0, 1, 2$ are globally defined on the sphere and these 6 degrees of freedom correspond to the Lorentz group SO(3,1). Thus Poincaré is a subgroup of BMS. The other modes induce local conformal transformations on the sphere. The conformal factor that arises after the holomorphic conformal transformation above is

$$\alpha = \left(\frac{n}{2} - \frac{\zeta_u\bar{\zeta}_u}{1+\zeta_u\bar{\zeta}_u}\right)\zeta_u^{n-1}\,. \tag{4.29}$$

Near $\mathcal{I}^+$, the generators of superrotations are thus

$$\xi_n = \left(\frac{n}{2} + \frac{\zeta_u\bar{\zeta}_u}{1+\zeta_u\bar{\zeta}_u}\right)\zeta_u^{n-1}\left(\Omega\partial_\Omega + u\partial_u\right) + \zeta_u^n\partial_{\zeta_u} \tag{4.30}$$

and their anti-holomorphic counterparts.

The description at $\mathcal{I}^-$ is analogous. The ingoing Eddington-Finkelstein coordinates are (v, r, θ, ϕ_v), where $v = t + r_*$ is the advanced time and $\phi_v = \phi + \hat{r}$ is the comoving angular coordinate. The metric is obtained from (4.23) by simply making $u \to v$ and $dr \to -dr$. Past null infinity is again located at $\Omega = 0$, its null vector is $\hat{n}^a = \partial_v^a$ and the base space is the sphere, parametrized by (θ, ϕ_v). The ASG is another copy of the BMS group; we have supertranslations generated by $\alpha'(\theta, \phi_v)\partial_v^a$ and superrotations as in equation (4.30), but with u replaced by v.

4.3 Large Diffeomorphisms

Let us now solve the equations for the large diffeos – the traceless transverse gauge preserving residual transformations (2.69),

$$\nabla^2 \xi^a = 0\,, \quad \nabla_a \xi^a = 0\,, \tag{4.31}$$

that asymptote to the BMS group generators. In a Ricci-flat background, this problem can be cast into the solutions of the vector potential of a spin-1 perturbation of the Kerr metric. These satisfy the Maxwell's equations

$$\nabla^a F_{ab} = 0\,, \quad \nabla_{[a} F_{bc]} = 0\,. \tag{4.32}$$

The electromagnetic field tensor is written as the exterior derivative of the vector potential, $F_{ab} = \nabla_a A_b - \nabla_b A_a$, where it is defined up to U(1) gauge transformations $A_a \sim A_a + \nabla_a \chi$. Equations (4.32) generate solutions for (4.31) if we choose the Lorenz gauge $\nabla_a A^a = 0$. More explicitly, just note that

$$\nabla^a F_{ab} = \nabla^2 A_b - \nabla^a \nabla_b A_a = \nabla^2 A_b - \nabla_b(\nabla^a A_a) - R_b{}^c A_c\,, \tag{4.33}$$

reducing to the vector Laplacian in a Ricci-flat background if Lorenz gauge is imposed.

There is a simpler approach, which is to look for scalar modes. Suppose $\xi_a = \nabla_a \Phi$. The divergence condition implies that Φ satisfies Laplace's equation

$$\nabla^2 \Phi = 0\,. \tag{4.34}$$

It turns out that this is the only constraint on the scalar field, for

$$\nabla^2 \nabla_a \Phi = \nabla_a \nabla^2 \Phi - R_a{}^b \nabla_b \Phi \tag{4.35}$$

is automatically zero. From the point of view of spin-1 perturbation, these scalar modes are the trivial solutions with vanishing electromagnetic fields. In what follows, we will show that they can only generate the spatial translations of the BMS group and then we will explore the boundary conditions in the general case of the vector modes.

4.3.1 Scalar Modes

We need to impose appropriate boundary conditions on the scalar field Φ in order for the vector field $\xi^a = g^{ab}\nabla_b\Phi$ approach a generator of the BMS group. Since the inverse metric is of order Ω^2 at null infinity, the scalar field must be of order Ω^{-1} for ξ^a be finite and, in fact, the only order zero component will be in the null direction. Thus, in principle, only supertranslations can be generated by scalar modes. The boundary condition on Φ is

$$\Phi = -\Omega^{-1}\alpha(\theta,\phi) + \mathcal{O}(1) \ , \tag{4.36}$$

meaning that $\xi^a = \alpha(\theta,\phi)\partial_u^a + \mathcal{O}(\Omega)$. Note that translations in flat space can be expressed as gradients $(\zeta^\mu)_a = \nabla_a(x^\mu)$, while rotations cannot. Hence it seems reasonable that only supertranslations can be described by a scalar mode.

