UNIVERSIDADE FEDERAL DE PERNAMBUCO
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

JANDEILSON SANTOS DA SILVA

CONTROL RESULTS FOR KORTEWEG-DE VRIES TYPE SYSTEMS

Recife

2025

JANDEILSON SANTOS DA SILVA

CONTROL RESULTS FOR KORTEWEG-DE VRIES TYPE SYSTEMS

Thesis submitted to the Graduate Program in Mathematics of the Federal University of Pernambuco, as a partial requirement for the degree of Doctor of Philosophy in Mathematics

Concentration area: Analysis

Advisor: Prof. Roberto de Almeida Capistrano-Filho

Recife

2025

.Catalogação de Publicação na Fonte. UFPE - Biblioteca Central

Silva, Jandeilson Santos da.
Control results for Korteweg-De Vries type systems / Jandeilson Santos da Silva. - Recife, 2025.
188 f.: il.

UNIVERSIDADE FEDERAL DE PERNAMBUCO, Centro de Ciências Exatas e da Natureza, Programa de Pós-Graduação em Matemática.
Orientação: Roberto de Almeida Capistrano-Filho.
Inclui referências e apêndices.

1. Equações dispersivas; 2. Controlabilidade na fronteira; 3. Comprimentos críticos; 4. Método do retorno; 5. Grafos estrelados; 6. Abordagem flatness. I. Capistrano-Filho, Roberto de Almeida. II. Título.

UFPE-Biblioteca Central

JANDEILSON SANTOS DA SILVA

Control results for Korteweg-de Vries type equations

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

Aprovado em: 11/07/2025

BANCA EXAMINADORA

Prof. Dr. Roberto de Almeida Capistrano Filho (Orientador)
Universidade Federal de Pernambuco

Prof. Dr. Cilon Valdez Ferreira Perusato (Examinador Interno)
Universidade Federal de Pernambuco

Prof. Dr. Hugo Parada Ríos (Examinador Externo)
Université Toulouse III-Paul Sabatier - France

Prof. Dr. Chulkwang Kwak (Examinador Externo)
Ewha Womans University - Korea

Prof. Dr. Fernando Andrés Gallego Restrepo (Examinador Externo)
Universidad Nacional de Colombia - Sede Manizales

Dedicated to my parents, sisters, and son.

ACKNOWLEDGEMENTS

I thank God, the author of yet another achievement in my life.

I am deeply grateful to my parents, João and Tereza, and to my sisters, my steadfast supporters and encouragers at every stage. To my son, João Sérgio, for being my greatest source of perseverance. To my entire family, whose wisdom and teachings have guided me along this journey. You are the foundation of it all.

I sincerely thank my friends who, from the beginning, cheered me on and truly rejoiced with me every step of the way.

To my fellow doctoral colleagues, with whom I shared moments of learning, relaxation, and countless coffees, without which the calculations would not have been possible. More broadly, I am grateful to all the friends brought into my life by mathematics.

To my advisor, Professor Roberto Capistrano-Filho, whose guidance throughout this doctoral journey fostered tremendous learning and enabled a meaningful integration into mathematical research. I am sincerely thankful for all the doors you have opened.

To Professors Aldo Trajano and Severino Horácio, my advisors during my undergraduate and master's studies, who were great motivators in my mathematical journey and from whom I learned so much.

I thank all the professors of Dmat-UFPE who contributed to my academic development.

My thanks also go to the members of the examination committee, Professors Chulkwang Kwak, Cilon Perusato, Fernando Gallego, and Hugo Parada, for their valuable suggestions and contributions to the improvement of this work.

Finally, I express my gratitude to UFPE and CNPq for enabling the completion of this thesis through financial and institutional support.

RESUMO

Esta tese investiga a controlabilidade por condições de contorno para sistemas dispersivos governados por equações do tipo Korteweg-de Vries (KdV). O objetivo principal é conduzir o estado do sistema por meio de controles aplicados nas fronteiras. Inicialmente, focamos na conhecida equação de KdV em um domínio limitado com condições de contorno puramente de Neumann e utilizando um único controle. Uma dificuldade central surge quando o comprimento do domínio espacial é crítico, tornando o sistema linear associado incontrolável. Para contornar esse obstáculo, utilizamos o método do retorno, que permite estabelecer a controlabilidade do sistema não linear.

O segundo problema considera a equação de KdV em um grafo em forma de estrela, modelado como um sistema de N equações do tipo KdV definidas nos intervalos $(0, \ell_j)$, acopladas por uma condição no nó central. Demonstramos a controlabilidade do sistema por meio de N controles de fronteira, que podem ser de Neumann, de Dirichlet ou uma combinação de ambos. Através de uma análise espectral detalhada para cada configuração de controle, identificamos os conjuntos de comprimentos críticos correspondentes.

Por fim, estudamos a controlabilidade da equação de KdV de quinta ordem, também conhecida como Equação de Kawahara, utilizando dois controles de fronteira. Neste caso, adotamos o método flatness — uma abordagem não convencional que dispensa o uso de desigualdades de observabilidade. Esse método consiste em parametrizar as variáveis de estado e de controle por meio dos chamados "flat outputs", as quais são funções pertencentes a espaços de Gevrey. Dentro desse arcabouço, abordamos dois problemas principais: alcançar a controlabilidade nula e caracterizar o conjunto de estados alcançáveis a partir do estado zero, identificando assim um espaço funcional no qual é possível obter controlabilidade exata.

Palavras-chaves: Equações dispersivas; Controlabilidade na fronteira; Comprimentos críticos; Método do retorno; Grafos estrelados; Abordagem flatness.

ABSTRACT

This thesis investigates boundary controllability for dispersive systems governed by the Korteweg-de Vries (KdV) type equation. The main goal is to steer the system's state using boundary controls. We first focus on the well-known KdV equation in a bounded domain with purely Neumann boundary conditions and a single control input. A central difficulty arises when the spatial domain length is critical, rendering the associated linear system uncontrollable. To address this, we employ the return method to establish controllability of the nonlinear system.

The second problem considers the KdV equation on a star-shaped graph, modeled as a system of N KdV-type equations defined on intervals $(0, \ell_j)$, coupled through a condition at the central node. We demonstrate controllability using N boundary controls, which may be Neumann, Dirichlet, or a combination of both. We identify the corresponding sets of critical lengths through detailed spectral analysis for each boundary configuration.

Lastly, we explore the controllability of the fifth-order KdV equation, also known as the Kawahara equation, using two boundary controls. Here, we adopt the flatness method—a nonstandard approach that bypasses the need for an observability inequality. This method expresses the state and control variables in terms of so-called “flat outputs” in Gevrey spaces. Within this framework, we address two key problems: achieving null controllability and characterizing the set of states reachable from zero, thereby identifying a functional space where exact controllability holds.

Keywords: Dispersive equations; Boundary controllability; Critical lengths; Return method; Star-graphs; Flatness approach.

LIST OF FIGURES

LIST OF TABLES

CONTENTS

1 INTRODUCTION

In this thesis, we are concerned with the controllability property of dispersive systems governed by partial differential equations of the KdV type. More specifically, we want to know whether a given system can be driven from any initial data to any pre-established final data, through the appropriate choice of a control acting internally or at the boundary of the domain.

The results obtained here provide controllability properties for the KdV equation posed on a bounded interval or a star graph, and for the Kawahara equation also on a bounded domain. Before presenting these results, it is interesting to bring some considerations about the emergence of these dispersive models.

1.1 HISTORICAL ASPECTS ABOUT DISPERSIVE EQUATIONS

The systems addressed here arise as models for solitary waves, a phenomenon first observed in 1834 by Scottish naval engineer John Scott Russell. He was working at the Union Canal connecting Edinburgh with Glasgow in Scotland, and was impressed by a wave traveling slowly without changing its shape and speed, which he called "Wave of Translation" (RUSSELL, 1844). With this fascination, for approximately the next ten years, Russell carried out several experiments to better understand this phenomenon. His empirical findings led him to the following relationship:

$$c^2 = g(h + a)$$

where c is the velocity of the solitary wave, a is the maximum amplitude above the water surface, h is the finite depth, and g is the acceleration of gravity. His discoveries revolutionized 19th-century naval engineering and earned him the Gold Medal of the Royal Society of Edinburgh in 1837.

However, until that point, Russell's studies were still viewed with some skepticism by the scientific community, and people had difficulty describing his observations. Although he continued his research for a while, he decided to stop his experiments when he saw contradictions between his results and the work of famous mathematicians, such as G.B. Airy's theory of water waves (AIRY, 1845) and G.G. Stokes' theory of shallow water waves (STOKES, 1847). The reason for so much disagreement is that Russell's observations were a consequence of nonlinear effects, which were disregarded at the time. An analytical demonstration of the existence of uniformly traveling waves would then be necessary, which challenged the mathematical community in the following years.

It took a few decades until, in 1871, the French mathematician Joseph Valentin Boussinesq published a work where nonlinear terms were considered, and the solutions obtained had characteristics similar to those of the solitary waves observed originally by Russell. Boussinesq studied a model of long, incompressible, and rotating-free waves in a shallow channel with rectangular cross-section, disregarding friction along the boundary, arriving at the following equation

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\left[\frac{3h^2}{2H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right]$$

where (t, x) are the coordinates of the fluid particle at time t, h is the amplitude of the wave, H is the height of the water in equilibrium and g is the gravitational constant (BOUSSINESQ, 1871). Unfortunately, these Boussinesq results did not receive due attention in England.

Five years later, the English physicist Lord Rayleigh confirmed Russell's results, independently of Boussinesq, by giving a solitary wave profile. He assumed the existence of a stationary wave vanishing at infinity and considered only the spatial dependence, which led him to the equation

$$\left(\frac{dh}{dx}\right)^2 + \frac{3}{H^3}h^2(h - h_0) = 0$$

with the same previous notation and h_0 being the crest of the wave (RAYLEIGH, 1876). The solution to this equation can be expressed explicitly by

$$h(x) = h_0 \mathrm{sech}^2\left(\sqrt{\frac{h_0}{4H^3}}x\right).$$

In the same paper, Rayleigh gave credit for the results obtained there to Boussinesq (1871):

> I have lately seen a memoir by M. Boussinesq (1871, *Comptes Rendus*, Vol. LXXII.), in which is contained a theory of the solitary wave very similar to that of this paper. So far as our results are common, the credit of priority belongs, of course, to M. Boussinesq (RAYLEIGH, 1876).

Finally, in 1895, two Dutch mathematicians provided the most well-known model for the waves observed by Russell, which is still widely studied today. Diederick Johannes Korteweg and his doctoral student Gustav de Vries included the effect of surface tension in Rayleigh's equation, obtaining the famous KdV equation.

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial \xi}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \xi^2}\right), \tag{1.1}$$

where η denotes the surface elevation above the equilibrium level l, α is a small arbitrary constant related to the motion of the liquid, g is the gravitational constant and $\sigma = \frac{l^3}{3} - \frac{Tl}{\rho g}$, with surface capillary tension T and density ρ (KORTEWEG; VRIES, 1895). Performing the change of variables

$$t = \sqrt{\frac{g}{l\sigma}}\tau, \quad x = -\frac{1}{\sqrt{\sigma}}\xi, \quad u = \left(\frac{1}{2}\eta + \frac{1}{3}\alpha\right)$$

in (1.1) we obtain the KdV equation in its best-known form

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.2}$$

Below we provide a brief presentation of the dispersive models addressed in this thesis, which arise from improvements in (1.2), aiming at generalizations or greater fidelity to the physical context.

1.2 DISPERSIVE WAVE EQUATIONS

In physical terms, a wave is called dispersive when the phase velocity is different from the group velocity, which also means that the phase velocity is not constant; it depends on the frequency. Thus, different frequencies travel at different speeds. Roughly speaking, it is not a wave but a group or packet of waves within which each wave travels at its speed, different from the speed of the group as a whole. As time evolves, these different waves disperse in the medium, resulting in a single rise of water dividing into wave-trains.

In general, these waves are modeled mathematically by differential equations of the form

$$F(\partial_x, \partial_t,)u(x,t) = 0 \tag{1.3}$$

where F is a polynomial in two variables. As Linares and Ponce (LINARES; PONCE, 2015) explains, we try to find solutions u satisfying

$$u(x,t) = Ae^{i(kx-\omega t)}$$

where A is the amplitude, k is the wave number and ω is the frequency, which happens if and only if

$$F(ik, -iw) = 0.$$

This equation is called the dispersion relation and, in several models, it allows us to write ω as a real function of k, namely

$$\omega = \omega(k).$$

In these terms, the differential equation (1.3) (so the waves modeled by it) is called dispersive when $w''(k) \neq 0$. By definition, the phase velocity v_p and the group velocity v_g are given by

$$v_p = \frac{\omega}{k} \quad \text{and} \quad v_g = \frac{d\omega}{dk}.$$

The relationship $w''(k) \neq 0$, which defines a dispersive wave, means that the group velocity v_g is not constant and consequently the phase velocity v_p is also not constant, exactly as we mentioned initially.

In this work, we address problems concerning two models of dispersive equations: the KdV equation and the Kawahara equation.

1.2.1 The KdV equation

The KdV equation, given in (1.1) and (1.2), is one of the best-known dispersive models in the literature and can be considered a prototype for models of waves traveling in a rectangular channel of relatively shallow depth. Actually, as Benjamin, Bona, and Mahony (1972) point out, by appropriately redefining the dependent and independent variables in (1.1), the KdV equation can be obtained in the form

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.4}$$

According to Bona and Winther (1983), equation (1.4) is a more appropriate model for waves in a uniform channel than equation (1.2), due to physical issues associated with the temporal variable.

1.2.2 The Kawahara equation

In the KdV equation (1.2), the coefficient of the dispersion term (third-order derivative term) can vary according to the medium and can be very small or even zero. In such cases, we must consider higher-order dispersion terms to balance the effect of the nonlinearity. This is what occurs in the propagation of the magneto-acoustic wave in a cold collision-free plasma, where the third-order derivative disappears and is replaced by a fifth-order derivative, which was done in (KAKUTANI; ONO, 1969). The resulting equation is of the form

$$u_t + \frac{3}{2}uu_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0, \tag{1.5}$$

where α and β are constants representing the dispersion effect and may be either positive or negative. Equation (1.5) is due to Hasimoto (HASIMOTO, 1970) and Kawahara (KAWAHARA, 1972) and known as the generalized KdV equation or the Kawahara equation.

1.3 MAIN RESULTS AND STRUCTURE OF THE WORK

One of the main objectives when studying models represented by differential equations is to obtain qualitative properties that carry useful information about the physical context being modeled. Among the most desired properties, the possibility of making predictions about the behavior of the system under study or even controlling this system stands out. The latter is the core of control theory, where the aim is to manipulate the system so that, at a given time, it is in a predetermined state.

Our goal here is to investigate the controllability property of the KdV equation posed on a bounded domain or a star graph, and of the Kawahara equation on a bounded domain.

1.3.1 Boundary controllability of the Korteweg-de Vries equation: The Neumann case

The first problem studied in this thesis concerns the controllability for the KdV equation under boundary conditions purely Neumann

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & \text{in } (0,L)\times(0,T), \\ u_{xx}(0,t) = u_{xx}(L,t) = 0, & \text{in } (0,T), \\ u_x(L,t) = h(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{1.6}$$

where $h(t)$ will be considered as a control input. Recently, Caicedo, Capistrano-Filho and Zhang (2017) proved that system (1.6) is exactly controllable around an equilibrium $u \equiv c$, provided the length L of the spatial domain does not belong to $\mathcal{R}_c$, where $c \neq -1$ and

$$\mathcal{R}_c := \left\{ \frac{2\pi}{\sqrt{3(c+1)}} \sqrt{m^2 + ml + l^2};\ m, l \in \mathbb{N}^* \right\} \cup \left\{ \frac{m\pi}{\sqrt{c+1}};\ m \in \mathbb{N}^* \right\}.$$

Naturally, the following question arises:

Question $\mathcal{A}_1$: *Is the system* (1.6) *exactly controllable around c when $L \in \mathcal{R}_c$?*

In our first work, we give a positive answer to this question, which consists of proving the theorem below.

Theorem 1.1 *Let $T > 0$, $c \neq -1$ and $L \in \mathcal{R}_c$. The system* (1.6) *is exactly controllable around c in $L^2(0, L)$, that is, there exists $\delta > 0$ such that, for every $u_0, u_T \in L^2(0, L)$ with*

$$\|u_0 - c\|_{L^2(0,L)}, \|u_T - c\|_{L^2(0,L)} < \delta$$

it is possible to find $h \in L^2(0, T)$ such that the corresponding solution of (1.6) *satisfies*

$$u(\cdot, 0) = u_0 \quad \textit{and} \quad u(\cdot, T) = u_T.$$

The difficulty in proving this result lies in the fact that, when $L \in \mathcal{R}_c$, the linearization of (1.6) around c is not exactly controllable, so a standard argument using only fixed point theorems does not work in this case. We need to combine this technique with the so-called *return method*. To do this, it is necessary to establish the following auxiliary result, which says to us that the set of critical lengths $\mathcal{R}_c$ is sensitive to small disturbances in equilibrium c.

Theorem 1.2 *Let $T > 0$, $c \neq -1$ and $L \in \mathcal{R}_c$. There exists $\varepsilon_c > 0$ such that, for every $d \in (c - \varepsilon_c, c + \varepsilon_c) \backslash \{c\}$, $d \neq -1$, we have $L \notin \mathcal{R}_d$. Consequently, the linearization of* (1.6) *around d is exactly controllable; and the nonlinear system* (1.6) *is exactly controllable around the steady state d in $L^2(0, L)$.*

1.3.2 Boundary observation of the KdV equation on graphs

The second problem addressed in this work concerns the KdV equation posed on a star-graph $\mathcal{T} := \bigcup_{j=1}^{N} e_j$, where e_j are the edges.

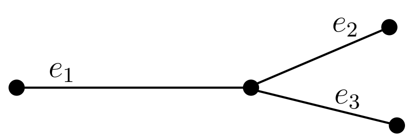

Figure 1 – Star-graph with 3 edges

Naturally identifying each edge with an interval $I_j = [0, \ell_j]$, the pivot space for data is

$$\mathbb{L}^2(\mathcal{T}) := \prod_{j=1}^{N} L^2(0, \ell_j)$$

and our system can be written as follows

$$\begin{cases} \partial_t u_j(t,x) + \partial_x u_j(t,x) + \partial_x^3 u_j(t,x) = 0, & t \in (0,T),\ x \in (0,\ell_j),\ j = 1,\ldots,N, \\ u_j(t,0) = u_1(t,0), & t \in (0,T),\ \forall j = 2,\ldots,N, \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) + g_0(t), & t \in (0,T), \\ u_j(t,\ell_j) = p_j(t), \quad \partial_x u_j(t,\ell_j) = g_j(t), & t \in (0,T),\ j = 1,\ldots,N, \\ u_j(0,x) = u_j^0(x), & x \in I_j, \end{cases} \tag{1.7}$$

where $\alpha > \frac{N}{2}$. This system arose with the proposal of the KdV equation as a model for blood pressure waves.

Controllability properties for a system of type (1.7) were considered for the first time, as far as we know, in (AMMARI; CRÉPEAU, 2018), where the authors proved the exact controllability of (1.7) by using $n+1$ controls, namely $g_0, g_1 ..., g_N$, provided that $\#\{\ell_j \in \mathcal{N}\} \leq 1$, where $\mathcal{N}$ is the critical set for a single KdV equation with Dirichlet boundary conditions and Neumann control, introduced by Rosier in (ROSIER, 1997). Recently, Parada (PARADA, 2024) obtained null controllability to the system (1.7) employing a mixed configuration of $2(N-2)$ Neumann and Dirichlet controls, $g_2, ..., g_N, p_2, ..., p_N$. Here, our goal is to obtain the exact controllability for (1.7) with a smaller number of controls, that is, we are concerned with answering the following question.

Question $\mathcal{B}_1$: *Given $T > 0$ and $u^0, u^T \in L^2(0,L)$, can one find a number of control inputs p_j, g_j less than $2(N-2)$ such that the corresponding solution $u(x,t)$ of* (2.2) *satisfies*

$$u(x,T) = u^T(x)? \tag{1.8}$$

Specifically, we want to use only N controls, which leads us to situations with critical lengths whose characterization depends on the number of Neumann and Dirichlet controls used, and involves the sets

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \ : k,\ l\ \in \mathbb{N}^* \right\}, \tag{1.9}$$

introduced in (ROSIER, 1997) as the set of critical lengths for a single KdV equation with Dirichlet boundary conditions and one Neumann control, the set $\mathcal{N}^*$ given by

$$\mathcal{N}^* = \left\{ L \in \mathbb{R}^+ \setminus \{0\}; \begin{array}{l} \exists (a,b) \in \mathbb{C}^2 \text{ such that } ae^a = be^b = -(a+b)e^{-(a+b)} \\ \text{and } L^2 = -\left(a^2 + ab + b^2\right) \end{array} \right\}, \quad (1.10)$$

introduced in (GLASS; GUERRERO, 2010) as the set of critical lengths for a single KdV equation with one Dirichlet control and, finally, the set

$$\mathcal{N}^\dagger = \left\{ L \in \mathbb{R}^+ \setminus \{0\}; \begin{array}{l} \exists (a,b) \in \mathbb{C}^2 \text{ such that } a^2e^a = b^2e^b = (a+b)^2e^{-(a+b)} \\ \text{and } L^2 = -\left(a^2 + ab + b^2\right) \end{array} \right\}, \quad (1.11)$$

a new critical set in the literature for the KdV equation, introduced in this work.

Theorem 1.3 *Let $T > 0$ and $u^0, u^T \in \mathbb{L}^2(\mathcal{T})$. Consider $\alpha = N$, $\ell_j = L$ for $j = 1, \dots, N$ and denote by $m \in \{1, ..., N\}$ the number of Neumann controls used.*

1. *If $N = 2$, then for any $L > 0$, there exist controls $(g_1, 0) \in [L^2(0,T)]^2$ and $(0, p_2) \in [L^2(0,T)]^2$ such that the unique solution u of* (1.7) *satisfies* (1.8)*.*

2. *If $N \geq 3$ and $m = 1$ then, there exist controls $(g_1, 0, ..., 0) \in [L^2(0,T)]^N$ and $(0, p_2, ..., p_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (1.7) *satisfies* (1.8)*, if and only if $L \notin \mathcal{N}^*$.*

3. *If $N \geq 3$ and $m = N-1$ then, there exist controls $(g_1, ..., g_{N-1}, 0) \in [L^2(0,T)]^N$ and $(0, ..., 0, p_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (1.7) *satisfies* (1.8)*, if and only if $L \notin \mathcal{N}$.*

4. *If $N > 3$ and $1 < m < N-1$ then, there exist controls $(g_1, ..., g_m, 0, ..., 0) \in [L^2(0,T)]^N$ and $(0, ..., 0, p_{m+1}, ..., p_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (1.7) *satisfies* (1.8)*, if and only if $L \notin \mathcal{N} \cup \mathcal{N}^*$.*

5. *If, $N \geq 2$, then there exist controls $(g_1, ..., g_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (1.7) *satisfies* (1.8)*, if and only if $L \notin \mathcal{N} \cup \mathcal{N}^*$.*

6. *If, $N \geq 2$, then there exist controls $(p_1, ..., p_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (1.7) *satisfies* (1.8)*, if and only if $L \notin \mathcal{N}^* \cup \mathcal{N}^\dagger$.*

As is well known in the literature, proving controllability for system (1.7) is equivalent to establishing an observability inequality for the adjoint system. To do this, we will use the same strategy as Rosier (1997), combining the multiplier method with contradiction and compactness arguments.

