Universidade Federal de Pernambuco
Centro de Ciências Exatas e da Natureza
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Tese de Doutorado

# Hankel and sub-Hankel determinants

## (a detailed study of their polar ideals)

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Universidade Federal de Pernambuco

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## (a detailed study of their polar ideals)

Tese apresentada ao programa de Pós-Graduação em Matemática Pura da Universidade Federal de Pernambuco, como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática Pura.

Orientador: Prof. Dr. Aron Simis

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**HANKEL AND SUB HANKEL DETERMINANTS**

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**Abstract**

The theme of this thesis is the theory of homaloidal determinants. The focus is on the homological properties of the determinant of a generic Hankel matrix and of one of its degenerations as a method of studying their homaloidal behavior. In characteristic zero we show that the first has nonvanishing Hessian (hence its polar map defines a dominant rational map) but it is non-homaloidal. The case of the degenerate determinant has been proved to be homaloidal in [3]); we determine the ideal theoretic and numerical invariants of the corresponding gradient (polar) ideal, as well as its homological nature. These can in turn be used to simplify a few passages in the proofs of [3]. All results draw on some nontrivial underlying commutative algebra and the nature of its use is one of the assets of this thesis.

**Keywords:** Hankel matrix. Homaloidal polinomials. Polar map. Gradient ideal. Hessian. Linear syzygies. Symmetric algebra. Rees algebra. Plücker relations.

**Resumo**

Os resultados desta tese se enquadram na teoria dos polinômios homaloidais, com ênfase no caso de determinantes. O objetivo principal é o estudo das propriedades homológicas do determinante da matriz genérica de Hankel e de uma de suas degenerações, como um método de abordar o seu comportamento de natureza homalóide. No caso da matriz de Hankel genérica, em característica zero, concluimos que o Hessiano do determinante é não nulo (equivalentemente, o mapa polar associado é dominante), mas o determinante não é homalóide. No caso degenerado, sabe-se que o determinante é homalóide (provado por Cilibert-Russo-Simis [3]); aqui, determinamos os invariantes numéricos e homológicos do respectivo ideal gradiente (polar), esses podendo ser usados para simplificar algumas passagens no argumento de [3]. Os principais resultados da tese são baseados em ferramentas não triviais da álgebra comutativa e a natureza do uso dessas ferramentas é um dos recursos importantes desta tese.

**Palavras-chave:** Matriz de Hankel. Polinômio homalóide. Mapa polar. Ideal gradiente. Hessiano. Sizigias lineares. Álgebra simétrica. Álgebra de Rees. Equações de Plücker.

# Contents

# Chapter 1

# Introduction

The subject of Cremona transformations is a classical chapter of algebraic geometry. However, the classification of such maps in projective space $\mathbb{P}^n$ is well understood only for $n \leq 2$. In higher dimension the structure of the Cremona group is far from being fully understood, and classification is poorly known. An important class of Cremona maps of $\mathbb{P}^n$ consists of the so-called *polar maps*, i.e., rational maps $\nabla f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by the partial derivatives of a form $f \in R := k[\mathbf{x}] = k[x_0, \ldots, x_n]$ over a field of characteristic zero. The hypersurface $V(f)$ is called a *homaloidal hypersurface* if $\nabla f$ is a Cremona transformation of $\mathbb{P}^n$, in which case, by extension, $f$ is also dubbed a *homaloidal polynomial.*

One measure of the complexity of a hypersurface $V(f)$ can be read off $\nabla f$. Thus, e.g., the hypersurface with vanishing Hessian is considered as the simplest, and a homaloidal hypersurface simplest among those for which the Hessian is not identically zero – recall that $\nabla f$ being dominant (equivalently, in characateristic zero, the Hessian of $f$ not vanishing) is a necessary condition for homaloidness.

The simplest example of a homaloidal hypersurface $V(f)$ is a smooth quadric: in this case the polar map $\nabla f$ is an invertible linear map. This is also the only case of a reduced homaloidal polynomial if $n = 1$.

The plane case of homaloidal hypersurfaces has been settled by Dolgachev in [5]:

**Theorem.** A reduced homaloidal curve $V(f) \subset \mathbb{P}^2$ has degree $\leq 3$.

From this it is fairly easy to show that the curve is one of the following:

(i) A smooth conic;

(ii) The union of three distinct non-concurrent lines;

(iii) The union of a smooth conic with one of its tangent lines.

This result motivated experts to ask whether an analogue would hold for higher dimensions. This question has been negatively settled in [3, Theorem 3.13], by showing that for any $n \geq 3$ and any $d \geq 2n-3$ there exist irreducible homaloidal hypersurfaces of degree $d$. These examples are dual hypersurfaces to fairly elaborated projections of scrolls, which leads one to ask whether for certain special structured classes of homaloidal hypersurfaces it is reasonable to expect that the degree stays beneath the number of variables.

Clearly, the notion of "structured" is quite foggy. One would need some sort of anchor on the Dolgachev result. For example, in all cases of plane reduced homaloidal curves the minimal generating syzygies of the gradient ideal are linear. This kind of restriction might have an impact in higher dimension, but it is not clear which exactly.

One sort of modified question is open as far as we know:

**Question 1.0.1.** Let $\mathcal{M}$ denote a square matrix over $k[\mathbf{x}] = k[x_0, \ldots, x_n]$ whose entries are forms of equal degrees and such that $f := \det(\mathcal{M})$ is an irreducible homaloidal polynomial. Is $\deg(f) \leq n+1$?

It looks like the question is wide open even if the entries are linear. In [3, Section 4] a class of examples has been studied in which the answer to the last question is affirmative. Unfortunately, there are scarcely any general methods for studying, much less recognizing, such polynomials.

The main motivation of this work is the hope to shed additional light into this facet of the theory. One goal of this thesis is to consider "sufficiently" structured matrices of linear forms. Even for those there seems to be no comprehensive study of the homaloidal behavior of the corresponding determinants. Still, one advantage of dealing with these matrices is that they are often 1-generic in the sense of [9, Definition-Proposition 1.1]. An important case study is the class of generic catalectic matrices; these have been shown to be 1-generic (see [21, Proposition 2.1]). Their study from the homaloidal point of view is included in [18].

A major underlying problem in this context is to understand the properties of the so-called *gradient ideal* (or *singular ideal*) of the polynomial $f$, that is, the ideal $J = J(f) \subset R$ generated by the partial derivatives of $f$. Regardless of whether $f$ is homaloidal, a particular question asks when $J$ has an irrelevant primary component, i.e., when it is not saturated. Of course, if $\mathrm{Proj}R/(f)$ is smooth its gradient ideal will itself be an irrelevant primary ideal (in addition, $f$ will not be homaloidal unless $\deg(f) \leq 2$). One can easily cook up families of irreducible plane curves whose gradient ideals have an irrelevant component. It is much harder to exhibit an irreducible polynomial in $f \in k[x_0, x_1, x_2]$ such that its gradient ideal has no irrelevant component (i.e., such that its gradient ideal is perfect of codimension 2).

Here is a brief description of the contents of the thesis.

Chapter 2 is about preliminaries: we review a few basic notions about homogeneous ideals, with emphasis on their homological features on one hand, such as syzygies and the main associated algebras, and on the other hand, on their role in birational map theory.

Chapters 3 and 4 contain the core of the results and are each subdivided in several sections.

Chapter 3 deals with the homological information on the generic square Hankel matrix of arbitrary size. Thus, let $\mathcal{H}_n$ stand for the generic $n \times n$ Hankel matrix and let $P \subset R$ denote its ideal of submaximal minors (i.e., $(n-1)$-minors). We prove that $P$ is the minimal primary component of the gradient ideal $J = J(f) \subset R$ of $f := \det \mathcal{H}_n$. We have used quite a bit of arguments, ranging from multiplicities to initial ideals to Plücker relations (straightening laws) of Hankel maximal minors via the Gruson–Peskine change of matrix trick. The only remaining associated (necessarily embedded) prime of $R/J$ is the obvious candidate $Q := I_{n-2}(\mathcal{H}_n)$ – defining the singular locus of the determinantal variety $V(P)$. This guess is equivalent to the expected equality $J : P = Q$. For a conjecture about Hankel determinantal ideals and their reductions see [18, conjecture 3.15]. If this conjecture proves to be affirmative then one can conclude that the Hankel determinant is not homaloidal via the criterion stated in Proposition 2.2.3 and the result proved in the thesis to the effect that the *linear syzygy rank* of $J$ is $3 < 2n-2$ for $n \geq 3$.

In the case where $n = 3$ we are able to solve this conjecture in the affirmative, see 3.2.1. The most laborious is the one about the linear type property which, for this value of $n$, allows to apply some well established criteria.

In Chapter 4 we focus on a certain degeneration of the generic square Hankel matrix in which some entries are replaced by zeros in strategic places. One of the features of this degeneration is to get the partial derivatives to be as close as possible to the submaximal minors. This was originally introduced in [3, Section 4.1] under the designation *sub-Hankel matrix*. The basic equations governing the homological side of the corresponding gradient ideal were developed, putting in evidence a kind of telescopic structure, with subideals being codimension 2 perfect ideals. Here we complete the homological information on the gradient ideal. We have a theorem which shows that the gradient ideal is of linear type. By some propositions and techniques we resolve the minimal free resolution, the associated primes, multiplicity of gradient ideal and so forth.

In the Appendix we give a resumé of circulant matrices, showing that the determinant of a circulant matrix is a reduced homaloidal polynomial. This matrix is a classical object much studied from various viewpoints (see [25]). Over a field containing the roots of unity of the right degree the circulant determinant is the reduced product of linear factors, thus retrieving the usual reciprocal involution. Other than this, the Appendix includes some material used in the results. Some of this is routine, but there is a section on matrices whose entries are either variables or zeros which is not very well-known. This result has been obtained during the thesis work unaware of a previous proof given in [10].

# Chapter 2

# Ideal theoretic methods of birational maps

The aim of this chapter is to introduce the algebraic tool required throughout. Though the overall objective is to detect homaloidal determinants and their properties, there is a richness of notions from commutative algebra that has lately been used in birational maps (see [6], [22], [23], [20], [21], [19]). Often these notions are useful for the geometric result.

## 2.1 Recap of basic homological tools

Let $I$ be an ideal in a ring $R$. The symmetric algebra of $I$ is defined to be the tensor algebra of $I$ modulo symmetrization. It is naturally $\mathbb{N}$-graded as an $R$-algebra, denoted $\mathcal{S}(I) = \bigoplus_{j\geq 0} \mathcal{S}^j(I)$, where $\mathcal{S}^j(I)$ stands for the $j$th symmetric power of $I$. The *Rees algebra* is the $\mathbb{N}$-graded $R$-algebra $\mathcal{R}(I) := \bigoplus_{j\geq 0} I^j \simeq R[It] \subset R[t]$. Clearly, there is a natural surjective $R$-algebra homomorphism

$$\alpha : \mathcal{S}(I) \to \mathcal{R}(I). \tag{2.1}$$

A more concrete way of looking at $\mathcal{S}(I)$ arises from a free module presentation of $I$, as follows. Given such a presentation $0 \to Z \to R^n \to I \to 0$, we derive from the universal property of symmetrization the following algebra presentation

$$\mathcal{S}(I) \simeq \frac{\mathcal{S}(R^n)}{\mathcal{L}} \simeq \frac{R[T_1, \cdots, T_n]}{\mathcal{L}},$$

where $\mathcal{L}$ is the ideal generated by the linear forms $r_{1j}T_1 + \cdots + r_{nj}T_n$ corresponding to the generators (syzygies) $z = (r_{1j}, \cdots, r_{nj})^t \in Z$ of $I$ with respect to the chosen set of generators.

The Rees algebra has a similar algebra presentation

$$\mathcal{R}(I) \simeq \frac{R[T_1, \cdots, T_n]}{\mathcal{J}},$$

where $\mathcal{J}$ is defined to be the kernel of the $R$-algebra surjection $R[\mathbf{T}] := R[T_1, \cdots, T_n] \twoheadrightarrow R[It]$, with $T_i \mapsto f_i t$. Thus, $\mathcal{J}$ is generated by all forms $F(T_1, \cdots, T_n) \in R[\mathbf{T}]$ such that $F(f_1, \cdots, f_n) = 0$. Note that $\mathcal{J}$ is a homogenous ideal in the standard grading of $R[\mathbf{T}]$.

If $R = k[\mathbf{x}]$ is a standard graded ring over a field $k$ then $\mathcal{J}$ is a bihomogeneous ideal in the standard bigraded $k$-algebra $k[\mathbf{x}, \mathbf{T}]$. One has, namely

$$\mathcal{J} = \bigoplus_{(i,j)} \mathcal{J}_{(i,j)} \supset (\bigoplus_{i} \mathcal{J}_{(*,1)}) = \mathcal{L}.$$

Going back to the natural map (2.1), the ideal $I$ is said to be of *linear type* if $\alpha$ is an isomorphism. The terminology itself was introduced by L. Robbiano and G. Valla, but the earliest example of an ideal of linear type is found in [17]. Every ideal generated by a regular sequence has this property (see [13]) for further details).

An ideal $I \subset R$ of linear type satisfies the Artin–Nagata condition $G_\infty$ stating that the minimal number of generators of $I$ locally at any prime $p \in \mathrm{Spec}\,(R/I)$ is at most the height of $p$. This was first observed in the work of Herzog–Simis–Vasconcelos (see, e.g., [14, Proposition 2.4]). It is known that this condition is equivalent to a condition in terms of a free presentation

$$R^m \xrightarrow{\varphi} R^{n+1} \longrightarrow I \longrightarrow 0$$

of $I$ and the Fitting ideals of $I$, namely:

$$\mathrm{ht}\,(I_t(\varphi)) \geq \mathrm{rank}(\varphi) - t + 2, \quad \text{for} \quad 1 \leq t \leq \mathrm{rank}(\varphi), \tag{2.2}$$

where $I_t(\varphi)$ denotes the ideal generated by the $t \times t$ minors of a representative matrix of $\varphi$ and ht denotes the height of an ideal (see, e.g., [14, Lemma 8.2], [24, §1.3]). Because of its formulation in terms of Fitting ideals, the condition has been dubbed as $(F_1)$.

Suppose that $R$ is a standard graded ring and that $I$ is generated by forms $\mathbf{f} = f_0, \cdots, f_n$ of a fixed degree $s$. In this case, $I$ is more precisely given by means of a free graded presentation

$$R(-(s+1))^\ell \oplus \sum_{j \geq 2} R(-(s+j)) \xrightarrow{\varphi} R(-s)^{n+1} \to I \to 0$$

for suitable shifts $-(s+j)$ and rank $\ell \geq 0$. Of much interest in this work is the value of $\ell$, so let us state in which form. We call the image of $R(-(s+1))^\ell$ by $\varphi$ the LINEAR PART of $\varphi$ – often denoted $\varphi_1$ – and say that $\varphi$ has MAXIMAL LINEAR RANK if its linear part has rank $n$ $(= \mathrm{rank}(\varphi))$. Clearly, the latter condition is trivially satisfied if the two coincide, in which case $I$ is said to have LINEAR PRESENTATION (or is LINEARLY PRESENTED).

A quite notable fact, whenever $R = k[\mathbf{x}] = k[x_0, \ldots, x_n]$ is a standard graded polynomial ring over an infinite field $k$, is the case where $I$ happens to be of linear type and generated by $n+1$ forms of the same degree. Then $I$ has maximal analytic spread and hence ($k$ infinite) is generated by analytically independent forms (with respect to the irrelevant maximal ideal) of the same degree. Then these forms are algebraically independent elements over $k$, hence define a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. This will be a cornerstone of many an argument to follow.

A useful technique to get information about an ideal is its initial ideal. The following theorem summarizes some of its basic properties.

**Theorem 2.1.1.** [12, Theorem 3.3.4.] *Let $I \subset R$ be a homogeneous ideal and $<$ a graded monomial order on $R$. Then*

(*a*) $\dim R/I = \dim R/in_<(I)$;

(*b*) $\operatorname{depth} R/I \geq \operatorname{depth} R/in_<(I)$.

(*c*) $\operatorname{reg} R/I \leq \operatorname{reg} R/in_<(I)$;

Among the most important numerical invariants of a homogeneous ideal is its multiplicity. A close notion is the multiplicity of an $\mathfrak{m}$-primary ideal in a Noetherian local ring $(R, \mathfrak{m})$. We recall some basics related to these notions which are used in the subsequent chapters.

**Definition 2.1.2.** Let $M$ be a graded $R$-module whose graded components $M_n$ have finite length for all $n$. The numerical function

$$H(M, -) : \mathbb{Z} \to \mathbb{Z}$$

$$H(M, n) = \ell(M_n)$$

for all $n \in \mathbb{Z}$ is the *Hilbert Function* of $M$, and

$$H_M(t) := \sum_{n \in \mathbb{Z}} H(M, n) t^n$$

is the *Hilbert series* of $M$.

The Hilbert series of a graded module can be expressed in terms of its graded free resolution.

**Lemma 2.1.3.** [1, Lemma 4.1.13] *Let $M$ be a finite graded $R-$module of finite projective dimension, and let*

$$0 \to \bigoplus_j R(-j)^{\beta_{pj}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0j}} \to M \to 0$$

*be a graded free resolution of $M$. Then*

$$H_M(t) = S_M(t) H_R(t)$$

*where $S_M(t) = \sum_{i,j} (-1)^i \beta_{ij} t^j$. In particular, if $R = k[x_1, \cdots, x_n]$ is the polynomial ring over the field $k$, then*

$$H_M(t) = S_M(t)/(1-t)^n.$$

**Corollary 2.1.4.** [1, Corollary 4.1.14] *Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring over a field $k$, and $M$ be a finite graded $R-$module of dimension $d$. Then*

$$S_M^{(n-d+i)}(1) = (-1)^{n-d} (n-d+i)! e_i.$$

If $I \subset R$ is a homogeneous ideal, one says that $R/I$ has a *pure resolution* of type $(d_1, \cdots, d_p)$ if its graded free minimal resolution has the form

$$0 \to R(-d_p)^{\beta_p} \to \cdots \to R(-d_1)^{\beta_1} \to R \to R/I \to 0.$$

Note that $d_1 < d_2 < \cdots < d_p$.

