UNIVERSIDADE FEDERAL DE PERNAMBUCO
DEPARTAMENTO DE FÍSICA – CCEN
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA

TESE DE DOUTORADO

BLACK HOLE SCATTERING, ISOMONODROMY AND HIDDEN SYMMETRIES

FÁBIO MAGALHÃES DE NOVAES SANTOS

Recife
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FÁBIO MAGALHÃES DE NOVAES SANTOS

Tese apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutor em Física.

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To the future that I long and to the past that I miss,
I hope we can keep living on the present.

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Resumo

Espalhamento de campos ao redor de buracos negros é uma importante problema tanto na área de astrofísica, na detecção de ondas gravitacionais, como em aplicações teóricas, por exemplo, em AdS/CFT e gravidade quântica. Nesta tese, estudamos aspectos teóricos do espalhamento de campos em torno de buracos negros Kerr-NUT-(A)dS em 4 dimensões (buraco negro girante, com carga topológica tipo NUT e imerso em um espaço-tempo de curvatura escalar constante). Após separação da equação de Klein-Gordon, reduzimos o problema a um espalhamento unidimensional na variável radial. Em particular, estudamos o espalhamento de um campo escalar conformemente acoplado ao espaço-tempo ($\xi = \frac{1}{6}$), pois nesse caso o número de pontos singulares reduz de 5 para 4. A equação resultante é Fuchsiana do tipo Heun, equação mais geral do que a hipergeométrica, com 4 pontos singulares regulares, e suas soluções não existem em termos de funções elementares. A maior parte dos estudos nessa área de espalhamento são aproximados ou puramente numéricos. Encontramos, então, uma expressão analítica para os coeficientes de espalhamento em termos da monodromias das soluções. Estes coeficientes dependem de traços de monodromias compostas. Usamos a teoria de deformações isomonodrômicas para encontrar esses coeficientes através das soluções assintóticas da equação de Painlevé VI. Em particular, estudamos o espalhamento no caso Kerr-dS em detalhe. Além disso, discutimos certos resultados interessantes no contexto da descrição dual dos estados de um buraco negro em termos de uma teoria de campos conforme, a chamada dualidade Kerr/CFT. Modos conformemente acoplados em Kerr-AdS extremal sugerem uma descrição em termos de uma teoria de campo conforme para a frequência no limite superradiante, além da região próxima do horizonte.

Palavras-chave: Espalhamento de Campos. Buracos Negros. Painlevé VI. Deformações Isomonodrômicas. Simetrias Escondidas. Kerr-NUT-(A)dS.

Abstract

Scattering of fields around black holes is an important problem in astrophysics - in the detection of gravitationals waves - as well in purely theoretical applications, for example, in AdS/CFT and quantum gravity. In this thesis, we study the scattering of scalar fields around Kerr-NUT-(A)dS black holes in 4 dimensions (a rotating black hole, with topological NUT charge and embedded in a spacetime with constant scalar curvature). After separation of variables in the Klein-Gordon equation, we simplify the problem to a 1-dimensional scattering by an effective potential in the radial direction. In particular, we study conformally coupled modes in this background ($\xi = \frac{1}{6}$) because, in this case, the number of singular points decreases from 5 to 4. The resulting Fuchsian equation belong to the Heun class, more general than the hypergeometric equation, and it is well-known that their solutions are not generically expressible in terms of elementary functions. Most studies in scattering theory are approximate or purely numerical. Here we find an analytic expression for the scattering coefficients in terms of the monodromies of the solutions. We show how it is possible to find the scattering coefficients using monodromies of the differential equation without having the explicit solution. These coefficients depend on composite traces of monodromies. Thus, we use the theory of isomonodromic deformations to find these coefficients via Painlevé VI asymptotics. In particular, we study Kerr-dS scattering in some detail. Furthermore, we discuss certain interesting results for the dual conformal field theory description of black hole states, the so-called Kerr/CFT duality. Conformally coupled modes of extremal Kerr-AdS black hole suggest a CFT description for frequency at the superradiant bound, beyond the near-horizon region.

Keywords: Black Holes. Scattering Theory. Isomonodromic Deformations. Painlevé VI. Hidden Symmetries. Kerr-NUT-(A)dS.

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CHAPTER 1

Introduction

"You cannot pass! I am a servant of the Secret Fire, wielder of the Flame of Anor. The dark fire will not avail you, Flame of Udun! Go back to the shadow. You shall not pass!"

—GANDALF

Black holes are compact, localized, gravitational systems which have several interesting physical properties [1, 2]. Their defining property is the existence of a definite region where no causal physical excitation can escape. This special region is called the *event horizon*. From the point of view of an external observer, a black hole is very much like a star as its gravitational field pulls every light and matter around it towards its center. However, because of the event horizon, no light can classically escape the inner region so that is why we call this compact object black. As both theory and experiment overwhelmingly suggest that it is not possible for two physical systems interact through faster than light signals, the region beyond the event horizon is believed to be completely inaccesible to an external observer.

Of course we can try to safely study black holes from some distance, for example, by throwing a light ray towards it and seeing how much light is reflected by its gravitational potential. The results obtained are neatly described by the generalization of classical scattering theory to curved spacetimes [3, 4]. This classical picture is what one usually has in mind when searching in the sky for astrophysical objects which ought to be described by a black hole. Direct detection of these elusive objects has not been achieved yet because actual black holes are very “dirty” objects. However, astronomers can infer the existence of black holes via indirect ways, like gravitational waves or electromagnetic signals produced by these objects. Typical candidates for black hole detection are binary systems, formed by a black hole and a companion infalling star, and black holes in the center of galaxies. Binary systems present rapid accretion of mass of the inbound star, producing powerful X-ray signals like in Cygnus X-1, a binary which is widely believed to have a black hole with around 10 $M_\odot$[1]. In the center of galaxies we expect to encounter supermassive black holes with $10^5 - 10^{14} M_\odot$ and may also be detected through indirect ways. Finally, there has always been speculation about the existence of primordial black holes permeating the universe since very early times, being formed by evolution of inhomogeneities after the big bang. There is even a suggestion that those could explain the abundance of dark matter in the universe. However, there are several constraints on their existence today, like black hole evaporation, for those very small, and gravitational lensing, for those very large, so even if they exist, their detection would be extremely hard [5].

Another possibility to probe the structure of a black hole is to throw into it a measuring apparatus which sends periodic signals to an outside observer. The problem with this approach is that the signals will take longer and longer to reach the external observer because of the gravitational redshift as the apparatus approaches the event horizon. For the external observer, the device will take an infinite amount of time to reach the event horizon, despite the finite amount of proper time necessary for the infalling device to reach and cross the horizon.

The fate of an infalling observer following a geodesic into a typical black hole is a more delicate matter. The classical answer is that nothing unusual will happen because there is no curvature singularity at the event horizon. Of course, depending on the size of the black hole, the dragging force might be too large and destroy completely the infalling device or astronaut, but the standard point of view is that there is no classical experiment one may do to discover if it has crossed the event horizon. However, there has been some dispute about this classical argument which is typically referred to as the *firewall paradox*. Almheiri *et al* [6] argue that if we consider two entangled states, one falling into and another going away from a black hole, to preserve unitary evolution of these states, as demanded by quantum mechanics, the infalling state will suffer a "burning" fate while trying to cross the event horizon corresponding to, so much as one says that it will have to cross a "firewall" to reach the black hole interior. Burning here means that the infalling observer will detect modes with arbitrarily high frequencies while crossing the horizon.

The considerations above suggest that, although we have a good classical understanding of these gravitational systems, we have a hard time understanding its quantum structure. However, there has been a lot of progress during the last 50 years about the quantum nature of a black hole. The first thing one can do is study semiclassical processes where we have a classical black hole metric coupled to a quantum field [7]. Several things can be learnt with this approach, one of

[1] $M_\odot = 1.98 \times 10^{30}$ kg is the solar mass.

the most important ones is that black holes spontaneously emit thermal radiation - the Hawking radiation. In particular, black holes can lose energy in this process and therefore evaporate, shrinking until there is nothing left. At least, that is what one can conclude from a first view. However, as the black hole shrinks in size, quantum gravity effects become important in the approach of the singularity, so the safe answer is that we do not know what is the final fate of these objects. If the black hole completely evaporates, it raises several questions about unitarity and entropy. The fact that these objects emit radiation implies that they are thermodynamical systems and thus possess entropy along with their temperature. We shall comment more about this below.

Finally, there has been very important advances and applications of black holes in recent years. Most notably, we comment about two of the most important ones: microcanonical entropy calculations and gauge/gravity duality. The entropy of a black hole is proportional to the event horizon area, differently from what we expect from conventional thermodynamical systems. This made some authors suggest that the "information" of the black hole interior is holographically encoded on the horizon surface, making contact with the related topic of holographic dualities [8, 9]. Strominger and Vafa made an impressive calculation matching the black hole entropy, counting the number of states of a system of D-branes in string theory with conserved charges equal to those of the black hole [10]. This particular calculation corresponded to the entropy of an extremal black hole in supergravity, but soon followed other examples in less symmetric backgrounds (see, for example, the works of Ashoke Sen and collaborators [11, 12, 13, 14]). Therefore, string theory seems to be the best candidate up to now to describe black holes microstates.

AdS/CFT[2] related dualities have been abundantly[3] studied today since the seminal paper of Maldacena [15] with applications in several areas like studies on the Quark-Gluon Plasma (QGP) [16, 17, 18], holographic superconductors and condensed matter systems (also called AdS/CMT) [19, 20]. There is also suggestions for atomic physics as well [21]. For general reviews see [22, 23, 24, 25]. In its most general form, it states that a D-dimensional conformal gauge theory with a large number of fields at strong coupling (no gravity) should be better described by a gravitational theory at weak coupling in (D+1)-dimensions. So it is usually referred as *weak-strong holographic duality*. It is truly amazing because it relates the non-perturbative sector of a non-gravitational theory with another one with gravitational states at perturbative level! This means that classical gravity can solve several questions about these

[2] Anti-de Sitter / Conformal Field Theory

[3] In INSPIRE HEP database, this paper already has 9946 citations!

dual theories which were very hard to probe before. So it is not a surprise that it draw so much attention from the physics community. Furthermore, if we reverse the duality, we can also try to assess non-perturbative properties of gravity itself. As we are most interested in black holes here, we would like to use the duality in this other direction to try to understand better our objects of study non-perturbatively. For a review on what AdS/CFT can say about black holes, see [26, 27, 28, 29].

The closest this duality has reached to describing a physical system in the lab, as far as we know, is in the case of AdS/QCD and the quark-gluon plasma (QGP). There are models in the literature describing quark confinement and chiral symmetry breaking mechanisms which are important in this context. Also, seen as a macroscopic plasma, transport properties for the QGP have also been calculated. The most well-known calculation is that of the viscosity-to-entropy ratio, which in the simplest holographic models is given by an universal factor $1/4\pi$ [30, 31]. Violations of this bound have been discovered theoretically for models with higher-order curvature corrections and suggestions have been made when this could happen [32].

AdS/CFT has also inspired another formulation of the correspondence with respect to black holes in purely geometrical terms, the Kerr/CFT correspondence [33]. This proposal tries to find the dual CFT theory of a black hole by understanding better the hidden symmetries in black holes solutions in the near-horizon and extremal limits of these metrics. We review this correspondence in details in chapter 4 of this thesis, as it will be important for our applications.

The applications related to QGP and condensed-matter systems typically have a black hole in the center of AdS spacetime and the temperature of the horizon is directly related to thermal states of the dual theory. Another important ingredient for understanding thermalization processes in these systems is the calculation of *quasinormal modes*, which correspond to the poles of the retarded Green's functions in the dual theory. In terms of the black holes, quasinormal modes are certain special resonances in the spectrum in which the transmission amplitude of the mode diverges and the reflection amplitude is zero. Those are also related to questions about stability of the background itself [34, 35, 36, 37]. Therefore, the understanding of black holes and its scattering properties is very important for AdS/CFT applications and probably for a more fundamental understanding of the duality itself.

1.1 Summary of Thesis Chapters

After this bird's eye view on the black hole literature, we now come back to the main matter of this thesis: scattering theory of fields around black holes. The study of scattering of fields around black holes is very important for astrophysical applications, stability criteria of gravitational solutions, AdS/CFT applications as well and to gain insight on the quantum description of a black hole. Therefore, it is interesting to gain more analytical control about the structure of scattering amplitudes in this context.

In this thesis, we present our original work [38] in this direction with respect to the mathematical physics of this problem and the physical implications thereof. Inspired by the recent work of Castro, Maloney, Lapan and Rodriguez [39, 40] on $D = 4$ Kerr black hole scattering from monodromy, we generalize this proposal by applying it to $D = 4$ Kerr-NUT-(A)dS black holes. We discover that this approach is much more natural in this context and, in the particular case of a conformally coupled scalar field, we reduce the main problem to the study of a Heun equation, a Fuchsian equation with 4 regular singular points. This is done in chapter 2. We also discuss (A)dS spheroidal harmonics in this chapter, elucidating the equation structure and presenting partial numerical results for its eigenvalues.

To understand the monodromy group of Heun equation, we delve into full mathematical discourse in chapter 3. There we present the current understanding of monodromy representations, its symplectic and algebraic structure, as well as its connection with recent developments in 2D conformal field theory. Further, we present the main tool of our trade here: the theory of isomonodromic deformations and Painlevé transcendents, which are non-linear equivalents of classical special functions. Here we see how the scattering problem can be seen as integrable, at least in principle. However, numerical work must be done to reach a quantitative answer and we present a numerical result on the integration of Painlevé VI equation.

In chapter 4, we present the main results of this thesis, in particular, the generic structure of scattering amplitudes between two regular singular points with ingoing/outgoing boundary conditions. We further propose that this structure must be valid in any dimension and the same idea of our method can be implemented. We briefly discuss the physical case of scattering in Kerr-dS spacetime, considering superradiance and scattering between different asymptotic regions. We also propose a connection of isomonodromic deformations of our system with the hidden symmetry encountered in the Kerr/CFT correspondence. Finally, we show that the extremal limit of a conformally coupled wave equation in Kerr-(A)dS spacetime gives a hypergeometric equation in two cases: (i) in the superradiant limit and (ii) when the rotation parameter is equal to the AdS radius (only for the AdS case).

After the presentation of all the main material, we discuss and review our results in chapter 5, also proposing new directions and perspectives for our work: generalization to higher dimensions, structure of quasinormal modes and recovery of previous results via confluence.

1.1.1 Summary of Original Results

- Section 2.2.1 of scattering on $(A)dS_2 \times S^2$ spacetimes using monodromies is an original presentation of this subject.

- Section 2.3 presents our work on the setup for the scattering analysis of massless conformally coupled scalar field in Kerr-NUT-(A)dS background. We show that $r = \infty$ is a removable singularity in this case, reducing the radial part of Klein-Gordon equation to a Heun equation. This result is equivalent to the one in [41] for zero spin in the Teukolsky equation, although here we do for arbitrary non-minimal coupling and show that this removability only happens when $\xi = 1/6$. This clarifies the origin of this removability in the solution space. We also present an unpublished analysis on (A)dS scalar spheroidal harmonics and some numerical results for its eigenvalues C_ℓ using Leaver's continued-fraction method [42].

- The numerical results for Painlevé VI integration in section 3.4 are original and still unpublished.

- The whole of chapter 4, except the review on Kerr/CFT correspondence in section 4.2, consists of original results, most of it still unpublished. The structure of generic scattering and scattering through the interior of a black hole has been discussed in [38]. The specifics of Kerr-dS scattering and the relation between Kerr/CFT correspondence is still unpublished.

Now we turn to a brief introduction on the main properties of black holes and its thermodynamics as a warm-up for our thesis.

1.2 Classical Description of Black Holes

An astrophysical black hole spacetime can be nicely illustrated by the conformal diagram of matter collapsing to form a black hole. The grey area in figure 1.1 corresponds to a collapsing star. In the process, an event horizon is formed at radius $r = 2M$, a region of no escape even for light. This causal diagram (also called conformal diagram or Penrose diagram) describes the global and causal structure of the spacetime. It is conformal as it preserves angles and every light ray travels at 45 degrees. The most important thing about those diagrams is that they give a way to draw an infinite spacetime into a finite piece of paper. The whole spacetime fits in a finite picture because we compactify coordinates by making a transformation of the type $x \to \tan^{-1}(x)$. To understand how to draw a Penrose diagram, check appendix A.

The exterior part of Figure 1.1, the region outside the black hole, has a casual structure similar to Minkowski spacetime, as seen in appendix A. The main difference now is that light rays inside the interior region $r < 2M$, beyond the dashed line, never reach i^+. This means that the black hole is outside of causal contact with the exterior universe. Furthermore, the wiggling line corresponds to the black hole *singularity* at $r = 0$.

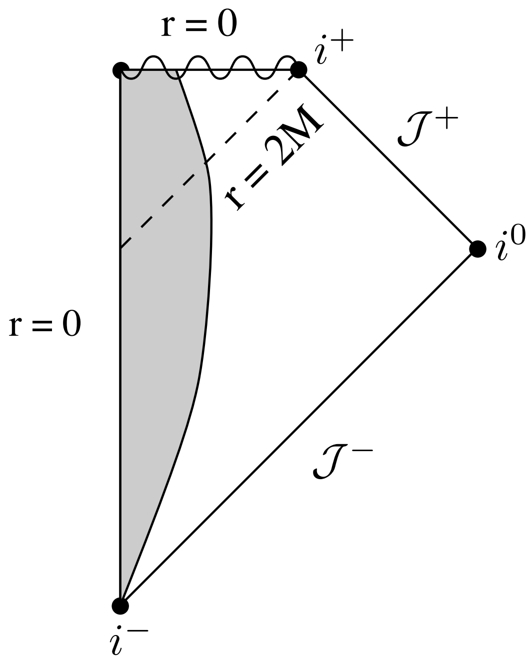

Figure 1.1: Matter collapsing into a black hole. Adapted from [43].

In principle, the event horizon has nothing special locally; the problem really lies at the singularity. What happens there, apart from having a very high spacetime curvature nearby, nobody knows. The resolution of the singularity is one of the biggest problems related to quantum gravity. But we notice that the problem is not with the singularity itself, as electrodynamics have also singularities called point charges. In a nutshell, quantum electrodynamics solves this problem with renormalization, whereas general relativity does not allow this procedure. This is due to the intrinsic nonlinearity of the theory, as gravitons scatters gravitons. But not just that, as quantum chromodynamics is also non-linear although renormalizable. It is also a matter of the strength of its perturbative interactions. The natural scale of quantum gravity is the Planck mass $M_{Pl}^2 = \hbar c/8\pi G \sim 10^{18}$ GeV, with G being Newton's gravitational constant. To finish this quantum digression, we mention that non-renormalizable theories just stop to make sense at its natural UV scale, because only there perturbation theory breaks down.

1.2.1 Schwarzschild Metric and Classical Trajectories

Let us focus on a particular black hole, the Schwarzschild black hole, which is the unique[4] asymptotically flat, spherically symmetric solution of general relativity outside a source of mass M_{BH}. This solution also describes the exterior region of non-rotating spherical astrophysical bodies. The Schwarzschild metric in coordinates (t, r, θ, φ) is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \tag{1.1}$$

where $M = GM_{BH}/c^2$ and M_{BH} is the black hole mass. In the following, we use natural units where $G = c = \hbar = 1$, which thus implies that $M = M_{BH}$. Here and in the following we use signature $(-,+,+,+)$ for the metric $g_{\mu\nu}$ and Einstein's summation convention - two equal indices, one above, other below, means summation over all values of the index. The metric (1.1) describes how we measure distances in this particular static frame outside the black hole. However, there is no collapsing matter here, so we call this an *eternal* black hole. Further, when $r \to \infty$, we recover the usual Minkowski metric, so this is an *asymptotically flat* spacetime. This is a very important statement, as it is not generally true in general relativity, and because we have a good notion of observables and conserved quantities at infinity. So we can think of it as a spherical body lying inside Minkowski spacetime. Thus, it is easier, for example, to define thermodynamic variables for this system. Finally, notice that there is a coordinate singularity at

[4]If we consider the cosmological constant $\Lambda \neq 0$, there are other possibilities. See, for example, [44, 45].

$r = 2M$; this corresponds to the event horizon mentioned above. To explain why this is a point of no return, we have to talk about geodesics.

Geodesics are paths extremizing spacetime distance. In general relativity, test masses follow geodesics in the absence of any external force, which in our earthbound beings context is equivalent to saying that inertial observers are those in free fall (the equivalence principle). So if we choose coordinates $x^\mu = (t, r, \theta, \varphi)$ and extremize the action

$$S = \int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu} = \int d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{1.2}$$

we obtain the *geodesic equation*

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{1.3}$$

where the *Christoffel symbol* is given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}(\partial_\nu g_{\rho\alpha} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}). \tag{1.4}$$

The geodesics norm $\dot{x}^\mu \dot{x}_\mu$ is equal to -1 for timelike geodesics, +1 for spacelike and 0 for null geodesics. In terms of *abstract tensor notation*[5] [1], if u^a is the generator of orbits $x^\mu(\tau)$, the geodesic equation is equivalent to $u^b \nabla_b u^a = 0$, corresponding to the parallel transport of the vector along its own direction. The symbol ∇_a is the *covariant derivative*, and we can take the geodesic equation as a definition of it.

The easiest way to understand the necessity of a covariant derivative is the following: consider a vector $u = u^a e_a$ tangent to some trajectory and written in some basis e_a. If we try to take a partial derivative of u, we get $\partial_a u = \partial_a(u^b e_b) = \partial_a u^b e_b + u^b \partial_a e_b$. By supposition, the derivative of the basis vector must also be a vector in the same vector space, so it must be a linear combination of the basis vectors and we define $\partial_a e_b = C^c_{\ ab} e_c$. So we define $\partial_a u = \nabla_a u^b e_b = (\partial_a u^b + C^b_{ac} u^c) e_b$, that is,

$$\nabla_a u^b = \partial_a u^b + C^b_{ac} u^c. \tag{1.5}$$

In particular, we get the Christoffel connection above if we choose the compatibility condition $\nabla_a g_{bc} = 0$. A consequence of this is that the norm of a geodesic vector is preserved along its own geodesics. Finally, if we choose a particular coordinate system, we have that $u^\mu = \frac{dx^\mu}{d\tau}$ and we get the geodesic equation (1.3) above.

[5] An abstract tensor is written with latin indices, $T_{ab...c}$, if we do not specify a basis. We write with greek indices $T_{\mu\nu...\rho}$ the tensor components in a specific basis.

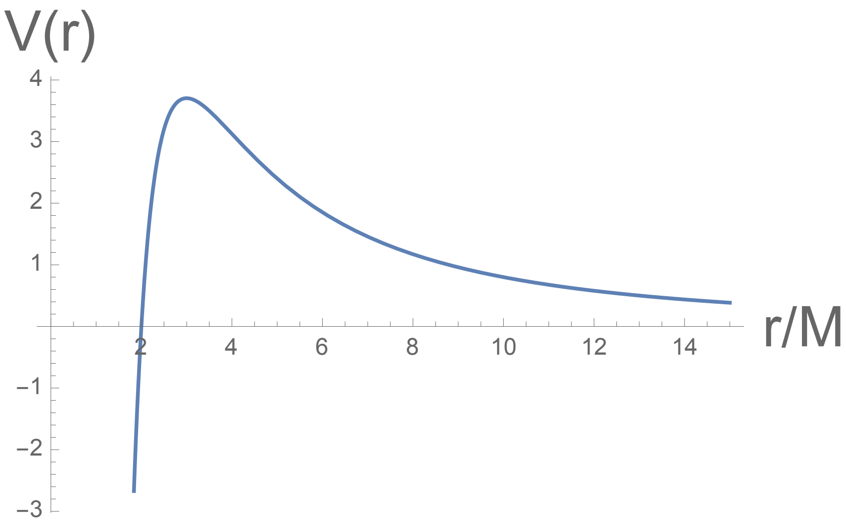

Figure 1.2: Typical form of effective potential for Schwarzschild null geodesics. The crossing point is $r = 2M$ and there is an unstable bounded orbit at $r = 3M$.

To study classical trajectories in this spacetime, we also have to consider conserved quantities, in this case, energy E and angular momentum L, to define initial conditions. In general relativity, we express conserved quantities in terms of *Killing vectors*, that is, generators of isometries of the metric. Mathematically, we say that if ξ^a is a Killing vector if the *Lie derivative* of the metric vanishes, i.e., if $\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0$. In this case, the quantity $P = u_a \xi^a$ is conserved along u^a geodesics. For Schwarzschild metric, we have two Killing vectors, $t^a = \partial_t^a$ and $\varphi^a = \partial_\varphi^a$, and we define $t^a u_a = -E$ and $\varphi^a u_a = L$. Instead of using the geodesic equations, we can use $\dot{x}^a \dot{x}_a = 0$ to find null geodesics in the Schwarzschild case. If we restrict to the equatorial plane $\theta = \pi/2$ in the coordinates above we get

$$\frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} = \frac{1}{2}E^2, \tag{1.6}$$

which is the conservation of energy for a test particle of unit mass in a effective potential

$$V(r) = \frac{L^2}{2r^2} - \frac{ML^2}{r^3}, \tag{1.7}$$

consisting of a centrifugal barrier and a gravitational term. There is no Newtonian term as we are considering trajectories of light rays so the gravitational term only appears as a general relativity effect.

The effective potential $V(r)$ is zero at $r = 2M$ and has an *unstable* bounding orbit at $r = 3M$, where the potential has a maximum (see figure 1.2). So we can ask what happens if we send an

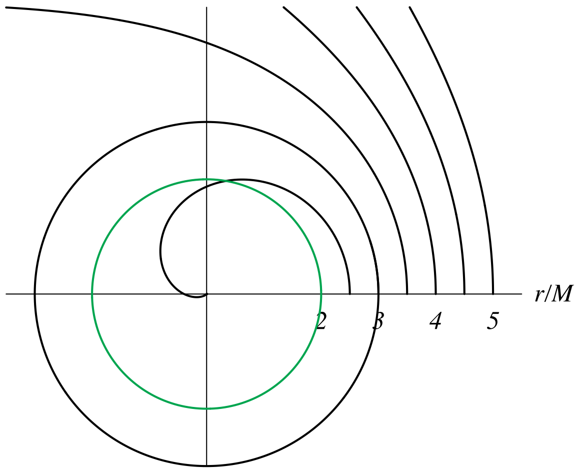

Figure 1.3: Light trajectories near a Schwarzschild black hole with different impact parameters as a function of r/M. The photon orbit is a circular closed orbit at $r = 3M$. Inside this orbit, every axially thrown light ray falls into the black hole. The innermost circle is the event horizon. Retrieved from [4].

astronaut falling into the black hole sending light signs in the positive φ-direction, as shown in Figure 1.3. As $\dot{\varphi} = L/r^2$, we have that

$$\frac{d\varphi}{dr} = \frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2}, \tag{1.8}$$

with $r(t)$ a solution of (1.6) and $b = L/E$ the *apparent impact parameter* as measured from infinity. If we draw these orbits for several values of the impact parameters, we get figure 1.3. What we see first is the bending of light rays near a compact object, one of the first experimental tests that confirmed general relativity as a consistent theory of gravity. We see the photon orbit at $r = 3M$ and that every light ray inside this orbit never reaches spatial infinity again. Another interesting thing is that if the astronaut turns on its rockets staying stationary nearby the event horizon and continues to throw light rays changing its direction, we have that *almost* all light rays are going to fall into the black hole (see figure 1.4). At $r = 2M$ there is no way a light ray can escape anymore and it stays at most orbiting at this radius.

Of course, as light rays are the fastest things in the universe, massive particles are also completely lost after crossing the event horizon. If an observer tries to follow a stationary trajectory at a fixed radius, its geodesic vector u^a is proportional to ∂_t and we now see that its norm becomes zero exactly at the event horizon. No massive particle can follow this trajectory. In fact, as ∂_t is a Killing vector, we have that the event horizon is generated by the orbits of this

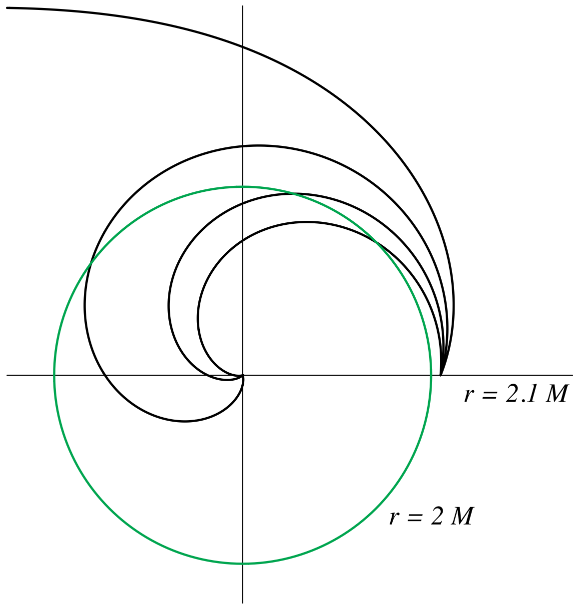

Figure 1.4: Light trajectories at $r = 2.1M$ thrown at different angles. Retrieved from [4].

vector, so we also call it a *Killing horizon*. This concept will be important below.

To finish our conceptual introduction about black holes, we can ask some questions about the fate of a massive observer falling into the black hole. We argued that there is no geodesic trajectory in which those observers remain stationary at the event horizon, but can they follow a non-geodesic path and remain stationary? This means being accelerated with respect to stationary paths at infinity, which gives our standard of a straight path. If ξ^a is the stationary Killing vector, we want to follow trajectories of $u^a = \xi^a/V$ with $V = \sqrt{-\xi_a \xi^a}$, such that $u_a u^a = -1$. The local acceleration necessary to stay on this path is $a^b = u^c \nabla_c u^b = \nabla^b \ln V$. This diverges at the event horizon, as $V \to 0$ there. So it seems not possible either. However, if we tie the observer to a rope and slowly lower him down from infinity, we can ask what is the necessary force to keep him stationary with respect to an equally stationary spaceship holding him from infinity. One can thus show that the necessary force at infinity $F_\infty = VF$ is actually finite because the *redshift factor* V regularizes it. Therefore, we define

$$\kappa = \lim_{r \to 2M} (Va) \tag{1.9}$$

as the *surface gravity* at the horizon, in analogy to the local gravitational acceleration g on the Earth's surface.

But can we actually see the observer reaching the horizon from a standpoint far away the

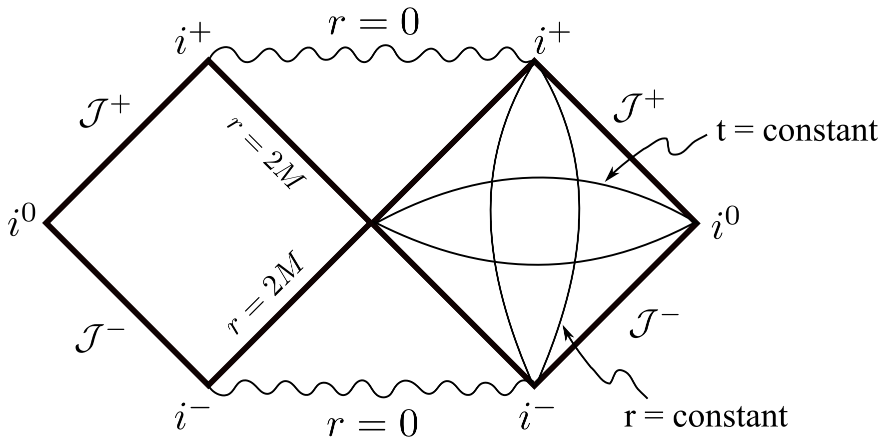

Figure 1.5: Conformal diagram of the maximal extension of the Schwarzschild black hole. Adapted from [43].

event horizon? Unfortunately, as we saw earlier, light rays become more and more trapped as the observer falls into the black hole and its light rays take more and more time to reach out an outside observer in the process, so we actually never see the last light signal. The observer becomes "frozen" from our perspective. That is what is usually called the *gravitational redshift*.

From an infalling observer point of view, nothing unusual seems to happen when crossing the horizon in this classical description. The singularity in the horizon is just a coordinate one; we can actually redefine our coordinates and *extend* the metric inside the horizon. That is one of the main lessons of general relativity: the physics we see depends on the reference frame we are. The only point where coordinates always fail is the $r = 0$ singularity, consisting in a true curvature singularity. The conformal diagram of the maximally extended Schwarzschild spacetime is shown in figure 1.5 for comparison with the case of gravitational collapse in figure 1.1.

1.2.2 Kerr Metric and Energy Extraction from Black Holes

Are there more general black holes in general relativity? The answer is yes. As stated in [2] and proved by Hawking [46], every stationary black hole obeying vacuum or eletrovac Einstein's equations have a Killing horizon. Therefore, it might happen that the timelike Killing vector ξ^a fails to be null at the horizon, so we have to add another Killing vector in order to obtain the Killing horizon generator. The most general asymptotically flat 4-dimensional black hole is the Kerr-Newman black hole, which is a generalization of (1.1) with angular momentum J and electric charge Q. Therefore, we can also have a rotating contribution to the Killing horizon generator and thus we define

$$\chi^a = \xi^a + \Omega_H \psi^a, \tag{1.10}$$

where the constant Ω_H is interpreted as the *horizon angular velocity* as we typically choose coordinates that $\xi = \partial_t$ and $\psi = \partial_\phi$.

In fact, there are also other charges we can add to a black hole, as for example a NUT charge, but its addition brings strange behaviour, like closed timelike curves, so we do not usually mention it. For an extensive review of exact solutions of general relativity, see [47]. On the other hand, these metrics have applications in AdS/CFT, as the NUT charge corresponds to particular rotating plasmas [48]. As a matter of fact, in this thesis we are going to focus on Kerr-NUT-(A)dS metrics [49], corresponding to rotating black holes with a NUT charge in asymptotically (A)dS spacetime. We shall discuss these in the next chapter.

Roy Kerr [50] found the most general vacuum rotating black hole in 4 dimensions with mass M and angular momentum $J = Ma$, with its metric in Boyer-Lindquist coordinates given by

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)d\phi - a dt)^2 + \rho^2 d\theta^2 \,, \tag{1.11}$$

where

$$\Delta = r^2 - 2Mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta. \tag{1.12}$$

This black hole has two horizons given by the roots of $\Delta = (r - r_+)(r - r_-) = 0$, that is,

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{1.13}$$

and when $a = 0$ we recover the Schwarzschild metric corresponding to a non-rotating black hole, as we have studied above. The larger root r_+ - the outer horizon - is an event horizon, while the smaller root r_- - the inner horizon - is a *Cauchy horizon* (as it does not correspond to a conformal boundary of spacetime). Those two are coordinate singularities, just like the Schwarzschild case. The inner horizon is actually unstable under small perturbations [51], so we do not expect it to exist in physically realistic gravitational collapse, only for some finite period of time while the collapse is still occurring. Nevertheless, we expect that this type of black hole approximately describes the exterior region of astrophysical black holes.

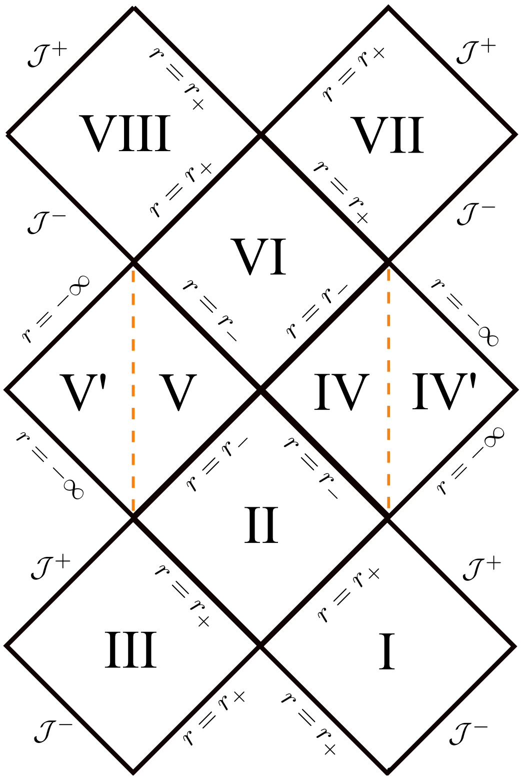

Figure 1.6: Penrose diagram of maximally extended Kerr black hole along the symmetry axis $\theta = 0$. The yellow dashed lines correspond to the black hole singularity $r = 0$. The diagram repeats itself indefinitely both in the upward and downward directions.

The curvature singularity of the Kerr metric actually lies at $\rho = 0$, which implies in $r = 0$ and $\theta = \pi/2$. It has a ring-like topology when the metric is written in Kerr-Schild form [1]. Because of that, it is possible to pass through the singularity, avoiding the ring, and reach another asymptotically flat region with boundary at $r = -\infty$. This can be seen in the conformal diagram of the maximally extended Kerr solution for $\theta = 0$ in Figure 1.6. In the bottom of the Kerr diagram, we have two asymptotically flat regions, I and III. Observers can start their trajectories at region I, enter the black hole region II and then reach the singularity located at one of the yellow dashed lines in regions IV or V. However, in contrast with the Schwarzschild case, there are two ways an observer can avoid the singularity. One way was mentioned above, by crossing the ring-like singularity, an observer can reach regions IV′ or V′. Another possibility is the observer to come out of the black hole passing through region VI and reaching an asymptotically flat region VII or VIII different from the starting one. So it is possible to escape from the a black hole interior but only in an alternative universe! Of course, this shall not be taken so seriously, as we mentioned earlier, this intricate interior structure is unstable to small perturbations. However, it might be interesting to study scattering between different asymptotic regions in other theoretical context, as we do in subsection 4.1.2 of this thesis.

There are several physical properties in the Kerr black hole worth investigating from a fundamental point of view. First, an observer trying to stay in a non-rotating stationary frame has a hard time as it is dragged along with the horizon rotation even before it reaches the horizon! This is called the *dragging of inertial frames*. The explanation for this is that the Killing vector ξ^a becomes spacelike inside the *ergosphere*, defined as the region

$$r_+ < r < M + (M^2 - a^2 \cos^2\theta)^{1/2}. \tag{1.14}$$

A stationary observer with tangent vector $u^a = \nabla^a t/N$, with $N^2 = \nabla_a t \nabla^a t$, always has a zero angular momentum as $L = \psi^a u_a = 0$ but necessarily has a non-zero angular velocity given by

$$\Omega = \frac{d\varphi}{dt} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}. \tag{1.15}$$

Those are the natural stationary observers in the Kerr metric, which rotate everywhere with respect to infinity. There is no need of torque to start this rotation, as it is a property of the spacetime, so that is why the angular momentum is conserved. This is related to the fact that (1.10) represents the Killing vector of a rotating horizon. In particular, we define the horizon angular velocity as (1.15) calculated at $r = r_+$, giving

$$\Omega_H = \frac{a}{r_+^2 + a^2}. \tag{1.16}$$

How faster can this black hole rotate? From (1.13), we see that at most $a = M$ and the two horizons coalesce, $r_+ = r_- = M$, rotating with $\Omega_H = 1/2M$. If we calculate the *linear* velocity associated to its angular velocity, we get $v_H = a/r_+ = 1$, corresponding to the light speed and the maximal velocity the horizon can rotate. The Kerr solution thus degenerates to an *extremal* Kerr black hole and we will study more about this solution in section 4.2 below.

Another important thing about the Kerr black hole is that it is actually possible to extract energy from it in a classical situation by a so-called *Penrose process*. This relies on the fact that the effective potential of Kerr geodesics has a region where modes with *negative* energy can exist lying inside the ergoregion (1.14), particularly between r_+ and $r = 2M$ in the equatorial plane [4]. In the context of general relativity, as every matter and energy couples with gravity, negative energy has a definite meaning of retrieving energy from the black hole. This can be accomplished by throwing a particle with some definite energy and momentum which explodes inside this particular region, with one of the pieces with negative energy. The other piece can have enough energy to climb back the gravitational potential and come out the ergoregion with additional energy. Two important commentaries follow this process.

First, the energy extracted from the black hole lowers its angular momentum J and its mass M, but there is a limit for the maximum energy which one can retrieve. As the process depends on the ergoregion, it stops when the angular momentum reaches zero and, at this point, the mass is given by Christodoulou's *irreducible mass* $M_{\text{irr}}^2 = \frac{1}{2}(M^2 + (M^4 - J^2)^{1/2})$. It happens to be the case that the area of the event horizon is proportional to this mass, $A_H = 4\pi(r_+^2 + a^2) = 16\pi M_{\text{irr}}^2$. As the variation $\delta M_{irr} > 0$, this suggests that $\delta A_H > 0$ in a Penrose process. This result is actually a confirmation of *Hawking's area theorem*, stating that the area of a black hole event horizon must always increase in a classical process, given certain reasonable energy conditions [46, 1].

Second, although it might seem that we should be building energy extraction machines around every rotating black hole we find, it has been shown that this is not very practical [1]. This is because the actual break up of the particle thrown into the black hole must be very precise and at relativistic velocities, so we do not hope to be using this in the coming future.

Finally, there is a wave analog of the Penrose process called *superradiant scattering*. As in the case of particles, we can send an incoming wave towards a black hole which scatters with a higher amplitude than its incident flux. This happens if the wave has a frequency in the range $0 < \omega < m\Omega_H$. One can show that in this case there will be a negative energy flux into the horizon, resulting in energy extraction from the black hole and increased amplitude of the outgoing wave. We shall discuss more about this in chapter 4 for Kerr-dS black holes. In particular, we notice that this phenomenon happens for scalar, vector and tensor fields, but actually does not happens for fermions, as the energy flux is always positive definite in this case [1].