1.3.3 Control of Kawahara equation using flat outputs

Our third problem concerns the controllability of the Kawahara equation on a bounded domain. Precisely, we are interested in the control properties of the system

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = 0, & (x,t) \in (-1,0) \times (0,T), \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t \in (0,T), \\ u(-1,t) = h_1(t), \quad u_x(-1,t) = h_2(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (-1,0), \end{cases} \tag{1.12}$$

where $u(t,\cdot)$ is the state at time $t \in (0,T)$ and h_1, h_2 are the controls. The problem of internal controllability for the Kawahara equation has been widely studied in recent years, as can be seen for example in (ZHANG; ZHAO, 2012), (ZHAO; ZHANG, 2015), (CAPISTRANO-FILHO; GOMES, 2021), (PAZOTO; SOTO, 2023), (CAPISTRANO-FILHO; SOUSA; GALLEGO, 2023) e (AHAMED; MONDAL, 2025). However, the boundary controllability for the linear equation in (1.12) is still a challenging and developing topic. Motivated by this, we are interested in the following problem.

Question $\mathcal{C}_1$: *Is the system* (1.12) *null controllable in* $L^2(0,L)$*, that is, given* $T > 0$ *and* $u_0 \in L^2(-1,0)$ *can we find controls* h_1, h_2 *(in suitable spaces) such that the solution of* (1.12) *satisfies* $u(T,\cdot) = 0$*?*

In Chapter refchap.flat-kawa we will give a positive answer to this question using the flatness approach, introduced in (FLIESS et al., 1995) for finite-dimensional systems and in (LAROCHE; MARTIN; ROUCHON, 2000) for the case of infinite dimension. This is a method in which the state and control variables are explicitly expressed as a function of an output variable (called the flat variable) and its derivatives. In our case, the method allows us to trace trajectories of the system that end in the null state, that is, zero is a state that can be reached from any initial state. Naturally, the question arises as to which states can be reached from zero, which motivates a second problem.

Question $\mathcal{D}_1$: *Can we find a space $\mathcal{R}$ with the property that, if the final data $u_1 \in \mathcal{R}$ then one can get control functions h_1, h_2 such that the solution u of the system* (1.12) *with $u_0 = 0$ satisfies $u(\cdot, T) = u_1$?*

Using the same approach, we will present a particular space of states reachable from the null state, answering $\mathcal{D}_1$. It is important to emphasize that the problem of determining all directions that can be reached from zero by system (1.12) constitutes a difficult task whose complete realization means fully understanding the behavior of the Kawahara equation on bounded domains from the point of view of control theory.

1.3.4 Structure of the thesis

We conclude this introductory chapter with a brief overview of the organization of the results presented in the subsequent chapters. In Chapter 2, we address the control problem for the Korteweg-de Vries (KdV) equation under purely Neumann boundary conditions, providing answers to Question $\mathcal{A}_1$. Chapter 3 is devoted to the study of the KdV equation on star-shaped graphs. In response to Question $\mathcal{B}_1$, we present several scenarios in which controllability is achieved using N controls, depending on the nature of the spatial domain length L. In Chapter 4, we apply the flatness approach to establish the controllability of the Kawahara equation on a bounded domain, employing two boundary controls and thereby answering Question $\mathcal{C}_1$. Furthermore, we exhibit an example of a reachable state space from zero, which yields a satisfactory answer to Question $\mathcal{D}_1$. In Chapter 5 we provide a perspective for continuing the investigation of the problems addressed here, with some open questions that may encourage new work. The Appendix A addresses elements of the theories that underpin this work, particularly about functional spaces, semigroup theory, and control theory. Finally, Appendix B brings some important characteristics about $\mathcal{N}^\dagger$.

2 BOUNDARY CONTROLLABILITY OF THE KORTEWEG-DE VRIES EQUATION: THE NEUMANN CASE

In this chapter, we are interested in the exact controllability property for the KdV equation posed on an interval $[0, L]$ under Neumann boundary conditions, when L belongs to the *critical lengths set*

$$\mathcal{R}_c := \left\{ \frac{2\pi}{\sqrt{3(c+1)}}\sqrt{m^2 + ml + l^2};\ m, l \in \mathbb{N}^* \right\} \cup \left\{ \frac{m\pi}{\sqrt{c+1}};\ m \in \mathbb{N}^* \right\},$$

with $c \neq -1$. Precisely, we show that the system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & \text{in } (0, L) \times (0, T), \\ y_{xx}(0, t) = y_{xx}(L, t) = 0, & \text{in } (0, T), \\ y_x(L, t) = h(t), & \text{in } (0, T), \\ y(x, 0) = y_0(x), & \text{in } (0, L), \end{cases} \tag{2.1}$$

is exactly controllable arround c in $L^2(0, L)$ for the critical case, i.e., when $L \in \mathcal{R}_c$, using only a control input, namely $h(t)$. The result is achieved using the *return method* together with a fixed point argument.

2.1 SETTING PROBLEM AND MAIN RESULTS

The control problem for the KdV equation was presented in a pioneering work of Rosier (ROSIER, 1997) that studied the following system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0, L) \times (0, T), \\ u(0, t) = 0,\ u(L, t) = 0,\ u_x(L, t) = g(t), & \text{in } (0, T), \\ u(x, 0) = u_0(x), & \text{in } (0, L), \end{cases} \tag{2.2}$$

where the boundary value function $g(t)$ is considered as a control input. Precisely, the author addressed the following control problem for the system (2.2):

Question $\mathcal{A}_2$: *Given $T > 0$ and $u_0, u_T \in L^2(0, L)$, can one find an appropriate control input $g(t) \in L^2(0, T)$ such that the corresponding solution $u(x, t)$ of the system* (2.2) *satisfies*

$$u(x, 0) = u_0(x) \quad \textit{and} \quad u(x, T) = u_T(x)? \tag{2.3}$$

From the answer given by Rosier to this question, the phenomenon of critical lengths for the KdV equation arose. He proved that the linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = g(t), & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L), \end{cases} \tag{2.4}$$

associated to (2.2), is exactly controllabe if, and only if, $L \notin \mathcal{N}$, where

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2}\ : k,\ l\ \in \mathbb{N}^* \right\}.$$

In this sense, the positive real numbers $L \in \mathcal{N}$ are called *critical lengths* for the problem (2.4).

Definition 2.1 *A spatial domain* $(0,L)$ *is called critical for the system* (2.4) *if its domain length* L *belongs to* $\mathcal{N}$*.*

Thus, if L is not a critical length, the system (2.4) is controllable, so, using a point fixed theorem, we get the property of local controllability for the nonlinear system (2.2) (ROSIER, 1997). When $L \in \mathcal{N}$, this approach does not work to prove the controllability for (2.2). However, Rosier was still able to prove that the space of directions unreachable from zero for (2.4) has finite dimension. Precisely, given $L \in \mathcal{N}$, there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $\mathcal{M} = \mathcal{M}(L)$, such that for every nonzero state $\psi \in \mathcal{M}$ and $g \in L^2(0,T)$, the solution of (2.4) corresponding to g and $u_0 = 0$ satisfies $u(\cdot,T) \neq \psi$.

The question of the behavior of (2.2) under the critical length regime waited until 2004 to begin to be unraveled, when Coron and Crépeau (CORON; CRÉPEAU, 2004) proved the controllability of such a system for $L = 2k\pi$. Using the power series expansion method, they proved that the effect of nonlinearity gives the system back the possibility of reaching the directions missed by the linear problem, that is, the space $\mathcal{M}(L)$ is reachable for (2.2). After that, the work of Cerpa and Crépeau (CERPA, 2007; CERPA; CRÉPEAU, 2009) extended this result to any critical length and sufficiently large time. In summary, we have the following result.

Theorem 2.1 (Coron, Crépeau and Cerpa) *For any* $L \in \mathcal{N}$*, there exists* $T_L > 0$ *such that, for any* $T > T_L$*, the nonlinear system* (2.2) *is locally exactly controllable at time* T*.*

The above theorem remains valid in T_L, as justified in (CERPA, 2007), where a description of T_L is also given. Recently, Coron, Koenig, and Nguyen (CORON; KOENIG; NGUYEN, 2024) proved that small-time local controllability does not hold in general for system (2.2) when $L \in \mathcal{N}$ and the data are taken in more regular spaces than L^2. Specifically, they identified a class of critical lengths for which controllability does not hold, as stated below.

Theorem 2.2 *Let $k, l \in \mathbb{N}^*$ be such that $2k + l \in 3\mathbb{N}^*$. Assume that*

$$L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}.$$

Then system(2.2) *is not small-time locally null-controllable with controls in H^1 and initial and final datum in $H^3(0, L) \cap H^1_0(0, L)$. Precisely, there exist $T_0 > 0$ and $\varepsilon_0 > 0$ such that, for all $\delta > 0$, one can find $y_0 \in H^3(0, L) \cap H^1_0(0, L)$ with $\|y_0\|_{H^3(0,L)} < \delta$ such that for all $u \in H^1(0, T_0)$ with $\|u\|_{H^1(0,T_0} < \varepsilon_0$ and $u(0) = y_0'(L)$, we have*

$$y(\cdot, T_0) \equiv 0,$$

where $y \in C([0, T_0]; H^3(0, L)) \cap L^2([0, T_0]; H^4(0, L))$ is the unique solution of (2.2).

The problem (2.1) was addressed by Caicedo, Capistrano-Filho, and Zhang in (CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017), where the authors established results of well-posedness, regularity, and complete characterization of the set of critical lengths associated with the problem. Their result can be read as follows.

Theorem 2.3 (Caicedo, Capistrano-Filho and Zhang, 2017) *Let $T > 0$, $c \neq -1$ and $L \notin \mathcal{R}_c$. There exists $\delta > 0$ such that for any $y_0, y_T \in L^2(0, L)$ with*

$$\|y_0 - c\|_{L^2(0,L)} < \delta \quad \text{and} \quad \|y_T - c\|_{L^2(0,L)} < \delta$$

one can find $h \in L^2(0, T)$ such that the system (2.1) *admits a unique solution*

$$y \in \mathcal{Z}_T := C\left([0, T]; L^2(0, L)\right) \cap L^2\left(0, T; H^1(0, L)\right)$$

satisfying $y(x, T) = y_T(x)$.

As in (ROSIER, 1997), the first step is to obtain a control result for the linear system, namely,

$$\begin{cases} v_t + (1 + c)v_x + v_{xxx} = 0, & \text{in } (0, L) \times (0, T), \\ v_{xx}(0, t) = v_{xx}(L, t) = 0, & \text{in } (0, T), \\ v_x(L, t) = h(t), & \text{in } (0, T), \\ v(x, 0) = v_0(x), & \text{in } (0, L). \end{cases} \tag{2.5}$$

Precisely, the authors in (CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017) proved the following result.

Theorem 2.4 (Caicedo, Capistrano-Filho and Zhang, 2017) *For $c \neq -1$, the linear system* (2.5) *is exactly controllable in the space $L^2(0,L)$ if and only if $L \notin \mathcal{R}_c$. Otherwise, that is, if $c = -1$, the system* (2.5) *is not exactly controllable in the space $L^2(0,L)$ for any $L > 0$.*

With this in hand, they extend the result for the nonlinear system (2.1) using a fixed point argument, achieving Theorem 2.3 whenever $L \notin \mathcal{R}_c$. In this context and following the spirit of the results about (2.2), a natural question appears:

Question $\mathcal{B}_2$: *Is the system* (2.1) *exactly controllable around $c \neq -1$ when L is a critical length? In other terms, given $T > 0$, $L \in \mathcal{R}_c$ and $u_0, u_T \in L^2(0,L)$ close enough to c, can one find an appropriate control input $h \in L^2(0,T)$ such that the solution y of the system* (2.1)*, corresponding to h and y_0, satisfies $y(\cdot,T) = y_T$?*

The main result in this chapter provides an affirmative answer to the Question $\mathcal{B}_2$. Precisely, we have the following:

Theorem 2.5 *Let $T > 0$, $c = 0$ and $L \in \mathcal{R}_0$. Then, system* (2.1) *is exactly controllable around the origin 0 in $L^2(0,L)$, that is, there exists $\delta > 0$ such that, for every $y_0, y_T \in L^2(0,L)$ with*

$$\|y_0\|_{L^2(0,L)}, \|y_T\|_{L^2(0,L)} < \delta$$

it is possible to find $h \in L^2(0,T)$ such that the corresponding solution of (2.1) *satisfies $y(\cdot,T) = y_T$.*

This previous result can be generalized for any $c \neq -1$, giving us Theorem 1.1, whose proof is analogous to that of Theorem 2.5. To prove them we need an auxiliary property that ensures that, for c near enough to 0 (small perturbations of 0), the system (2.1) is exactly controllable in a neighborhood of c in $L^2(0,L)$ for $L \in \mathcal{R}_0$. In general, for d close enough to $c \neq -1$ one has $L \notin \mathcal{R}_d$ so that the system (2.5) corresponding to d is exactly controllable. In other words, the set of critical lengths is sensitive to small disturbances in equilibrium c, and the result can be read as follows.

Theorem 2.6 *Let $T > 0$, $c \neq -1$ and $L \in \mathcal{R}_c$. There exists $\varepsilon_c > 0$ such that, for every $d \in (c-\varepsilon_c, c+\varepsilon_c)\backslash\{c\}$, $d \neq -1$, we have $L \notin \mathcal{R}_d$. Consequently, the linear system* (2.5)*, with $c = d$, is exactly controllable; and the nonlinear system* (2.1) *is exactly controllable around the steady state d in $L^2(0, L)$, that is, there exists $\delta_d > 0$ such that, for any $y_0, y_T \in L^2(0, L)$ with*

$$\|y_0 - d\|_{L^2(0,L)} < \delta_d \text{ and } \|y_T - d\|_{L^2(0,L)} < \delta_d,$$

one can find $h \in L^2(0, T)$ such that the system (2.1) *admits a unique solution $y \in \mathcal{Z}_T$ satisfying $y(\cdot, T) = y_T$.*

2.2 HEURISTIC AND CHAPTER'S OUTLINE

The proof of Theorem 2.6 is based on the topological properties of real numbers together with Theorem 2.3. Moreover, with this in hand, both results stated in the previous paragraph (Theorems 2.5 and 1.1) rely on the so-called *return method* together with the fixed point argument.

It is important to point out that the return method was introduced by J.M. Coron in (CORON, 1992) (see also (CORON, 1993)) and has been used by several authors to prove control results in the critical lengths for the KdV-type equation (see, for instance, (CRÉPEAU, 2001; CORON; CRÉPEAU, 2004; CERPA, 2007; CERPA; CRÉPEAU, 2009)). This method consists of building particular trajectories of the system (2.1) starting and ending at some equilibrium such that the linearization of the system around these trajectories has good properties. Here, we use a combination of this method with a fixed point argument, successfully applied in (GLASS, 2008). We mention that this method can be applied together with quasi-static deformations and power series expansion. We refer the reader to the nice book of Coron (CORON, 2020) for more details of the method.

Concerning the construction of solutions to the Theorems 2.5 and 1.1 we follow the following procedure: In the first time, we construct a solution that starts from y_0 and reaches at time $T/3$ a state which is in some sense close to d (which is yet to be defined). Then we construct a solution (close to the state solution d), which starts at time $2T/3$ from the previous state. In the last step, we bring the latter state to 0 *via* a function y_2, as we can see in Figure 2 below. For details of this construction, see the characterization of the function y in (2.27).

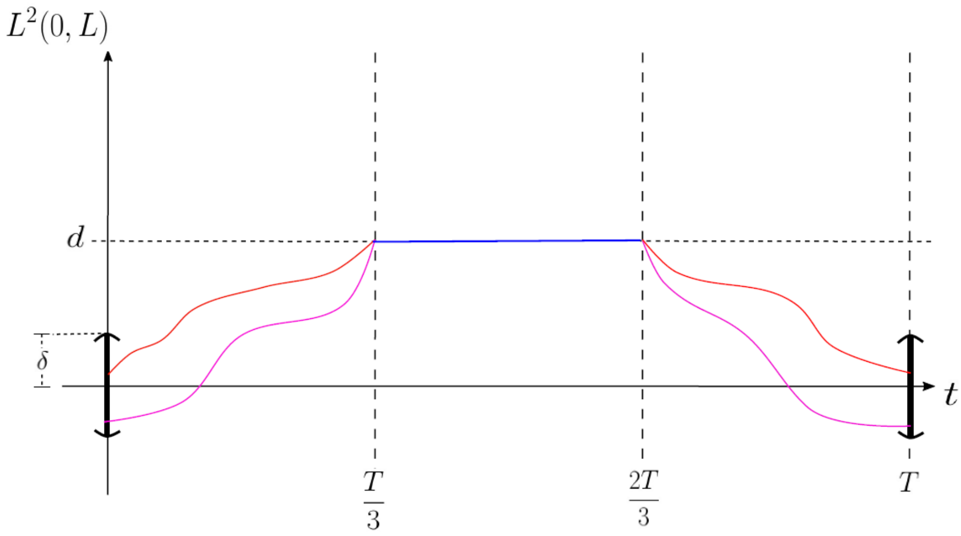

Figure 2 – Solutions driving states close to 0 to constants and *vice-versa*

We finish this section with an outline of this chapter, which consists of three parts, including this introduction. Section 2.3 gives an overview of the well-posedness of the system (2.1). Section 2.4 is devoted to proving carefully the controllability of the system (2.1) when $L \in \mathcal{R}_c$. Precisely, in the first part of Section 2.4, we deal with the proof of Theorem 2.6. In the second part, we prove the construction of the function y mentioned before, and finally, in the third part of Section 2.4, we use these previous results to achieve Theorem 2.5.

2.3 OVERVIEW OF THE WELL-POSEDNESS THEORY

In this section, we review the well-posedness theory for the KdV equation. The results presented here can be found in (BONA; SUN; ZHANG, 2003; CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017; KRAMER; RIVAS; ZHANG, 2013). For that, consider $L > 0$ and $T_0, T_1 \in \mathbb{R}$ with $T_0 < T_1$. We define the space

$$\mathcal{Z}_{T_0,T_1} := C\left([T_0, T_1]; L^2(0, L)\right) \cap L^2\left([T_0, T_1]; H^1(0, L)\right)$$

which is a Banach space with the following norm

$$\|y\|_{\mathcal{Z}_{T_0,T_1}} := \max_{t\in[T_0,T_1]} \|y(\cdot, t)\|_{L^2(0,L)} + \left(\int_{T_0}^{T_1} \|y(\cdot, t)\|^2_{H^1(0,L)} dt\right)^{1/2}.$$

For any $T > 0$ we denote $\mathcal{Z}_{0,T}$ simply by $\mathcal{Z}_T$.

Additionally, let $T > 0$ be given and consider the space

$$\mathcal{H}_T := H^{-\frac{1}{3}}(0, T) \times L^2(0, T) \times H^{-\frac{1}{3}}(0, T)$$

with a norm

$$\|(h_1,h_2,h_3)\|_{\mathcal{H}_T} := \|h_1\|_{H^{-\frac{1}{3}}(0,T)} + \|h_2\|_{L^2(0,T)} + \|h_3\|_{H^{-\frac{1}{3}}(0,T)}.$$

The next proposition, showed in (CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017, Proposition 2.5), provides the well-posedness to the following system

$$\begin{cases} u_t + u_{xxx} = f, & \text{in } (0,L)\times(0,T), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ u(x,0) = u_0, & \text{in } (0,L). \end{cases} \tag{2.6}$$

Proposition 2.1 (Caicedo, Capistrano-Filho and Zhang, 2017) *Let $T>0$ be. For any $v_0 \in L^2(0,L)$, $\boldsymbol{h} = (h_1,h_2,h_3) \in \mathcal{H}_T$ and $f \in L^1(0,T,L^2(0,L))$, the IBVP* (2.6) *admits a unique mild solution $u \in \mathcal{Z}_T$, which satisfies*

$$\partial_x^j u \in L_x^\infty(0,L;H^{(1-j)/3}(0,T)),\ j=0,1,2.$$

Moreover, there exists $C_1 > 0$ such that

$$\|u\|_{\mathcal{Z}_T} + \sum_{j=0}^{2}\left\|\partial_x^j u\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T))} \le C_1\left(\|u_0\|_{L^2(0,L)} + \|\boldsymbol{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right).$$

Remark 2.1 *We highlight that the constant C_1 in the above result depends on T. However, for any $\theta \in (0,T]$, the estimates in the Proposition 2.1 hold with the same constant C_1 corresponding to T.*

In fact, let $u_0 \in L^2(0,L)$, $\boldsymbol{h} \in \mathcal{H}_\theta$, $f \in L^1(0,\theta;L^2(0,L))$ and $u \in \mathcal{Z}_\theta$ the solution of (2.6) *corresponding to these datas. We extend $\boldsymbol{h}$ and f to $[0,T]$ (we will also denote these extensions by $\boldsymbol{h}$ and f) putting*

$$\boldsymbol{h} = 0 \text{ in } (\theta,T] \quad \text{and} \quad f = 0 \text{ in } (\theta,T].$$

Now, denote by $\tilde{u} \in \mathcal{Z}_T$ the corresponding solution of (2.6)*. Then, from Proposition 2.1, we have $\tilde{u}\big|_{[0,\theta]} = u$ and*

$$\begin{aligned} \|u\|_{\mathcal{Z}_\theta} \le \|\tilde{u}\|_{\mathcal{Z}_T} &\le C_1\left(\|u_0\|_{L^2(0,L)} + \|\boldsymbol{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right) \\ &= C_1\left(\|u_0\|_{L^2(0,L)} + \|\boldsymbol{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right). \end{aligned}$$

Using the Proposition 2.1, we can get properties for the linear problem

$$\begin{cases} y_t + (ay)_x + y_{xxx} = f, & \text{in } (0,L)\times(0,T), \\ y_{xx}(0,t) = h_1(t),\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,L), \end{cases} \tag{2.7}$$

where $a \in \mathcal{Z}_T$ is given. To do this, the following lemma will be very useful and was proved in (KRAMER; ZHANG, 2010, Lemma 3).

Lemma 2.1 (Kramer, Zhang (KRAMER; ZHANG, 2010)) *There exists a constant $C > 0$ such that*

$$\int_0^T \|uv_x(\cdot,t)\|_{L^2(0,L)}dt \leq C\left(T^{\frac{1}{2}} + T^{\frac{1}{3}}\right)\|u\|_{\mathcal{Z}_T}\|v\|_{\mathcal{Z}_T},$$

for every $u, v \in \mathcal{Z}_T$.

With these previous results in hand, the following proposition gives us the well-posedness of the general system (2.7), which will be used several times, so, for the sake of completeness, we will give the proof.

Proposition 2.2 *For any $y_0 \in L^2(0,L)$, $\boldsymbol{h} = (h_1, h_2, h_3) \in \mathcal{H}_T$ and $f \in L^1(0,T,L^2(0,L))$, the IBVP (2.7) admits a unique mild solution $y \in \mathcal{Z}_T$, which satisfies*

$$\partial_x^j y \in L_x^\infty(0,L;H^{(1-j)/3}(0,T)),\ j = 0,1,2,$$

and

$$\|y\|_{\mathcal{Z}_T} \leq C_2\left(\|y_0\|_{L^2(0,L)} + \|\boldsymbol{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right),$$

for some positive constant C_2 which depends only on T and $\|a\|_{\mathcal{Z}_T}$. In addition, the solution y possesses the following sharp trace estimates

$$\sum_{j=0}^{2}\left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T))} \leq C_2\left(\|y_0\|_{L^2(0,L)} + \|\boldsymbol{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right).$$

In particular, the map $(y_0, \boldsymbol{h}, f) \mapsto y$ is Lipschitz continuous.