**Theorem 2.1.5.** [1, Theorem 4.1.15] *Suppose $R/I$ is Cohen-Macaulay and has a pure resolution of type $(d_1, \cdots, d_p)$. Then*

$$e(R/I) = \frac{1}{p!} \prod_{i=1}^{p} d_i.$$

We now record the basics on the "local" Hilbert-Samuel function.

Let $(R, \mathbf{m})$ be a Noetherian local ring and let $M$ be a finitely generated $R$-module. The *Hilbert-Samuel function* of $M$ is

$$\chi_M(n) : n \mapsto \ell(M/\mathbf{m}^{n+1}M),$$

which is given by a polynomial:

$$\chi_M(n) = \frac{e(M)}{d!}n^d + \text{lower order terms},$$

for $n >> 0$.

One has $d = \dim(M)$, while the integer $e(M)$ is called the *multiplicity* of $M$. This will be mainly applied in the case where $M = R$, in which case one also says that $e(R)$ is the multiplicity of $\mathfrak{m}$ on $R$. We note that there is a generalization of this notion for $\mathfrak{m}$-primary ideals $I$ (instead of $\mathfrak{m}$). In such a generalization, the analogous integer obtained is called the multiplicity of $I$ on $R$ (or $M$, for a module).

**Proposition 2.1.6.** *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finite $R$-modules of the same dimension then $e(M) = e(M') + e(M'')$.*

**Proposition 2.1.7.** (Associativity formula) *Let $\{P_1, \cdots, P_t\}$ denote the set of minimal prime ideals of the $R$-module $M$ such that* $\dim(R/P) = \dim M$. *Then*

$$e(M) = \sum_{i=1}^{t} e(R/P_i)\ell(M_{P_i}),$$

*where $\ell(M_{P_i})$ stands for the length of $M_{P_i}$ over $R_{P_i}$.*

## 2.2 Ideal theoretic tools in birational maps

Let $k$ be an arbitrary field . For the purpose of the full geometric picture we may have to assume $k$ to be algebraically closed. Later we may have to assume that $\mathrm{char}(k) = 0$, but for the present $k$ may have arbitrary characteristic. We denote by $\mathbb{P}^n = \mathbb{P}^n_k$ the $n$th projective space. Let $X \subset \mathbb{P}^n$ be a projective subvariety, which will be assumed to be integral (i.e., reduced and irreducible or, equivalently, its largest defining ideal $I(X) \subset k[\mathbf{x}] = k[x_0, \ldots, x_n]$ is prime). Thus, $X$ admits a *function field* $k(X)$ and one has the main dimension theorem of these preliminaries, namely, $\dim X = \mathrm{tr.deg.}_k k(X)$, where $\dim X$ is a combinatorial-like dimension defined in terms of lengths of chains of subvarieties.

Let $\mathbb{P}^m$ be yet another projective space. We wish to consider rational maps with "domain" $X$ and target $\mathbb{P}^m$. The quotation marks indicate that there will be some difficulties as to what will be exactly the domain of these maps. Of course the terminology comes from having these maps defined in terms of rational functions on the domain, i.e., fractions of polynomial functions thereof. As is known, there are two different fields around in the theory of projective varieties: the "big" field of fractions of the homogeneous coordinate ring $k[X] = k[\mathbf{x}]/I(X)$ of $X$ and the "small" field of fractions of an affine piece of $X$. While the first gives perfectly defined regular functions on the structural affine cone over $X$, it is the latter that is called the function field of

$X$ because it makes sense to say for its elements that they induce locally well-defined functions on $X$ - while the elements of $k[X]$ do not.

The basic notion is that of an *m-rational datum* on $X$, i.e, a collection of fractions

$$f_1/f_0, \ldots, f_m/f_0 \in k(X),$$

where $f_0, f_1, \ldots, f_m \in k[X]$ are forms of the same degree, with $f_0 \neq 0$. In a slightly imprecise way, the above datum induces a map with target $\mathbb{A}^m \subset \mathbb{P}^m$ and domain a certain subset of closed points of $X$, namely, given $p = (a_0 : \cdots : a_n) \in X$, such that $f_0(p) = f(a_0, \cdots, a_n)) \neq 0$, the image of $p$ is the point with affine coordinates $(f_1(p)/f_0(p), \ldots, f_m(p)/f_0(p)) \in \mathbb{A}_k^m$. If one thinks of points in an older language, $P = (\bar{x}_0 : \cdots : \bar{x}_n)$ can be thought of as being the generic point of $X$, where $\bar{x}_i$ denotes the residue of the variable $x_i$ in $k[X]$. In this version, the image of the generic point is the "affine point" $(f_1(P)/f_0(P), \ldots, f_m(P)/f_0(P)) = (f_1/f_0, \ldots, f_m/f_0)$. Thus the coordinates of the image of the generic point of $X$ generate a $k$-subfield of $k(X)$ and can be thought of as the generic point of a subvariety whose affine coordinate ring is $k[f_1/f_0, \ldots, f_m/f_0]$.

The map thus obtained is called a *rational map* – hence the usage of the notation $F\colon X \dashrightarrow \mathbb{P}^m$ to indicate such a map, where the dotted tail of the arrow reminds us of the partial domain of the map.

We next highlight the main features of the concept.

- The finitely generated $k$-domain $k[f_1/f_0, \ldots, f_m/f_0] \subset k(f_1/f_0, \ldots, f_m/f_0) \subset k(X)$ is the affine coordinate ring of a uniquely defined subvariety of $\mathbb{A}^m \subset \mathbb{P}^m$. The closure of the latter $Y \subset \mathbb{P}^m$ is called the *image* of the rational map $F$. By abuse, we also say that the rational map $F$ is defined by the homogeneous datum $f_0, f_1, \ldots, f_m$ and frequently use the notation $F = (f_0 : f_1 : \ldots : f_m)$ which as we will indicate below conveys the lack of uniqueness of choosing these forms.

- We emphasize en passant that the image of a rational map on an integral projective subvariety is an integral projective subvariety of the target. As such, its homogeneous coordinate ring is isomorphic to the subring $k[f_0, \ldots, f_m] \subset k[X]$ as graded $k$-algebras - we say that one obtains the first from the second by a degree renormalization. Thus, although the dealing with rational maps is rather subtle, the image of such maps is a rather easily understood object in algebraic terms.

- One says that $F$ is *birational onto its image* if there exists a rational map $G\colon Y \dashrightarrow \mathbb{P}^n$ whose image is $X$ such that $F$ and $G$ are inverses to each other as field maps, i.e., if they define a $k$-isomorphism of fields $k(X) \simeq k(Y)$ and its inverse.

- One can also express the latter behavior at the level of the homogeneous data, namely, if $F = (f_0 : f_1 : \ldots : f_m)$ and $G = (g_0 : g_1 : \ldots : g_n)$ then $F$ is birational onto its image with inverse map $G$ if and only if the following relations are satisfied:

$$(f_0(g_0, g_1, \ldots, g_n) \colon \ldots : f_m(g_0, g_1, \ldots, g_n)) = (\bar{y_0} \colon \ldots : \bar{y_m})$$

  and

$$(g_0(f_0, f_1, \ldots, f_m) \colon \ldots : g_n(f_0, f_1, \ldots, f_m)) \equiv (\bar{x_0} \colon \ldots : \bar{x_n})$$

  as tuples of homogeneous coordinates in $\mathbb{P}^m_{K(Y)}$ (resp. $\mathbb{P}^n_{K(X)}$) where $K(Y)$ is the field of fractions of $k[Y]$ (resp. of $k[X]$).

- If $d$ is the common degree of $f_0, f_1, \ldots, f_m$, it is not difficult to see that this is the case if and only if the integral domains $k[f_0, f_1, \ldots, f_m] \subset k[k[X]_d] = k[(\mathbf{x})_d]/I(X)_d$ have the same field of fractions.

- There is nothing unique about the choice of $f_0, f_1, \ldots, f_m$, as one learns quite early in a first course. Nevertheless, any other choice of such a representative of the map $F$, say, $f'_0, f'_1, \ldots, f'_m$, necessarily satisfies the condition that the $2 \times 2$ minors of the matrix

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_m \\ f'_0 & f'_1 & \cdots & f'_m \end{pmatrix}$$

  vanish (as elements of the homogeneous coordinate ring $k[X]$ of $X$). Trivial remark as it is it represents a notable clarification in the theory.

- Note that the latter condition also means that the above matrix has rank 1 over the field of fractions $K(X)$ of $k[X]$, i.e., the two row vectors are proportional over this field. This justifies the previous notation $F = (f_0 : \cdots : f_m)$ (rigorously, an element of $\mathbb{P}^m_{K(X)}$).

- When considering the *base locus* of a rational map there is a potential confusion as to whether one is looking at the projective scheme defined by the forms $\mathbf{f}$ or more accurately at the homogeneous ideal generated by these forms. In the present approach we will need to be fairly precise about the ideal of defining equations of the associated Rees algebra of the ideal version. Moreover, the subalgebra $k[f_0, \ldots, f_m]$ which should be left intact as it defines the image of the given map. Also we will be using the subalgebra and the defining ring of the blowup on the same foot, hence one had better keep the Rees algebra intact as well.

Quite a bit of the nature of the statements and their proofs appeal to the syzygies of an ideal representative of $F$, however the very format of the conditions do not directly involve them. This is because the condition for birationality rather requires dwelling on the "hidden side" of the defining forms of the map, i.e., on the generators of the Rees algebra. Namely, the theory hinges on the existence of sufficiently many defining Rees equations of bidegree $(1, s)$, for $s \geq 1$ and the corresponding Jacobian matrix with respect to the $\mathbf{x}$-variables - a so-called *weak Jacobian dual* matrix - hence there will be no restriction on the field characteristic, a non-negligible point in birational geometry as is well-known.

In terms of this matrix $\Psi$, a criterion has undergone various successive formulations and improvement ([22], [23], [6]). In its most general form it states:

**Theorem 2.2.1.** [6, theorem 2.18] *Let $\mathcal{F} : X \dashrightarrow \mathbb{P}^n$ be a rational map, where $X \subset \mathbb{P}^m$ is non-degenerated. The following are equivalent:*

(a) *$\mathcal{F}$ is birational onto the image.*

(b) $\operatorname{rank}(\Psi) = m$.

*Moreover, when these conditions hold then the coordinates of the inverse to $\mathcal{F}$ are the* (*ordered, signed*) *$m$-minors of an arbitrary $m \times (m+1)$ submatrix rank $m$ of a weak Jacobian dual matrix of $\mathcal{F}$.*

Though flexible and computationally effective, the criterion depends on being able to compute generators of the defining ideal of a Rees algebra. Fortunately, one can often trade the full information about these generators for certain sufficient conditions directly related to the base ideal of the rational map.

**Theorem 2.2.2.** [6, Theorem 3.2] *Let $\mathcal{F} : \mathbb{P}^m \dashrightarrow \mathbb{P}^n$ be a rational map, given by $n+1$ forms $\mathbf{f} = \{f_0, \cdots, f_n\}$ of a fixed degree. If $\dim(k[\mathbf{f}]) = m+1$ and the linear rank of the base ideal $(\mathbf{f})$ is $m$ (maximal possible) then $\mathcal{F}$ is birational onto its image.*

When the defining forms $f_i$s of the map generate an ideal of linear type, then they are algebraically independent and, in particular, $m = n$. Consequently $\dim(k[\mathbf{f}]) = \operatorname{trdeg}(k[\mathbf{f}]) = m+1$. Applying in the case where $I$ is the base ideal of a polar map, one gets:

**Proposition 2.2.3.** [6, Proposition 3.4] *Let $f \in k[\mathbf{x}]$ denote a square-free homogeneous polynomial of degree $d \geq 3$, let $I \subset k[\mathbf{x}]$ stand for its gradient ideal. If $I$ is of linear type then the following conditions are equivalent:*

(a) *$f$ is homaloidal*

(b) *$I$ has maximal linear rank.*

*Moreover, if these conditions take place, the inverse map $\mathcal{F}^{-1}$ has degree $\leq m$.*

Note that the ideal $I$ has codimension at least two since $f$ is assumed to be squarefree. In particular, the partial derivatives give the unique representative of codimension $\geq 2$ of the polar map.

# Chapter 3

# Hankel Matrices

A Hankel matrix is a special case of the so-called catalecticant. As such it inherits many of the features of the latter and, in addition, it profits from the fact that it is a symmetric matrix (a property no other catalecticant has).

Sitting on the junction of these classes of matrices, no wonder it features many special properties. To get a grip of this junction, let us start with the form of the catalecticant,

Let $m \geq 2$ and $1 \leq r \leq m$ be given integers. Let $R = k[X_0, \ldots, X_n]$ be a polynomial ring with $n+1 = (m-1)(r+1)+1$ variables. The *$r$-leap $m \times m$ generic catalecticant* is the matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 & \ldots & X_{m-1} \\ X_r & X_{r+1} & X_{r+2} & \ldots & X_{m+r-1} \\ X_{2r} & X_{2r+1} & X_{2r+2} & \ldots & X_{m+2r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{(m-1)r} & X_{(m-1)r+1} & X_{(m-1)r+2} & \ldots & X_{(m-1)r+(m-1)} \end{pmatrix}$$

The extreme values $r = 1$ and $r = m$ yield, respectively, the ordinary Hankel matrix and the generic matrix. Note that the corresponding determinant will have low degree ($= m$) as compared to the dimension of the ring and still involve all variables.

A fundamental result proved in [9] tells us that the generic Hankel matrix is 1-generic. As a consequence, the determinant of a square such matrix is irreducible and so is any of its subdeterminants. A much stronger result for the Hankel matrix is that its ideals of minors of arbitrary size are prime ideals, All these results and more are proved in [9]; the last result is [9, Proposition 4.3].

## 3.1 Basics

Consider the $n \times n$ Hankel matrix:

$$\mathcal{H} = \mathcal{H}(n) := \begin{pmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ x_{n-1} & x_n & x_{n+1} & \ldots & x_{2n-3} & x_{2n-2} \end{pmatrix}$$

Let $J \subset R = k[x_0, \ldots, x_{2n-2}]$ be the gradient ideal of $\det(\mathcal{H})$, i.e., the ideal generated by the partial derivatives $f_i := \partial \det(\mathcal{H})/\partial x_i$ of $\det(H)$.

The first obvious question is about the codimension of $J$. By Proposition 5.3.1, we know that $J \subset I_{n-1}(\mathcal{H})$, the ideal generated by all minors of $\mathcal{H}$ of the size $n-1$. On the hand, by the trick in ([11, Lemme 2.3]), the latter ideal of minors coincide with the ideal of maximal minors of an $(n-1)\times(n+1)$ Hankel matrix. Since the latter is a specialization of the $(n-1)\times(n+1)$ generic matrix, the codimension of $J_{n-1}(\mathcal{H})$ is 3. Therefore, the codimension of $J$ is at most 3.

The following result shows that it is exactly 3, with an extra precision.

**Lemma 3.1.1.** *Let $\mathcal{H}$ denote the generic $n \times n$ Hankel matrix and let $J$ denote the gradient ideal of* $\det(\mathcal{H})$. *Then $J$ is a codimension* 3 *ideal contained in $P := I_{n-1}(\mathcal{H})$.*

**Proof.** By Proposition 5.3.1 $J$, $J \subset P$, hence $J$ has codimension at most 3. We consider the initial ideal of $J$ in the reverse lexicographic order. Once more, using Proposition 5.3.1 with $\mathcal{M} = \mathcal{H}$ and $R = k[x_0, \ldots, x_{2n-2}]$, direct inspection shows that $\mathrm{in}(f_0) = x_n^{n-1}$, $\mathrm{in}(f_{2n-2}) = x_{n-2}^{n-1}$, $\mathrm{in}(f_{n-1}) = n x_{n-1}^{n-1}$. Clearly then $\mathrm{in}(J)$ has codimension at least 3. Therefore, $J$ has codimension at least 3 as well. □

**Remark 3.1.2.** Observe that $\{f_0, f_{2n-2}, f_{n-1}\}$ is itself a regular sequence. This follows from Lemma 2.1.1 since $\mathrm{in}(f_0, f_{2n-2}, f_{n-1}) \supset (\mathrm{in}(f_0), \mathrm{in}(f_{2n-2}), \mathrm{in}(f_{n-1}))$, thus implying that the ideal $(f_0, f_{2n-2}, f_{n-1})$ has codimension 3 and therefore $\{f_0, f_{2n-2}, f_{n-1}\}$ is a regular sequence as we are in a graded case.

## 3.2 The $3 \times 3$ Hankel matrix

In this part we focus on the size $m = 3$. The reason to do this will become clear later.

Let

$$\mathcal{H} = \mathcal{H}_3 = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

**Proposition 3.2.1.** (characteristic zero) *Let $\mathcal{H}$ denote the generic* $3 \times 3$ *Hankel matrix in the variables $x_0, \ldots, x_4$ and let $J \subset R := k[x_0, \ldots, x_4]$ denote the gradient ideal of* $\det \mathcal{H}$.