One might wonder if there is a deeper connection between thermodynamical processes like Penrose's one with geometrical quantities of the black hole. For sure there is, as we will see in the following. However, before finishing this section, we present our main object of study in this thesis, the Kerr-(A)dS black hole metric

$$ds^2 = -\frac{\Delta_r}{\rho^2\chi^4}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Delta_\theta}{\rho^2\chi^4}\left(adt - (r^2+a^2)d\phi\right)^2 + \rho^2\left(\frac{d\theta^2}{\Delta_\theta} + \frac{dr^2}{\Delta_r}\right), \tag{1.17}$$

where $\chi^2 = 1 + a^2/L^2$, with the dS radius $L^2 = 3/\Lambda$, and

$$\Delta_\theta = 1 + \frac{a^2}{L^2}\cos^2\theta, \quad \Delta_r = (r^2+a^2)(1 - \frac{r^2}{L^2}) - 2Mr, \quad \rho^2 = r^2 + a^2\cos^2\theta. \tag{1.18}$$

We have chosen the dS radius, but one can write in terms of the AdS radius just by a Wick rotation in the radius $L \to iL$. We can recover the Kerr metric but making $L \to \infty$. The coordinate singularities of this metric are now given by the root of a 4th order polynomial $\Delta_r = 0$. For a certain range of black hole parameters, we can find 4 real roots in the dS case, which we call (r_{--}, r_-, r_+, r_C), and 2 real roots in the AdS case, called $(r_-, r_+, \zeta, \bar{\zeta})$. Again we have an inner and outer horizon in both cases, r_- and r_+. In the AdS case, the other two roots are not physical, living in the complex plane. In the dS case, one of the remaining roots is not physical, r_{--} is a negative number, but r_C is what we call a *cosmological event horizon* [52]. To clarify the causal structure, we show the Penrose diagram of Kerr-dS for $\theta = 0$ in figure 1.7. Now this diagram continues indefinitely in all directions. The yellow dashed line again represents the

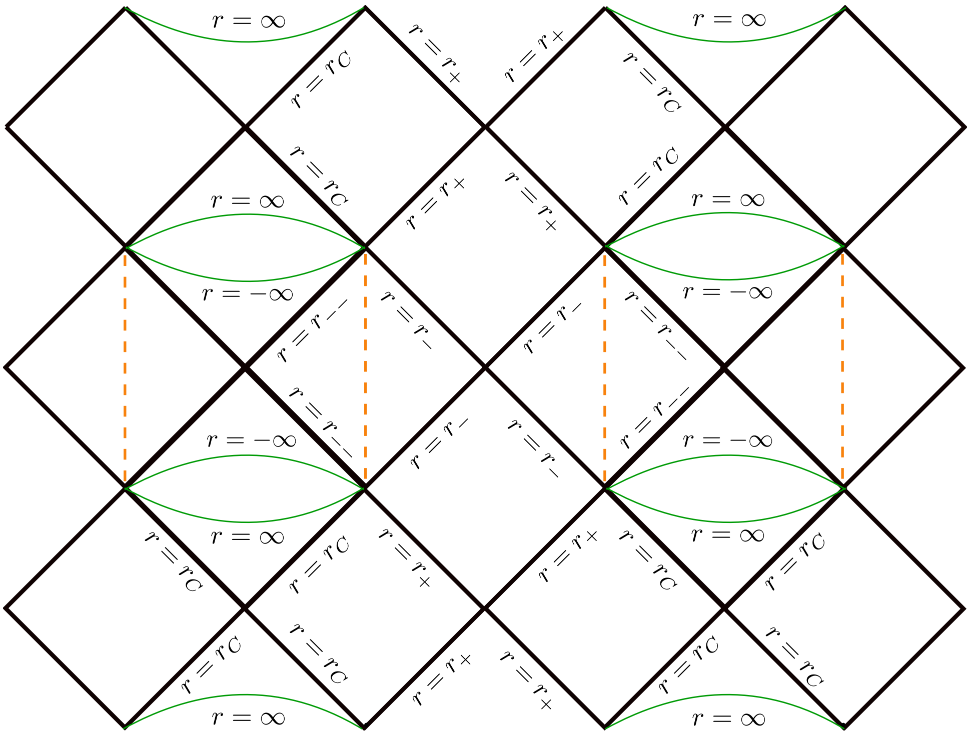

Figure 1.7: Causal diagram of maximally extended Kerr-dS black hole for $\theta = 0$.

black hole singularity, which can be crossed into an alternate universe. The matter of scattering between different initial regions is even worse now, as there are even more regions in which a wave can scatter to. In fact, we are actually interested in the scattering between the cosmological horizon r_C and the black hole event horizon r_+. We postpone a more detailed discussion of the Kerr-(A)dS metric to chapter 4.

1.3 Black Holes as Thermal Systems

To finish this chapter, we give a brief review of black hole thermodynamics. The discussion above has shown that black holes carry mass, angular momentum and other types of charges. The *no-hair theorem* of general relativity [1] actually states that every black hole in vacuum Einstein-Maxwell theory is the same except for its global charges[6] (M, J, Q). These seem like good thermodynamical variables as they are well-defined at infinity. We have also seen that there are some classical processes in which energy can be extracted from these systems. What is left to have a thermodynamical description? Temperature and entropy, of course.

The final fate of every classical gravitational system after a long time should always be a black hole, as proved by several singularity theorems in the 1960s [46, 1]. Thus they are natural stationary states of general relativity. Stationary black holes should always present a Killing horizon $\mathcal{K}$ and the physical reason was already discussed: light rays are trapped on them so they must generate the horizon. For a generic rotating black hole, we have null generators χ^a of the type (1.10). Null vectors are both tangent and normal to the null surface they generate, as their norm is zero. So we expect that the gradient of its norm is also tangent to the null surface, i.e.,

$$\nabla^b(\chi^a\chi_a) = -2\kappa\chi^b, \tag{1.19}$$

where this equation is valid only at $\mathcal{K}$. One can actually show using this equation, Frobenius theorem and some algebra that, on the horizon,

$$\kappa = \lim_{r\to r_H}(Va), \tag{1.20}$$

which is exactly the equation for the surface gravity (1.9) in a more general context. Here, a^c corresponds to the acceleration of χ^a orbits and $a = (a^c a_c)^{1/2}$. Finally, by doing some more work and using Einstein's equation, one can show that κ is actually constant over the whole Killing horizon $\mathcal{K}$. So we have found a quantity to represent our *zeroth law of black hole thermodynamics*, the constancy of temperature at equilibrium. Therefore, we propose that the temperature of the black hole horizon (which is the part we have access to from the outside) must be proportional to κ.

To prove a first law of thermodynamics, we start with a Gauss-like definition of mass in the

[6] We might also include a *fixed* cosmological constant Λ here too. For considering Λ as a variable thermodynamical quantity, see [53]. For stationary black holes with hair, see [54]

vacuum case (no matter outside)

$$M = -\frac{1}{8\pi} \int_{\mathcal{H}} \epsilon_{abcd} \nabla^c \xi^d, \tag{1.21}$$

where $\mathcal{H}$ represents the 2-sphere at the intersection of two Killing horizons. M is thus the conserved charge associated to the timelike Killing vector ξ^a. By explicit calculation, one can show that [1]

$$M = \frac{1}{4\pi} \kappa A_H + 2\Omega_H J_H, \tag{1.22}$$

which can be interpreted below as a generalized Gibbs-Duhem relation of thermodynamics. This actually suggests that M must be a homogeneous function of A_H and J_H. Finally, by some more amount of work, we can calculate the actual variation of the energy M and obtain the *first law of black hole thermodynamics*

$$\delta M = \frac{1}{8\pi} \kappa \delta A_H + \Omega_H \delta J_H. \tag{1.23}$$

As we saw that κ is the black hole temperature, A_H must be related to the black hole entropy. This matches very well with the aforementioned Hawking's area theorem, in which $\delta A_H > 0$ in classical processes, corresponding to the *second law of thermodynamics*. The third law of thermodynamics, stating that $S \to 0$ as $T \to 0$ is not respected by black holes, as extremal ones have zero temperature in this sense but non-zero entropy. However, this indicates a degenerate ground state of the theory, as it also happens in ordinary quantum systems [2].

Summarizing, we got the following table representing the laws of black hole thermody-

Law	Context
Zeroth	$\kappa = 0$ over horizon of stationary black hole
First	$dE = \frac{\kappa}{8\pi} dA + \Omega_H dJ$
Second	$\delta A \geq 0$ in any classical process
Third	Impossible to achieve $\kappa = 0$ by physical process

namics. We can also state a third law by noticing that it must be impossible to achieve $T_H = 0$ by a classical process. This classical thermodynamical picture was first proposed by [55, 56]. However, up to this point, it seems just a mathematical analogy, although we have seen that we can actually respect these laws in energy extraction processes. The most mysterious quantities are the black hole temperature and entropy. A classical black hole does not radiate so how come a black hole thermalizes with other systems? Furthermore, if we have an entropy, why is it proportional to the black hole area and not to its volume[7]? Finally, as we seek a microscopic description of a black hole, what are the microscopic degrees of freedom being counted by this entropy, if there are any?

The best way to convince ourselves that black holes really are "conventional" thermodynamical systems is by turning on a quantum field outside the black hole. The result of this experiment is that black holes actually thermally radiate at a temperature $k_B T_H = \frac{\hbar\kappa}{2\pi c}$ - the Hawking radiation [57]. For a Schwarzschild black hole, we have that $\kappa = 1/4M$ and thus

$$T_H \approx 6\times 10^{-8}\left(\frac{M_\odot}{M}\right) K, \tag{1.24}$$

so for a typical black hole of a few orders of magnitude of the solar mass, this temperature is still lower that the cosmic microwave background temperature of $T_{CMB} = 2.7\ K$, hindering its direct detection.

In conclusion, curiously, black holes radiate exactly as a blackbody (or as a greybody, if we consider the transmitted amplitude). There are several equivalent ways to show this, as for example by canonically quantizing the field and studying its density matrix in the region outside the black hole. If we trace out modes outside this region, we obtain a thermal distribution with temperature T_H. Another way is to go to the Euclidean QFT and study the definition of KMS[8] states in terms of Euclidean time periodicity of the Green's function [7]. A related way to this one is also to study the removal of conical singularities in the near-horizon metric [1]. Finally, there is a much easier way suggested by [39] using monodromies and Euclidean time periodicity, which we shall show in chapter 4 of this thesis.

On the other hand, while these procedures seem well-defined at first sight, there is a more complicated problem on how to define a regular vacuum for the quantum field. What happens is that the Fock space in curved spacetimes cannot be univocally defined, as the particle content of the state depends intrinsically on which basis we have chosen to expand the operators. In curved space, there is a whole class of unitarily inequivalent options [7]. In the case of a Schwarzschild black hole, we have typically three physically reasonable vacua: the Unruh vacuum, the Hartle-Hawking vacuum and the Boulware vacuum. The first one is appropriate to describe the spherical collapse where there is a time-asymmetric thermal flux from the hole. This also happens to be the appropriate vacuum to study in scattering problems, as it describes flux of quanta through the horizon. The Hartle-Hawking vacuum is the one describing a thermal bath of particles in equilibrium with the black hole temperature, so it is a steady state. Finally,

[7]One reason is that the black hole volume is not even well-defined because of the singularity.

[8]Kubo-Martin-Schwinger.

the Boulware vacuum coincides with Minkowski vacuum at $I^{\pm}$ and thus corresponds to the vacuum around a static star, as it detects no particles around it.

In this chapter, we have seen the richness of black holes discussing their classical and semiclassical properties. This sets up the pace and spirit of our work, as we want to understand as much as we can about these mysterious and exciting physical systems. Now, we go beyond the geometric optics approximation and start our endeavour in black hole scattering theory.

CHAPTER 2

Black Hole Scattering Theory

"The main of life is composed ... of meteorous pleasures which dance before us and are dissipated".

—SAMUEL JOHNSON

Our main interest in this thesis is to obtain analytic scattering data of waves impinging a rotating black hole in (A)dS spacetime. We thus start this chapter with a brief discussion of classical scattering theory of waves around asymptotically flat black holes. Throughout this work, we are going to focus mostly on the scattering of scalar waves for simplicity. The discussion of higher spin fields does not changes qualitatively from spin zero case, as can be seen in the Teukolsky formalism of gravitational perturbations [58].

The success of the standard formalism to treat wave scattering is limited by the inability to find exact solutions in terms of elementary functions. As we will see later, the Klein-Gordon equation for Kerr-NUT-(A)dS black holes, after separation of variables, falls into a class of ODEs whose solutions are transcendental functions more general than the hypergeometric function. This makes the analysis of the scattering problem much more difficult. In section 2.2, we introduce the monodromy technique discussed in [39] to obtain scattering coefficients for asymptotically flat black holes, paving the path to applying this technique to more general Kerr-NUT-(A)dS black holes.

There are two kinds of scattering data we are typically interested in black hole physics: classical and semiclassical data. Classical wave scattering is conventionally described by an incoming plane-wave from spatial infinity which is scattered by the black hole [3, 4]. The *transmission coefficient*, $\mathcal{T}(\omega)$, denotes how much of the incoming radiation was absorbed by the black hole, and the *reflection coefficient*, $\mathcal{R}(\omega)$, is how much radiation is reflected back to spatial infinity. Semiclassical scattering appears when we consider radiative effects due to vacuum fluctuations of a field outside the black hole. As discussed in the introduction, black holes emit blackbody radiation with Hawking temperature T_H depending on the black hole thermodynamical variables. This scattering has different boundary conditions with respect to the classical case. We have outgoing waves at the horizon which are scattered by the gravitational potential around the black hole, so $\mathcal{R}(\omega)$ refers to radiation reflected back to the horizon and $\mathcal{T}(\omega)$ is how much radiation is transmitted to the region outside the black hole. Therefore, one can show that the mean number of particles $\langle n(\omega)\rangle$ emitted via Hawking radiation is given by

$$\langle n(\omega)\rangle = \frac{\gamma(\omega)}{e^{\omega/T_H} \pm 1}, \tag{2.1}$$

where we have minus (plus) sign if the particle is a boson (fermion). The coefficient $\gamma(\omega) = |\mathcal{T}(\omega)|^2$ is called the *greybody factor*, which is just the probability for an outgoing wave to reach infinity [59]. One of our main aims in this work is to obtain exact results for the greybody factor in asymptotically (A)dS cases. Results valid for some particular ranges of ω have been obtained for d-dimensional static black holes in [59]. For asymptotic analysis and quasinormal modes for asymptotically flat rotating black holes, see [60]. There are few papers addressing scattering by Kerr-(A)dS black holes [61, 41, 62, 63, 36], in comparison with what is done on static backgrounds. However, there is an extensive literature about the near-horizon extremal Kerr metric, including the (A)dS case, in relation with the Kerr/CFT correspondence [64, 65, 66, 67].

In the asymptotically flat case, we might also be interested in the *absorption cross-section*

$$\sigma(\omega) = \gamma(\omega)|\Psi|^2,$$

where $|\Psi|^2$ is the projection of an incoming plane wave into the asymptotic spherical wave. In asymptotically (A)dS cases, there is no accepted definition of cross-section because there are no plane-waves or well-defined one particle states at spatial infinity. Actually, AdS spacetimes have a timelike boundary at $r = \infty$ and good observables at the infinity boundary are the correlation functions of the dual CFT, according to the AdS/CFT duality. In the global dS case, the asymptotic regions are in the past and future of the spacetime, so there is no notion of spatial infinity. However, in dS static patch, we can use the cosmological horizon as an equivalent notion for spatial infinity but, then again, asymptotics are different from those in the flat case. Therefore, the notion of cross-sections only comes about in asymptotically flat cases and so we shall not discuss this subject in the following.

2.1 Classical Wave Scattering by Black Holes

We want to study small perturbations of a $D = 4$ black hole spacetime with respect to field disturbances. In principle, spin s field perturbations $\delta\phi$ are coupled with metric perturbations δg_{ab} through Einstein's equation. The most general and clean formalism to treat linear perturbations of the gravitational field is the Teukolsky formalism [58, 51]. For higher dimensions, there is no general formalism for gravitational perturbations. For spherically symmetric solutions, the most used formalism is the gauge invariant Kodama-Ishibashi formalism [68].

To set the stage for our work, we start with Einstein-Hilbert action coupled to some matter Lagrangian $\mathcal{L}_m(\Phi, \nabla\Phi)$ in a D-dimensional spacetime (M, g_{ab})

$$S = \frac{1}{16\pi G}\int_M d^D x\,\sqrt{-g}\,(R - 2\Lambda) + \int_M d^D x\,\sqrt{-g}\,\mathcal{L}_m, \tag{2.2}$$

where Λ is the cosmological constant and G is Newton's gravitational constant. In the following, we shall always stick to natural units $G = c = \hbar = 1$, unless strictly necessary. In principle, the matter field Φ represent a field, or a set of fields, of any spin. Varying the action (2.2) with respect to the metric g_{ab}, we get Einstein's equation

$$R_{ab} - \frac{1}{2}(R - 2\Lambda)g_{ab} = 8\pi T_{ab}, \tag{2.3}$$

where T_{ab} is the energy-momentum tensor associated to Φ, defined by

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{ab}}. \tag{2.4}$$

One of the most important things of general relativity is invariance under diffeomorphisms and this necessarily implies local conservation of the stress tensor [1]

$$\nabla_a T^{ab} = 0.$$

Now, we perturb the fields by $g_{ab} = g_{ab}^{BG} + h_{ab}$ and $\Phi = \Phi^{BG} + \phi$, where BG denotes the background state in which we are perturbing. In general, after substitution of these expansions into (2.3), we will have a set of coupled partial differential equations for the perturbations (h_{ab}, ϕ) depending on the background solutions. For $D = 4$, generic perturbations decouple for any spin s, which is the main result of the Teukolsky formalism [58]. For general D, there is no general formalism and research is still under progress [69, 70].

For the special case of scalar perturbations in a vacuum D-dimensional spacetime, one can show that gravitational perturbations decouple from scalar ones and we can thus safely discard backreaction[1] from small disturbances. Therefore, now and in the rest of this thesis, we will study simply Klein-Gordon equation for scalar perturbations ϕ in some fixed metric background g_{ab}^{BG}

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\phi) = 0. \tag{2.5}$$

For a generic background spacetime, we have no hope of solving this partial differential equation (PDE), as space and time can be interwoven in complicated ways. However, as we shall see later, there is a very general class of spacetimes which allow separable solutions of (2.5). In those separable cases, our job is to find solutions for ordinary differential equations (ODEs) and impose boundary conditions for the scattering problem of interest. Typically, we have a radial equation and an angular equation after separation of variables. We are thus left only with an equivalent one-dimensional scattering problem for some effective potential $V(r)$ in the radial equation. In those simple cases, solving the scattering problem amounts to finding how much of an incident wave flux towards the black hole horizon was transmitted inside the black hole and how much was reflected back. Given the transmission and reflection coefficients $\mathcal{T}(\omega)$ and $\mathcal{R}(\omega)$, we can thus calculate an associate absorption cross-section $\sigma(\omega)$ when appropriate.

Summarizing, we can always choose a radial coordinate r such that the resulting ordinary differential equation for the radial perturbation $\phi_\omega(r)$ is written in self-adjoint form

$$\partial_r(P(r)\partial_r\phi_\omega) - Q(r)\phi_\omega = 0 \tag{2.6}$$

By a coordinate transformation $dr^* = dr/P(r)$, we can put (2.6) into a Schrödinger-like form

$$\frac{d^2\phi_\omega}{dr^{*2}} + (\omega^2 - V(r))\phi_\omega = 0, \tag{2.7}$$

where ω is complex in general[2]. Thus our problem reduces to a one-dimensional scattering of an incoming wave from $r^* \to \infty$ (which may correspond to spatial infinity or another horizon, depending on the case). Typical potentials $V(r^*)$ for $d = 4$ Schwarzschild black hole are shown in figure 2.1. From the figure it is clear that $V(r) \to 0$ for $r^* \to \pm\infty$. Thus, in the classical scattering, we have an IN mode if we impose ingoing boundary conditions at the horizon [59],

$$\begin{aligned} \phi_\omega^{IN} &\sim e^{-i\omega r^*} - \mathcal{R}e^{i\omega r^*}, \qquad && r^* \to +\infty, \\ \phi_\omega^{IN} &\sim \mathcal{T}e^{-i\omega r^*}, && r^* \to -\infty, \end{aligned} \tag{2.8}$$

[1]From a QFT point of view, backreaction means loop corrections on the metric fluctuations due to fluctuations of the matter field.

[2]In the Kerr case, the potential V also depends on ω.

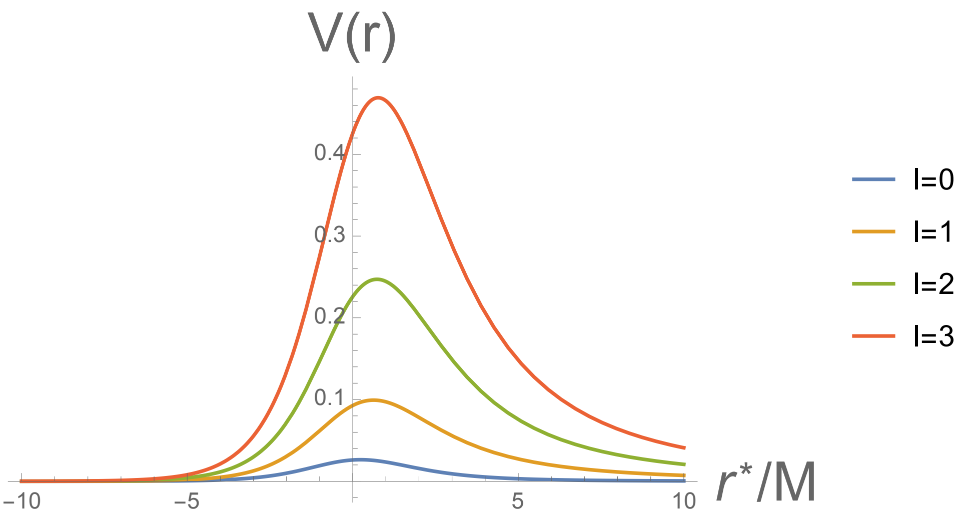

Figure 2.1: Typical potentials in the d=4 Schwarzschild scalar field scattering for different values of angular momentum number *l*.

A linearly independent solution is obtained by making $\omega \to -\omega$, that is, $\phi_{-\omega}$, which obeys

$$\begin{aligned} \phi^{IN}_{-\omega} &\sim e^{i\omega r^*} - \tilde{\mathcal{R}} e^{-i\omega r^*}, \qquad && r^* \to +\infty, \\ \phi^{IN}_{-\omega} &\sim \tilde{\mathcal{T}} e^{i\omega r^*}, && r^* \to -\infty. \end{aligned} \tag{2.9}$$

To assess the amount of scattered waves, we introduce the flux of radiation per unit area

$$W := \frac{1}{2i}\left(\phi_{-\omega}\frac{d\phi_\omega}{dr^*} - \phi_\omega \frac{d\phi_{-\omega}}{dr^*}\right), \tag{2.10}$$

which is essentially the Wronskian between the two solutions. As is well known from quantum mechanics (and we will show below), this current is conserved in different constant r^* surfaces. At the horizon, $W_{hor} = -\omega\mathcal{T}\tilde{\mathcal{T}}$, representing the flux entering the black hole. In the asymptotic region, $W_{asy} = W_{in} - W_{out} = -\omega - (-\omega\mathcal{R}\tilde{\mathcal{R}}) = \omega(\mathcal{R}\tilde{\mathcal{R}} - 1)$. Because the flux is conserved, $W_{asy} = W_{hor}$ and thus

$$\mathcal{R}\tilde{\mathcal{R}} + \mathcal{T}\tilde{\mathcal{T}} = 1. \tag{2.11}$$

The greybody factor is defined to be the ratio between the horizon flux and the incoming flux

$$\gamma(\omega) = \frac{W_{hor}}{W_{in}} = \mathcal{T}(\omega)\tilde{\mathcal{T}}(\omega). \tag{2.12}$$

If $\omega \in \mathbb{R}$, then we have $\phi^{IN}_{-\omega} = (\phi^{IN}_{\omega})^*$, implying that $\tilde{\mathcal{R}} = \mathcal{R}^*$, $\tilde{\mathcal{T}} = \mathcal{T}^*$. Therefore, we have that

$$|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1 \quad \text{and} \quad \gamma(\omega) = |\mathcal{T}(\omega)|^2. \tag{2.13}$$

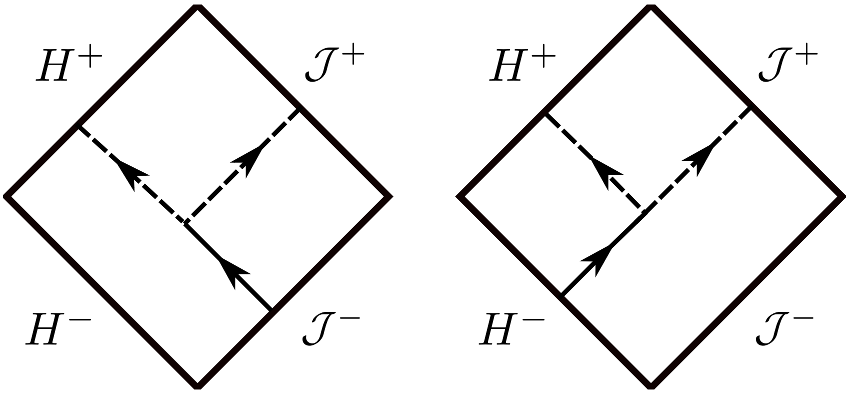

Figure 2.2: Conformal diagrams of IN modes (left) and UP modes (right).

The semiclassical scattering has an outgoing mode at the horizon, called the UP (or OUT) mode,

$$\begin{aligned} \phi_{-\omega}^{UP} &\sim \mathcal{T}' e^{i\omega r^*}, & r^* &\to +\infty, \\ \phi_{-\omega}^{UP} &\sim e^{i\omega r^*} - \mathcal{R}' e^{-i\omega r^*}, & r^* &\to -\infty, \end{aligned} \tag{2.14}$$

and the mode ϕ'_ω obeys

$$\begin{aligned} \phi_{\omega}^{UP} &\sim \tilde{\mathcal{T}}' e^{-i\omega r^*}, & r^* &\to +\infty, \\ \phi_{\omega}^{UP} &\sim e^{-i\omega r^*} - \tilde{R}' e^{i\omega r^*}, & r^* &\to -\infty. \end{aligned} \tag{2.15}$$

Since the space of solutions has dimension 2, we can express each solution in terms of two others. Using (2.8) and (2.9), we find that

$$\begin{pmatrix} \phi_{\omega}^{UP} \\ \phi_{-\omega}^{UP} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}}{\mathcal{T}} \\ \frac{\tilde{\mathcal{R}}}{\tilde{\mathcal{T}}} & \frac{1}{\tilde{\mathcal{T}}} \end{pmatrix} \begin{pmatrix} \phi_{\omega}^{IN} \\ \phi_{-\omega}^{IN} \end{pmatrix}, \tag{2.16}$$

representing the change of basis matrix between ingoing and outgoing solutions for the different types of scattering problems. We notice that

$$R' = -\frac{\mathcal{T}}{\tilde{\mathcal{T}}}\tilde{R}, \quad \mathcal{T}' = \mathcal{T}, \quad \tilde{\mathcal{R}}' = -\frac{\tilde{\mathcal{T}}}{\mathcal{T}}\mathcal{R}, \quad \tilde{\mathcal{T}}' = \tilde{\mathcal{T}}, \tag{2.17}$$

which implies that $\mathcal{T}'\tilde{\mathcal{T}}' = \mathcal{T}\tilde{\mathcal{T}}$ and, therefore, the greybody factor $\gamma(\omega)$ is the same for both scattering problems.

An equivalent way to treat scattering problems is to use IN and UP modes with a different normalization [4]

$$\phi_{\omega}^{IN} \sim \begin{cases} A_{in}(\omega)e^{-i\omega r^*} + A_{out}(\omega)e^{i\omega r^*}, & r^* \to +\infty, \\ e^{-i\omega r^*}, & r^* \to -\infty. \end{cases} \tag{2.18}$$

and

$$\phi_{-\omega}^{UP} \sim \begin{cases} e^{i\omega r^*}, & r^* \to +\infty \\ B_{out}(\omega)e^{i\omega r^*} + B_{in}(\omega)e^{-i\omega r^*}, & r^* \to -\infty. \end{cases} \tag{2.19}$$

These modes are illustrated in the conformal diagram of an asymptotically flat region in figure 2.2. Using the conserved current (2.10), we obtain the greybody factor as

$$\gamma(\omega) = 1 - \frac{A_{out}(\omega)A_{out}(-\omega)}{A_{in}(\omega)A_{in}(-\omega)},$$

and for real frequencies we have just

$$\gamma(\omega) = 1 - \left|\frac{A_{out}(\omega)}{A_{in}(\omega)}\right|^2,$$

2.2 Monodromies and Scattering Data

In the previous section, we have seen the standard way to treat scattering data in curved spacetimes. Even in the simple Schwarzschild case, although separable, the radial equation is solved neither in terms of elementary functions nor in terms of hypergeometric functions. Actually, it is a confluent Heun equation with 2 regular singular points at $r = 0$ and $r = 2M$, and one irregular singular point at $r = \infty$. This irregularity complicates the scattering analysis because of *Stokes phenomena* in the asymptotic expansions at infinity. Despite good results for small frequency and asymptotic imaginary frequencies using the matching technique [59], we want to have more analytic control of the scattering data in these problems. For that matter, we introduce in this section the more powerful *monodromy technique* for scattering problems presented in [40, 39]. This approach sheds new light into the generic structure of scattering in curved spacetimes and will allow us to obtain more information about the scattering data of Kerr-NUT-(A)dS black holes.

Monodromies associated to singular points of ODEs have been used in the literature to calculate greybody factors at large imaginary frequencies and quasinormal modes [71, 72, 60, 59]. The recent work of Castro *et al* [40, 39] has shown how to obtain the scattering matrix of a scalar field in the Kerr black hole background using monodromies. The method relies on the monodromies obtained around singular points of the radial part of (2.5). In the following, we review the monodromy technique as presented in [40, 39]. For a review of the standard Frobenius analysis of ODEs, we refer the reader to appendix B.

Given a self-adjoint radial equation extended to the complex plane

$$\partial_z(U(z)\partial_z\psi(z)) - V(z)\psi(z) = 0\,, \tag{2.20}$$

we can rewrite it in terms of a SL$(2,\mathbb{C})$ gauge potential $A = A(z)dz$

$$(\partial_z - A(z))\Phi(z) = 0\,, \quad A(z) = \begin{pmatrix} 0 & U^{-1} \\ V & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Psi_1^{(1)} & \Psi_1^{(2)} \\ \Psi_2^{(1)} & \Psi_2^{(2)} \end{pmatrix},$$

where Φ is the fundamental matrix formed by two linearly independent solutions of (2.20), $\Psi^{(i)} = (\Psi_1^{(i)}, \Psi_2^{(i)})^T$, $i = 1,2$. We recover (2.20) by setting $\Psi_1 \equiv \psi$ and $\Psi_2 \equiv U\partial_z\psi$. We can study the behaviour of Φ around singular points of (2.20) by writing the formal solution around a counterclockwise loop γ

$$\Phi_\gamma(z) = \mathcal{P}\exp\left(\oint_\gamma A\right)\Phi(z) =: \Phi(z)M_\gamma\,,$$

where $\mathcal{P}$ means path-ordering and M_γ is called the **monodromy matrix** associated to γ with base point z. Given that A is meromorphic, the branch points of Φ are related to the poles of A and these points are always represented on a conjugacy class of the monodromy group, which are in turn related to the singular points of (2.20). Let $\{z_i\}$, $i = 1,...,n$, be the set of all branch points of Φ and M_i be the corresponding monodromy matrix of a fundamental loop enclosing z_i. The connection A has thus a pole on z_i and it is always possible to write $A(z) = (z-z_i)^{-R_i-1}A^i(z)$ in a particular gauge, where $A^i(z)$ is regular at the pole and $R_i \in \mathbb{N}_0$ is called the *Poincaré Rank* of z_i. We have to make sure that this pole does not correspond to a *removable singularity*, otherwise, the pole may be removed by an appropriate gauge transformation. A wise move is to put (2.20) in an *irreducible canonical form*, by means of convenient *homographic* and *homotopic transformations*, in which only non-removable singularities appear [73]. See also appendix C for an explicit example of these transformations.

If the pole z_i has $R_i = 0$, then it is a *regular singular point*, while if $R_i > 0$ it is an *irregular singular point*. For a regular singular point, the conjugacy class of M_i is given by

$$M_i \simeq \exp\oint_{\gamma_i}(z-z_i)^{-1}A^i(z) = e^{2\pi i A^i(z_i)} \simeq \begin{pmatrix} e^{-2\pi\alpha_i} & 0 \\ 0 & e^{2\pi\alpha_i} \end{pmatrix}, \tag{2.21}$$

where the last matrix is in a convenient parametrization for the self-adjoint gauge Tr$A = 0$. Notice that, in general, $\alpha_i \in \mathbb{C}$. In a generic gauge, we have that

$$M_i \simeq \begin{pmatrix} e^{2\pi i\rho_1^i} & 0 \\ 0 & e^{2\pi i\rho_2^i} \end{pmatrix} = \exp(i\pi\rho_0^i)\,\mathbb{1} + \exp(-i\pi\theta_i\sigma^3)\,, \tag{2.22}$$

where $\rho_{1,2}^i$ correspond to the *Frobenius coefficients* associated to the singular point z_i,

$$\rho_0^i = \rho_1^i + \rho_2^i, \quad \theta_i := \rho_2^i - \rho_1^i, \tag{2.23}$$

and σ^i denotes the Pauli matrices. In the following, we will interchangeably refer to α_i and θ_i as the *monodromy coefficients* associated to z_i. In the self-adjoint gauge, $\theta_i = -2i\alpha_i$.

An irregular singular point will also belong to an equivalence class like (2.21) but its monodromy is just part of the true monodromy. In general, one has to account also for *Stokes phenomena* on irregular singular points [39], which is akin to the asymptotic sectors appearing in the discussion of Airy equation. We will explain more about irregular singular points when we talk about scattering by Kerr black holes below.

A special case is when the monodromy eigenvalues are equal, that is, when $e^{4\pi\alpha_i} = 1$. In this case we have a *resonant* singularity and one of the solutions of (2.20) may have a logarithmic branch cut (cf. appendix B). The exception is when we have an *apparent singularity*, in which the logarithm term is not present [73]. An apparent singularity may also be removable, i.e. no more poles existing in the ODE, but not necessarily. The role of apparent singularities will be very important in what follows and will be addressed in more details in the next chapter.

One important property of the monodromy matrices is that the monodromy of a loop enclosing all singular points, including the point at infinity, must be trivial

$$M_1 M_2 \dots M_n = \mathbb{1}. \tag{2.24}$$

We will use this property in the folowing to find the scattering matrix. Our goal is to find an appropriate SL$(2,\mathbb{C})$ representation of the monodromy group respecting the above relation.

The choice of basis of Φ is not unique and $\Phi \to \Phi g$ with $g \in GL(2,\mathbb{C})$ generates a solution in another basis[3]. Let $\psi_{1,2}$ be an arbitrary basis of solutions of (2.20) with fundamental matrix Φ and let $\psi_i^{in/out}$ be a basis of purely ingoing/outgoing states such that

$$\psi_i^{out}(z) = (z-z_i)^{i\alpha_i}(1+O(z-z_i)), \quad \psi_i^{in}(z) = (z-z_i)^{-i\alpha_i}(1+O(z-z_i)). \tag{2.25}$$

The definition of in and out basis for complex ω and α and the physical interpretation will depend on the sign of Re$(\omega\alpha_i)$. A general solution can be written as a linear combination of the in/out basis. Therefore, the fundamental matrix Φ can be written as

$$\Phi(z) = \Phi_i(z)\,g_i = \left(\Phi_0^i + O(z-z_i)\right)\begin{pmatrix} (z-z_i)^{i\alpha_i} & 0 \\ 0 & (z-z_i)^{-i\alpha_i} \end{pmatrix} g_i\,, \tag{2.26}$$

where Φ_0^i is a constant invertible matrix and $g_i \in GL(2,\mathbb{C})$ connects the in/out basis with another basis. In the Φ_i basis, the monodromy is diagonal. This can be easily seen by making $(z-z_i) \to$

[3]Notice that a general gauge transformation is a left action: $\Phi \to g(z)\Phi$ with $A \to g^{-1}Ag - g^{-1}\partial_z g$. For more details, see appendix C.

$(z - z_i)e^{2\pi i}$ representing the new branch after following a loop around z_i. The monodromy M_i on a generic basis is thus in the equivalence class

$$M_i = g_i^{-1} \begin{pmatrix} e^{-2\pi\alpha_i} & 0 \\ 0 & e^{2\pi\alpha_i} \end{pmatrix} g_i =: g_i^{-1} D_i g_i, \tag{2.27}$$

where $D_i = e^{-2\pi\alpha_i\sigma^3}$ and, of course, the matrix $g_i \in GL(2,\mathbb{C})$ diagonalizes the monodromy. This means that, if v_i are left eigenvectors of M_i, we have

$$v_i M_i g_i^{-1} = v_i g_i^{-1} D_i \tag{2.28}$$

$$\lambda_i (v_i g_i^{-1}) = (v_i g_i^{-1}) D_i \tag{2.29}$$

Therefore, we have that $(v_i g_i^{-1}) = (1 \;\; 0)$ and the rows of g_i are given by the left eigenvectors of M_i.

The *connection matrix* between two singular points is defined as the change of basis matrix between the local solutions at these points

$$\mathcal{M}_{i\to j} = \Phi_i^{-1}\Phi_j = g_i g_j^{-1}, \tag{2.30}$$

where we used that $\Phi = \Phi_i g_i = \Phi_j g_j$ in the last equality. To relate this matrix with conventional scattering problems like (2.16), we have to define a standard normalization. This can be achieved by the Klein-Gordon inner product

$$\langle\psi_1,\psi_2\rangle := \frac{W(\psi_1^*,\psi_2)}{2i\omega} = \frac{U(z)}{2i\omega}\left(\psi_1^*\partial_z\psi_2 - \psi_2\partial_z\psi_1^*\right), \tag{2.31}$$

where $W(\psi_1,\psi_2)$ is the Wronskian between two solutions of (2.20). Notice that this gives a well-defined inner product only when the ODE is real because then ψ^* is also a solution. The norm of a solution ψ is thus $||\psi||^2 := \langle\psi,\psi\rangle$. We can also show that W is independent of z because

$$\begin{aligned} W(\psi_1,\psi_2) &= \det\Phi(z) \\ &= \det\exp\left(\int^z A\right)\det\Phi(z_0) = \exp\left(\int^z \operatorname{Tr} A\right)\det\Phi(z_0) \\ &= \det\Phi(z_0), \end{aligned}$$

because $\operatorname{Tr} A = 0$ in the self-adjoint gauge[4]. Finally, we parametrize the connection matrix as

$$\mathcal{M}_{i\to j} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}}{\mathcal{T}} \\ \frac{\mathcal{R}^*}{\mathcal{T}^*} & \frac{1}{\mathcal{T}^*} \end{pmatrix}, \qquad |\mathcal{R}|^2 + |\mathcal{T}|^2 = 1 \,. \tag{2.32}$$

[4]It will also be invariant in the ODE normal form, but *not* in the canonical form.

Compare this with the discussion we have made to reach (2.16). Our main object of study in the following is thus the connection matrix (2.32).

In the case of real frequencies, (2.20) is real, and if all α_i are real, we have a well-defined norm and (2.32) is defined up to phases. Despite of that, the greybody factor is uniquely defined, as the phases cancel. In the more general case $\omega \in \mathbb{C}$, which happens in the study of quasinormal modes [34], the connection matrix is not so simple and it is defined up to normalization rescaling of the solutions[5]

$$\mathcal{M}_{i\to j} = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}}{\mathcal{T}} \\ \frac{\tilde{\mathcal{R}}}{\tilde{\mathcal{T}}} & \frac{1}{\tilde{\mathcal{T}}} \end{pmatrix} \begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix}. \tag{2.33}$$

According to Castro *et al* [39], these extra terms play an important role in studying the analytical structure of the connection matrix with respect to quasinormal modes of Kerr black holes. Their importance is also related to the appearance of an irregular singularity and plane-wave asymptotics in the asymptotically flat case. On the other hand, for asymptotically (A)dS spaces this ambiguity seems immaterial because all singular points are regular. In any case, we will leave the discussion about quasinormal modes for future work and they will not be addressed in this thesis.

When we have only two regular singular points, the monodromy matrices are inverse of each other and, therefore, the scattering matrix (2.32) is equal to identity. However, if one of the two singular points is irregular, we can still have non-trivial scattering, in parallel with what happens in Coulomb scattering. A simple case where the scattering coefficients can be obtained directly from monodromy data is when we have 3 regular singular points, i.e., in the hypergeometric class of ODEs. In the self-adjoint gauge, the monodromy matrices are $SL(2,\mathbb{C})$ matrices, i.e.,

$$\det(M_i) = 1\,, \qquad \operatorname{tr}(M_i) = 2\cosh(2\pi\alpha_i)\,, \qquad M_i \neq \mathbf{1}\,, \qquad \text{for} \quad i = 1,2,3\,, \tag{2.34}$$

and one possible choice of basis is [40, 39, 74]

$$M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 2\cosh(2\pi\alpha_1) \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 2\cosh(2\pi\alpha_2) & e^{2\pi\alpha_3} \\ -e^{-2\pi\alpha_3} & 0 \end{pmatrix}, \tag{2.35}$$

$$M_3 = \begin{pmatrix} e^{2\pi\alpha_3} & 0 \\ 2(e^{-2\pi\alpha_3}\cosh(2\pi\alpha_1) - \cosh(2\pi\alpha_2)) & e^{-2\pi\alpha_3} \end{pmatrix}. \tag{2.36}$$

[5] If we make $\Psi^{(i)} \to c_i \Psi^{(i)}$ or $\Phi \to \Phi\,\mathrm{diag}(c_1, c_2)$, with c_i two arbitrary constants, we still have the same two solutions of the ODE.