In the tetrad formalism, the Laplacian can be written as

$$\nabla^2\Phi = \eta^{\mu\nu}e^a_\mu e^b_\nu\nabla_a\nabla_b\Phi = \eta^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - \eta^{\mu\nu}\eta^{\rho\sigma}\lambda_{\mu\rho\nu}\nabla_\sigma\Phi \,, \tag{4.37}$$

where $\nabla_\mu = e^a_\mu\nabla_a = (D,\overline{D},\delta,\bar{\delta})$ are the directional derivatives along the tetrad vectors. The Ricci rotation coefficients can be read off the spin coefficients (4.17) by noting that complex conjugation is equivalent to interchange the indices 3 and 4, since $\overline{m} = (m)^*$. The result is

$$\begin{aligned}\nabla^2\Phi = -D\overline{D}\Phi - \overline{D}D\Phi + \delta\bar{\delta}\Phi + \bar{\delta}\delta\Phi \\ + (\bar{\gamma} + \bar{\gamma}^* - \bar{\mu} - \bar{\mu}^*)D\Phi + (\bar{\rho} + \bar{\rho}^* - \bar{\epsilon} - \bar{\epsilon}^*)\overline{D}\Phi \\ - (\bar{\tau}^* - \bar{\pi} + \bar{\alpha} - \bar{\beta}^*)\delta\Phi - (\bar{\tau} - \bar{\pi}^* + \bar{\alpha}^* - \bar{\beta})\bar{\delta}\Phi \,.\end{aligned} \tag{4.38}$$

Using the expressions (4.15) for the directional derivatives and (4.17) for the spin coefficients, we may write the Laplacian operator in terms of the Boyer-Lindquist coordinates:

$$\nabla^2\Phi = \frac{1}{\Sigma}\left[\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{1}{\Delta}\left((r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\phi}\right)^2 + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\phi}\right)^2\right] . \quad (4.39)$$

This operator is separable and the equation can be split in two confluent Heun equations for r and θ. Expanding the solution in terms of the eigenvalues of ∂_t and ∂_ϕ – the frequency ω and angular momentum m – and an angular quantum number l, the general expression for the scalar field is

$$\Phi_{\omega,l,m}(t,r,\theta,\phi) = \mathrm{e}^{-i\omega t}\left(C_+h^+_{\omega,l,m}(r) + C_-h^-_{\omega,l,m}(r)\right){}_0S_{lm}(\cos\theta)\mathrm{e}^{im\phi} , \quad (4.40)$$

where $h^\pm$ are confluent Heun functions [35] and ${}_0S_{lm}$ are the scalar spheroidal harmonics [36]. The radial function have the asymptotic behavior (see [37])

$$h^\pm_{\omega,l,m} = \frac{\mathrm{e}^{\pm i\omega r_*}}{r}\left(1 + \mathcal{O}\left(r^{-1}\right)\right) , \quad (4.41)$$

where r_* is the tortoise coordinate (4.21). Taking into account the time dependence, we see that the solutions have either an explicit dependence on u at $\mathcal{I}^+$ or on v at $\mathcal{I}^-$, thus violating our boundary condition (4.36).

The only possibility is to look at solutions with zero frequency. In this case, the angular part of the Laplacian corresponds to the Laplacian on the sphere whose solutions are the spherical harmonics Y_l^m with eigenvalue $-l(l+1)$. The radial part of the scalar field $\Phi_{l,m}(r,\theta,\phi) = R_{l,m}(r)Y_l^m(\theta,\phi)$ now satisfies the differential equation

$$\left[\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} + \left(\frac{a^2m^2}{\Delta} - l(l+1)\right)\right]R_{l,m} = 0 . \quad (4.42)$$