(a) *Let $P := I_2(\mathcal{H}) \subset R$; then $P$ and $\mathfrak{m} := (x_0, \ldots, x_4)$ are the only associated primes of $R/J$ and $P$ is the minimal primary component of $J$.*

(b) *$J$ is an ideal of of linear type and the syzygy matrix of $J$ has linear rank* 3; *in particular,* $\det \mathcal{H}$ *is* not *homaloidal.*

**Proof.** By the principle ([11, Lemme 2.3]) mentioned earlier, in the Hankel case the ideal $P$ generated by the $2 \times 2$ minors of $\mathcal{H}$ coincides with the ideal generated by the maximal minors of the Hankel matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}, \tag{3.1}$$

which is the defining prime ideal of the normal quartic curve in $\mathbb{P}^4$.

(a) The contents of this item are that the gradient ideal of $\det(\mathcal{H})$ defines the same projective scheme as the normal quartic. However, we still need the algebraic details to show this. Set $f := \det \mathcal{H}$, and $f_i := \partial f / \partial x_i$. Note that these are

$$\{f_4 = \Delta_{12}, f_3 = -1/2\Delta_{13}, \mathbf{f_2} = \mathbf{\Delta_{14}} + \mathbf{3\Delta_{23}}, f_1 = -1/2\Delta_{24}, f_0 = \Delta_{34}\}, \tag{3.2}$$

where $\Delta_{ij}$ is the 2-minor of (3.1) with $i$th and $j$th columns ($i < j$). Clearly, $P$ is a minimal prime of $R/J$ since $J$ has codimension 3 by Proposition 3.1.1.

We need the following result:

CLAIM: $\mathfrak{m}P \subset J$.

It suffices to show that, for $i = 0, \ldots, 4$, either $x_i\Delta_{23} \in J$ or $x_i\Delta_{14} \in J$. The following relations take care of this:

$$\begin{aligned} x_0\Delta_{23} &= -x_2f_4 - 1/2x_1f_3 \\ x_1\Delta_{23} &= -x_3f_4 + 1/2x_2f_3 \\ x_2\Delta_{14} &= x_0f_0 - 1/2x_3f_3 \\ x_3\Delta_{23} &= 1/2x_3f_2 + 1/4x_4f_3 - 1/4x_2f_1 \\ x_4\Delta_{14} &= x_4f_2 + 3x_2f_0 + 3/2x_3f_1 \end{aligned}$$

Now, from the claim follows that $\mathfrak{m}$ is an associated prime of $R/J$ and, furthermore, that $P \subset J : \mathfrak{m} \subset J : \mathfrak{m}^\infty$. But since $P$ is a minimal prime of $R/J$ of maximal dimension, it follows that $P = J : \mathfrak{m}^\infty$. This means that the primary decomposition of $J$ away from $\mathfrak{m}$ is $P$, so we are done for this item.

(b) To show the linear type property we use the criterion of [14, Theorem 9.1], informally known as the "$(F_1)$ + sliding depth" criterion. Now, $(F_1)$ (also known as $(G_\infty)$ is the following:

CLAIM: $\mu(J_Q) \leq \operatorname{ht} Q$, for every prime ideal $Q$.

To prove this assertion we have to drill quite a bit through the syzygies of the generators of $J$. Since $P$ has the expected codimension (= 3), its presentation matrix is linear, as a piece of the Eagon–Northcott complex. Noting that the syzygies derive from the Laplace relations coming from the four $2 \times 3$ submatrices of (3.1), and ordering the minors to adjust to the order in which they appear along the partials as in (3.2), one obtains the following matrix:

$$\mathcal{S} = \begin{pmatrix} x_2 & x_3 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ -x_1 & -x_2 & 0 & 0 & x_3 & x_4 & 0 & 0 \\ 0 & 0 & -x_1 & -x_2 & -x_2 & -x_3 & 0 & 0 \\ x_0 & x_1 & 0 & 0 & 0 & 0 & x_3 & x_4 \\ 0 & 0 & x_0 & x_1 & 0 & 0 & -x_2 & -x_3 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & x_1 & x_2 \end{pmatrix}.$$

Let $\mathcal{S}_j$ $(1 \leq j \leq 8)$ denote the $j$th column of this matrix. By this preliminaries we can see:

$$\begin{pmatrix} \Delta_{12} & \Delta_{13} & \Delta_{14} & \Delta_{23} & \Delta_{24} & \Delta_{34} \end{pmatrix} \begin{pmatrix} x_2 & x_3 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ -x_1 & -x_2 & 0 & 0 & x_3 & x_4 & 0 & 0 \\ 0 & 0 & -x_1 & -x_2 & -x_2 & -x_3 & 0 & 0 \\ x_0 & x_1 & 0 & 0 & 0 & 0 & x_3 & x_4 \\ 0 & 0 & x_0 & x_1 & 0 & 0 & -x_2 & -x_3 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & x_1 & x_2 \end{pmatrix} = \mathbf{0}.$$

Suppose that $\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}$ is a syzygy of $J = (f_0, f_1, f_2, f_3, f_4)$. Therefore $a_0f_0 + a_1f_1 + a_2f_2 + a_3f_3 + a_4f_4 = 0$. So that

$$a_0\Delta_{34} + a_1(-1/2\Delta_{24}) + a_2(3\Delta_{23} + \Delta_{14}) + a_3(-1/2\Delta_{13}) + a_4\Delta_{12} = 0.$$

Therefore $\begin{pmatrix} a_4 & -1/2a_3 & \mathbf{a_2} & \mathbf{3a_2} & -1/2a_1 & a_0 \end{pmatrix}$ is a syzygy of $J$ then it must be K-linear combination of columns of $\mathcal{S}$. Conversely any $K-$linear combination of the columns of $\mathcal{S}$ such that the entry in the fourth coordinate is 3 times the entry in the third one gives a syzygy of $J$.

As one can see $a_4, a_3, a_1$ and $a_0$ can be considered independently from each-other but for choosing $a_2$ we have to be careful. (The critical point is $\Delta_{14} + 3\Delta_{23} = f_2$.) Hence in the 3th and 4th rows of $\mathcal{S}$ we have to choose columns with the same variable. Therefore we just have these combinations:

$$-\mathcal{S}_6 + 3\mathcal{S}_7, 3\mathcal{S}_2 - \mathcal{S}_3, \mathcal{S}_5 - \mathcal{S}_4.$$

We can go further and explicitly derive the linear syzygies.

$$\begin{aligned} 3\mathcal{S}_2 - \mathcal{S}_3 &= 3x_3\Delta_{12} - 3x_2\Delta_{13} + 3x_1\Delta_{23} - x_3\Delta_{12} + x_1\Delta_{14} - x_0\Delta_{24} \\ &= 2x_3\Delta_{12} - 3x_2\Delta_{13} + x_1(3\Delta_{23} + \Delta_{14}) - x_0\Delta_{24} = 0 \end{aligned}$$

Now by replacing $\Delta_{ij}$ by the correspondent partial derivative we will have :

$$2x_3f_4 + 3/2x_2f_3 + x_1f_2 + 1/2x_0f_1 = 0$$

That is

$$4x_3f_4 + 3x_2f_3 + 2x_1f_2 + x_0f_1 = 0.$$

Consequently we get linear syzygy

$$\begin{pmatrix} 0 & x_0 & 2x_1 & 3x_2 & 4x_3 \end{pmatrix}.$$

Similarly another linear syzygies is:

$$-\mathcal{S}_4 + \mathcal{S}_5 = 2x_0f_0 + x_1f_1 - x_3f_3 - 2x_4f_4 = 0$$

or in the other words is

$$\begin{pmatrix} 2x_0 & x_1 & 0 & -x_3 & -2x_4 \end{pmatrix}$$

and

$$-\mathcal{S}_6 + 3\mathcal{S}_7 = 4x_1f_0 + 3x_2f_1 + 2x_3f_2 + x_4f_3 = 0$$

or

$$\begin{pmatrix} 4x_1 & 3x_2 & 2x_3 & x_4 & 0 \end{pmatrix}.$$

In this way one gets that the following linear syzygies

$$\begin{pmatrix} 0 & -2x_0 & 4x_1 \\ x_0 & -x_1 & 3x_2 \\ 2x_1 & 0 & 2x_3 \\ 3x_2 & x_3 & x_4 \\ 4x_3 & 2x_4 & 0 \end{pmatrix}$$

generate the linear part of the entire module of syzygies of (3.2). The rank of this matrix is clearly 3, hence this is the value of the linear rank of $J$. We now contend that, together with the Koszul syzygies, they suffice to check the stated property of $J$. For this, we note that this property is equivalent to a property of the Fitting ideals of $J$, and can be stated as follows:

$$\operatorname{ht} I_t(\varphi) \geq \operatorname{rank}(\varphi) - t + 2 = 4 - t + 2 = 6 - t, \quad \text{for } 1 \leq t \leq 4.$$

Obviously, it suffices to show these estimates for the above submatrix $\psi$. For $t = 1, 2$ the linear syzygies so far described immediately give the required inequalities. For $t = 3, 4$, we use the matrix of Koszul syzygies where one knows that the codimension of any of its Fitting ideals is $\geq \mathrm{ht}\,(J) = 3$. So much for the property $(F_1)$.

The sliding depth condition reads as

CLAIM: $\mathrm{depth}\,(H_i)_Q \geq \mathrm{ht}\,(Q) - \mu(J_Q) + i$, for every prime $Q \supset J$ with $0 \leq i \leq \mu(J_Q) - \mathrm{ht}\,(J_Q)$. Here $H_i$ denotes the $i$th Koszul homology module on the generators of $J$

Note that if $Q \supset J$ then necessarily $Q \supset P$. We first assume that $Q = \mathfrak{m}$, in which case $\mu(J_\mathfrak{m}) = \mu(J) = 5$, $\mathrm{ht}\,(J_\mathfrak{m}) = \mathrm{ht}\,(J) = 3$, hence $\mu(J_\mathfrak{m}) - \mathrm{ht}\,(J_\mathfrak{m}) \leq 2$. In this case, our range for $i$ is $0 \leq i \leq 2$. For $i = 2$ the inequality is automatic since $H_2$ is Cohen–Macaulay and it is trivial if $i = 0$. For $i = 1$ we need to show that $H_1$ has depth at least 1. For this it suffices to show that $J$ is a syzygetic ideal since we then have an exact sequence $0 \to H_1 \longrightarrow (R/J)^5 \longrightarrow J/J^2 \to 0$, proving that $H_1$ has homological dimension at most 3 over $R$ and hence, depth at least 1. The syzygetic calculation is equivalent to showing that any syzygy of $J$ with entries in $J$ belongs to the module of Koszul syzygies of $J$. This computation has been carried out with *Macaulay* (still simpler than carrying out the computation of the entire Rees ideal.)

Next assume that $Q \subsetneq \mathfrak{m}$, hence $\mu(J_Q) \leq \mathrm{ht}\,(Q) \leq 4$ (by (a)). Clearly, $\mathrm{ht}\,(J_Q) = \mathrm{ht}\,(P_Q) = 3$. By part (a), we know that $J_Q = P_Q$. Therefore, $\mu(J_Q) - \mathrm{ht}\,(J_Q) = \mu(P_Q) - \mathrm{ht}\,(P_Q) = 3 - 3 = 0$ since $R/P$ is an isolated singularity. Then $i = 0$ is the only case. One has $\mathrm{depth}\,(H_0)_Q = \mathrm{depth}\,(R_Q/P_Q) = 2 = \mathrm{ht}\,(Q) - \mu(J_Q) + 0$.

Thus, $J$ is an ideal of linear type.

Since we have also proved above that the linear rank of the syzygies of $J$ is $\leq 3$, we are done for the supplementary assertion by Proposition 2.2.3. $\square$

**Remark 3.2.2.** We will prove later that in arbitrary size the generic Hankel matrix has linear rank 3.

## 3.3 Hankel matrix of arbitrary size

The generic Hankel matrix of size $n \times n$ is the symmetric matrix

$$\mathcal{H} := \begin{pmatrix} x_0 & x_1 & \dots & x_{n-1} \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_n & \dots & x_{2n-2} \end{pmatrix}$$

One expects results analogous to those of Proposition 3.2.1, but there will be some differences. For example, if $m \geq 4$ the ring $R/J$ will no longer have depth zero. However, the main bulk of the properties in the case $n = 3$ will hold true.

For the proofs we will use the following preliminaries. We emphasize that throughout the following discussion, $R = k[x_0, \dots, x_{2n-2}]$.

### 3.3.1 A fundamental primary component of $J$

We will use the following result, possibly well-known.

**Lemma 3.3.1.** *Set $P := I_{n-1}(\mathcal{H})$. Then $P$ is a prime ideal and the multiplicity of $R/P$ is*

$$e(R/P) = \frac{1}{3!}(n-1)n(n+1).$$

**Proof.** By [9, Proposition 4.3], the ideal $P$ is prime and has maximal possible height. On the other hand, by the same principle of [11, Lemme 2.3], $P$ is the ideal of the maximal minors of the Hankel matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_n \\ x_1 & x_2 & \dots & x_{n+1} \\ \vdots & \vdots & \dots & \vdots \\ x_{n-2} & x_{n-1} & \dots & x_{2n-2} \end{pmatrix}$$

Therefore, this ideal has height $n-(n-2)+1 = 3$. It follows that the Eagon–Northcott complex resolves $R/P$; since this complex is well-known to be a pure $(n-1)$-resolution one can apply Theorem 2.1.5 to get the desired expression. □

**Lemma 3.3.2.** *Letting* $\text{in}(J) \subset R$ *denote the initial ideal of $J$ in the revlex monomial order and writing $J' \subset \text{in}(J)$ for the subideal generated by the elements $\{\text{in}(f_0), \dots, \text{in}(f_{2n-2})\}$, one has an inequality $e(R/J') < 2e(R/P)$.*

**Proof.** Let us compute $e(R/J')$ in a direct way. One can see that $J'$ is generated by the monomials

$$G = (x_{n-2}^{n-1}, x_{n-2}^{n-2}x_{n-1}, \dots, x_{n-1}^{n-1}, x_{n-1}^{n-2}x_n, \dots, x_n^{n-1}). \tag{3.3}$$

Writing $J''$ for the ideal of $S := k[x_{n-2}, x_{n-1}, x_n]$ generated by these monomials, one has $J' = J''R$, hence $e(R/J') = e(S/J'')$.

Since $J''$ is a $m-$primary ideal in $S$, one has $e(S/J'') = dim_k(S/J'')$. Set $x = x_{n-2}, y = x_{n-1}, z = x_n$ for easier reading. We partition the $k$-vector basis of $(S/J'')$ into disjoint sets according to powers of $y$. Namely, let $A_i$ denote the set of basis elements whose $y$-degree is $i-1$, for $1 \le i \le n-1$. One can easily seen that $A_i$ the set of all monomiala $x^\alpha y^\beta z^\gamma$ such that $\beta = i-1$, $\alpha < n-i$ and $\gamma < n-i$; to see this note that if $\alpha \ge n-i$ then the monomial has degree $\ge i-1+n-i = n-1$, implying it is an element of $G$ in (3.3). Explicitly:

$$A_i = \{y^{i-1}, y^{i-1}z, \dots, y^{i-1}z^{n-i-1}, y^{i-1}x, y^{i-1}xz, \dots, y^{i-1}xz^{n-i-1}, \dots, y^{i-1}x^{n-i-1}, y^{i-1}x^{n-i-1}z, \\ \dots, y^{i-1}x^{n-i-1}z^{n-i-1}\}.$$

Summing up over $1 \le i \le n-1$, one gets: $dim_k(S/J') = \sum_{i=1}^{n-1} |A_i| = \sum_{i=1}^{n-1}(n-i)^2 = (n-1)^2 + (n-2)^2 + \dots + 1$. Therefore, the multiplicity of $J'$ is $\sum_{i=1}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6}$.

Consequently one has;

$$e(R/J') = \frac{1}{6}(n-1)n(2n-1) < \frac{1}{3}(n-1)n(n+1) = 2e(R/P),$$

as required. □

**Proposition 3.3.3.** *$P$ is the $P$-primary part of $J$.*

*Proof.* Let $\mathcal{P} \subset R$ denote the $P$-primary part of $J$. Since $\mathcal{P}$ is the contraction of $\mathcal{P}_P$ to $R$, it suffices to show that $J_P = P_P$, i.e., that $\ell(J_P) = \ell(P_P)$.

By the associativity formula for multiplicities, one has $e(R/J) \geq e(R/P)\ell(R_P/J_P)$, where $\ell$ denotes length. Since $J' = (\text{in}(f_0), \ldots, \text{in}(f_{2n-2})) \subset \text{in}(J)$ also has codimension 3, using Lemma 3.3.2 yields:

$$e(R/P)\ell(R_P/J_P) \leq e(R/J) = e(R/\text{in}(J)) \leq e(R/J') < 2e(R/P).$$

Clearly this is only the case if $\ell(R_P/J_P) = 1$; since in the short exact sequence

$$0 \to P_P/J_P \to R_P/J_P \to R_P/P_P \to 0$$

the rightmost module is nonzero, we must have $\ell(P_P/J_P) = 0$, as stated. □

### 3.3.2 The linear rank of the Hankel gradient Ideal

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The goal of this part is to establish the linear rank of the Hankel gradient ideal in arbitrary dimension. For convenience we use the notation $H$ instead of $\mathcal{H}$ for the generic Hankel matrix, and also use $n$ instead of $m$.

Let

$$H = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ldots & \vdots \\ x_{n-1} & x_n & \cdots & x_{2n-2} \end{pmatrix}$$

By Proposition 5.3.1 we have the following relation:

$$f_i := \frac{\partial(\det(H))}{\partial x_i} = \sum_{k+l=i+2} \Delta_l^k, \quad 0 \leq i \leq 2n-2, \tag{3.4}$$

where $\Delta_l^k$ is the determinant of the minor of $H$ obtained by deleting the $l$th row and $k$th column. By the symmetry of matrix $H$ we have $\Delta_l^k = \Delta_k^l$ – this will often be used to simplify the work.