According to (2.30), we then have, for example,

$$\mathcal{M}_{2\to1} = \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix} \begin{pmatrix} \sinh\pi(\alpha_1-\alpha_2-\alpha_3) & \sinh\pi(\alpha_1+\alpha_2+\alpha_3) \\ \sinh\pi(\alpha_1+\alpha_2-\alpha_3) & \sinh\pi(\alpha_1-\alpha_2+\alpha_3) \end{pmatrix} \begin{pmatrix} d_3 & \\ & d_4 \end{pmatrix} . \tag{2.37}$$

We then choose the d_i to fix the same normalization as (2.32) and the transmission coefficient is thus given by[6]

$$|\mathcal{T}|^2 = \frac{\sinh(2\pi\alpha_1)\sinh(2\pi\alpha_2)}{\sinh\pi(-\alpha_1+\alpha_2+\alpha_3)\sinh\pi(\alpha_1-\alpha_2+\alpha_3)} . \tag{2.38}$$

This formula can be checked to be correct for scalar scattering on BTZ black hole [75] and $(A)dS_2\times S^2$ spacetime. We will detail the latter case as an example below. This will make explicit the relation between monodromies and boundary conditions.

2.2.1 Scattering on $(A)dS_2\times S^2$ spacetimes

The metric of a comoving patch of $(A)dS_2\times S^2$ spacetime is given by

$$ds^2 = -f(\rho)dt^2 + f^{-1}(\rho)d\rho^2 + L^2 d\Omega_2^2, \tag{2.39}$$

with $f(\rho) = 1 - \frac{\rho^2}{L^2}$ and $L^2 = \frac{3}{\Lambda}$ is the (A)dS curvature radius. We redefine ρ to absorb the curvature radius by making $\rho \to L\rho$. In the following we shall focus on the dS case[7]. The radial part of Klein-Gordon equation for a non-minimally coupled scalar field is then given by

$$\frac{d}{d\rho}\left((1-\rho^2)\frac{du_{l\omega}}{d\rho}\right) + \left(\beta(\beta+1) - \frac{\mu^2}{1-\rho^2}\right)u_{l\omega} = 0 \tag{2.40}$$

where

$$\mu = \pm i\omega, \quad \beta = -1/2 + i\lambda, \tag{2.41}$$

$$\lambda = \pm\sqrt{(l+1/2)^2 + d}, \quad d = 4\xi - 1/2. \tag{2.42}$$

The ODE (2.40) has 3 regular singular points at $\rho = \pm1$ and $\rho = \infty$ and its solutions, $u_{l\omega}$, are Legendre functions, which belong to the hypergeometric class.

Let $u_{lm}^{(\text{in})}$ be a basis of solutions such that

$$u_{lm}^{(\text{in})} \sim \begin{cases} (\rho+1)^{-i\omega/2} & \text{as} \quad \rho \to -1, \\ A_{lm}^{(\text{in})}(\rho-1)^{-i\omega/2} + A_{lm}^{(\text{out})}(\rho-1)^{i\omega/2} & \text{as} \quad \rho \to 1. \end{cases} \tag{2.43}$$

[6]Note that we need to take care of the signs to obtain a positive definite transmission coefficient.

[7]To get rid of L, we also make $t \to Lt$ and L is then just a constant scale factor in the metric. The AdS case may be recovered by letting $L \to iL$.

This basis represents the scattering between $\rho = \pm 1$, with an ingoing condition at $\rho = -1$. According to the asymptotic behaviour of the Legendre functions, we have (see, for example, [76])

$$A^{(\text{in})}_{lm} = \frac{\Gamma(1-i\omega)\Gamma(-i\omega)}{\Gamma(1+\beta-i\omega)\Gamma(-\beta-i\omega)}\,, \quad A^{(\text{out})}_{lm} = \frac{\Gamma(1-i\omega)\Gamma(i\omega)}{\Gamma(1+\beta)\Gamma(-\beta)} = \frac{\sinh(i\pi\beta)}{\sinh(\pi\omega)}\,. \tag{2.44}$$

The greybody factor for real frequencies is

$$\gamma_l(\omega) = 1 - \left|\frac{A^{(\text{out})}_{lm}}{A^{(\text{in})}_{lm}}\right|^2 = 1 - |\mathcal{R}|^2 = |\mathcal{T}|^2, \tag{2.45}$$

where $\mathcal{R},\mathcal{T}$ are the reflection and transmission coefficients, respectively. Using (2.44) and properties of the Gamma function, it is straightforward to see that

$$\gamma_l(\omega) = \frac{\sinh^2(\pi\omega)}{\cosh\pi(\lambda-\omega)\cosh\pi(\lambda+\omega)}. \tag{2.46}$$

This result appears even more clearly if we use the monodromy technique. The monodromy coefficients are given by

$$\alpha_\pm = \pm\frac{\omega}{2}\,, \quad \alpha_\infty = -\frac{i}{2} + \lambda\,, \tag{2.47}$$

and, choosing the monodromy matrices (2.35) to represent the scattering between $\rho = \pm 1$, we have that

$$M_+ = \begin{pmatrix} 0 & -1 \\ 1 & 2\cosh(2\pi\alpha_+) \end{pmatrix}, \qquad M_- = \begin{pmatrix} 2\cosh(2\pi\alpha_-) & e^{2\pi\alpha_\infty} \\ -e^{-2\pi\alpha_\infty} & 0 \end{pmatrix}. \tag{2.48}$$

Therefore, the transmission coefficient is given by (2.38)

$$\begin{aligned}
\gamma_l(\omega) &= \frac{\sinh(2\pi\alpha_+)\sinh(2\pi\alpha_-)}{\sinh\pi(-\alpha_+ + \alpha_- + \alpha_\infty)\sinh\pi(\alpha_+ - \alpha_- + \alpha_\infty)} \\
&= \frac{-\sinh^2(2\pi\alpha_+)}{\sinh\pi(\alpha_\infty - 2\alpha_+)\sinh\pi(2\alpha_+ + \alpha_\infty)} \\
&= \frac{-\sinh^2(\pi\omega)}{\sinh[\pi(\lambda-\omega) - i\frac{\pi}{2}]\sinh[\pi(\lambda+\omega) - i\frac{\pi}{2}]} \\
&= \frac{\sinh^2(\pi\omega)}{\cosh\pi(\lambda-\omega)\cosh\pi(\lambda+\omega)}\,,
\end{aligned}$$

which is exactly the same result as (2.46).

We can understand better the calculation above if we write

$$\Phi = \begin{pmatrix} u^{(\text{up})}_{lm}(\rho) & u^{(\text{in})}_{lm}(\rho) \\ (1-\rho^2)\partial_\rho u^{(\text{up})}_{lm}(\rho) & (1-\rho^2)\partial_\rho u^{(\text{in})}_{lm}(\rho) \end{pmatrix}. \tag{2.49}$$

We know that,

$$u_{lm}^{(\text{in})}(\rho) = \Gamma(1-\mu)P_\beta^\mu(-\rho)\,, \quad u_{lm}^{(\text{up})}(\rho) = \Gamma(1-\mu)P_\beta^\mu(\rho)\,, \tag{2.50}$$

where $P_\beta^\mu(\rho)$ is the Legendre function, so, near $\rho = -1$, we have that

$$\Phi \sim \begin{pmatrix} 2^{i\omega/2}A_{lm}^{(\text{out})} & 2^{-i\omega/2}A_{lm}^{(\text{in})} \\ -i\omega\, 2^{i\omega/2} & i\omega\, 2^{-i\omega/2} \end{pmatrix} \begin{pmatrix} (1+\rho)^{-i\omega/2} & 0 \\ 0 & (1+\rho)^{i\omega/2} \end{pmatrix} \begin{pmatrix} 1 & 1/A_{lm}^{(\text{out})} \\ 1 & 0 \end{pmatrix} \tag{2.51}$$

and, if we write $\Phi = \Phi_- g_-$, we obtain

$$g_- = \begin{pmatrix} 1 & 1/A_{lm}^{(\text{out})} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\sinh(\pi\omega)}{\sinh(i\pi\beta)} \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad M_- = \begin{pmatrix} e^{2\pi(\omega/2)} & 0 \\ 2\sinh(\pi\alpha_\infty) & e^{-2\pi(\omega/2)} \end{pmatrix}. \tag{2.52}$$

The same thing can be done near $\rho = +1$

$$g_+ = \begin{pmatrix} 0 & 1 \\ 1/A_{lm}^{(\text{out})} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\sinh(\pi\omega)}{\sinh(i\pi\beta)} & 1 \end{pmatrix} \quad \Rightarrow \quad M_+ = \begin{pmatrix} e^{-2\pi(\omega/2)} & 2\sinh(\pi\alpha_\infty) \\ 0 & e^{2\pi(\omega/2)} \end{pmatrix}. \tag{2.53}$$

Note that this basis is different from (2.48). Calculating the connection matrix as $\mathcal{M}_{-\to+} = g_- g_+^{-1}$, we obtain that

$$\begin{aligned} \gamma_l(\omega) &= \frac{\sinh^2 \pi\omega}{\sinh^2 \pi\omega - \sinh^2(i\pi\beta)} \\ &= \frac{\sinh^2 \pi\omega}{\sinh^2 \pi\omega + \cosh^2(\pi\lambda)} \\ &= \frac{\sinh^2(\pi\omega)}{\cosh\pi(\lambda-\omega)\cosh\pi(\lambda+\omega)}. \end{aligned} \tag{2.54}$$

This result shows that the hypergeometric class, with 3 singular points, has all its representations parameterized by the local monodromy coefficients. We shall see in the next chapter what happens in more general cases.

2.2.2 Kerr and Schwarzschild Black Holes

Castro *et al* have shown how to calculate scattering coefficients for Kerr and Schwarzschild black holes [39] and also the importance of these results to the Kerr/CFT correspondence [40]. The main motivation of our work is to extend their results to (A)dS asymptotics. Here we briefly review the results of [39] as a starting point to our considerations.

The Kerr metric in Boyer-Lindquist coordinates is given by

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left(adt - (r^2+a^2)d\phi\right)^2, \tag{2.55}$$

where

$$\Delta_r = r^2 - 2Mr + a^2 = (r-r_+)(r-r_-), \quad \Sigma = r^2 + a^2\cos^2\theta. \tag{2.56}$$

When $a = 0$, we recover the standard Schwarzschild metric (1.1). For a minimally coupled massless scalar field ψ, we use separation of variables

$$\psi(t,r,\theta,\phi) = e^{-i\omega t}e^{im\phi}S(\theta)R(r) \tag{2.57}$$

in the KG equation (2.5). After a convenient redefinition of the separation constant $C_\ell \to C_\ell + a^2\omega^2 - 2ma\omega$, the angular equation reduces to

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta}\right]S(\theta) = -C_\ell S(\theta)\,. \tag{2.58}$$

Its solutions are usually called *scalar spheroidal harmonics*, as they are the natural generalization of spherical harmonics. Its eigenvalues C_ℓ can be calculated perturbatively for low frequencies or numerically [77, 39]. On the other hand, the radial equation is given by

$$\left[\partial_r(\Delta_r\partial_r) + 2ma\omega - a^2\omega^2 + \frac{\left(\omega(r^2+a^2) - am\right)^2}{\Delta_r}\right]R(r) = C_\ell R(r). \tag{2.59}$$

Using that $r^2 + a^2 = \Delta_r + 2Mr$ and the residue theorem to expand terms into partial fractions, we can arrive at the radial equation given in [39]

$$\left[\partial_r(\Delta_r\partial_r) + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)} + (r^2 + 2M(r+2M))\omega^2\right]R(r) = C_\ell R(r)\,. \tag{2.60}$$

Both angular and radial equations are of *confluent Heun type* because both have 2 regular singular points and 1 irregular point [78, 73]. The radial equation, for example, has two regular points at $r = r_\pm$ and an irregular one at $r = \infty$. The Frobenius coefficients can be found by substitution of

$$R(r) \sim (r-r_\pm)^{i\alpha_\pm}[1 + O(r-r_\pm)] \tag{2.61}$$

into (2.59) or (2.60), giving

$$\alpha_\pm = \frac{\omega(r_\pm^2 + a^2) - am}{\Delta_r'(r_\pm)} = \pm\frac{2Mr_\pm\omega - am}{r_+ - r_-}. \tag{2.62}$$

Irregular singular points do not have convergent series representations because the recurrence relations cannot be solved in terms of an initial seed. This irregularity now complicates the direct application of the monodromy technique as we did above in the $(A)dS_2 \times S^2$ case. Instead of a Taylor expansion, we have to content ourselves with a Laurent expansion at $r = \infty$

$$\Phi_\infty = \Bigg(\sum_{n=-\infty}^{\infty} r^n \Phi_n^\infty \Bigg) \begin{pmatrix} r^{-i\alpha_{\rm irr}} & 0 \\ 0 & r^{i\alpha_{\rm irr}} \end{pmatrix}. \tag{2.63}$$

The monodromy of this solution is $e^{2\pi\alpha_{\rm irr}}$, but now we cannot calculate $\alpha_{\rm irr}$ just by inserting the expansion above in the ODE. However, there are formal asymptotic expansions, also called *Thomé solutions* [73], which can be used as WKB approximations near the irregular point (see Appendix B). In the Kerr case, $r = \infty$ is a rank-1 irregular point [73], thus the asymptotic expansion is given by

$$R(r) \sim e^{\mp i\omega r} r^{\mp i\lambda - 1}[1 + O(r^{-1})], \tag{2.64}$$

with $\lambda = 2M\omega$, obtained again after substitution into the radial equation (2.60). The fundamental matrix in this plane-wave basis is thus

$$\Phi_{\rm pw} = \Bigg(\sum_{n=0}^{\infty} r^{-n} \Phi_n^{\rm pw} \Bigg) \begin{pmatrix} e^{-i\omega r} r^{-i\lambda-1} & 0 \\ 0 & e^{i\omega r} r^{i\lambda-1} \end{pmatrix}. \tag{2.65}$$

The coefficient λ does not represent the *true* monodromy of $r = \infty$ because, as we remarked before, (2.65) does not truly represent the solution at this point; we have now the complication of *Stokes phenomena*, similarly to what happens in the Airy equation [76, 79]. This means that (2.65) is only valid on a wedge[8] around $r = \infty$ bounded by two Stokes rays with fixed $\arg(z)$. Crossing one Stokes ray gives a different asymptotic expansion, which is related to (2.65) by a *Stokes matrix*. Thus if $e^{2\pi i\Lambda_0}$ represent the monodromy matrix of the asymptotic expansion (2.65) and M_∞ represent the true monodromy associated to (2.63), it is possible to show that for a rank-1 irregular point

$$M_\infty = e^{2\pi\Lambda_0} S_{-1} S_0, \tag{2.66}$$

where S_i represent Stokes matrices

$$S_{-1} = \begin{pmatrix} 1 & 0 \\ C_{-1} & 1 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 1 & C_0 \\ 0 & 1 \end{pmatrix}, \tag{2.67}$$

[8] For a rank $R > 0$ irregular point, we have $2R$ wedges dividing the asymptotic region.

and the C_i are called *Stokes multipliers*. For more details, see [39]. From the trace of (2.66), we have that

$$\alpha_{\text{irr}} = \frac{1}{2\pi}\cosh^{-1}\left[\cosh(2\pi\lambda) + e^{2\pi\lambda}C_0C_{-1}/2\right]. \tag{2.68}$$

In other words, we have at first two options to calculate the true monodromy α_{irr}: perturbatively in $a\omega$ for low-frequencies or numerically by calculating the Stokes multipliers C_i. These two approaches are pursued in the appendices of [39].

At this point, we emphasize that when we have irregular points, series expansions are typically *not* good representations of analytic functions. If we had an integral representation for the confluent Heun equation, as we have for hypergeometric functions, the monodromy could be obtained directly from it! Unfortunately, there are no known integral representations in terms of elementary functions for non-trivial Heun functions. The common belief in the literature is that it is not possible to obtain simple representations in the general case for these special functions. Another approach to obtain good representations of irregular functions is *summation* of the asymptotic expansion, i.e., to find another function which matches the true function in some non-zero radius around the irregular point. The theory of *multisummability* actually can obtain holomorphic summations of asymptotic formal series which solve *any* ODE [80]. Albeit being a very interesting subject, we shall not delve into these matters as, in this thesis, we are avoiding irregular points by the introduction of a cosmological constant. On the other hand, one of our important conclusions here is that, in several cases, we might avoid completely to deal with irregular points and asymptotic expansions by *confluence* procedures to obtain the less general results. This means that we expect to recover the correct scattering coefficients when we take $\Lambda \to 0$ below[9].

The Schwarzschild case is obtained by making $a = 0$ in the formulas above. The radial equation is still of confluent Heun type with regular singular points at $r = 0$ and $r = 2M$ and one irregular at infinity. The monodromy coefficients are $\alpha_+ = 2M\omega$ and $\alpha_- = 0$. This latter case thus implies that $r = 0$ is a resonant singular point and a logarithm term must appear in the series expansion around it. Furthermore, while $\lambda = 2M\omega$ as in the Kerr case, α_{irr} is different and must also be numerically calculated using Stokes multipliers.

The monodromy analysis reviewed here can be applied to higher spin cases using the Teukolsky formalism [39], but for a matter of simplicity we shall not address this here. Notice that this has not been done yet in the literature using the monodromy approach.

[9] This is a work in progress with prof. Bruno Cunha.

2.3 Kerr-NUT-(A)dS Black Holes

The monodromy approach has proved very useful to obtain analytical information about black hole scattering data. We may wonder how much generalization of this approach can we make? What do we get for black holes in (A)dS spacetimes and higher-dimensional black holes?

An important property which simplified considerably the scattering analysis was the separability of KG equation into a radial and angular part. In the following, we review results from Frolov and Kubizňak [81, 82] showing that any D-dimensional spacetime possessing a *principal conformal Killing-Yano tensor* has a separable KG equation. In particular, spacetimes presenting this algebraic structure, for a convenient choice of coordinates, have a *Kerr-NUT-(A)dS metric*, which is the most general higher-dimensional black hole metric with spherical topology. Oota and Yasui have also shown that Dirac and gravitational perturbations are also separable for this class of spacetimes [83, 84].

In the following, we apply the monodromy approach to scattering of scalar massless fields by rotating black holes in (A)dS spacetimes, specifically to $D = 4$ Kerr-NUT-(A)dS black holes. We will find an important complication for the direct use of this technique with respect to the monodromy group which appears in this case. To overcome this difficulty, we will use the theory of isomonodromic perturbations and Painlevé VI asymptotics, presented only in the next chapter. This section consists mostly of our original work[10] on the aforementioned subject presented in [38] and also some unpublished results on the numerical computation of (A)dS spheroidal harmonics eigenvalues.

2.3.1 Killing-Yano Tensors and Separability

In $D = 4$ spacetime dimensions, Teukolsky [58] has shown, using the Newman-Penrose formalism, that Kerr field equations are separable for any spin $s \leq 2$ perturbations. This important property comes about because of the existence of a *hidden* Killing tensor, that is, an object which does not comes about naturally from the isometries of the Kerr metric. Furthermore, this Killing tensor can be shown to be the *square* of a Killing-Yano tensor.

For the reader's sake, we remind that if a D-dimensional spacetime has a *Killing vector* ξ^a, then the metric g_{ab} is invariant under isometries generated by this vector field, i.e., the Lie

[10] We notice that our approach is very similar to the one in [41] for the spin zero case, but just slightly more general as we calculate with non-minimal coupling with the metric and also allow for a NUT charge. Electric and magnetic charges could also be included in the scalar case.

derivative of the metric is zero[11], $\mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)} = 0$. Moreover, for a geodesic field u^a, the quantity $P = \xi_a u^a$ is conserved along geodesics generated by u^a. A symmetric generalization of a Killing vector is a *Killing tensor* K_{ab}, which, given that

$$P_K = K_{ab}u^a u^b \tag{2.69}$$

is a conserved quantity on geodesics, we can show using the geodesic equation that

$$\nabla_{(c}K_{ab)} = 0. \tag{2.70}$$

A *Killing-Yano tensor* (KY) f_{ab}, on the other hand, is an antisymmetric generalization of a Killing vector[12]. If $P_a = f_{ab}u^b$ is parallel propagated along the u^a geodesic, that is, if $u^c\nabla_c P_a = 0$, then we can show that

$$\nabla_{(c}f_{a)b} = 0. \tag{2.71}$$

The interesting thing about KY tensors is that we can always construct a Killing tensor from it

$$K_{ab} = f_{ac}f^c{}_b. \tag{2.72}$$

It is also interesting to us study *conformal Killing vectors* ξ^a, generators of conformal symmetries $\mathcal{L}_\xi g_{ab} = \Omega(x)g_{ab}$, i.e., $2\nabla_{(a}\xi_{b)} = \Omega g_{ab}$. Taking the trace of this equation shows that $\Omega = \nabla_a\xi^a/D$. In this case, $P = \xi_a u^a$ is conserved only for null geodesics. We can thus define a *conformal Killing tensor* K_{ab} if it obeys

$$\nabla_{(c}K_{ab)} = g_{(ca}\tilde{K}_{b)}, \quad \tilde{K}_b = \frac{2}{D+2}(\nabla^a K_{ab} + \frac{1}{2}\nabla_b K), \tag{2.73}$$

where (2.69) is conserved only for null geodesics. Finally, a *conformal Killing-Yano tensor* (CKY) is an antisymmetric h_{ab} satisfying

$$\nabla_{(a}h_{b)c} = g_{ab}\eta_c - g_{c(a}\eta_{b)}, \quad \eta_a = \frac{1}{D-1}\nabla^b h_{ab}. \tag{2.74}$$

Of course, $K_{ab} = h_{ac}h^c{}_b$ is a conformal Killing tensor in this case. All tensorial definitions above admit higher order versions [85].

A 2-form h_{ab} is a principal conformal Killing-Yano tensor (PCKY) if it is a closed and non-degenerate conformal Killing-Yano tensor (CKY). Consider a spacetime (M, g) with $D = 2n+\varepsilon$

[11]For a tensor of any order, we use parenthesis to denote indices symmetrization, i.e., $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$, and brackets to denote antisymmetrization, i.e., $A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$, where we always divide by the number of permutations of the indices.

[12]This means that f is a totally antisymmetric tensor.

dimensions allowing a PCKY, where $\varepsilon = 0, 1$, to distinguish between even and odd dimensions. The existence of such structure results into a tower of $n-1$ Killing-Yano tensors, which implies n Killing tensors if we include the metric tensor. Those Killing tensors can then be used to construct $n+\varepsilon$ commuting Killing vectors. Thus a spacetime with a PCKY has $D = 2n+\varepsilon$ conserved quantities. This is sufficient for integration of the geodesic equation (1.3), but it is also enough for complete separability of Klein-Gordon, Dirac and gravitational perturbation equations [81, 85].

Following [81], we can choose canonical coordinates $\{\psi_i, x_\mu\}$, where ψ_k, $k = 1, \ldots, n-1+\epsilon$ correspond to Killing vector affine parameters and x_μ, $\mu = 1, \ldots, n$ are the PCKY eigenvalues. For us, ψ_0 is the time coordinate, ψ_k are azimuthal coordinates and x_μ stand for radial and latitude coordinates. In such coordinates, the generic metric of (M, g) which allows for a PCKY can be written as

$$ds^2 = \sum_{\mu=1}^{n}\left[\frac{dx_\mu^2}{Q_\mu} + Q_\mu\left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k\right)^2 - \frac{\epsilon c}{A^{(n)}}\left(\sum_{k=0}^{n-1} A^{(k)} d\psi_k\right)^2\right] \tag{2.75}$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad A_\mu^{(j)} = \sum_{\substack{\nu_1<\cdots<\nu_j \\ \nu_i \neq \mu}} x_{\nu_1}^2 \ldots x_{\nu_j}^2, \quad A^{(j)} = \sum_{\nu_1<\cdots<\nu_j} x_{\nu_1}^2 \ldots x_{\nu_j}^2, \tag{2.76}$$

$$U_\mu = \prod_{\nu\neq\mu}(x_\nu^2 - x_\mu^2), \quad X_\mu = \sum_{k=\epsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\epsilon} + \frac{\epsilon c}{x_\mu^2}. \tag{2.77}$$

The polynomial X_μ is obtained by substituting the metric (2.75) into the D-dimensional Einstein equations. The metric with proper signature is recovered when we set $r = -ix_n$ and the mass parameter $M = (-i)^{1+\epsilon} b_n$. This metric is one of the possible forms of the Kerr-NUT-(A)dS metric described in [49].

As we mentioned above, one of the most interesting properties of the Kerr-NUT-(A)dS metric is separability of field equations. Consider the massive Klein-Gordon equation

$$(\Box - m^2)\Phi = 0. \tag{2.78}$$

Its solution can be decomposed as

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu) \prod_{k=0}^{n+\epsilon-1} e^{i\Psi_k \psi_k} \tag{2.79}$$

and substitution in (2.78) gives

$$(X_\mu R'_\mu)' + \epsilon \frac{X_\mu}{x_\mu} R'_\mu + \left(V_\mu - \frac{W_\mu^2}{X_\mu} \right) R_\mu = 0, \tag{2.80}$$

where

$$W_\mu = \sum_{k=0}^{n+\epsilon-1} \Psi_k (-x_\mu^2)^{n-1-k}, \quad V_\mu = \sum_{k=0}^{n+\epsilon-1} \kappa_k (-x_\mu^2)^{n-1-k}, \tag{2.81}$$

and κ_k and Ψ_k are separation constants. For more details, see, for example, [85].

We shall focus in the $D = 4$ case for the rest of the thesis. In this case, we choose coordinates $(x_1, x_2, \psi_0, \psi_1)$, where x^μ, $\mu = 1,2$, represent the PCKY eigenvalues and ψ_i, $i = 0,1$, are the Killing parameters of the 2 associated Killing vectors. Now if we set $(x_1, x_2, \psi_0, \psi_1) \equiv (p, ir, t, \phi)$, the metric (2.75) is written as

$$ds^2 = -\frac{Q(r)}{r^2+p^2}(dt + p^2 d\phi)^2 + \frac{r^2+p^2}{Q(r)} dr^2 + \frac{P(p)}{r^2+p^2}(dt - r^2 d\phi)^2 + \frac{r^2+p^2}{P(p)} dp^2 , \tag{2.82}$$

where $P(p)$ and $Q(r)$ are 4^{th} order polynomials given by [86]

$$P(p) = -\frac{\Lambda}{3} p^4 - \epsilon p^2 + 2np + k, \tag{2.83a}$$

$$Q(r) = -\frac{\Lambda}{3} r^4 + \epsilon r^2 - 2Mr + k, \tag{2.83b}$$

$$\epsilon = 1 - (a^2 + 6b^2)\frac{\Lambda}{3} , \quad k = (a^2 - b^2)(1 - b^2 \Lambda), \quad n = b\left[1 + (a^2 - 4b^2)\frac{\Lambda}{3}\right]. \tag{2.83c}$$

The parameters are the black hole mass M, angular momentum to mass ratio a, cosmological constant Λ, and the NUT parameter b. To make contact with the physically meaningful Kerr-NUT-(A)dS metric, we set

$$p = b + a\cos\theta, \quad \chi^2 = 1 + \Lambda a^2/3, \tag{2.84}$$

and make the substitution $\phi \to \phi / a\chi^2$ and $t \to (t - \frac{(a+b)^2}{a}\phi)/\chi^2$, in this order. If we set $b = 0$ after this, we have the usual Kerr-(A)dS metric in Chambers-Moss coordinates [61, 86]. If we further set $\Lambda = 0$, we obtain the Kerr metric in Boyer-Lindquist coordinates (2.55).

After these changes of coordinates, the Kerr-NUT-(A)dS metric can be written as [86]

$$\begin{aligned} ds^2 = &-\frac{Q}{\rho^2\chi^4}\left[dt - (a\sin^2\theta + 4b\sin^2\tfrac{\theta}{2})d\phi\right]^2 + \frac{\rho^2}{Q}dr^2 \\ &+ \frac{\tilde{P}\sin^2\theta}{\rho^2\chi^4}\left(adt - (r^2 + (a+b)^2)d\phi\right)^2 + \frac{\rho^2}{\tilde{P}}d\theta^2, \end{aligned} \tag{2.85}$$

and if we set the dS radius $L^2 = 3/\Lambda$, we have

$$\tilde{P}(\theta) = 1 + \frac{4ab}{L^2}\cos\theta + \frac{a^2}{L^2}\cos^2\theta, \quad \rho^2 = r^2 + (a+b)^2\cos^2\theta. \tag{2.86}$$

In the rest of this thesis, we are going to dismiss the discussion with a NUT charge and set $b = 0$ in the following. However, here we briefly mention the gravitational interpretation of the NUT charge.

2.3.2 Physical Interpretation of NUT Charge

As discussed in [87], the Kerr-NUT black hole ($\Lambda = 0$ above) has an intricate global structure. There are two types of singularities, coordinate singularities and determinant singularities. At $Q = 0$ or $\rho = 0$ we have coordinate singularities, and at $\theta = 0, \pi$, the metric determinant vanishes and we have the other type of singularity. As we have seen before, the roots of $Q = 0$ are just apparent singularities and can be removed by change of coordinates. The roots of $\rho = 0$ now have two possibilities: when $a^2 < b^2$, there is no curvature singularity at this point, while in the other case, we have two values of θ which are singular. For $b = 0$, the determinant singularities would be the usual chart failure in the poles of a 2-sphere. When $b \neq 0$, these correspond to degeneracies of spherical coordinates in a 3-sphere. This imposes the identification

$$(\phi, t) \to (\phi + (n_1 + n_2)2\pi, t + (n_1 - n_2)4\pi b), \quad (n_1, n_2) \in \mathbb{Z}. \tag{2.87}$$

In the region $r > r_+$, the time periodicity implies the existence of closed timelike curves. This also happens for Kerr-NUT-(A)dS metrics for radii larger than the larger root of Q [88].

We did not find a more detailed analysis of the global structure of Kerr-NUT-(A)dS metrics as done in [87]. For example, in the interior region of Kerr-NUT spacetime, $r_- < r < r_+$, there are well-defined timelike orbits and people give a name to this region of its own, the Kerr-Taub space. The Kerr-Taub space can be interpreted as a closed, inhomogeneous electromagnetic-gravitational wave undergoing gravitational collapse [87]. Another interesting thing mentioned in [87] is that there is another possible identification of coordinates, due to Bonnor [89], which avoids closed timelike curves in the exterior region. In this identification, the Kerr-NUT spacetime is seen as a superposition of a Kerr black hole and a massless source of angular momentum located at $\theta = \pi$. This suggests that the dual CFT interpretation of the NUT charge is related to some type of rotation of the dual plasma [48]. For more details about the NUT charge and recent applications, see [90, 91, 92].

2.3.3 (A)dS Spheroidal Harmonics

As before, the KG equation is separated into an angular and a radial part. First, we shall address the angular equation, which is obtained via (2.80) setting $\mu = 1$. For Kerr-NUT-(A)dS metric in Chambers-Moss coordinates, we have

$$\partial_p(P(p)\partial_p S(p)) + \left(-4\Lambda\xi p^2 - \frac{(\Psi_0 p^2 - \Psi_1)^2}{P(p)}\right) S(p) = -C_\ell S(p)\,,$$
$$\Psi_0 = \omega\chi^2, \quad \Psi_1 = \left(\omega(a+b)^2 - am\right)\chi^2\,. \tag{2.88}$$

Our interest in this equation is to obtain its eigenvalues C_ℓ, which are critical in the determination of the monodromy group of the radial equation, as we shall see below. As there is no analytical approach available, here we obtain the eigenvalues numerically for several values of (ω, m). Giammatteo and Moss [37] have also noticed that Padé approximants give a very good approximation for C_ℓ as a function of ω in the Kerr-AdS case, but this approach also relies on numerics.

Now let us specialize to the Kerr-(A)dS case (b=0). In this case, $n = 0$, thus

$$P(p) = (a^2 - p^2)(1 - L^{-2}p^2) = L^{-2}(p^2 - a^2)(p^2 - L^2), \tag{2.89}$$

where $L^2 = -3/\Lambda$ is the $(A)dS$ radius. The resulting angular equation is

$$\partial_p(P(p)\partial_p S) + \left(-4\Lambda\xi p^2 - \frac{\chi^4}{P(p)}\left[(a^2 - p^2)\omega - ma\right]^2\right) S = -C_\ell S, \tag{2.90}$$

where $\chi^2 = (1 - a^2/L^2)$. Equation (2.90) has 5 regular singular points at $p = \{\pm a, \pm L, \infty\}$ and the local coefficients are given by

$$\theta_{\pm a} = \mp m, \quad \theta_{\pm L} = \pm L\left(\omega\chi^2 + maL^{-2}\right) \tag{2.91}$$

and

$$\theta_\infty = \sqrt{9 - 48\xi}. \tag{2.92}$$

The first important thing to notice about (2.90) is that its coefficients are all even functions of p. This suggests a reduction of singular points using the quadratic change of variables $x = p^2$, resulting in

$$\partial_x^2 S + \left(\frac{1/2}{x} + \frac{1}{x - a^2} + \frac{1}{x - L^2}\right)\partial_x S + \\ + \frac{1}{4xP(x)}\left(-4\Lambda\xi x + C_\ell - \frac{\chi^4}{P(x)}\left[(a^2 - x)\omega - ma\right]^2\right) S = 0, \tag{2.93}$$

with $P(x) = \alpha^2(x-a^2)(x-L^2)$. Now (2.93) has only 4 regular singular points at $x = \{0, a^2, L^2, \infty\}$. The point $x = 0$ has $\theta_0 = 1/2$ and we know that $x = p^2 = 0$ is actually a regular point of $S(p)$, as it is not a singular point of (2.90). This singularity appears because of the branching structure of the square root and is usually called an *elementary singularity* as it can always be removed by a quadratic transformation [93, 73]. A quadratic transformation cuts the local coefficients by half, therefore

$$\theta_\infty = \frac{1}{2}\sqrt{9-48\xi}, \tag{2.94}$$

and, if $\xi = 1/6$, we also have $\theta_\infty = 1/2$. This allows for a further reduction of (2.90) to a *Magnus-Winkler-Ince* equation (see Appendix C.3). However, this is not very useful for our purposes.

There is a more interesting reduction of (2.90) which makes it possible to connect with other spheroidal wave equations. First, we set $p = a\cos\theta$ and obtain

$$P(\theta) = a^2\sin^2\theta(\chi^2 + (1-\chi^2)\sin^2\theta). \tag{2.95}$$

After redefining $C_\ell \to C_\ell + \chi^2(a^2\omega^2 - 2ma\omega)$ in (2.90), we arrive at the following angular equation

$$\begin{aligned} &\frac{1}{\sin\theta}\partial_\theta[\sin\theta(\chi^2+(1-\chi^2)\sin^2\theta)\partial_\theta S(\theta)] \\ &\quad+\Bigg(12\xi(1-\chi^2)\cos^2\theta + a^2\omega^2\chi^2(\chi^2\cos^2\theta+(1-\chi^2)\sin^2\theta) \\ &\quad-2ma\omega\chi^2(1-\chi^2)\sin^2\theta - \frac{m^2\chi^4}{\sin^2\theta(\chi^2+(1-\chi^2)\sin^2\theta)}\Bigg)S(\theta) = -C_\ell S(\theta). \end{aligned} \tag{2.96}$$

Note that the limit $\chi^2 \to 1$ corresponds to the Kerr angular equation (2.58). Now if we make $u = \cos\theta$, we arrive at a simplified form of (2.96)

$$\partial_u(1-u^2)(1-\hat{a}^2u^2)\partial_u S(u) + \left(Au^2 + B + C_\ell - \frac{m^2(1-\hat{a}^2)^2}{(1-u^2)(1-\hat{a}^2u^2)}\right)S(u) = 0, \tag{2.97}$$

$$A = 12\xi\hat{a}^2 + (a\omega)^2(1-\hat{a}^2)(1-2\hat{a}^2) + 2ma\omega\hat{a}^2(1-\hat{a}^2),$$

$$B = a\omega(a\omega - 2m)\hat{a}^2(1-\hat{a}^2),$$

where $\hat{a} = a/L$ is the rotation parameter in units of the AdS radius. This equation has 5 regular singular points at $u = \pm 1, u = \pm\hat{a}^{-1}$ and $u = \infty$. We call solutions of (2.97) *scalar (A)dS spheroidal harmonics*. This is appropriate because we recover the scalar spheroidal harmonics equation by making $\hat{a} = 0$ in (2.97). The dS case is obtained just by making $L \to iL$.

In order to study scattering coefficients in Kerr-(A)dS, we need the eigenvalues C_ℓ coming from regularity conditions on solutions of (2.97). Here we restrict to the case $\xi = 1/6$ because it is more interesting to us. A very simple and direct way to calculate the eigenvalues is to use Leaver's continued-fraction method [77, 42]. First, we reduce the number of singularities to 4 by a quadratic transformation $z = u^2$. If we also let

$$S(z) = (z-1)^{m/2}(z-\hat{a}^{-2})^{m\hat{a}/2}\hat{S}(z), \tag{2.98}$$

we put the angular equation into a Heun form

$$\partial_z^2\hat{S} + \left(\frac{1/2}{z} + \frac{1+m}{z-1} + \frac{1+m\hat{a}}{z-\hat{a}^{-2}}\right)\partial_z\hat{S} + \left[\frac{\hat{A}_1}{z} + \frac{\hat{A}_2}{z-1} + \frac{\hat{A}_3}{z-\hat{a}^{-2}}\right]\hat{S} = 0, \tag{2.99}$$

with

$$\hat{A}_1 = \frac{1}{4}\left[\left(\hat{a}^2-1\right)\left(m^2-\hat{a}^2(m+c)^2\right) - m\left(\hat{a}^3+1\right) + C_\ell\right],$$

$$\hat{A}_2 = \frac{1}{4(\hat{a}^2-1)}\left[c^2\left(\hat{a}^2-1\right)^2 + m(m+1)\left(\hat{a}^2(2\hat{a}+3)-1\right) + 2\hat{a}^2 + C_\ell\right],$$

$$\hat{A}_3 = \frac{1}{4(\hat{a}^2-1)}\left[c^2\left(\hat{a}^2-1\right)^3 + 2c\hat{a}^2\left(\hat{a}^2-1\right)^2 m + \hat{a}^2\left((\hat{a}-2)(\hat{a}+1)^2 m(m\hat{a}+1) - C_\ell - 2\right)\right],$$

where $c \equiv a\omega$ and $\sum_{i=1}^{3} A_i = 0$. If we take $\hat{a} \to 0$ in (2.99), we get a reduced confluent Heun equation

$$\partial_z^2\hat{S} + \left(\frac{1/2}{z} + \frac{1+m}{z-1}\right)\partial_z\hat{S} + \left[\frac{\hat{B}_1}{z} + \frac{\hat{B}_2}{z-1}\right]\hat{S} = 0, \tag{2.100}$$

with

$$B_1 = \frac{1}{4}[C_\ell - m(m+1)], \qquad B_2 = -\frac{1}{4}[C_\ell - m(m+1) + c^2]. \tag{2.101}$$

This equation is a reduction obtained from the Kerr angular equation doing a similar quadratic and a homotopic transformation as above. The eigenvalues of (2.100) can be obtained by Leaver's method so we can use these as seeds for (2.99), in the same way as spherical harmonics eigenvalues are used as seeds for the Kerr case. In fact, the eigenvalues of (2.100) are equivalent to the Kerr angular equation in oblate spheroidal harmonics form, as we can see in the plot below of C_ℓ as a function of $c = a\omega$. Compare with the values obtained in [77].

Given a Heun equation in canonical form

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)y' + \frac{\alpha_+\alpha_- z - q}{z(z-1)(z-t)}y = 0, \tag{2.102}$$

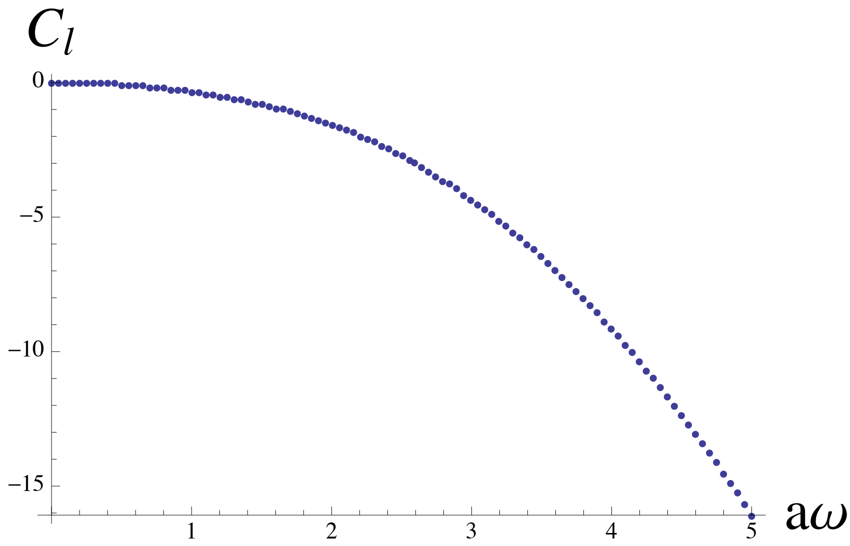

Figure 2.3: Kerr angular eigenvalues for $\ell = m = 0$ obtained from eq. (2.100)

the three-term recurrence relation resulting from substitution of $y(z) = \sum_{n=0}^{\infty} g_n z^n$ is given by

$$-(Q_0 + q)g_0 + R_0 g_1 = 0, \tag{2.103a}$$

$$P_n g_{n-1} - (Q_n + q)g_n + R_n g_{n+1} = 0, \quad (n > 0) \tag{2.103b}$$

with

$$\begin{aligned} P_n &= (n-1+\alpha_+)(n-1+\alpha_-), \\ Q_n &= n((t+1)(n-1+\gamma)+t\delta+\epsilon), \\ R_n &= t(n+1)(n+\gamma). \end{aligned} \tag{2.104}$$

This is justified because we want augmented convergence at $z = 1$ and the series at $z = 0$ has convergence radius one.