The large r behavior is now either r^l or r^{-l-1}, and hence only the $l = 1$ modes satisfy our boundary condition. The equation above has three regular singular points at r_+, r_- and infinity and the general solution is a hypergeometric function. The critical exponents at both horizons are

$$\pm i\frac{am}{r_+ - r_-} .$$

The solution that diverges as r^l at infinity is

$$R_{l,m}(r) = -r^l \left(\frac{r-r_+}{r-r_-}\right)^{-\frac{iam}{r_+-r_-}} {}_2F_1\left(-\frac{2iam}{r_+-r_-} - l, -l; -2l; \frac{r_+-r_-}{r-r_-}\right) . \tag{4.43}$$

Using (4.22), we can write the scalar field in terms of the comoving angular coordinate $\phi_u = \phi - \hat{r}$ as

$$\Phi_{l,m}(r,\theta,\phi_u) = -{}_2F_1\left(-\frac{2iam}{r_+-r_-} - l, -l; -2l; \frac{r_+-r_-}{r-r_-}\right) r^l Y_l^m(\theta,\phi_u) . \tag{4.44}$$

Note that we could write the solution above making explicit the other critical exponent at the horizon; this would lead to the expression

$$\Phi_{l,m}(r,\theta,\phi_v) = -{}_2F_1\left(+\frac{2iam}{r_+-r_-} - l, -l; -2l; \frac{r_+-r_-}{r-r_-}\right) r^l Y_l^m(\theta,\phi_v) , \tag{4.45}$$

written in terms of the ingoing comoving angular coordinate $\phi_v = \phi + \hat{r}$. Thus the analysis above is equally applicable to future and past null infinities; the scalar modes induce changes in both.

4.3.2 Vector modes

As argued in the beginning of this section, solutions of the Maxwell's equations in Ricci-flat spaces for the vector potential in the Lorenz gauge yield diffeomorphisms that preserve the traceless-transverse gauge of metric perturbations. There remains the problem of finding solutions that satisfy the boundary condition necessary for matching a generator of the ASG. In this subsection we shall investigate the existence of such boundary conditions.

Let us start by writing Maxwell's equation in the tetrad formalism. The components of the field strength tensor are

$$F_{\mu\nu} = e^a_\mu e^b_\nu F_{ab} = \nabla_\mu A_\nu - \nabla_\nu A_\mu - \eta^{\rho\sigma} A_\rho (\lambda_{\mu\sigma\nu} - \lambda_{\nu\rho\mu}) . \tag{4.46}$$

There are 6 independent components in four dimensions, which are organized in 3 complex components in the NP formalism; namely

$$\phi_0 = F_{13} , \quad \phi_1 = \frac{1}{2}(F_{12} + F_{34}) , \quad \phi_2 = F_{42} . \tag{4.47}$$

In terms of the vector potential, these components are

$$
\begin{aligned}
\phi_0 &= (D - \overline{\rho}^* + \overline{\epsilon}^* - \overline{\epsilon})A_3 - (\delta + \overline{\pi}^* - \overline{\alpha}^* - \overline{\beta})A_1 + \overline{\kappa}A_2 - \overline{\sigma}A_4\,, \\
\phi_1 &= \frac{1}{2}\left[(D - \overline{\rho}^* + \overline{\epsilon}^* + \overline{\epsilon})A_2 - (\delta + \overline{\pi}^* - \overline{\alpha}^* + \overline{\beta})A_4 + \overline{\mu}A_1 - \overline{\pi}A_3\right] \\
&\quad + \frac{1}{2}\left[(\overline{\delta} + \overline{\beta}^* - \overline{\tau}^* - \overline{\alpha})A_3 - (\overline{D} + \overline{\mu}^* - \overline{\gamma}^* - \overline{\gamma})A_1 + \overline{\rho}A_2 - \overline{\tau}A_4\right]\,, \\
\phi_2 &= (\overline{\delta} + \overline{\beta}^* - \overline{\tau}^* + \overline{\alpha})A_2 - (\overline{D} + \overline{\mu}^* - \overline{\gamma}^* + \overline{\gamma})A_4 + \overline{\nu}A_1 - \overline{\lambda}A_3\,.
\end{aligned}
\tag{4.48}
$$