We now consider the Hankel matrix of size $(n-1) \times (n+1)$:

$$\text{GP} = \begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_{n+1} \\ \vdots & \vdots & \ldots & \vdots \\ x_{n-2} & x_{n-1} & \cdots & x_{2n-2} \end{pmatrix}$$

A maximal minor of GP is denoted, as usual, by $(n-1)$-tuple of the indices of its columns. We next express the partial derivatives in terms of the maximal minors of GP. For this one resorts to both [11, Lemme 2.3] and [4, Corollary 2.2(a)], where the latter contains the details of a generalization of the Gruson–Peskine trick.

**Lemma 3.3.4.** *The following holds, for* $0 \leq j \leq 2n-2$*:*

$$f_j = \begin{cases} \sum_{i=0}^{j/2}(j+1-2i)[1, \cdots, \widehat{(i+1)}, \cdots, \widehat{(j+2-i)}, \cdots, n+1] & j < n, \\ \sum_{i=1}^{n-j/2}(2n+1-j-2i)[1, \cdots, \widehat{(i+1+j-n)}, \cdots, \widehat{(n+2-i)}, \cdots, n+1] & j \geq n. \end{cases} \tag{3.5}$$

*Proof.* In the proof of [4, Corollary 2.2(a)] one has to replace the parameters $h, t$ and $k$ with $r, n-1$ and $n-r$, respectively, and observe that $(s-(i+1)+n+1-(r+s-i)) = n-r$ and $(s-(i+s+r-1-n)+n+1-(n+2-i)) = n-r$. Then, in that argument, in each case one is to choose all possible $e(I) = e_1, ..., e_{n-1}$ such that $e_i = 1$ if $i \in I$ and $e_i = 0$ if $i \notin I$ and $I \subseteq \{1, ..., n-1\}, \mid I \mid = n-r$.

Then one has

$$\Delta_r^s = \sum_{i=0}^{r-1} [1, \cdots, \widehat{(i+1)}, \cdots, s, \cdots, \widehat{(r+s-i)}, \cdots, n+1] \tag{3.6}$$

if $s-2 < n-r$, and

$$\Delta_r^s = \sum_{i=1}^{n-s+1} [1, \cdots, \widehat{(i+s+r-1-n)}, \cdots, s, \cdots, \widehat{(n+2-i)}, \cdots, n+1] \tag{3.7}$$

if $s-2 \geq n-r$. The result now follows easily from these formulas. □

As a consequence any linear syzygy $L$ of the partial derivatives induces a linear syzygy of the maximal minor of GP, whose coordinates can be read off Lemma 3.3.4) as $k$-linear combinations of the coordinates of $L$. The idea is to revert somehow this situation, by establishing a system of linear equations which is to be conveniently solved. Thus, one needs the structure of the linear syzygies of the maximal minors of GP. As one knows, the ideal of maximal minors of GP is linearly presented and, in fact, has a linear resolution given by the classical Eagon–Northcott complex ([7]). This complex is better visualized as a symmetrization of the Koszul complex:

$$0 \to \bigwedge^{n+1} R^{n+1} \otimes_R Sym^2(R^{n-1})^* \to \bigwedge^{n} R^{n+1} \otimes_R Sym^1(R^{n-1})^* \stackrel{\text{EN}}{\to} \bigwedge^{n-1} R^{n+1} \otimes_R R \to I_{n-1}(\text{GP}) \to 0 \tag{3.8}$$

For our convenience, we will introduce the following convention: for $0 \leq i \leq n$, write $b(x_i) := (x_i \ x_{i+1} \ \ldots \ x_{n-2+i})$. Thinking of $b(x_i)$ as a symbol, the presentation of $I_{n-1}(\text{GP})$ can be thought of as the $n$th differential of the Koszul complex $K_\bullet(b(x_0), \cdots, b(x_n); R)$.

To explain this, one refers to (3.8). Let $\partial_j^n$ be the $n$th differential in the Koszul complex $K_\bullet(x_{j-1}, \cdots, x_{j-1+n})$, then $\text{EN} = (\partial_1^n | \cdots | \partial_{n-1}^n)$ as a multi-function. EN is represented by a matrix with $(n-1)(n+1)$ columns and $\binom{n+1}{n-1}$ rows. The shape of all matrices $\partial_i^n$s is the same except possibly by a shift in the indices. That is, By adding 1 to the indices of the first column of $\partial_1^n$ we obtain the first column of $\partial_2^n$ and so on. Therefore if we rearrange the presentation of EN such that for all $i$ all of the $i$th columns of $\partial_1^n$,...,$\partial_{n-1}^n$ are put together, then $\text{EN} = n$th differential of the Kozul complex $K_\bullet(b(x_0), \cdots, b(x_n); R)$, as claimed.

Next, a partial derivative $f_i$ individualizes a certain subset of maximal minors of GP according to (3.5). Denote by $L_i$ the corresponding subset of rows in the presentation matrix, which can now be written in the following way:

$$
\left[\begin{array}{cccccccccc}
\hline
-b(x_{n-1}) & -b(x_n) & & & & & & & & \\
\hline
b(x_{n-2}) & b(0) & -b(x_n) & & & & & & & \\
\hline
-b(x_{n-3}) & b(0) & b(0) & -b(x_n) & & & & & & \\
b(0) & b(x_{n-2}) & b(x_{n-1}) & & & & & & & \\
\hline
 & & & & & \vdots & & & & \\
\hline
-b(x_1) & & & & & & & & -b(x_n) & b(0) \\
 & b(x_2) & & & & & & b(x_{n-1}) & b(0) & b(0) \\
 & & -b(x_3) & & & & -b(x_{n-2}) & b(0) & b(0) & b(0) \\
 & & & \ddots & & \iddots & & & & \\
 & & & & b(x_{n/2}) & b(x_{3n/2+1}) & & & & \\
\hline
b(x_0) & & & & & & & & & -b(x_n) \\
b(0) & -b(x_1) & & & & & & & b(x_{n-1}) & b(0) \\
b(0) & b(0) & b(x_2) & & & & & -b(x_{n-2}) & b(0) & b(0) \\
 & & & \ddots & & & \iddots & & & \\
 & & & & b(x_{n/2-1}) & b(0) & b(x_{n/2+1}) & & & \\
\hline
b(0) & b(x_0) & & & & & & & & b(x_{n-1}) \\
b(0) & b(0) & -b(x_1) & & & & & & -b(x_{n-2}) & b(0) \\
b(0) & b(0) & b(0) & b(x_2) & & & & b(x_{n-3}) & b(0) & b(0) \\
 & & & & \ddots & & \iddots & & & \\
 & & & & & b(x_{n/2-1}) & b(x_{n/2}) & & & \\
\hline
 & & & & & \vdots & & & & \\
\hline
 & & & & & & b(x_0) & b(0) & b(0) & b(x_3) \\
 & & & & & & & -b(x_1) & -b(x_2) & b(0) \\
\hline
 & & & & & & & b(x_0) & b(0) & -b(x_2) \\
\hline
 & & & & & & & & b(x_0) & b(x_1) \\
\hline
\end{array}\right]
$$

where the blocks correspond to the descending sequence

$$L_{2n-2}, L_{2n-3}, L_{2n-4}, \ldots, L_n, L_{n-1}, L_{n-2}, \ldots, L_2, L_1, L_0.$$

The blank spaces are blocks of zeros – in some places we write $b(0)$ to emphasize such a block.

For example, in the case $n = 4$, $L_0 \leftrightarrow \{[3,4,5]\}$, $L_1 \leftrightarrow \{[2,4,5]\}$, $L_2 \leftrightarrow \{[2,3,5],[1,4,5]\}$, $L_3 \leftrightarrow \{[2,3,4],[1,3,5]\}$, $L_4 \leftrightarrow \{[1,3,4],[1,2,5]\}$, $L_5 \leftrightarrow \{[1,2,4]\}$, $L_6 \leftrightarrow \{[1,2,3]\}$.

A useful fact is that the number of rows of the block corresponding to $L_i$ is given by

$$\sharp L_i = \sharp L_{2n-2-i} = \lfloor (i+2)/2 \rfloor \ \text{ and } \ \sharp L_{n-1} = \lfloor (n+1)/2 \rfloor,$$

for $i = 0, \cdots, n-2$.

After these preliminaries, we are ready to state the main result of this part.

**Theorem 3.3.5.** *For any $n$, the linear rank of $J$ is 3.*

*Proof.* Let $(a_0\ a_1\ \ldots\ a_{2n-2})^t$ denote a linear syzygy of $J = (f_0, f_1, ..., f_{2n-2})$. Lemma 3.3.4 allows to replace each $f_i$ with its expression in terms of maximal minors of GP. Introduce a notation for the field coefficients in those expressions:

$$l_{ij} := \begin{cases} 2n+1-j-2i & j \geq n, 1 \leq i \leq n - j/2, \\ j+1-2i & 0 \leq j < n, 0 \leq i \leq j/2 \end{cases} \tag{3.9}$$

Then the column matrix $(l_{ij}a_j)_{\substack{0\leq j<n,\,0\leq i\leq j/2\\ j\geq n,\,1\leq i\leq n-j/2}}$ is a syzygy of the maximal minors of GP.

Note that EN has $(n+1)(n-1)$ columns consisting of $(n+1)$ blocks, each of these containing of $(n-1)$ columns. Let $C_k^t$ denote the $((t-1)(n-1)+k)$-th column of EN, for $1 \leq t \leq n+1$ and $1 \leq k \leq n-1$. Write

$$(l_{ij}a_j) = \sum_{\forall(t,k)} c_k^t C_k^t, \tag{3.10}$$

for certain base field elements $c_k^t$. Similarly, let the $k$th row of the block corresponding to $L_t$ be denoted by $R_k^t$. Then, for $0 \leq m \leq 2n-2$), we have a system of linear equations derived from (3.10):

$$\mathcal{S}_m := \begin{cases} l_{1m}a_m &= \sum_{(t,k)} c_k^t(R_1^m \cap C_k^t), \\ \vdots & \\ l_{i'm}a_m &= \sum_{(t,k)} c_k^t(R_{i'}^m \cap C_k^t) \end{cases} \tag{3.11}$$

where $i' = \lfloor n-(m/2) \rfloor$ if $m \geq n$ and $i' = \lfloor (m+2)/2 \rfloor$ otherwise.

Since $R_1^m \cap C_k^t$ is either zero or some variable, all coefficients above are base field elements.

Next write

$$a_m = a_{m,0}x_0 + \cdots + a_{m,2n-2}x_{2n-2},\ 0 \leq m \leq 2n-2,$$

where the coefficients are base field elements.

Substituting in (3.11) and taking in account that on any equation of $\mathcal{S}_m$ the coefficient of $x_i$ on the left must be the same as the coefficient of $x_i$ on the right we are led to the following systems of equations, one for each $i = 0, \cdots, 2n-2$:

$$\begin{cases} l_{1,m}a_{m,i}x_i &= \sum c_k^t x_i, \quad (t,k) \text{ is such that } R_1^m \cap C_k^t = x_i \\ \vdots & \\ l_{i',m}a_{m,i}x_i &= \sum_{(t,k)} c_k^t x_i \quad (t,k) \text{ is such that } R_{i'}^m \cap C_k^t = x_i \end{cases}$$

The driving idea is that if we find the coefficients $c_k^t$ we will know the $a_m$'s as well. We next establish several relations among these coefficients that will shorten the work.

**Claim 1.** $c_j^i = 0$ for all $i, j$ with $i + j \leq n-1$ and, by an argument of symmetry, $c_j^i = 0$ for all $i, j$ with $i + j \geq n+3$.

First notice that in $\mathcal{S}_{n-i}$ we have $l_{2,n-i}a_{n-i,0}x_0 = 0$ since $\forall(t,k)\ \ x_0 \notin R_2^{n-i} \cap C_k^t$. Therefore $a_{n-i,0} = 0$ and $l_{1,n-i}a_{n-i,0}x_0 = 0 = c_1^i x_0$. Hence $c_1^i = 0$ for $1 \leq i \leq n-2$.

Now we show that if $c_k^t = 0$ and $t + k \leq n-1$ then $c_{k+1}^{t-1} = 0$. By the system $\mathcal{S}_{n-t+1}$ we have $l_{2,n-t+1}a_{n-t+1,k}x_k = c_k^t x_k = 0$,therefore $a_{n-t+1} = 0$. Hence $l_{1,n-t+1}a_{n-t+1,k}x_k = c_{k+1}^{t-1}x_k = 0$

and, consequently, $c_{k+1}^{t-1} = 0$. We thus obtain

$$\begin{cases} c_1^{n-2} = c_2^{n-3} = \cdots = c_{n-2}^1 = 0 \\ c_1^{n-3} = c_2^{n-4} = \cdots = c_{n-3}^1 = 0 \\ \vdots \\ c_1^1 = 0 \end{cases}$$

**Claim 2.** We show that either $c_j^i = 0$ for all $i+j=n$ or else $c_j^i \neq 0$ for $i+j=n$ and are uniquely determined.

By the system $\mathcal{S}_{n-1}$ we have $l_{1,n-1}a_{n-1,n-2}x_{n-2} = c_{n-1}^1 x_{n-2}$ and $l_{2,n-1}a_{n-1,n-2}x_{n-2} = c_{n-2}^2 x_{n-2}$, therefore we have $c_{n-1}^1 = \frac{l_{1,n-1}}{l_{2,n-1}} c_{n-2}^2$. Hence by the same method in the systems $\mathcal{S}_{n-2}$, $\mathcal{S}_{n-3}, \ldots, \mathcal{S}_2$ we have the following:

$$\begin{cases} c_{n-1}^1 = \frac{l_{1,n-1}}{l_{2,n-1}} c_{n-2}^2 \\ c_{n-2}^2 = \frac{l_{1,n-2}}{l_{2,n-2}} c_{n-3}^3 \\ \vdots \\ c_2^{n-2} = \frac{l_{1,2}}{l_{2,2}} c_1^{n-1} \end{cases}$$

As one can see $c_j^i$ for $i+j=n$ are either zero or are uniquely determined. By symmetry, the same result holds for $i+j=n+2$, i.e either all $c_j^i$ for $i+j=n+2$.

**Claim 3.** One can see that either all $c_j^i$ for $i+j=n+1$ must be zeros or all $c_j^i$ for $i+j=n+1$ are non-zero and uniquely determined.

By the system of equation $\mathcal{S}_{n-1}$ we have $l_{1,n-1}a_{n-1,n-1}x_{n-1} = 0$, since $x-n-1 \neq R_{n-1}^1 \cap C_k^t$ for any $(t,k)$. Consequently, $a_{n-1,n-1} = 0$. By the system of equations in $\mathcal{S}_{n-1}$ we have $l_{k,n-1}a_{n-1,n-1}x_{n-1} = (-c_{n-k+1}^k + c_{k-1}^{n-k+2})x_{n-1} = 0$ for $1 < k \leq \lfloor (n+1)/2 \rfloor$ . Therefore we have:

$$\begin{cases} c_{n-1}^2 = c_1^n \\ c_{n-2}^3 = c_2^{n-1} \\ \vdots \\ c_{\lfloor (n+1)/2 \rfloor +1}^{\lfloor (n+1)/2 \rfloor} = c_{\lfloor (n+1)/2 \rfloor -1}^{\lfloor (n+1)/2 \rfloor +2} \end{cases} \tag{3.12}$$

By the system of equations in $\mathcal{S}_{n-2}$ one has:

$$\begin{cases} c_{n-1}^2 = \frac{l_{1,n-2}}{l_{2,n-2}}(c_{n-2}^3 + c_1^n) \\ c_{n-1}^2 = \frac{l_{1,n-2}}{l_{3,n-2}}(c_{n-3}^4 + c_2^{n-1}) \\ \vdots \\ c_{n-1}^2 = \frac{l_{1,n-2}}{l_{t-1,n-2}}(c_k^t + c_{t-2}^{k+2}) \\ c_{n-1}^2 = \frac{l_{1,n-2}}{l_{t,n-2}}(c_k^{t+1} + c_{t-1}^{k+1}) \\ \vdots \\ c_{n-1}^2 = \frac{l_{1,n-2}}{l_{\lfloor (n+1)/2 \rfloor, n-2}}(c_{\lfloor (n+1)/2 \rfloor}^{\lfloor (n+1)/2 \rfloor +1} + c_{\lfloor (n+1)/2 \rfloor -1}^{\lfloor (n+1)/2 \rfloor +2}) \end{cases} \tag{3.13}$$

Hence by (3.12) and (3.13) one can see if $c_{n-1}^2 \neq 0$ then all $c_j^i$ for $i+j=n+1$ are non-zero and are uniquely determined.

Now we show that if $c_k^t = 0$ for one $(t,k)$ such that $t+k = n+1$ then all $c_j^i$ for $i+j = n+1$ are zero.

Suppose that $t \leq k$ and $c_k^t = 0$. By the system of equations in $\mathcal{S}_{n-t}$ we have:

$$\begin{cases} l_{1,n-t}a_{n-t,n-t}x_{n-t} = c_k^t x_{n-t} = 0 \\ l_{2,n-t}a_{n-t,n-t}x_{n-t} = (c_{k-1}^{t+1} + c_{t-1}^{k+1})x_{n-t} \end{cases} \tag{3.14}$$

From these equations we have $a_{n-t,n-t} = 0$, therefore $c_{k-1}^{t+1} + c_{t-1}^{k+1}$ must be zero. Since by (3.12) $c_k^t = c_{t-1}^{k+1} = 0$ hence $c_{k-1}^{t+1} = 0$. By (3.13) $c_{n-1}^2 = 0$, thus all $c_j^i$ for $i+j = n+1$ are zero.

So much for the structure of the coefficients $c_k^t$. We now apply these findings to derive the linear syzygies of $J$.