For (2.99) we have that

$$\gamma = 1/2, \quad \delta = 1+m, \quad \epsilon = 1+m\hat{a}, \quad t = \hat{a}^{-2},$$
$$q = -d^{-2}\hat{A}_1, \quad \alpha_+\alpha_- = \hat{A}_2 + \hat{a}^{-2}\hat{A}_3$$

and, using Fuchs relation $\gamma+\delta+\epsilon = \alpha_+ + \alpha_- + 1$ and the expressions above, we have

$$\alpha_\pm = \frac{1}{2}\left(\frac{3}{2}+(1+d)m\right) \pm \frac{1}{2}\sqrt{\left(\frac{3}{2}+(1+d)m\right)^2 - 4(\hat{A}_2+\hat{a}^{-2}\hat{A}_3)}. \tag{2.105}$$

We notice that (2.103) diverges if we make simply $\hat{a} \to 0$. However, if we multiply (2.103) by $\hat{a}^2$, we have a well-defined confluence to the (2.100) recurrence relations. We made a *Mathe-*

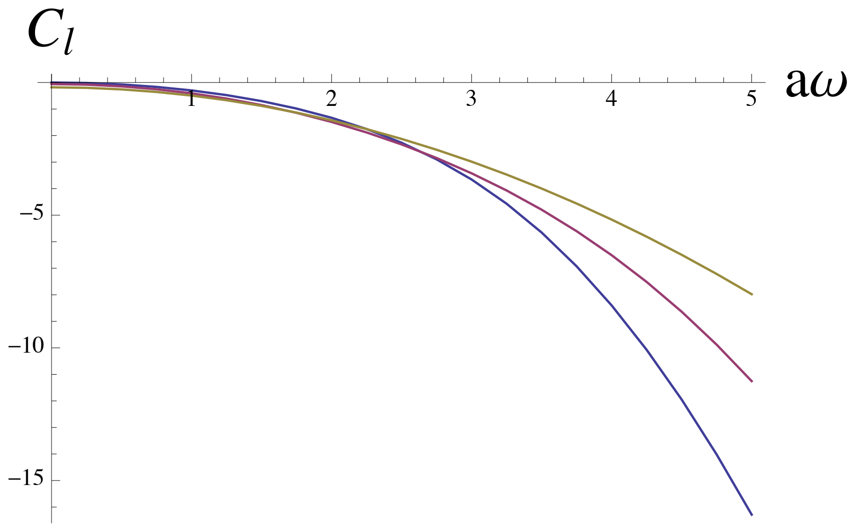

Figure 2.4: Kerr-AdS angular eigenvalues $l = m = 0$ as a function of $a\omega$ for $\hat{a} = 0$ (blue), $\hat{a} = 0.3$ (magenta) and $\hat{a} = 0.8$ (yellow).

matica notebook[13] to numerically calculate angular eigenvalues for $l = m = 0$. In figure 2.5, we see the behaviour of C_ℓ as a function of $\hat{a} \in (0, 1)$, as we increase $a\omega$.

2.3.4 Kerr-NUT-(A)dS case

Let $\psi(t, \phi, r, \theta) = e^{-i\omega t} e^{im\phi} R(r) S(\theta)$ be a solution of the Klein-Gordon equation for $D = 4$ Kerr-NUT-(A)dS in Chambers-Moss coordinates. The radial equation resulting from this solution is

$$\partial_r(Q(r)\partial_r R(r)) + \left(V_r(r) + \frac{W_r^2}{Q(r)}\right) R(r) = 0\,, \tag{2.106}$$

where

$$Q(r) = -\frac{\Lambda}{3} r^4 + \epsilon r^2 - 2Mr + k, \tag{2.107a}$$

$$\epsilon = 1 - (a^2 + 6b^2)\frac{\Lambda}{3}\,, \quad k = (a^2 - b^2)(1 - b^2\Lambda), \tag{2.107b}$$

[13]Our algorithm is still not working very well for values of $l \neq m$.

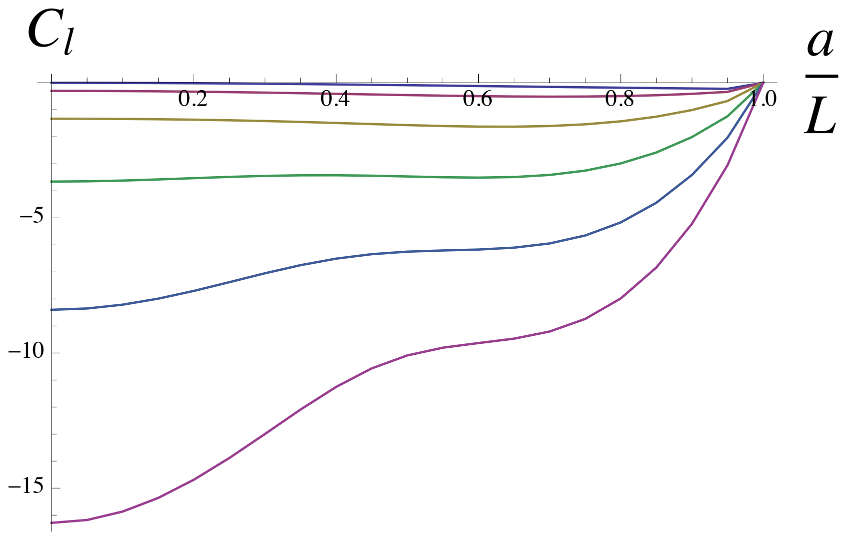

Figure 2.5: Kerr-AdS angular eigenvalues $l = m = 0$ as a function of $\hat{a}$ as we increase $a\omega$ (first blue line near the axis) from zero up to $a\omega = 5$ (purple line starting at $C_\ell \approx -16$).

and

$$V_r = \kappa_0 r^2 + \kappa_1, \quad W_r = \Psi_0 r^2 + \Psi_1, \tag{2.108a}$$

$$\kappa_0 = -4\Lambda\xi, \quad \kappa_1 = -C_l, \tag{2.108b}$$

$$\Psi_0 = \omega\left(1 + \frac{\Lambda a^2}{3}\right), \quad \Psi_1 = a\left(\omega\frac{(a+b)^2}{a} - m\right)\left(1 + \frac{\Lambda a^2}{3}\right). \tag{2.108c}$$

The parameter ξ is the coupling constant between the scalar field and the Ricci scalar. Typical values of the parameter ξ are minimal coupling $\xi = 0$ and conformal coupling $\xi = 1/6$. The separation constant between the angular and radial equations is C_ℓ. The angular equation has essentially the same form as the radial one, associated to the problem of finding the eigenvalues of a second order differential operator with four regular singular points, in which case correspond to unphysical values for the latitude coordinates.

In the following, we assume that all roots of $Q(r)$ are distinct and there are two real roots at least. When $\Lambda \to 0$, two of those roots match the Kerr horizons (r_+, r_-) and the other two diverge, leaving us with an irregular singular point of index 1 at infinity (and, therefore, a confluent Heun equation [39]). The characteristic coefficients – solutions for the indicial equations – of the finite singularities r_i are

$$\rho_i^\pm = \pm i\left(\frac{\Psi_0 r_i^2 + \Psi_1}{Q'(r_i)}\right), \quad i = 1, \ldots, 4 \tag{2.109}$$

and for $r = \infty$ we have

$$\rho_\infty^\pm = \frac{3}{2} \pm \frac{1}{2}\sqrt{9-48\xi}. \tag{2.110}$$

These coefficients give the local asymptotic behaviour of waves approaching any of the singular points, for example, one of the black hole horizons.

In this form, equation (2.106) has 5 regular singular points, including the point at infinity. It is possible to show that the point at infinity is actually an apparent singularity when $\xi = 1/6$ and can be further removed by a gauge transformation. In that case, (2.106) can be cast into a Heun type equation with 4 regular singular points given by the roots of $Q(r) = -\frac{\Lambda}{3}\prod_{i=1}^{4}(r-r_i)$. This is done in the following. An equivalent result has been reported by [41] for massless perturbations of spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ for Kerr-(A)dS (the so-called Teukolsky master equation) and for $s = 0, \frac{1}{2}$ for Kerr-Newman-(A)dS. One can show that Teukolsky master equation reduces to conformally coupled Klein-Gordon equation for scalar perturbations, being those perturbations of the Weyl tensor. Our computation below show this for the spin zero case, because of the explicit non-minimal coupling, and can be straightforwardly extended for higher spin cases as done in [41]. In conclusion, our result is not new, but our approach shows that the conformal coupling is the origin of the triviality of $r = \infty$. The reduction to Heun has also been shown true for $s = \frac{1}{2}, 1, 2$ perturbations of all type-D metrics with cosmological constant[14] [94].

2.3.5 Heun equation for Conformally Coupled Kerr-NUT-(A)dS

For $\xi = 1/6$, it is possible to transform (2.106) with 5 regular singular points into a Heun equation with only 4 regular points. This is because $r = \infty$ in (2.106) becomes a removable singularity. In this section, we apply the transformations used in [94] for a scalar field, adapting the notation for our purposes[15], and we also calculate the difference between characteristic exponents, θ_k, for each canonical form we obtain. As it turns out, these exponents are more useful for us because they are invariant under translations $z \to az + b$ and homotopic transformations [73], which preserve the monodromy properties of a ODE, as will be seen in section 4.

By making the homographic transformation

$$z = \frac{r-r_1}{r-r_4}\frac{r_2-r_4}{r_2-r_1}\,, \tag{2.111}$$

[14] Notice that [94] does not refer explicitly to the scalar case in their paper. However, our eq. (2.106) with $\xi = 1/6$ can be obtained just by setting $s = 0$ in eq. (11) of [94], so their result is also equivalent to ours.

[15] With respect to the parameters of [94], we must set $a = 0$ and, in a non-trivial change, their term $2g_4w^2$ must be equated to $-4\Lambda\xi r^2$ to obtain the non-minimally coupled case.

we map the singular points as

$$(r_1, r_2, r_3, r_4, \infty) \quad \mapsto \quad (0, 1, t_0, \infty, z_\infty) \tag{2.112}$$

with

$$z_\infty = \frac{r_2 - r_4}{r_2 - r_1}\,, \quad t_0 = \frac{r_3 - r_1}{r_3 - r_4} z_\infty. \tag{2.113}$$

Typically we set the relevant points for the scattering problem to $z = 0$ and $z = 1$, but we can consistently choose any two points to study. We note at this point that, for the de Sitter case, t_0 is a real number, which can be taken to be between 1 and ∞, whereas for the anti-de Sitter case, it is a pure phase $|t_0| = 1$. Now, we define

$$\sigma_\pm(r) \equiv \Psi_0 \, r \pm \Psi_1\,, \tag{2.114}$$

$$f(r) \equiv 4\xi\Lambda r^2 + C_l\,, \tag{2.115}$$

and

$$d_k^{-1} \equiv -\frac{\Lambda}{3} \prod_{\substack{j=1 \\ j\neq k}}^{3} (r_k - r_j) = \frac{Q'(r_k)}{r_k - r_4}. \tag{2.116}$$

Then, eq. (2.106) transforms to

$$\frac{d^2R}{dz^2} + p(z)\frac{dR}{dz} + q(z)R = 0, \tag{2.117a}$$

$$\begin{aligned} p(z) &= \frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-t_0} - \frac{2}{z-z_\infty}, \\ q(z) &= \frac{F_1}{z^2} + \frac{F_2}{(z-1)^2} + \frac{F_3}{(z-t_0)^2} + \frac{12\xi}{(z-z_\infty)^2} + \frac{E_1}{z} + \frac{E_2}{z-1} + \frac{E_3}{z-t_0} + \frac{E_\infty}{z-z_\infty}, \end{aligned} \tag{2.117b}$$

where

$$F_k = \left(\frac{d_k\sigma_+(r_k^2)}{r_k - r_4}\right)^2 = \left(\frac{\Psi_0 r_k^2 + \Psi_1}{Q'(r_k)}\right)^2, \tag{2.118a}$$

$$E_\infty = \frac{12\xi}{z_\infty(r_4 - r_1)}\left(\sum_{k=1}^{3} r_k - r_4\right), \tag{2.118b}$$

$$E_k = \frac{d_k}{z_k - z_\infty}\left\{f(r_k) - \frac{2\Lambda}{3}\frac{d_k^2}{r_k - r_4}\sigma_+(r_k^2)\left[\sigma_-(r_k^2)\sum_{j\neq k}^{3} r_j - 2r_k\sigma_-\left(\prod_{j\neq k}^{3} r_j\right)\right]\right\}. \tag{2.118c}$$

The θ_k for the finite singularities $z_k = \{0, 1, t_0\}$ can be obtained by plugging $R(z) \sim (z - z_k)^{\theta_k/2}$ into (2.117)

$$\theta_k = 2\sqrt{-F_k} = 2i\left(\frac{\Psi_0 r_k^2 + \Psi_1}{Q'(r_k)}\right), \quad (k = 1, 2, 3) \tag{2.119}$$

and for the singularity at $z = \infty$,

$$\theta_\infty = 2i\sqrt{12\xi + E_2 + t_0 E_3 + z_\infty E_\infty - \sum_{k=1}^{3} \frac{\theta_k^2}{4}}. \tag{2.120}$$

It is possible to show that θ_∞ does not depend on C_ℓ. Finally, for $z = z_\infty$ we have

$$\theta_{z_\infty} = \sqrt{9 - 48\xi}. \tag{2.121}$$

When the difference of any two characteristic exponents is an integer, we have a *resonant* singularity. This happens in (2.121) for $\xi = \{0, 5/48, 1/6, 3/16\}$. Thus, we have a logarithmic behaviour near z_∞, except for $\xi = 1/6$ because, in this case, it is also a removable singularity, as will be seen briefly. The property of being removable only happens if θ_{z_∞} is an integer different from zero. If $\theta_{z_\infty} = 0$, we always have a logarithmic singularity. For more on this subject, see [74, 73, 95].

To finish this section, we now show that (2.117) can be transformed into a Heun equation when $\xi = 1/6$. First, we make the homotopic transformation

$$R(z) = z^{-\theta_0/2}(z-1)^{-\theta_1/2}(z-t_0)^{-\theta_t/2}(z-z_\infty)^{\beta} y(z). \tag{2.122}$$

The transformed ODE is now given by

$$\frac{d^2y}{dz^2} + \hat{p}(z)\frac{dy}{dz} + \hat{q}(z)y = 0, \tag{2.123}$$

where

$$\hat{p}(z) = \frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t_0} + \frac{2\beta-2}{z-z_\infty}, \tag{2.124}$$

$$\hat{q}(z) = \frac{\hat{E}_1}{z} + \frac{\hat{E}_2}{z-1} + \frac{\hat{E}_3}{z-t_0} + \frac{\hat{E}_\infty}{z-z_\infty} + \frac{\hat{F}_\infty}{(z-z_\infty)^2}, \tag{2.125}$$

with

$$\hat{E}_k = \frac{d_k}{z_k - z_\infty} f(r_k) + \sum_{j\neq k}^{3} \frac{\theta_k + \theta_j}{2(z_j - z_k)} + \frac{\theta_k(1-\beta)+\beta}{z_k - z_\infty} \,, \tag{2.126a}$$

$$\hat{E}_\infty = \frac{12\xi}{z_\infty(r_4 - r_1)} \left(\sum_{k=1}^{3} r_k - r_4 \right) - \sum_{k=1}^{3} \frac{\theta_k(1-\beta)+\beta}{z_k - z_\infty} \,, \tag{2.126b}$$

$$\hat{F}_\infty = \beta^2 - 3\beta + 12\xi \,. \tag{2.126c}$$

Note that $\hat{F}_\infty = 0$ is the indicial polynomial associated with the expansion at $z = z_\infty$. Thus it is natural to choose β to be one of the characteristic exponents setting $\hat{F}_\infty = 0$. However, to completely remove $z = z_\infty$ from (2.123), we need that $\beta = 1$ in (2.124). This further constraints $\xi = 1/6$ because of (2.126c). Now, we still need to check that $\hat{E}_\infty$ can be set to zero. The coefficients $\hat{E}$ above are simplified by noticing that

$$\sum_{k=1}^{4} \theta_k = 0, \qquad \sum_{k=1}^{4} \theta_k r_k = \frac{6i\Psi_0}{\Lambda}, \tag{2.127}$$

$$\sum_{j\neq k}^{3} \frac{\theta_k + \theta_j}{2(z_j - z_k)} = -2i \left(\frac{d_k}{z_k - z_\infty} \right) \left(\frac{\Psi_0 r_4 r_k + \Psi_1}{r_k - r_4} \right), \quad (k = 1,2,3) \tag{2.128}$$

where we used the residue theorem to show these identities. This implies that, for $\beta = 1$ and $\xi = 1/6$,

$$\hat{E}_k = \frac{d_k}{z_k - z_\infty} \left[f(r_k) - 2i \left(\frac{\Psi_0 r_4 r_k + \Psi_1}{r_k - r_4} \right) \right] + \frac{1}{z_k - z_\infty} \,, \tag{2.129}$$

$$\hat{E}_\infty = \frac{1}{(r_4 - r_1) z_\infty} \sum_{k=1}^{4} r_k \,. \tag{2.130}$$

The polynomial (2.83b) has no third-order term, so this means that the sum of all of its roots is zero. Therefore, $\hat{E}_\infty = 0$ generically if $\beta = 1$. This completes our proof that (2.123) is a Fuchsian equation with 4 regular singular points, also called Heun equation.

Summing up, the radial equation of conformally coupled scalar perturbations of Kerr-NUT-(A)dS black hole can be cast as a Heun equation in canonical form

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_{t_0}}{z-t_0} \right) y' + \left(\frac{\kappa_1\kappa_2}{z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)} \right) y = 0, \tag{2.131}$$

with coefficients

$$\theta_k = 2i\left(\frac{\Psi_0 r_k^2 + \Psi_1}{Q'(r_k)}\right), \quad k = 1,2,3,4 \tag{2.132}$$

$$K_0 = -\hat{E}_3, \qquad t_0 = \frac{r_3 - r_1}{r_3 - r_4}\frac{r_2 - r_4}{r_2 - r_1}, \tag{2.133}$$

where we make the correspondence $k \in \{1,2,3,4\} \sim \{0,1,t_0,\infty\}$.

The values of θ_k obey Fuchs relation, fixing $\kappa_{1,2}$ via $\theta_0 + \theta_1 + \theta_{t_0} + \kappa_1 + \kappa_2 = 2$ and $\kappa_2 - \kappa_1 = \theta_\infty$. Also, in terms of (2.129) we have that $\kappa_1\kappa_2 = \hat{E}_2 + t_0\hat{E}_3 = 1 + \theta_4$. These follow from the regularity condition at infinity, $\sum_{i=1}^3 \hat{E}_i = 0$. The set of 7 parameters $(\theta_0, \theta_1, \theta_{t_0}, \kappa_1, \kappa_2; t_0, K_0)$ define the Heun equation and its fundamental solutions. By Fuchs relation, we see that the minimal defining set has 6 parameters. For more details about Heun equation, we refer to [78, 73].

In the Kerr-NUT-(A)dS case, we note the importance of K_0 indexing the solutions because the only dependence on C_ℓ comes from it. As mentioned before, the local Frobenius behaviour of the solutions do not depend on C_ℓ, but this dependence will come about in the parametrization of the monodromy group done below.

The appearance of the extra singularity t_0 in (2.131) makes things more complicated than the hypergeometric case. First, the coefficients of the series solution obey a three-term recurrence relation [78], which is not easily tractable to find explicit solutions [96]. Second, there is no known integral representation of Heun functions in terms of elementary functions, which hinders a direct treatment of the monodromies. Therefore, we need to look for an alternative approach to solve the connection problem of Heun equation. In the next chapter, we will see how the isomonodromic deformation theory [97, 98, 99] can shed light on this problem.

Chapter 3

Differential Equations, Isomonodromy and Painlevé Transcendents

There is no scientific discoverer, no poet, no painter, no musician, who will not tell you that he found ready made his discovery or poem or picture – that it came to him from outside, and that he did not consciously create it from within.

—WILLIAM KINGDON CLIFFORD

We now know that the problem of finding scattering coefficients for a scalar field conformally coupled to a Kerr-NUT-(A)dS black hole reduces to finding monodromy representations for the Heun equation. Mathematically, we want to solve Heun's connection problem - how to connect two fundamental solutions - using monodromy representations of Heun functions. This problem has no general solution in the literature and it becomes even worse for 5 or more singular points. This resides on the fact that Heun functions are still not very well understood compared to what is known about hypergeometric functions [78, 73].

As suggested by the study of hypergeometric functions, the most general way to define a so-called special function is via its singularity structure. This is the point of view defended, for example, in [73] and has a long history passing through the works of Riemann, Poincaré, Fuchs, Painlevé, Ince and several others. In fact, from this point of view, we can say we know a function if we know its local expansions (which can be asymptotic) near its singular points and how to connect these expansions with each other. Said in another way, the global behaviour of a function is simply obtained by "gluing" local expansions converging up to the next singular point. That is the essence of the *connection problem* we mentioned before. As discussed in section 2.2, this problem is intrinsically related to the *direct monodromy problem*:

> *Given a differential equation with n singular points, find an* SL(2, $\mathbb{C}$) *monodromy representation associated to its singular points. If there are irregular points, also find the Stokes matrices.*

Therefore, if we solve the monodromy problem of Heun equation, we can find the scattering data. However, this problem has no explicit solution in the literature because, unlike the hypergeometric case, the monodromy group of the 4-punctured sphere depends on global data not accesible by local Frobenius expansions. As we shall see, the monodromy matrices also depend on composite traces

$$m_{ij} \equiv \mathrm{Tr}\, M_i M_j \simeq \mathrm{Tr}\, \mathcal{P} \exp\left(\oint_{\gamma_{ij}} A(z) dz \right), \tag{3.1}$$

where γ_{ij} is a path enclosing both singular points z_i and z_j. These traces are not as easy to compute as the single traces and in fact have no simple expression in the Heun case. Because of our limitation with calculating these traces, we need to look into another direction.

The issue of finding monodromy representations of Fuchsian differential equations is deeply related to the classical Riemann-Hilbert problem in the theory of ordinary differential equations, also called the *inverse monodromy problem*:

> *Given an irreducible* SL(2, $\mathbb{C}$) *representation* ρ *of the fundamental group of the n-punctured Riemann sphere, find a Fuchsian differential equation which has* ρ *as its monodromy representation.*

It is well-known that this problem has a negative solution in general. Poincaré calculated that, given a set of n singular points $S = \{z_1, z_2, ..., z_n\}$, the dimension $M(S)$ of the space of irreducible monodromy representations of the n-punctured sphere is larger than the dimension $E(S)$ of the space of irreducible Fuchsian equations with n singular points in S. Specifically, as we show below, for second order ODEs, we have that $M(S) - E(S) = n - 3$. Therefore, only for the hypergeometric case, $n = 3$, the number of parameters match and we have a unique correspondence between representations of the monodromy group and the Fuchsian ODE.

The Riemann-Hilbert problem has a long story, back from Riemann, Hilbert and Poincaré, passing through partial answers by Plemelj [100] and Röhrl [101], and, finally, reaching the most general answer given 20 years ago by Bolibruch (see, for example, the book [102]). By the time Hilbert proposed this problem as the 21st of his famous list, he probably had a feeling that the theory of ordinary differential equations was almost complete, with only this problem left for closure. The classical Riemann-Hilbert problem has since then been expanded to a more general Riemann-Hilbert approach and a lot of cross-fertilization between different areas has occurred, specially in integrable systems [103]. The Riemann-Hilbert approach has also been successfully applied to the isomonodromic deformation theory, developed mainly by the Japanese school in the seminal series of papers by Miwa, Jimbo and Ueno [97, 98, 99]. One of the main results of these works is to show that the monodromy problem of Fuchsian systems is related to the connection problem of Painlevé VI solutions. An important corollary of Jimbo and collaborators' work is that Painlevé VI equation is integrable in the sense that its integration constants are directly related to the monodromy data of a 4-point Fuchsian system [104]. In the following, we shall review the isomonodromic deformation theory and show how its results are useful in order to obtain the scattering coefficients in black hole scattering.

The essential point is the connection between Painlevé VI asymptotics and monodromy data. This has been known since Jimbo's work [104], but, as far as we know, has not been put into practice to solve the monodromy problem of the Heun equation until recently in the context of 2-dimensional conformal field theories by Litvinov *et al* [105]. Also there has been several advances in $c = 1$ Liouville theory related to Painlevé VI connection formulas [106, 107, 108, 109]. These works in CFT and Liouville theory give some hints on how to generalize our approach to higher-dimensions and how to obtain more analytical information about the composite traces we are interested to organize the monodromy data. We shall explore these lines of inquire in the conclusions of this thesis. But now, let us talk about the monodromy group of Heun equation.

3.1 Monodromy Group of Heun Equation

Here we study the monodromy group of the 4-punctured sphere in more mathematical detail. First, let us review some definitions for the Riemann sphere with n holes. Given a set $S = \{z_1, z_2, ..., z_n\}$ of n points, we define the n-punctured Riemann sphere as $D = \mathbb{CP}^1 \setminus S$. Here D represents the domain of a Fuchsian ODE in the complex plane. Given a point $b \in D$, we want to study the set of loops $L(D, b)$ with base point b. Given any two loops $\gamma_1, \gamma_2 \in \mathbb{CP}^1$, we say that γ_1 is *homotopically* equivalent to γ_2, that is, $\gamma_1 \simeq \gamma_2$, if and only if one loop can be continuously deformed to match the other[1]. A consequence of this equivalence is that a loop not enclosing a singular point is homotopic to a point. The singular set S promotes topological obstructions to continuous deformations of loops to a point. The set of all equivalence classes of loops in

[1]The operation of continuously deforming a path (or a loop) is called a *homotopy*.

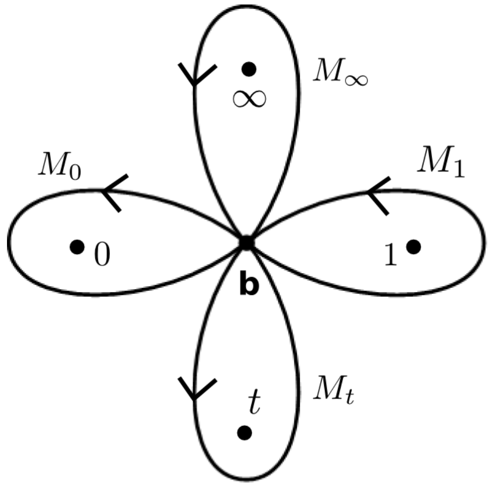

Figure 3.1: Bouquet of 4 loops defining the 4-punctured monodromy group with base point *b*.

$L(D,b)$ forms a group $\pi_1(D,b)$ and is called the *fundamental group*[2] of D. Thus, we write the fundamental group as the quotient space $\pi_1(D,b) = L(D,b)/\simeq$. For more details, see [74].

The monodromy group is defined by the group homomorphism

$$\rho : \pi_1(D,b) \to \mathrm{GL}(2,\mathbb{C}) \tag{3.2}$$

where, in our context, the $\mathrm{GL}(2,\mathbb{C})$ matrices act on the space of fundamental solutions Φ of a Fuchsian ODE. This defines a $\mathrm{GL}(2,\mathbb{C})$ representation of the homotopy group and we refer to this representation as the *monodromy group* $\mathcal{M}(D)$ of the n-punctured sphere[3]. Therefore, given a representative loop $\gamma \in \pi_1(D,b)$, the analytic continuation of Φ around γ is given by $\Phi_\gamma = \Phi M_\gamma$, where $M_\gamma \in \mathrm{GL}(2,\mathbb{C})$ is the monodromy matrix, as we have seen in section 2.2. In fact, the $\mathrm{GL}(2,\mathbb{C})$ determinant is just a gauge artifact, and in the following we restrict to $\mathrm{SL}(2,\mathbb{C})$ representations of the fundamental group.

The set of all representations of the monodromy group modulo overall conjugation

$$\mathcal{R}_{\mathcal{M}} = \mathrm{Hom}(\pi_1(D,b), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C}) \tag{3.3}$$

is called the *moduli space*[4] of $\mathrm{SL}(2,\mathbb{C})$ representations of $\mathcal{M}(D)$. This space can be parameterized by trace invariants constructed from the monodromy generators [110, 111, 112]. We

[2]Also known as the first homotopy group, because loops are 1-dimensional objects. The second homotopy group is about equivalence classes of 2-spheres and so on.

[3]We can also study other types of topological spaces with non-trivial genus, boundaries and cross-caps, as the 1-torus or the Klein bottle.

[4]The moduli space here is essentially the parameter space of all equivalence classes of a set under some equivalence relation.

shall describe this construction below. The moduli space $\mathcal{R}_{\mathcal{M}}$ is also called in the literature the *character variety* of the n-punctured sphere. This concept has important applications in hyperbolic geometry and in the theory of deformations of hyperbolic structures [111]. We can also define the *relative* character variety, which represents the moduli space $\mathcal{R}_{\mathcal{M}}$ with fixed elementary monodromies, $\mathcal{R}^*_{\mathcal{M}}$. This restricted moduli space is isomorphic to the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$ connections $\mathcal{A}_n$ over the n-punctured sphere. Atiyah and Bott [113] have shown that the latter space has a natural symplectic structure, the Atiyah-Bott symplectic form

$$\Omega = \frac{1}{2\pi i}\int_{\Sigma} \mathrm{Tr}(\delta A \wedge \delta A), \tag{3.4}$$

showing an important relation of the monodromy group with gauge theories and symplectic geometry (see also [114, 110, 112]). This will give us further insights below but, for now, let us focus on the algebraic properties of the monodromy group and its moduli space.

3.1.1 Moduli Space of Monodromy Representations

The monodromy group can be defined as a free group[5] $G_n = \langle M_1, ..., M_{n-1}\rangle$ with $n-1$ generators M_i. The matrix M_n can be obtained by the identity (2.24). Each M_i represent the monodromy of elementary loops enclosing a singular point z_i. We want to obtain, if possible, an explicit parameterization of $\mathcal{M}(D)$ in terms of invariant characters. This is not a simple task in general, although there are some hints on how to obtain such parameterization explicitly in the literature [109, 111]. In fact, the most direct way to parametrize G_n is in terms of its trace invariants. How many trace invariants do we need to characterize G_n? We have $n-1$ $\mathrm{SL}(2,\mathbb{C})$ generators with $3(n-1)$ complex parameters. Monodromy representations are equivalent by conjugation of a $\mathrm{SL}(2,\mathbb{C})$ matrix which has 3 free parameters. Therefore, the number of inequivalent representations of G_n is given by the dimension of the quotient space $\mathrm{SL}(2,\mathbb{C})^{n-1}/\mathrm{SL}(2,\mathbb{C})$, that is $\dim G_n = 3(n-1) - 3 = 3n - 6$. If we fix n elementary traces $m_i = \mathrm{Tr}\, M_i$, we have that the dimension of the relative moduli space is $2(n-3)$. This counting will be important below.

The classification of 2-dimensional representations of a free group with 2 generators G_2 is well-known and given, for example, in [74]. The parameterization there presented is equivalent to the one presented in the final paragraph of section 2.2. All representations of G_2 are

[5] A free group is formed by the set of all words constructed by concatenation of its generators modulo conjugation.

characterized by the eigenvalues of M_i, which can be recast in terms of the traces

$$\begin{aligned} m_i &= \operatorname{Tr} M_i = 2\cos(\pi\theta_i), \quad i = 1,2, \\ m_{12} &= \operatorname{Tr}(M_1 M_2) = 2\cos(\pi\sigma_{12}), \quad \sigma_{12} \in \mathbb{C} \end{aligned} \tag{3.5}$$

and those are further constrained by the identity [111, 112]

$$W_3(m_1,m_2,m_{12}) \equiv m_1^2 + m_2^2 + m_{12}^2 - m_1 m_2 m_{12} - \kappa - 2 = 0, \tag{3.6}$$

where $\kappa = \operatorname{Tr}(M_1 M_2 M_1^{-1} M_2^{-1})$. This identity follows directly from the trace properties and the basic identity $\operatorname{Tr}(g)\operatorname{Tr}(h) = \operatorname{Tr}(gh) + \operatorname{Tr}(gh^{-1})$, a consequence of the Cayley-Hamilton theorem for $SL(2,\mathbb{C})$ matrices [111]. Now we see that, fixed the monodromies m_i, the composite trace m_{12} is determined by relation (3.6). Therefore, the moduli space of the hypergeometric monodromy group with fixed monodromies has no extra parameters. In other words, the local Frobenius expansions completely determine the monodromy group.

In the case of interest, we want to study the monodromy group of Heun equation, a free group with 3 generators $G_3 = \langle M_1, M_2, M_3\rangle$, parameterized by the trace parameters

$$m_i = \operatorname{Tr} M_i = 2\cos(\pi\theta_i), \quad i = 1,2,3, \tag{3.7}$$

$$m_4 = \operatorname{Tr} M_4 = \operatorname{Tr}(M_1 M_2 M_3) = 2\cos(\pi\theta_4), \tag{3.8}$$

with the second equality following from the monodromy identity (2.24), and composite traces coordinates

$$m_{ij} = \operatorname{Tr}(M_i M_j) = 2\cos(\pi\sigma_{ij}), \quad i,j = 1,2,3, \tag{3.9}$$

where $m_{ij} = m_{ji}$ and $i \neq j$. These traces obey the *Fricke-Jimbo relation*

$$\begin{aligned} &W_4(m_1,m_2,m_3,m_{13},m_{23},m_{12},m_4) = \\ &\quad m_{13}m_{23}m_{12} + m_{13}^2 + m_{23}^2 + m_{12}^2 - m_{13}(m_2 m_4 + m_1 m_3) - m_{23}(m_1 m_4 + m_2 m_3) \\ &\quad - m_{12}(m_3 m_4 + m_1 m_2) + m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_1 m_2 m_3 m_4 - 4 = 0. \end{aligned} \tag{3.10}$$

Therefore, fixed the monodromies m_i, the relative moduli space $\mathcal{R}^*_{\mathcal{M}}$ is parameterized[6] only by 2 coordinates m_{ij}, as one of the composite traces is fixed by (3.10). Now we see that the knowledge of just the local monodromies m_i is not sufficient to determine a representation of the 4-point monodromy group.

[6] Geometrically, (3.10) defines a 6-dimensional hypersurface on the character variety, isomorphic to $\mathbb{C}^7$, and a 2-dimensional hypersurface on the relative moduli space, which is isomorphic to $\mathbb{C}^3$ [111].

We now can ask what is the most general parameterization of the 4-point monodromy group? A natural attempt is to remember that, in the irreducible case, each monodromy generator is diagonalizable

$$M_i = g_i^{-1} e^{i\pi\theta_i\sigma^3} g_i, \quad (i = 1,2,3) \tag{3.11}$$

and as the $g_i \in \mathrm{SL}(2,\mathbb{C})$, we can parametrize them as

$$g_i = \exp(i\psi_i\sigma^3/2)\exp(i\phi_i\sigma^1/2)\exp(i\varphi_i\sigma^3/2), \tag{3.12}$$

where $(\psi_i, \phi_i, \varphi_i)$ are taken as complex parameters. Plugging this into (3.11) shows that φ_i can be set to zero. Therefore, each matrix depends on 2 parameters (ψ_i, ϕ_i), giving a total of 6 parameters. Furthermore, by simple matrix multiplication we get that

$$\begin{aligned} m_{ij} = 2\cos(\theta_i/2)\cos(\theta_j/2) + 2\sin(\theta_i/2)\sin(\theta_j/2) \\ \times\Big(\cos(\phi_i/2)\cos(\phi_j/2) + \sin(\phi_i/2)\sin(\phi_j/2)\cos((\psi_i - \psi_j)/2)\Big), \end{aligned} \tag{3.13}$$

so there is an extra redundancy $\psi_i \to \psi_i + a$, where a is a constant, allowing us to set one of the ψ_i to zero. One usually uses the extra gauge freedom to make M_4 diagonal, making the trace coordinates explicit in the generators M_i. Given this $\mathrm{SL}(2,\mathbb{C})$ parameterization of each M_i, we could try to relate the 6 parameters (ϕ_i, ψ_i) with six independent traces (θ_i, σ_{ij}), but the relations are somewhat complicated for a simple approach.

Following [104], we now address a clever way to express the general parameterization of *irreducible* representations of the monodromy group in terms of trace coordinates. We notice that Jimbo's parameterization has been revised by Boalch [115] making a very small correction in eq. (1.8) of Jimbo's paper and also giving an explicit derivation of that formula. Here, we follow Jimbo to write the monodromy parameterization below. *En passant*, we notice that the construction found in [110] is a more sofisticate and detailed description of Jimbo's parameterization and gives further insight into its algebro-geometric construction. There is possible to find a derivation of the following result.

We start by defining the monodromy matrices M_k with $k = 0, 1, t, \infty$, representing the 4 singular points on the Riemann sphere fixed by a homographic transformation, obeying

$$M_\infty M_1 M_t M_0 = 1. \tag{3.14}$$

Here we choose a basis where M_∞ is diagonal. Let $\sigma := \sigma_{0t}$ represent one of the composite traces. Restricting to irreducible representations, we suppose that

$$\theta_0 \pm \theta_t \pm \sigma, \quad \theta_0 \pm \theta_t \mp \sigma, \quad \theta_\infty \pm \theta_1 \pm \sigma, \quad \theta_\infty \pm \theta_1 \mp \sigma \tag{3.15}$$

are not equal to even integers[7]. Therefore,

$$M_1 = \frac{1}{i\sin(\pi\theta_\infty)}\begin{pmatrix} \cos(\pi\sigma)-e^{-i\pi\theta_\infty}\cos(\pi\theta_1) & -2e^{-i\pi\theta_\infty}\sin\frac{\pi}{2}(\theta_\infty+\theta_1+\sigma)\sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma) \\ 2e^{i\pi\theta_\infty}\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma)\sin\frac{\pi}{2}(\theta_\infty-\theta_1-\sigma) & -\cos(\pi\sigma)+e^{i\pi\theta_\infty}\cos(\pi\theta_1) \end{pmatrix}, \tag{3.16a}$$

$$CM_0C^{-1} = \frac{1}{i\sin(\pi\sigma)}\begin{pmatrix} e^{i\pi\sigma}\cos(\pi\theta_0)-\cos(\pi\theta_t) & 2s\sin\frac{\pi}{2}(\theta_0+\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t+\sigma) \\ -2s^{-1}\sin\frac{\pi}{2}(\theta_0+\theta_t+\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t-\sigma) & -e^{-i\pi\sigma}\cos(\pi\theta_0)+\cos(\pi\theta_t) \end{pmatrix}, \tag{3.16b}$$

$$CM_tC^{-1} = \frac{1}{i\sin(\pi\sigma)}\begin{pmatrix} e^{i\pi\sigma}\cos(\pi\theta_t)-\cos(\pi\theta_0) & -2se^{-i\pi\theta_\infty}\sin\frac{\pi}{2}(\theta_0+\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t+\sigma) \\ 2s^{-1}e^{-i\pi\sigma}\sin\frac{\pi}{2}(\theta_0+\theta_t+\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t-\sigma) & -e^{-i\pi\sigma}\cos(\pi\theta_t)+\cos(\pi\theta_0) \end{pmatrix}, \tag{3.16c}$$

where

$$C = \begin{pmatrix} \sin\frac{\pi}{2}(\theta_\infty-\theta_1-\sigma) & \sin\frac{\pi}{2}(\theta_\infty+\theta_1+\sigma) \\ \sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma) & \sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma) \end{pmatrix} \tag{3.17}$$

is a change of basis matrix which diagonalize M_0M_t. Jimbo's parametrization also depend on the parameter s given by

$$\begin{aligned} 4s\sin\frac{\pi}{2}(\theta_0+\theta_t-\sigma)\sin\frac{\pi}{2}(\theta_0-\theta_t+\sigma)\sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma)\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma) \\ = i\sin(\pi\sigma)\cos(\pi\sigma_{01})+\cos(\pi\theta_t)\cos(\pi\theta_1)+\cos(\pi\theta_\infty)\cos(\pi\theta_0)+ \\ e^{i\pi\sigma}(i\sin(\pi\sigma)\cos(\pi\sigma_{1t})-\cos(\pi\theta_t)\cos(\pi\theta_\infty)-\cos(\pi\theta_0)\cos(\pi\theta_1)). \end{aligned} \tag{3.18}$$

We now see that the parametrization of the Heun monodromy group is much more complicated than the hypergeometric case. The generators depend not only on the local monodromies θ_i but also on composite monodromies σ_{ij}. As we have seen before, it is easy to find the θ_i directly from the ODE but the parameters σ_{ij} are still elusive to us. The question that remains is: how are the composite traces m_{ij} related to the parameters of Heun equation?

3.1.2 Symplectic Structure of Moduli Space

Going back to the starting discussion above, we see that the monodromy matrices depend both on the loop and on the chosen basis of solutions Φ. Changing the base point b of the loops implies a change of basis on the monodromy representation. If we change the basis by some $SL(2,\mathbb{C})$ matrix, $\Phi_2 = \Phi_1 g_{12}$, we have that $M_2 = g_{12}^{-1}M_1 g_{12}$. As every two monodromy representations are conjugate by some $g \in SL(2,\mathbb{C})$, we have that the conjugacy class of the monodromy representation is uniquely determined by the ODE parameters. This means that, fixed

[7]For reducible cases, the parametrization needs to be slightly modified [104].

the monodromy coefficients θ_i, the moduli space of a Fuchsian ODE with fixed monodromy is isomorphic to the relative moduli space of monodromy representations of the same ODE. Let us be more explicit about this.