Proof: Let $y_0 \in L^2(0,L)$, $\mathbf{h} \in \mathcal{H}_T$ and $f \in L^1(0,T;L^2(0,L))$ be given. Consider θ satisfying $0 < \theta \leq T$ and define the map $\Gamma : \mathcal{Z}_\theta \to \mathcal{Z}_\theta$ in the next way: for $y \in \mathcal{Z}_\theta$, put Γy being the solution of

$$\begin{cases} u_t + u_{xxx} = f - (ay)_x, & \text{in } (0,L)\times(0,\theta), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,\theta), \\ u(x,0) = y_0(x), & \text{in } (0,L). \end{cases}$$

Consider the set

$$B = \{y \in \mathcal{Z}_\theta;\ \|y\|_{\mathcal{Z}_\theta} \leq r\},$$

with $r > 0$ to be determined later. From Proposition 2.1 and Lemma 2.1 we have for any $y \in B$ the following estimate

$$\begin{aligned}\|\Gamma y\|_{\mathcal{Z}_\theta} \leq & C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))} + \|(ay)_x\|_{L^1(0,\theta;L^2(0,L))}\right) \\ \leq & C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right) \\ & + 2C_1C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right)\|a\|_{\mathcal{Z}_\theta}\|y\|_{\mathcal{Z}_\theta} \\ \leq & C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right) \\ & + 2C_1C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right)\|a\|_{\mathcal{Z}_T}\|y\|_{\mathcal{Z}_\theta}.\end{aligned} \tag{2.8}$$

Choosing

$$r = 2C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right)$$

and θ satisfying

$$2C_1C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right)\|a\|_{\mathcal{Z}_T} < \frac{1}{2}, \tag{2.9}$$

the inequality (2.8) give us

$$\|\Gamma y\|_{\mathcal{Z}_\theta} \leq \frac{r}{2} + \frac{r}{2} = r,$$

that is, $\Gamma(B) \subset B$. Furthermore, $\Gamma y - \Gamma w$ solves

$$\begin{cases} u_t + u_{xxx} = [a(-y+w)]_x, & \text{in } (0,L)\times(0,\theta), \\ u_{xx}(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & \text{in } (0,\theta), \\ u(x,0) = 0, & \text{in } (0,L). \end{cases}$$

So, from Proposition 2.1, Lemma 2.1 and inequality (2.9), we have

$$\begin{aligned}\|\Gamma y - \Gamma w\|_{\mathcal{Z}_\theta} &\leq C_1\|[a(y-w)]_x\|_{L^1(0,\theta;L^2(0,L))} \\ &\leq 2C_1C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right)\|a\|_{\mathcal{Z}_\theta}\|y-w\|_{\mathcal{Z}_\theta} \\ &\leq 2C_1C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right)\|a\|_{\mathcal{Z}_T}\|y-w\|_{\mathcal{Z}_\theta} \\ &< \frac{1}{2}\|y-w\|_{\mathcal{Z}_\theta}.\end{aligned}$$

Thus, $\Gamma : B \to B$ is a contraction so that, by Banach's fixed point theorem, Γ has a fixed point $y \in B$ which is a solution to the problem (2.7) in $[0, \theta]$, corresponding to data $(y_0, \mathbf{h}, f)$. Additionally, inequalities (2.8) and (2.9) yields that

$$\|y\|_{\mathcal{Z}_\theta} \leq C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))} \right) + \frac{1}{2} \|y\|_{\mathcal{Z}_\theta}$$

and, therefore,

$$\|y\|_{\mathcal{Z}_\theta} \leq 2C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))} \right).$$

Since θ depends only on a, with a standard continuation extension argument, the solution y can be extended to interval $[0, T]$ and the following estimate holds

$$\|y\|_{\mathcal{Z}_T} \leq C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))} \right), \tag{2.10}$$

for some suitable constant $C_2 > 0$ which only depends on T and $\|a\|_{\mathcal{Z}_T}$. Therefore, it follows from (2.10) that the map $(y_0, \mathbf{h}, f) \mapsto y$ is Lipschitz continuous and, as a consequence of this, we have the uniqueness of the solution y in $\mathcal{Z}_T$. The sharp trace estimates follow as a consequence of the Proposition 2.1, showing the proposition. □

Now, we will study the well-posedness of the nonlinear problem, namely:

$$\begin{cases} y_t + (ay)_x + y_{xxx} + yy_x = f, & \text{in } (0,L) \times (0,T), \\ y_{xx}(0,t) = h_1(t), \ y_x(L,t) = h_2(t), \ y_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,L). \end{cases} \tag{2.11}$$

The result can be read as follows.

Proposition 2.3 *Let $T > 0$, $a \in \mathcal{Z}_T$ and $\lambda > 0$ be given. For every $y_0 \in L^2(0,L)$, $\boldsymbol{h} = (h_1, h_2, h_3) \in \mathcal{H}_T$ and $f \in L^1(0,T,L^2(0,L))$ satisfying*

$$\|y_0\|_{L^2(0,L)} + \|\boldsymbol{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))} < \lambda,$$

there exists $T^* \in (0,T]$ (denpending only on λ) and a unique mild solution $y \in \mathcal{Z}_{T^*}$ of (2.11) which possesses the hidden regularities

$$\partial_x^j y \in L_x^\infty(0,L; H^{(1-j)/3}(0,T^*)), \ j = 0,1,2.$$

The corresponding solution map $(y_0, \boldsymbol{h}, f) \mapsto S(y_0, \boldsymbol{h}, f)$ is Lipschitz continuous, that is, there exists a positive constant $L = L(\lambda)$ such that, for every

$$(y_0, \boldsymbol{h}, f_0), (y_1, \boldsymbol{g}, f_1) \in L^2(0,L) \times \mathcal{H}_T \times L^1(0,T;L^2(0,L))$$

with

$$\|y_0\|_{L^2(0,L)} + \|\boldsymbol{h}\|_{\mathcal{H}_T} + \|f_0\|_{L^1(0,T;L^2(0,L))} < \lambda,$$
$$\|y_1\|_{L^2(0,L)} + \|\boldsymbol{g}\|_{\mathcal{H}_T} + \|f_1\|_{L^1(0,T;L^2(0,L))} < \lambda,$$

we have

$$\begin{aligned}\|S(y_0,\boldsymbol{h},f_0) - S(y_1,\boldsymbol{g},f_1)\|_{\mathcal{Z}_{T^*}} \leq & L_\lambda \Big(\|y_0 - y_1\|_{L^2(0,L)} \\ & + \|\boldsymbol{h} - \boldsymbol{g}\|_{\mathcal{H}_{T^*}} + \|f_0 - f_1\|_{L^1(0,T^*;L^2(0,L))}\Big).\end{aligned}$$

In addition,

$$\sum_{j=0}^{2} \left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T^*))} \leq C_3 \left(\|y_0\|_{L^2(0,L)} + \|\boldsymbol{h}\|_{\mathcal{H}_{T^*}} + \|f\|_{L^1(0,T^*;L^2(0,L))}\right)$$

for some constant $C_3 > 0$.

Proof: We will proceed as follows: First, we will show the existence of a solution and obtain the desired estimates. Secondly, we will get an estimate that provides the uniqueness of the solution and guarantees that S is locally Lipschitz continuous.

To do that, consider $y_0 \in L^2(0,L)$, $\mathbf{h} \in \mathcal{H}_T$ and $f \in L^1(0,T;L^2(0,L))$ with

$$\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_T} + \|f_0\|_{L^1(0,T;L^2(0,L))} < \lambda.$$

Let θ satisfying $0 < \theta \leq T$ and define the map $\Gamma : \mathcal{Z}_\theta \to \mathcal{Z}_\theta$ in the following way: For $y \in \mathcal{Z}_\theta$, pick Γy as solution of the following problem

$$\begin{cases} u_t + (au)_x + u_{xxx} = f - yy_x, & \text{in } (0,L)\times(0,\theta), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,\theta), \\ u(x,0) = y_0(x), & \text{in } (0,L). \end{cases}$$

Consider the set

$$B = \{y \in \mathcal{Z}_\theta;\ \|y\|_{\mathcal{Z}_\theta} \leq r\},$$

with $r > 0$ to be determined later. Using Proposition 2.2 and Lemma 2.1 we obtain, for any $y \in B$,

$$\begin{aligned}\|\Gamma y\|_{\mathcal{Z}_\theta} &\leq C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))} + \|yy_x\|_{L^1(0,\theta;L^2(0,L))}\right) \\ &\leq C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right) + C_2 C \left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) \|y\|_{\mathcal{Z}_T}^2 \qquad (2.12) \\ &\leq C_2\lambda + C_2 C \left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) \|y\|_{\mathcal{Z}_T}^2.\end{aligned}$$

Choosing

$$r = 4C_2\lambda$$

and θ satisfying

$$C_2C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) r < \frac{1}{4}, \tag{2.13}$$

inequality (2.12) give us

$$\|\Gamma\|_{\mathcal{Z}_T} \leq \frac{r}{4} + \frac{r}{4} = \frac{r}{2} < r,$$

that is, $\Gamma(B) \subset B$. Moreover given $y, w \in B$, $\Gamma y - \Gamma w$ solves

$$\begin{cases} u_t + (au)_x + u_{xxx} = -yy_x + ww_x, & \text{in } (0,L)\times(0,\theta), \\ u_{xx}(0,t) = 0,\ u_x(L,t) = 0,\ u_{xx}(L,t) = 0, & \text{in } (0,\theta), \\ u(x,0) = 0, & \text{in } (0,L). \end{cases}$$

Therefore, from Proposition 2.2,

$$\|\Gamma y - \Gamma w\|_{\mathcal{Z}_\theta} \leq C_2\|yy_x - ww_x\|_{L^1(0,\theta;L^2(0,L))}.$$

Note that

$$yy_x - ww_x = \frac{1}{2}\Big[(y+w)_x(y-w) + (y+w)(y-w)_x\Big].$$

Then, thanks to the Lemma 2.1, we get that

$$\begin{aligned} &\|yy_x - ww_x\|_{L^1(0,\theta;L^2(0,L))} \\ &\leq \frac{1}{2}\Big[\|(y+w)_x(y-w)\|_{L^1(0,\theta;L^2(0,L))} + \|(y+w)(y-w)_x\|_{L^1(0,\theta;L^2(0,L))}\Big] \\ &\leq \frac{1}{2}\left[C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right)\|y+w\|_{\mathcal{Z}_\theta}\|y-w\|_{\mathcal{Z}_\theta} + C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right)\|y+w\|_{\mathcal{Z}_\theta}\|y-w\|_{\mathcal{Z}_\theta}\right] \\ &= C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right)\|y+w\|_{\mathcal{Z}_\theta}\|y-w\|_{\mathcal{Z}_\theta} \\ &\leq 2C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) r\|y-w\|_{\mathcal{Z}_\theta} \end{aligned}$$

and, using (2.13) yields

$$\|\Gamma y - \Gamma w\|_{\mathcal{Z}_\theta} \leq 2C_2C\left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) r\|y-w\|_{\mathcal{Z}_\theta} < \frac{1}{2}\|y-w\|_{\mathcal{Z}_\theta}.$$

Hence, $\Gamma : B \to B$ is a contraction so that, by Banach's fixed point theorem, Γ has a fixed point $y \in B$ which is a solution to the problem (2.11) in $[0,\theta]$, corresponding to data $(y_0, \mathbf{h}, f)$. Due to the inequalities (2.12) and (2.13), we have

$$\|y\|_{\mathcal{Z}_\theta} \le C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right) + \frac{1}{4}\|y\|_{\mathcal{Z}_\theta}.$$

So we can write

$$\|y\|_{\mathcal{Z}_\theta} \le \tilde{C}_3 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right), \tag{2.14}$$

for some suitable constant $\tilde{C}_3 > 0$ which only depends on T and a.

Using one more time the Proposition 2.2 together with Lemma 2.1, we obtain

$$\begin{aligned}
&\sum_{j=0}^{2} \left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,\theta))} \\
&\le C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))} + \|yy_x\|_{L^1(0,\theta;L^2(0,L))}\right) \\
&\le C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right) + C_2 C \left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) \|y\|_{\mathcal{Z}_\theta}^2.
\end{aligned}$$

Since $y \in B$ we have

$$C_2 C \left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) \|y\|_{\mathcal{Z}_\theta}^2 \le C_2 C \left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) r \|y\|_{\mathcal{Z}_\theta}$$

and from (2.13) we get

$$\sum_{j=0}^{2} \left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,\theta))} \le C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right) + \frac{1}{4}\|y\|_{\mathcal{Z}_\theta}.$$

Thus, by (2.14) it follows that

$$\sum_{j=0}^{2} \left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,\theta))} \le C_3 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right),$$

where $C_3 := C_2 + \tilde{C}_3/4$.

Now, consider

$$(y_0, \mathbf{h}, f_0), (y_1, \mathbf{g}, f_1) \in L^2(0,L) \times \mathcal{H}_T \times L^1(0,T;L^2(0,L))$$

with

$$\begin{aligned}
\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_T} + \|f_0\|_{L^1(0,T;L^2(0,L))} < \lambda, \\
\|y_1\|_{L^2(0,L)} + \|\mathbf{g}\|_{\mathcal{H}_T} + \|f_1\|_{L^1(0,T;L^2(0,L))} < \lambda.
\end{aligned}$$

Write **h** $= (h_1, h_2, h_3)$ and **g** $= (g_1, g_2, g_3)$. Let y and u be solutions of

$$\begin{cases} y_t + (ay)_x + y_{xxx} + yy_x = f_0, & \text{in } (0,L)\times(0,\theta), \\ y_{xx}(0,t) = h_1(t),\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = h_3(t), & \text{in } (0,\theta), \\ y(x,0) = y_0(x) & \text{in } (0,L), \end{cases}$$

and

$$\begin{cases} u_t + (au)_x + u_{xxx} + uu_x = f_1, & \text{in } (0,L)\times(0,\theta), \\ u_{xx}(0,t) = g_1(t),\ u_x(L,t) = g_2(t),\ u_{xx}(L,t) = g_3(t), & \text{in } (0,\theta), \\ u(x,0) = y_1(x) & \text{in } (0,L), \end{cases}$$

respectively. Then, $w = y - u$ solves the problem

$$\begin{cases} w_t + \left[\left(a + \frac{1}{2}(y+u)\right) w\right]_x + w_{xxx} = f_0 - f_1, & \text{in } (0,L)\times(0,\theta,) \\ w_{xx}(0,\cdot) = h_1 - g_1,\ w_x(L,\cdot) = h_2 - g_2,\ w_{xx}(L,\cdot) = h_3 - g_3, & \text{in } (0,\theta), \\ w(x,0) = y_0(x) - y_1(x), & \text{in } (0,L). \end{cases}$$

From Proposition 2.2 it follows that

$$\|y-u\|_{\mathcal{Z}_\theta} \leq D\left(\|y_0 - y_1\|_{L^2(0,L)} + \|\mathbf{h}-\mathbf{g}\|_{\mathcal{H}_\theta} + \|f_0 - f_1\|_{L^1(0,\theta;L^2(0,L))}\right), \tag{2.15}$$

where D is a constant which depends on θ and $\|a + \frac{1}{2}(y+u)\|_{\mathcal{Z}_T}$. But, using (2.14) we have

$$\begin{aligned} \|a + \frac{1}{2}(y+u)\|_{\mathcal{Z}_T} \leq & \|a\|_{\mathcal{Z}_T} + \frac{1}{2}\left(\|y\|_{\mathcal{Z}_T} + \|u\|_{\mathcal{Z}_T}\right) \\ \leq & \|a\|_{\mathcal{Z}_T} + \frac{\tilde{C}_3}{2}\left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_T} + \|f_0\|_{L^1(0,T;L^2(0,L))}\right) \\ & + \frac{\tilde{C}_3}{2}\left(\|y_1\|_{L^2(0,L)} + \|\mathbf{g}\|_{\mathcal{H}_T} + \|f_1\|_{L^1(0,T;L^2(0,L))}\right) \end{aligned}$$

and, consequently,

$$\|a + \frac{1}{2}(y+u)\|_{\mathcal{Z}_T} \leq \|a\|_{\mathcal{Z}_T} + \tilde{C}_3\lambda.$$

Hence, the constant D can be chosen depending only on θ and $\|a\|_{\mathcal{Z}_T}$ (and also on λ), so that, (2.15) gives us the uniqueness of the solution which turns the solution map S well-defined. Moreover, from (2.15)

$$\begin{aligned} \|S(y_0, \mathbf{h}, f_0) - S(y_1, \mathbf{g}, f_1)\|_{\mathcal{Z}_\theta} \leq & D\left(\|y_0 - y_1\|_{L^2(0,L)}\right. \\ & \left. + \|\mathbf{h}-\mathbf{g}\|_{\mathcal{H}_T} + \|f_0 - f_1\|_{L^1(0,T;L^2(0,L))}\right), \end{aligned}$$

which concludes the proof. □

Finally, the following Lemma, whose proof can be found in (ROSIER, 1997), will be useful in the next section.

Lemma 2.2 (Rosier, 1997) *If $y \in \mathcal{Z}_T$ then $yy_x \in L^1(0,T;L^2(0,L))$ and the map*

$$\begin{aligned} \mathcal{Z}_T &\longrightarrow L^1(0,T;L^2(0,L)) \\ y &\longmapsto yy_x, \end{aligned}$$

is continuous. More precisely, for every $y, z \in \mathcal{Z}_T$ we have that

$$\begin{aligned} \|yy_x - zz_x\|_{L^1(0,T;L^2(0,L))} \leq & C_4 \Big(\|y\|_{L^2(0,T;H^1(0,L))} \\ & + \|z\|_{L^2(0,T;H^1(0,L))} \Big) \|y - z\|_{L^2(0,T;H^1(0,L))}, \end{aligned}$$

where C_4 is a positive constant that depends only on L.

Remark 2.2 *We end this section with the following remarks.*

1. *For every $a \in \mathcal{Z}_T$, the Proposition 2.2 give us the well-definition for the solution operator*

$$\Lambda_a : L^2(0,L) \times \mathcal{H}_T \times L^1(0,T;L^2(0,L)) \to \mathcal{Z}_T$$

where, for each $(y_0, \boldsymbol{h}, f) \in L^2(0,L) \times \mathcal{H}_T \times L^1(0,T;L^2(0,L))$, $\Lambda_a(y_0, \boldsymbol{h}, f)$ is the corresponding solution to the problem (2.7). *Furthermore, the Proposition 2.3 guarantees that Λ_a is a bounded linear operator.*

2. *Of course, the constant $C_2 > 0$ in the Proposition 2.2 depends on $\|a\|_{\mathcal{Z}_T}$. However, given $M > 0$, the same constant C_2 can be used for every $a \in \mathcal{Z}_T$ with $\|a\|_{\mathcal{Z}_T} \leq M$.*

2.4 BOUNDARY CONTROLLABILITY IN THE CRITICAL LENGTH

In this section, we study the controllability of the system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & \text{in } (0,L) \times (0,T), \\ y_{xx}(0,t) = 0, \ y_x(L,t) = h(t), \ y_{xx}(L,t) = 0, & \text{in } (0,T), \\ y(x,0) = y_0(x) & \text{in } (0,L), \end{cases} \tag{2.16}$$

around $c \neq -1$ when L is a critical length, that is, L belongs to the set

$$\mathcal{R}_c := \left\{ \frac{2\pi}{\sqrt{3(c+1)}} \sqrt{m^2 + ml + l^2} \ ; \ m, l \in \mathbb{N}^* \right\} \cup \left\{ \frac{m\pi}{\sqrt{c+1}}; \ m \in \mathbb{N}^* \right\}.$$

We will use the return method together with the fixed point argument to ensure the controllability of the system (2.16) when $L \in \mathcal{R}_c$. Before presenting the proof of the main result, let us give a preliminary result that is important in our analysis.

2.4.1 An auxiliary result

As mentioned earlier, the proof of the main result in this chapter is based on the return method, an approach in which we rely on the already known controllability of some system to achieve the controllability of the desired nonlinear system. The initial idea is that this auxiliary system is the linearization of the nonlinear system itself around one of its trajectories. However, it may happen that these linearizations are not useful, so that, to try to apply this method, we have to rely on systems that are not necessarily linearizations of the one studied. See, for instance (CORON, 1993).

In our case, we want to study the controllability of (2.16) around an equilibrium $y \equiv c$ when $c \neq -1$ and $L \in \mathcal{R}_c$. Hence, the natural system to think of as auxiliary is the linearization of (2.16) around this equilibrium, namely,

$$\begin{cases} y_t + (1+c)y_x + y_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ y_{xx}(0,t) = 0,\ y_x(L,t) = h(t),\ y_{xx}(L,t) = 0, & \text{in } (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,L), \end{cases} \tag{2.17}$$

But as seen before, when $L \in \mathcal{R}_c$, the linear system (2.17) is not exactly controllable. So, in this section, we first give the proof of Theorem 2.6, which, among other things, provides an idea of systems on which we can rely to apply the return method.

Proof of Theorem 2.6: Let $c \neq -1$ and $L \in \mathcal{R}_c$. We will split the proof into two cases.

First case: $L = 2\pi\frac{\sqrt{j^2+jk+k^2}}{\sqrt{3(c+1)}}$ for some $(j,k) \in \mathbb{N}^* \times \mathbb{N}^*$.

Consider $d \neq -1$ and assume that $L \in \mathcal{R}_d$. Then

$$L = 2\pi\frac{\sqrt{m^2+ml+l^2}}{\sqrt{3(d+1)}};\ m,l \in \mathbb{N}^* \tag{2.18}$$

or

$$L = \frac{m\pi}{\sqrt{d+1}};\ m \in \mathbb{N}^*. \tag{2.19}$$

If (2.18) is the case, we have that

$$2\pi\frac{\sqrt{j^2+jk+k^2}}{\sqrt{3(c+1)}} = 2\pi\frac{\sqrt{m^2+ml+l^2}}{\sqrt{3(d+1)}},$$

which implies

$$d = \frac{(c+1)(m^2+ml+l^2)}{j^2+jk+k^2} - 1$$

with $m, l \in \mathbb{N}^*$.

Otherwise, if (2.19) holds, so

$$2\pi \frac{\sqrt{j^2+jk+k^2}}{\sqrt{3(c+1)}} = \frac{m\pi}{\sqrt{d+1}}$$

giving that

$$d = \frac{3m^2(c+1)}{4(j^2+jk+k^2)} - 1$$

where $m \in \mathbb{N}^*$. Therefore, if $L \in \mathcal{R}_d$ then we necessarily have $d \in \mathcal{A}_1 \cup \mathcal{B}_1$. Here,

$$\mathcal{A}_1 := \left\{ \frac{(c+1)(m^2+ml+l^2)}{j^2+jk+k^2} - 1,\ m, l \in \mathbb{N}^* \right\}$$

and

$$\mathcal{B}_1 := \left\{ \frac{3m^2(c+1)}{4(j^2+jk+k^2)} - 1;\ m \in \mathbb{N}^* \right\}.$$

We are now in a position to prove that $\mathcal{A}_1 \cup \mathcal{B}_1$ is discrete. To do that, consider $x, y \in \mathcal{A}_1$ such that $x \neq y$ in the form

$$x = \frac{(c+1)(m_1^2+m_1l_1+l_1^2)}{j^2+jk+k^2} - 1$$

and

$$y = \frac{(c+1)(m_2^2+m_2l_2+l_2^2)}{j^2+jk+k^2} - 1,$$

with $m_1, l_1, m_2, l_2 \in \mathbb{N}^*$. Note that

$$x - y = \frac{c+1}{j^2+jk+k^2} \left[(m_1^2+m_1l_1+l_1^2) - (m_2^2+m_2l_2+l_2^2) \right].$$

Since $x \neq y$ we have

$$(m_1^2+m_1l_1+l_1^2) \neq (m_2^2+m_2l_2+l_2^2).$$

Thus

$$\left| (m_1^2+m_1l_1+l_1^2) - (m_2^2+m_2l_2+l_2^2) \right| \geq 1$$

so that

$$d(x,y) \geq \frac{c+1}{j^2+jk+k^2},\ \forall x, y \in \mathcal{A}_1,\ x \neq y.$$

Analogously, we get

$$d(x,y) \geq \frac{3(c+1)}{4(j^2+jk+k^2)},\ \forall\ x, y \in \mathcal{B}_1, x \neq y.$$

Now, let $x \in \mathcal{A}_1$ and $y \in \mathcal{B}_1$ with $x \neq y$, as follow:

$$x = \frac{(c+1)(m^2+ml+l^2)}{j^2+jk+k^2} - 1$$

and

$$y = \frac{3p^2(c+1)}{4(j^2+jk+k^2)} - 1,$$

where $m, l, p \in \mathbb{N}^*$. Observe that

$$\begin{aligned} x - y =& \frac{(c+1)(m^2+ml+l^2)}{j^2+jk+k^2} - \frac{3p^2(c+1)}{4(j^2+jk+k^2)} \\ =& \frac{4(c+1)(m^2+ml+l^2)}{4(j^2+jk+k^2)} - \frac{3p^2(c+1)}{4(j^2+jk+k^2)} \\ =& \frac{c+1}{4(j^2+jk+k^2)}\left[4(m^2+ml+l^2) - 3p^2\right]. \end{aligned}$$

Since $x \neq y$ we have that $4(m^2+ml+l^2)$ and $3p^2$ are distinct natural numbers so

$$\left|4(m^2+ml+l^2) - 3p^2\right| \geq 1$$

and, consequently, we have

$$d(x,y) \geq \frac{c+1}{4(j^2+jk+k^2)}.$$

From all the above, we conclude that

$$d(x,y) \geq \frac{c+1}{4(j^2+jk+k^2)}, \ \forall\, x, y \in \mathcal{A}_1 \cup \mathcal{B}_1, \ x \neq y,$$

which implies that $\mathcal{A}_1 \cup \mathcal{B}_1$ is discrete.