We start from claim 2. Suppose that $c_{n-1}^1 \neq 0$. Without loss of generality we may assume that $c_{n-1}^1 = 1$, since this does not affect the degree of a syzygy. As seen above, every coefficient $c_j^i$ with $i+j = n$ is uniquely determined in terms of $c_{n-1}^1$. From $\mathcal{S}_m$, for $1 \leq m \leq 2n-2$ we have:

$$a_m = \frac{1}{l_{1,m}} \sum_{i+j=n} c_j^i (R_1^m \cap C_j^i) = \frac{1}{l_{1,m}} \sum_{i+j=n} c_j^i x_{m-1} = \alpha_m x_{m-1}$$

where

$$\alpha_m = \frac{1}{l_{1,m}} \sum_{i+j=n} c_j^i \text{ is an element of the base field,}$$

and $a_0 = 0$. This affords the linear syzygy

$$(0 \quad \alpha_1 x_0 \quad \alpha_2 x_1 \quad \cdots \quad \alpha_{2n-2} x_{2n-3}).$$

Still by claim 2, a similar argument holds in case $i+j = n+2$. This entails the following linear syzygy:

$$(\beta_0 x_1 \quad \beta_1 x_2 \quad \cdots \quad \beta_{2n-3} x_{2n-2} \quad 0),$$

for uniquely determined $\beta_i \in k$.

These two syzygies are clearly independent, i.e., they give a linear submatrix of rank 2 of the entire linear syzygies.

Now consider claim 3. Suppose that $c_n^1 \neq 0$. Again without loss of generality one can assume that $c_n^1 = 1$. All $c_j^i$ where $i+j = n+1$ are uniquely determined in terms of $c_n^1$ by (3.12) and (3.13). In the system of equation $\mathcal{S}_m$, for $0 \leq m \leq 2n-2$ we have:

$$a_m = \frac{1}{l_{1,m}} \sum_{i+j=n+1} c_j^i (R_1^m \cap C_j^i) = \frac{1}{l_{1,m}} \sum_{i+j=n+1} c_j^i x_m = \gamma_m x_m$$

where

$$\gamma_m = \frac{1}{l_{1,m}} \sum_{i+j=n+1} c_j^i \text{ is an element of the base field .}$$

Therefore we conclude that the only additional linear syzygy is the following:

$$(\gamma_0 x_0 \quad \gamma_1 x_1 \quad \cdots \quad \gamma_{2n-2} x_{2n-2}).$$

This implies that the linear rank of $J$ is at most 3. Actually, an additional little effort establishes that the three syzygies give rank 3. □

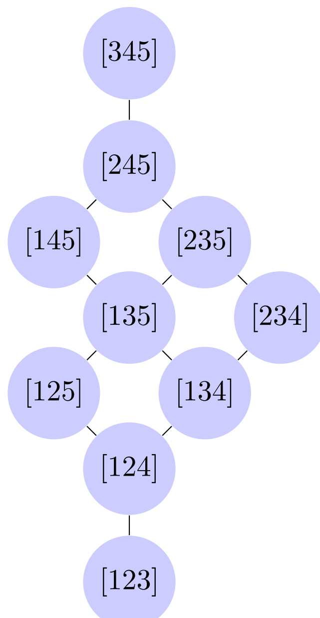
Sufficient computational evidence motivates the following

**Conjecture 3.3.6.** *The gradient ideal of the generic Hankel determinant is of linear type.*

If the conjecture is affirmative, it would follow from the previous theorem and the criterion of Proposition 2.2.3 that $det(H)$ is not homaloidal.

### 3.3.3 The radical of the Hankel gradient ideal

In the previous subsection we have seen that a partial derivative $f_i$ corresponds to a set of maximal minors of the $(n-1)\times(n+1)$ Hankel matrix GP. The set of maximal minors of GP is endowed with the usual partial order, by which $[i_1,...,i_{n-1}] \le [j_1,...,j_{n-1}] \Leftrightarrow i_1 \le j_1,...,i_{n-1} \le j_{n-1}$. For example, for $n = 4$ we have:




In this diagram minors on the same horizontal line are not comparable. For more details see [2, chapter 4]. The maximal minors corresponding to $f_j$ – i.e., those that are summands with nonzero coefficient in (3.5) – are incomparable with respect to the partial order defined above and coincide with the incomparable minors along one of the horizontal lines of the above diagram.

The main result of this part will use the theory of Plücker relations (see the Appendix for a brief resumé) in order to produce suitable equations of integral dependence.

**Proposition 3.3.7.** *Let* $P = I_{n-1}(H)$ *then* $P = \sqrt{J}$.

*Proof.* By Proposition 3.3.3, $P$ is the $P-$primary component of $J$, hence $\sqrt{J} \subseteq P$. It is enough to show the generators of $P$ belong to $\sqrt{J}$.

Since $P = I_{n-1}(H) = I_{n-1}(\mathrm{GP})$, we can partition the elements of $P$ into subsets, each associated to a partial derivative $f_j$ for $j = 0, \cdots, 2n-2$, as explained above. By a descending recursion on $j = 2n-2, \cdots, n$ we show that all minors in this range belong to $\sqrt{J}$. A symmetric argument will complete the proof for $j = 0, \cdots, n-1$.

The first step holds since $f_{2n-2} = [1, 2, ..., n-1]$ and $f_{2n-3} = 2[1, 2, ..., n-2, n]$ are already in $J$. We explain the next inductive step to make clear the procedure. The next row of minors in the diagram are the one corresponding to $f_{2n-4}$. Consider the following Plücker relation associated to the product $[1,...,n-3,n-1,n][1,...,n-2,n+1]$:

$$\begin{array}{rcl}[1,...,n-3,n-1,n][1,...,n-2,n+1] & = & 1/2[1,...,n-3,n-1,n+1]f_{2n-3} \\ & - & f_{2n-2}[1,...,n-3,n,n+1].\end{array} \tag{3.15}$$

We derive that the following equation is satisfied by $[1,...,n-3,n-1,n]$:

$$X^2-1/3X(3[1,...,n-3,n-1,n]+[1,...,n-2,n+1])+1/3[1,...,n-3,n-1,n][1,...,n-2,n+1]=0. \tag{3.16}$$

By (3.5) $f_{2n-4}=3[1,...,n-3,n-1,n]+[1,...,n-2,n+1]$ and by (3.15) the constant coefficient of (3.16) is in $J$. Hence $[1,...,n-3,n-1,n]\in\sqrt{J}$.

Now suppose that $j<2n-4$ and that, for $j<t\leq 2n-2$, any minor associated to $f_t$ belongs to $\sqrt{J}$. Let $[1,...,i+\widehat{1+j}-n,...,n\widehat{+2-}i,...,n+1]$ be a minor associated to $f_j$, where $1\leq i\leq [n-j/2]$ and $j\geq n$. We are going to show that $[1,...,i+\widehat{1+j}-n,...,n\widehat{+2-}i,...,n+1]\in\sqrt{J}$. Clearly this minor satisfies the following equation:

$$X^2-\frac{1}{2n+1-j-2i}Xf_j+\frac{1}{2n+1-j-2i}g=0,$$

where

$$g=\sum_{i\neq l=1}^{[n-j/2]}[1,...,i+\widehat{1+j}-n,...,n\widehat{+2-}i,...,n+1][1,...,l+\widehat{1+j}-n,...,n\widehat{+2-}l,...,n+1].$$

For any $i,l$ such that $1\leq i<l\leq[n-j/2]$, consider the following Plücker relation:

$$\begin{array}{c}[1,...,i+\widehat{1+j}-n,...,n+2-l,...,n\widehat{+2-}i,...,n+1][1,...,i+1+j-n,...,l+\widehat{1+j}-n,...,n\widehat{+2-}l,...,n+1] \\ -[1,...,i+1+j-n,...,n\widehat{+2-}l,...,n\widehat{+2-}i,...,n+1][1,...,i+\widehat{1+j}-n,...,l+\widehat{1+j}-n,...,n+2-l,...,n+1]+ \\ [1,...,i+\widehat{1+j}-n,...,n\widehat{+2-}l,...,n+2-i,...,n+1][1,...,i+1+j-n,...,l+\widehat{1+j}-n,...,n+2-l,...,n\widehat{+2-}i,...,n+1] \\ =0.\end{array}$$

In the above equation the red minors are less than the $[1,...,i+\widehat{1+j}-n,...,n\widehat{+2-}i,...,n+1]$ and belong to $f_t$ for some $t>j$, hence by the hypotheses of induction they belong to $\sqrt{J}$. Therefore $g\in\sqrt{J}$. The result now follows by induction. □

**Corollary 3.3.8.** $P=I_{n-1}(H)$ *is the minimal component of the gradient ideal of* $det(H)$.

*Proof.* This follows from the previous proposition and from Proposition 3.3.3. □

**Corollary 3.3.9.** $e(R/J)=e(R/P)=1/6(n-1)n(n+1)$.

*Proof.* By 3.3.7 $P$ is the only minimal prime of $J$, therefore by associativity formula $e(R/J)=e(R/P)l(R_P/J_P)$. As $l(R_P/J_P)=1$ we are done. □

**Corollary 3.3.10.** $\mathrm{ht}(J:P)\geq 4$.

*Proof.* Consider the following short exact sequence

$$0\to\frac{P}{J}\to\frac{R}{J}\to\frac{R}{P}\to 0.$$

Since $e(R/J)=e(R/P)$ and they have the same dimension, $\dim(P/J)<2n-1-3$ which implies the result. □

### 3.3.4 The Hessian of the Hankel determinant

In this section we show that the polar map defined by the determinant of Hankel matrix is dominant. Since we are in characteristic zero, this assertion is equivalent to the following one:

**Proposition 3.3.11.** *The determinant of the Hankel matrix has non-vanishing Hessian.*

*Proof.* It suffices to show that, upon evaluating the variables at suitable base field elements, the Hessian determinant does not vanish. We will show that a suitable such specialization of the Hessian determinant is a pure power of $x_{n-1}$ with nonzero coefficient.

Recall that the partial derivatives of $f$ are sums of $(n-1)$-minors of $H$ (3.4). For the second partial derivatives $\partial^2 f/\partial x_i \partial x_j$ we have an analogous expression in terms of the $(n-2)$-minors, namely, $\partial^2 f/\partial x_i \partial x_j$ is the sum of minors obtained by omitting rows and columns containing $x_i$ and $x_j$. Noting that $deg(\partial^2 f/\partial x_i \partial x_j) = n-2$, one sees that a pure power of $x_{n-1}$ in an entry of the Hessian matrix has exponent $n-2$.

Now, in the Hankel matrix the entries along the main anti-diagonal are the variable $x_{n-1}$ repeated; for convenience of the next argument denote by $a_i\,(= x_{n-1})$ this entry on the $i$th row, $0 \leq i \leq n-1$. Clearly, saying that an $(n-2)$-minor effectively involves the term $x_{n-1}^{n-2}$ is tantamount to involving $n-2$ of the $a_i$s.

Consider the minor corresponding to a given choice

$$a_0, \cdots, \widehat{a_i}, \cdots, \widehat{a_j}, \cdots, a_{n-1}$$

where $i < j$, with $a_i$ and $a_j$ omitted:

$$\begin{pmatrix}
x_0 & x_1 & \cdots & \cdots & \cdots & & x_{n-1} \\
x_1 & x_2 & \cdots & \cdots & \cdots & \cdots & x_n \\
\vdots & \vdots & \ldots & \cdots & \cdots & \cdots & \vdots \\
x_i & \cdots & x_\alpha & \cdots & x_{n-1} & \cdots & x_{n-1+i} \\
\vdots & & \vdots & \cdots & \ldots & \cdots & \vdots \\
x_j & \cdots & x_{n-1} & \cdots & x_\beta & \cdots & x_{n-1+j} \\
\vdots & & & \cdots & \vdots & \ldots & \vdots \\
x_{n-1} & x_n & \cdots & \cdots & \cdots & & x_{2n-2}
\end{pmatrix} \tag{3.17}$$

$$= (-1)^{n+1}(n-1)x_{n-1}^{n-2} + \text{terms of smaller degree in } \; x_{n-1}$$

Note that (3.17) is a term of $f_{\alpha\beta} = \partial^2 f/\partial x_\alpha \partial x_\beta$, and of $f_{(n-1)(n-1)}$ as well. Now, if $f_{\alpha\beta}$ has a term which is a pure power of $x_{n-1}$ then upon choosing $\alpha$, the index $\beta$ will be uniquely determined, and vice versa. Therefore by (3.4) we have

$$\begin{aligned}
f_{\alpha\beta} &= \sum_\alpha (-1)^{n+1}(n-1)x_{n-1}^{n-2} + \text{terms of smaller degree in } \; x_{n-1} \\
&= (\alpha+1)(-1)^{n+1}(n-1)x_{n-1}^{n-2} + \text{terms of smaller degree in } \; x_{n-1} \\
&= c_{\alpha\beta}x_{n-1}^{n-2} + \text{terms of smaller degree in } \; x_{n-1},
\end{aligned}$$

with $c_{\alpha\beta} \neq 0$.

Ready inspection of (3.17) yields that the only values of $\alpha$ and $\beta$ such that $f_{\alpha\beta}$ has a term which is a pure power of $x_{n-1}$ for this choice of $a_i, a_j$ are such that:

$$j - i = n - 1 - \alpha = 1 - n + \beta, \qquad \text{for } \ j - i = 1, \cdots, n-1.$$

Equivalently, $\alpha + \beta = 2n - 2$. The corresponding second partial derivatives $f_{\alpha\beta}$ are exactly the entries along the main anti-diagonal of the Hessian matrix, and no other entries on the Hessian matrix involves pure powers of $x_{n-1}$:

$$\text{Hess} = \det \begin{pmatrix} & & \cdots & & f_{0,(2n-2)} \\ & & \cdots & f_{1,(2n-3)} & \\ & \cdots & f_{2,(2n-4)} & & \\ \vdots & \vdots & \ldots & & \vdots \\ f_{(2n-2),0} & \cdots & & & \end{pmatrix}_{2n-1 \times 2n-1}$$

Evaluating at $(0, 0, \cdots, 0, x_{n-1}, 0, \cdots, 0)$ yields:

$$\text{Hess}(0, 0, \cdots, 0, x_{n-1}, 0, \cdots, 0) = \left( \prod_{\alpha+\beta=2n-2} c_{\alpha\beta} \right) x_{n-1}^{(n-2)(2n-1)} \neq 0,$$

since every $c_{\alpha\beta} \neq 0$. □

# Chapter 4

# Degeneration of a Hankel Matrix

In this section we deal with the determinant of a so-called sub-Hankel matrix, considered in [3, Section 4]. This matrix is a certain degeneration of the generic Hankel matrix where some entries are replaced by zeros. For consistency and appropriate referencing, we will keep the same notation as in [3].

## 4.1 Sub-Hankel Matrices: the gradient ideal

Let $x_0, \ldots, x_n$ be variables over a field $k$ and set

$$M^{(n)} = M^{(n)}(x_0, \ldots, x_n) = \begin{pmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_n \\ x_2 & x_3 & x_4 & \ldots & x_n & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ x_{n-2} & x_{n-1} & x_n & \ldots & 0 & 0 \\ x_{n-1} & x_n & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Note that the matrix has two tags: the upper index $(n)$ indicates the size of the matrix, while the variables enclosed in parentheses are the total set of variables used in the matrix. This detailed notation was introduced in [3] as several of these matrices were considered with variable tags throughout. However, here we omit the list of variables if they are sufficiently clear from the context.

This matrix will be called a GENERIC SUB–HANKEL MATRIX; more precisely, $M^{(n)}$ is the GENERIC SUB–HANKEL MATRIX OF ORDER $n$ ON THE VARIABLES $x_0, \ldots, x_n$.

Throughout we fix a polynomial ring $R = k[\mathbf{x}] = k[x_0, \ldots, x_n]$ and will assume that $k$ is a field of characteristic zero. We will denote by $f^{(n)}(x_0, \ldots, x_n)$ the determinant of $M^{(n)}(x_0, \ldots, x_n)$ for any $r \geq 1$, and we set $f^{(0)} = 1$.

In [3, Section 4] the main objective was to prove that this determinant is homaloidal and explain the geometric contents of the corresponding polar map. Here we turn ourselves to the algebraic-homological behavior of the ideal $J \subset R$ generated by its partial derivatives.

A common feature between the two approaches is a systematic use of a recurrence using the subideal $J_i \subset J$ generated by the first $i+1$ partial derivatives further divided by the gcd of these derivatives. We need the following results drawn upon [3]. Throughout we set $f := f^{(n)}$.

**Lemma 4.1.1.** ([3, Lemma 4.2]) *If $n \geq 2$, then*

(i) *For $0 \leq i \leq n-1$, one has*

$$\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_i} \in k\left[x_{n-i}, \ldots, x_n\right] \tag{4.1}$$

*and the g.c.d. of these partial derivatives is $x_n^{n-i-1}$*

(ii) *For any $i$ in the range $1 \leq i \leq n-1$, the following holds:*

$$x_n \frac{\partial f}{\partial x_i} = -\sum_{k=0}^{i-1} \frac{2i-k}{i}\, x_{r-i+k} \frac{\partial f}{\partial x_k}. \tag{4.2}$$

*Moreover,*

$$x_n \frac{\partial f}{\partial x_n} = (n-1)x_0 \frac{\partial f}{\partial x_0} + (n-2)x_1 \frac{\partial f}{\partial x_1} + \cdots + x_{n-2} \frac{\partial f}{\partial x_{n-2}} \tag{4.3}$$

**Proposition 4.1.2.** ([3, Proposition 4.3 and its proof]) *For every $1 \leq i \leq n-1$, the ideal $J_i$ is perfect and linearly presented with recurrent Hilbert–Burch matrix of the form*

$$\varphi(J_i) = \left(\begin{array}{c|c} 2\,x_{n-i} & \\ \frac{2i-1}{i}\,x_{n-i+1} & \\ \vdots & \varphi(J_{i-1}) \\ \frac{i+1}{i}\,x_{n-1} & \\ \hline x_n & \mathbf{0} \end{array}\right)$$

We next prove a few additional results not obtained in [3].