The divergence of the potential reads

$$
\begin{aligned}
\nabla_a A^a &= e^a_\mu e^b_\nu \nabla_a A_b = \eta^{\mu\nu}\nabla_\mu A_\nu - \eta^{\mu\nu}\eta^{\rho\sigma}A_\rho\lambda_{\mu\sigma\nu} \\
&= -\left[(D - \overline{\rho}^* + \overline{\epsilon}^* + \overline{\epsilon})A_2 - (\delta + \overline{\pi}^* - \overline{\alpha}^* + \overline{\beta})A_4 + \overline{\mu}A_1 - \overline{\pi}A_3\right] \\
&\quad + \left[(\overline{\delta} + \overline{\beta}^* - \overline{\tau}^* - \overline{\alpha})A_3 - (\overline{D} + \overline{\mu}^* - \overline{\gamma}^* - \overline{\gamma})A_1 + \overline{\rho}A_2 - \overline{\tau}A_4\right]\,.
\end{aligned}
\tag{4.49}
$$

Hence Lorenz gauge implies that both terms in square brackets in the expression for ϕ_1 in (4.48) are equal. Finally, the 8 Maxwell's equations are written in the form of 4 complex equations [30]:

$$
\begin{aligned}
(\overline{\delta} - 2\overline{\alpha} + \overline{\pi})\phi_0 &= (D - 2\overline{\rho})\phi_1 + \overline{\kappa}\phi_2\,, \\
(D + 2\overline{\epsilon} - \overline{\rho})\phi_2 &= (\overline{\delta} + \overline{\pi})\phi_1 - \overline{\lambda}\phi_0\,, \\
(\overline{D} - 2\overline{\gamma} + \overline{\mu})\phi_0 &= (\delta - 2\overline{\tau})\phi_1 + \overline{\sigma}\phi_2\,, \\
(\delta + 2\overline{\beta} - \overline{\tau})\phi_2 &= (\overline{D} + 2\overline{\mu})\phi_1 - \nu\phi_0\,.
\end{aligned}
\tag{4.50}
$$

Let us consider the ASG at $\mathcal{I}^+$. In order to find out the boundary conditions the coefficients A_μ of the vector potential must satisfy, let us first write the tetrad in term of outgoing coordinates (u, r, θ, ϕ_u):

$$
\begin{aligned}
l &= \partial_r\,, \\
n &= -\frac{\Delta}{2\Sigma}\partial_r + \frac{r^2 + a^2}{\Sigma}\partial_u + \frac{a}{\Sigma}\partial_{\phi_u}\,, \\
m &= \frac{1}{\sigma\sqrt{2}}\left(ia\sin\theta\,\partial_u + \partial_\theta + \frac{i}{\sin\theta}\partial_{\phi_u}\right)\,, \\
\overline{m} &= (m)^*\,.
\end{aligned}
\tag{4.51}
$$

The expression for the vector $A^a = \eta^{\mu\nu} A_\mu e^a_\nu$ in this coordinate system is thus

$$A^a = \left[-A_1 + a\sin^2\theta \left(-A_1 \frac{a}{\Sigma} + \frac{i}{\sqrt{2}\sin\theta} \left(\frac{A_4}{\sigma} - \frac{A_3}{\sigma^*} \right) \right) \right] \partial^a_u + \left(A_1 \frac{\Delta}{2\Sigma} - A_2 \right) \partial^a_r + \frac{1}{\sqrt{2}} \left(\frac{A_4}{\sigma} + \frac{A_3}{\sigma^*} \right) \partial^a_\theta + \left(-A_1 \frac{a}{\Sigma} + \frac{i}{\sqrt{2}\sin\theta} \left(\frac{A_4}{\sigma} - \frac{A_3}{\sigma^*} \right) \right) \partial^a_{\phi_u} \quad (4.52)$$

and the Lorenz gauge condition is

$$0 = \frac{\partial}{\partial u} \left[-A_1 + a\sin^2\theta \left(-A_1 \frac{a}{\Sigma} + \frac{i}{\sqrt{2}\sin\theta} \left(\frac{A_4}{\sigma} - \frac{A_3}{\sigma^*} \right) \right) \right] + \frac{1}{\Sigma} \frac{\partial}{\partial r} \left[\Sigma \left(A_1 \frac{\Delta}{2\Sigma} - A_2 \right) \right] + \frac{1}{\Sigma\sin\theta} \frac{\partial}{\partial\theta} \left[\frac{\Sigma\sin\theta}{\sqrt{2}} \left(\frac{A_4}{\sigma} + \frac{A_3}{\sigma^*} \right) \right] + \frac{\partial}{\partial\phi} \left(-A_1 \frac{a}{\Sigma} + \frac{i}{\sqrt{2}\sin\theta} \left(\frac{A_4}{\sigma} - \frac{A_3}{\sigma^*} \right) \right) . \quad (4.53)$$