Note that $c \in \mathcal{A}_1 \cup \mathcal{B}_1$ since

$$c = \left[\frac{(c+1)(j^2+jk+k^2)}{j^2+jk+k^2} - 1\right] \in \mathcal{A}_1.$$

As any point in $\mathcal{A}_1 \cup \mathcal{B}_1$ is isolated, there exists $\epsilon_1 > 0$ such that

$$(c-\epsilon_1, c+\epsilon_1) \cap [\mathcal{A}_1 \cup \mathcal{B}_1] = \{c\}.$$

Therefore for $d \in (c-\epsilon_1, c+\epsilon_1)\backslash\{c\}$ we have that $d \notin \mathcal{A}_1 \cup \mathcal{B}_1$ and, therefore, $L \notin \mathcal{R}_d$.

Second case: $L = \frac{k\pi}{\sqrt{c+1}}$ for some $k \in \mathbb{N}^*$.

Let $d \neq -1$ and suppose that $L \in \mathcal{R}_d$, that is, (2.18) or (2.19) holds. If (2.18) holds, then

$$\frac{k\pi}{\sqrt{c+1}} = 2\pi\frac{\sqrt{m^2+ml+l^2}}{\sqrt{3(d+1)}}$$

giving that

$$d = \frac{(c+1)(m^2+ml+l^2)}{3k^2} - 1,$$

with $m, l \in \mathbb{N}^*$. If (2.19) is the case, thus

$$\frac{k\pi}{\sqrt{c+1}} = \frac{m\pi}{\sqrt{d+1}}$$

so that

$$d = \frac{(c+1)m^2}{k^2} - 1.$$

Therefore, if $L \in \mathcal{R}_d$ then we have necessarily $d \in \mathcal{A}_2 \cup \mathcal{B}_2$ where

$$\mathcal{A}_2 := \left\{ \frac{(c+1)(m^2+ml+l^2)}{3k^2} - 1;\ m, l \in \mathbb{N}^* \right\}$$

and

$$\mathcal{B}_2 := \left\{ \frac{(c+1)m^2}{k^2} - 1;\ m \in \mathbb{N}^* \right\}.$$

As done before, taking $x, y \in \mathcal{A}_2$ with $x \neq y$, such that

$$x = \frac{(c+1)(m_1^2+m_1l_1+l_1^2)}{3k^2} - 1$$

and

$$y = \frac{(c+1)(m_2^2+m_2l_2+l_2^2)}{3k^2} - 1,$$

where $m_1, l_1, m_2, l_2 \in \mathbb{N}^*$,yields that

$$x - y = \frac{c+1}{3k^2}\left[(m_1^2+m_1l_1+l_1^2) - (m_2^2+m_2l_2+l_2^2)\right]$$

and, as $x \neq y$ we have

$$\left|(m_1^2+m_1l_1+l_1^2) - (m_2^2+m_2l_2+l_2^2)\right| \geq 1.$$

Consequently,

$$d(x,y) \geq \frac{c+1}{3k^2}, \forall x, y \in \mathcal{A}_2,\ x \neq y.$$

In an analogous way,

$$d(x,y) \geq \frac{c+1}{k^2},\ \forall x, y \in \mathcal{B}_2,\ x \neq y$$

and

$$d(x,y) \geq \frac{c+1}{3k^2},\ \forall x \in \mathcal{A}_2,\ y \in \mathcal{B}_2,\ x \neq y.$$

Therefore,

$$d(x,y) \geq \frac{c+1}{3k^2},\ \forall x, y \in \mathcal{A}_2 \cup \mathcal{B}_2,\ x \neq y.$$

Proceeding as in the first case, we conclude that there exists $\epsilon_2 > 0$ such that, for every $d \in (c-\epsilon_2, c+\epsilon_2)\backslash\{c\}$ we have $d \notin \mathcal{A}_2 \cup \mathcal{B}_2$ so that $L \notin \mathcal{R}_d$. Considering $\epsilon_c := \min\{\epsilon_1, \epsilon_2\}$, thanks thanks to the (CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017, Proposition 3.6) and Theorem 2.3, the proof is completed. □

2.4.2 Construction of the trajectories

In this subsection, for simplicity, we will consider $c = 0$. Given $T > 0$ and $L \in \mathcal{R}_c$, note that by Theorem 2.6 there exist $\epsilon_0 > 0$ such that for every $d \in (0, \epsilon_0)$, the system (2.17) is exactly controllable around d. Henceforth, we use C, C_1, C_2 and C_3 to denote the positive constants given in Proposition 2.3, corresponding to T. One can see that, for every $\tau \in [0, T]$, the same constants can be used to apply the corresponding results for $\|\cdot\|_{\mathcal{Z}_\tau}$. To simplify some notations, we will define $\tau = \frac{T}{3}$.

The first result of this subsection ensures that we can construct solutions for the system (2.16) which starts close to 0 (left-hand side of the spatial domain) and achieves some non-null equilibrium in a certain time $T/3$. The result is shown by a fixed-point argument.

Proposition 2.4 *There exist $\delta_1 > 0$ such that, for every $d \in (0, \delta_1)$ and $y_0 \in L^2(0, L)$ with $\|y_0\|_{L^2(0,L)} < \delta_1$, there exists $h_1 \in L^2(0, T/3)$ such that, the solution of (2.16) for $t \in [0, T/3]$, satisfies*

$$y(\cdot, T/3) = d.$$

Proof: Let $\delta_1 \in (0, \epsilon_0)$ be a number to be chosen later. Consider $d \in (0, \delta_1)$ and $y_0 \in L^2(0, L)$ satisfying

$$\|y_0\|_{L^2(0,L)} < \delta_1.$$

For $\varepsilon \in (0, \epsilon_0)$ such that

$$C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|\varepsilon\|_{\mathcal{Z}_T} < \delta_1 \quad \text{and} \quad C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|\varepsilon\|^2_{\mathcal{Z}_T} < \delta_1, \tag{2.20}$$

where $C > 0$ is the positive constant given in Lemma 2.1, (CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017, Proposition 3.6) guarantees the existence of a bounded linear operator

$$\Psi^\varepsilon : L^2(0, L) \times L^2(0, L) \to L^2(0, \tau)$$

such that, for any $u_0, u_\tau \in L^2(0,L)$, the solution u of

$$\begin{cases} u_t + (1+\varepsilon)u_x + u_{xxx} = 0, & \text{in } (0,L)\times(0,\tau), \\ u_{xx}(0,t) = 0,\ u_x(L,t) = h(t),\ u_{xx}(L,t) = 0, & \text{in } (0,\tau), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases}$$

with the control $h = \Psi^\varepsilon(u_0, u_\tau)$ satisfies $u(\cdot,\tau) = u_\tau$.

We will denote, for simplicity, the operator $\Lambda_{1+\varepsilon}$ (given in Proposition 2.2 and Remark 2.2 with $a = 1+\varepsilon$) by Λ_ε. Observe that, if y is solution for (2.16) for some control h, then y is a solution of

$$\begin{cases} y_t + (1+\varepsilon)y_x + y_{xxx} = -yy_x + \varepsilon y_x, & \text{in } (0,L)\times(0,\tau), \\ y_{xx}(0,t) = 0,\ y_x(L,t) = h(t),\ y_{xx}(L,t) = 0, & \text{in } (0,\tau), \\ y(x,0) = y_0(x) & \text{in } (0,L), \end{cases}$$

that is,

$$y = \Lambda_\varepsilon(y_0, h, -yy_x + \varepsilon y_x) = \Lambda_\varepsilon(y_0, h, 0) + \Lambda_\varepsilon(0,0,-yy_x + \varepsilon y_x). \tag{2.21}$$

Let $y \in \mathcal{Z}_\tau$ and $h_y \in L^2(0,\tau)$ given by

$$h_y = \Psi^\varepsilon\Big(y_0, d - \Lambda_\varepsilon(0,0,-yy_x + \varepsilon y_x)(\cdot,\tau)\Big).$$

Define the map $\Gamma : \mathcal{Z}_\tau \to \mathcal{Z}_\tau$ by

$$\Gamma y = \Lambda_\varepsilon(y_0, h_y, 0) + \Lambda_\varepsilon(0,0,-yy_x + \varepsilon y_x).$$

Note that if Γ has a fixed point y, then, from the above construction, it follows that y is a solution of (2.16) with the control h_y. Moreover, from (2.21) we have

$$y = \Lambda_\varepsilon(y_0, h_y, 0) + \Lambda_\varepsilon(0,0,-yy_x + \varepsilon y_x)$$

so, by definitions of $\Lambda_\varepsilon,\ h_y$ and Ψ^ε, we get that

$$\begin{aligned} y(\cdot,\tau) &= \Lambda_\varepsilon(y_0, h_y, 0)(\cdot,\tau) + \Lambda_\varepsilon(0,0,-yy_x + \varepsilon y_x)(\cdot,\tau) \\ &= d - \Lambda_\varepsilon(0,0,-yy_x + \varepsilon y_x)(\cdot,\tau) + \Lambda_\varepsilon(0,0,-yy_x + \varepsilon y_x)(\cdot,\tau) \\ &= d, \end{aligned}$$

and our problem would be solved. So we will focus our efforts on showing that Γ has a fixed point in a suitable metric space.

To do that, let B be the set

$$B = \{y \in \mathcal{Z}_\tau; \ \|y\|_{\mathcal{Z}_\tau} \leq r\},$$

with $r > 0$ to be chosen later. By (2.21) and Proposition 2.2 (together with Remark 2.2) we have, for $y \in B$, that

$$\begin{aligned}
\|\Gamma y\|_{\mathcal{Z}_\tau} &\leq \|\Lambda_\varepsilon(y_0, h_y, 0)\|_{\mathcal{Z}_\tau} + \|\Lambda_\varepsilon(0,0,-yy_x + \varepsilon y_x)\|_{\mathcal{Z}_\tau} \\
&\leq C_2 \left(\|y_0\|_{L^2(0,L)} + \|h_y\|_{L^2(0,\tau)} + \| - yy_x + \varepsilon y_x\|_{L^1(0,\tau;L^2(0,L))}\right).
\end{aligned}$$

From Lemma 2.1 and Young's inequality, we ensure that

$$\begin{aligned}
\|yy_x - \varepsilon y_x\|_{L^1(0,\tau,L^2(0,L))} &= \|(y-\varepsilon)y_x\|_{L^1(0,\tau,L^2(0,L))} \\
&\leq C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|y-\varepsilon\|_{\mathcal{Z}_\tau}\|y\|_{\mathcal{Z}_\tau} \\
&\leq C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\frac{1}{2}\left(\|y-\varepsilon\|^2_{\mathcal{Z}_\tau} + \|y\|^2_{\mathcal{Z}_\tau}\right) \\
&\leq C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\frac{1}{2}\left[(\|y\|_{\mathcal{Z}_\tau} + \|\varepsilon\|_{\mathcal{Z}_\tau})^2 + \|y\|^2_{\mathcal{Z}_\tau}\right] \\
&\leq C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\frac{1}{2}\left(\|y\|^2_{\mathcal{Z}_\tau} + 2\|y\|_{\mathcal{Z}_\tau}\|\varepsilon\|_{\mathcal{Z}_\tau} + \|\varepsilon\|^2_{\mathcal{Z}_\tau} + \|y\|^2_{\mathcal{Z}_\tau}\right) \\
&\leq C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\frac{1}{2}\left(\|y\|^2_{\mathcal{Z}_\tau} + \|y\|^2_{\mathcal{Z}_\tau} + \|\varepsilon\|^2_{\mathcal{Z}_\tau} + \|\varepsilon\|^2_{\mathcal{Z}_\tau} + \|y\|^2_{\mathcal{Z}_\tau}\right).
\end{aligned}$$

Thus,

$$\|yy_x - \varepsilon y_x\|_{L^1(0,\tau,L^2(0,L))} \leq C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\frac{3}{2}\|y\|^2_{\mathcal{Z}_\tau} + C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|\varepsilon\|^2_{\mathcal{Z}_\tau}$$

and, by (2.20) we obtain

$$\|yy_x - \varepsilon y_x\|_{L^1(0,\tau,L^2(0,L))} \leq C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\frac{3}{2}\|y\|^2_{\mathcal{Z}_\tau} + \delta_1,$$

that is,

$$\|yy_x - \varepsilon y_x\|_{L^1(0,\tau,L^2(0,L))} \leq \overline{C}\|y\|^2_{\mathcal{Z}_\tau} + \delta_1,$$

where

$$\overline{C} := \frac{3C}{2}\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right).$$

In this way

$$\begin{aligned}\|h_y\|_{L^2(0,\tau)} &= \|\Psi^\varepsilon\Big(y_0, d-\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)\Big)\|_{L^2(0,\tau)}\\ &\leq \|\Psi^\varepsilon\| \left(\|y_0\|_{L^2(0,L)} + \|d\|_{L^2(0,L)} + \|\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)\|_{L^2(0,L)}\right)\\ &\leq \|\Psi^\varepsilon\|\delta_1 + \|\Psi^\varepsilon\| d\sqrt{L} + \|\Psi^\varepsilon\|\|\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)\|_{\mathcal{Z}_\tau}\\ &\leq \|\Psi^\varepsilon\|\delta_1 + \|\Psi^\varepsilon\|\delta_1\sqrt{L} + \|\Psi^\varepsilon\| C_2\|-yy_x+\varepsilon y_x\|_{L^1(0,\tau;L^2(0,L))}\\ &\leq \|\Psi^\varepsilon\|\delta_1 + \|\Psi^\varepsilon\|\delta_1\sqrt{L} + \|\Psi^\varepsilon\| C_2\overline{C}\|y\|^2_{\mathcal{Z}_\tau} + \|\Psi^\varepsilon\| C_2\delta_1\\ &= \left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\delta_1 + C_2\overline{C}\|\Psi^\varepsilon\| r^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Gamma y\|_{\mathcal{Z}_\tau} &\leq C_2\delta_1 + C_2\left[\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\delta_1 + C_2\overline{C}\|\Psi^\varepsilon\| r^2\right] + C_2\left(\overline{C}r^2+\delta_1\right)\\ &= \left[2C_2 + C_2\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\right]\delta_1 + \left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\overline{C}r^2.\end{aligned}$$

Choosing

$$r = 2\left[2C_2 + C_2\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\right]\delta_1$$

and δ_1 small enough such that

$$\left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\overline{C}r < \frac{1}{2}, \quad 2\left(C_2^2\|\Psi^\varepsilon\| + C_2\right) r < \frac{1}{2}, \quad \left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\delta_1 < \frac{1}{2}, \tag{2.22}$$

yields that

$$\|\Gamma\|_{\mathcal{Z}_\tau} \leq \frac{r}{2} + \frac{r}{2} = r \implies \Gamma(B) \subset B.$$

Additionally, observe that for $y, w \in B$, Proposition 2.2 give us

$$\begin{aligned}\|\Gamma y - \Gamma w\|_{\mathcal{Z}_\tau} =& \|\Lambda_\varepsilon(0, h_y - h_w, 0) + \Lambda_\varepsilon(0,0,-yy_x + ww_x + \varepsilon y_x - \varepsilon w_x)\|_{\mathcal{Z}_\tau}\\ \leq& C_2\|h_y - h_w\|_{L^2(0,\tau)} + C_2\|yy_x - ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ &+ C_2\|\varepsilon(y_x - w_x)\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

Since

$$\begin{aligned}h_y - h_w &= \Psi^\varepsilon\left(0, -\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau) + \Lambda_\varepsilon(0,0,-ww_x+\varepsilon w_x)(\cdot,\tau)\right)\\ &= \Psi^\varepsilon\left(0, \Lambda_\varepsilon(yy_x - \varepsilon y_x - ww_x + \varepsilon w_x)(\cdot,\tau)\right),\end{aligned}$$

we have again from Proposition 2.2 that

$$\begin{aligned} C_2\|h_y - h_w\|_{L^2(0,\tau)} \leq & C_2^2\|\Psi^\varepsilon\|\|yy_x - \varepsilon y_x - ww_x + \varepsilon w_x\|_{L^1(0,\tau;L^2(0,L))} \\ \leq & C_2^2\|\Psi^\varepsilon\|\|yy_x - ww_x\|_{L^1(0,\tau;L^2(0,L))} \\ & + C_2^2\|\Psi^\varepsilon\|\|\varepsilon y_x - \varepsilon w_x\|_{L^1(0,\tau;L^2(0,L))}. \end{aligned}$$

Putting these two previous inequalities together, we find that

$$\begin{aligned} \|\Gamma y - \Gamma w\|_{\mathcal{Z}_\tau} \leq & \left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\|yy_x - ww_x\|_{L^1(0,\tau;L^2(0,L))} \\ & + \left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\|\varepsilon(y_x - w_x)\|_{L^1(0,\tau;L^2(0,L))}. \end{aligned}$$

From Lemmas 2.1 and 2.2, together with the choices (2.20) and (2.22), it follows that

$$\begin{aligned} \|\Gamma y - \Gamma w\|_{\mathcal{Z}_\tau} \leq & 2\left(C_2^2\|\Psi^\varepsilon\| + C_2\right) r\|y - w\|_{\mathcal{Z}_\tau} \\ & + \left(C_2^2\|\Psi^\varepsilon\| + C_2\right) C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|\varepsilon\|_{\mathcal{Z}_\tau}\|y - w\|_{\mathcal{Z}_\tau} \\ \leq & \left[2\left(C_2^2\|\Psi^\varepsilon\| + C_2\right) r + \left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\delta_1\right]\|y - w\|_{\mathcal{Z}_\tau} \\ \leq & \|y - w\|_{\mathcal{Z}_\tau} \end{aligned}$$

Therefore, $\Gamma : B \to B$ is a contraction so that, by Banach's fixed point theorem, Γ has a fixed point $y \in B$, concluding the proof. □

The second result of this subsection ensures the construction of solutions for the system (2.16) on $[2T/3, T]$ starting in one non-null equilibrium and ending near 0.

Proposition 2.5 *There exists $\delta_2 > 0$ such that, for every $d \in (0, \delta_2)$ and $y_T \in L^2(0, L)$ satisfying $\|y_T\|_{L^2(0,L)} < \delta_2$, there exists $h_2 \in L^2(2T/3, T)$ such that, the solution of* (2.1) *for $t \in [2T/3, T]$ satisfies*

$$y(\cdot, 2T/3) = d \quad \textit{and} \quad y(\cdot, T) = y_T.$$

Proof: Let $\delta_2 \in (0, \epsilon_0)$ be a number to be chosen later. Consider $d \in (0, \delta_2)$ and $y_T \in L^2(0, L)$ satisfying

$$\|y_T\|_{L^2(0,L)} < \delta_2.$$

Let $\varepsilon \in (0, \epsilon_0)$ be such that

$$C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|\varepsilon\|_{\mathcal{Z}_T} < \delta_2 \quad \text{and} \quad C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|\varepsilon\|_{\mathcal{Z}_T}^2 < \delta_2, \tag{2.23}$$

where $C > 0$ is the positive constant given in Lemma 2.1. If z is a solution to the problem

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x = 0, & \text{in } (0,L)\times(0,\tau), \\ z_{xx}(0,t) = 0,\ z_x(L,t) = h(t),\ z_{xx}(L,t) = 0 & \text{in } (0,\tau), \\ z(x,0) = d, & \text{in } (0,L), \end{cases} \tag{2.24}$$

then z is a solution of

$$\begin{cases} z_t + (1+\varepsilon)z_x + z_{xxx} = -zz_x + \varepsilon z_x, & \text{in } (0,L)\times(0,\tau), \\ z_{xx}(0,t) = 0,\ z_x(L,t) = h(t),\ z_{xx}(L,t) = 0, & \text{in } (0,\tau), \\ z(x,0) = d, & \text{in } (0,L), \end{cases}$$

that is,

$$z = \Lambda_\varepsilon(d, h, -zz_x + \varepsilon z_x) = \Lambda_\varepsilon(d,h,0) + \Lambda_\varepsilon(0,0,-zz_x + \varepsilon z_x). \tag{2.25}$$

Given $z \in \mathcal{Z}_\tau$, let $h_z \in L^2(0,\tau)$ defined by

$$h_z = \Psi^\varepsilon\Big(d, y_T - \Lambda_\varepsilon(0,0,-zz_x + \varepsilon z_x)(\cdot,\tau)\Big).$$

Now, consider the map $\Gamma : \mathcal{Z}_\tau \to \mathcal{Z}_\tau$ given by

$$\Gamma z = \Lambda_\varepsilon(d, h_z, 0) + \Lambda_\varepsilon(0,0,-zz_x + \varepsilon z_x).$$

Once again, if Γ has a fixed point z, from the above construction, it follows that z is a solution of (2.24) with the control h_z. Moreover, from (2.25) we have

$$z = \Lambda_\varepsilon(d, h_z, 0) + \Lambda_\varepsilon(0,0,-zz_x + \varepsilon z_x)$$

so, by definitions of $\Lambda_\varepsilon,\ h_z$ and Ψ^ε, it follows that

$$z(\cdot,0) = \Lambda_\varepsilon(d,h_z,0)(\cdot,0) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,0) = d + 0 = d$$

and

$$\begin{aligned} z(\cdot,\tau) &= \Lambda_\varepsilon(d,h_z,0)(\cdot,\tau) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau) \\ &= y_T - \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau) \\ &= y_T. \end{aligned}$$

Hence our issue would be solved defining $y : [0,L]\times[2T/3,T] \to \mathbb{R}$ by

$$y(x,t) = z(x, t - 2T/3).$$

Now, our focus is to show that Γ has a fixed point in a suitable metric space. To do that, consider the set B given by

$$B = \{z \in \mathcal{Z}_\tau;\ \|z\|_{\mathcal{Z}_\tau} \leq r\},$$

with $r > 0$ to be chosen later. By (2.25) and Proposition 2.2 (together with Remark 2.2) we have, for $z \in B$, that

$$\begin{aligned}\|\Gamma z\|_{\mathcal{Z}_\tau} &\leq \|\Lambda_\varepsilon(d, h_z, 0)\|_{\mathcal{Z}_\tau} + \|\Lambda_\varepsilon(0, 0, -zz_x + \varepsilon z_x)\|_{\mathcal{Z}_\tau} \\ &\leq C_2 \left(\|d\|_{L^2(0,L)} + \|h_z\|_{L^2(0,\tau)} + \|-zz_x + \varepsilon z_x\|_{L^1(0,\tau;L^2(0,L))}\right).\end{aligned}$$