**Lemma 4.1.3.** *Keeping the above notation, set further $P = (x_{n-1}, x_n) \subset R = k[\mathbf{x}]$. Then*

(i) *$P$ is the radical of $J$ and all the ideals $J_i$, $1 \leq i \leq n-1$ are $P$-primary;*

(ii) *The $R_P$-module $R_P/(J_i)_P$ has length $\binom{i+1}{2}$;*

(iii) *The ideal of $R_P/(J_{n-1})_P$ generated by the forms $x_n, x_{n-1}^{n-1}$ has length $\binom{n-1}{2}$.*

**Proof.** (i) This is clear from the form of these ideals: any prime ideal containing any of these has to contain $P$, which is clearly the unique minimal prime thereof. By Proposition 4.1.2, each $J_i$ is perfect, hence $R/J_i$ is Cohen–Macaulay, thus implying that that $P$ is the only associated prime of $J_i$.

(ii) By the primary case of the associativity formula for the multiplicities, one has

$$e(R/J_i) = \ell(R_P/(J_i)_P)\, e(R/P) = \ell(R_P/(J_i)_P),$$

since $P$ is generated by linear forms. On the other hand, from Proposition 4.1.2 one has the graded free resolution

$$0 \to R(-(i+1))^i \to R(-i)^{i+1} \to R \to R/J_i \to 0 \tag{4.4}$$

Applying the multiplicity formula for Cohen–Macaulay rings with pure resolution ([15]), one derives in this case $e(k[\mathbf{x}]/J_i) = \binom{i+1}{2}$, as required.

(iii) As pointed out previously, one has $(x_n, J_{n-1}) = (x_n, x_{n-1}^{n-1})$. On the other hand,

$$(x_n, J_{n-1})/J_{n-1} \simeq (x_n)/(x_n) \cap J_{n-1} \simeq (R/(J_{n-1} : x_n))\,(1) \tag{4.5}$$
$$= (R/J_{n-2})\,(1), \tag{4.6}$$

where the equality $(J_{n-1} : x_n) = J_{n-2}$ follows from Proposition 4.1.2. Therefore

$$\ell\left((x_n, x_{n-1}^{n-1})_P/(J_{n-1})_P\right) = \ell(R/J_{n-2})_P = \binom{n-1}{2},$$

by part (ii). □

## 4.2 The minimal free resolution

Minimal graded free resolutions are useful tools for studying graded modules as they determine the Hilbert series, the Castelnuovo-Mumford regularity and other invariants of the module. In this part we obtain the minimal graded free resolution of the gradient ideal $J$.

We start with the following auxiliary isomorphism:

**Lemma 4.2.1.** *With the previous notation, one has*

$$J/J_{n-1} \simeq \frac{R}{(x_n, x_{n-1}^{n-1})}\,(-(n-1)).$$

**Proof.** The following isomorphisms of $R$-graded modules are immediate:

$$\frac{J}{J_{n-1}} = \frac{\left(J_{n-1}, \frac{\partial f}{\partial x_n}\right)}{J_{n-1}} \simeq \frac{\left(\frac{\partial f}{\partial x_n}\right)}{J_{n-1} \bigcap \left(\frac{\partial f}{\partial x_n}\right)} \simeq \frac{R}{\left(J_{n-1} \colon \frac{\partial f}{\partial x_n}\right)}\,(n-1)\,. \tag{4.7}$$

We claim that $\left(J_{n-1} \colon \frac{\partial f}{\partial x_n}\right) = (x_n, J_{n-1})$. Once this is proved, we will have $\left(J_{n-1} \colon \frac{\partial f}{\partial x_n}\right) = (x_n, x_{n-1}^{n-1})$ because it follows easily from the structure of $f$ and its derivatives that $(x_n, J_{n-1}) = (x_n, x_{n-1}^{n-1})$.

Now, by (4.3), $(x_n, J_{n-1}) \subset \left(J_{n-1} \colon \frac{\partial f}{\partial x_n}\right)$ as trivially $J_{n-1} \subset \left(J_{n-1} \colon \frac{\partial f}{\partial x_n}\right)$.

For the reverse inclusion, we proceed as follows.

Let $r \in (J_{n-1} : \partial f/\partial x_n)$ and write $P := (x_{n-1}, x_n)$. Since $J_{n-1}$ is a $P$-primary ideal and $\partial f/\partial x_n \notin J_{n-1}$ then $r \in P = (x_n, x_{n-1})$. Next rewrite $r = r(x_0, ..., x_n)$ as

$$r = x_n h(x_0, ..., x_n) + r'(x_0, ..., x_{n-1}) \in P,$$

where $x_n$ divides no term on the second summand. Then $r'(x_0, ..., x_{n-1}) \in (x_{n-1})$, so let $l \in \mathbb{N}$ be such that $r'(x_0, ..., x_{n-1}) = x_{n-1}^l r''(x_0, ..., x_{n-1})$ and $x_{n-1}$ does not divide $r''(x_0, ..., x_{n-1})$. It follows that $r''(x_0, ..., x_{n-1}) \notin P$.

Recall that $(x_n, J_{n-1}) = (x_n, x_{n-1}^{n-1})$. Since $J_{n-1}$ is P–primary and $r'' x_{n-1}^l \partial f/\partial x_n \in J_{n-1}$ then $x_{n-1}^l \partial f/\partial x_n \in J_{n-1}$. Thus, for the required reverse inclusion it is enough to show that $l \geq n-1$. Write

$$x_{n-1}^l \frac{\partial f}{\partial x_n} = \sum_{i=0}^{n-1} r_i \frac{\partial f}{\partial x_i}, \tag{4.8}$$

where $r_i \in R$.

Writing $r_i = x_n g_i(x_0, ..., x_n) + g_i'(x_0, ..., x_{n-1})$ and drawing upon (4.2) yields

$$\begin{aligned} x_{n-1}^l \partial f/\partial x_n &= \sum_{i=0}^{n-1} r_i \partial f/\partial x_i = \sum_{i=0}^{n-1} (x_n g_i(x_0, ..., x_n) + g_i'(x_0, ..., x_{n-1}))\partial f/\partial x_i \\ &= \sum_{i=0}^{n-1} g_i(x_0, ..., x_n)(-\sum_{k=0}^{i-1} \frac{2i-k}{i} x_{n-i+k} \frac{\partial f}{\partial x_k}) + \sum_{i=0}^{n-1} g_i'(x_0, ..., x_{n-1})\partial f/\partial x_i. \end{aligned}$$

By repeating this process for $g_i$ and so forth, after finitely many steps we may suppose that the coefficients of $\partial f/\partial x_i$ in (4.8) do not involve $x_n$.

Multiplying both sides of (4.8) by $x_n$ and using (4.3) yields a syzygy of the ideal $J_{n-1}$:

$$((n-1)x_{n-1}^l x_0 - x_n r_0)\frac{\partial f}{\partial x_0} + \cdots + (x_{n-2}x_{n-1}^l - x_n r_{n-2})\frac{\partial}{\partial x_{n-2}} - x_n r_{n-1}\frac{\partial f}{\partial x_{n-1}} = 0. \tag{4.9}$$

Thinking of this syzygy as a column vector $K$, we can write

$$K = \alpha_1 C_1 + \cdots + \alpha_{n-1} C_{n-1} \tag{4.10}$$

for suitable $\alpha_i \in k[x_0, \cdots, x_n]$, where $C_i$ denotes the $i$th column of the syzygy matrix of $J_{n-1}$ as in Proposition 4.1.2:

$$\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & \dots & 2x_{n-1} \\ \frac{2n-3}{n-1}x_2 & \frac{2n-5}{n-2}x_3 & \frac{2n-7}{n-3}x_4 & \dots & x_n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{n+2}{n-1}x_{n-3} & \frac{n}{n-2}x_{n-2} & \frac{n-2}{n-3}x_{n-1} & \dots & 0 \\ \frac{n+1}{n-1}x_{n-2} & \frac{n-1}{n-2}x_{n-1} & x_n & \dots & 0 \\ \frac{n}{n-1}x_{n-1} & x_n & 0 & \dots & 0 \\ x_n & 0 & 0 & \dots & 0 \end{pmatrix}_{n\times(n-1)}$$

This affords the following relations by looking at the last two rows:

$$x_{n-2}x_{n-1}^l - x_n r_{n-2} = -r_{n-1}\frac{n}{n-1}x_{n-1} + \alpha_2 x_n \quad \text{and} \quad \alpha_1 = -r_{n-1}.$$

Since we already assumed $r_i \in k[x_0, ..., x_{n-1}]$ for all $i$, then $\alpha_2 = -r_{n-2}$. Therefore

$$-r_{n-1}\frac{n}{n-1}x_{n-1} = x_{n-2}x_{n-1}^l.$$

In particular $l \geq 1$ and hence $r_{n-1} = -\frac{n-1}{n}x_{n-2}x_{n-1}^{l-1}$.

Inspecting the $(n-2)$'th row yields:

$$2x_{n-1}^l x_{n-3} - x_n r_{n-3} = -\frac{n+1}{n-1} r_{n-1} x_{n-2} - \frac{n-1}{n-2} r_{n-2} x_{n-1} + \alpha_3 x_n.$$

Since $r_{n-1}, r_{n-2} \in k[x_0, ..., x_{n-1}]$ then $\alpha_3 = -r_{n-3}$. Substituting for $r_{n-1}$ obtains

$$\begin{aligned} 2x_{n-1}^l x_{n-3} &= -\frac{n+1}{n-1} r_{n-1} x_{n-2} - \frac{n-1}{n-2} r_{n-2} x_{n-1} \\ &= \frac{n+1}{n} x_{n-2}^2 x_{n-1}^{l-1} - \frac{n-1}{n-2} r_{n-2} x_{n-1} \end{aligned}$$

Then necessarily $l \geq 2$ and furthermore

$$r_{n-2} = \frac{n-2}{n-1}(-2x_{n-1}^{l-1} x_{n-3} + \frac{n+1}{n} x_{n-2}^2 x_{n-1}^{l-2}) = x_{n-1}^{l-2} s_{n-2}, \tag{4.11}$$

for some $s_{n-i} \in k[x_0, ..., x_{n-1}]$.

Since $x_n$ appears exactly once on each row of the syzygy matrix below the first one, the argument inducts yielding that for every $1 \leq i \leq n-2$ one has $\alpha_i = -r_{n-i}$ and $r_{n-i} = x_{n-1}^{l-i} s_{n-i}$ with $s_{n-i} \in k[x_0, ..., x_{n-1}]$ and $l \geq i$. In particular, $l \geq n-2$ and $r_2 = x_{n-1}^{l-n+2} s_2$.

Finally, from the first row of $K$, we have:

$$(n-1)x_0 x_{n-1}^l - x_n r_0 = -2x_1 r_{n-1} - 2x_2 r_{n-2} - ... - 2x_{n-2} r_2 - 2x_{n-1} r_1.$$

Hence $r_0 = 0$. Rearranging yields

$$-2x_1 r_{n-1} - 2x_2 r_{n-2} - ... - 2x_{n-1} r_1 - (n-1)x_0 x_{n-1}^l = 2x_{n-2} r_2 = 2x_{n-2} x_{n-1}^{l-n+2} s_2. \tag{4.12}$$

Since the left hand side is divisible by $x_{n-1}$ so is the right hand side. Thus $l - n + 2 > 0$. In other words, $l \geq n-1$, as desired. □

**Theorem 4.2.2.** *The minimal graded resolution of $R/J$ has the form*

$$0 \to R(-(2n-1)) \to R(-n)^n \oplus R(-2(n-1)) \xrightarrow{\varphi} R^{n+1}(-(n-1)) \to R.$$

*In particular, $R/J$ is almost Cohen–Macaulay, but not Cohen–Macaulay.*

**Proof.** By Lemma 4.2.1 we have a free minimal resolution

$$\mathcal{C}\colon \quad 0 \to R(-(2n-1)) \to R(-n) \oplus R(-2(n-1)) \to R(-(n-1)) \to J/J_{n-1} \to 0,$$

On the other hand, (4.4) gives a resolution

$$\mathcal{J}_{n-1}\colon \quad 0 \to R(-n)^{n-1} \to R(-(n-1))^n \to R \to R/J_{n-1} \to 0$$

Since the inclusion $J/J_{n-1} \subset R/J_{n-1}$ induces a map of complexes $\mathcal{C} \to \mathcal{J}_{n-1}$, the resulting mapping cone is a resolution of $R/J$ ([8, Exercise A3.30]):

$$0 \to R(-(2n-1)) \to R(-n)^n \oplus R(-2(n-1)) \xrightarrow{\varphi} R^{n+1}(-(n-1)) \to R \to R/J \to 0, \tag{4.13}$$

(where the right end tail $R \oplus J/J_{n-1} \to R/J_{n-1} \to 0$ has been replaced by $R \to R/J \to 0$). Moreover, since for every relevant index $i$, the shifts of $(\mathcal{J}_{n-1})_i$ are strictly smaller than those of $(\mathcal{C})_i$, it follows by [loc. cit.] that (4.13) is minimal. In particular, one reads from it that the linear part $\varphi_1$ has rank $n$, hence maximal. □

Let us dwell a little more on the details of the mapping cone in the previous proof. It is of the form

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & R^{n-1} & \xrightarrow{\varphi_{n-1}} & R^n & \xrightarrow{(\frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_{n-1}})} & R & \longrightarrow & \frac{R}{J_{n-1}} & \longrightarrow & 0 \\
 & & g_4\uparrow & & g_3\uparrow & & g_2\uparrow & & g_1\uparrow & & \\
0 & \longrightarrow & R & \xrightarrow{(x_{n-1}^{n-1},-x_n)^t} & R^2 & \xrightarrow{(x_n,x_{n-1}^{n-1})} & R & \longrightarrow & J/Jn-1 & \longrightarrow & 0
\end{array}$$

Note that $g_2$ is multiplication by $\partial f/\partial x_n$ and the induced map $g_3$ is given by the following $n \times 2$ matrix:

$$g_3 = \begin{pmatrix} (n-1)x_0 & r_0 \\ (n-2)x_1 & r_1 \\ \vdots & \\ x_{n-2} & r_{n-2} \\ 0 & r_{n-1} \end{pmatrix}.$$

We next find out the entries of $g_4$. By the commutativity of the mapping cone diagram, we have

$$\varphi_{n-1} \circ g_4 = g_3 \circ \begin{pmatrix} x_{n-1}^{n-1} \\ -x_n \end{pmatrix} = \begin{pmatrix} (n-1)x_0 & r_0 \\ (n-2)x_1 & r_1 \\ \vdots & \\ x_{n-2} & r_{n-2} \\ 0 & r_{n-1} \end{pmatrix} \begin{pmatrix} x_{n-1}^{n-1} \\ -x_n \end{pmatrix} = \begin{pmatrix} (n-1)x_{n-1}^{n-1}x_0 - x_n r_0 \\ (n-2)x_{n-1}^{n-1}x_1 - x_n r_1 \\ \vdots \\ x_{n-2}x_{n-1}^{n-1} - x_n r_{n-2} \\ -x_n r_{n-1} \end{pmatrix}$$

where the rightmost matrix is the syzygy $K$ in (4.9), viewed as a column vector, where $l = n-1$. By reasoning as in the argument that ensues (4.10), one gets that the $i$th entry of $g_4$ is $-r_{n-i}$, $1 \le i \le n-1$. As a result, the leftmost map in (4.13) is $\psi := (x_{n-1}^{n-1}, -x_n, g_4^t) = (x_{n-1}^{n-1}, -x_n, -r_{n-1}, \dots, -r_1)$.

**Corollary 4.2.3.** *With the above notation we have:*

(i) *The Hilbert function of the $R/J$ is*

$$H(t) = S(t)/(1-t)^{n+1}$$

*where*

$$S(t) = 1 - (n+1)t^{n-1} + nt^n + t^{2n-2} - t^{2n-1}.$$

(ii) *The multiplicity of $R/J$ is*

$$e_0 = (n-1)(n-2)/2.$$

(iii) $\operatorname{reg}(R/J) = 2n-4$.

These invariants are computed easily from the resolution of $R/J$ by the usual methods.