Given a generic Fuchsian ODE with n finite singular points, we can always transform it to its *normal form* (see Appendix B), that is

$$\psi''(z)+T(z)\psi(z)=0, \qquad T(z)=\sum_{i=1}^{n}\left(\frac{\delta_i}{(z-z_i)^2}+\frac{c_i}{z-z_i}\right), \tag{3.19a}$$

$$\sum_{i=1}^{n}c_i=0\,,\quad \sum_{i=1}^{n}(c_iz_i+\delta_i)=0\,,\quad \sum_{i=1}^{n}(c_iz_i^2+2\delta_iz_i)=0, \tag{3.19b}$$

where (3.19b) are the necessary and sufficient conditions for $z=\infty$ to be a regular point. Because of (3.19b), there are only $n-3$ independent c_i, and we can also fix 3 of the z_i to be $0,1$ and ∞, by a homographic transformation. The monodromy parameters are defined by $\delta_i=(1-\theta_i^2)/4$, thus, if we fix these, we can parametrize Fuchsian equations by $2(n-3)$ complex numbers (c_i,z_i). We have that δ_i and z_i are local parameters, depending only on the local behaviour of the solutions of (3.19a), and the c_i, called *accessory parameters*, have global properties not probed locally. The accessory parameters are usually related to spectral parameters of differential equations [78, 73]. The angular eigenvalue C_ℓ dependence appears exactly in the accessory parameter of Heun equation (2.131), and that is why it did not appear in the Frobenius coefficients θ_i. The δ_i are related to elementary monodromies, so the remaining parameters c_i,z_i must be non-trivially related to the composite monodromies (3.9). In fact, (c_i,z_i) form canonical coordinates on the moduli space of flat SL$(2,\mathbb{C})$ connections $\mathcal{A}_n$ mentioned before, serving as Darboux coordinates to express the Atiyah-Bott symplectic form

$$\Omega=\sum_{i=1}^{n-3}dc_i\wedge dz_i. \tag{3.20}$$

$\mathcal{A}_n$ is isomorphic to the moduli space of monodromy representations with fixed θ_i, thus we expect there must be some canonical transformation relating (c_i,z_i) to the composite traces parameters σ_{ij}. Nekrasov *et al* [112] have found such transformation and, recently, Litvinov *et al* [105] have shown how the parameters of the Heun equation are related to the monodromy parameters. This relationship was thus used in [105] to find an equivalent way to address the monodromy problem in terms of classical conformal blocks of 2-dimensional CFTs. Their approach can be understood both in terms of symplectic geometry as well as in terms of isomonodromic deformation theory, which will be our next topic of discussion. To end this section, we briefly show how to construct canonical coordinates directly related to the monodromy traces.

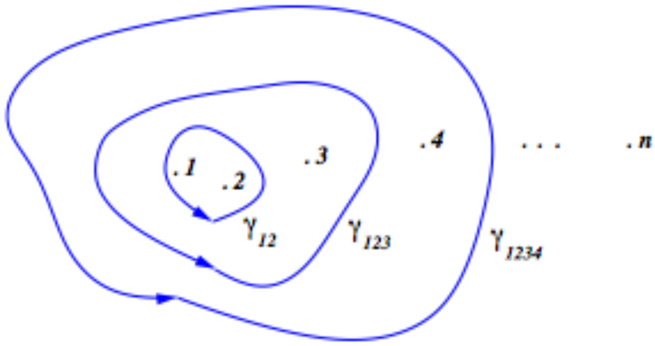

Figure 3.2: Composite loops associated to accessory parameters. Retrieved from [105].

First, let us consider the monodromy group of n singular points and $n-1$ generators M_k. For each loop $\gamma_{12...k}$ enclosing k singular points, we address a monodromy matrix $M_{12...k} = M(\gamma_{12...k})$ (see Figure 3.2). Litvinov *et al* [105] thus define the non-trivial composite traces

$$\mathrm{Tr}(M_{12}) = -\cos(\pi\nu_1), \quad \mathrm{Tr}(M_{123}) = -2\cos(\pi\nu_2), \; ... \; \mathrm{Tr}(M_{12...n-2}) = -2\cos(\pi\nu_{n-3}), \tag{3.21}$$

which fixes the conjugacy classes of those monodromy matrices. However, we notice that to fix the whole monodromy group we still need $n-3$ other traces. Nekrasov *et al* [112] have shown that canonically conjugate coordinates $\mu_1, ..., \mu_{n-3}$ can be found in a geometric construction of an n-gon in SL$(2,\mathbb{C})$ such that

$$\Omega = \sum_{i=1}^{n-3} d\nu_i \wedge d\mu_i \tag{3.22}$$

is the symplectic structure of the moduli space. Moreover, there is a generating function $W(\nu, z)$ connecting both sets of canonical coordinates

$$\mu_i = \frac{\partial W}{\partial \nu_i}, \qquad c_i = \frac{\partial W}{\partial z_i}, \tag{3.23}$$

where $W(\nu,z) = W_0(\nu) + f_\nu(z)$, $W_0(\nu)$ is related to the classical limit of the 3-point structure functions and $f_\nu(z)$ represents the *classical conformal blocks* of 2D CFT [105].

The connection between differential equations and classical conformal blocks arise because general conformal blocks of chiral vertex operators obey certain special differential equations representing null-vector conditions in the moduli space of the n-sphere [105]. In the classical limit, the resulting equation is a Fuchsian ODE with n singular points equivalent to (3.19a) where the accessory parameters c_i relate with the classical conformal blocks as above. Therefore, in the Heun case $n = 4$, knowing the accessory parameter c fixes the composite monodromy parameter ν in terms of $z = t$ through the knowledge of the classical conformal block

$f_\nu(t)$. Expansions of classical conformal blocks are well-known [116, 117] and can be used to connect ν with c. This suggests a way to obtain analytical results for ν and thus our scattering problem. However, the relationship between these parameters is not very trivial and we leave this discussion for further work.

Summarizing, to fully describe the monodromy group of Heun equation, we need to obtain two composite traces from the three possibilities m_{0t}, m_{1t} and m_{01}, because one of them is constrained by Fricke-Jimbo relation (3.10). Analytical results are still out of reach, so we need another approach. The theory of isomonodromic deformations comes to our aid and will give further insight into this problem. But before introducing this theory, we present Painlevé transcendents in the next section, as they will be useful later.

3.2 Painlevé Transcendents

Painlevé VI equation is the most general rational second-order nonlinear equation whose solution has no movable branch points - the *Painlevé property* - and no movable essential singularities. It emerged as an attempt of the mathematician Paul Painlevé to extend the definition of special functions to the non-linear realm. The ability to be extended over the whole punctured complex plane by knowledge of local information is an important property of special functions. This property is a direct consequence of Painlevé transcendents, the solutions of Painlevé equations, having only movable poles. A simple example of an ODE whose solution has a *movable branch point* is

$$my'y^{m-1} = 1, \quad m \in \mathbb{N}. \tag{3.24}$$

The solution of this equation is $y(t) = (t-c)^{1/m}$, where c is an arbitrary complex constant depending on the *initial condition*, having an algebraic branch point at $t = c$. That is why Painlevé was actually interested in the Painlevé property in the first place, to find the nonlinear equivalent of special functions being defined by connection formulas. This can only be achieved if we can define the transcendental function only from information of the ODE itself, which is not the case when we have movable branch points.

In the beginning of the 20th century, Painlevé [118] studied equations of the type

$$\frac{d^2y}{dt^2} = R\left(t, y, \frac{dy}{dt}\right), \tag{3.25}$$

with R meromorphic in y and rational in $\frac{dy}{dt}$. He discovered that there are actually 50 equations obeying the Painlevé property but only 6 of them, usually called P_J, $J = I, II, ..., VI$, are not integrable by quadrature or reducible to some linear equation. In fact, Painlevé had found only the first three simplest types because of errors in his calculations, but his student, B. Gambier, later found the other three [74].

Painlevé transcendents have important applications in several areas of mathematics and physics. As we can imagine, the literature on Painlevé equations is very extensive, being more than 100 years old. Painlevé equations appear as ODE reductions of several nonlinear partial differential equations (PDEs) like Kortweg-de Vries (KdV) equation (P_I and P_{II}), of nonlinear Schrödinger equation (P_{IV}), of Ernst equation (P_{VI}) and Sine-Gordon equation (P_{III}). In particular, all Painlevé equations can be obtained as one-dimensional reductions of SL$(2,\mathbb{C})$ anti-self-dual Yang-Mills equations in 4 complex dimensions, restricted by the so-called Painlevé subgroups of the conformal group [119]. There are also very interesting applications in statistical physics and quantum field theory, and some of the most important are cited in [120]: The two-point correlation functions for the two-dimensional Ising model [121], for the one-dimensional impenetrable Bose-gas at zero temperature [122], and for the one-dimensional isotropic XY-model [123]. Applications in topological field theory can be found in [124], and also on the partition function of 2D quantum gravity models [125, 126]. This last application is connected to random matrix theory and orthogonal polynomials, the Tracy-Widom distribution of eigenvalues (which has important relations with percolation and growth processes) and Fredholm and Toeplitz determinants [127, 128, 129]. For example, Fredholm determinants can be calculated in terms of the solution of the sigma-form of P_V. The usage of Painlevé in fluid dynamics, particularly in the study of time-evolving surfaces in Hele-shaw flow with surface tension, can be found in [130]. Finally, there is also a very interesting application in number theory: there is analytical and numerical evidence that the distribution of zeros of the ζ-function on the line $\mathrm{Re}\, z = \frac{1}{2}$ follows the same behaviour as the distribution of eigenvalues of the Gaussian Unitary Ensemble in random matrix theory [131]. For a review of Painlevé transcendents and its modern approaches, see, for example, [132, 120].

As we mentioned above, Painlevé transcendents are best understood in terms of the theory of isomonodromic deformations, as its connection problem can be solved in terms of the monodromy data of an associated Fuchsian system. In the next subsections, we briefly review some properties of Painlevé transcendents and the theory of isomonodromic deformations based on the presentation of [74]. For the reader interested in learning about these subjects more thoroughly, we suggest to start reading the accessible book by Iwasaki *et al* [74] and then the more extensive treatise by Novokshenov *et al* [120], which will introduce the reader to the more advanced literature.

3.2.1 Painlevé Equations

We start this section by listing the 6 Painlevé equations mentioned in the introduction:

$$P_I: \frac{d^2y}{dt^2} = 6y^2 + t, \tag{3.26a}$$

$$P_{II}: \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha, \tag{3.26b}$$

$$P_{III}: \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{\delta}{y} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3, \tag{3.26c}$$

$$P_{IV}: \frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{\beta}{y} + 2(t^2 - \alpha)y + 4ty^2 + \tfrac{3}{2}y^3, \tag{3.26d}$$

$$\begin{aligned} P_V: \frac{d^2y}{dt^2} &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} \\ &\quad + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}, \end{aligned} \tag{3.26e}$$

$$\begin{aligned} P_{VI}: \frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha - \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \left(\tfrac{1}{2} - \delta\right)\frac{t(t-1)}{(y-t)^2}\right), \end{aligned} \tag{3.26f}$$

where $\alpha, \beta, \gamma, \delta$ are complex constants. By an appropriate rescaling of y and t, we can reduce the number of parameters of P_{III} by two and P_V by one, so those have 2 and 3 independent parameters, respectively. The P_{VI} equation has a non-standard notation here, following [74], which can be obtained from the standard one by taking $\beta \to -\beta$ and $\delta \to \frac{1}{2} - \delta$.

For generic parameters, the solutions of (3.26) have critical points at $t = 0, 1, \infty$. We also have singular behaviour when $y(t_0) = 0, 1, t_0, \infty$ for $t_0 \in \mathbb{CP}^1 \backslash \{0, 1, \infty\}$, and we say these points are *regular* if the coefficients of the right-hand side of (3.26) have a simple pole at these points. In P_{VI}, all critical points are regular, but in the other cases we have one irregular point at $y = \infty$.

Solutions of Painlevé equations are transcendental functions expressible not even in terms of integrals of hypergeometric functions, for example. This means that there is no simple classical representation of these functions. Although they do not share this property with respect to classical special functions, Painlevé functions can be defined in terms of its connection formulas, similar to the discussion we made in the previous chapter. This remarkable result was first presented in the 1980's by Jimbo [104], using the theory of isomonodromic deformations of [97, 98, 99]. The complete table of connection formulas for the P_{VI} equation were obtained rather recently [133, 134]. For more details on the Painlevé connection problem and its extensive literature, see [120]. We shall discuss more about connection formulas when we introduce isomonodromic deformations.

3.2.2 Confluence Limits

An important thing to remember about Painlevé equations is that we can obtain all P_J ($J = I,...,V$) by certain limiting processes on P_{VI}. These limits are obtained by confluence of singular points and scaling transformations on the P_{VI} parameters, and can be found in [74]. In the the diagram below, each arrow indicates a possible Painlevé reduction. Below each P_J

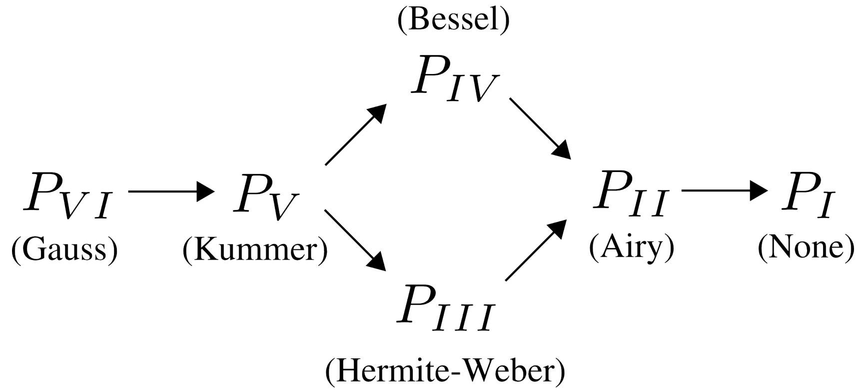

Figure 3.3: Diagram of Painlevé reductions and the corresponding restrictions to hypergeometric cases.

type, except for P_I, we show that this diagram also works for confluent reductions of Gauss's hypergeometric equation. There are two ways to relate Painlevé solutions with hypergeometric solutions. The first one, described in [74], is that, for certain restrictions on the Painlevé parameters $(\alpha,\beta,\gamma,\delta)$, we have that the P_J have solutions $y_J(t)$ of the form

$$y_J(t) = f_J(t)\frac{d}{dt}\log(g_J(t)u_J(t)), \quad J = II,...,VI \tag{3.27}$$

where $f_J(t), g_J(t)$ are simple rational functions, and $u_J(t)$ is a solution of an equation of hypergeometric class. This matches with the relations shown in Figure 3.3. The other way to make contact with the hypergeometric class is by linearization of some Painlevé equations with $\beta = 0$. By linearization we mean that, given a P_J equation

$$y''(t) = F(y, y', t)\Big|_{\beta=0}, \tag{3.28}$$

we take *only* the linear part of the right-hand side

$$F(y, y', t)\Big|_{\beta=0} \sim \frac{\partial F|_{\beta=0}}{\partial y}\Bigg|_{y=0, y'=0} y + \frac{\partial F|_{\beta=0}}{\partial y'}\Bigg|_{y=0, y'=0} y'. \tag{3.29}$$

The following linearizations are obtained in this way [73]

- $P_{VI} \to$ Gauss hypergeometric equation;
- $P_V \to$ Confluent hypergeometric equation $\sim$ Kummer;
- $P_V|_{\delta=0, \gamma=-2} \to$ Reduced confluent hypergeometric equation $\sim$ Bessel;
- $P_{IV} \to$ Biconfluent hypergeometric equation $\sim$ Hermite;
- $P_{II} \to$ Reduced Biconfluent hypergeometric equation $\sim$ Airy.

Only P_{III} and P_I do not enter in such correspondence.

3.2.3 Symmetries

Discrete symmetries of Painlevé equations have been studied by Painlevé [118] and, more recently, by Okamoto [135]. These symmetries naturally act on the space of parameters $V = (\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. So, for example, in the case of P_{VI}, if we change the variables $(t, y) \to (1 - t, 1 - y)$, we have that $(\alpha, \beta, \gamma, \delta) \to (\alpha, \gamma, \beta, \delta)$. Therefore, a symmetry of P_{VI} is given by a *birational transformation* $T: (t, y) \to (t_1, y_1)$ and an affine transformation $\ell : V \to V$ which is just a permutation of V. If we consider an element of the group G of symmetries as a pair $\sigma = (T, \ell)$, we can show that G has 3 generators $\sigma_i = (T_i, \ell_i)$ given by

$$\begin{aligned}
&T_1 : y \to 1 - y, \quad t \to 1 - t, && l_1 : (\alpha, \beta, \gamma, \delta) \to (\alpha, \gamma, \beta, \delta), \\
&T_2 : y \to \frac{1}{y}, \quad t \to \frac{1}{t}, && l_2 : (\alpha, \beta, \gamma, \delta) \to (\beta, \alpha, \gamma, \delta), \\
&T_3 : y \to \frac{y - t}{1 - t}, \quad t \to \frac{t}{t - 1}, && l_3 : (\alpha, \beta, \gamma, \delta) \to (\alpha, \delta, \gamma, \beta).
\end{aligned}$$

By the action of l_i on the parameter space, we see that G is isomorphic to the symmetric group S_4. Also notice that G leaves the set of critical points $\{0, 1, t, \infty\}$ invariant.

Okamoto has also discovered that there are *Bäcklund transformations* on the variables $(q, p; \alpha, \beta, \gamma, \delta)$ keeping the Painlevé Hamiltonian structure invariant [135]. This symmetry generates an infinite number of new solutions of Painlevé Hamiltonian system, and can be understood as canonical transformations on the phase space. These transformations can also be understood in geometrical terms [110]. Now let us talk about Painlevé Hamiltonian structure.

3.2.4 Hamiltonian Structure

Painlevé equations have a Hamiltonian structure and we can rewrite each P_J as a Hamiltonian system $\mathcal{H}_J$ of canonically conjugate variables $(q(t), p(t))$ and Hamiltonian H_J. This can be explicitly proved by analyzing series expansions around the critical points $t = 0, 1, \infty$. In fact, suppose we give an initial condition to P_{VI} at the point $t_0 \in \mathbb{CP}^1 \backslash \{0, 1, \infty\}$ such that $y(t_0) = \zeta$, with $\zeta \in \{0, 1, \infty, t_0\}$. This is a *singular initial value problem*, because P_{VI} equation is singular at those points. Notice that this kind of behaviour does not appear in linear differential equations. For generic P_{VI} parameters, one can show that there are two 1-parameter families of solutions which are analytic at t_0 satisfying $y(t_0) = \zeta$. Using those series expansions, one can do a kind of resummation of the series derivative to obtain the canonically conjugate variable and, thus, the Hamiltonian system valid around t_0 for some chosen ζ. This can be done for each of the four ζ values and, after "gluing" the 4 solutions, we have the complete Hamiltonian system for all possible singular initial conditions. This has to be done because, although the solution is analytic for a certain choice of $y(t_0) = \zeta$, the series expansion converges only up to the next singular point. This entails a somewhat complicate construction, outlined in [74]. From the theory of isomonodromic deformations, this Hamiltonian structure appears more naturally, as we shall see later.

There is an even easier way to obtain Painlevé Hamiltonians discovered by Slavyanov [136, 137], which is summarized by this simple statement:

> *Painlevé equations are the classical analogues of Heun equations.*

This means that for Heun equation and each of its confluent reductions we have an associated Painlevé equation. Here, we mention only the case of P_{VI} because all other cases can be obtained from it by confluence. Considering the Heun equation (2.131) as a Schrödinger equation, we can make $q = z$ and $p = i\partial_z$ to obtain its associated Hamiltonian[8]

$$K(q,p,t) = \frac{1}{t(t-1)}\Big[q(q-1)(q-t)p^2 - \Big(\theta_0(q-1)(q-t) + \theta_1 q(q-t) + (\theta_t-1)q(q-1)\Big)p + \kappa_1\kappa_2(q-t)\Big]. \tag{3.30}$$

The Hamiltonian system $\mathcal{H}_{VI}$ generated by K is thus given by

$$\frac{dq}{dt} = \frac{\partial K}{\partial p} = \frac{1}{t(t-1)}\Big[2q(q-1)(q-t)p - \Big(\theta_0(q-1)(q-t) + \theta_1 q(q-t) + (\theta_t-1)q(q-1)\Big)\Big], \tag{3.31a}$$

$$\frac{dp}{dt} = -\frac{\partial K}{\partial q} = -\frac{1}{t(t-1)}\Big[(q(q-1)+(q-1)(q-t)+q(q-t))p^2 - ((\theta_0+\theta_1)(q-t)+(\theta_0+\theta_t-1)(q-1)+(\theta_1+\theta_t-1)q)p + \kappa_1\kappa_2\Big]. \tag{3.31b}$$

After combining both equations, one can show that $q(t)$ obeys P_{VI} equation (3.26f) with

$$\alpha = \tfrac{1}{2}\theta_\infty^2, \quad \beta = \tfrac{1}{2}\theta_0^2, \quad \gamma = \tfrac{1}{2}\theta_1^2, \quad \delta = \tfrac{1}{2}\theta_t^2. \tag{3.32}$$

Another way is to make a Legendre transformation on $K(q,p,t)$ to obtain the Lagrangian and then find the Euler-Lagrange equations of motion, corresponding to the classical equations of motion of this Hamiltonian system. The Hamiltonian systems of the other P_J can also be obtained by taking appropriate confluent limits of $\mathcal{H}_{VI}$, analogously to the diagram in Figure 3.3 [74].

3.3 Isomonodromic Deformations of Fuchsian Systems

In this section, we review the theory of isomonodromic deformations and its relation with Painlevé's connection problem. The theory of isomonodromic deformations has been fully developed in the early 80's by Jimbo, Miwa and Ueno [97, 98, 99] and it has been a very fruitful tool in mathematical physics, with applications in gauge theory and integrability. However, the seeds of this theory come from a long time ago since the pioneer work of Fuchs, along with the contributions of Garnier [138] and Schlesinger [139]. Essentially, this theory is about how to deform certain parameters of Fuchsian systems such that its monodromy data is preserved. Jimbo and collaborators have found a way to construct such deformations of meromorphic

[8]We rescale and redefine Heun's parameters to conveniently match with P_{VI} equation.

linear systems with n singular points, generalizing earlier works in less generic settings. For a review on this subject, we refer to [74, 120].

As it must be clear by now, we want to address here Fuchsian systems with $n = 4$ singular points. It happens that isomonodromic deformation equations in this case can be recast as the Painlevé VI equation [74]. How this comes about will be discussed below.

3.3.1 Fuchsian Systems, Apparent Singularities and Isomonodromy

We have seen in section 3.1 that the number of parameters defining a second-order Fuchsian ODE is equal to the number of parameters defining a monodromy representation of the ODE's singular points. In the $n = 4$ case, we have that the ODE paramaters $\mathcal{E} = \{c, t; \theta_i\}$ map to the trace coordinates $\mathcal{M} = \{\sigma_{ij}; \theta_i\}$ and we have 6 irreducible parameters on both sides. In general, the space of irreducible Fuchsian equations $\mathcal{E}$ has $3n-6$ parameters, as we have seen in section 3.1.2, matching the dimension of $\mathcal{M}$, the moduli space of $\mathrm{SL}(2,\mathbb{C})$ representations of the n-point monodromy group, explained in section 3.1.1.

Let us do the parameter counting now in another way. Consider a Fuchsian ODE with n finite singular points $\{z_i\}$, given by

$$\partial_z^2 y + P(z)\partial_z y + Q(z) y = 0. \tag{3.33}$$

Being a Fuchsian ODE implies that there is a gauge such that

$$P(z) = \sum_{i=1}^{n} \frac{A_i}{z - z_i}, \tag{3.34a}$$

$$Q(z) = \sum_{i=1}^{n} \left(\frac{B_i}{(z-z_i)^2} + \frac{C_i}{z - z_i} \right), \tag{3.34b}$$

with $3n$ complex parameters (A_i, B_i, C_i) and

$$\sum_{i=1}^{n} A_i = 2, \quad \sum_{i=1}^{n} C_i = 0, \quad \sum_{i=1}^{n} (z_i C_i + B_i) = 0, \quad \sum_{i=1}^{n} (z_i^2 C_i + z_i B_i) = 0, \tag{3.35}$$

are the 4 conditions implying that $z = \infty$ is a regular point. Therefore, we have $3n-4$ complex parameters parametrizing (3.33) with fixed z_i. However, we can also introduce the singular points into our counting, giving $4n-4$ parameters. These can be reduced by 3 homographic transformations, fixing 3 singular points to $\{0, 1, \infty\}$, and $n-1$ homotopic transformations, one for each finite z_i (see appendix C). Therefore, we are left with $4n-4-(n+2) = 3n-6$ irreducible parameters.

As we have seen in section 2.2, Fuchsian systems have a more natural relation with monodromies by means of a flat connection. So it is interesting also to count the number of parameters of these systems. Consider thus a Fuchsian system of differential equations

$$\partial_z \mathcal{Y}(z) = A(z)\mathcal{Y}(z), \qquad A(z) = \sum_{i=1}^{n} \frac{A_i}{z - z_i}, \tag{3.36}$$

where $\mathcal{Y}(z) = (y_1(z) \;\; y_2(z))^T$ and the GL$(2,\mathbb{C})$ matrices A_i obeying the regularity condition

$$\sum_{i=1}^{n} A_i = 0. \tag{3.37}$$

In general, this system has $4n-4$ parameters, without considering the z_i, and if we also quotient out an SL$(2,\mathbb{C})$ conjugation, we are left with $4n-7$ parameters. Comparing with our discussion above, we have $n-3$ extra parameters than in the Fuchsian ODE with $3n-4$. Now what is the interpretation of these parameters in terms of (3.33)?

Let us write

$$A(z) = \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix}. \tag{3.38}$$

It is easy to verify that $y_1(z)$ obeys the differential equation (3.33) with

$$P(z) = -\partial_z \log A_{12} - \operatorname{Tr} A(z), \tag{3.39a}$$

$$Q(z) = \det A(z) - \partial_z A_{11} + A_{11}\partial_z \log A_{12}. \tag{3.39b}$$

The choice of TrA and A_{12} mostly determine which ODE gauge we are considering. However, there is an additional interesting thing about A_{12}; its zeros correspond to *apparent singularities* of the ODE in an appropriate gauge. For the sake of the reader, we remember that apparent singularities are the ones having trivial monodromy although appearing as poles in (3.33). Now, this definition becomes clear if we make a one-to-one correspondence between the Fuchsian ODE with the Fuchsian system (3.36), because zeros of A_{12} do not contribute to residues of $A(z)$.

It is simpler if we restrict to SL$(2,\mathbb{C})$ monodromy representations, obtainable by taking Tr$A = 0$, which implies that

$$A(z) = \begin{pmatrix} B(z) & C(z) \\ D(z) & -B(z) \end{pmatrix}, \tag{3.40}$$

and in terms of the partial fraction decomposition in (3.36), we have that Tr$A_i = 0$. The monodromy matrices are thus given by

$$M_i \simeq \exp\left(\oint_{\gamma_i} \frac{A_i}{z - z_i} dz \right) = e^{2\pi i A_i} \simeq \begin{pmatrix} e^{-i\pi\theta_i} & 0 \\ 0 & e^{i\pi\theta_i} \end{pmatrix}, \tag{3.41}$$

similarly to our definition in section 2.2. Thus the constant matrices A_i are 2×2 traceless matrices with $\det A_i = -\theta_i^2/4$. This gauge choice thus correspond to a self-adjoint form of (3.33).

We now consider $z = \infty$ a singular point and we define $A_\infty = -\sum_{i=1}^{n} A_i$. Under an SL(2,$\mathbb{C}$) conjugation, we choose a gauge in which this point has diagonal monodromy, that is $A_4 = -\frac{\theta_\infty}{2}\sigma^3$. This gauge choice thus implies that $C(z)$ and $D(z)$ in (3.40) decay as $1/z^2$ as $z\to\infty$, so both have the form

$$\frac{p_{n-3}(z)}{\prod_{i=1}^{n-1}(z-z_i)}, \tag{3.42}$$

where $p_{n-3}(z)$ is a polynomial of order $n-3$. So if this polynomial has $n-3$ simples zeros, those will correspond to apparent singularities of (3.33). However, we notice that this is no trivial assumption, because there is some amount of work to deductively show this to be the case for the corresponding Fuchsian ODE. For a complete demonstration of the statements made here in the generic n-point case, we refer to [74].

It is easier to understand the relation between Fuchsian systems and ODEs if we start by making some assumptions about the ODE. Let us focus on the $n = 4$ case for simplicity. We want to study deformations of our Fuchsian ODE which preserve the monodromies θ_i. With this goal in mind, we introduce an apparent singularity at $z = \lambda$ in its Riemann scheme

$$\begin{pmatrix} z=z_i & z=\infty & z=\lambda \\ 0 & \alpha & 0 \\ \theta_i & \alpha+\theta_\infty & 2 \end{pmatrix}, \tag{3.43}$$

with $z_i = 0,1,t$ for $i = 0,1,t$. Fuchs relation implies that

$$\alpha = -\frac{1}{2}\left(\sum_{i=1}^{3}\theta_i + \theta_\infty - 1\right). \tag{3.44}$$

The Riemann scheme above also implies that (3.33) is in a natural canonical form. This can be explicitly shown by imposing this Riemann scheme in (3.34a). Therefore, we can write the Fuchsian ODE in terms of $(\theta_i;\lambda,\mu,K)$ as

$$P(z) = \sum_{i=1}^{3}\frac{1-\theta_i}{z-z_i} - \frac{1}{z-\lambda} \tag{3.45a}$$

$$Q(z) = \frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \tag{3.45b}$$

where K and μ are new ODE parameters. We say that an ODE with the above form is of *Garnier type* [138, 74]. For our purposes, we may also call it a *deformed Heun equation*, because it is essentially a Heun equation with one extra apparent singularity. Notice that this equation is similar to what we have encountered in the Kerr-NUT-(A)dS radial equation, starting with 5 singular points and removing an apparent singularity when $\xi = 1/6$. However, here we have a singularity with local coefficients $(0,2)$ and in the radial equation we had a singularity of type $(0,1)$, so what follows concerns a different type of apparent singularity as a deformation of the Heun equation (2.131).

The point $z = \lambda$ will be an apparent singularity if K is a specific rational function of λ, μ and t. This means that (3.33) with coefficients (3.45) has a regular series solution at $z = \lambda$ if and only if

$$K(\lambda,\mu,t) = \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) + (\theta_t-1)\lambda(\lambda-1)\}\mu + \kappa(\lambda-t)]. \quad (3.46)$$

This is explicitly shown in appendix B. For a Garnier ODE with $n+3$ singular points and n apparent singularities similar formulas also follow and we refer to [74] for this general case.

We notice that $K(\lambda,\mu,t)$ above corresponds exactly to the P_{VI} Hamiltonian (3.30). Therefore, we can think of the Hamiltonian flow $(\lambda(t),\mu(t))$ generated by $K(\lambda,\mu,t)$ to correspond to an *isomonodromic deformation* of the Garnier ODE, and we say these form a *Garnier system* $\mathcal{G}_1$. This evolution is isomonodromic because the local Frobenius coefficients of $z = 0,1,t$ do not depend on λ,μ and K, so we can change them as long as $z = \lambda$ still remains an apparent singularity. Summarizing, we are left with a 3-parameter family of ODEs $E_\theta(\lambda,\mu,t)$ with fixed monodromy, parametrized by t and the initial conditions $(\lambda(t_0),\mu(t_0))$. So the extended phase space of the Garnier system has a symplectic structure given by

$$\Phi^*\Omega = d\mu \wedge d\lambda - dK \wedge dt, \quad (3.47)$$

where Φ^* represents the pullback of the Atiyah-Bott symplectic form to the extended phase space (μ,λ,K,t).

Now, for completeness, we want to find a flat connection

$$A(z) = \sum_{i=1}^{3} \frac{A_i}{z-z_i}, \qquad A_i = \begin{pmatrix} A^i_{11} & A^i_{12} \\ A^i_{21} & A^i_{22} \end{pmatrix} \quad (3.48)$$

corresponding to the Garnier ODE above such that $z = \lambda$ have trivial monodromies. In this case, it is convenient to choose a gauge where $\mathrm{Tr} A_i = \theta_i$. Comparing (3.45) with (3.39), we find that

$$A_{12}(z) = k\frac{z-\lambda}{z(z-1)(z-t)}, \quad k \in \mathbb{C}, \quad (3.49)$$

and, thus, the apparent singularity correspond to a simple zero of A_{12}, as we expected. By the partial fraction expansion (3.48) and (3.39), we see that a pole at $z = \lambda$ only comes from A_{12}, therefore

$$\mu = \operatorname*{Res}_{z=\lambda} Q(z) = \operatorname*{Res}_{z=\lambda} \frac{A_{11}(z)}{z-\lambda} = \sum_{i=1}^{3} \frac{A_{11}^{i}}{\lambda - z_i}. \tag{3.50}$$

In a similar fashion, we can show that

$$\begin{aligned} K &= -\operatorname*{Res}_{z=t} Q(z) \\ &= \frac{A_{11}^{t}}{\lambda - t} + \frac{A_{11}^{0} + A_{11}^{t} - \theta_0\theta_t}{t} + \frac{A_{11}^{1} + A_{11}^{t} - \theta_1\theta_t}{t-1} \\ &\qquad + \frac{1}{t}\operatorname{Tr} A_0 A_t + \frac{1}{t-1}\operatorname{Tr} A_1 A_t. \end{aligned} \tag{3.51}$$

This shows the explicit relation between the Garnier ODE and the Fuchsian system. Thus, in principle, we can also study isomonodromic deformations of an associated Fuchsian system and connect with Garnier systems [74]. The deformation equations of a Fuchsian system form what is typically called a *Schlesinger system*. This system is also integrable and its hamiltonian structure is connected to the Garnier system by a canonical transformation. Therefore, both systems are equivalent from this point of view. The Schlesinger system is important for the study of the generic n-point system and, as we will discuss below, is also important for the generic scattering in d dimensions because there we have more than 4 singular points. In this case, solutions of the Garnier system do not have the Painlevé property, in contrast with solutions of the Schlesinger system.

The whole point of discussing isomonodromic flows was to use this method to find the monodromy parameters of Heun equation. The Garnier ODE with (3.45) can be related to our Heun equation by taking $\lambda = t_0$ at $t = t_0$. This gives an initial condition for our hamiltonian flow.

Now let us relate the coefficients of the Garnier ODE with Heun equation (2.131). First, consider the Garnier ODE written as

$$\begin{aligned} &\partial_z^2 y + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda} \right) \partial_z y \\ &\qquad + \left(\frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)} \right) y = 0. \end{aligned} \tag{3.52}$$

So if we set $\lambda(t_0) = t_0$ with $t_0 \neq 0, 1, \infty$, we have that $\theta_t = \theta_{t_0} - 1$ and

$$K(\lambda = t_0, \mu_0, t_0) - \mu_0 = -\mu_0\theta_t = K_0 \quad \Rightarrow \quad \mu_0 = \frac{K_0}{1-\theta_{t_0}}. \tag{3.53}$$

So by using the specific values of K_0 and θ_i, we have an initial condition for the isomonodromic flow.

3.4 Schlesinger System Asymptotics and Painlevé VI

Now that we know how to connect Heun equation with isomonodromic deformations, we need to understand how to recover the trace parameter σ from Painlevé asymptotics. In this section, we review Jimbo's work [104] solving Painlevé VI connection problem by means of the monodromy data of a Fuchsian system with 4 singular points. A full tabulation of P_{VI} connection formulas in terms of monodromy data has been given in [133], completing Jimbo's results. A review on several P_{VI} properties is given in [134].

First, consider a Fuchsian system

$$\partial_z Y(z,t) = A(z,t)Y(z,t), \qquad A(z,t) = \frac{A_0(t)}{z} + \frac{A_1(t)}{z-1} + \frac{A_t(t)}{z-t}, \tag{3.54}$$

with 4 singular points at $z = 0, 1, t, \infty$ and $Y(z,t)$ a fundamental matrix of this system. We explicitly suppose that the system depends on t as a deformation parameter. A Fuchsian system of this type is called a *Schlesinger system* [139]. We choose a gauge where $\mathrm{Tr} A_\mu = 0$ and $\mathrm{Tr} A_\mu^2 = -\theta_\mu^2/4$, $\mu = 0, 1, t, \infty$. The behaviour at infinity is fixed by

$$A_\infty \equiv -(A_0 + A_1 + A_t) = -\frac{\theta_\infty}{2}\sigma^3, \tag{3.55}$$

and we suppose that θ_μ, $\mu = 0, 1, t, \infty$ are not integers[9]. We introduce a new differential system with respect to the deformation parameter

$$\partial_z Y(z,t) = A(z,t)Y(z,t), \tag{3.56}$$

$$\partial_t Y(z,t) = B(z,t)Y(z,t). \tag{3.57}$$

These two equations form a Lax pair and the system is *Frobenius integrable*[10] if and only if

$$\partial_t A - \partial_z B + [A, B] = 0. \tag{3.58}$$

Suppose that the Schlesinger system admits *isomonodromic* solutions, i.e., solutions whose monodromy matrices do not depend on t. Therefore, one can show that [74]

$$B(z,t) = -\frac{A_t(t)}{z-t}. \tag{3.59}$$

[9] For comparison with our convention for the Garnier system, it is convenient to take $\theta_\infty \to \theta_\infty - 1$.

[10] That is, it obeys $\partial_t \partial_z Y = \partial_z \partial_t Y$.

In this case, the *Schlesinger equations* of (3.56), obtained by plugging (3.59) into (3.58) are given by

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}, \qquad \frac{dA_t}{dt} = \frac{[A_0, A_t]}{t} + \frac{[A_1, A_t]}{t-1}. \tag{3.60}$$

These equations also present a hamiltonian structure [114] and we thus say they form a Schlesinger system $\mathcal{S}_1$ [74]. As we have seen above, if we let the component A_{12} to be in the form (3.49), then $\lambda(t)$ obeys P_{VI} equation with

$$\alpha = \tfrac{1}{2}\theta_\infty^2, \quad \beta = \tfrac{1}{2}\theta_0^2, \quad \gamma = \tfrac{1}{2}\theta_1^2, \quad \delta = \tfrac{1}{2}\theta_t^2. \tag{3.61}$$

This result can also be obtainable from Schlesinger equations, but it is not so straightforward to reach directly [98, 74]. A more convenient way to do this is by relating the Schlesinger system with the Garnier system. We change to the canonical gauge where $\mathrm{Tr} A_i = 0$ and $\det A_i = -\theta_i^2/4$. This is obtained simply by doing $A_i \to A_i + \frac{\theta_i}{2}\mathbb{1}$, which corresponds to a gauge transformation with $U(z) = T(z)\mathbb{1}$ and $T(z) = \prod_{i=1}^{3}(z - z_i)^{\theta_i/2}$ (see appendix C). In this new gauge, we already found a relation between the connection $A(z)$ and the Garnier system in the last section.

We want to study Schlesinger system asymptotics to obtain the asymptotics of $\lambda(t)$. If we obtain the asymptotic expansions of $A_i(t)$ at the critical points $t = 0, 1, \infty$, we can relate those with $A_{12}(t)$ and find the asymptotics of $\lambda(t)$. We need to study only the $t = 0$ asymptotics, because the other cases can be obtained by homographic transformations. In the following, we review the work of Jimbo in [104].

Let the monodromy group be parametrized as in (3.16). The matrices C_i diagonalizing each M_i are the connection matrices of the Riemann-Hilbert problem

$$Y(z,t) = (G_0(t) + O(z))\, z^{\theta_0\sigma^3/2} C_0 \qquad (z \to 0) \tag{3.62}$$

$$= (G_1(t) + O(z-1))\, z^{\theta_1\sigma^3/2} C_1 \qquad (z \to 1) \tag{3.63}$$

$$= (G_t(t) + O(z-t))\, z^{\theta_t\sigma^3/2} C_t \qquad (z \to t) \tag{3.64}$$

$$= (1 + O(z^{-1}))\, z^{-\theta_\infty\sigma^3/2} \qquad (z \to \infty). \tag{3.65}$$

We find the connection matrices, for example, by diagonalizing the monodromies (3.16). However, how did we find the monodromy parametrization (3.16)? We shall see in the following.

In the $t \to 0$ approximation, Schlesinger equations (3.60) reduce to

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \quad \frac{dA_1}{dt} = -[A_t, A_1], \quad \frac{dA_t}{dt} \approx \frac{[A_0, A_t]}{t} + O(t^0). \tag{3.66}$$

This means that, near $t = 0$, A_t and A_0 have a logarithmic divergence

$$A_0 \approx t^{\Lambda} A_0^0 t^{-\Lambda}, \quad \text{and} \quad A_t \approx t^{\Lambda} A_t^0 t^{-\Lambda} \quad , \text{where } \Lambda = A_0^0 + A_t^0, \tag{3.67}$$

whereas A_1 has a continuous limit as $t \to 0$. In terms of the fundamental matrix $Y(z,t)$ in (3.54) the system splits into an equation for $Y_0(z) = \lim_{t\to 0} Y(z,t)$ and another for $Y_1(z) = \lim_{t\to 0} t^{-\Lambda} Y(tz,t)$

$$\frac{dY_0}{dz} = \left(\frac{\Lambda}{z} + \frac{A_1^0}{z-1}\right) Y_0, \qquad \frac{dY_1}{dz} = \left(\frac{A_0^0}{z} + \frac{A_t^0}{z-1}\right) Y_1. \tag{3.68}$$

This is necessary because Y_0 addresses the smooth part of the connection, while Y_1 represents the singular parts. Each problem gives a hypergeometric connection whose solution is described in Appendix D. Assuming the general case where there is no integer difference between the exponents, the solutions are

$$Y_0 = Y\Big(\tfrac{1}{2}(\theta_\infty - \theta_1 - \sigma),\ -\tfrac{1}{2}(\theta_\infty + \theta_1 + \sigma),\ 1-\sigma;\ z\Big) z^{-\sigma/2}(z-1)^{-\theta_1/2}, \tag{3.69}$$

$$Y_1 = G_1 Y\Big(-\tfrac{1}{2}(\theta_0 + \theta_t + \sigma),\ -\tfrac{1}{2}(\theta_0 + \theta_t - \sigma),\ 1-\theta_0;\ z\Big) C_1 z^{-\theta_0/2}(z-1)^{-\theta_t/2}, \tag{3.70}$$

where $Y(a,b;c;z)$ represents a hypergeometric fundamental solution (see Appendix D) and

$$G_1 = G_{abc}^{(0)} \begin{pmatrix} 1 & 0 \\ 0 & -\hat{s}^{-1} \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -\hat{s}^{-1} \end{pmatrix} C_{abc}^{(0)} \tag{3.71}$$

with

$$\begin{aligned} \hat{s} = {} & \frac{\Gamma(1-\sigma)^2 \Gamma(\frac{1}{2}(\theta_0 + \theta_t + \sigma) + 1)\Gamma(\frac{1}{2}(-\theta_0 + \theta_t + \sigma) + 1)}{\Gamma(1+\sigma)^2 \Gamma(\frac{1}{2}(\theta_0 + \theta_t - \sigma) + 1)\Gamma(\frac{1}{2}(-\theta_0 + \theta_t - \sigma) + 1)} \times \\ & \frac{\Gamma(\frac{1}{2}(\theta_\infty + \theta_1 + \sigma) + 1)\Gamma(\frac{1}{2}(-\theta_\infty + \theta_1 + \sigma) + 1)}{\Gamma(\frac{1}{2}(\theta_\infty + \theta_1 - \sigma) + 1)\Gamma(\frac{1}{2}(-\theta_\infty + \theta_1 - \sigma) + 1)} s, \end{aligned} \tag{3.72}$$

and the parameter s is given by (3.18).