As in the proof of the Proposition 2.4,

$$\|zz_x - \varepsilon z_x\|_{L^1(0,\tau,L^2(0,L))} \leq \overline{C}\|z\|^2_{\mathcal{Z}_\tau} + \delta_2.$$

Moreover,

$$\begin{aligned}\|h_z\|_{L^2(0,\tau)} &= \|\Psi^\varepsilon\Big(d, y_T - \Lambda_\varepsilon(0,0,-zz_x + \varepsilon z_x)(\cdot,\tau)\Big)\|_{L^2(0,\tau)} \\ &\leq \|\Psi^\varepsilon\| \left(\|d\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} + \|\Lambda_\varepsilon(0,0,-zz_x + \varepsilon z_x)(\cdot,\tau)\|_{L^2(0,L)}\right) \\ &\leq \|\Psi^\varepsilon\| d\sqrt{L} + \|\Psi^\varepsilon\|\delta_2 + \|\Psi^\varepsilon\|\|\Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)\|_{\mathcal{Z}_\tau} \\ &\leq \|\Psi^\varepsilon\|\delta_2 + \|\Psi^\varepsilon\|\delta_2\sqrt{L} + \|\Psi^\varepsilon\| C_2 \|-zz_x + \varepsilon z_x\|_{L^1(0,\tau;L^2(0,L))} \\ &\leq \|\Psi^\varepsilon\|\delta_2 + \|\Psi^\varepsilon\|\delta_2\sqrt{L} + \|\Psi^\varepsilon\| C_2\overline{C}\|z\|^2_{\mathcal{Z}_\tau} + \|\Psi^\varepsilon\| C_2\delta_2 \\ &= \left(1 + \sqrt{L} + C_2\right)\|\Psi^\varepsilon\|\delta_2 + C_2\overline{C}\|\Psi^\varepsilon\| r^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Gamma z\|_{\mathcal{Z}_\tau} &\leq C_2\delta_2\sqrt{L} + C_2\left[\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\delta_2 + C_2\overline{C}\|\Psi^\varepsilon\| r^2\right] + C_2\left(\overline{C}r^2 + \delta_2\right) \\ &= \left[C_2\sqrt{L} + C_2\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\| + C_2\right]\delta_2 + \left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\overline{C}r^2.\end{aligned}$$

Choosing

$$r = 2\left[C_2\sqrt{L} + C_2\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\| + C_2\right]\delta_2$$

and δ_2 small enough such that

$$\left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\overline{C}r < \frac{1}{2}, \quad 2\left(C_2^2\|\Psi^\varepsilon\| + C_2\right) r < \frac{1}{2}, \quad \left(C_2^2\|\Psi^\varepsilon\| + C_2\right)\delta_1 < \frac{1}{2}, \tag{2.26}$$

we get that

$$\|\Gamma z\|_{\mathcal{Z}_\tau} \leq \frac{r}{2} + \frac{r}{2} = r \implies \Gamma(B) \subset B.$$

Furthermore, observe that for $z, w \in B$, Proposition 2.2 give us

$$\begin{aligned}\|\Gamma z-\Gamma w\|_{\mathcal{Z}_\tau} =&\|\Lambda_\varepsilon(0,h_z-h_w,0)+\Lambda_\varepsilon(0,0,-zz_x+ww_x+\varepsilon z_x-\varepsilon w_x)\|_{\mathcal{Z}_\tau}\\ \leq& C_2\|h_z-h_w\|_{L^2(0,\tau)}+C_2\|zz_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ &+C_2\|\varepsilon(z_x-w_x)\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

Since

$$\begin{aligned}h_z-h_w&=\Psi^\varepsilon\left(0,-\Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau)+\Lambda_\varepsilon(0,0,-ww_x+\varepsilon w_x)(\cdot,\tau)\right)\\ &=\Psi^\varepsilon\left(0,\Lambda_\varepsilon(zz_x-\varepsilon z_x-ww_x+\varepsilon w_x)(\cdot,\tau)\right),\end{aligned}$$

we have, again from Proposition 2.2, that

$$\begin{aligned}C_2\|h_z-h_w\|_{L^2(0,\tau)}\leq& C_2^2\|\Psi^\varepsilon\|\|zz_x-\varepsilon z_x-ww_x+\varepsilon w_x\|_{L^1(0,\tau;L^2(0,L))}\\ \leq& C_2^2\|\Psi^\varepsilon\|\|zz_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ &+C_2^2\|\Psi^\varepsilon\|\|\varepsilon z_x-\varepsilon w_x\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

Then,

$$\begin{aligned}\|\Gamma z-\Gamma w\|_{\mathcal{Z}_\tau}\leq&\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\|zz_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ &+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\|\varepsilon(z_x-w_x)\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

From Lemmas 2.1 and 2.2, together with (2.23) and (2.26), it follows that

$$\begin{aligned}\|\Gamma z-\Gamma w\|_{\mathcal{Z}_\tau}\leq&2\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)r\|z-w\|_{\mathcal{Z}_\tau}\\ &+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)C\left(\tau^{\frac{1}{2}}+\tau^{\frac{1}{3}}\right)\|\varepsilon\|_{\mathcal{Z}_\tau}\|z-w\|_{\mathcal{Z}_\tau}\\ \leq&\left[2\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)r+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\delta_2\right]\|z-w\|_{\mathcal{Z}_\tau}\\ \leq&\|z-w\|_{\mathcal{Z}_\tau}.\end{aligned}$$

Therefore, $\Gamma : B \to B$ is a contraction so that, by Banach's fixed point theorem, Γ has a fixed point $z \in B$, which concludes our proof. □

2.4.3 Controllability on $\mathcal{R}_c$

We are in a position to prove Theorems 2.5 and 1.1. For the sake of simplicity, we will give the proof of the case $L \in \mathcal{R}_0$ (Theorem 2.5), and the case $L \in \mathcal{R}_c$ (Theorem 1.1) follows similarly.

Proof of Theorem 2.5: Let δ_1 and δ_2 be positive real numbers given in Propositions 2.4 and 2.5, respectively. Define $\delta := \min\{\delta_1, \delta_2\}$ and consider $d \in (0, \delta)$ and $y_0, y_T \in L^2(0, L)$ such that

$$\|y_0\|_{L^2(0,L)}, \|y_T\|_{L^2(0,L)} < \delta.$$

From Proposition 2.4 there exists $h_1 \in L^2(0, T/3)$ such that, the solution $y_1 \in \mathcal{Z}_{T/3}$ of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & \text{in } (0, L) \times (0, T/3), \\ y_{xx}(0, t) = 0, \ y_x(L, t) = h_1(t), \ y_{xx}(L, t) = 0, & \text{in } (0, T/3), \\ y(x, 0) = y_0(x), & \text{in } (0, L), \end{cases}$$

satisfies satisfies

$$y_1(x, T/3) = d.$$

On the other hand, thanks to the Proposition 2.5, there exists $h_2 \in L^2(2T/3, T)$ such that, the solution $y_2 \in \mathcal{Z}_{2T/3,T}$ of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & \text{in } (0, L) \times (2T/3, T), \\ y_{xx}(0, t) = 0, \ y_x(L, t) = h_2(t), \ y_{xx}(L, t) = 0, & \text{in } (2T/3, T), \\ y(x, 2T/3) = d, & \text{in } (0, L), \end{cases}$$

satisfies

$$y_2(x, T) = y_T.$$

Defining $y : [0, L] \times [0, T] \to \mathbb{R}$ by

$$y = \begin{cases} y_1, & \text{in } [0, T/3], \\ d, & \text{in } [T/3, 2T/3], \\ y_2, & \text{in } [2T/3, T], \end{cases} \tag{2.27}$$

we have that $y \in \mathcal{Z}_T$ and y is solution of (2.16) driving y_0 to y_T at time T, by using the control

$$h = (t) \begin{cases} h_1(t), & t \in [0, T/3], \\ 0, & t \in [T/3, 2T/3], \\ h_2(t), & t \in [2T/3, T], \end{cases}$$

showing that the system (2.16) is exactly controllable, and the proof is completed. □

2.5 CONCLUSIONS AND PERSPECTIVES

In this chapter, we successfully proved the local controllability of the purely Neumann KdV system (2.1), in the regime of critical lengths. Note that there is no time or critical length restriction in our main result, Teorema 1.1, which differs from the results obtained for the system with Dirichlet conditions, where the small-time controllability is not valid for a class of critical lengths (see Coron 2024).

An important factor that made it possible to use the return method to address the problem for any critical length was the fact that, for any $L > 0$, we know a class of stationary solutions of system (2.1) in terms of elementary functions, namely, the constant functions. In general, determining such trajectories may not be a simple task, and, in the case of the problem with Dirichlet conditions (2.2), these trajectories may vary according to the length $L > 0$. One can see, for example, in Crépeau (2001), all the effort involved in determining a family of trajectories for (2.2) in the particular case $L = 2k\pi$.

3 BOUNDARY OBSERVATION OF THE KDV EQUATION ON GRAPHS

This chapter is devoted to the study of a KdV equation posed on a star graph. Therefore, before anything else, we will make some considerations about this type of domain. For more details or a deeper reading regarding quantum graphs, we suggest (BERKOLAIKO; KUCHMENT, 2013; ANGULO; CAVALCANTE, 2013). The results of this chapter were developed in a joint work with Professor Hugo Parada[1].

3.1 METRIC AND QUANTUM GRAPHS

Recall that a graph consists of a set of points (called vertices) and a set of segments (called edges) connecting these points. It is somewhat intuitive that, to treat this object from an analytical point of view, the first step is to try to endow it with a topology. To do that, the basic idea is simple: we will identify the edges with an interval and define the distance between two points of the graph as the length of the shortest path connecting them. With this natural metric, we obtain what is called a metric graph. From there, common objects such as Lebesgue or Sobolev spaces are defined intuitively.

Defining a differential operator on a graph essentially consists of providing each edge with a differential operator, which is also done intuitively, using differential operators on the intervals with which we identify each edge. A graph equipped with a differential operator is called a quantum graph. These objects have been frequently used in the literature to model physical phenomena, for example, as a model of free electrons in organic molecules, as model systems in quantum chaos, in the study of waveguides, as a limit to the shrinkage of thin wires, in photonic crystals and Anderson localization – the absence of wave diffusion in a disordered medium – or mesoscopic physics, to theoretically understand nanotechnology.

3.2 DISPERSIVE SYSTEMS ON GRAPH STRUCTURE

In recent years, the study of nonlinear dispersive models on metric graphs has gained significant attention from mathematicians, physicists, chemists, and engineers (see (BERKOLAIKO; KUCHMENT, 2013; BLANK; EXNER; HAVLICEK, 2008; BURIONI et al., 2001; KUCHMENT,

[1] Université Paul Sabatier, Institut de Mathématiques de Toulouse, 118 Route de Narbonne, 31062 Toulouse, France. *Email address*: hugo.parada@math.univ-toulouse.fr.

2004; MUGNOLO, 2015) for details and further references). A primary framework for modeling these phenomena is a particular case of star graphs called $\mathcal{Y}$-junction, a metric graph with N half-lines of the form $(0, +\infty)$ that connect at a common vertex $\nu = 0$. On each edge, a nonlinear equation is defined, such as the nonlinear Schrödinger equation in the works of Adami *et al.* (ADAMI et al., 2014; ADAMI et al., 2016) and Angulo and Goloshchapova (ANGULO; GOLOSHCHAPOVA, 2018b; ANGULO; GOLOSHCHAPOVA, 2018a)).

Another example is the Benjamin–Bona–Mahony (BBM) equation. Bona and Cascaval (BONA; CASCAVAL, 2008) established local well-posedness in the Sobolev space H^1, while Mugnolo and Rault (MUGNOLO; RAULT, 2014) demonstrated the existence of traveling waves for the BBM equation on graphs. Using an alternative approach, Ammari and Crépeau (AMMARI; CRÉPEAU, 2019) derived results for well-posedness and stabilization of the BBM equation in a star-shaped network with bounded edges.

In the areas of control theory and inverse problems, notable contributions have been made. Ignat *et al.* (IGNAT; PAZOTO; ROSIER, 2011) investigated the inverse problem for the heat and Schrödinger equations on a tree structure. Later, Baudouin and Yamamoto (BAUDOUIN; YAMAMOTO, 2015) introduced a unified and simplified approach to the inverse problem of coefficient determination. The introduction of nonlinearities in dispersive models of such networks creates a rich field for exploring soliton propagation and nonlinear dynamics. A key challenge in this analysis is the vertex, where the star graph may exhibit bifurcation or multi-bifurcation behavior, especially in more complex graph structures.

3.2.1 The KdV equation on star graphs

Recently, Crépeau and Sorine (CRÉPEAU; SORINE, 2007)) and Chuico (CHUIKO et al., 2016) found that the KdV equation serves as a good model for studying arterial pressure waves in large arteries, which suggests the study of the KdV equation in domains described by star graphs and has prompted several works over the past few years. For example, regarding the well-posedness theory, Cavalcante (2018) studied the local well-posedness for the Cauchy problem of the Korteweg-de Vries equation on a metric star graph with a negative half-line and two positive half-lines meeting at a common vertex $\nu = 0$ (the so-called $\mathcal{Y}$-junction). About control and stability theory, Ammari and Crépeau (2018), Cerpa, Crépeau and Moreno (2020); Cerpa, Crépeu and Valein (2020); Parada, Crépeau and Prieur (2022a) and (2022b) proved results on stabilization and boundary controllability for the KdV equation on star-shaped graphs.

As pointed out in (ANGULO; CAVALCANTE, 2013), the study of the KdV equation in star graphs is still underdeveloped and presents some difficulties. The main one is associated with bifurcations, as mentioned above. In other words, determining the boundary conditions to be considered at the vertex $x = 0$, in the most appropriate way both for the physical context and for a mathematical study, is still a challenge.

3.3 SETTING OF PROBLEM AND FUNCTIONAL FRAMEWORK

The main objective of this chapter is to investigate the exact controllability of the linear KdV equation on a star-shaped network. The equation governing the dynamics is given by

$$\begin{cases} \partial_t u_j(t,x) + \partial_x u_j(t,x) + \partial_x^3 u_j(t,x) = 0, & t \in (0,T),\ x \in (0,\ell_j),\ j = 1,\dots,N, \\ u_j(t,0) = u_1(t,0), & t \in (0,T),\ \forall j = 2,\dots,N, \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) + g_0(t), & t \in (0,T), \\ u_j(t,\ell_j) = p_j(t), \quad \partial_x u_j(t,\ell_j) = g_j(t), & t \in (0,T),\ j = 1,\dots,N, \\ u_j(0,x) = u_j^0(x), & x \in I_j := (0,l_j), \end{cases} \tag{3.1}$$

where $N \geq 2$ is an integer, $\alpha > \frac{N}{2}$, $u(t,\cdot)$ is the state at time t and p_j, g_0, g_j are the control inputs. The conditions at the central node are motivated by previous studies (AMMARI; CRÉPEAU, 2018; CAVALCANTE, 2018; PARADA; CRÉPEAU; PRIEUR, 2022a; PARADA; CRÉPEAU; PRIEUR, 2022b). Following (PARADA; CRÉPEAU; PRIEUR, 2022b), u_j represents the dimensionless, scaled deflection from the rest position, and v_j denotes the velocity along a branch j of long water waves. Thus, we have:

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j + u_j \partial_x u_j = 0, & x \in (0,\ell_j),\ t \in (0,T),\ j = 1,\dots,N, \\ v_j = u_j - \frac{1}{6}u_j^2 + 2\partial_x^2 u_j, & x \in (0,\ell_j),\ t \in (0,T),\ j = 1,\dots,N. \end{cases} \tag{3.2}$$

Assuming the water level at the central node is constant and the net flux is zero, the following natural conditions arise

$$u_j(t,0) = u_1(t,0), \quad t \in (0,T),\ j = 2,\dots,N,$$

and

$$\sum_{j=1}^{N} u_j(t,0) v_j(t,0) = 0, \quad t \in (0,T).$$

Linearizing (3.2) around zero yields the following boundary conditions

$$u_j(t,0) = u_1(t,0), \quad t \in (0,T),\ j = 2,\dots,N,$$

and

$$\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\frac{N}{2} u_1(t,0), \quad t \in (0,T).$$

With this context, we introduce some notations used throughout this chapter. Given $N \geq 2$, consider a star-shaped network $\mathcal{T}$ with N edges described by intervals $I_j = (0, l_j)$, $l_j > 0$ for $j = 1,\dots,N$. Denoting by $e_1, ..., e_N$ the edges of $\mathcal{T}$ one have $\mathcal{T} = \bigcup_{j=1}^{N} e_j$. From now on, we will consider:

i. The vector u is given by

$$u = (u_1, ..., u_N) \in \mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^{N} L^2(0, l_j)$$

and final and initial data is

$$u^0 = (u_1^0, ..., u_N^0) \in \mathbb{L}^2(\mathcal{T}) \quad \text{and} \quad u^T = (u_1^T, ..., u_N^T) \in \mathbb{L}^2(\mathcal{T}).$$

ii. The inner product in $\mathbb{L}^2(\mathcal{T})$ will be given by

$$(u,z)_{\mathbb{L}^2(\mathcal{T})} = \sum_{j=1}^{N} \int_0^{l_j} u_j z_j dx, \quad u, z \in \mathbb{L}^2(\mathcal{T}).$$

Moreover,

$$\mathbb{H}^s(\mathcal{T}) = \prod_{j=1}^{N} H^s(0, l_j), \quad s \in \mathbb{R}.$$

iii. Define also the following spaces:

$$\mathbb{H}_0^k(\mathcal{T}) = \prod_{j=1}^{N} H_0^k(0, l_j),\ k \in \mathbb{N}$$

and

$$H_r^k(0, l_j) = \left\{ v \in H^k(0, l_j), v^{(i-1)}(l_j) = 0,\ 1 \leq i \leq k \right\},\ k \in \mathbb{N},$$

where $v^{(m)} = \frac{dv}{dx^m}$ and the index r is related to the null right boundary conditions. In addition,

$$\mathbb{H}_r^k(\mathcal{T}) = \prod_{j=1}^{N} H_r^k(0, l_j) \quad \text{and} \quad \|u\|^2_{\mathbb{H}_r^k(\mathcal{T})} = \sum_{j=1}^{n} \|u_j\|_{H^k(0,l_j)}\ k \in \mathbb{N}.$$

iv. Consider the following characterization:

$$H_r^{-1}(0,l_j) = \left(H_r^1(0,l_j)\right)'$$

as the dual space of $H_r(0,l_j)$ with respect to the pivot space, $L^2(0,l_j)$ and $\mathbb{H}_r^{-1}$ denotes the cartesian product of $H_r^{-1}(0,l_j)$.

v. Let

$$\mathbb{H}_e^k(\mathcal{T}) = \left\{u = (u_1, ..., u_N) \in \mathbb{H}_r^k(\mathcal{T});\ u_1(0) = u_j(0),\ j = 1, \ldots, N\right\},\ k \in \mathbb{N},$$

and

$$\mathbb{B} = C([0,T], \mathbb{L}^2(\mathcal{T})) \cap L^2(0,T,\mathbb{H}_e^1(\mathcal{T})),$$

with the norm

$$\|u\|_{\mathbb{B}} := \|u\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))} + \|u\|_{L^2(0,T,\mathbb{H}_e^1(\mathcal{T}))} = \max_{t\in[0,T]} \|u\|_{\mathbb{L}^2(\mathcal{T})} + \left(\int_0^T \|u(t,\cdot)\|^2_{\mathbb{H}_e^1} dt\right)^{\frac{1}{2}}.$$

vi. Finally, we need to introduce spaces that are paramount for our main results. For $m = 0, \ldots, N$, consider

$$X_m = \prod_{j=1}^m L^2(0,l_j) \times \prod_{i=m+1}^N H_0^1(0,l_i), \qquad X_0 = \prod_{i=1}^N H_0^1(0,l_i), \qquad X_N = \prod_{j=1}^N L^2(0,l_j), \tag{3.3}$$

and

$$Y_m = \prod_{j=1}^m H^1(0,l_j) \times \prod_{i=m+1}^N H^2(0,l_i), \qquad Y_0 = \prod_{i=1}^N H^2(0,l_i), \qquad Y_N = \prod_{i=1}^N H^1(0,l_j). \tag{3.4}$$

endowed with the usual Hilbertian norms.

3.3.1 Background of control theory of a single KdV

As expected, given the behavior of the KdV equation on bounded intervals, the investigation of controllability issues for system (3.1) gives rise to the phenomenon of critical lengths. Hence, it is convenient to briefly review the control of the KdV equation posed on intervals, as far as is convenient for the presentation of our results.

As seen in section 2.1, Rosier (1997) proved that the linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = g(t), & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L), \end{cases} \tag{3.5}$$

is exactly controllabe if, and only if, $L \notin \mathcal{N}$, where $\mathcal{N}$ is defined by

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2}\ : k,\ l\ \in \mathbb{N}^* \right\}. \tag{3.6}$$

and called the critical lengths set for the system (3.5). But it is important to point out that the existence and the characterization of a critical length set as in (3.6) are directly related to the number of controls used and their positions in the boundary conditions. For example, Rosier 1997 clarifies that the use of a second control, in the Dirichlet condition on the right side of the boundary, returns the exact controllability of (3.5). Precisely, the system

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ u(0,t) = 0,\ u(L,t) = h(t),\ u_x(L,t) = g(t), & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L), \end{cases} \tag{3.7}$$

is exactly controllable. Hence, does not exist critical length for the system (3.7) and it is controllable for any $L > 0$.

On the other hand, Glass and Guerrero (2010) proved that the use of only the Dirichlet boundary control in (3.7) gives rise to a set of critical lengths different from $\mathcal{N}$. Precisely, he proved that the system

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ u(0,t) = 0,\ u(L,t) = h(t),\ u_x(L,t) = 0, & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L), \end{cases} \tag{3.8}$$

is exactly controllable if, and only if, L does not belong to the countable set $\mathcal{N}^*$ described by

$$\mathcal{N}^* = \left\{ L \in \mathbb{R}^+ \setminus \{0\};\ \begin{array}{l} \exists (a,b) \in \mathbb{C}^2 \text{ such that } ae^a = be^b = -(a+b)e^{-(a+b)} \\ \text{and } L^2 = -\left(a^2+ab+b^2\right) \end{array} \right\}. \tag{3.9}$$

In addition to these last two sets, investigation of the KdV equation on star graphs will lead us to a new set of critical lengths, namely,

$$\mathcal{N}^\dagger = \left\{ L \in \mathbb{R}^+ \setminus \{0\};\ \begin{array}{l} \exists (a,b) \in \mathbb{C}^2 \text{ such that } a^2e^a = b^2e^b = (a+b)^2e^{-(a+b)} \\ \text{and } L^2 = -\left(a^2+ab+b^2\right) \end{array} \right\} \tag{3.10}$$

As we will prove in the Appendix B, the set $\mathcal{N}^\dagger$ is a countable, non-empty new critical set in the literature of controllability of KdV equations.