## 4.3 The associated primes

**Lemma 4.3.1.** $Q = (x_n, x_{n-1}, x_{n-2}) \in \mathrm{Ass}(R/J)$.

*Proof.* $\mathrm{ht}(Q) = 3$. by [8, corrolary20.14(a)] it is enough to show that $Q \supseteq I(\psi)$ where $I(\psi) = I_{rank(\psi)}(\psi) = I_1(\psi)$ which is equal to the ideal generated by entries of $\psi$, which are

$$x_{n-1}^{n-1}, x_n, -r_{n-1}, ..., -r_1.$$

Recall from the proof of Proposition 4.2.1 that $r_{n-i} = x_{n-1}^{n-i-1} s_{n-i}$ where $s_{n-i} \in k[x_0, ..., x_{n-1}]$ for all $1 \le i \le n-1$. Therefore $r_i \in (x_{n-1}) \subseteq Q$ for $2 \le i \le n-1$. It just remains to show that $r_1 \in Q$. By manipulating equation (4.12), where $l = n-1$, we have

$$x_{n-1}r_1 = -x_1 x_{n-1}^{l-1} s_{n-1} - x_2 x_{n-1}^{l-2} s_{n-2} - ... - x_{n-3} x_{n-1}^2 s_3 - \frac{n-1}{2} x_0 x_{n-1}^l - x_{n-2} x_{n-1} s_2,$$

then

$$r_1 = -x_1 x_{n-1}^{l-2} s_{n-1} - x_2 x_{n-1}^{l-3} s_{n-2} - ... - x_{n-3} x_{n-1} s_3 - \frac{n-1}{2} x_0 x_{n-1}^{l-1} - x_{n-2} s_2$$

As one can see, except the last term which is divided by $x_{n-2}$ the others terms have factor $x_{n-1}$. Thus $r_1 \in (x_{n-1}, x_{n-2})$. Consequently $Q \supseteq I(\psi)$, as desired. □

**Proposition 4.3.2.** $\mathrm{Ass}(R/J) = \{P, Q\}$.

*Proof.* Suppose that $q \in Ass(\frac{R}{J})$. Since $\mathrm{pd}(R/J)_q$ is finite, then the Auslander-Buchsbaum theorem says that

$$\mathrm{depth}\, R_q = \mathrm{pd}((\frac{R}{J})_q) + \mathrm{depth}\,(q_q, (\frac{R}{J})_q).$$

In this formula $\mathrm{depth}\,(q_q, (\frac{R}{J})_q) = 0$, for $q \in \mathrm{Ass}(\frac{R}{J})$, and $\mathrm{pd}((\frac{R}{J})_q) \le 3$ by the resolution of $R/J$. Therefore $\mathrm{depth}\, R_q \le 3$. Since $R$ is a Cohen Macaulay ring the latter is equal to $\mathrm{ht}(q)$.

Consequently if $q \in \mathrm{Ass}(\frac{R}{J})$ then $\mathrm{ht}(q) \le 3$. As $rad(J) = P = (x_n, x_{n-1})$ the only prime divisor of height 2 of $J$ is $P$; while, by [8, corrolary20.14(a)], an associated prime of height 3 of $R/J$ must contain $I(\psi) = (x_{n-1}^{n-1}, x_n, -r_{n-1}, ..., -r_1)$.

Recall from the proof of Proposition 4.2.1 that $r_{n-i} = x_{n-1}^{n-i-1} s_{n-i}$ where $s_{n-i} \in k[x_0, ..., x_{n-1}]$ for all $1 \le i \le n-1$. As an extra piece of information, we look at the inductive structure of $s_i$ in the proof of Proposition 4.2.1 and Equation (4.11). One can prove inductively that for $1 \le i \le n-2$, $s_{n-i} = x_{n-1} t_{n-i} + x_{n-i}^i$ where $t_{n-i} \in k[x_0, ..., x_{n-1}]$ .

Therefore a prime ideal $q$ of height 3 belongs to $\mathrm{Ass}(R/J)$ if and only if $q \supseteq (x_n, x_{n-1}, x_{n-2}s_2)$. The latter is equal to $(x_n, x_{n-1}, x_{n-2}(x_{n-1}t_2 + x_{n-2}^{n-2})) = (x_n, x_{n-1}, x_{n-2}^{n-1})$. It then follows that $q = (x_n, x_{n-1}, x_{n-2}) = Q$.

□

**Corollary 4.3.3.** *Set $P := \sqrt{J} = (x_{n-1}, x_n)$. Then the $P$-primary component of $J$ is $J_{n-2}$.*

**Proof.** Since $J_{n-2}$ is a $P$-primary ideal (Proposition 4.1.3 (i)), it is equivalent to show the equality $J_P = (J_{n-2})_P$. Since $J \subset J_{n-2}$ we will be done by showing the equality of lengths $l(\frac{R_P}{J_P}) = l(\frac{R_P}{(J_{n-2})_P})$.

Now, with the present data, by the associativity formula one has $l(\frac{R_P}{J_P}) = e(R/J)$ – the multiplicity of $R/J$. By 4.2.3 we have $e(R/J) = (n-1)(n-2)/2 = \binom{n-1}{2}$. But the latter coincides with $l(R_P/(J_{n-2})_P)$ by Lemma 4.1.3 (ii). □

## 4.4 The linear type property

**Theorem 4.4.1.** *$J$ is of linear type.*

**Proof.** By definition, we have to show that the natural surjective $R$-homomorphism $\mathcal{S}_R(J) \twoheadrightarrow \mathcal{R}_R(J)$ from the symmetric algebra of $J$ to its Rees algebra is injective. One knows that this is the case if and only if $\mathcal{S}_R(J)$ is a domain.

Set $\mathcal{S}_R(J) \simeq R[y_0, \cdots, y_n]/\mathcal{L}$, where $\mathcal{L}$ is the ideal generated by the 1-forms coming from the syzygies of $J$. We will argue as follows: since $x_n$ belongs to the radical of $J$, $J_{x_n}$ is the unit ideal in $R_{x_n}$. Now, suppose one shows that $x_n$ is a non zero-divisor modulo $\mathcal{L}$. Then $\mathcal{L}_{x_n}$ is the defining ideal of the symmetric algebra of $J_{x_n} = R_{x_n}$, hence it is the zero ideal in a polynomial ring over a domain. In particular, it is a prime ideal, hence so must be $\mathcal{L}$. Therefore $\mathcal{S}_R(J)$ is a domain, thus the structural homomorphism is injective as observed.

By a quirk, it will be easier to show first that $y_n$ is non-zero divisor modulo $\mathcal{L}$ and then that $(y_n, x_n)$ is a regular sequence modulo $\mathcal{L}$. In this case, since $\mathcal{S}_R(J)$ is a positively graded ring any permutation of a regular sequence is a regular sequence, hence $(x_n, y_n)$ is a regular sequence as well, in particular $x_n$ is a non-zero divisor over $\mathcal{S}_R(J)$.

**Step 1.** $y_n$ is non-zero divisor modulo $\mathcal{L}$.

Let $h = h(\mathbf{x}, \mathbf{y}) \in R[y_0, \ldots, y_n]$ be such that $y_n h(\mathbf{x}, \mathbf{y}) \in \mathcal{L}$. Say,

$$y_n h = \sum_{i=1}^{n+1} h_i g_i,$$

for suitable $h_i = h_i(\mathbf{x}, \mathbf{y}) \in R[y_0, \ldots, y_n]$, where

$$g_i = 2x_{n-i}y_0 + \frac{2i-1}{i} x_{n-i+1}y_1 + \cdots + x_n y_i \quad (1 \leq i \leq n-1)$$

and

$$g_n = (n-1)x_0y_0 + (n-2)x_1y_1 + \cdots + x_ny_n, \quad g_{n+1} = \sum_{j=1}^{n-1} r_j y_j + x_{n-1}^{n-1} y_n$$

generate $\mathcal{L}$ (from (4.13)), with $r_i$ being as in (4.9).

Decompose further $h_i = h_i' + y_n h_i''$, with $h_i' \in R[y_0, ..., y_{n-1}]$. Then

$$y_n h(\mathbf{x}, \mathbf{y}) = y_n \sum_{i=1}^{n-1} h_i'' g_i + y_n h_n'' g_n + h_n' x_n y_n + y_n h_{n+1}'' g_{n+1} + h_{n+1}' x_{n-1}^{n-1} y_n.$$

Since in the right hand side the terms not divisible by $y_n$ must vanish, we get

$$h = \left( \sum_{i=1}^{n-1} h_i'' g_i + h_n'' g_n + h_{n+1}'' g_{n+1} \right) + h_n' x_n + h_{n+1}' x_{n-1}^{n-1}.$$

To show that $h \in \mathcal{L}$ it is then enough to check that

$$h_n' x_n + h_{n+1}' x_{n-1}^{n-1} \in \mathcal{L}.$$

Now a form in $\mathcal{L}$ must vanish when evaluated at $y_i \mapsto \partial f/\partial x_i$ – the generators of $J$. Letting $\partial\mathbf{f}$ denote these partial derivatives we have $\frac{\partial f}{\partial x_n} h(\mathbf{x}, \partial\mathbf{f}) = 0$, hence $h(\mathbf{x}, \partial\mathbf{f}) = 0$. Retrieving in terms of the expression of $h$ implies that

$$h'_n(\mathbf{x}, \partial\mathbf{f})\, x_n + h'_{n+1}(\mathbf{x}, \partial\mathbf{f})\, x_{n-1}^{n-1} = 0.$$

Since $h'_n, h'_{n+1} \in R[y_0, ..., y_{n-1}]$, the form $h'_n x_n + h'_{n+1} x_{n-1}^{n-1}$ belongs to the defining ideal of $\mathcal{R}_R(J_{n-1})$. By [3, Proposition 4.3], $J_{n-1}$ is of linear type, hence $h'_n x_n + h'_{n+1} x_{n-1}^{n-1}$ belongs to the defining ideal of $\mathcal{S}_R(J_{n-1})$, which is a subideal of $\mathcal{L}$ by the general theory developed in [3]. This shows the contention.

**Step 2.** $x_n$ is non-zero divisor modulo $(\mathcal{L}, y_n)$.

One has $(\mathcal{L}, y_n) = (g_1, \ldots, g_{n-1}, g, h, y_n)$, where $g = (n-1)x_0y_0 + (n-2)x_1y_1 + ... + x_{n-2}y_{n-2}$ and $h = \sum_{j=1}^{n-1} r_j y_j$. Then

$$\frac{R[y_0, \ldots, y_n]}{(\mathcal{L}, y_n)} \simeq \frac{R[y_0, \ldots, y_{n-1}]}{(g_1, \ldots, g_{n-1}, g, h)}$$

hence we are to show that $x_n$ is a nonzerodivisor on the rightmost ring. Let then $\kappa = \kappa(\mathbf{x}, \mathbf{y} \setminus y_n)$ be a form in $R[y_0, \ldots, y_{n-1}]$ such that $x_n\kappa \in (g_1, \ldots, g_{n-1}, g, h)$. Write $x_n\kappa = \sum_{j=1}^{n-1} \mu_j g_j + \mu g + \nu h$, for suitable forms $\mu_j, \mu, \nu \in R[y_0, \ldots, y_{n-1}]$. Evaluating $y_i \mapsto f_{x_i} := \partial f/\partial x_i$ for $i = 0, \cdots, n-1$, and taking in account the shape of $g$ and $h$, we have

$$\begin{aligned} x_n\kappa(\mathbf{x}, f_{x_0}, \ldots, f_{x_{n-1}}) &= \mu(\mathbf{x}, f_{x_0}, \ldots, f_{x_{n-1}})g(\mathbf{x}, f_{x_0}, \ldots, f_{x_{n-1}}) \\ &+ \nu(\mathbf{x}, f_{x_0}, \ldots, f_{x_{n-1}})h(\mathbf{x}, f_{x_0}, \ldots, f_{x_{n-1}}) \\ &= -\mu(\mathbf{x}, f_{x_0}, \ldots, f_{x_{n-1}})x_n\, f_{x_n} - \nu(\mathbf{x}, f_{x_0}, \ldots, f_{x_{n-1}})x_{n-1}^{n-1}\, f_{x_n}. \end{aligned}$$

On the other hand, by the shape of $f_{x_n}$, one has $\gcd(x_n, f_{x_n}) = 1$. It follows that

$$\kappa(\mathbf{x}, f_{x_0}, \ldots, f_{x_{n-1}}) = \delta\, f_{x_n},$$

for some $\delta \in R$. Pulling back to the $\mathbf{y}$-variables tells us that $\kappa - \delta\, y_n$ vanishes on the partial derivatives, and hence it belongs to affine ideal $\tilde{\mathcal{J}}$ of all polynomials vanishing on the partial derivatives. This ideal is prime because we can consider $\partial\mathbf{f} := (f_{x_0}, \ldots, f_{x_n})$ as a point in $K^{n+1}$, where $K$ denotes the field of fractions of $R$ and consider the ideal of $K[\mathbf{y}]$ vanishing on $\partial\mathbf{f}$ and then contract to $R[\mathbf{y}]$. We note that the Rees ideal $\mathcal{J}$ is the largest homogeneous ideal contained in $\tilde{\mathcal{J}}$.

Now, multiplying $\kappa - \delta\, y_n$ by $x_n$ and using that $x_n\kappa \in \mathcal{L} \subset \mathcal{J} \subset \tilde{\mathcal{J}}$, it follows that $x_n\, \delta\, y_n \in \tilde{\mathcal{J}}$. Since this element is (trivially) homogeneous in $\mathbf{y}$, it must belong to the Rees ideal, hence $\delta = 0$. Therefore, $\kappa \in \tilde{\mathcal{J}}$. But, since $\kappa$ is assumed to be homogeneous, it belongs to the Rees ideal. Thus, we have $\kappa \in \mathcal{J} \cap R[y_0, \ldots, y_{n-1}]$. But this means that $\kappa$ belongs to the Rees ideal $\mathcal{J}'$ of $J_{n-1}$. Since the latter is of linear type by we conclude as above that $\kappa \in \mathcal{L}$, as was to be shown. □

# Chapter 5

# Appendix

## 5.1 The Eagon-Northcott Complex

**The complex associated with a matrix**

Consider a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sr} \end{pmatrix} \qquad (s \leq r),$$

where the elements $a_{ij}$ belong to $R$. The subdeterminants of $A$ of order $s$ will generate an ideal. This ideal will be denoted by $I_s(A)$.

Let $K$ be the exterior algebra generated by $X_1, X_2, \cdots, X_r$, then the $k$th row of $A$ determines a differentiation $\Delta_k$ on $K$; in fact

$$\Delta_k(X_{i_1} X_{i_2} \cdots X_{i_n}) = \sum_{p=1}^{n} (-1)^{p+1} a_{k i_p} X_{i_1} \cdots \widehat{X_{i_p}} \cdots X_{i_n}.$$

Furthermore, one easily verifies that

$$\Delta_k \Delta_h + \Delta_h \Delta_k = 0 \qquad (h \neq k). \tag{5.1}$$

Next let $Y_1, Y_2, \cdots, Y_s$ be $s$ new symbols and, in the polynomial ring $R[Y_1, Y_2, \cdots, Y_s]$, denote by $\Phi_n$ the $R$-module consisting of all forms of degree $n$. We are now ready to describe the complex $C^A$. It has component modules $C_0^A, C_1^A, \cdots, C_{r-s+1}^A$, where

$$\begin{cases} C_{q+1}^A = K_{s+q} \otimes_R \Phi_q \qquad (q = 0, 1, \cdots, r-s) \\ C_0^A = R \end{cases}$$

and therefore, as a complex, it has the form

$$\cdots \to 0 \to C_{r-s+1}^A \xrightarrow{d} C_{r-s}^A \to \cdots \to C_1^A \xrightarrow{d} C_0^A \to 0 \to \cdots.$$

In order to explain how the differentiation homomorphism $d$ works, we observe that if $q \geq 0$, then $C_{q+1}^A$ has an $R$-base consisting of the elements

$$X_{i_1}\cdots X_{i_{s+q}}\otimes Y_1^{\nu_1}\cdots Y_s^{\nu_s}\qquad (1\le i_1<\cdots<i_{s+q}\le r;\quad \nu_1+\cdots+\nu_s=q).$$

When $q>0$ the result of operating with $d$ on a member of such a basis is given by the formula

$$d(X_{i_1}\cdots X_{i_{s+q}}\otimes Y_1^{\nu_1}\cdots Y_s^{\nu_s})=\sum_{j^*}\Delta_j(X_{i_1}\cdots X_{i_{s+q}})\otimes Y_1^{\nu_1}\cdots Y_j^{\nu_j-1}\cdots Y_s^{\nu_s},$$

where the asterisk attached to the summation sign means that we sum only over those values of $j$ for which $\nu_j>0$. For $q=0$ the formula

$$d(X_{i_1}\cdots X_{i_s}\otimes 1)=\det\begin{pmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_s}\\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_s}\\ \vdots & \vdots & \cdots & \vdots\\ a_{si_1} & a_{si_2} & \cdots & a_{si_s}\end{pmatrix}$$

is to be employed. With the use of (5.1), it is a straightforward matter to verify that $d^2=0$.

$C^A$ is a free complex and

$$d(C_1^A)=I_s(A).$$

When $s=1$, that is when the matrix consists of a single row, $C_q^A$ $(q\ge 1)$ has the products

$$X_{i_1}\cdots X_{i_q}\otimes Y_1^{q-1}\qquad (i_1<i_2<\cdots<i_q)$$

as an $R$-base and the differentiation becomes

$$d(X_{i_1}\cdots X_{i_q}\otimes Y_1^{q-1})=\Delta_1(X_{i_1}\cdots X_{i_q})\otimes Y_1^{q-2}$$

for $q>1$, while $d(X_{i_1}\otimes 1)=a_{1i_1}$. This shows that, when the matrix has only a single row, $C^A$ is essentially the Koszul complex generated by the row.

After these preliminaries, let $M$ be an $R$-module and put

$$C^{M^A}=C^A\otimes M,$$

where $\otimes$ without any subscript denotes a tensor product over $R$. $C^{M^A}$ is a complex

$$\cdots\to 0\to C_{r-s+1}^{M^A}\xrightarrow{d}C_{r-s}^{M^A}\to\cdots\to C_1^{M^A}\xrightarrow{d}C_0^{M^A}\to 0\to\cdots.$$

**Corollary 5.1.1.** [7] *Let $R$ be a Noetherian ring and $M$ a finitely generated $R-$module for which $I_s(A)M\ne M$. If now $gr(I_s(A),M)=r-s+1$, then*

$$0\to C_{r-s+1}^{M^A}\xrightarrow{d}C_{r-s}^{M^A}\to\cdots\to C_1^{M^A}\xrightarrow{d}M\to M/I_s(A)M\to 0$$

*is an exact sequence.*

## 5.2 Plücker Relations

The following essentials are extracted from [16, Section 2].