We will use a slightly different boundary condition for supertranslations, where we fix some subleading order terms, cf. [38]:

$$\xi = \alpha\,\partial_u + \frac{1}{2}\Box\alpha\,\partial_r - \frac{1}{r}\frac{\partial\alpha}{\partial\theta}\partial_\theta - \frac{1}{r\sin\theta}\frac{\partial\alpha}{\partial\phi_u}\partial_{\phi_u} + \dots \quad (4.54)$$

where $\Box$ is the Laplacian on the sphere. See also [17]. The tetrad components of the vector potential satisfying the asymptotic condition above, in the Lorenz gauge, are

$$\begin{aligned} A_1 &= -\alpha P - \frac{a\sin\theta}{2\Sigma}\frac{\partial\alpha}{\partial\phi}\frac{\partial F}{\partial r}\,, \\ A_2 &= -\alpha\frac{\Delta P}{2\Sigma} - \frac{a\Delta\sin\theta}{4\Sigma^2}\frac{\partial\alpha}{\partial\phi}\frac{\partial F}{\partial r} - \frac{F}{2\Sigma}\Box\alpha\,, \\ A_3 &= -\frac{1}{\sigma\sqrt{2}}\left(\frac{1}{2}\frac{\partial\alpha}{\partial\theta}\frac{\partial F}{\partial r} + ia\sin\theta\alpha P + i\frac{r^2+a^2}{2\Sigma}\frac{\partial\alpha}{\partial\phi}\frac{\partial F}{\partial r} \right)\,, \\ A_4 &= (A_3)^*\,. \end{aligned} \quad (4.55)$$

Here $\alpha = \alpha(\theta, \phi_u)$, $P = P(r, \theta)$ and $F = F_r(r) + F_\theta(\theta)$. Near infinity, these functions must behave as $P = 1 + \mathcal{O}(r^-1)$ and $F = r^2 + \mathcal{O}(r)$. In the simplest case, when $\alpha = P = 1$, the vector potential corresponds to the time translation ∂^a_u and the field strength tensor components are

$$\phi_0 = 0\,, \quad \phi_1 = -\frac{M}{(\sigma^*)^2}\,, \quad \phi_2 = 0\,. \quad (4.56)$$

One may recognize this as the electromagnetic field due to an electric charge M in Kerr [39]. This is just the conserved charge associated with time translation in the Kerr black hole, cf. (2.70). The case for a general supertranslation is not as direct and we shall save the analysis for future work.

The boundary conditions for a superrotation at future null infinity is given in (4.30); in outgoing coordinates, it reads

$$\xi_n = \zeta_u^{n-1}\left[\frac{1}{2}(n-1-\cos\theta)\left(u\partial_u - r\partial_r\right) - \frac{\sin\theta}{2}\partial_\theta - \frac{i}{2}\partial_{\phi_u}\right] + \ldots \tag{4.57}$$

Writing a set of expressions for the tetrad components of the vector potential satisfying this boundary condition in the Lorenz gauge is not a trivial task for a superrotation. We postpone the general analysis for future work as well, but give a description of the simplest of these symmetries as follows. From (4.57), we can see that azimuthal rotation is generated by the combination $i(\xi_1 - \bar{\xi}_1)$, which is an isometry and automatically satisfy the equations for a large diffeo. The field strengths are

$$\phi_0 = i\sqrt{2}\sin\theta\,, \quad \phi_1 = \frac{aM\sin^2\theta}{(\sigma^*)^2} + \frac{a + ir\cos\theta}{\sigma^*}\,, \quad \phi_2 = -i\frac{\Delta\sin\theta}{(\sigma^*)^2\sqrt{2}}\,. \tag{4.58}$$

The charge associated with this isometry is less direct, however an explicit calculation of (2.70) can be done where one needs to integrate the F^{ur} component of the electromagnetic tensor on a sphere at infinity. This component can be expressed in terms of the ϕ_i above, and the charge obtained is $J = aM$, the angular momentum of the Kerr black hole.