3.3.2 Main results

Controllability results for (3.1) were first addressed in (AMMARI; CRÉPEAU, 2018), where the authors proved that (3.1) is exactly controllable using $N+1$ controls $g_0, g_1, g_2, ..., g_N$ or even N controls $g_0, g_2, ..., g_N$, provided that $\#\{l_j \in \mathcal{N}\}$.More recently, Parada (2024) obtained null controllability of system (3.1) by removing the central node control and using $2(N-2)$ Neumann and Dirichlet controls. Based on this, we are interested in finding answers to the following question:

Question $\mathcal{A}_3$: *Is the system* (3.1) *controllable using a smaller number of controls and without the use of a control g_0 at the central node? More specifically, considering $g_0 = 0$, given $u^0, u^T \in \mathbb{L}^2(\mathcal{T})$ under which configurations of the controls $(p_1, ..., p_N)$ and $(g_1, ..., g_N)$, with some of the p_j's and g_j's possibly equal to zero, the solution u of* (3.1) *with initial data u^0 satisfies*

$$u(T, \cdot) = u^T? \tag{3.11}$$

Our proposal here is to remove the control at the central node and continue working in a mixed configuration, but reducing the number of controls used and varying their quantity between Neumann or Dirichlet, and, as extreme cases of this variation, we analyze the situations where only Neumann controls or only Dirichlet controls are used. We will present six new possibilities that ensure boundary controllability properties for the system (3.1), and that exhibit a relation of the length l_j with a critical set, that is, in some appropriated set of boundary conditions the controllability holds if and only if the length l_j avoids certain values or for all $l_j > 0$. Precisely we consider $g_0 = 0$ and, given $0 \leq m \leq N$, we put Neumann controls on the first m edges and Dirichlet controls on the remaining $N-m$ edges, that is, $p_j = 0$ for $j = 1, \ldots, m$ and $g_j = 0$ for $j = m+1, \ldots, N$, resulting in the following control system

$$
\begin{cases}
\partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & t \in (0,T),\ x \in (0,l_j),\ j = 1,\dots,N, \\
u_j(t,0) = u_1(t,0), & t \in (0,T),\ j = 2,\dots,N, \\
\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0), & t \in (0,T), \\
u_j(t,l_j) = 0, \qquad \partial_x u_j(t,l_j) = g_j(t), & t \in (0,T),\ j = 1,\dots,m, \\
u_j(t,l_j) = p_j(t), \qquad \partial_x u_j(t,l_j) = 0, & t \in (0,T),\ j = m+1,\dots,N, \\
u_j(0,x) = u_j^0(x), & x \in (0,l_j),\ j = 1,\dots,N.
\end{cases}
\tag{3.12}
$$

For $m = 0$, we omit the fourth equation from the system above, whereas for $m = N$, we omit the fifth equation.

With this in mind, we will analyze the problem under the assumption $\alpha = N$ and $l_j = L$ for $j = 1,\dots,N$ in six situations (with different results). The first one considers $N = 2$ and $m = 1$, and it is illustrated in Figure 3. We prove that the corresponding system (3.12) is exactly controllable for any $L > 0$.

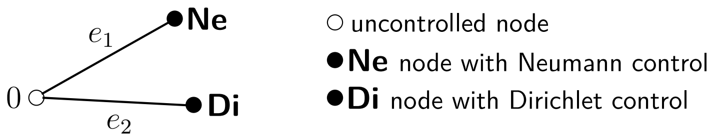

Figure 3 – Network with 2 edges and mixed controls

The second situation considers $N \geq 3$, $m = 1$, and it is illustrated in Figure 4. In this case, we can prove the exact controllability to the corresponding system (3.12) when $L \notin \mathcal{N}^*$, where $\mathcal{N}^*$ is defined by (3.9).

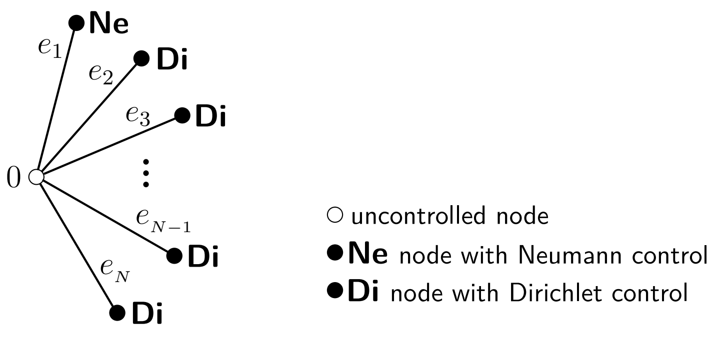

Figure 4 – Network with N edges for $m = 1$

In the third case, we consider $N \geq 3$ and $m = N - 1$, as illustrated in Figure 5. For this situation, the exact controllability to the corresponding system (3.12) holds if and only if $L \notin \mathcal{N}$, where $\mathcal{N}$ is defined by (3.6).

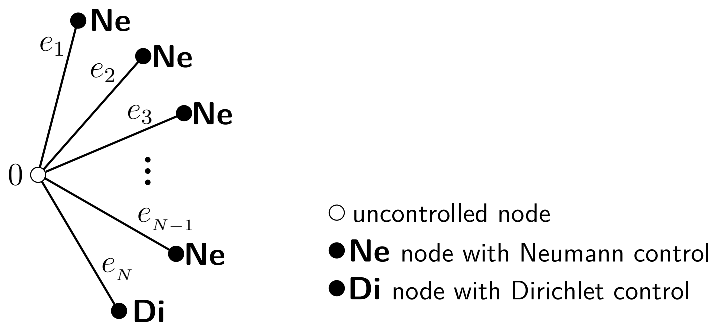

Figure 5 – Network with N edges for $m = N - 1$

In the fourth case, we study the problem under the configuration $N > 3$ and $1 < m < N - 1$. Figure 6 illustrates this situation for $m = 2$. In this case, we can prove the exact controllability to the corresponding system (3.12) when $L \notin \mathcal{N} \cup \mathcal{N}^*$.

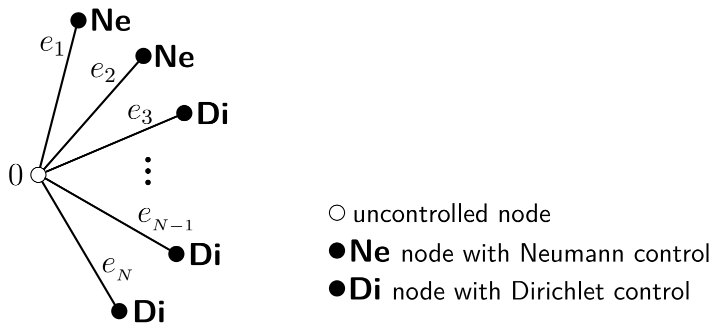

Figure 6 – Network with N edges for $m = 2$

In the full Neumann case, namely, $N \geq 2$ and $m = N$, illustrated in Figure 7, we can prove the controllability if $L \notin \mathcal{N} \cup \mathcal{N}^*$.

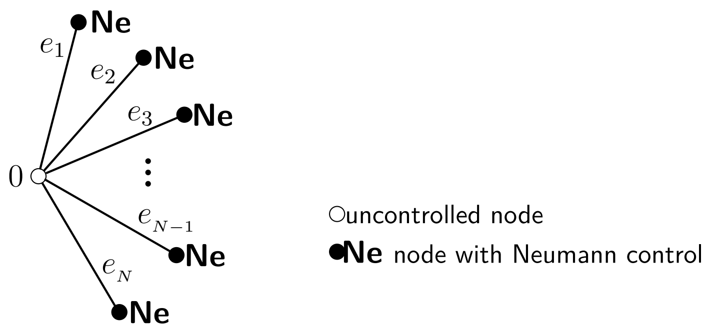

Figure 7 – Network with N edges with control on Neumann condition

Finally, in the full Dirichlet case, that is, $N \geq 2$ and $m = 0$, illustrated in Figure 8, we can prove the controllability if $L \notin \mathcal{N}^* \cup \mathcal{N}^\dagger$, where $\mathcal{N}^\dagger$ is defined by (3.10).

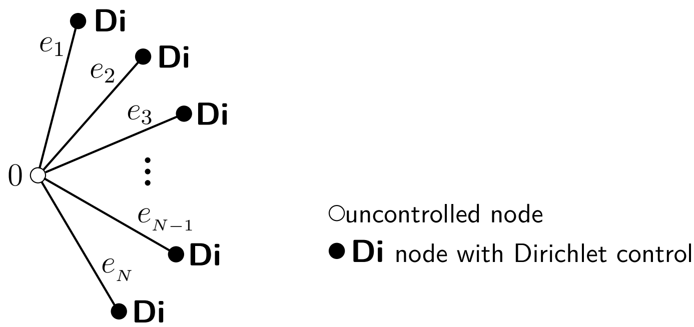

Figure 8 – Network with N edges with control on Neumann condition

With this in hand, the main result of this work is given below and mixes the Neumann and Dirichlet boundary conditions (see Figs. 3, 4, 5, and 6).

Theorem 3.1 *Let $T > 0$ and $u^0, u^T \in \mathbb{L}^2(\mathcal{T})$. Consider $\alpha = N$, $m = 0, 1, ..., N$ and $l_j = L$ for $j = 1, \ldots, N$.*

1. *If $N = 2$, then for any $L > 0$, there exist controls $(g_1, 0) \in [L^2(0,T)]^2$ and $(0, p_2) \in [L^2(0,T)]^2$ such that the unique solution u of* (3.12) *satisfies* (3.11).

2. *If $N \geq 3$ and $m = 1$ then, there exist controls $(g_1, 0, ..., 0) \in [L^2(0,T)]^N$ and $(0, p_2, ..., p_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (3.12) *satisfies* (3.11), *if and only if $L \notin \mathcal{N}^*$.*

3. *If $N \geq 3$ and $m = N-1$ then, there exist controls $(g_1, ..., g_{N-1}, 0) \in [L^2(0,T)]^N$ and $(0, ..., 0, p_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (3.12) *satisfies* (3.11), *if and only if $L \notin \mathcal{N}$.*

4. *If $N > 3$ and $1 < m < N-1$ then, there exist controls $(g_1, ..., g_m, 0, ..., 0) \in [L^2(0,T)]^N$ and $(0, ..., 0, p_{m+1}, ..., p_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (3.12) *satisfies* (3.11), *if and only if $L \notin \mathcal{N} \cup \mathcal{N}^*$.*

5. *If $N \geq 2$ and $m = N$, then there exist controls $(g_1, ..., g_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (3.12) *satisfies* (3.11), *if and only if $L \notin \mathcal{N} \cup \mathcal{N}^*$.*

6. *If $N \geq 2$ and $m = 0$, then there exist controls $(p_1, ..., p_N) \in [L^2(0,T)]^N$ such that the unique solution u of* (3.12) *satisfies* (3.11), *if and only if $L \notin \mathcal{N}^* \cup \mathcal{N}^\dagger$.*

For a better description, we can summarize our results in the following table.

	Neumann	Dirichlet	Critical lengths
$N=2$	1	1	$\emptyset$
$N\geq 3$	1	$N-1$	$\mathcal{N}^*$
$N\geq 3$	$N-1$	1	$\mathcal{N}$
$N>3$	m	$N-m$	$\mathcal{N}^*\cup\mathcal{N}$
$N\geq 2$	N	0	$\mathcal{N}^*\cup\mathcal{N}$
$N\geq 2$	0	N	$\mathcal{N}^*\cup\mathcal{N}^\dagger$

Table 1 – Relation between the placement of the controls and the critical lengths.

We will provide a brief overview of our approach. To establish well-posedness, the classical semigroup theory typically used for the KdV equation in networks, as seen in various studies (AMMARI; CRÉPEAU, 2018; CERPA; CRÉPEAU; MORENO, 2020; PARADA; CRÉPEAU; PRIEUR, 2022a; PARADA; CRÉPEAU; PRIEUR, 2022b), proves to be less effective for handling Dirichlet controls. Instead, studies like (CAPISTRANO-FILHO; GALLEGO; PAZOTO, 2019; CARREÑO; LOYOLA, 2023; GLASS; GUERRERO, 2010; GUILLERON, 2014) mainly rely on "solutions by transposition" combined with interpolation methods. However, these techniques are still underdeveloped for the KdV equation on networks, where the network structure introduces challenging trace terms in the computations. These terms are addressed in a recent work by the second author (PARADA, 2024) and in this work.

For the control problem, it is well known in the literature (LIONS, 1988b) that proving the exact controllability of the system (3.12) is equivalent to establishing an inequality of *observability* for the adjoint system associated with (3.12), namely,

$$\begin{cases} -\partial_t\varphi_j-\partial_x\varphi_j-\partial_x^3\varphi_j=0, & t\in(0,T),\ x\in(0,l_j),\ j=1,\dots,N,\\ \varphi_j(t,0)=\varphi_1(t,0), & t\in(0,T),\ j=2,\dots,N,\\ \displaystyle\sum_{j=1}^N\partial_x^2\varphi_j(t,0)=(\alpha-N)\varphi_1(t,0), & t\in(0,T)\\ \varphi_j(t,l_j)=\partial_x\varphi_j(t,0)=0, & t\in(0,T),\ j=1,\dots,N,\\ \varphi_j(T,x)=\varphi_j^T(x), & x\in(0,l_j),\ j=1,\dots,N. \end{cases} \tag{3.13}$$

In our terms, the desired observability inequality is given by

$$\|\varphi^T\|_{X_m}^2\leq\tilde{C}\left(\sum_{j=1}^m\|\partial_x\varphi_j(\cdot,l_j)\|_{L^2(0,T)}^2+\sum_{j=m+1}^N\|\partial_x^2\varphi_j(\cdot,l_j)\|_{L^2(0,T)}^2\right),\ \forall\varphi^T\in X_m,$$

where $\varphi(t, x)$ is the solution of (3.13) associated to φ^T. To achieve this, we generally employ the multiplier method and compactness arguments, which reduce the task to demonstrating a unique continuation property for the state operator.

In our context, proving this unique continuation property involves analyzing the spectral problem associated with the relevant linear operator. Specifically, after applying the Fourier transform, the question becomes one of determining when a certain quotient of entire functions remains entire. We introduce a polynomial function $q : \mathbb{C} \to \mathbb{C}$ and a family of functions

$$N_\alpha : \mathbb{C} \times (0, \infty) \to \mathbb{C},$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ and each restriction $N_\alpha(\cdot, L)$ is entire for any $L > 0$. We then define a family of functions $f_\alpha(\cdot, L)$ by

$$f_\alpha(\mu, L) = \frac{N_\alpha(\mu, L)}{q(\mu)},$$

within its maximal domain. The problem then reduces to finding $L > 0$ such that there exists $\alpha \in \mathbb{C} \setminus \{0\}$ for which $f_\alpha(\cdot, L)$ is entire. In some works, such as (CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017; CAPISTRANO–FILHO; PAZOTO; ROSIER, 2019; ROSIER, 1997) and references therein, this approach provides an explicit characterization of the set of critical lengths when it exists. However, in many cases, this set cannot be explicitly determined (ARARUNA; CAPISTRANO-FILHO; DORONIN, 2012; CAPISTRANO-FILHO et al., 2023; CAPISTRANO-FILHO; GALLEGO; PAZOTO, 2019; CERPA; RIVAS; ZHANG, 2013).

In this context, the contributions of this work are twofold. First, we examine the well-posedness of the system (3.12) under mixed boundary conditions, specifically, Dirichlet and Neumann boundary conditions. Second, we provide, for the first time, a detailed analysis of a star graph structure when the function f_α, relevant to our scenarios, is entire. This analysis results in a characterization of critical sets $\mathcal{N}$ defined by (3.6), $\mathcal{N}^*$ defined by (3.9) or $\mathcal{N} \cup \mathcal{N}^*$, resembling those established by Rosier (ROSIER, 1997) and by Glass and Guerrero (GLASS; GUERRERO, 2010). Additionally, we identify a new critical set, denoted as $\mathcal{N}^\dagger$. Due to the increased complexity of the functions N_α in our setting compared to these earlier works, our approach involves a refined and meticulous adaptation of the analysis of these functions.

The next two sections are organized as follows: In Section 3.4, we conduct a detailed analysis of the well-posedness of problem (3.12) and (3.13), incorporating relevant results from the single KdV equation. Building on this, the objective of Section 3.5 is to establish the observability inequality for system (3.12) with boundary controls acting under mixed boundary conditions, specifically Dirichlet and Neumann conditions, as well as under full Neumann boundary conditions and full Dirichlet boundary conditions. Here, we carefully examine the function f_α associated with our setting, leading to Theorem 3.1.

3.4 WELL-POSEDNESS ANALYSIS

In this section, we examine the well-posedness of the problem

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j = 0, & t \in (0,T),\ x \in (0,l_j),\ j = 1,\dots,N, \\ u_j(t,0) = u_1(t,0), & t \in (0,T),\ j = 2,\dots,N, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0), & t \in (0,T) \\ u_j(t,l_j) = p_j(t), \qquad \partial_x u_j(t,l_j) = g_j(t), & t \in (0,T),\ j = 1,\dots,N, \\ u_j(0,x) = u_j^0(x), & x \in (0,l_j),\ j = 1,\dots,N, \end{cases} \tag{3.14}$$

as established in (PARADA, 2024), along with its corresponding adjoint problem. To achieve this, we verify the Kato smoothing property for the solution traces, which is essential for deriving an observability inequality that characterizes critical lengths. Additionally, we investigate the regularity of solutions to the adjoint system when the data belong to regular spaces. Note that the positive constants C introduced here may vary from line to line.

3.4.1 Weak solutions

The next proposition was proved in (PARADA, 2024), and it gives the well-posedness to the problem

$$\begin{cases} -\partial_t v_j - \partial_x v_j - \partial_x^3 v_j = f_j, & t \in (0,T),\ x \in (0,l_j),\ j = 1,\dots,N, \\ v_j(t,0) = v_1(t,0), & t \in (0,T),\ j = 2,\dots,N, \\ \displaystyle\sum_{j=1}^{N} \partial_x^2 v_j(t,0) = (\alpha - N) v_1(t,0), & t \in (0,T), \\ v_j(t,l_j) = \partial_x v_j(t,0) = 0, & t \in (0,T),\ j = 1,\dots,N, \\ v_j(T,x) = 0, & x \in (0,l_j),\ j = 1,\dots,N, \end{cases} \tag{3.15}$$

for $f = (f_1, ..., f_N) \in L^2(0,T,\mathbb{L}^2(\mathcal{T}))$. This result is useful in establishing the notion of a solution for (3.14).

Proposition 3.1 *If $f \in L^2(0,T,\mathbb{L}^2(\mathcal{T}))$ there exists a unique solution*

$$v \in \mathbb{B}_1 := C([0,T],\mathbb{H}^1_r(\mathcal{T})) \cap L^2(0,T,\mathbb{H}^2(\mathcal{T}))$$

of (3.15). *Moreover, there exists $C_1 > 0$ such that*

$$\|v\|_{\mathbb{B}_1} + \sum_{j=1}^{N} \|\partial_x^2 v_j(\cdot,l_j)\|_{L^2(0,T)} + \sum_{j=1}^{N} \|\partial_x v_j(\cdot,l_j)\|_{H^{\frac{1}{3}}(0,T)} \leq C_1 \|f\|_{L^2(0,T,\mathbb{L}^2(\mathcal{T})}.$$

Assume $u_j, p_j, g_j, v_j, f_j \in C^\infty([0,T]\times[0,l_j])$ and $u_j^0 \in C^\infty[0,l_j]$. Suppose that $u = (u_1,...,u_N)$ is a solution of (3.14) subject to the data p_j, g_j, u_0^j and $v = (v_1,...,v_N)$ is a solution of (3.15) subject to $f = (f_1,...,f_N)$. Multiplying (3.15) by u_j, integrating by parts and using the boundary conditions in (3.14) and (3.15) we obtain

$$\sum_{j=1}^{N}\int_0^{l_j} u_j^0(x)v_j(0,x) - \sum_{j=1}^{N}\int_0^T p_j(t)\partial_x^2 v_j(t,l_j) + \sum_{j=1}^{N}\int_0^T g_j(t)\partial_x v_j(t,l_j) = \sum_{j=1}^{N}\int_0^T\int_0^{l_j} f_j u_j.$$

Identifying $u_j^0 \in C^\infty([0,l_j]) \subset L^2(0,l_j)$ with the functional $(u_j^0)^* \in H_r^{-1}(0,l_j)$ given by

$$(u_j^0)^*(w) = \int_0^{l_j} u_j^0 w, \quad \forall w \in H_r^{-1}(0,l_j),$$

we can write

$$\int_0^{l_j} u_j^0(x)v_j(0,x) = \langle u_j^0, v_j(0,\cdot)\rangle_{H_r^{-1}(0,l_j)\times H_r^1(0,l_j)}.$$

Now identifying $g_j \in C^\infty([0,T]) \subset L^2(0,T)$ with the functional $g_j^* \in H^{-\frac{1}{3}}(0,T)$ given by

$$g_j^*(h) = \int_0^T g_j h, \quad \forall h \in H^{\frac{1}{3}}(0,T)$$

we can write

$$\int_0^T g_j(t)\partial_x v_j(t,l_j) = \langle g_j, \partial_x v_j(\cdot,l_j)\rangle_{H^{-\frac{1}{3}}(0,T)\times H^{\frac{1}{3}}(0,T)}.$$

Therefore we have

$$\begin{aligned}\sum_{j=1}^{N}\int_0^T\int_0^{l_j} f_j u_j =& \sum_{j=1}^{N}\langle u_j^0, v_j(0,\cdot)\rangle_{H_r^{-1}(0,l_j)\times H_r^1(0,l_j)} - \sum_{j=1}^{N}\int_0^T p_j(t)\partial_x^2 v_j(t,l_j) \\ &+ \sum_{j=1}^{N}\langle g_j, \partial_x v_j(\cdot,l_j)\rangle_{H^{-\frac{1}{3}}(0,T)\times H^{\frac{1}{3}}(0,T)}.\end{aligned}$$

This motivates the next definition, giving us a weak solution for the system (3.14).

Definition 3.1 *Given* $T > 0$, $u_0 = (u_1^0, u_2^0, \cdots, u_N^0) \in \mathbb{H}_r^{-1}(\mathcal{T})$, $p \in \left(L^2(0,T)\right)^N$ *and* $g \in (H^{-\frac{1}{3}}(0,T))^N$ *a solution by transposition of* (3.14) *is a function* $u \in L^2(0,T,\mathbb{L}^2(\mathcal{T}))$ *satisfying*

$$\begin{aligned}\sum_{j=1}^{N}\int_0^T\int_0^{l_j} f_j u_j =& \sum_{j=1}^{N}\langle u_j^0, v_j(0,\cdot)\rangle_{H_r^{-1}(0,l_j)\times H_r^1(0,l_j)} - \sum_{j=1}^{N}\int_0^T p_j(t)\partial_x^2 v_j(t,l_j)\\ &+ \sum_{j=1}^{N}\langle g_j, \partial_x v_j(\cdot,l_j)\rangle_{H^{-\frac{1}{3}}(0,T)\times H^{\frac{1}{3}}(0,T)},\end{aligned}$$

for all $f \in L^2(0,T,\mathbb{L}^2(\mathcal{T}))$, *where* v *is the solution of* (3.15) *corresponding to* f.

The next result provides the well-posedness of (3.14), proved in (PARADA, 2024).

Theorem 3.2 *Let* $T > 0$ *be given. For all* $u^0 \in \mathbb{H}_r^{-1}(\mathcal{T})$, $p \in \left(L^2(0,T)\right)^N$ *and* $g \in (H^{-\frac{1}{3}}(0,T))^N$, *there exists a unique solution by transposition for* (3.14).

3.4.2 The adjoint system

The adjoint system associated with (3.14) is given by (3.13). The next result ensures that the system (3.13) admits a unique solution.

Proposition 3.2 *For any* $\varphi^T \in \mathbb{L}^2(\mathcal{T})$ *the system* (3.13) *admits a unique solution* $\varphi \in \mathbb{B}$ *which satisfies*

$$\|\varphi\|_{\mathbb{B}} \leq C\|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})}, \tag{3.16}$$

for $C > 0$, *and*

$$\sum_{j=1}^{N}\|\partial_x\varphi_j(\cdot,l_j)\|_{L^2(0,T)}^2 \leq \|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})}^2, \tag{3.17}$$

for all $\varphi^T \in \mathbb{L}^2(\mathcal{T})$.