Now, each A_i^0 and Λ are traceless SL$(2,\mathbb{C})$ matrices

$$A_i^0 = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \tag{3.73}$$

constrained by $\det A_i^0 = -\theta_i^2/4$, $\det \Lambda = -\sigma^2/4$ and (3.55). This is enough to determine the A_i^0

up to two parameters σ and s. Using this procedure, we arrive at

$$\Lambda + \frac{1}{2}\sigma\mathbb{1} = \frac{1}{4\theta_\infty}\begin{pmatrix} (-\theta_\infty - \theta_1 + \sigma)(\theta_\infty - \theta_1 - \sigma) & (-\theta_\infty - \theta_1 + \sigma)(\theta_\infty + \theta_1 + \sigma) \\ (\theta_\infty - \theta_1 + \sigma)(\theta_\infty - \theta_1 - \sigma) & (\theta_\infty - \theta_1 + \sigma)(\theta_\infty + \theta_1 + \sigma) \end{pmatrix}; \tag{3.74a}$$

$$A_1^0 + \frac{1}{2}\theta_1\mathbb{1} = \frac{1}{4\theta_\infty}\begin{pmatrix} -(\theta_\infty - \theta_1)^2 + \sigma^2 & (\theta_\infty + \theta_1)^2 - \sigma^2 \\ -(\theta_\infty - \theta_1)^2 + \sigma^2 & (\theta_\infty + \theta_1)^2 - \sigma^2 \end{pmatrix}; \tag{3.74b}$$

$$A_0^0 + \frac{1}{2}\theta_0\mathbb{I} = G_1\frac{1}{4\sigma}\begin{pmatrix} (\theta_0 - \theta_t + \sigma)(\theta_0 + \theta_t + \sigma) & (\theta_0 - \theta_t + \sigma)(-\theta_0 - \theta_t + \sigma) \\ (\theta_0 - \theta_t - \sigma)(\theta_0 + \theta_t + \sigma) & (\theta_0 - \theta_t - \sigma)(-\theta_0 - \theta_t + \sigma) \end{pmatrix}G_1^{-1}; \tag{3.74c}$$

$$A_t^0 + \frac{1}{2}\theta_t\mathbb{I} = G_1\frac{1}{4\sigma}\begin{pmatrix} (\theta_t + \sigma)^2 - \theta_0 & -(\theta_t - \sigma)^2 + \theta_0^2 \\ (\theta_t + \sigma)^2 - \theta_0 & -(\theta_t - \sigma)^2 + \theta_0^2 \end{pmatrix}G_1^{-1}. \tag{3.74d}$$

To obtain the monodromy parameterization (3.16), we have to notice that

$$M_i = \exp(2\pi i B_i) = \cos(i\pi\theta_i) + 2B_i\frac{\sin(\pi\theta_i)}{\theta_i}, \tag{3.75}$$

where B_i is a traceless SL(2, $\mathbb{C}$) matrix with $\operatorname{Tr} B_i^2 = \theta_i^2/4$. The B_i are not exactly the A_i because of (3.41). Using the constraints of the monodromy group and some amount of algebra we can arrive at (3.16). However, the most straightforward way to arrive at (3.16) is by following the steps in [110].

For the asymptotics as $t \to 1$, one just need to change $\theta_0 \to \theta_1$, $\lambda(t) \to \lambda(t) - 1$ and $\sigma = \sigma_{1t}$. Finally, the asymptotic formula for the Painlevé transcendent itself, as in [133], is

$$\lambda(t) \simeq 1 + \frac{(\theta_t - \theta_1 + \sigma)(\theta_t + \theta_1 + \sigma)(\theta_\infty + \theta_0 + \sigma)}{4\sigma^2(\theta_\infty + \theta_0 - \sigma)\hat{s}}(1-t)^{1-\sigma}(1 + O(t^\sigma, t^{1-\sigma})), \tag{3.76}$$

assuming, as always, $0 \leq \operatorname{Re}\sigma < 1$.

Summarizing, the asymptotic behaviour of P_{VI} near its critical points is given by

$$\lambda(t) = \begin{cases} a_0 t^{1-\sigma_{0t}}(1 + O(t^\delta)), & |t| < r, \\ 1 + a_1(1-t)^{1-\sigma_{t1}}(1 + O((1-t)^\delta), & |t-1| < r, \\ a_\infty t^{\sigma_{01}}(1 + O(t^{-\delta})), & |1/t| < r, \end{cases} \tag{3.77}$$

where δ is a small positive number, $r > 0$ is a sufficiently small radius, $a_i \neq 0$ are functions of monodromy data and $0 \leq \operatorname{Re}\sigma_{ij} < 1$. In the particular case $\operatorname{Re}\sigma = 0$, there are three leading terms in the asymptotic expansions. At $t = 0$, we have, for example,

$$\lambda(t) = a_0 t^{1-\sigma_{0t}} + \frac{4A^2}{a_0}t^{1+\sigma_{0t}} + Bt + O(t^2), \tag{3.78}$$

with A and B being specific functions of monodromy data. If we set $\sigma_{0t} = 2i\nu_{0t}$, $\nu \in \mathbb{R}$, this formula can be rewritten as

$$\lambda(t) = t\big(A\sin(2\nu\log t + \phi) + B\big) + O(t^2), \qquad a_0 = \frac{A}{2i}e^{i\phi}, \tag{3.79}$$

still showing an oscillatory behaviour near the critical point.

Depending on the scattering problem we are interested, we can try to numerically obtain σ_{ij} by interpolation of P_{VI} solution near its critical points. In the next page, we show two example plots of the behaviour of P_{VI} near $t = 1$ for initial values obtained from Heun equation for the Kerr-dS case. By interpolation of this behaviour, with P_{VI} asymptotics above, we can obtain a value for σ_{1t}. If we want to plot the greybody factor as a function of $a\omega$, we need to obtain this monodromy parameter for each value by interpolation. That is our proposal to obtain the scattering coefficients via isomonodromy.

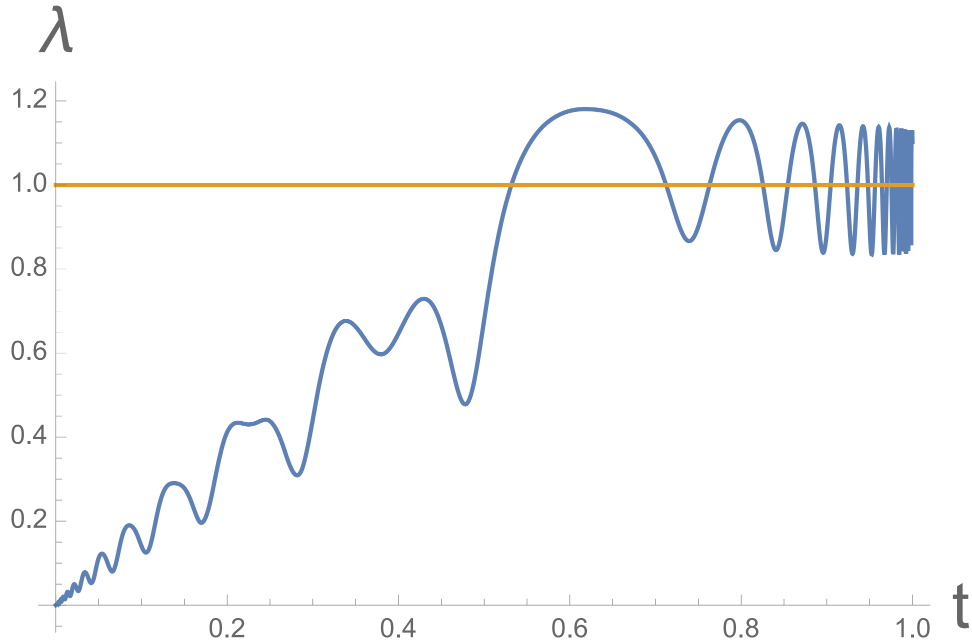

Figure 3.4: Behaviour of P_{VI} for a particular initial condition obtained from Kerr-dS Heun equation.

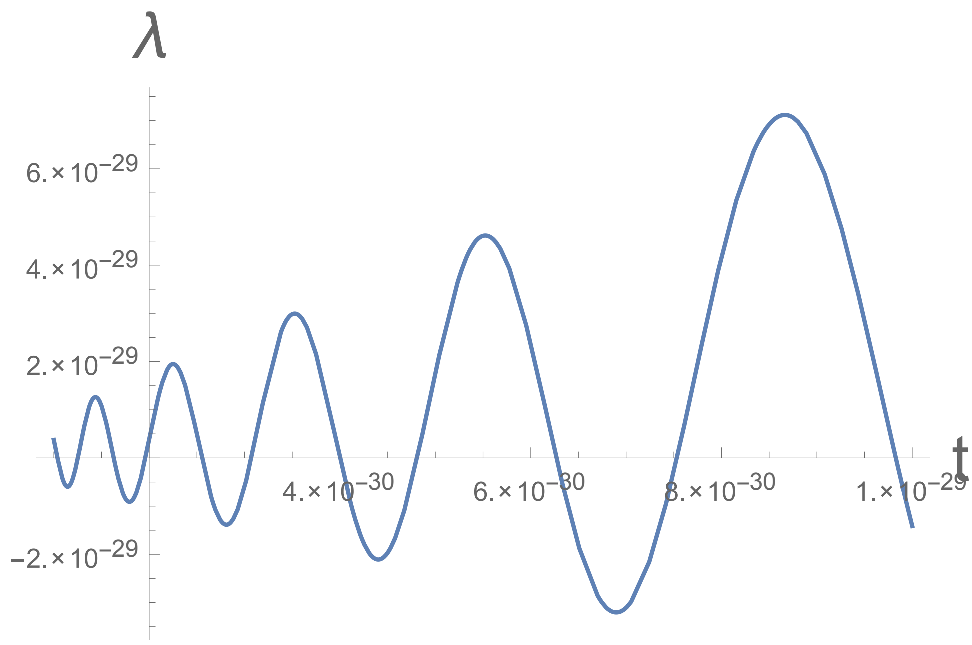

Figure 3.5: A zoom-in of the asymptotic region very close to $z = 0$ from fig. 3.4. This behaviour suggests a linear decaying oscillation and thus $\operatorname{Im}\sigma_{0t} = 0$. The fit must be done using (3.79)

CHAPTER 4

Scattering Theory and Hidden Symmetries

Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy.

—RICHARD FEYNMAN

After a long discussion about the relation between the monodromy group of Heun equation, isomonodromic deformations and Painlevé transcendents, we can finally propose a solution for the scattering problem we have set up. In this chapter, we show the explicit scattering matrix for a conformally coupled scalar field on a Kerr-(A)dS background. To obtain the scattering coefficients, we need to do some numerical work to find the $(A)dS$ angular eigenvalues, as discussed in section 2.3.3, and to find the composite trace parameter σ by numerical integration of P_{VI}, as discussed in 3.4. In the other hand, the plots obtained in the previous chapter for P_{VI} depend on the angular eigenvalue because the initial conditions depend on the parameters of Heun equation. As our numerical procedure still needs improvement with respect to this part, we postpone accurate numerical results for future work. So here we limit ourselves to discuss theoretical details of our proposal.

The NUT charge physically changes some properties of the black hole solution, but just slightly with respect to the black hole scattering problem, so, for a matter of simplicity, we set the NUT charge to zero in the following discussion. Of course, in AdS/CFT applications with a NUT charge, quasinormal modes and superradiant scattering will depend on the NUT charge. For more on the relation of NUT charge and AdS/CFT, see [48, 140, 141, 142, 90, 91, 92].

The scattering problem for Kerr-NUT-(A)dS black holes also presents a hidden symmetry which has not been noticed before in this context. The isomonodromic flow connects a 1-parameter family of Fuchsian equations with the same monodromy; thus, with the same scattering data. In particular, at the critical points of the flow, the differential equation becomes a hypergeometric equation, a fact essential in Jimbo's work revised in the last chapter. This allows us to relate the scattering of an arbitrary Kerr-NUT-(A)dS black hole with an *extremal* black hole solution in two special cases. We conjecture that this might explain some facts about the so-called *Kerr/CFT correspondence*, which we shall review in the end of this chapter [143, 33, 67].

4.1 Kerr-(A)dS Scattering

In this section, we discuss how to obtain scattering coefficients from monodromies for waves scattering on Kerr-(A)dS black holes. For convenience, we repeat here the relevant equations. The Kerr-(A)dS metric in Chambers-Moss coordinates is given by

$$ds^2 = -\frac{\Delta_r(r)}{(r^2+p^2)\chi^4}\left(dt - \frac{(a^2-p^2)}{a}d\phi\right)^2 + \frac{\Delta_p(p)}{(r^2+p^2)\chi^4}\left(dt - \frac{(r^2+a^2)}{a}d\phi\right)^2 + \frac{r^2+p^2}{\Delta_p(p)}dp^2 + \frac{r^2+p^2}{\Delta_r(r)}dr^2 \tag{4.1}$$

where $\chi^2 = 1 + \Lambda a^2/3 = 1 + a^2/L^2$, with the dS radius given by $L^2 = \Lambda/3$, and

$$\Delta_p(p) = -\frac{\Lambda}{3}p^4 - \epsilon p^2 + a^2, \tag{4.2a}$$

$$\Delta_r(r) = -\frac{\Lambda}{3}r^4 + \epsilon r^2 - 2Mr + a^2, \tag{4.2b}$$

$$\epsilon = 1 - \frac{a^2\Lambda}{3} = 1 - \frac{a^2}{L^2}\,, \tag{4.2c}$$

We have changed the notation here to match the usual notation in the literature for (4.2) (see, for example, [36]). The Kerr-AdS metric can be more appropriately written if we make $L \to iL$ and we change the expressions below conveniently when necessary. The 4 roots of $\Delta_r = 0$ correspond to the singular points of the metric. In the Kerr-dS case, we are interested when there are 4 real roots $(r_-, r_H, r_C, -r_H - r_- - r_C)$ and, in the Kerr-AdS case, we always have at least two complex roots and two real horizons $(r_H, r_C, \zeta, \bar{\zeta})$.

For each horizon, we define the Killing vectors $\xi_{H,C} = \partial_t + \Omega(r_{H,C})\partial_\phi$ such that they are null at each respective horizon. This entails to the constants $\Omega_{H,C} = \Omega(r_{H,C})$ being the angular velocities of each horizon. In particular, as discussed in the intro for the Kerr metric, this induces a frame-dragging effect near the event horizon, as no observer can stay stationary with respect to ∂_t and is forced to co-rotate with the horizon. The angular velocity and temperatures of the event and cosmological horizons for an observer following $\xi_{H,C}$ orbits are given by

$$\Omega_{H,C} = \frac{a}{r_{H,C}^2 + a^2}, \quad T_H = \pm\frac{\Delta_r'(r_H)}{4\pi\chi^2(r_H^2+a^2)}, \quad T_C = \mp\frac{\Delta_r'(r_C)}{4\pi\chi^2(r_C^2+a^2)}, \tag{4.3}$$

in which we choose the sign of $T_{H,C}$ to both be positive temperatures, which depends on the sign of Λ. Notice, however, that these quantities are not the appropriate thermodynamic variables, because this frame is rotating at infinity and at the cosmological horizon. So there is no notion of asymptotic timelike observer to define stationary quantities as in flat spacetime and we cannot write a proper first law of thermodynamics for these variables [144].

For asymptotically AdS spacetimes, we can go to a non-rotating frame at infinity and define the thermodynamical variables with respect to this frame [145, 144, 36]. As shown by Gibbons *et al* in [144], the proper thermodynamical variables are

$$\Omega_h = \chi^2\Omega_H + \frac{a}{L^2}, \quad T_h = \chi^2 T_H, \tag{4.4}$$

corresponding to a non-rotating frame at spatial infinity. Asymptotically dS spacetimes have an asymptotic SO(1,4) isometry which allows us to define a invariant vacuum under its isometries. In fact, one can show that there is a timelike Killing vector inside the cosmological horizon, and thus we can define our thermodynamical variables with respect to the orbits of this vector [52, 146].

Now we go back to the scattering problem. The radial equation of interest from section 2.3.5 is

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_{t_0}}{z-t_0}\right)y' + \left(\frac{1+\theta_\infty}{z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)}\right)y = 0, \tag{4.5}$$

where

$$t_0 = \frac{r_3-r_1}{r_3-r_4}\frac{r_2-r_4}{r_2-r_1}, \quad z_\infty = \frac{r_2-r_4}{r_2-r_1} \tag{4.6}$$

and

$$\theta_k = 2i\chi^2\left(\frac{\omega(r_k^2+a^2)-am}{\Delta_r'(r_k)}\right) = \frac{i}{2\pi}\left(\frac{\omega-\Omega_k m}{T_k}\right), \quad k = 0,1,t_0,\infty, \tag{4.7}$$

$$K_0 = -\frac{1}{t-z_\infty}\left[1 + \frac{r_3-r_4}{\Delta_r'(r_3)}\left(-\frac{2}{L^2}r_3^2 + C_\ell\right) - 2i\chi^2\frac{\omega(r_3r_4+a^2)-am}{\Delta_r'(r_3)}\right]. \tag{4.8}$$

Here we notice that, to numerically obtain C_ℓ it is convenient to make $C_\ell \to C_\ell + \chi^2(a^2\omega^2 - 2ma\omega)$ and put the angular equation in the (A)dS spheroidal harmonic form.

We want explicit expressions for the scattering coefficients between two regular singular points. The monodromy parametrization found by Jimbo was important to understand the structure of irreducible representations of the 4-point monodromy group. However, if we want to obtain scattering coefficients between two regular singular points from monodromy, we can use a simpler parametrization. First, for a regular singular point z_i, we know that $M_i = g_i^{-1} e^{i\pi\theta_i\sigma_3} g_i$, where

$$g_i = \exp(i\psi_i\sigma_3/2)\exp(\phi_i\sigma_1/2)\exp(i\varphi_i\sigma_3/2), \tag{4.9}$$

is a parametrization of SL(2, $\mathbb{C}$). We may set $\varphi_i = 0$ because they are immaterial for the monodromy matrices[1]. We want to solve a scattering problem which is purely ingoing or outgoing at one of the singular points. This means that one of the monodromies is in diagonal form. Therefore, the scattering matrix in this case is given by

$$\mathcal{M}_{i\to j} = g_i = \begin{pmatrix} e^{i\psi_i/2}\cosh(\phi_i/2) & ie^{-i\psi_i/2}\sinh(\phi_i/2) \\ -ie^{i\psi_i/2}\sinh(\phi_i/2) & e^{-i\psi_i/2}\cosh(\phi_i/2). \end{pmatrix} = \begin{pmatrix} 1/\mathcal{T} & \mathcal{R}/\mathcal{T} \\ \mathcal{R}'/\mathcal{T}' & 1/\mathcal{T}' \end{pmatrix}. \tag{4.10}$$

In our approach, we obtain the composite traces $m_{ij} = \operatorname{Tr} M_i M_j = 2\cos\pi\sigma_{ij}$ via Painlevé VI asymptotics. It is easy to check that, in the case of interest where M_j is diagonal, we have that

$$m_{ij} = 2\cos\pi\theta_i\cos\pi\theta_j - 2\sin\pi\theta_i\sin\pi\theta_j\cosh\phi_i, \tag{4.11}$$

which can be used to find

$$\mathcal{T}\mathcal{T}' = \cosh^{-2}(\phi_i/2) = \frac{2\sin\pi\theta_i\sin\pi\theta_j}{\cos\pi(\theta_i-\theta_j) - \cos\pi\sigma_{ij}}. \tag{4.12}$$

Notice that using the identity $\cos(a) - \cos(b) = 2\sin(b+a)/2\sin(b-a)/2$, we can rewrite the above formula as

$$\mathcal{T}\mathcal{T}' = \frac{\sin\pi\theta_i\sin\pi\theta_j}{\sin\frac{\pi}{2}(\sigma_{ij}+\theta_i-\theta_j)\sin\frac{\pi}{2}(\sigma_{ij}-\theta_i+\theta_j)}. \tag{4.13}$$

In this interesting form, we recover the hypergeometric case (2.38) if we notice that $\sigma_{ij} = \theta_k$ for $k \neq i, j$, following from the monodromy group identity.

The formula (4.13) will correspond to the classical transmission coefficient (greybody factor) when the parameters are purely real or imaginary, such that we have the interpretation of normalized fluxes of particles across some tunneling barrier. This parametrization will be enough for the Kerr-dS case, because we are typically interested in the scattering between the outer black hole horizon r_+ and the cosmological horizon r_C. In this case the θ_i are all purely imaginary. We also notice that there is no restriction on the number of singular points of the radial ODE for this greybody factor be derived. Therefore, in principle, this formula should work for higher dimensional (A)dS black holes for scattering between two regular singular points with ingoing/outgoing conditions.

[1]However, we notice that these might appear as overall constants in the fundamental matrix $\Phi = \Phi_i g_i$.

In the other hand, as discussed in chapter 2, spatial infinity, $r = \infty$ in the original coordinates of Klein-Gordon equation (2.106), is a removable singularity in the conformally coupled case and then is not a singular point of our Heun equation. The parametrization thus seems not convenient for Kerr-AdS scattering because we want scattering coefficients between the black hole outer horizon r_+ and spatial infinity $r = \infty$. Spatial infinity is mapped to the point $z = z_\infty$ so the fundamental matrix is given by $\Phi = \Phi_i g_i = \Phi_{z_\infty} g_{z_\infty}$. But z_∞ is just a regular point of our solution and Φ has a Taylor expansion around it in any basis. This implies that g_{z_∞} is just a matter of gauge and we can set it to unit. Therefore, (4.10) is also valid for Kerr-AdS, but in this case the boundary conditions are not so clear. We could start with a generic parametrization of the 5-point monodromy group with $M_{r=\infty}$ diagonal and then specialize to the case where $M_{r=\infty}$ is trivial. However, a general parametrization of the 5-point monodromy group is not available in the literature right know, so this problem is still open.

4.1.1 Greybody Factor for Kerr-dS

Here we restrict the discussion about our scattering formula for the Kerr-dS black hole. This black hole exists only when its outer horizon temperature $T_H \geq 0$ and $a/L < 1$, where $L^2 = 3/\Lambda$ is the de Sitter radius and a is the rotation parameter. This puts restrictions for the values of (Λ, M, a) in which there is a regular black hole solution [61]. These conditions correspond to the case where $Q(r)$ has 4 real roots (r_{--}, r_-, r_H, r_C), where $r_{--} \equiv -r_- - r_H - r_C$. From (4.3) we have that $T_H = K_H(r_C - r_H)$ and $T_C = K_C(r_C - r_H)$, with $K_{H,C} > 0$ for $r_C > r_H > r_-$. Therefore, we have that the extremality condition $T_H = 0$ is equivalent to $T_C = 0$ if $r_H = r_C$ and the extremal solution is actually an equilibrium solution, as the horizon temperatures are equal. For nonzero temperatures of both horizons, we always expect some heat flow in semiclassical processes.

These statements are graphically understood in Figure 4.1 showing $\Delta_r(r)$ for the particular case $L = 5$, $a = 0.5$ and $M = 0.8$. The negative root is not shown in the plot, thus the smallest root is r_- and the other two correspond to r_H and r_C. Looking at the plot, we see that the condition for $\Delta_r'(r_H) = 0$ can correspond to the merging of both singular points r_H and r_C. Of course, the same can happen if we merge r_H and r_-, which is another type of extremality. For more on Kerr-dS thermodynamics, see, for example, the pioneer work of Gibbons and Hawking [52].

We want to study the scattering between the black hole horizon r_H and the cosmological

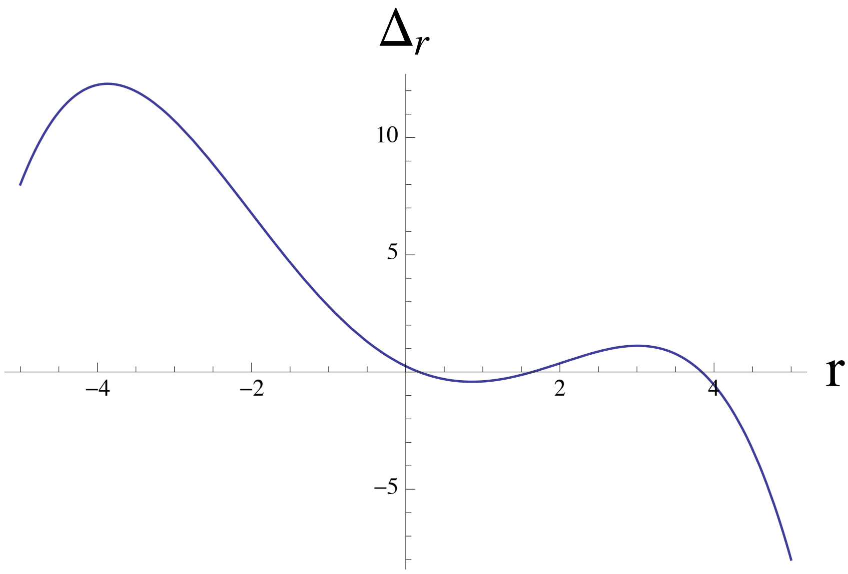

Figure 4.1: A plot of $\Delta_r(r)$ showing the singular points of the metric. The derivative $\Delta_r'(r)$ gives the behaviour of horizon temperatures.

horizon r_C. The Frobenius coefficients at these points are

$$\theta_{H,C} = \pm\frac{i}{2\pi}\left(\frac{\omega - \Omega_{H,C} m}{T_{H,C}}\right). \tag{4.14}$$

We choose opposite signs to θ_H and θ_C because while one of the waves is ingoing, the other is outgoing, and we have a radiation flow between both horizons. The transmission coefficient (4.12) is thus given by

$$\gamma_\ell(\omega, m) = |\mathcal{T}|^2 = \frac{\sinh(\frac{\omega-\Omega_H m}{2T_H})\sinh(\frac{\omega-\Omega_C m}{2T_C})}{\sinh\frac{1}{2}\left(\frac{\omega-\Omega_H m}{2T_H} + \frac{\omega-\Omega_C m}{2T_C} + \nu_{HC}\right)\sinh\frac{1}{2}\left(\frac{\omega-\Omega_H m}{2T_H} + \frac{\omega-\Omega_C m}{2T_C} - \nu_{HC}\right)}, \tag{4.15}$$

where we redefined $\pi\sigma_{HC}(\omega,\ell,m) = i\nu_{HC}(\omega,\ell,m)$. This is one of our main results in this thesis as we suggest that this structure must be the same even for higher-dimensional cases of scattering between two regular singular points. All the nontrivial information of the greybody factor in this form is coming from ν_{HC}, as its behaviour will strongly depend on global properties of the solutions. We believe for now that is as good as it gets with respect to having an explicit formula for the transmission coefficient.

For the greybody factor to be real and positive, we need to ensure several constraints. First, ν_{HC} must be real for γ_ℓ to be real. So physically we expect $\mathrm{Re}\,\sigma_{HC} = 0$ and the corresponding asymptotic behaviour for P_{VI} is given in the previous chapter[2]. Further, as $\Omega_H \geq \Omega_C$, for the

[2]We hope this can be checked in the future by perturbative expansions.

numerator to be positive we need either $\omega > \Omega_H m$ or $\omega < \Omega_c m$. For the intermediate case, $\Omega_H m > \omega > \Omega_C m$, we expect that the greybody factor is going to be negative, an indication of *superradiant scattering*. Of course, this depends also on the sign of the denominator of (4.15). Let

$$A(\omega,m) \equiv \frac{\omega - \Omega_H m}{2T_H} + \frac{\omega - \Omega_C m}{2T_C}. \tag{4.16}$$

It is best to write the denominator as $\cosh A - \cosh \nu_{HC}$. The denominator is positive if $|A| > |\nu_{HC}|$. We cannot guarantee this is always true inside the range $\Omega_H m > \omega > \Omega_C m$ without proper knowledge of ν_{HC}. By analysing the flux of radiation on both horizons, the authors of [147] support the suggested superradiant regime. However, as A necessarily becomes negative inside this regime, if ν_{HC} is real, we actually expect a divergence when $|A| = |\nu_{HC}|$ in the greybody factor for a real frequency! This is technically not a quasinormal mode, as the frequency is real. However, we notice that this equality actually corresponds to a *reducible* case of monodromy representations, so this limit has to be analysed more carefully with respect to the discussion in [104].

So we cannot prove superradiance occurs for the whole range $\Omega_H m > \omega > \Omega_C m$ and this deserve a more detailed analysis in the future. Two possibilities to obtain more information about the behaviour of σ_{HC} are to study perturbative expansions for ω near the superradiant boundaries or to perturbatively calculate σ_{HC} via P_{VI} asymptotics. In the following, we stick to the superradiant regime suggested above based on the evidences of [147].

What about the asymptotic behaviour of our expression? For low frequencies $\omega \to 0$, γ_ℓ goes to a constant if we neglect the ω dependence on ν_{HC} and $A(0,m) > \nu_{HC}$. If $\nu_{HC} \to (\frac{\Omega_H m}{2T_H} - \frac{\Omega_C m}{2T_C})$, we have that $\gamma_\ell \to 1$. If $\nu_{HC} \to \pm A$, the greybody factor diverges, as discussed above, but this does not seem physically reasonable. Those seem to be the easy guesses. Thus it seems reasonable to expect that γ_ℓ tends to a constant for small frequency when $m \neq 0$. This would correspond to a fully delocalized wave with finite angular momentum and with a nonzero probability to tunnel between the horizons.

In the $m = 0$ case, we might expect to obtain the familiar result of the Schwarzschild case where $\gamma_\ell \to 0$ for low frequencies, as the Compton length of the radiation is much larger than the black hole radius and the black hole becomes transparent to the radiation. However, in Schwarzschild-dS case, a curious behaviour happens with the greybody factor of a massless *minimally coupled* scalar field when $l = m = 0$: as $\omega \to 0$, γ_0 goes to a constant [148, 149, 150]. The authors of [149] suggest an explanation: a zero energy mode is fully delocalized between both horizons and thus there is a finite probability for this mode tunnel between the horizons, as we suggested above. This can also be obtained by this expression if ν_{HC} goes to zero in this limit in an appropriate way, as (4.15) should also work for the minimally coupled case. In the conformally coupled case though, we expect that γ_0 goes to zero for small frequency and $m = 0$.

For arbitrarily high frequencies, $\omega \to \infty$, we expect that $\gamma_l \to 1$, as this limit corresponds to zero impact parameter and very small wavelength, so all radiation must be absorbed. For very large frequency, we have

$$\gamma_\ell \to \frac{e^{\omega T_{HC}}}{e^{|\nu_{HC}|} + e^{\omega T_{HC}}} \to 1, \qquad T_{HC} \equiv \frac{1}{T_H} + \frac{1}{T_C} \tag{4.17}$$

if ν_{HC} grows slower than a power of ω.

Our scattering expression is very interesting because shows explicitly that the transmission coefficient is zero in the superradiant case $\omega = \Omega_H m$, corresponding to a pole in the scattering matrix. In fact, the poles of the scattering matrix are the complex zeros of (4.15)

$$\omega = \Omega_{H,C} m + 2\pi i n T_{H,C}, \tag{4.18}$$

for n an integer, in agreement with [151, 61]. For radiation with frequencies in the superradiant regime, the black hole can actually transfer part of its angular momentum to the wave, which scatters back to infinity with more energy than it initially had. If one puts a “mirror” surrounding the black hole, one can actually create what is called a *black hole bomb* [152, 153, 154]. The pressure of the radiation can increase without bound inside this contraption, thus the bomb alias. In fact, the scalar field mass can act similarly as a mirror because of its interaction with the gravitational potential of the black hole.

Summarizing, we expect the following regimes for the greybody factor (4.15)

$$\begin{cases} \omega > \Omega_H m \quad \text{or} \quad \Omega_C m > \omega & \text{Normal scattering} \\ \quad \Omega_H m > \omega > \Omega_C m & \text{Superradiant scattering} \end{cases} \tag{4.19}$$

These conditions can be related to the impact parameter classically given by $b = \mathcal{L}/\mathcal{E} \sim \ell/\omega$, with $\mathcal{L}$ and $\mathcal{E}$ the total angular momentum and energy of the incoming radiation. In fact, we have that

$$\frac{\omega}{m} = \frac{\omega}{\ell}\frac{\ell}{m} \sim \frac{1}{b}\frac{\mathcal{L}}{\mathcal{L}_z} \tag{4.20}$$

which is of order $1/b$ for an axisymmetric incoming wave. If b is very small, the radiation will be absorbed with high probability while if b is very large we expect no absorption. For a fine-tuned impact parameter, we expect superradiant scattering.

Finally, quasinormal modes correspond to complex frequencies such that γ_l diverges, that is, at the poles of (4.15). Taking a look at (4.12), we see that the poles are obtained from $\sigma_{HC} = \theta_H - \theta_C + 2n$, corresponding to the transcendental equation

$$\nu_{HC}(\omega, \ell, m) = A(\omega, m) + 2\pi i n, \qquad n \in \mathbb{Z}. \tag{4.21}$$

This is similar in structure with known results about quasinormal modes [34].

4.1.2 Scattering Through the Black Hole Interior

To finish this section, we comment on one of our speculations in [38] about scattering inside eternal black holes. Up to this point, we have studied the external patch of Kerr-(A)dS black holes in Chambers-Moss rotating coordinates. As in the Schwarzschild and Kerr case, we can study what is the global structure of Kerr-(A)dS spacetimes by analytic continuation of our coordinates to its maximally extended spacetime [151, 61]. The result is that these eternal black holes have an infinite number of asymptotically distinct regions, as shown in figure 4.2 for the Kerr-AdS black hole[3]. We have been studying region I up to now and the hyperbolic lines shown there correspond to trajectories of constant radius. An external observer in this region can never reach the black hole unless it gives up being in constant r paths. Transversing the outer horizon r_+ implies change to appropriate Kruskal-like coordinates. We are then in region II and again we can go through the inner horizon to reach region III. This is a quite curious statement but has no useful application in everyday life, as astrophysical black holes have not this kind of infinite global structure (see introduction). However, if we are interested to understand what is the CFT dual of an eternal black hole and try to use this duality to describe physical systems in the lab, this might be an important point. If we have infinite different asymptotic regions, which one shall we choose to define our dual CFT? Or should we have an infinite number of CFT vacua? These kind of questions often do not appear because the literature usually considers Schwarzschild and extremal black holes, both having a finite casual diagram.

In terms of monodromies, we can start with a ingoing/outgoing state Φ_∞ at $r = \infty$ and follow the closed path γ shown in figure 4.3, enclosing both inner and outer horizon. Avoiding the discussion of the location of the branch cuts, we expect the result of following this path is

$$\Phi_{\infty,\gamma} = \Phi_\infty M_+ M_-, \tag{4.22}$$

[3] The Kerr-dS black hole global structure is a little bit more complicated, infinite also horizontally. For mor on that, see [52, 61, 155].

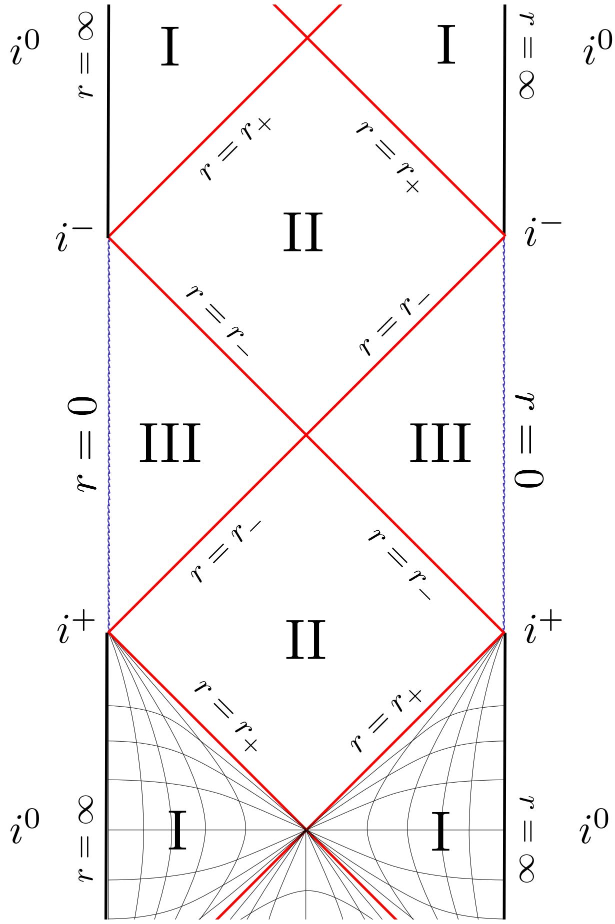

Figure 4.2: Causal diagram for the maximally extended Kerr-AdS black hole for $\theta = \pi/2$. For each different asymptotic region we assign a different Hilbert space.

so the connection matrix in this case is exactly the composition of two monodromy matrices. If repeat this n times, the connection matrix will be the n-th power of this process.

How does AdS/CFT cope with this is a very interesting question which we do not have a precise answer yet[4]. For a related discussion about this, see [29]. One speculation we can make is the following. Consider that each asymptotic region I has an associated Hilbert space $\mathcal{H}_I$. If we study only local propagation in one of those Hilbert space at low energies, it seems safe to discard the contributions from the scattering from other regions. However, low energy means larger spreading in the wave function so one might wonder if those infinite Hilbert spaces should contribute to the S-matrix. Classically, the contributions from these regions

[4]This question has actually been indirectly suggested by Prof. Jorge Zanelli in the IFT Quantum Gravity school last year, in which this author was present. Prof. Zanelli asked Prof. Juan Maldacena about this and received no definitive answer.

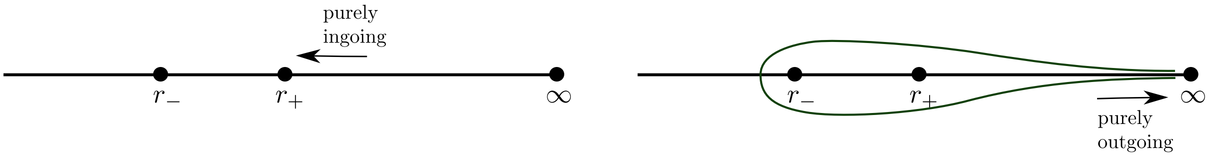

Figure 4.3: The schematics of scattering. On the left hand side, the constrain that the solution is "purely ingoing" at $r = r_+$. On the right-hand side, the monodromy associated with a solution that emerges at a different region I.

should interfere destructively in the path-integral, but if we consider quantum corrections, those might become important. This also touches the delicate matter of unitary evolution in quantum gravity, which is also a tricky subject as the firewall diatribe suggests [6].

4.2 Kerr/CFT Correspondence and Hidden Symmetries

The Kerr/CFT correspondence is a proposal put forward by [33] as an attempt to describe the CFT dual theory of an extremal Kerr black hole, without relying to string theory. Extremal black holes are well-known for their nice properties, as its hidden conformal symmetry near the horizon, related to the fact that extremal black holes are defined as zero temperature, $T_H = 0$, black objects. These are also BPS states in supergravity theories so they play an important role as supersymmetric vacua. Thus, it seems a very good place to start looking for a CFT dual of a black hole. The Kerr/CFT correspondence relies on this hidden conformal structure of extremal black holes to connect with an appropriate CFT description allow us to organize the spectrum of excitations of these black holes. Below, we review this subject as presented in [143]. For an extensive review, see [156].

We start by recalling the Kerr black hole metric

$$ds^2 = -\frac{\Delta}{\rho^2}(d\hat{t} - a\sin^2\theta d\hat{\phi})^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\sin^2\theta}{\rho^2}((\hat{r}^2 + a^2)d\hat{\phi} - ad\hat{t})^2 + \rho^2 d\theta^2 \,, \tag{4.23}$$

with

$$\Delta = \hat{r}^2 - 2M\hat{r} + a^2, \quad \rho^2 = \hat{r}^2 + a^2\cos^2\theta, \tag{4.24}$$

where M is the black hole mass, J its angular momentum and $a = J/M$ is their ratio. In the extremal limit, we make $a = M$ and therefore

$$r_{\pm} = a = M, \quad S = 2\pi M^2 = 2\pi J,$$
$$T_H = 0, \quad \Omega_H = \frac{1}{2M} \, .$$

where S is the black hole entropy. The extremal angular velocity actually corresponds to a horizon rotating at the speed of light. Therefore, we expect that the dual CFT must be chiral. It is very curious that these black holes have finite entropy while having zero temperature, indicating that they are degenerate vacua. The Near-Horizon Extremal Kerr metric (NHEK) [157] is obtained by changing the coordinates to

$$r = \frac{\hat{r} - M}{\lambda M}, \quad t = \frac{\lambda \hat{t}}{2M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M},$$

in the Kerr metric and, taking $\lambda \to 0$, we obtain the NHEK metric

$$ds^2 = 2\Omega^2 J \left[\frac{dr^2}{r^2} + d\theta^2 - r^2 dt^2 + \Lambda^2 (d\phi + r dt)^2 \right],$$
$$\Omega^2 = \frac{1 + \cos^2\theta}{2}, \quad \Lambda = \frac{2\sin\theta}{1 + \cos^2\theta} \, .$$

This metric represents, for each fixed θ, a deformed AdS_3 spacetime, corresponding more precisely to a Hopf fibration $AdS_2 \ltimes S^1$. Thus the $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ isometry of AdS_3 breaks down to $\mathrm{SL}(2,\mathbb{R}) \ltimes U(1)$ isometry. The symbol $\ltimes$ represents a semi-direct product, meaning that only the right hand set is a normal subgroup. This means that a ϕ rotation represents a pure isometry but the other isometries mix rotations with $\mathrm{SL}(2,\mathbb{R})$ transformations. So we can say the extremal Kerr metric has a *hidden conformal symmetry* in its near-horizon limit.