The same analysis can be done with ingoing coordinates to study the large diffeos acting at past null infinity. It should be noted that the time translation is now generated ∂_v and azimuthal rotation by ∂_{ϕ_v}. The charges associated with such symmetries are the same.

4.4 Near horizon

The general idea that one can map near horizon diffeomorphism to those acting non trivially at infinity can also be applied to the Kerr black hole. We did not find yet the most general solutions of the large diffeos in this case nor the behavior of the vector modes near horizon. In what follows we shall only find what diffeos at the event horizon are mapped to the scalar modes defining translations at infinity.

As discussed in the end of subsection 4.3.1, the two solutions obtained with ingoing and outgoing coordinates are equal and, in fact, act in both null infinities. From a near horizon perspective, this is even more clear; the critical exponents of the radial part of the scalar modes,

$$\pm i\frac{am}{r_+ - r_-}\,,$$

imply the following behavior for the translation modes:

$$\Phi_{1,m} = \left(C_+ Y_1^m(\theta,\phi_v) + C_- Y_1^m(\theta,\phi_u)\right) + \mathcal{O}\left(r - r_+\right)\,. \tag{4.59}$$

Here $C_\pm$ are the connection coefficients of the hypergeometric equation. In the case $m = 0$, corresponding to a z-axis translation, both terms contribute equally in the future and past null event horizons, since there is no azimuthal dependence. However, for the other translations, generated by the $m = \pm 1$ modes, only one term will contribute in each case. Near the future horizon, the outgoing coordinate ϕ_u winds indefinitely and thus the C_- term averages to zero. Similar reasoning implies that only the C_- term acts at the past event horizon. More general diffeos besides the above, time translations and azimuthal rotations, are expected to be included in the vector modes of the previous section.

5 Conclusion

In this work we proposed to use the gauge fixing procedure for perturbations in classical general relativity to associate asymptotic symmetries to bulk diffeomorphism. The idea was to relax the boundary conditions of the residual diffeomorphisms, that left the trace and divergence of metric perturbations invariant, to include the possibility of having a non trivial action at infinity, i.e., the diffeomorphism should act as an asymptotic symmetry generator at infinity.

The large diffeomorphisms, defined this way, act on the entire space and, in the presence of a black hole, associates an asymptotic symmetry to a near-horizon diffeomorphism. In the three dimensional BTZ black hole, the complete integrability of Einstein's equations allowed us to find exact solutions for the large diffeomorphisms that shared some characteristics with the asymptotic symmetries at spatial infinity. In the four dimensional case, the situation was not as simple; special solutions were found that map spatial translations at null infinity to the bulk, however a general solution for the supertranslations and superrotations was not found. Instead, we presented partial calculation of the more general setting which may give rise to large diffeomorphisms related to the rest of the BMS group.

Equipped with these solutions, one may work in the opposite way; an horizon isometry can be written as a specific combination of large diffeos. This will allow the computation of charges associated with black hole isometries as combinations of the ASG charges on phase space. Note that the association is not unique; we chose the traceless-transverse gauge in the definition of the large diffeos and different choices shall result in different descriptions. However, we expect the physics to be the same, i.e., the algebra of charges should be invariant. This non-uniqueness on the canonical generators of transformations in the phase space is a key piece in the appearance of central terms in the algebra of charges in Classical Mechanics [40]. An example, in the context of gravitation, was given by Brown and Henneaux in three dimensions [7]. A positive result for the appearance of central terms could be enable us to derive the Bekenstein-Hawking entropy in four dimensions.

This is not a finished work and much needs to be done. In Kerr, we intend to finish the calculations for the large diffeos. A closed solution valid throughout the bulk is not necessary, only the asymptotic behavior near horizon matching the asymptotic behavior at infinity, much like a scattering problem. We shall then investigate the charges associated with the event horizon isometries in this context. We first work with the BTZ black hole trying to reproduce the Bekenstein-Hawking entropy, a task already done with the charges of the asymptotic isometries at spatial infinity. We will then apply to the four dimensional case and look for possible hints, such as the existence of central extensions, on the algebra of observables that may aid us in the task of counting the black hole entropy.

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