Proof: Define the operator $Au = -\partial_x u - \partial_x^3 u$ with domain

$$\mathcal{D}(A) = \left\{u \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^1(\mathcal{T}),\ \sum_{j=1}^{N}\partial_x^2 u_j(0) = -\alpha u_1(0)\right\}$$

where

$$\partial_x^k u = \left(\partial_x^k u_1, ..., \partial_x^k u_N\right).$$

Observe that A is the operator associated with (3.14) when $p_j = g_j = 0$. The adjoint operator of A is given by $A^*v = \partial_x v + \partial_x^3 v$ with domain

$$\mathcal{D}(A^*) = \left\{v \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_e^1(\mathcal{T});\ \partial_x v_j(0) = 0,\ j = 1,\dots,N,\ \ \sum_{j=1}^{N}\partial_x^2 v_j(0) = (\alpha - N)v_1(0)\right\}.$$

A^* is the operator associated with (3.13). According to (AMMARI; CRÉPEAU, 2018), A is closed, A and A^* are dissipative, so A^* generates a strongly semigroup of contractions on $\mathbb{L}^2(\mathcal{T})$ which will be denoted by $\{S(t)\}_{t\geq 0}$. From the semigroup theory it follows that, for any $\varphi^T \in \mathbb{L}^2(\mathcal{T})$ the problem (3.13) has a unique (mild) solution $\varphi = S(T-\cdot)\varphi^T \in C([0,T], \mathbb{L}^2(\mathcal{T}))$. Note that

$$\|\varphi\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))} = \|S(T-\cdot)\varphi^T\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))} \leq \|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})}, \ \forall \varphi^T \in \mathbb{L}^2(\mathcal{T}).$$

To see that $\varphi \in L^2(0,T,\mathbb{H}^1_e(\mathcal{T}))$ and to obtain the estimates (3.16) and (3.17), first suppose $\varphi^T \in \mathcal{D}(A^*)$. In this case, from semigroup theory, we have

$$\varphi \in C^1([0,T], \mathbb{L}^2(\mathcal{T})) \cap C([0,T], \mathcal{D}(A^*)).$$

Assume $q_j \in C^\infty([0,T]\times[0,l_j])$ with $q_j(t,0) = q_1(t,0)$. Multiplying the first equation of (3.13) by $q\varphi_j$, integrating by parts, and using the boundary conditions, we obtain

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^N \int_0^{l_j} (q_j\varphi_j^2)(T,x) + \frac{3}{2}\sum_{j=1}^N \int_0^T \int_0^{l_j} \partial_x q_j \partial_x \varphi_j^2 =& \frac{1}{2}\sum_{j=1}^N \int_0^T \int_0^{l_j} \varphi_j^2(\partial_t q_j + \partial_x q_j + \partial_x^3 q_j) \\ &+ \left(\alpha - \frac{N}{2}\right)\int_0^T q_1(t,0)\varphi_1^2(t,0) \\ &+ \frac{1}{2}\sum_{j=1}^N \int_0^T q_j(t,l_j)\partial_x\varphi_j^2(t,l_j) \\ &+ \frac{1}{2}\sum_{j=1}^N \int_0^T \partial_x^2 q_j(t,0)\varphi_1^2(t,0) \\ &+ \frac{1}{2}\sum_{j=1}^N \int_0^{l_j} (q_j\varphi_j^2)(0,x).\end{aligned}$$

Choosing $q_j = 1$ we get

$$\frac{1}{2}\sum_{j=1}^N \int_0^{l_j} \varphi_j^2(T,x) = \left(\alpha - \frac{N}{2}\right)\int_0^T \varphi_1^2(t,0) + \frac{1}{2}\sum_{j=1}^N \int_0^T \partial_x\varphi_j^2(t,l_j) + \frac{1}{2}\sum_{j=1}^N \int_0^{l_j} \varphi_j^2(0,x).$$

Thus,

$$\sum_{j=1}^N \|\partial_x\varphi_j(\cdot,l_j)\|^2_{L^2(0,T)} \leq \|\varphi^T\|^2_{\mathbb{L}^2(\mathcal{T})},$$

that is, the map

$$\begin{aligned}\delta : \mathcal{D}(A^*) : \ &\to \ L^2(0,T) \\ \varphi^T \ &\mapsto \ \partial_x\varphi(\cdot,l_j)\end{aligned}$$

is continuous. Now, choosing $q_j = x$ it follows that

$$\int_0^T \|\varphi_x(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} \leq \left(\frac{T+2M}{3}\right) \|\varphi^T\|^2_{\mathbb{L}^2(\mathcal{T})},$$

where $M = \max_{1\leq j\leq N} l_j$. Then, (3.16) holds for every $\varphi^T \in \mathcal{D}(A^*)$ and consequently the map

$$\begin{aligned} \Gamma^* : (\mathcal{D}(A^*), \|\cdot\|_{\mathbb{L}^2(\mathcal{T})}) &\to (\mathbb{B}, \|\cdot\|_{\mathbb{B}}) \\ \varphi^T &\mapsto \Gamma^*\varphi^T = \varphi = S(T-\cdot)\varphi^T \end{aligned}$$

is continuous. By an argument of density we extend the maps Γ^* and δ to $(\mathbb{L}^2(\mathcal{T}), \|\cdot\|_{\mathbb{L}^2(\mathcal{T})})$. For each $\varphi^T \in \mathbb{L}^2(\mathcal{T})$ we refer to $\delta(\varphi^T)$ when write $\partial_x\varphi(t,l_j)$ and, in this sense, (3.17) holds for every $\varphi^T \in \mathbb{L}^2(\mathcal{T})$. Moreover, since

$$\Gamma\varphi^T = S(T-\cdot)\varphi^T \quad \forall \varphi^T \in \mathcal{D}(A^*),$$

by density and continuity, it follows that

$$\Gamma\varphi^T = S(T-\cdot)\varphi^T, \quad \forall \varphi^T \in \mathbb{L}^2(\mathcal{T})$$

and thus (3.16) was verified. □

3.4.3 Trace estimates: A review for a single KdV

Now we proceed to prove that the solutions of (3.13) possess the sharp Kato smoothing property, namely

$$\partial_x^k \varphi_j \in L_x^\infty\left(0, l_j; H^{\frac{1-k}{3}}(0,T)\right), \ k = 0,1,2, \quad j = 1,\dots,N.$$

Let us review some results for the single KdV equation to see it. Precisely, the first result is given by (BONA; SUN; ZHANG, 2003, Propositions 2.7-2.9) and (CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017, Proposition 2.2).

Proposition 3.3 *Let $L, T > 0$ be given. For any $(h_1,h_2,h_3) \in H^{\frac{1}{3}}(0,T) \times H^{\frac{1}{3}}(0,T) \times L^2(0,T)$ there exists a unique solution $v \in \mathcal{Z}_T := C([0,T];L^2(0,L)) \cap L^2(0,L;H^1(0,L))$ to the problem*

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v = 0, & t \in (0,T), \ x \in (0,L), \\ v(t,0) = h_1(t), \quad v(t,L) = h_2(t), \quad \partial_x v(t,L) = h_3(t), & t \in (0,T), \\ v(0,x) = 0, & x \in (0,L) \end{cases} \tag{3.18}$$

which possesses the sharp Kato smoothing property

$$\partial_x^k v \in L_x^\infty\left(0, L; H^{\frac{1-k}{3}}(0,T)\right), \ k = 0,1,2.$$

Furthermore, there exists $C > 0$ such that

$$\|v\|_{\mathcal{Z}_T} + \sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k v(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \le C\left(\|h_1\|_{H^{\frac{1}{3}}(0,T)} + \|h_2\|_{H^{\frac{1}{3}}(0,T)} + \|h_3\|_{L^2(0,T)}\right).$$

Henceforth we denote by $W_b(t)\vec{h}$ the solution of (3.18) corresponding to $\vec{h} = (h_1, h_2, h_3)$. Considering this, (BONA; SUN; ZHANG, 2003, Proposition 2.1) gives us the following proposition.

Proposition 3.4 *Let $L, T > 0$ be given. For any $v_0 \in L^2(0,L)$ there exists a unique solution $v \in \mathcal{Z}_T$ for the problem*

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v = 0, & t \in (0,T), \ x \in (0,L), \\ v(t,0) = 0, \quad v(t,L) = 0, \quad \partial_x v(t,L) = 0, & t \in (0,T), \\ v(0,x) = v_0(x), & x \in (0,L), \end{cases} \tag{3.19}$$

which is given by $v = W(\cdot)v_0$ where $\{W(t)\}_{t\ge 0}$ is the C_0-semigroup of contractions generated in $L^2(0,L)$ by the operator $A_L v = -\partial_x v - \partial_x^3 v$ with domain

$$D(A_L) = \{w \in H^3(0,L); \ v(0) = v(L) = \partial_x v(L) = 0\}.$$

Moreover, there exists a constant $C > 0$ such that

$$\|v\|_{\mathcal{Z}_T} \le C\|v_0\|_{L^2(0,L)} \quad \textit{and} \quad \|\partial_x v(\cdot,0)\|_{L^2(0,T)} \le C\|v_0\|_{L^2(0,L)}.$$

We will show that the solution $v = W(\cdot)v_0$ of (3.19) possesses the Kato smoothing property. According to (KRAMER; RIVAS; ZHANG, 2013), the linear KdV equation

$$\begin{cases} \partial_t z + \partial_x z + \partial_x^3 z = 0, & t \in \mathbb{R}_+, \ x \in \mathbb{R} \\ z(0,x) = z_0(x) \end{cases}$$

has a unique solution, given by

$$z(t,x) = (W_{\mathbb{R}} z_0)(x) := c\int_{\mathbb{R}} e^{i(\xi^3-\xi)t} e^{ix\xi} \hat{z_0}(\xi) d\xi$$

where $c \in \mathbb{R}$ and $\hat{z_0}$ denotes the Fourier transform of z_0. The result can be read below.

Proposition 3.5 (Kramer, Rivas and Zhang (2013), Lemma 2.9) *There exists a constant $C > 0$ such that, for every $z_0 \in L^2(\mathbb{R})$,*

$$\sum_{k=0}^{2} \sup_{x\in\mathbb{R}} \|\partial_x^k W_{\mathbb{R}}(\cdot)v_0(x)\|_{H_t^{\frac{1-k}{3}}(\mathbb{R})} \le C\|v_0\|_{L^2(\mathbb{R})}.$$

Additionally, (BONA; SUN; ZHANG, 2003) showed the following:

Proposition 3.6 (Bona, Sun and Zhang (2003), Proposition 2.13) *Let $L, T > 0$ be given. There exists a constant $C > 0$ such that, for every $v_0 \in L^2(0,L)$, the solution $v = W(\cdot)v_0$ of (3.19) satisfies*

$$\sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k v(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C\|v_0\|_{L^2(0,L)}.$$

Combining the previous propositions, precisely, Propositions 3.3 and 3.6, and (CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017, Proposition 2.6), the next results are verified.

Proposition 3.7 *Consider $L, T > 0$. For every $v_0 \in L^2(0,L)$ and $(h_1,h_2,h_3) \in \mathcal{H}_T := H^{\frac{1}{3}}(0,T)\times H^{\frac{1}{3}}(0,T)\times L^2(0,T)$ there exists a unique solution $v \in \mathcal{Z}_T := C([0,T];L^2(0,L))\cap L^2(0,L;H^1(0,L))$ to the problem*

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v = 0, & t\in(0,T),\ x\in(0,L), \\ v(t,0) = h_1(t), \quad v(t,L) = h_2(t), \quad \partial_x v(t,L) = h_3(t), & t\in(0,T), \\ v(0,x) = v_0(x), & x\in(0,L), \end{cases}$$

which possesses the sharp Kato smoothing property

$$\partial_x^k v \in L_x^\infty\left(0,L;H^{\frac{1-k}{3}}(0,T)\right),\ k=0,1,2.$$

Furthermore, there exists $C > 0$ such that

$$\|v\|_{\mathcal{Z}_T} + \sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k v(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C\left(\|v_0\|_{L^2(0,L)} + \|h\|_{\mathcal{H}_T}\right).$$

Corollary 3.1 *Let $T, L > 0$ be given. Given $v^T \in L^2(0,L)$ and $h \in H^{\frac{1}{3}}(0,T)$ there exists a unique solution $v \in \mathcal{Z}_T$ for the problem*

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v = 0, & t\in(0,T),\ x\in(0,L), \\ v(t,0) = h(t), \quad v(t,L) = \partial_x v(t,0) = 0, & t\in(0,T), \\ v(T,x) = v^T(x), & x\in(0,L) \end{cases}$$

which satisfies

$$\partial_x^k v \in L_x^\infty\left(0,L;H^{\frac{1-k}{3}}(0,T)\right),\ k=0,1,2$$

and

$$\|v\|_{\mathcal{Z}_T} + \sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k v(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C\left(\|v^T\|_{L^2(0,L)} + \|h\|_{H^{\frac{1}{3}}(0,T)}\right),$$

for some constant $C > 0$.

Proposition 3.8 *Given* $T, L > 0$, $v^T \in L^2(0,L)$ *and* $h \in H^{-\frac{1}{3}}(0,T)$ *there exists a unique solution* $v \in \mathcal{Z}_T$ *of the problem*

$$\begin{cases} \partial_t v + \partial_x v + \partial_x^3 v = 0, & t \in (0,T),\ x \in (0,L), \\ v(t,L) = \partial_x v(t,0) = 0, \quad \partial_x^2 v(t,0) = h(t), & t \in (0,T), \\ v(T,x) = v^T(x), & x \in (0,L) \end{cases}$$

which satisfies

$$\partial_x^k v \in L_x^\infty \left(0, L; H^{\frac{1-k}{3}}(0,T)\right),\ k = 0,1,2.$$

and

$$\|v\|_{\mathcal{Z}_T} + \sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k v(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C\left(\|v^T\|_{L^2(0,L)} + \|h\|_{H^{-\frac{1}{3}}(0,T)}\right).$$

for some positive constant C.

3.4.4 Trace estimates: For KdV in star-graph

We are now in a position to prove trace estimates to (3.13). The first result ensures the hidden regularity.

Proposition 3.9 *For* $T > 0$ *and every* $\varphi^T \in \mathbb{L}^2(\mathcal{T})$, *the solution* φ *of* (3.13) *posses the hidden regularity*

$$\partial_x^k \varphi_j \in L_x^\infty \left(0, l_j; H^{\frac{1-k}{3}}(0,T)\right),\ k = 0,1,2, \quad j = 1,\dots,N.$$

Moreover, there exists $C > 0$ *such that*

$$\sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k \varphi_j(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C\|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})},\ j = 1,\dots,N.$$

Proof: Our proof will be split into two cases. We divide it into two cases.

Case 1: $N \in \mathbb{N}$ and $l_j = L$, for every $j = 1,\dots,N$.

For each $j = 1,\dots,N$ define $\xi_j = \varphi_1 - \varphi_j$. Note that

$$\xi_j \in \mathcal{Z}_T = C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

and its solves

$$\begin{cases}\partial_t\xi_j+\partial_x\xi_j+\partial_x^3\xi_j=0, & t\in(0,T),\ x\in(0,L),\\ \xi_j(t,0)=\xi_j(t,L)=\partial_x\xi_j(t,0)=0, & t\in(0,T),\\ \xi_j(T,x)=\varphi_1^T(x)-\varphi_j^T(x), & x\in(0,L).\end{cases}$$

Thanks to Corollary 3.1 we get $\partial_x^k\xi_j\in L_x^\infty\left(0,L;H^{\frac{1-k}{3}}(0,T)\right)$, for $k=0,1,2$, and

$$\|\xi_j\|_{\mathcal{Z}_T}+\sum_{k=0}^{2}\sup_{x\in(0,L)}\|\partial_x^k\xi_j(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)}\leq C\left(\|\varphi_1^T-\varphi_j^T\|_{L^2(0,L)}\right)\leq C\|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})} \quad (3.20)$$

where $C>0$ is a constant.

Now, since $\varphi\in L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))$ and $H^1(0,L)\hookrightarrow C([0,L])$

$$\int_0^T|\varphi_1(t,0)|^2dt\leq c\|\varphi\|^2_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))}$$

for some $c>0$. From Proposition 3.2 it follows that

$$\int_0^T|\varphi_1(t,0)|^2dt\leq\tilde{C}\|\varphi^T\|^2_{\mathbb{L}^2(\mathcal{T})},$$

for $\tilde{C}>0$. Then $\varphi_1(\cdot,0)\in L^2(0,T)\hookrightarrow H^{-\frac{1}{3}}(0,T)$ and by the last inequality

$$\|\varphi_1(\cdot,0)\|_{H^{-\frac{1}{3}}(0,T)}\leq C\|\varphi_1(\cdot,0)\|_{L^2(0,T)}\leq C\|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})}, \quad (3.21)$$

for some constant $C>0$.

Defining $\psi=\sum_{j=1}^{N}\varphi_j$ we conclude that

$$\psi\in\mathcal{Z}_T=C([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))$$

and its solves the system

$$\begin{cases}\partial_t\psi+\partial_x\psi+\partial_x^3\psi=0, & t\in(0,T),\ x\in(0,L),\\ \psi(t,L)=\partial_x\psi(t,0)=0,\quad \partial_x^2\psi(t,0)=(\alpha-N)\varphi_1(t,0), & t\in(0,T),\\ \psi(T,x)=\sum_{j=1}^{N}\varphi_j^T(x), & x\in(0,L).\end{cases}$$

Then, the Proposition 3.8 gives us $\partial_x^k\psi\in L_x^\infty\left(0,L;H^{\frac{1-k}{3}}(0,T)\right)$ for $k=0,1,2$ and

$$\|\psi\|_{\mathcal{Z}_T}+\sum_{k=0}^{2}\sup_{x\in(0,L)}\|\partial_x^k\psi(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)}\leq C\left(\sum_{j=1}^{N}\|\varphi_j^T\|_{L^2(0,L)}+|\alpha-N|\|\varphi_1(\cdot,0)\|_{H^{-\frac{1}{3}}(0,T)}\right),$$

for some positive constant C. Using (3.21) we get another constant, still denoted by $C > 0$ such that

$$\|\psi\|_{\mathcal{Z}_T} + \sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k \psi(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C(|\alpha - N|)\|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})}. \tag{3.22}$$

Observe that

$$\sum_{j=1}^{N} \xi_j + \psi = \sum_{j=1}^{N} (\varphi_1 - \varphi_j) + \sum_{j=1}^{N} \varphi_j = N\varphi_1,$$

that is,

$$\varphi_1 = \frac{1}{N}\left(\sum_{j=1}^{N} \xi_j + \psi\right).$$

But we know that $\psi, \xi_j \in L_x^\infty\left(0, L; H^{\frac{1-k}{3}}(0,T)\right)$ for $j = 1, \ldots, N$, so we conclude that

$$\partial_x^k \varphi_1 \in L_x^\infty\left(0, L; H^{\frac{1-k}{3}}(0,T)\right),$$

for $k = 0, 1, 2$. Moreover,

$$\begin{aligned}\sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k \varphi_1(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq &\frac{1}{N}\left(\sum_{j=1}^{N}\sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k \xi_j(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)}\right.\\ &\left. + \sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k \psi(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)}\right).\end{aligned}$$

Due to the inequalities (3.20) and (3.22) it follows that

$$\sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k \varphi_1(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C\|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})} \tag{3.23}$$

where $C = C(\alpha, N) > 0$. Now, since φ_j is the solution of

$$\begin{cases}\partial_t \varphi_j + \partial_x \varphi_j + \partial_x^3 \varphi_j = 0, & t \in (0,T),\ x \in (0,L),\\ \varphi_j(t,0) = \varphi_1(t,0),\ \varphi_j(t,L) = \partial_x \varphi_j(t,0) = 0, & t \in (0,T),\\ \varphi_j(T,x) = \varphi_j^T(x), & x \in (0,L),\end{cases}$$

as $\varphi_1(t,0) \in H^{\frac{1}{3}}(0,T)$, Corollary 3.1 ensures that $\partial_x^k \varphi_j \in L_x^\infty\left(0, L; H^{\frac{1-k}{3}}(0,T)\right)$ for $k = 0, 1, 2$ and using (3.23) we get

$$\sum_{k=0}^{2} \sup_{x\in(0,L)} \|\partial_x^k \varphi_j(\cdot,x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C\|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})},$$

for $C > 0$ showing this case.

Case 2: $N \in \mathbb{N}$ and arbitrary lengths $l_j > 0,\ j = 1, \ldots, N$.

Define $L = \max_{j=1,\ldots,N} l_j$ and

$$\tilde{\varphi}_j^T(x) := \begin{cases} \varphi_j^T(x), & x \in (0, l_j), \\ 0, & x \in (0, L) \backslash (0, l_j). \end{cases}$$

Denote by $\tilde{\varphi} \in C([0,T], \mathbb{L}^2(\tilde{\mathcal{T}})) \cap L^2(0, T, \mathbb{H}_e^1(\tilde{\mathcal{T}}))$ the solution of (3.12) associated to data $\tilde{\varphi}^T = (\tilde{\varphi}_1^T, ..., \tilde{\varphi}_N^T) \in \mathbb{L}^2(\tilde{\mathcal{T}}) = (L^2(0, L))^N$. From the previous case, we have

$$\sum_{k=0}^{2} \sup_{x \in (0,L)} \|\partial_x^k \tilde{\varphi}_j(\cdot, x)\|_{H^{\frac{1-k}{3}}(0,T)} \leq C \|\tilde{\varphi}^T\|_{\mathbb{L}^2(\tilde{\mathcal{T}})}$$

It follows from the definition that $\|\tilde{\varphi}^T\|_{\mathbb{L}^2(\tilde{\mathcal{T}})} = \|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})}$. Also from the definition we have $\tilde{\varphi}_j^T = \varphi_j^T$ in $(0, l_j)$ so φ_j and $\tilde{\varphi}_j$ solve the system

$$\begin{cases} \partial_t \varphi_j + \partial_x \varphi_j + \partial_x^3 \varphi_j = 0, & t \in (0,T),\ x \in (0, l_j), \\ \varphi_j(t, 0) = \varphi_1(t, 0) = \varphi_j(t, l_j) = \partial_x \varphi_j(t, 0) = 0, & t \in (0,T), \\ \varphi_j(T, x) = \varphi_j^T(x), & x \in (0, l_j). \end{cases}$$

Then by uniqueness of solution we obtain $\tilde{\varphi}_j = \varphi_j$ in $(0,T) \times (0, l_j)$. Consequently, for $j = 1, \ldots, N$,

$$\begin{aligned} \sum_{k=0}^{2} \sup_{x \in (0,l_j)} \|\partial_x^k \varphi_j(\cdot, x)\|_{H^{\frac{1-k}{3}}(0,T)} &= \sum_{k=0}^{2} \sup_{x \in (0,l_j)} \|\partial_x^k \tilde{\varphi}_j(\cdot, x)\|_{H^{\frac{1-k}{3}}(0,T)} \\ &\leq \sum_{k=0}^{2} \sup_{x \in (0,L)} \|\partial_x^k \tilde{\varphi}_j(\cdot, x)\|_{H^{\frac{1-k}{3}}(0,T)} \\ &\leq C \|\tilde{\varphi}^T\|_{\mathbb{L}^2(\tilde{\mathcal{T}})} \\ &= C \|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})}. \end{aligned}$$

which concludes case 2 and, consequently, the proof of Proposition 3.9. □

To finish this subsection, we establish that the traces $\partial_x^2 \varphi_j(\cdot, 0), \partial_x^2 \varphi_j(\cdot, l_j)$ belong to $L^2(0,T)$ and depend continuously on the initial data.

Proposition 3.10 *Given $\varphi^T \in \mathbb{L}^2(\mathcal{T})$, we have that $\partial_x^2 \varphi_j(\cdot, x) \in L^2(0,T)$, for any $x \in [0, l_j]$. Moreover, the following inequality holds*

$$\int_0^T (\partial_x^2 \varphi_j(t, x))^2 dt \leq C \|\varphi^T\|_{\mathbb{L}^2(\mathcal{T})}^2,\ \forall\ \varphi^T \in \mathbb{L}^2(\mathcal{T}),$$

for some positive constant C.