General Relativity is a diffeomorphism invariant theory: diffeomorphisms are seen as gauge transformations. However, solutions of Einstein's equations can have symmetries which are not "pure gauge" in this context. Let us suppose that these symmetries preserve the asymptotic boundary conditions of the solution. We define the Asymptotic Symmetry Group (ASG) as the quotient of Allowed Diffeomorphisms over Trivial Diffeomorphisms. Representatives of this group are diffeomorphisms preserving the asymptotic conditions but changing the bulk structure. This has been studied by Brown and Henneaux [158] in the context of AdS_3 and is a very important paper in general relativity as it shows that not every diffeomorphism is a pure gauge transformation in this theory. This has been acknowledged by Dirac a long time ago [159]. Every attempt to canonically quantize gravity should pass through this point.

For each diffeomorphism ζ, we associate a conserved charge Q_ζ via $\mathcal{L}_\zeta \Phi = \{Q_\zeta, \Phi\}$, where $\{.,.\}$ represents Dirac brackets. ASG diffeomorphisms preserve the boundary conditions and have finite boundary charges. Trivial diffeomorphisms have null charges at the boundary. In general, the asymptotic charges algebra can present a central charge [160]

$$\{Q_\zeta, Q_\eta\} = Q_{[\zeta,\eta]} + c_{\zeta\eta}, \tag{4.25}$$

where

$$c_{\zeta\eta} = \frac{1}{8\pi G} \int_{\partial\Sigma} K_\eta[\mathcal{L}_\zeta g, g] \tag{4.26}$$

with $g_{ab} = \bar{g}_{ab} + h_{ab}$ being the metric expansion in background plus its asymptotic part, $\partial\Sigma$ is the boundary of a spacelike hypersurface and the 2-form is given by

$$\begin{aligned} K_\zeta(h,g) = -\frac{1}{4}\epsilon_{\alpha\beta\mu\nu} dx^\alpha \wedge dx^\beta \Big[\zeta^\nu D^\mu h - \zeta^\nu D_\sigma h^{\mu\sigma} + \zeta_\sigma D^\nu h^{\mu\sigma} + \frac{1}{2} h D^\nu \zeta^\mu \\ + \frac{1}{2} h^{\sigma\nu}(D^\mu \zeta_\sigma - D_\sigma \zeta^\mu) \Big], \end{aligned} \tag{4.27}$$

where D_μ is the covariant derivative restricted to Σ. The full details and formulas of this connection between symmetries and asymptotic charges can be seen in [161, 162].

In the NHEK geometry, we suppose that $g_{ab} = \bar{g}_{ab} + h_{ab}$, and we choose the following asymptotic boundary conditions

$$\begin{aligned} h_{tt} &= O(r^2), \quad h_{t\phi} = h_{\phi\phi} = O(1), \\ h_{\phi r} &= h_{\theta\theta} = h_{\theta\phi} = h_{\theta t} = O(r^{-1}), \\ h_{rr} &= O(r^{-3}), \quad h_{tr} = h_{\theta r} = O(r^{-2}) \, . \end{aligned}$$

These boundary conditions where introduced *ad hoc* in [33] but has been physically justified by a renormalization group flow reasoning given by Cunha and Queiroz [163]. The most general diffeomorphism ζ preserving the above conditions, via $\mathcal{L}_\zeta g_{ab} = h_{ab}$, is given by

$$\zeta(\epsilon) = \epsilon(\phi)\partial_\phi - r\epsilon'(\phi)\partial_r \, .$$

If we choose a Fourier representation $\zeta_n \equiv \zeta(-e^{-in\phi})$, we obtain the *Witt algebra*

$$[\zeta_n, \zeta_m] = i(n-m)\zeta_{n+m} \, .$$

Using these diffeomorphisms, we can calculate an expression for the central charge using (4.26)

$$c_{\zeta_n\zeta_m} = -iJ(m^3 + 2m)\delta_{n+m},$$

and, defining the adimensional operators

$$\hbar L_n = Q_{\zeta_n} + \frac{3J}{2}\delta_n,$$

the algebra (4.25) of the asymptotic charges Q_ζ results in a copy of the Virasoro algebra [164, 165]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{J}{\hbar}m(m^2-1)\delta_{n+m}\ .$$

From this algebra, we see that the central charge of the dual CFT is

$$c_L = 12J/\hbar.$$

In the following, we make $\hbar = 1$.

Now suppose there exists a quantum field in thermal equilibrium with the black hole with Boltzmann weight $\exp\left(-\frac{(\omega-\Omega_H m)}{T_H}\right)$. The Frolov-Thorne vacuum [166] is an attempt at generalizing this state in thermal equilibrium for Kerr (up to the speed of light surface). However, it is shown in [167, 168] that the Frolov-Thorne vacuum is not well-defined for bosons due to an infrared divergence. On the other hand, for fermions this vacuum is well-defined [169].

The extremal limit $T_H \to 0$ of this weight seems to diverge, but one has to notice that, in near-horizon coordinates,

$$e^{-i\omega\hat{t}+im\hat{\phi}} = e^{-in_R t + in_L\phi}, \qquad n_L \equiv m, \quad n_R \equiv \frac{2M\omega - m}{\lambda}, \tag{4.28}$$

and we can rewrite

$$\exp\left(-\frac{\omega-\Omega_H m}{T_H}\right) = \exp\left(-\frac{n_L}{T_L} - \frac{n_R}{T_R}\right), \tag{4.29}$$

with

$$T_L = \frac{r_+ - M}{2\pi(r_+ - a)}, \quad T_R = \frac{r_+ - M}{2\pi r_+\lambda}. \tag{4.30}$$

In the extremal limit $r_+ = M = a$, we have that $T_R = 0$ and $T_L = 1/2\pi$. This temperature can be associated to the action of a $U(1)$ symmetry generated by ∂_ϕ and relates to the chiral sector. Notice that this limit only makes sense for modes in superradiant limit $\omega = \Omega_H m$.

Now, the most important consequence of this whole discussion. Suppose that we have a 2D chiral thermal CFT at temperature $T_L = 1/2\pi$ and central charge $c_L = 12J$. According to Cardy formula [170], the CFT entropy is

$$S_{\text{CFT}} = \frac{\pi^2}{3}c_L T_L = 2\pi J = S_{BH},$$

corresponding exactly to the Kerr black hole entropy! That is an important indication that we can organize the spectrum of states of a black hole using a 2D CFT description. As a caveat, this formula is valid if $T_L \gg \Delta_0$, where Δ_0 is the CFT lightest energy gap, but in general it is expected that black holes should have such mass gap [171].

Despite the apparently incidental correspondence between S_{CFT} and S_{BH} from this geometrical point of view, the Kerr extremal scattering amplitudes of a spin zero probe field also match with the correlation functions of a chiral CFT, suggesting that the correspondence holds even semiclassically [143]. That is one of the reasons why we discussed the $AdS_2 \times S^2$ case in section 2.2.1, as a related but simpler case of this underlying structure of near-horizon limits of extremal black holes. In fact, one can show that this near-horizon structure appears even for higher-dimensional black holes [172, 173, 174].

Being the correspondence purely geometric, it is not known if this construction is maintained in the full quantum theory. Some challenges in this area are:

1. Find the dual CFT in the extremal and non-extremal case ($T_H \neq 0$);
2. Exhibit specific models of the duality in String Theory (ST);
3. Show that the duality is maintained even non-perturbatively;
4. Extend the results to more general gravitational cases;

The case for the validity of item 1 has been made mostly on the analysis of scattering fields around the non-extremal Kerr geometry, which was shown to present two copies of $SL(2,\mathbb{R})$ consistent with the presence of a Virasoro algebra [175, 176, 66, 177]. However, the asymptotic symmetry group techniques do not work in this non-extremal case. There has been some progress on the construction of string theory realizations of the NHEK geometry, but the resulting CFTs, called dipole CFTs, are non-relativistic and show non-local properties making them hard to analyze [178, 179]. The item 3 above stands about the question of quantum integrability of the conjectured duality, if it goes beyond low energy, weakly coupled states, but in principle one needs more information about item 2 to understand this.

Our work can shed some light on the CFT structure of scattering coefficients, as discussed in [40]. There, the authors have shown that the scattering amplitude (2.38) for a minimally coupled massless scalar field in the Kerr metric can be rewritten as

$$\mathcal{T}\mathcal{T}' = \frac{\sinh 2\pi(\omega_L + \omega_R)\sinh(2\pi\alpha_{\rm irr})}{\sinh\pi(\omega_L - \alpha_{\rm irr})\sinh\pi(\omega_R + \alpha_{\rm irr})}, \tag{4.31}$$

where $\omega_L = \alpha_+ - \alpha_-$ and $\omega_R = \alpha_+ + \alpha_-$, so we have a similar interpretation of left and right moving modes as in the discussion above for a non-extremal black hole. First, let us pick an eigenfunction $\psi_{\omega,m}(r)$ of the scalar wave equation. If we take $(r-r_H) \to e^{2\pi i}(r-r_H)$ around the horizon, we know that this solution picks a monodromy $e^{-2\pi\alpha_H}$, with $\alpha_H = (\omega - \Omega_H m)/4\pi T_H$. However, if consider the full wave function $e^{-i\omega t + im\phi}\psi_{\omega,m}(r)$, we see that it is monodromy-invariant if we make the identification $t \sim t + i/T_H$ and $\phi \sim \phi + i\Omega_H/T_H$ and we encircle the horizon twice. This is exactly the kind of periodicity we expect to have a thermal Green function, defining a KMS state in the theory. Therefore we see that the black hole temperature is also justifiable thermodynamically in this way via monodromies. But what about the left and right modes? If we choose conjugate variables as t_L, t_R, the identification we made above corresponds to $(t_L, t_R) \sim (t_L, t_R) + 2\pi i(1,1)$ and the corresponding temperatures are

$$T_L = \frac{r_+ + r_-}{4\pi a}, \quad T_R = \frac{r_+ - r_-}{4\pi a}, \tag{4.32}$$

which simplify to the Kerr/CFT case in the extremal limit $r_+ = r_- = a$.

So (4.31) seems to indicate a thermal state of a 2D CFT with a left- and right-moving sectors with energies $\omega_{L,R}$ and temperatures $T_{L,R}$. The irregular monodromy $\alpha_{\rm irr}$ should correspond to the *conformal weight* of the CFT operator dual to the scalar mode. However, as noticed by [40], this CFT description is only valid for low energies because only then $\alpha_{\rm irr}$ is approximately constant. In general, as we know, this coefficient depends on ω and m, which implies a non-trivial coupling between the two CFT sectors, complicating its description. This suggests that we should expect a CFT dual only for the low-energy limit.

On the other hand, as we described above, the scattering amplitude $\mathrm{SL}(2,\mathbb{C})$ structure is pervasive for stationary black holes in any dimension, because it is a result of the group properties of the monodromy group and connection formulas. In particular, we have derived equation (4.15) for the first time for the Kerr-dS black hole, whose isometry group is not the same as in the AdS case. This indicates two opposite points of view. Maybe the $\mathrm{SL}(2,\mathbb{C})$ structure appearing in scattering theory of black holes is just a coincidence and there is no deep CFT connection in general. Those are just two different theories with a similar group structure. However, the extremal case of this conjectured duality is a lot more compelling that just a mathematical coincidence because this structure also appears geometrically in the asymptotic symmetry group. This suggests that small deformations of this picture should be valid for near-extremal black holes. In any case, extremal black holes are well described by D-branes systems in string theory.

Now we present another kind of symmetry which is surprisingly related to the hidden symmetry we just discussed above in the context of isomonodromic deformations.

4.2.1 Isomonodromic Flows and Hidden Symmetry

In the previous chapter, we studied how the theory of isomonodromic flows allow us to solve the monodromy problem of differential equations via Painlevé asymptotics. An important ingredient that makes Jimbo's derivation work is that the isomonodromic flow of a 4-point Fuchsian system has a fixed point at a 3-point system belonging to the hypergeometric class. This suggests another hidden symmetry for black hole scattering which might be related to the Kerr/CFT correspondence [38]. This happens because the hypergeometric limit of the isomonodromic flow is actually correspondent to an extremal limit of the original black hole metric.

Consider the Garnier type ODE

$$\partial_z^2 y + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\tilde{\theta}_t}{z-t} - \frac{1}{z-\lambda}\right)\partial_z y + \left(\frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}\right) y = 0. \quad (4.33)$$

We want to check the limit $t \to 1$ considering the critical behaviour of isomonodromic variables $(\lambda(t), \mu(t))$. These variables are solutions of the Garnier system

$$\frac{d\lambda}{dt} = \frac{dK}{d\mu}, \quad \frac{d\mu}{dt} = -\frac{dK}{d\lambda}. \quad (4.34)$$

Assuming that $0 < \mathrm{Re}\,\sigma < 1$, we can plug the critical behaviour $\lambda \to 1 + A(\sigma,\psi)(t-1)^{1-\sigma}$ into (4.34) to discover the critical behaviour of μ. We thus get the leading behaviour (see (3.31))

$$\mu(t) \to \frac{\theta_0}{2} + \frac{1}{2A(\sigma,\psi)}(\theta_1 + \tilde{\theta}_t - \sigma)(t-1)^{-1+\sigma}. \quad (4.35)$$

Note that $\sigma \equiv \sigma_{1t}$ in both formulas. With these asymptotic formulas, we can calculate the $t \to 1$ limit of (4.33), which is given by

$$\partial_z^2 y + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1-\tilde{\theta}_t}{z-1}\right)\partial_z y + \left(\frac{\kappa}{z(z-1)} + \frac{L_1}{z(z-1)^2}\right) y = 0, \quad (4.36)$$

with

$$L_1 = \lim_{t\to 1}\left[\lambda(\lambda-1)\mu - t(t-1)K\right]. \quad (4.37)$$

Careful calculation of the above formula using the asymptotics of λ and μ gives

$$\begin{aligned} L_1 &= \frac{1}{4}(\theta_1 + \tilde{\theta}_t - \sigma)(\theta_1 + \tilde{\theta}_t - \sigma + 2) \\ &= \frac{1}{4}(\theta_1 + \theta_t - \sigma - 1)(\theta_1 + \theta_t - \sigma + 1) \\ &= \frac{1}{4}[(\theta_1 + \theta_t - \sigma)^2 - 1], \end{aligned}$$

where we used that $\tilde{\theta}_t = \theta_t - 1$ to match Heun equation. This limit is unique up to different boundary conditions on $\lambda(t)$. We notice that for $\mathrm{Re}\,\sigma = 0$, this derivation is slightly different because other terms become relevant in the asymptotic expansion.

Now we compare this isomonodromic limit with the extremal limit of a Kerr-AdS black hole for definiteness. Consider the radial equation of conformally coupled scalar massless perturbations of the Kerr-AdS metric in Heun form

$$\partial_z^2 y + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t}\right)\partial_z y + \left(\frac{\kappa}{z(z-1)} - \frac{t(t-1)h}{z(z-1)(z-t)}\right)y = 0. \tag{4.38}$$

We shall take $(r_1 = \zeta, r_2 = r_+, r_3 = r_-, r_4 = \bar{\zeta}), \mapsto (z = 0, z = 1, z = t, z = \infty)$ which then implies that

$$t = \frac{(r_+ - \bar{\zeta})(r_- - \zeta)}{(r_+ - \zeta)(r_- - \bar{\zeta})}, \quad t - 1 = \frac{(r_+ - r_-)(\bar{\zeta} - \zeta)}{(r_+ - \zeta)(r_- - \bar{\zeta})}, \quad t - z_\infty = \frac{(r_+ - \bar{\zeta})(\bar{\zeta} - \zeta)}{(r_+ - \zeta)(r_- - \bar{\zeta})}, \tag{4.39}$$

In this convention t is a pure phase. We also have that

$$\theta_0 = -\frac{6i\chi^2}{\Lambda}\frac{\omega(\zeta^2 + a^2) - am}{(\zeta - r_+)(\zeta - r_-)(\zeta - \bar{\zeta})}, \tag{4.40a}$$

$$\theta_1 = -\frac{6i\chi^2}{\Lambda}\frac{\omega(r_+^2 + a^2) - am}{(r_+ - \zeta)(r_+ - \bar{\zeta})(r_+ - r_-)}, \tag{4.40b}$$

$$\theta_t = \frac{6i\chi^2}{\Lambda}\frac{\omega(r_-^2 + a^2) - am}{(r_- - \zeta)(r_- - \bar{\zeta})(r_+ - r_-)}. \tag{4.40c}$$

and

$$t(t-1)h = \frac{t(t-1)}{t - z_\infty}\left(1 - \frac{3(t - z_\infty)f(r_+)}{\Lambda(t-1)|r_+ - \zeta|^2}\right) - \frac{1}{2}\left[(t-1)(\theta_0 + \theta_t) + t(\theta_1 + \theta_t)\right] \tag{4.41}$$

Note also that $\theta_\infty = -\bar{\theta}_0$ for ω, m real.

If we take the extremal limit $r_+ \to r_- \equiv r_H$, we have that $t \to 1$ and θ_1, θ_t and h diverge. In fact, we have that $\theta_1 = -\tilde{\theta}_1/(t-1)$ and $\theta_t = \tilde{\theta}_t/(t-1)$. Although the θ's diverge separately, their sum $\theta_1 + \theta_t \to 0$, implying that (4.38) converges. Moreover, $t(t-1)h$ has a finite limit, as one can check in (4.41). To properly calculate the confluence limit, we multiply (4.38) by $z(z-1)(z-t)$ and let $t \to 1$. This entails to

$$\partial_z^2 y + \left(\frac{1-\theta_0}{z} + \frac{2}{z-1} + \frac{\tilde{\theta}_t}{(z-1)^2}\right)\partial_z y + \left(\frac{\tilde{L}_1 - \kappa_0}{z(z-1)} + \frac{\tilde{L}_1}{(z-1)^2}\right)y = 0. \tag{4.42}$$

with

$$\theta_0 = \frac{6i\chi^2}{\Lambda}\frac{\omega(\zeta^2 + a^2) - am}{(\zeta - r_H)^2(\zeta - \bar{\zeta})}, \quad \tilde{\theta}_t = \frac{6i(r_H^2 + a^2)}{\Lambda|r_H - \zeta|^4}\chi^2(\omega - \Omega_H m), \quad \kappa_0 = \frac{1}{4}[(\theta_0 - 2)^2 - \theta_\infty^2], \tag{4.43}$$

and

$$\tilde{L}_1 \equiv \lim_{t\to 1} t(t-1)h = \frac{3f(r_H)}{\Lambda|r_H-\zeta|^2} + \frac{6i(r_H^2+a^2)\,\mathrm{Im}\,\zeta}{\Lambda|r_H-\zeta|^4}\chi^2(\omega-\Omega_H m). \tag{4.44}$$

This is a confluent Heun equation in non-canonical form and corresponds to the wave equation of a conformally coupled scalar field in the extremal Kerr-AdS background. More generally, the Teukolsky master equation for this metric should also reduce to a confluent Heun in the extremal case. However, this still is not a hypergeometric equation. We need to constraint $\tilde{\theta}_t = 0$ in order to make $z = 1$ a regular singular point. This can happen in at least two cases: (i) in the superradiant limit $\omega = \Omega_H m$, and (ii) in the rotation limit $a = L$, in which $\chi^2 = 0$. Case (i) corresponds to the $\Lambda \neq 0$ generalization of a reduction first noticed in [180]. As discussed above, the modes in the near-horizon region correspond exactly to the original superradiant modes of the full metric.

Case (ii) is also intriguing because it is an upper bound in the rotation parameter a. In the coordinates we are using, this corresponds to the Einstein universe in the boundary of Kerr-AdS rotating at the speed of light. This is a degenerate limit of the Kerr-AdS metric and, according to [181, 182], it can also be studied in a conformal field theory in a rotating Einstein universe. This is similar to a extremal limit because the horizon is also rotating at the speed of light in that case. However, [181] argue that this cannot be made finite by a scaling limit as in the extremal case.

As far as we know, those particular extremal limits have not been acknowledged in the literature [183], although there are some works in the near-horizon extremal case, most notably [67]. This is probably the case because, if we take the extremal limit of the original radial equation with 5 points (2.123), we get an equation with 4 singular points in the Heun class. It seems that the work by Suzuki *et al* [41, 62, 63] has not been widely acknowledged, as they show that one of the singular points is actually removable and thus we get a hypergeometric equation. Summing up, the connection problem can thus be explicitly solved in the same terms as before. This suggests that conformal scalar excitations of a Kerr-(A)dS black hole can also be described by a CFT, along the lines of the Kerr/CFT correspondence, as we have a hypergeometric connection.

The new hidden symmetry we mentioned in the beginning is slightly different but may be also related to the Kerr/CFT correspondence. The extremal limit taken above is not isomonodromic, as the θ_i change. But if we see this limit as the result of an isomonodromic flow, we have the curious result that a whole 1-parameter class of black holes have the scattering properties equivalent to a particular extremal case. This confluence symmetry of scattering

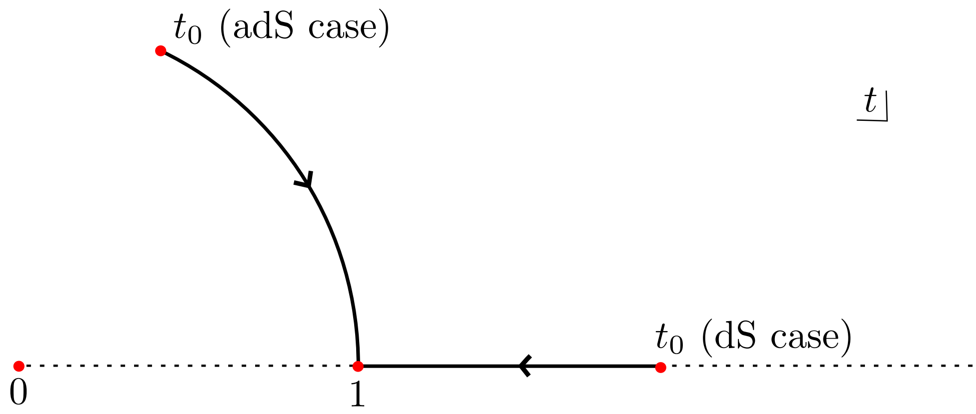

Figure 4.4: The physical space of Kerr-(A)dS isomonodromic flow. The solid lines describe a family of solutions with the same monodromy and thus with same scattering amplitudes.

amplitudes is depicted in 4.4.

Chapter 5

Conclusions and Perspectives

"The young man knows that he is irretrievably lost. This is no town of cats, he finally realizes. It is the place where he is meant to be lost. It is another world, which has been prepared especially for him. And never again, for all eternity, will the train stop at this station to take him back to the world he came from."

—HARUKI MURAKAMI, 1Q84

Black holes are very interesting systems with applications ranging from astrophysics to dual descriptions of conformal field theories, as in the ubiquitous AdS/CFT correspondence. First relegated to just mathematical curiosities, they have changed from being exotic theoretical constructs to be abundantly used (and abused) in several areas of modern physics. However, both observationally and theoretically, these systems still withhold a lot of mysteries, both in the sense of being *black* (no light come out of it) and of being *holes* (everything falling there is irretrievably lost). The biggest mystery is definitely the one of quantum gravity. Not just the singularities, but the intrinsic nonlinearity and fundamental nature of gravity and spacetime play key roles in the non-renormalizable aspect of general relativity, hindering the advances in the canonical quantization program. String theoretical advances have increased a lot our understanding of gravity, specially in the case of black holes. Vafa and Strominger [10] have calculated the entropy of a supersymmetric black hole by counting the number of microscopic configurations with same charges for a particular system of D-branes and obtained the horizon area proportionality in the geometrical limit. Several calculations along these lines have appeared confirming this counting for other types of black holes with less supersymmetry, as in the works of Ashoke Sen and collaborators [11, 12, 13, 14]. Most and foremost, AdS/CFT calculations have improved dramatically how to characterize a gravitational system in terms of a gauge dual. In this framework, black holes appear as duals of thermal plasmas on the gauge side. However, this correspondence is highly entangled and nonlocal, so questions about unitary evolution and the fate of a infalling observer, which require a geometrical and local interpretation, still need a better understanding [6].

All this advancements are sometimes blocked by the lack of a more thorough understanding of general relativity itself, its symmetries and solutions. Sometimes there is no consensus what is the correct physical picture of certain situations involving a gravitational system, as in the case of the fate of an infalling apparatus or the final state of black hole evaporation. However, some other times the lack of understanding is due to our limited knowledge of the mathematical physics involved. That is exactly the case this thesis revolves, in which we explored a general method for finding scattering coefficients in the very general class of Kerr-NUT-(A)dS black holes in 4 dimensions, as developed in our paper [38]. We have shown in chapter 2 what we believe to be the best way to describe scattering in this case, using the *monodromy technique* of Castro *et al* [39]. This technique allowed us to represent transmission coefficients for the wave equation radial part in terms of only monodromy data: local Frobenius coefficients and Stokes coefficients. In fact, without regarding to approximations, this is the *only* way we know how to obtain a closed expression for these scattering amplitudes. In essence, we substitute the problem of finding an explicit solution of the ODE in question (generically transcendental, thus the approximations) to a simpler set up we can actually control better than other analytical methods. We have shown that in the conformally coupled case, the radial wave equation has a removable singularity and thus reduces to the special case of a Heun equation with 4 regular singular points. This lacks the complication of an irregular singular point, as in Kerr scattering, but the Heun monodromy group has a richer structure, demanding more effort to understand than the hypergeometric case. Finally, we have also described in more details the angular equation, the *(A)dS* spheroidal harmonics, being a natural generalization of spheroidal harmonics. We have shown that this can also be reduced to a Heun equation, valid for any value of the scalar coupling. Incidentally, the resulting equation has also been named *Magnus-Winkler-Ince* equation, which is a Hill type equation with important applications in the literature of periodic systems [184].

Although we can always find explicit $SL(2,\mathbb{C})$ representations of the n-point monodromy group, it is not that easy to understand its general structure. Castro *et al* have mentioned the Kerr-AdS case in appendix B of [40] as not being possible to tackle using only monodromies. In chapter 3, we tried to convince the reader that this is actually possible. We have described recent developments in the understanding of generic $SL(2,\mathbb{C})$ monodromy group representations, as described in [110, 112], deeply related to the works of Fricke and Klein [111] and Jimbo

[104]. We have shown how to parametrize the monodromy representations in terms of trace coordinates. In particular, monodromy groups with $n > 3$ depend explicitly on the composite traces $m_{ij} = \mathrm{Tr}\, M_i M_j = 2\cosh\sigma_{ij}$ and there lies the core of our problem. The symplectic structure of the monodromy group has also been presented in relation to the space of flat SL$(2,\mathbb{C})$ connections, which has important applications in 2D conformal field theory, in particular in the calculation of classical conformal blocks expansions [105] and the relation of $c = 1$ correlation functions and tau functions [107, 106, 109]. These developments are related to the recently discovered AGT relation [185] where instanton partition functions of certain $\mathcal{N} = 2$ gauge theories were are related to Liouville conformal blocks. It is one of our plans to continue this line of study with the experience got in this thesis. In particular, we hope to address the problem of irregular conformal blocks, in relation to irregular singular points of Fuchsian systems [186].

The most important work to us enlightening the 4-point monodromy group structure is due to Jimbo [104], which connects Fuchsian systems with 4 singular points with Painlevé transcendents via the theory of monodromy preserving deformations, developed by the same author, Miwa and Ueno in a series of very important papers [97, 98, 99]. This theory was further worked out to be applicable to the more general n-point case, so-called *Schlesinger flows*. The addition of more singular points does not changes very much the conceptual picture of isomonodromic flows and thus we focused on the particular $n = 4$ case. This is the natural place to start the study of these kind of flows, the related Fuchsian system presents one extra parameter, corresponding to an apparent singularity in the Fuchsian ODE, serving as a deformation function $\lambda(t)$. It turns out that the integrability conditions for the deformed Fuchsian system is exactly P_{VI} equation for $\lambda(t)$.

We have also discussed important properties of Painlevé transcendents, as its symmetries, hamiltonian structure and confluences. An important fact appeared here: Painlevé functions are the classical equations of motion of Heun hamiltonians. Both special functions are also related to hypergeometric functions by some special limits and confluences, forming a very powerful triad in mathematical physics. In particular, via confluence, we can obtain every P_J equation, each directly related to a confluent reduction of Heun equation.

Finally, we presented how Schlesinger system asymptotics solve the P_{VI} connection problem in terms of the Fuchsian system monodromies. Painlevé asymptotics depend explicitly on the composite monodromy parameter σ, paving the way to a solution of our original scattering problem. The solution depends on the initial condition of P_{VI} flow, which is given by our Heun equation of interest. In particular, we presented a plot of P_{VI} near one of its critical points by inputing initial conditions from Kerr-dS scattering. Although this numerical computation relies on the numerical evaluation of angular eigenvalues C_ℓ, the behaviour seemed very robust in the examples we tried. We computed the eigenvalues of (A)dS spheroidal harmonics via Leaver's continued fraction method [42] with partial success, obtaining confident results only for the $l = m$ case. We plan to tackle this problem as soon as possible and obtain trustable numerical results for applications.

After the complete discussion of the mathematical physics involved in the scattering problem, we turned to theoretical applications. In chapter 4, we presented a general formula for the scattering amplitude between two regular singular points of an ODE with n singular points. This equation follows from the monodromy technique: scattering problems typically impose ingoing/outgoing boundary conditions at some singular point, implying a diagonalizable monodromy, and by the monodromy group structure, we can find an explicit parametrization for scattering amplitudes in terms of monodromy data. However, the typical scattering problem in asymptotically flat spacetimes usually imposes plane-wave boundary conditions at spatial infinity, and in this case our expression is just part of the solution. Plane-wave scattering is allowed when we have asymptotic plane-wave expansions, which occur when there is an irregular singular point. In the (A)dS cases, there is no irregular singular point and, thus, no plane-wave asymptotics.

A caveat resulting from having $r = \infty$ removed from our wave equation is that its monodromy becomes trivial and the singular point becomes regular. Therefore, the parameterization for this case is not so clear and we do not have a solution for this problem yet. Although the Kerr-AdS case should be also solved along the same lines, we postpone this case for the future when this matter becomes settled in our minds. Luckily, the Kerr-dS scattering happens between two singular points, the event horizon r_+ and the cosmological horizon r_C. We thus presented a very concise and interesting formula for the Kerr-dS scattering amplitude in (4.15). This equation suggests two boundaries for superradiant scattering, $\omega = m\Omega_H$ and $\omega = m\Omega_C$, however this still remains to be checked numerically. We also discussed several constraints we expect the scattering amplitude should follow. For high frequencies, we expect that the amplitude goes always to 1 and for low frequencies, we expect it can go to constant or to zero, depending on m. The zeros of the scattering amplitude also match results in the literature. Finally, we presented a transcendental equation for the quasinormal modes of Kerr-dS black holes. This should definitely be a matter of future digression, as quasinormal modes have an important role in the AdS/CFT correspondence and the stability analysis of black holes. In particular, most applications of gauge/gravity duality use static backgrounds and we need more understanding of rotating plasmas. For more about this, see [48, 141, 140, 36]. Other even less explored aspect is what is the gauge dual of NUT charge? There has been some studies about this in [142, 90, 91, 92].

We have also made prospective speculations about semiclassical descriptions of black holes. First we addressed some comments about scattering between different asymptotic regions in the Kerr-(A)dS maximally extended spacetime. Timelike trajectories exist in which radiation can transverse the black hole interior and reach the other side. Classically it seems justifiable to trace out the contributions of the infinite asymptotic regions, but we suggest these should be relevant for quantum regimes. This is also a problem for the precise definition of the asymptotic region in AdS/CFT in rotating backgrounds. Which asymptotic region shall we choose and what is the meaning of the others? These questions can also be relevant to the dual description of the black hole interior, as those are more smooth in these rotating backgrounds, and shed more light on matters of unitary evolution and final fate of evaporating black holes.

The Kerr/CFT correspondence is a very interesting suggestion for a CFT description of the near-horizon region of an extremal rotating black hole [33]. This correspondence relies on a hidden conformal symmetry, emerging in the near-horizon limit, suggesting a way to organize the spectrum of low-energy excitations of black holes. This conjecture is supported by scattering computations in the NHEK metric and for low-energy modes of non-extremal Kerr black hole. For higher frequencies, one does not expect such simple description, as the left and right-moving sector of the CFT should couple non-trivially. For the Kerr-AdS metric, we derived its extremal limit for a conformally coupled scalar field and we have shown that the radial equation reduces to a hypergeometric equation in two special limits (i) in the superradiant limit $\omega = \Omega_H m$ and (ii) in the rotation limit $a = L$. The first case must be related to the near-horizon extremal limit taken in [67]. This also suggests a CFT description for these black holes and definitely deserves a careful investigation in the near future.

Another kind of symmetry related to the Kerr/CFT correspondence appears in isomonodromic flows. The Schlesinger flow has a hypergeometric fixed point when we let $t \to 0, 1, \infty$. As the monodromy data is preserved by the flow, a whole class of different backgrounds having scattering amplitudes equivalent to the ones of an extremal case. Of course, this seems just another way of explaining the meaning of isomonodromic flows, but we imagine this observation can have interesting applications. However, up to now, we do not see how this could simplify even more what we have already discussed in this context.

Last, but not least, we suggested in the text that our scattering formula should work also for higher dimensional cases. The difference between Kerr-NUT-(A)dS black holes in higher dimensions lies in the polynomials Δ_r and Δ_p, which have higher orders and thus more roots. Therefore, our radial ODE will have more singular points and that is it. So we can still address scattering as we did in this thesis. However, the main difficulty is how to calculate the composite monodromy in this case. For a system with $n > 3$ points, there are actually $n - 3$ apparent singularities and thus $n - 3$ different isomonodromic flows. This is described in [74]. There it is shown that solutions of the Schlesinger system have the Painlevé property for any number of singular points. Therefore we should expect a related asymptotic structure. We do not know if this has been addressed in the literature so this seems a very interesting issue to be discussed later.

Another point, as we mentioned in the discussion about Painlevé VI equation, is that the other P_J equations can be obtained by certain confluence limits from P_{VI}. Those are directly related to Heun functions as each Painlevé equation correspond to a certain confluent form of Heun equation. Therefore, the singularity structure with respect to the regular singular points is essentially the same for both equations. This allow us to obtain results also about the Kerr black hole, whose Klein-Gordon equation is a confluent Heun equation, allowing us to extend the results of [39] via isomonodromic deformations of P_V. The asymptotics of P_V and P_{III} have also been worked out by Jimbo in his inspiring paper [104]. This latter case is actually related to the doubly confluent Heun equation, corresponding the extremal Kerr case.

In this thesis, we believe to have opened a whole avenue to treat very old problems with a new perspective from the theory of isomonodromic deformations. This mathematical insight is very effective to treat scattering problems in general and have also applications in several other areas of physics and mathematical physics. In particular to us, the application for wave scattering in black hole backgrounds suggests a new understanding of the underlying hidden symmetry of these important systems. We expect to have contributed to the quest of understanding more about general relativity and quantum gravity and we hope our work can be fruitfully enjoyed by other researchers in several areas.

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APPENDIX A

How to draw a spacetime on a finite piece of paper?

To understand how to draw a Penrose diagram, we present the simplest example: flat spacetime in 4 dimensions. The metric, which is the mathematical object used to measure spacetime distances, is given in cartesian coordinates (t, x, y, z) by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{A.1}$$

There is no easy way to draw this four-dimensional spacetime in a cartesian way, as our paper has only two dimensions. One thing we can do to simplify the task is to write this metric in terms of spherical coordinates (t, r, θ, φ)

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{A.2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric of the 2-sphere. Now, we can draw flat space with only two coordinates (t, r) and say that each point corresponds to a 2-sphere of radius r. Of course, we are missing the angular "action" which could happen to a particle, but that is okay for our purposes. Now we have two important details. First, $r = 0$ is not a sphere, so it is just a plain old point, or, as time passes by, it forms a straight line - a wordline. Second, and most importantly, we have a hard time to draw when either of the (t, r) coordinates go to infinity. Said in another way, how can we draw *infinity*? That is just a matter of *perspective*, as we shall see below.

A clever way to deal with this, is to think about spacetime in a *causal* and *conformal* way. The causal way comes about by introducing light-cone coordinates $u = t - r$ and $v = t + r$, with the domain $-\infty < u, v < \infty$ (see figure A.1). The figure depicts only a half-plane because $r > 0$ (or $u \leq v$). In these coordinates, the $t - r$ part of (A.2) becomes $ds^2 = -dudv$. Light rays follow lines of constant u or v and those divide spacetime into regions which are connected by signals moving slower than light speed (timelike) and faster than light speed (spacelike). Physical propagation must happen inside the light cone Now comes the conformal part. We want to bring the points at infinity to a finite region of spacetime *preserving light rays*. Every conformal transformation preserve light rays and we can achieve the desired transformation by

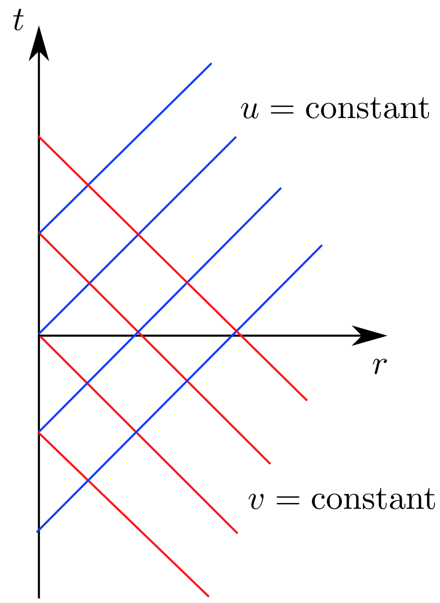

Figure A.1: Minkowski spacetime represented in the $t-r$ plane. Light rays are shown in light-cone coordinates (u, v). Adapted from [187].

choosing, for example, $U = \arctan u$ and $V = \arctan v$, implying that $-\frac{\pi}{2} < U, V < \frac{\pi}{2}$ with $U \leq V$. The metric can finally be written as

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}(-4dUdV + \sin^2(V-U)d\Omega^2), \tag{A.3}$$

and we see these coordinates fail exactly at the boundaries of the spacetime (see figure A.3). Finally, we define $U = (T-R)/2$ and $V = (T+R)/2$ to obtain a metric which is conformal to an *Einstein space*

$$ds^2 = \omega^{-2}(T,R)(-dT^2 + dR^2 + \sin^2 R\, d\Omega^2), \tag{A.4}$$

with $\omega(T,R) = \cos T + \cos R$, $0 < R < \pi$ and $|T| + R < \pi$.

Now that we have achieved our goal to fit the whole of Minkowski spacetime into a finite piece of paper, we can interpret the boundary of the spacetime. First, $T = -\pi, R = 0$ corresponds to a set of points at $t = -\infty$. We call it *past timelike infinity*, or i^-, and every massive particle starts its history there. Similarly, $T = \pi, R = 0$ is *future timelike infinity*, or i^+, and all massive particles end up here. The lines $U = -\pi/2$ and $V = \pi/2$ are respectively *past null infinity*, $\mathscr{J}^-$, and *future null infinity*, $\mathscr{J}^+$. Every light ray starts somewhere at $\mathscr{J}^-$ and ends at $\mathscr{J}^+$. Finally, the meeting point of these two lines is *spatial infinity*, i^0, and corresponds to $r = \infty$. All spacelike surfaces end up at i^0.

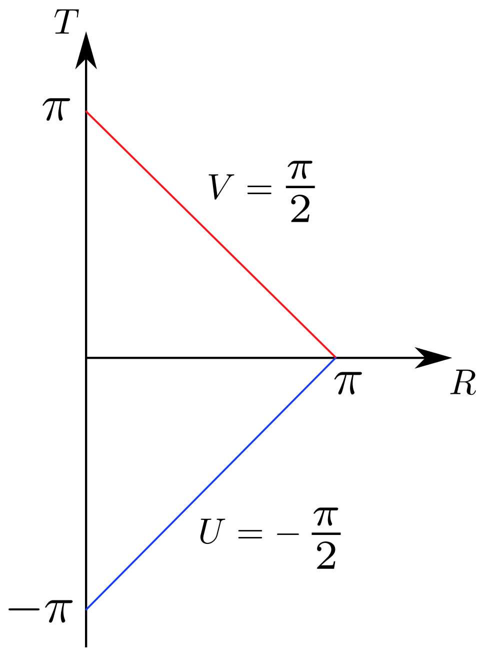

Figure A.2: Minkowski spacetime in compactified coordinates $U = \tanh u$ and $V = \tanh v$. The region depicted is $-1 < U, V < 1$ with $U \leq V$.

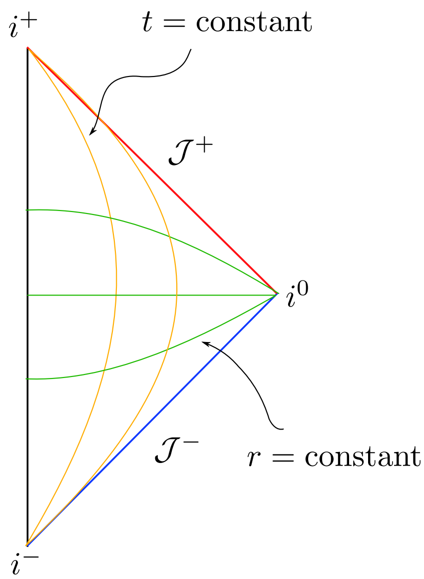

Figure A.3: Conformal diagram of Minkowski in the T-R plane.