A *linear space* $L$ in $\mathbb{P}^n$ is defined as the set of points $P = (p(0), \cdots, p(n))$ of $\mathbb{P}^n$ whose coordinates $p(j)$ satisfy a system of linear equations $\sum_{j=0}^{n} b_{\alpha j} p(j) = 0$ with $\alpha = 1, \cdots, (n-d)$. We say that $L$ has dimension $d$ if these $(n-d)$ equations are independent, that is if the $(n-d) \times (n+1)$ matrix of coefficients $(b_{\alpha j})$ has a nonzero $(n-d) \times (n-d)$-minor. By linear algebra, there are then $(d+1)$ points $P_i = (p_i(0), \cdots, p_i(n))$ in $L$ with $i = 0, \cdots, d$ which span $L$. Of course, we call $L$ a *line* if $d = 1$, a *plane* if $d = 2$ and a *hyperplane* if $d = (n-1)$. Following common usage, we call a $d$-dimensional linear space a $d$-*plane* for short.

For convenience, let us make the following convention. For any $(d+1) \times (n+1)$-matrix $(p_i(j))$ with $i = 0, \cdots, d$ and $j = 0, \cdots, n$, and any sequence of $d+1$ integers $j_0 \cdots j_d$ with $0 \leq j_\beta \leq n$, let us denote by $p(j_0 \cdots j_d)$ the determinant of the $(d+1) \times (d+1)$-matrix $(p_i(j_\beta))$ with $i, \beta = 0, \cdots, d$.

Fix a d-plane $L$ in $\mathbb{P}^n$. Pick $(d+1)$ points $P_i = (p_i(0), \cdots, p_i(n))$ with $i = 0, \cdots, d$ which span $L$, and form the $(d+1) \times (n+1)$-matrix $(p_i(j))$. At least one of the $(N+1)$ determinants $p(j_0 \cdots j_d)$ with $0 \leq j_0 < \cdots < j_d \leq n$ must be nonzero, where $N = \binom{n+1}{d+1} - 1$. So, when ordered lexicographically, these determinants define a point $(\cdots, p(j_0 \cdots j_d), \cdots)$ of $\mathbb{P}^N$. Therefore $L$ canonically gives rise to a point of $\mathbb{P}^N$. The coordinates $p(j_0 \cdots j_d)$ of this point are called the *Plücker coordinates* of $L$.

Suppose that $p(j_0 ... j_d)$ is a Plücker coordinate of a $d-$plane $L$ in $\mathrm{P}^n$. Therefore the following quadratic relations hold, see [16]:

$$\sum_{l=0}^{d+1} (-1)^l p(j_0 ... j_{d-1} k_l) p(k_0 ... \widehat{k_l} ... k_{d+1}) = 0, \tag{5.2}$$

where $j_0 ... j_{d-1}$ and $k_0 ... k_{d+1}$ are any sequences of integers with $0 \leq j_t, k_i \leq n$. Here $\widehat{k_l}$ means that the integer $k_l$ has been removed from the sequence.

In this work, we specifically need to compute Plücker equations for $(n-2)$-planes.

## 5.3 Matrices whose entries are variables

The following result has been obtained during the preparation of this thesis, without knowledge that it had a previous history in [10]. We give the statement and our independent proof.

**Proposition 5.3.1.** *Let $\mathcal{M}$ denote a square matrix over $R = k[x_0, \ldots, x_n]$ satisfying the following requirements:*

- *Every entry of $M$ is either 0 or $x_i$, for some $i = 0, \ldots, n$*
- *Any variable $x_i$ or 0 appears at most once on every row or column.*

*Let $f := \det(\mathcal{M}) \in R$. Then, for each $i = 0, \ldots, n$, the partial derivative $f_i$ of $f$ with respect to $x_i$ is the sum of the (signed) cofactors of the entry $x_i$ in all of its appearances on $\mathcal{M}$.*

*Proof.* By the Leibnitz formula for the determinant of an $l \times l$ matrix $M$, we have

$$\det(M) = \sum_{\sigma \in S_l} sgn(\sigma) \prod_{i=1}^{l} m_{i\sigma_i}.$$

Here the sum is computed over all permutations $\sigma$ of the set $\{1, 2, ..., l\}$ and $\sigma_i := \sigma(i)$. Note that $m_{i\sigma_i}$ is the entry at positions $(i, \sigma_i)$, where $i$ ranges from 1 to $l$. We divide $S_l$ into the following disjoint sets:

$S_l^0$ : all $l$-permutations such that $m_{j\sigma_j}$ is not $x_i$ for any $j = 1, \cdots, l$.
$S_l^1$ : all $l$-permutations such that $m_{j\sigma_j} = x_i$ for one and only one $j$.

$$\vdots$$

$S_l^l$ : the $l$-permutations which $m_{j\sigma_j} = x_i$ for all $j$ ranges from 1 to $l$.
Notice that some of the $\sigma^k$ may be empty.
By rewriting determinant of $M$ we have:

$$f = \det(M) = \sum_{\sigma^0 \in S_l^0} sgn(\sigma^0) \prod_{j=1}^{l} m_{j\sigma_j^0} + \sum_{\sigma^1 \in S_l^1} sgn(\sigma^1) \prod_{j=1}^{l} m_{j\sigma_j^1} + \sum_{\sigma^2 \in S_l^2} sgn(\sigma^2) \prod_{j=1}^{l} m_{j\sigma_j^2} \tag{5.3}$$

$$+ \cdots + \sum_{\sigma^l \in S_l^l} sgn(\sigma^l) \prod_{j=1}^{l} m_{j\sigma_j^l}. \tag{5.4}$$

We now introduce a new notation. Let $0 \leq k \leq l$ and $\sigma^k \in S_l^k$ such that $m_{j_t\sigma_{j_t}^k} = x_i$ for $t = 1, \cdots, k$. For any $t$ we set $\lambda^t(\sigma^k)$ to be the restriction of $\sigma^k$ to $\{1, \cdots, \hat{j}_t, \cdots, l\}$. Thus $\lambda^t(\sigma^k)$ is a bijection from $\{1, \cdots, \hat{j}_t, \cdots, l\}$ to $\{1, \cdots, \widehat{\sigma_{j_t}^k}, \cdots, l\}$. In other words, if we consider the submatrix obtained from removing the $j_t$-th row and $\sigma_{j_t}^k$-th column and renumbering the rows and columns, then $\lambda^t(\sigma^k)$ is a $(l-1)$-permutation which is applied to define the determinant of this submatrix.

For any positive integer $k$ set $\Lambda^k = \{\lambda^t(\sigma^k) : \sigma^k \in S_l^k \text{ and } t = 1, \cdots, k\}$.

Now we compute the partial derivative $\partial f / \partial x_i$ from both sides of (5.3) we have:

$$\partial f / \partial x_i = 0 + \sum_{\lambda^1 \in \Lambda^1} sgn(\lambda^1) \prod_{j=1}^{1-1} m_{j\lambda_j^1} + 2 \sum_{\lambda^2 \in \Lambda^2} sgn(\lambda^2) \prod_{j=1}^{1-1} m_{j\lambda_j^2} + 3 \sum_{\lambda^3 \in \Lambda^3} sgn(\lambda^3) \prod_{j=1}^{1-1} m_{j\lambda_j^3}$$

$$+ \cdots + l \sum_{\lambda^l \in \Lambda^l} sgn(\lambda^l) \prod_{j=1}^{l-1} m_{j\lambda_j^l}. \tag{5.5}$$

Now we want to compute summation of all cofactors with respect to the coordinates of $x_i$ in matrix $M$. By the definition of cofactor, we know that such a cofactor is the determinant of $(l-1) \times (l-1)$ submatrix of $M$ obtained by removing row and column which identified the position of $x_i$. Recall that we have at most one $x_i$ in each column. Consider $C_j^i$ to be the coordinate of the variable $x_i$ in the $j$th column. We know that maybe there is no $C_j^i$ because it can be happened that there is no $x_i$ in the column $j$. We denote $\Delta_{C_j^i}$ to be the cofactor corresponded to $C_j^i$.

Again, we introduce a new notation here for denoting $(l-1)$-permutations of the cofactor $\Delta_{C_j^i}$ where $j$ ranges between 1 and $l$.

Let ${}^j\Upsilon^0$ be the notation for all $(l-1)$-permutations corresponded to $\Delta_{C_j^i}$ such that the entry with respect to the position $(k, {}^j\Upsilon_k^0)$ is not $x_i$ for any $k = 1, \cdots, l-1$.

${}^j\Upsilon^1$ is the notation for all $(l-1)$-permutations corresponded to $\Delta_{C_j^i}$ such that one and only one of the entry with respect to the position $(k, {}^j\Upsilon_k^1)$ is $x_i$ for any $k = 1, \cdots, l-1$.

$\vdots$

${}^j\Upsilon^{l-1}$ is the notation for the $(l-1)$-permutations corresponded to $\Delta_{C_j^i}$ such that all entries with respect to the position $(k, {}^j\Upsilon_k^{l-1})$ is $x_i$ for any $k = 1, \cdots, l-1$.

By the definition of determinant we have :

$$\Delta_{C_j^i} = \sum_{\Upsilon} sgn(\Upsilon) \prod_k m_{k\,\Upsilon_k}.$$

To avoid the complexity of notation, we just denote the correspondent entry in the coordinate $(k, {}^j\Upsilon_k^i)$ by $m_{jik}$.

Now one can rewrite $\Delta_{C_1^i}, \Delta_{C_2^i}, \cdots, \Delta_{C_l^i}$ to the following distinct summation. We have:

$$\Delta_{C_1^i} = \sum_{{}^1\Upsilon^0} sgn({}^1\Upsilon^0) \prod_k m_{10k} + \sum_{{}^1\Upsilon^1} sgn({}^1\Upsilon^1) \prod m_{11k} + \cdots + \sum_{{}^1\Upsilon^{l-1}} sgn({}^1\Upsilon^{l-1}) \prod_k m_{1(l-1)k}.$$

$$\Delta_{C_2^i} = \sum_{{}^2\Upsilon^0} sgn({}^2\Upsilon^0) \prod_k m_{20k} + \sum_{{}^2\Upsilon^1} sgn({}^2\Upsilon^1) \prod m_{21k} + \cdots + \sum_{{}^2\Upsilon^{l-1}} sgn({}^2\Upsilon^{l-1}) \prod_k m_{2(l-1)k}.$$

$\vdots$

$$\Delta_{C_l^i} = \sum_{{}^l\Upsilon^0} sgn({}^l\Upsilon^0) \prod_k m_{l0k} + \sum_{{}^l\Upsilon^1} sgn({}^l\Upsilon^1) \prod m_{l1k} + \cdots + \sum_{{}^l\Upsilon^{l-1}} sgn({}^l\Upsilon^{l-1}) \prod_k m_{l(l-1)k}.$$

Now we want to consider the summation of all $\Delta_{C_1^i}, \Delta_{C_2^i}, \cdots, \Delta_{C_l^i}$. To this end, we sum up the above lines in vertical groups, that is:

$$\sum_j \Delta_{C_j^i} = \sum_{{}^1\Upsilon^0} sgn({}^1\Upsilon^0) \prod_k m_{10k} + \sum_{{}^2\Upsilon^0} sgn({}^2\Upsilon^0) \prod_k m_{20k} + \cdots + \sum_{{}^l\Upsilon^0} sgn({}^l\Upsilon^0) \prod_k m_{l0k}$$

$$+ \sum_{{}^1\Upsilon^1} sgn({}^1\Upsilon^1) \prod m_{11k} + \sum_{{}^2\Upsilon^1} sgn({}^2\Upsilon^1) \prod m_{21k} + \cdots + \sum_{{}^l\Upsilon^1} sgn({}^l\Upsilon^1) \prod m_{l1k}$$

$$+ \cdots$$

$$+\sum_{{}^1\Upsilon^{l-1}} sgn({}^1\Upsilon^{l-1})\prod_k m_{1(l-1)k}+\sum_{{}^2\Upsilon^{l-1}} sgn({}^2\Upsilon^{l-1})\prod_k m_{2(l-1)k}+\cdots+\sum_{{}^l\Upsilon^{l-1}} sgn({}^l\Upsilon^{l-1})\prod_k m_{l(l-1)k}.$$

Since the ring $R$ is a commutative ring, order in the multiplication is not important. Hence when we consider ${}^j\Upsilon^2$ which is the notation for all $(l-1)$-permutations corresponded to $\Delta_{C_j^i}$ such that two and only two of the entries with respect to the position $(k, {}^j\Upsilon_k^2)$ are $x_i$ for all $k = 1, \cdots, l-1$, By changing $j$ between 1 and $l$ , all permutations which contain exactly two $x_i$ are counted twice and for the case that we have exactly three $x_i$, all permutations that contain exactly three $x_i$ are counted three times and so on. Therefore One can see:

$$\sum_{\lambda^1\in\Lambda^1} sgn(\lambda^1)\prod_{j=1}^{1-1} m_{j\lambda_j^1} = \sum_{{}^1\Upsilon^0} sgn({}^1\Upsilon^0)\prod_k m_{10k}+\sum_{{}^2\Upsilon^0} sgn({}^2\Upsilon^0)\prod_k m_{20k}+\cdots+\sum_{{}^l\Upsilon^0} sgn({}^l\Upsilon^0)\prod_k m_{l0k}$$

$$2\sum_{\lambda^2\in\Lambda^2} sgn(\lambda^2)\prod_{j=1}^{1-1} m_{j\lambda_j^2} = \sum_{{}^1\Upsilon^1} sgn({}^1\Upsilon^1)\prod m_{11k}+\sum_{{}^2\Upsilon^1} sgn({}^2\Upsilon^1)\prod m_{21k}+\cdots+\sum_{{}^l\Upsilon^1} sgn({}^l\Upsilon^1)\prod m_{l1k}$$

$\vdots$

$$l\sum_{\lambda^l\in\Lambda^l} sgn(\lambda^l)\prod_{j=1}^{l-1} m_{j\lambda_j^l} = \sum_{{}^1\Upsilon^{l-1}} sgn({}^1\Upsilon^{l-1})\prod_k m_{1(l-1)k} + \sum_{{}^2\Upsilon^{l-1}} sgn({}^2\Upsilon^{l-1})\prod_k m_{2(l-1)k}$$
$$+\cdots+\sum_{{}^l\Upsilon^{l-1}} sgn({}^l\Upsilon^{l-1})\prod_k m_{l(l-1)k}.$$

Therefore

$$\partial f/\partial x_i = \sum_j \Delta_{C_j^i},$$

as was to be shown. □

## 5.4 Circulants

Circulant matrices are a classical subject with applications in many fields ([25]). A circulant can be looked at as a degeneration of a Hankel matrix.

Let $R$ be the polynomial ring $k[x_0, \ldots, x_n]$ over a field $k$. The $n \times n$ generic *circulant* is the matrix

$$C = \begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} & x_n \\ x_1 & x_2 & \ldots & x_n & x_0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ x_{n-1} & x_n & \ldots & x_{n-3} & x_{n-2} \\ x_n & x_0 & \ldots & x_{n-2} & x_{n-1} \end{pmatrix}$$

**Lemma 5.4.1.** *For every $i = 0, \ldots, n-1$, all (signed) cofactors of $C$ with respect to $x_i$ coincide.*

*Proof.* First note that in the circulant every variable entry repeats itself along the subdiagonal where it appears. Therefore, it suffices to prove the equalities $\Delta_0^j = \Delta_{j+1}^n$ and $\Delta_i^j = \Delta_{i+1}^{j-1}$ for any $0 \leq i \leq n-1$ and $1 \leq j \leq n$, where $\Delta_i^j$ denotes the (signed) cofactor obtained deleting the $i$th row and the $j$th column. Let $E_l^k$ (respectively, $C_l^k$)denote the elementary operation that permutes the $l$th and $k$th rows (respectively, columns).

**Step 1.** $\Delta_i^j = \Delta_{i+1}^{j-1}$.

$$\Delta_{i+1}^{j-1} \rightsquigarrow E_0^1 \rightsquigarrow E_1^2 \rightsquigarrow \cdots \rightsquigarrow E_{n-2}^{n-1} \rightsquigarrow C_{n-1}^0 \rightsquigarrow C_{n-1}^1 \rightsquigarrow \cdots \rightsquigarrow C_{n-1}^{n-2} = \Delta_i^j.$$

On each step the cofactor gets multiplied by $-1$; since the number of steps is even, the final sign does not change.

**Step 2.** $\Delta_0^j = \Delta_{j+1}^n$.

$$(\Delta_{j+1}^n)^t \rightsquigarrow C_0^1 \rightsquigarrow C_1^2 \rightsquigarrow \cdots \rightsquigarrow C_{n-2}^{n-1} = \Delta_0^j.$$

Note that the sign is preserved since $(-1)^{n-1}(-1)^{j+1+n} = (-1)^j$. □

**Proposition 5.4.2.** $\det C$ *is homaloidal.*

**Proof.** We show, namely:

(i) For every $i = 0, \ldots, n-1$, one has $f_i = (n+1)\triangle_0^i$.

(ii) $f$ is squarefree and the gradient ideal is perfect of codimension 2.

(iii) $f$ is homaloidal.

(i) This follows from Proposition 5.3.1 and Lemma 5.4.1.

(ii) From (i) we have that the gradient ideal of $f$ is generated by the maximal minors of (any) $(n+1) \times n$ submatrix of the circulant. Since the degrees are just right, the gradient is (perfect) of codimension 2. In particular, $f$ is squarefree.

(iii) From (i) and (ii), the gradient ideal has linear presentation. Therefore, it suffices to show that the syzygy matrix $\varphi$ of the gradient ideal satisfies the property (2.2) (see [22, Example 2.4]).

But note again that $\varphi$ can be any submatrix of $C$ of size $(n+1) \times n$. Close inspection shows that the initial ideal of $I_t(\varphi)$ in the graded reverse lexicographic order contains the pure powers $x_0^t, x_1^t, \ldots, x_{n-t+1}^t$. □

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