Appendix B

Frobenius Analysis of Ordinary Differential Equations

In this appendix, we review the Frobenius analysis of Fuchsian equations and the classification of these in terms of its singular points. For more details, see Slavyanov and Lay's book on special functions [73], focusing on second order ODEs, and Iwasaki *et al*'s book [74], which presents results for n-th order ODEs. Here we focus on second order ODEs, for simplicity.

Consider a linear, second order, ordinary differential equation (ODE) given by

$$L_z y(z) \equiv P_0(z)y''(z) + P_1(z)y'(z) + P_2(z)y(z) = 0, \tag{B.1}$$

where $P_k(z)$ are polynomials of order n_k in $z \in \mathbb{C}$. Suppose there are no common factors depending on z between the polynomials P_k. To include the point $z = \infty$ in the analysis, we define the ODE domain to be the Riemann sphere, $D = \mathbb{CP}^1$. The ODE is of second order so admits two linearly independent solutions and the most general solution is a linear combination of those.

Every point z_0 in some neighborhood of D having a well-defined initial value problem with regular initial conditions

$$y(z_0) = y_0, \quad y'(z_0) = y'_0, \tag{B.2}$$

is called an *ordinary* point of the ODE. The point at $z = \infty$ can be studied by making $z = 1/\zeta$ and focusing on $\zeta = 0$. In the neighborhood of ordinary points, the solutions are holomorphic.

The zeros of $P_0(z)$, denoted by z_i ($i = 1, ..., n$), are called *singular points* of (B.1). At these points, there is no well-defined initial value problem. However, although not holomorphic, the solutions might admit series expansions around the singular points. This is the main theme of Frobenius analysis of ODEs, stating the conditions (B.1) must satisfy such that admits series solutions around its singular points. In general, the solutions have Laurent series expansions around the singular points. We distinguish singular points that admit *finitely many* negative power terms in its Laurent expansion by saying those are *regular singular points*. Otherwise, we say it is a *irregular singular point*.

In the other hand, one can ask what are the conditions the polynomials $P_k(z)$ must obey such that (B.1) admits series solutions around $z = z_i$ of the type

$$y(z) = (z - z_i)^s \sum_{j=0}^{\infty} a_n (z - z_i)^n, \quad s \in \mathbb{C}, \tag{B.3}$$

where s is called the *characteristic exponent* or *Frobenius exponent* of the solution. According to Frobenius theorem, the conditions are

$$P(z) \equiv \frac{P_1(z)}{P_0(z)} \sim \frac{A_i}{z - z_i}, \tag{B.4a}$$

$$Q(z) \equiv \frac{P_2(z)}{P_0(z)} \sim \frac{B_i}{(z - z_i)^2} + \frac{C_i}{z - z_i}, \tag{B.4b}$$

that is, P has at most a first order pole at z_i and Q has at most a second order pole at z_i. A singular point obeying (B.4) is called a *Fuchsian singularity*. Equation (B.1) is called a *Fuchsian equation*, if each of its singular points are Fuchsian. Therefore, Frobenius theorem states that

A singular point z_i of (B.1) *is regular if and only if it is Fuchsian.*

So the knowledge of the ODE behaviour at its singular points allows to have important information about its solutions.

If $z = z_j$ is a regular singular point of this ODE, then the characteristic exponent s can be obtained, after substitution of (B.3) into (B.1), by the indicial equation

$$s(s-1) + p_j s + q_j = 0, \tag{B.5}$$

where

$$p_j = \text{Res}_{z=z_j} \frac{P_1(z)}{P_0(z)}, \quad q_j = \text{Res}_{z=z_j} (z - z_j) \frac{P_2(z)}{P_0(z)}. \tag{B.6}$$

At $z = \infty$, we have the indicial equation

$$s(s+1) + p_\infty s + q_\infty = 0, \tag{B.7}$$

where

$$p_\infty = -\text{Res}_{z=\infty} \frac{P_1(z)}{P_0(z)}, \quad q_\infty = \text{Res}_{z=\infty}\, z \frac{P_2(z)}{P_0(z)}. \tag{B.8}$$

These indicial equations can be easily obtained by studying the *Euler form* of (B.1). First, we make $z \to z + z_i$, such that the singular point of interest is now at $z = 0$. Further, we divide (B.1) by $P_0(z)$ and multiply it by z^2

$$z^2 y'' + z p(z) y' + q(z) y = 0, \tag{B.9}$$

where $p(z) \equiv zP(z)$ and $q(z) \equiv z^2 Q(z)$ are regular at $z = 0$ and, therefore,

$$p(z) = \sum_{m=0}^{\infty} p_m z^m, \quad q(z) = \sum_{m=0}^{\infty} q_m z^m. \tag{B.10}$$

If we define the *scale invariant* derivative $\theta = z\partial_z$, the ODE above becomes

$$\theta^2 y + (p(z) - 1)\theta y + q(z) y = 0. \tag{B.11}$$

Notice that $\theta z^n = n z^n$. Now we plug a Frobenius series $y = z^s \sum_{m=0}^{\infty} a_m z^m$ into (B.11) and we get, after a few manipulations,

$$\sum_{m=0}^{\infty} \left[((m+s)(m+s-1) + p_0(m+s) + q_0)\, a_m + \sum_{k=0}^{m-1} (p_{m-k}(k+s) + q_{m-k})\, a_k \right] z^m = 0, \tag{B.12}$$

and, for $m = 0$, we get exactly (B.5). Therefore, we have a series solutions if and only if

$$f(m+s) a_m + R_m = 0, \quad \forall\, m \geq 0, \tag{B.13}$$

where $R_0 = 0$ and

$$f(s) = s(s-1) + p_0 s + q_0, \tag{B.14}$$

$$R_m(a_{m-1}, a_m, ..., a_0; s) = \sum_{k=0}^{m-1} (p_{m-k}(k+s) + q_{m-k})\, a_k. \tag{B.15}$$

We also take $a_0 = 1$ for simplicity.

From this discussion, we see that the indicial equation has at most two distinct roots $s_{1,2}$, distinguishing both linearly independent Frobenius solutions at $z = z_i$. When the difference $s_1 - s_2$ is either equal to a positive integer N or equal to zero, we have a *resonant singularity*. This type of singularities are degenerate in the sense they typically present a logarithm term, as we show below. In the other hand, we have an analytic solution for $s = s_2$ if $R_N = 0$. In this case, we say that z_i is an *apparent singularity*.

However, if $R_N \neq 0$, we cannot solve for (B.13). In both resonant cases above, let us consider the equation

$$L_z y(s; z) = z^s f(s), \tag{B.16}$$

and its derivative with respect to s calculated at s_1

$$L_z \frac{\partial y}{\partial s}\bigg|_{s=s_1} = z^{s_1} f'(s_1). \tag{B.17}$$

If $s_1 = s_2$, we have that $f'(s_1) = 0$ and so

$$\left.\frac{\partial y}{\partial s}\right|_{s=s_1} = y(s_1;z)\log z + z^{s_1}\sum_{m=0}^{\infty} a'_m(s_1)z^m \tag{B.18}$$

is a solution of the ODE. In this case, one might think that the monodromy is trivial, $M \simeq e^{2\pi i}\mathbb{1}$, but by inspection, with a proper normalization, we have that

$$M \simeq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{B.19}$$

Now, if $s_1 - s_2 = N$, with $N > 0$, and $R_m \neq 0$, we take

$$y^* = z^{s_2}\sum_{m=0}^{\infty} a_m z^m, \tag{B.20}$$

where only a_N is not determined by (B.13). In fact, we have

$$L_z y^* = R_N(s_2)\, z^{s_2+N}. \tag{B.21}$$

As $s_1 = s_2 + N$, we can take

$$y_2 = f'(s_1)y^* - R_N(s_2)\left.\frac{\partial y}{\partial s}\right|_{s=s_1} \tag{B.22}$$

as the linearly independent solution in this case. In both cases discussed above, one of the solutions present a logarithm term, as we mentioned before, and its monodromy is equivalent to (B.19).

B.1 Apparent Singularity of Garnier System

In section 3.3.1, we have seen that a Garnier type ODE has an apparent singularity at $z = \lambda$ if and only if the function $K(\lambda,\mu,t)$ is in a specific form. Here we show this result. First, we remember that, for the Garnier ODE,

$$P(z) = \sum_{i=1}^{3}\frac{1-\theta_i}{z-z_i} - \frac{1}{z-\lambda} \tag{B.23a}$$

$$Q(z) = \frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}. \tag{B.23b}$$

First, we make $z \to z+\lambda$, such that we have (B.13) all over again. Now, we have that $p_0 = -1$, $q_0 = 0$ and $q_1 = \mu$. The indicial equation is thus $s(s-2) = 0$ and, if we pick $s_2 = 0$, equation (B.13) for $m = 1$ gives $a_1 = q_1 = \mu$. Now, for $m = 2$, we have that $f(2) = 0$ and thus $R_2(0) = \mu^2 + p_1\mu + q_2 = 0$. Given that

$$p_1 = \operatorname*{Res}_{z=0} z^{-2}p(z) = \operatorname*{Res}_{z=0} z^{-1}P(z+\lambda) = \sum_{i=1}^{3} \frac{1-\theta_i}{\lambda - z_i}, \tag{B.24}$$

$$q_2 = \operatorname*{Res}_{z=0} z^{-3}q(z) = \operatorname*{Res}_{z=0} z^{-1}Q(z+\lambda) = \frac{\kappa}{\lambda(\lambda-1)} - \frac{t(t-1)K}{\lambda(\lambda-1)(\lambda-t)} - \frac{2\lambda-1}{\lambda(\lambda-1)}\mu, \tag{B.25}$$

the constraint $\mu^2 + p_1\mu + q_2 = 0$ gives us exactly the formula (3.46) we were looking for K.

B.2 Gauge Transformations of Fuchsian Equations

Consider a general Fuchsian equation with n singular points $z = \{z_1, z_2, \ldots, z_n\}$,

$$\mathcal{L}_z y(z) \equiv D_z^2 y(z) + P(z)D_z y(z) + Q(z)y(z) = 0, \tag{B.26}$$

where $D_z = d/dz$ and P, Q are rational functions with poles at $z = z_i$. Fuchsian equations are preserved by general homographic transformations of the z coordinate

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{C}, \tag{B.27}$$

and also by s-homotopic transformations

$$y(z) = \prod_{i=1}^{n} (z-z_i)^{\rho_i} u(z), \quad \rho_i \in \mathbb{C}. \tag{B.28}$$

We applied both of these transformations on (2.106) in order to transform it into a canonical form with 5 singular points and, then, into a Heun equation with 4 singular points in the special case $\xi = 1/6$. If we set $T(z) = \prod_{i=1}^{n}(z-z_i)^{\rho_i}$, then the application of a s-homotopic transformation in (B.26) gives

$$\mathcal{L}_z(T(z)u(z)) = T(z)\hat{\mathcal{L}}_z u(z), \tag{B.29}$$

where

$$\hat{\mathcal{L}}_z = D_z^2 + (P+2A)D_z + (Q + A^2 + PA - B) \tag{B.30}$$

with

$$A = \sum_{i=1}^{n} \frac{\rho_i}{z-z_i}, \qquad B = \sum_{i=1}^{n} \frac{\rho_i}{(z-z_i)^2}. \tag{B.31}$$

In the discussion above, we can see that the Heun differential operator in canonical form can be written as

$$\mathcal{H}_z[p_1, p_2, p_3, a, b; t; q] = D_z^2 + \sum_{i=1}^{3} \frac{p_i}{z - z_i} D_z + \frac{abz - q}{\prod_{i=1}^{3}(z - z_i)}, \quad z_i = \{0, 1, t\}. \tag{B.32}$$

The application of (B.29) into Heun equation thus gives

$$D_z^2 + \sum_{i=1}^{3} \frac{p_i + 2\rho_i}{z - z_i} D_z + \sum_{i=1}^{3} \frac{\rho_i(\rho_i - 1 + p_i)}{(z - z_i)^2} + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\rho_i(\rho_j + p_j)}{(z - z_i)(z - z_j)} + \frac{abz - q}{\prod_{i=1}^{3}(z - z_i)}. \tag{B.33}$$

Note that this is not in canonical form anymore. To obtain a canonical form again, we must set to either values $\rho_i = \{0, 1 - p_i\} = \{0, -2i\alpha_i\}$. This of course changes the local coefficients of the solutions but, consists into a symmetry of Fuchsian equations and, moreover, of the connection matrices between singular points.

There are two other special forms of Heun equation which will be useful later. We can put Heun equation into *self-adjoint form* by choosing $\rho_i = (1 - p_i)/2 = -i\alpha_i$. Another useful form is the *Liouville normal form*, which is obtained by cancelling the first-order derivative term. This can be achieved by setting $\rho_i = -p_i/2$, which gives

$$\begin{aligned} \mathcal{N}_z &= D_z^2 + \sum_{i=1}^{3} \frac{p_i(2 - p_i)/4}{(z - z_i)^2} - \sum_{i \neq j}^{3} \frac{p_i p_j/4}{(z - z_i)(z - z_j)} + \frac{abz - q}{\prod_{i=1}^{3}(z - z_i)} \\ &= D_z^2 + \sum_{i=1}^{3} \frac{\delta_i}{(z - z_i)^2} + \frac{\alpha\beta z - \hat{q}}{z(z-1)(z-t)} \end{aligned} \tag{B.34}$$

where $\delta_i = (1 + 4\alpha_i^2)/4$ and

$$\alpha\beta = ab - (p_1 p_2 + p_1 p_3 + p_2 p_3)/2\,, \qquad \hat{q} = q + (p_1 p_2 t + p_1 p_3)/2. \tag{B.35}$$

We can rewrite (B.34) in terms of partial fractions using the definitions (2.126a)

$$\mathcal{N}_z = D_z^2 + \sum_{i=1}^{3} \left(\frac{\delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right) \tag{B.36}$$

where

$$c_i = \hat{E}_i - \frac{1}{2} \sum_{j \neq i} \frac{p_i p_j}{z_i - z_j}\,, \qquad \sum_{i=1}^{3} c_i = 0. \tag{B.37}$$

Interesting relations between coefficients can be obtained by Heun normal form with 4 finite singular points, that is

$$\psi''(w)+N(w)\psi(w)=0, \qquad N(w)=\sum_{i=1}^{4}\left(\frac{\delta_i}{(w-w_i)^2}+\frac{c_i}{w-w_i}\right), \tag{B.38a}$$

$$\sum_{i=1}^{3} c_i=0\,, \quad \sum_{i=1}^{3}(c_i w_i+\delta_i)=0\,, \quad \sum_{i=1}^{3}(c_i w_i^2+2\delta_i w_i)=0, \tag{B.38b}$$

where (B.38b) are the necessary conditions for $w=\infty$ be a regular point. Applying a homographic transformation to (B.38) such that $(w_1,w_2,w_3,w_4,\infty;w)\mapsto(0,1,t,\infty,z_\infty;z)$ and then letting $\psi\mapsto(z-z_\infty)^{-1}\psi$, we obtain

$$\psi''(z)+\tilde{N}(z)\psi(z)=0, \qquad \tilde{N}(z)=\sum_{i=1}^{3}\left(\frac{\delta_i}{(z-z_i)^2}+\frac{\tilde{c}_i}{z-z_i}\right), \tag{B.39}$$

such that

$$\tilde{c}_i=\frac{c_i w_{4i}^2-2\delta_i w_{4i}}{w_{14}z_\infty}\,, \qquad \sum_{i=1}^{3}\tilde{c}_i=0. \tag{B.40}$$

Note that

$$\sum_{i=1}^{3}\frac{c_i w_{4i}^2-2\delta_i w_{4i}}{z-z_i}=\frac{\tilde{c}_2+t\tilde{c}_3}{z(z-1)}+\frac{t(t-1)\tilde{c}_3}{z(z-1)(z-t)}\,. \tag{B.41}$$

Analyzing the behaviour of (B.39) at infinity, we may rewrite $\tilde{c}_2+t\tilde{c}_3$ as $\delta_4-(\delta_1+\delta_2+\delta_3)$.

APPENDIX C

Gauge Transformations of First Order Systems

C.1 From Self-Adjoint to Canonical Natural Form

A faster route to obtain Heun canonical form above from (2.106) is to work directly with the correspondent linear system. Consider again the equation

$$\partial_r(U(r)\partial_r\psi(r)) - V(r)\psi(r) = 0\,, \tag{C.1}$$

rewritten as

$$(\partial_r - A(r))\Psi(r) = 0\,, \quad A = \begin{pmatrix} 0 & U^{-1} \\ V & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi \\ U\partial_r\psi \end{pmatrix}.$$

Suppose that U and V are rational functions and (C.1) has n regular singular points $\{r_i\}$, $i = 1,\ldots,n$, with $r_n = \infty$. What happens with this system after a homographic transformation and a s-homotopic transformation? First, let us apply the homographic transformation

$$z = \frac{ar+b}{cr+d} \quad \Rightarrow \quad \partial_r = \frac{ad-bc}{(cr+d)^2}\partial_z := F^{-1}(r)\partial_z\,.$$

This implies a new gauge connection

$$\tilde{A} = \begin{pmatrix} 0 & \tilde{U}^{-1} \\ \tilde{V} & 0 \end{pmatrix} := \begin{pmatrix} 0 & FU^{-1} \\ FV & 0 \end{pmatrix} \tag{C.2}$$

with $\Psi = (\psi(z),\ \tilde{U}(z)\partial_z\psi(z))^T$. Now, let us apply a s-homotopic transformation in the field

$$\psi = G(z)\,\tilde{\psi} := \prod_{i=1}^{n}(z-z_i)^{-\rho_i/2}\,\tilde{\psi}, \quad \Rightarrow \quad \partial_z\psi = G\partial_z\psi + \partial_z G\psi = G(\partial_z + B)\psi,$$

where

$$B := -\sum_{i=1}^{n}\frac{\rho_i/2}{z-z_i}. \tag{C.3}$$

Thus,

$$\Psi \mapsto (T\tilde{\psi},\ T(\partial_z + B)\tilde{\psi})^T = \begin{pmatrix} T & 0 \\ \tilde{U}TB & T \end{pmatrix}\tilde{\Psi} := U(z)\tilde{\Psi},$$

$$\partial_z\Psi \mapsto U\partial_z\tilde{\Psi} + \partial_z U\tilde{\Psi},$$

and the new gauge potential is $\tilde{A}' = U^{-1}\tilde{A}U - U^{-1}\partial_z U$. Calculating all terms, we have that

$$\tilde{A}' = \begin{pmatrix} 0 & \tilde{U}^{-1} \\ \tilde{V} - \tilde{U}(B^2 + \partial_z B) - \partial_z \tilde{U} B & -2B \end{pmatrix}. \tag{C.4}$$

This can be further simplified by writing the linear system in terms of $\Psi = (\psi, \partial_z \psi)$, removing $\tilde{U}$ from its definition. Removing the tildes and commas for a more clean notation, we are left with a system in the form

$$\partial_z \Psi = A(z)\Psi, \quad A = \begin{pmatrix} 0 & 1 \\ \tilde{V}\tilde{U}^{-1} - (B^2 + \partial_z B - B\partial_z \log \tilde{U}) & -2B - \partial_z \log \tilde{U} \end{pmatrix}, \tag{C.5}$$

which implies in the following ODE for ψ

$$\partial_z^2 \psi + P(z)\partial_z \psi + Q(z)\psi = 0, \tag{C.6a}$$

$$P(z) = \partial_z \log \tilde{U} + 2B, \quad Q(z) = B^2 + \partial_z B + B\partial_z \log \tilde{U} - \tilde{V}\tilde{U}^{-1}. \tag{C.6b}$$

C.2 Rederivation of Heun Canonical Form

In the following, we rederive (2.131) applying the result above to the case $n = 5$ with appropriate functions and restricting to $\xi = 1/6$.

Let the homographic transformation $(r_1, r_2, r_3, r_4, \infty) \mapsto (0, 1, t_0, \infty, z_\infty)$ be given by

$$z = \frac{r_2 - r_4}{r_2 - r_1} \frac{r - r_1}{r - r_4} := z_\infty \frac{r - r_1}{r - r_4},$$

where $z_\infty = r_{24}/r_{21}$, $t_0 = z_\infty(r_{31}/r_{34})$ and $r_{ij} := r_i - r_j$. The inverse transformation will also be useful below

$$r = \frac{r_4 z - r_1 z_\infty}{z - z_\infty} \qquad \Rightarrow \qquad r - r_i = \frac{r_{4i} z + r_{i1} z_\infty}{z - z_\infty}, \quad i = 1, \ldots, 4. \tag{C.7}$$

This further implies on

$$\partial_r = \frac{r_{14} z_\infty}{(r - r_4)^2} \partial_z = -\frac{(z - z_\infty)^2}{r_{41} z_\infty} \partial_z \quad \Rightarrow \quad F^{-1}(z) = -\frac{(z - z_\infty)^2}{r_{41} z_\infty}.$$

In our case,

$$U = -\frac{\Lambda}{3} \prod_{i=1}^{4} (r - r_i) = -\frac{\Lambda}{3} (r_{41}^2 r_{42} r_{43} z_\infty) \frac{f(z)}{(z - z_\infty)^4}$$

where $f(z) = z(z-1)(z-t)$, such that

$$\tilde{U}(z) = F^{-1}(z)U(z) = \frac{\Lambda}{3}(r_{41}r_{42}r_{43})\frac{f(z)}{(z-z_\infty)^2}. \tag{C.8}$$

Also, given $\tilde{V} = FV$, we have

$$\tilde{V}\tilde{U}^{-1} = F\,\frac{W(4\Lambda\xi,\lambda_l)}{\tilde{U}} - \left(\frac{W(\Psi_0,\Psi_1)}{\tilde{U}}\right)^2, \tag{C.9}$$

where $W(C,D) := Cr^2 + D$, where C, D are constants.

Setting (C.3) as

$$B(z) = -\sum_{i=1}^{3}\frac{\rho_i/2}{z-z_i} + \frac{\beta}{z-z_\infty} \tag{C.10}$$

and using (C.8) in (C.6), we have

$$P(z) = \sum_{i=1}^{3}\frac{1-\rho_i}{z-z_i} + \frac{2\beta-2}{z-z_\infty}. \tag{C.11}$$

The other term of (C.6) requires a little more work. First, note that $Q(z)$ can be written as

$$Q(z) = \left(\sum_{i=1}^{3}\frac{\rho_i}{z-z_i}\right)^2 + \sum_{i=1}^{3}\frac{\left(\sum_{j\neq i}^{3}(\rho_i+\rho_j)/z_{ij} + [\beta+2(\beta-1)\rho_i]/z_{i\infty}\right)}{z-z_i}$$

$$-\frac{\sum_{i=1}^{3}[\beta+2(\beta-1)\rho_i]/z_{i\infty}}{z-z_\infty} + \frac{\beta^2-3\beta}{(z-z_\infty)^2} - \tilde{V}\tilde{U}^{-1}. \tag{C.12}$$

We now expand the terms in (C.9) into partial fractions by using that

$$\frac{W(C,D)}{f(z)} = \frac{C}{z(z-1)(z-t)}\left(\frac{r_4 z - r_1 z_\infty}{z-z_\infty}\right)^2 + \frac{D}{z(z-1)(z-t)}. \tag{C.13}$$

The first term above is expanded as

$$\frac{g(z)}{z(z-1)(z-t)(z-z_\infty)^2} = \sum_{i=1}^{3}\frac{(g/f')|_{z_i}}{z-z_i} + \frac{(g/f)'|_{z_\infty}}{z-z_\infty} + \frac{(g/f)|_{z_\infty}}{(z-z_\infty)^2}, \tag{C.14}$$

where we used the residue theorem to obtain the expansion coefficients. Using (C.7), (C.13) and (C.14), we obtain the first term of (C.9)

$$F\,\frac{W(4\Lambda\xi,\lambda_l)}{\tilde{U}} =$$

$$\sum_{i=1}^{3}\frac{(4\Lambda\xi r_i^2+\lambda_l)r_{14}z_\infty/(z_{i\infty}^2 U'(r_i))}{z-z_i} + \frac{12\xi(\sum_{i=1}^{3} r_i - r_4)/(r_{14}z_\infty)}{z-z_\infty} + \frac{12\xi}{(z-z_\infty)^2}. \tag{C.15}$$

The second term of (C.9) does not have z_∞ as a pole, so

$$-\left(\frac{W(\Psi_0,\Psi_1)}{\tilde{U}}\right)^2 = -\left(\sum_{i=1}^{3}\frac{(\Psi_0 r_i^2+\Psi_1)/U'(r_i)}{z-z_i}\right)^2 = \left(\sum_{i=1}^{3}\frac{\rho_i/2}{z-z_i}\right)^2, \tag{C.16}$$

if we choose

$$\rho_i = 2i\alpha_i = 2i\left(\frac{\Psi_0 r_i^2+\Psi_1}{U'(r_i)}\right). \tag{C.17}$$

Plugging (C.15) and (C.16) into (C.12), we finally obtain

$$Q(z) = \sum_{i=1}^{3}\frac{Q_i}{z-z_i}+\frac{Q_\infty}{z-z_\infty}+\frac{\beta^2-3\beta+12\xi}{(z-z_\infty)^2}, \tag{C.18a}$$

$$Q_i = \sum_{j\neq i}^{3}\frac{(\rho_i+\rho_j)/2}{z_{ji}}+\frac{\beta-(\beta-1)\rho_i}{z_{i\infty}}-\frac{r_{14}z_\infty}{z_{i\infty}^2}\left(\frac{4\Lambda\xi r_i^2+\lambda_l}{Q'(r_i)}\right), \tag{C.18b}$$

$$Q_\infty = -\frac{1}{r_{14}z_\infty}\left\{\sum_{i=1}^{3}(\beta-(\beta-1)\rho_i)(r_4-r_i)+12\xi(\sum_{i=1}^{3}r_i-r_4).\right\} \tag{C.18c}$$

Looking at (C.11) and (C.18a), we see that $z=z_\infty$ is removable if $\beta=1$ and $\xi=1/6$ as we already know, remembering also that $\sum_{i=1}^{4}r_i=0$ implies $Q_\infty=0$. Therefore, the final form we look for (C.6) is

$$\partial_z^2\psi+\sum_{i=1}^{3}\frac{1-\rho_i}{z-z_i}\,\partial_z\psi+\sum_{i=1}^{3}\frac{Q_i}{z-z_i}\psi=0, \tag{C.19}$$

with

$$Q_i = \frac{1}{2}\sum_{j\neq i}^{3}\frac{\rho_i+\rho_j}{z_{ji}}+\frac{1}{z_{i\infty}}-\frac{r_{14}z_\infty}{z_{i\infty}^2}\left(\frac{2\Lambda r_i^2/3+\lambda_l}{Q'(r_i)}\right). \tag{C.20}$$

It is easy to check that $\sum_{i=1}^{3}Q_i=0$, as it needs to be so that $z=\infty$ is also a regular singular point.

C.3 Reduction of Angular Equation for Kerr-(A)dS

As far as we know, it has not been noticed in the literature that Kerr-(A)dS angular equation (2.88) can be reduced to a confluent Heun equation when $\xi=1/6$, which is an even further reduction than in the radial equation. This comes about because the angular equation has two *elementary* singularities, which are regular and have $\theta=1/2$. These special singularities can be removed from the equation by a certain Schwarz-Christoffel transformation [93]. To prove this result, we first need to make a quadratic transformation in p, put the resulting equation in canonical form with $p=\infty$ at a finite point, revert the quadratic transformation and finally use a Schwarz-Christoffel transformation to remove two elementary singularities and result into a confluent Heun equation with 3 singular points. The trade off is that in the end the singular point at infinity will turn become irregular. However, from our perspective, this form is more useful because the 3 point monodromy group is known apart from the monodromy of the irregular singular point, which depends on Stokes data [39, 73].

So now let us specialize to the Kerr-AdS case. In this case, $n=0$, thus

$$P(p) = (a^2-p^2)(1-\alpha^2p^2) = \alpha^2(p^2-a^2)(p^2-\alpha^{-2}), \tag{C.21}$$

where $\alpha^2 = -\Lambda/3$. The resulting angular equation is

$$\partial_p(P(p)\partial_pS) + \left(-4\Lambda\xi p^2 + \lambda_l - \frac{\chi^4}{P(p)}\left[(a^2-p^2)\omega - ma\right]^2\right)S = 0, \tag{C.22}$$

where $\chi^2 = (1-\alpha^2a^2)$. Equation (C.22) has 5 regular singular points at $p = \{\pm a, \pm\alpha^{-1}, \infty\}$ and the local coefficients are given by

$$\theta_{\pm a} = \mp m, \quad \theta_{\pm\alpha^{-1}} = \pm\alpha^{-1}\left(\omega\chi^2 + ma\alpha^2\right) \tag{C.23}$$

and

$$\theta_\infty = \sqrt{9-48\xi}. \tag{C.24}$$

The first important thing to notice about (C.22) is that its coefficients are all even functions of p. This suggests a reduction of singular points using the quadratic change of variables $x=p^2$, resulting in

$$\partial_x^2S + \left(\frac{1/2}{x} + \frac{1}{x-a^2} + \frac{1}{x-\alpha^{-2}}\right)\partial_xS + \\ + \frac{1}{4xP(x)}\left(-4\Lambda\xi x + \lambda_l - \frac{\chi^4}{P(x)}\left[(a^2-x)\omega - ma\right]^2\right)S = 0, \tag{C.25}$$

with $P(x) = \alpha^2(x-a^2)(x-\alpha^{-2})$. Now (C.25) has only 4 regular singular points at $x = \{0, a^2, \alpha^{-2}, \infty\}$ and $x=0$ is a elementary singularity with $\theta_0 = 1/2$. Further, we know that $x = p^2 = 0$ is actually a regular point of $S(x)$, because of (C.22). A quadratic transformation cuts the local coefficients by half, therefore

$$\theta_\infty = \frac{1}{2}\sqrt{9-48\xi}, \tag{C.26}$$

and, if $\xi = 1/6$, we also have $\theta_\infty = 1/2$. Let us restrict to this case for now on.

We want to simultaneously get rid of both elementary singularities so it is useful to take $x = \infty$ into a finite point. With this in mind, we make another coordinate transformation

$$z = t^2 \frac{x}{x-\alpha^{-2}}, \quad t^2 = \frac{\alpha^2 a^2 - 1}{\alpha^2 a^2} = \frac{\chi^2}{\chi^2 - 1}, \tag{C.27}$$

which takes $(0, a^2, \infty, \alpha^{-2}) \mapsto (0, 1, t^2, \infty)$ and transforms (2.93) to

$$\partial_z^2 S + \left(\frac{1/2}{z} + \frac{1}{z-1} - \frac{1/2}{z-t^2} \right) \partial_z S + \\ + \left[-\frac{m^2/4}{(z-1)^2} + \frac{1/2}{(z-t^2)^2} + \frac{B_1}{z} + \frac{B_2}{z-1} + \frac{B_3}{z-t^2} \right] S = 0, \tag{C.28}$$

with

$$B_1 = -\frac{1}{4} \left[(\omega a - m)^2 \chi^2 - \frac{\lambda_l}{\chi^2} \right], \tag{C.29a}$$

$$B_2 = \frac{1}{4} \left[\frac{2\Lambda a^2}{3} - \lambda_l + m(2\omega a - m)\chi^2 \right], \tag{C.29b}$$

$$B_3 = -\frac{1}{4} \left[\frac{2\Lambda a^2}{3} + \frac{(1-\chi^2)}{\chi^2} \lambda_l + \omega^2 a^2 \chi^2 \right]. \tag{C.29c}$$

It is easy to check that $\sum_{i=1}^3 B_i = 0$ and thus $z = \infty$ is a regular singularity. Now we put (C.28) into canonical form making $S(z) = (z-1)^{-m/2} (z-t^2)^{1/2} \hat{S}(z)$,

$$\partial_z^2 \hat{S} + \left(\frac{1/2}{z} + \frac{1-m}{z-1} + \frac{1/2}{z-t^2} \right) \partial_z \hat{S} + \left[\frac{\hat{B}_1}{z} + \frac{\hat{B}_2}{z-1} + \frac{\hat{B}_3}{z-t^2} \right] \hat{S} = 0,$$

$$\begin{aligned} \hat{B}_1 &= B_1 + \frac{m}{4} + \frac{1-\chi^2}{4\chi^2}, \\ \hat{B}_2 &= B_2 - \frac{m\chi^2}{4} - \frac{m}{2}\left(1-\chi^2\right) + \frac{1-\chi^2}{2}, \\ \hat{B}_3 &= B_3 + \frac{m}{4}(1-\chi^2) - \frac{1-\chi^2}{4\chi^2} - \frac{1-\chi^2}{2}, \end{aligned} \tag{C.30}$$

and, of course, $\sum_{i=1}^3 \hat{B}_i = 0$.

To finish our demonstration, we make an inverse quadratic transformation $z = \zeta^2$ in (C.30)

$$\partial_\zeta^2 \hat{S} + \left(\frac{2(1-m)\zeta}{\zeta^2 - 1} + \frac{\zeta}{\zeta^2 - t^2} \right) \partial_z \hat{S} + \left(4\hat{B}_1 + 4\hat{B}_2 \frac{\zeta^2}{\zeta^2 - 1} + 4\hat{B}_3 \frac{\zeta^2}{\zeta^2 - t^2} \right) \hat{S} = 0, \tag{C.31}$$

which now has 5 regular singular points again at $\zeta = \{\pm 1, \pm t, \infty\}$. However, now $\zeta = \pm t$ are elementary and, therefore, may be removed by the Schwarz-Christoffel transformation

$$x = f(\zeta) = \int^{\zeta} \frac{du}{(u-t)^{1/2}(u+t)^{1/2}} = \cosh^{-1}(\zeta/t), \tag{C.32}$$

resulting in

$$\partial_x^2 \hat{S} + \left(\frac{2(1-m)\sinh x \cosh x}{\cosh^2 x - t^{-2}}\right)\partial_x \hat{S} + \\ + 4t^2\left(\hat{B}_1 \sinh^2 x + \hat{B}_2 \frac{\sinh^2 x \cosh^2 x}{\cosh^2 x - t^{-2}} + \hat{B}_3 \cosh^2 x\right)\hat{S} = 0. \tag{C.33}$$

The second line above can be simplified using that $\sum_{i=1}^{3} \hat{B}_i = 0$ giving

$$\partial_x^2 \hat{S} + \left(\frac{2(1-m)\sinh(x)\cosh(x)}{\cosh^2(x) - t^{-2}}\right)\partial_x \hat{S} + 4\left(\hat{B}_2 \frac{\sinh^2(x)}{\cosh^2(x) - t^{-2}} + t^2 \hat{B}_3\right)\hat{S} = 0. \tag{C.34}$$

Notice that, when $x \to \infty$, we have

$$\partial_x^2 \hat{S} + 2(1-m)\partial_x \hat{S} + 4\left(\hat{B}_2 + t^2 \hat{B}_3\right)\hat{S} = 0, \tag{C.35}$$

implying that infinity is actually an irregular singular point. We can go even one step further by noticing that, if we make $x = i\theta$ and simplify the coefficients of (C.35), we find that

$$\partial_\theta^2 \hat{S} + \frac{B \sin 2\theta}{1 + A\cos 2\theta}\partial_\theta \hat{S} + \frac{C + D\cos 2\theta}{1 + A\cos 2\theta}\hat{S} = 0, \tag{C.36}$$

with

$$A = \frac{t^2}{t^2 - 2} = \frac{\chi^2}{2 - \chi^2}, \quad B = 2A(m-1), \tag{C.37}$$

$$C = 4A[(t^2 - 2)\hat{B}_3 - \hat{B}_2], \quad D = 4A(\hat{B}_2 + t^2 \hat{B}_3). \tag{C.38}$$

This is called by Arscott [78] the *Magnus-Winkler-Ince* equation [184] and it has characteristics of a Hill-type equation, which is an important equation in the study of periodic forced oscillations. The equation (C.36) is of particular relevance because it is the most general linear second-order equation with periodic coefficients that can be solved by a trigonometric series with a three-term recursion relation [78]. Special cases of (C.36) are Mathieu equation and Lamé's equation.

APPENDIX D

Hypergeometric Connection Formulas

Here we review the connection formulas and fundamental matrix solution of a Hypergeometric differential system, as given in [104]. For more details on the derivation, see [74]. The Hypergeometric system in canonical form is given by

$$\frac{dY}{dz} = \left(\frac{L_0}{z} + \frac{L_1}{z-1}\right)Y \tag{D.1}$$

with local behaviour given by

$$Y(z) = G_0(1+O(z))z^{L_0} \qquad (z \to 0) \tag{D.2}$$

$$= G_1(1+O(z-1))(z-1)^{L_1} \qquad (z \to 1) \tag{D.3}$$

$$= (1+O(z^{-1}))z^{-L_\infty}, \qquad (z \to \infty), \tag{D.4}$$

where G_i are constant matrices, and the matrices L_i have $\det L_i = 0$ and $\operatorname{Tr} L_0 = 1-\gamma$, $\operatorname{Tr} L_1 = \gamma - \alpha - \beta - 1$. Also we fix

$$L_\infty = -(L_0 + L_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}. \tag{D.5}$$

With these conditions, the connection is explicitly given by

$$L_0 = \frac{1}{\beta-\alpha}\begin{pmatrix} -\alpha(\beta-\gamma+1) & \beta(\beta-\gamma+1) \\ -\alpha(\alpha-\gamma+1) & \beta(\alpha-\gamma+1) \end{pmatrix}, \qquad L_1 = \frac{1}{\beta-\alpha}\begin{pmatrix} \alpha(\alpha-\gamma+1) & -\beta(\beta-\gamma+1) \\ \alpha(\alpha-\gamma+1) & -\beta(\beta-\gamma+1) \end{pmatrix}. \tag{D.6}$$

These expressions follow either from direct relation with the hypergeometric differential equation or by an appropriate parameterization of the monodromy group.

The *hypergeometric fundamental solution* of this system is

$$Y(\alpha,\beta;\gamma;z) = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} z^{-\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)}. \tag{D.7}$$

and

$$\begin{aligned} Y_{11} &= {}_2F_1(\alpha, \alpha-\gamma+1; \alpha-\beta; \frac{1}{z}), \quad Y_{22} = {}_2F_1(\beta, \beta-\gamma+1; \beta-\alpha; \frac{1}{z}), \\ Y_{12} &= \frac{\beta(\beta-\gamma+1)}{(\beta-\alpha)(\beta-\alpha+1)}\frac{1}{z}\, {}_2F_1(\beta+1, \beta-\gamma+2; \beta-\alpha+2; \frac{1}{z}), \\ Y_{21} &= \frac{\alpha(\alpha-\gamma+1)}{(\alpha-\beta)(\alpha-\beta+1)}\frac{1}{z}\, {}_2F_1(\alpha+1, \alpha-\gamma+2; \alpha-\beta+2; \frac{1}{z}). \end{aligned} \tag{D.8}$$

The constants α,β and γ are given by

$$\alpha = \frac{1}{2}(\theta_\infty - \theta_1 - \sigma), \quad \beta = \frac{1}{2}(-\theta_\infty - \theta_1 - \sigma), \quad \gamma = 1 - \sigma. \tag{D.9}$$

The asymptotics of the hypergeometrics are

$$Y(\alpha,\beta,\gamma;z) = \begin{cases} G^{(0)}_{\alpha\beta\gamma}(1+O(z))z^{\begin{pmatrix} 1-\gamma & 0 \\ 0 & 0\end{pmatrix}} C^{(0)}_{\alpha\beta\gamma}, & z \to 0, \\ G^{(1)}_{\alpha\beta\gamma}(1+O(z-1))(z-1)^{\begin{pmatrix} \gamma-\alpha-\beta-1 & 0 \\ 0 & 0\end{pmatrix}} C^{(1)}_{\alpha\beta\gamma}, & z \to 1, \\ (1+O(z^{-1}))z^{\begin{pmatrix} -\alpha & 0 \\ 0 & -\beta\end{pmatrix}}, & z \to \infty, \end{cases} \tag{D.10}$$

where

$$G^{(0)}_{\alpha\beta\gamma} = \frac{1}{\beta-\alpha}\begin{pmatrix} \beta-\gamma+1 & \beta \\ \alpha-\gamma+1 & \alpha \end{pmatrix}, \qquad G^{(1)}_{\alpha\beta\gamma} = \frac{1}{\beta-\alpha}\begin{pmatrix} 1 & \beta(\beta-\gamma) \\ 1 & \alpha(\alpha-\gamma) \end{pmatrix}, \tag{D.11}$$

and the connection matrices are

$$C^{(0)}_{\alpha\beta\gamma} = \begin{pmatrix} e^{-\pi i(\alpha-\gamma+1)}\frac{\Gamma(\gamma-1)\Gamma(\alpha-\beta+1)}{\Gamma(\gamma-\beta)\Gamma(\alpha)} & e^{-\pi i(\beta-\gamma+1)}\frac{\Gamma(\gamma-1)\Gamma(\beta-\alpha+1)}{\Gamma(\gamma-\alpha)\Gamma(\beta)} \\ e^{-\pi i\alpha}\frac{\Gamma(1-\gamma)\Gamma(\alpha-\beta+1)}{\Gamma(1-\beta)\Gamma(\alpha-\gamma+1)} & -e^{-\pi i\beta}\frac{\Gamma(1-\gamma)\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)\Gamma(\beta-\gamma+1)} \end{pmatrix},$$

$$C^{(1)}_{\alpha\beta\gamma} = \begin{pmatrix} -\frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(\alpha-\beta+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\alpha)} & \frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\gamma+1)\Gamma(\beta)} \\ -e^{-\pi i(\gamma-\alpha-\beta-1)}\frac{\Gamma(\gamma-\alpha-\beta-1)\Gamma(\alpha-\beta+1)}{\Gamma(1-\beta)\Gamma(\gamma-\beta)} & e^{-\pi i(\gamma-\alpha-\beta-1)}\frac{\Gamma(\gamma-\alpha-\beta-1)\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)\Gamma(\gamma-\alpha)} \end{pmatrix}. \tag{D.12}$$