Proof: Consider the operator

$$P_j : H^3(0,l_j) \to L^2(0,l_j)$$

defined by $P_j w = \partial_x w + \partial_x^3 w$. The graph norm in $H^3(0,l_j)$ associated to P_j is

$$\|w\|_{P_j} = \|w\|_{L^2(0,l_j)} + \|P_j w\|_{L^2(0,l_j)}.$$

From (MARTIN et al., 2019, Lemma A.2) there exists a constant $d_1 > 0$ such that

$$\|w\|_{H^3(0,l_j)} \leq d_1 \|w\|_{P_j}, \ \forall w \in H^3(0,l_j), \ j = 1, \ldots, N. \tag{3.24}$$

On the other hand, given $v = (v_1, ..., v_N) \in \mathcal{D}(A^*)$ we have

$$\|v\|_{\mathcal{D}(A^*)} = \|v\|_{\mathbb{L}^2(\mathcal{T})} + \|\partial_x v + \partial_x^3 v\|_{\mathbb{L}^2(\mathcal{T})} \geq \|v\|_{\mathbb{L}^2(\mathcal{T})} = \left(\sum_{j=1}^N \|v_j\|^2_{L^2(0,l_j)}\right)^{\frac{1}{2}} \geq \|v_j\|_{L^2(0,l_j)}$$

as well as

$$\|v\|_{\mathcal{D}(A^*)} \geq \|\partial_x v + \partial_x^3 v\|_{\mathbb{L}^2(\mathcal{T})} = \left(\sum_{j=1}^N \|\partial_x v_j + \partial_x^3 v_j\|^2_{L^2(0,l_j)}\right)^{\frac{1}{2}} \geq \|\partial_x v_j + \partial_x^3 v_j\|_{L^2(0,l_j)}.$$

Therefore, this yields

$$\|v_j\|_{P_j} \leq 2\|v\|_{\mathcal{D}(A^*)}, \ j = 1, \ldots, N. \tag{3.25}$$

Let us first assume that $\varphi_j^T \in \mathcal{D}((A^*)^2)$.

Claim: We claim that $\partial_x^2 \varphi_j(t,\cdot) \in C([0,l_j])$ for every $t \in [0,T]$ and $\partial_x^2 \varphi_j(\cdot, x) \in C^1([0,T])$ for every $x \in [0,l_j]$.

Indeed, since $\varphi \in C([0,T], \mathcal{D}(A^*))$ we have $\partial_x^2 \varphi_j(t,\cdot) \in H^1(0,l_j) \hookrightarrow C([0,l_j])$. Hence, fixed $x \in [0,l_j]$, for some constant $d_2 > 0$ we get

$$\begin{aligned} |\partial_x^2 \varphi_j(t,x) - \partial_x^2 \varphi_j(t_0,x)| &\leq \|\partial_x^2 \varphi_j(t,\cdot) - \partial_x^2 \varphi_j(t_0,\cdot)\|_{C([0,l_j])} \\ &\leq d_2 \|\partial_x^2 \varphi_j(t,\cdot) - \partial_x^2 \varphi_j(t_0,\cdot)\|_{H^1(0,l_j)} \\ &\leq d_2 \|\varphi_j(t,\cdot) - \varphi_j(t_0,\cdot)\|_{H^3(0,l_j)} \end{aligned}$$

for any $t, t_0 \in [0,T]$. Using (3.24) and (3.25) we obtain

$$|\partial_x^2 \varphi_j(t,x) - \partial_x^2 \varphi_j(t_0,x)| \leq 2 d_1 d_2 \|\varphi_j(t,\cdot) - \varphi_j(t_0,\cdot)\|_{\mathcal{D}(A^*)},$$

from where we conclude that $\partial_x^2\varphi_j(\cdot,x) \in C([0,T])$. Analogously, $\partial_t\varphi \in C([0,T],\mathcal{D}(A^*))$ so

$$\begin{aligned}
|\partial_x^2\partial_t\varphi_j(t,x) - \partial_x^2\partial_t\varphi_j(t_0,x)| &\leq \|\partial_x^2\partial_t\varphi_j(t,\cdot) - \partial_x^2\partial_t\varphi_j(t_0,\cdot)\|_{C([0,l_j])} \\
&\leq d_2\|\partial_x^2\partial_t\varphi_j(t,\cdot) - \partial_x^2\partial_t\varphi_j(t_0,\cdot)\|_{H^1(0,l_j)} \\
&\leq d_2\|\partial_t\varphi_j(t,\cdot) - \partial_t\varphi_j(t_0,\cdot)\|_{H^3(0,l_j)} \\
&\leq 2d_1d_2\|\partial_t\varphi_j(t,\cdot) - \partial_t\varphi_j(t_0,\cdot)\|_{\mathcal{D}(A^*)}
\end{aligned}$$

and, consequently, $\partial_t\partial_x^2\varphi_j(\cdot,x) \in C([0,T])$, showing the claim.

Due to embeddings

$$H^1(0,T) \hookrightarrow H^{\frac{1}{3}}(0,T) \hookrightarrow L^2(0,T) \hookrightarrow H^{-\frac{1}{3}}(0,T)$$

we obtain, for each $x \in [0,l_j]$,

$$\int_0^T \partial_x^2\varphi_j^2(t,x)dt = \|\partial_x^2\varphi_j(\cdot,x)\|^2_{L^2(0,T)} \leq C\|\partial_x^2\varphi_j(\cdot,x)\|^2_{H^{\frac{1}{3}}(0,T)} = C\|\partial_x^2\varphi_j(\cdot,x)\|^2_{H^{-\frac{1}{3}}(0,T)}.$$

Using Proposition 3.9 it follows that

$$\int_0^T \partial_x^2\varphi_j^2(t,x)dt \leq C\|\varphi_j^T\|^2_{\mathbb{L}^2(\mathcal{T})}.$$

Finally, with an argument of density and continuity, we extend the result for every $\varphi_j^T \in \mathbb{L}^2(\mathcal{T})$. The result is thus achieved. □

3.4.5 Regularity for data in regular space

For this subsection it is convenient to remember the spaces defined in (3.3) and (3.4), namely, for $m = 0, ..., N$,

$$X_m = \prod_{j=1}^{m} L^2(0,l_j) \times \prod_{i=m+1}^{N} H_0^1(0,l_i) \quad \text{and} \quad Y_m = \prod_{j=1}^{m} H^1(0,l_j) \times \prod_{i=m+1}^{N} H^2(0,l_i).$$

With this spaces in hand, we have the following proposition.

Proposition 3.11 *If the final data, $\varphi^T \in X_m$ then the corresponding solution φ of the system* (3.13) *has the additional regularity $\varphi \in L^2(0,T;Y_m)$ with*

$$\|\varphi\|_{L^2(0,T;Y_m)} \leq C\|\varphi^T\|_{X_m}.$$

for some positive constant C. Furthermore, the estimate

$$\begin{aligned}
\|\varphi^T\|^2_{X_m} \leq{}& \frac{1}{T}\sum_{j=1}^{N}\int_0^T\int_0^{l_j}\varphi_j^2 + \left(\frac{1}{T}+\frac{6}{\sigma}\right)\sum_{j=m+1}^{N}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2 + 2\left(\alpha - \frac{N}{2}\right)\int_0^T\varphi_1^2(t,0) \\
&+ 2\sum_{j=m+1}^{N}\|\varphi_j^T\|^2_{L^2(0,l_j)} + \sum_{j=1}^{m}\int_0^T\partial_x\varphi_j^2(t,l_j) + \sum_{j=m+1}^{N}\int_0^T\partial_x^2\varphi_j^2(t,l_j),
\end{aligned}$$

is verified for every $\varphi^T \in X_m$*, where* $\sigma := \min_{1\le j\le N} l_j$.

Proof: Once we have that

$$\|\varphi\|^2_{L^2(0,T;Y_m)} = \int_0^T \|\varphi(t,\cdot)\|^2_{Y_m} dt = \int_0^T \sum_{j=1}^{m} \|\varphi_j(t,\cdot)\|^2_{H^1(0,l_j)} dt + \int_0^T \sum_{j=m+1}^{N} \|\varphi_j(t,\cdot)\|^2_{H^2(0,l_j)} dt$$
$$\le \|\varphi\|^2_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))} + \sum_{j=m+1}^{N} \int_0^T \|\varphi_j(t,\cdot)\|^2_{H^2(0,l_j)} dt,$$

from the first part of Proposition 3.2, it is sufficient to show that $\varphi_j \in L^2(0,T;H^2(0,l_j))$ for $j = m+1,\ldots,N$ with

$$\sum_{j=m+1}^{N} \|\varphi_j\|^2_{L^2(0,T;H^2(0,l_j))} \le \tilde{C}\|\varphi^T\|^2_{X_m} \tag{3.26}$$

for some constant $\tilde{C} > 0$.

To do this, first assume $\varphi^T \in \mathcal{D}(A^*) \cap X_m$, the result for all $\varphi^T \in X_m$ follows by an argument of density. In this case, it is clear that $\varphi_j(t,\cdot) \in H^2(0,l_j)$. Let $q_j \in C^\infty([0,T]\times[0,l_j])$ be given. Multiplying the adjoint system (3.13) by $q_j\partial_x^2\varphi_j$, integrating by parts, and using the boundary conditions, we get

$$\begin{aligned}\frac{1}{2}\int_0^T (q_j\partial_x^2\varphi_j^2)(t,l_j) + \frac{1}{2}\int_0^T (q_j\partial_x\varphi_j^2)(t,l_j) = -\frac{1}{2}\int_0^{l_j} (q_j\partial_x\varphi_j^2)(0,x) - \int_0^T (\partial_x q_j\partial_x\varphi_j\partial_x^2\varphi_j)(t,l_j)\\ -\frac{1}{2}\int_0^T\int_0^{l_j} \partial_x\varphi_j^2(\partial_t q_j + \partial_x q_j + \partial_x^3 q_j) + \frac{3}{2}\int_0^T\int_0^{l_j} \partial_x q_j\partial_x^2\varphi_j^2\\ +\frac{1}{2}\int_0^T q_j(t,0)\partial_x^2\varphi_j^2(t,0) + \frac{1}{2}\int_0^{l_j} q_j(T,x)\partial_x\varphi_j^2(T,x)\\ +\frac{1}{2}\int_0^T (\partial_x^2 q_j\partial_x\varphi_j^2)(t,l_j).\end{aligned} \tag{3.27}$$

Choosing $q_j = l_j - x$, it follows that

$$\begin{aligned}\frac{1}{2}\int_0^{l_j}(l_j - x)\partial_x\varphi_j^2(0,x) + \frac{3}{2}\int_0^T\int_0^{l_j}\partial_x^2\varphi_j^2 = \int_0^T (\partial_x\varphi_j\partial_x^2\varphi_j)(t,l_j) + \frac{1}{2}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2\\ +\frac{l_j}{2}\int_0^T \partial_x^2\varphi_j^2(t,0) + \frac{1}{2}\int_0^{l_j}(l_j - x)\partial_x\varphi_j^2(T,x).\end{aligned}$$

By Young's inequality and Propositions 3.2 and 3.10, we conclude that there exists $C > 0$ such that

$$\int_0^T (\partial_x\varphi_j\partial_x^2\varphi_j)(t,l_j) \le \frac{1}{2}\int_0^T \partial_x\varphi_j^2(t,l_j) + \frac{1}{2}\int_0^T \partial_x^2\varphi_j^2(t,l_j) \le C\|\varphi^T\|^2_{\mathbb{L}^2(\mathcal{T})}.$$

Defining $M = \max_{1\le j\le N} l_j$, for $j = m+1,\ldots,N$, and, again, using Propositions 3.2 and 3.10, it follows that exists another constant, still denote by $C > 0$, such that

$$\int_0^T\int_0^{l_j}\partial_x^2\varphi_j^2 \le C\|\varphi^T\|^2_{X_m}.$$

Combining this with the Proposition 3.2 we obtain, $\varphi_j \in L^2(0,T;H^2(0,l_j))$ and

$$\|\varphi_j\|_{L^2(0,T;H^2(0,l_j))} \leq C^* \|\varphi^T\|_{X_m}, \ \forall \ \varphi^T \in \mathcal{D}(A^*) \cap X_m,$$

where $C^* > 0$ is a constant and $j = m+1, \ldots, N$. Since $\mathcal{D}(A^*) \cap X_m$ is dense in X_m, the map

$$\begin{array}{rcl} \mathcal{D}(A^*) \cap X_m & \to & L^2(0,T;H^2(0,l_j)) \\ \varphi_T & \mapsto & \varphi_j \end{array}$$

extends continuously to the entire X_m with

$$\|\varphi_j\|_{L^2(0,T;H^2(0,l_j))} \leq C^* \|\varphi^T\|_{X_m}, \ \forall \varphi^T \in X_m$$

and therefore (3.26) holds with $\tilde{C} = (N-m)(C^*)^2$.

For the second part, in (3.27) choosing $q_j = t$ we get

$$\frac{1}{2}\int_0^T t\partial_x^2\varphi_j^2(t,l_j) + \frac{1}{2}\int_0^T t\partial_x\varphi_j^2(t,l_j) + \frac{1}{2}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2 = \frac{1}{2}\int_0^T t\partial_x^2\varphi_j^2(t,0) + \frac{T}{2}\int_0^{l_j}\partial_x\varphi_j^2(T,x),$$

where it follows that

$$\|\partial_x\varphi_j^T\|^2_{L^2(0,l_j)} \leq \frac{1}{T}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2 + \int_0^T\partial_x\varphi_j^2(t,l_j) + \int_0^T\partial_x^2\varphi_j^2(t,l_j), \ j = 1,\ldots,N. \tag{3.28}$$

On the other hand, by multiplying the adjoint system (3.13) by $q_j\varphi_j$, integrating by parts, and using the boundary conditions, we get

$$\begin{aligned} \frac{1}{2}\int_0^{l_j} q_j(T,x)\varphi_j^2(T,x) + \frac{3}{2}\int_0^T\int_0^{l_j}\partial_x q_j\partial_x\varphi_j^2 =& \frac{1}{2}\int_0^T \left(q_j(t,0) + \partial_x^2 q_j(t,0)\right)\varphi_1^2(t,0) \\ &+ \int_0^T q_j(t,0)\varphi_1(t,0)\partial_x^2\varphi_j(t,0) \\ &+ \frac{1}{2}\int_0^T q_j(t,l_j)\partial_x\varphi_j^2(t,l_j) \\ &+ \frac{1}{2}\int_0^T\int_0^{l_j}\varphi_j^2(\partial_t q_j + \partial_x q_j + \partial_x^3 q_j) \\ &+ \frac{1}{2}\int_0^{l_j} q_j(0,x)\varphi_j^2(0,x), \end{aligned} \tag{3.29}$$

for $j = 1,\ldots,N$. Choosing $q_j = x$ in (3.29), it follows that

$$\frac{l_j}{2}\int_0^T\partial_x\varphi_j^2(t,l_j) + \frac{1}{2}\int_0^T\int_0^{l_j}\varphi_j^2 + \frac{1}{2}\int_0^{l_j}x\varphi_j^2(0,x) = \frac{1}{2}\int_0^{l_j}x\varphi_j^2(T,x) + \frac{3}{2}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2$$

which implies, in particular, that

$$\int_0^T\partial_x\varphi_j^2(t,l_j) \leq \int_0^{l_j}\varphi_j^2(T,x) + \frac{3}{l_j}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2\ , \ j = 1,\ldots,N. \tag{3.30}$$

Now, assuming $q_j(t,0)=q_1(t,0)$ and summing on j in (3.29) we obtain

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{N}\int_0^{l_j} q_j(T,x)\varphi_j^2(T,x)+\frac{3}{2}\sum_{j=1}^{N}\int_0^T\int_0^{l_j}\partial_x q_j\partial_x\varphi_j^2 =&\frac{1}{2}\sum_{j=1}^{N}\int_0^T\int_0^{l_j}\varphi_j^2(\partial_t q_j+\partial_x q_j+\partial_x^3 q_j)\\ &+\left(\alpha-\frac{N}{2}\right)\int_0^T q_1(t,0)\varphi_1^2(t,0)\\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^T q_j(t,l_j)\partial_x\varphi_j^2(t,l_j)\\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^T \partial_x^2 q_j(t,0)\varphi_1^2(t,0)\\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^{l_j} q_j(0,x)\varphi_j^2(0,x).\end{aligned}$$

Now, choosing $q_j=t$ in (3.29), we also have that

$$\sum_{j=1}^{N}\|\varphi_j^T\|^2_{L^2(0,l_j)}\leq\frac{1}{T}\sum_{j=1}^{N}\int_0^T\int_0^{l_j}\varphi_j^2+2\left(\alpha-\frac{N}{2}\right)\int_0^T\varphi_1^2(t,0)+\sum_{j=1}^{N}\int_0^T\partial_x\varphi_j^2(t,l_j), \quad (3.31)$$

for $j=1,\dots,N$. Using (3.28) and (3.31) we get that

$$\begin{aligned}\|\varphi^T\|^2_{X_m}=&\sum_{j=1}^{m}\|\varphi_j^T\|^2_{L^2(0,l_j)}+\sum_{j=m+1}^{N}\|\varphi_j^T\|^2_{H_0^1(0,l_j)}\\ =&\sum_{j=1}^{m}\|\varphi_j^T\|^2_{L^2(0,l_j)}+\sum_{j=m+1}^{N}\|\partial_x\varphi_j^T\|^2_{L^2(0,l_j)}\\ \leq&\frac{1}{T}\sum_{j=1}^{N}\int_0^T\int_0^{l_j}\varphi_j^2+2\left(\alpha-\frac{N}{2}\right)\int_0^T\varphi_1^2(t,0)+\sum_{j=1}^{N}\int_0^T\partial_x\varphi_j^2(t,l_j)\\ &+\frac{1}{T}\sum_{j=m+1}^{N}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2+\sum_{j=m+1}^{N}\int_0^T\partial_x\varphi_j^2(t,l_j)+\sum_{j=m+1}^{N}\int_0^T\partial_x^2\varphi_j^2(t,l_j).\end{aligned}$$

Observe that from (3.30), the following holds

$$2\sum_{j=m+1}^{N}\int_0^T\partial_x\varphi_j^2(t,l_j)\leq 2\sum_{j=m+1}^{N}\int_0^{l_j}\varphi_j^2(T,x)+\sum_{j=m+1}^{N}\frac{6}{l_j}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2,$$

and consequently

$$\begin{aligned}\|\varphi^T\|^2_{X_m}\leq&\frac{1}{T}\sum_{j=1}^{N}\int_0^T\int_0^{l_j}\varphi_j^2+\left(\frac{1}{T}+\frac{6}{\sigma}\right)\sum_{j=m+1}^{N}\int_0^T\int_0^{l_j}\partial_x\varphi_j^2+2\left(\alpha-\frac{N}{2}\right)\int_0^T\varphi_1^2(t,0)\\ &+2\sum_{j=m+1}^{N}\|\varphi_j^T\|^2_{L^2(0,l_j)}+\sum_{j=1}^{m}\int_0^T\partial_x\varphi_j^2(t,l_j)+\sum_{j=m+1}^{N}\int_0^T\partial_x^2\varphi_j^2(t,l_j),\end{aligned}$$

giving the result. □

3.5 BOUNDARY OBSERVATION: NEUMANN AND DIRICHLET CONDITIONS

From now on, $\alpha = N$. Fixed $m = 0, \ldots, N$, we will consider Neumann boundary controls on the first m edges and Dirichlet boundary controls on the remaining ones. We only analyze the reachable states from the origin, that is, we will consider $u_j^0 = 0$ (the case where $u_0 \in \mathbb{L}^2(\mathcal{T})$ is arbitrary and $u^T = 0$ is done similarly and leads to the same observability inequality). In these terms, we have the following characterization of the controllability.

Lemma 3.1 *For $T > 0$, the controls $g_j \in L^2(0,T)$, $j = 1, \ldots, m$ and $p_j \in L^2(0,T)$, $j = m+1, \ldots, N$, drive $u_j^0 = 0$ to $u^T \in \mathbb{L}^2(\mathcal{T})$ if and only if*

$$\sum_{j=1}^{N} \int_0^{l_j} u_j^T \varphi_j^T = \sum_{j=1}^{m} \int_0^T \partial_x \varphi_j(t, l_j) g_j(t) - \sum_{j=m+1}^{N} \int_0^T \partial_x^2 \varphi_j(t, l_j) p_j(t), \tag{3.32}$$

for any $\varphi^T = (\varphi_1^T, \ldots, \varphi_N^T) \in \mathbb{L}^2(\mathcal{T})$ and φ solution of the system (3.13) *associated to φ^T.*

Proof: Let u be the solution of (3.12) corresponding to u_j^0. Given $\varphi^T \in \mathbb{L}^2(\mathcal{T})$, multiplying the first equation in (3.12) by the solution φ of (3.13), integrating by parts and using the boundary conditions, we obtain

$$\sum_{j=1}^{N} \int_0^{l_j} u_j(T,x) \varphi_j^T - \sum_{j=1}^{N} \int_0^{l_j} u_j^0(x) \varphi_j(0,x) = \sum_{j=1}^{m} \int_0^T \partial_x \varphi_j(t, l_j) g_j(t) - \sum_{j=m+1}^{N} \int_0^T \partial_x^2 \varphi_j(t, l_j) p_j(t)$$

so, for $u_j^0 = 0$ we have that

$$\sum_{j=1}^{N} \int_0^{l_j} u_j(T,x) \varphi_j^T(x) = \sum_{j=1}^{m} \int_0^T \partial_x \varphi_j(t, l_j) g_j(t) - \sum_{j=m+1}^{N} \int_0^T \partial_x^2 \varphi_j(t, l_j) p_j(t), \forall \varphi^T \in \mathbb{L}^2(\mathcal{T}). \tag{3.33}$$

If $u(T, \cdot) = u^T$ then (3.32) immediately follows from (3.33). Conversely, if (3.32) holds, thanks to (3.33) holds that

$$(u(T,\cdot), \varphi^T)_{\mathbb{L}^2(\mathcal{T})} = (u^T, \varphi^T)_{\mathbb{L}^2(\mathcal{T})}, \ \forall \varphi^T \in \mathbb{L}^2(\mathcal{T}),$$

and consequently $u(T, \cdot) = u^T$, showing the lemma. □

Consider the bilinear map $B : \mathbb{L}^2(\mathcal{T}) \times \mathbb{L}^2(\mathcal{T}) \to \mathbb{R}$ given by

$$B(\varphi^T, \psi^T) = \sum_{j=1}^{N} \int_0^T u_j(T,x) \psi_j^T(x),$$

with $u = (u_1, \ldots, u_N)$ being the solution of the system (3.12) with boundary controls $p_j = -\partial_x^2 \varphi_j(\cdot, l_j)$ and $g_j = \partial_x \varphi_j(\cdot, l_j)$, where $\varphi(t, \cdot) = S(T-t)\varphi^T$. Let us prove some properties of the bilinear map B.

(i) B is continuous.

Using the multiplier method, we obtain

$$B(\varphi^T,\psi^T)=\sum_{j=1}^{N}\int_0^T u_j(T,x)\psi_j^T(x)=\sum_{j=1}^{m}\int_0^T g_j(t)\psi_{jx}(t,l_j)-\sum_{j=m+1}^{N}\int_0^T p_j(t)\psi_{jxx}(t,l_j)$$
$$=\sum_{j=1}^{m}\int_0^T \partial_x\varphi_j(t,l_j)\psi_{jx}(t,l_j)+\sum_{j=m+1}^{N}\int_0^T \partial_x^2\varphi_j(t,l_j)\psi_{jxx}(t,l_j).$$

From the Propositions 3.2 and 3.10, we have that B is continuous, giving (i).

(ii) If B is coercive, then exact controllability holds, that is, $u(T,\cdot)=u^T$.

Indeed, assume for a moment that B is coercive. Then given $u^T\in\mathbb{L}^2(\mathcal{T})$, from the Lax-Milgram theorem, there exists $\varphi^T\in\mathbb{L}^2(\mathcal{T})$ such that

$$\sum_{j=1}^{N}\int_0^{l_j} u_j^T\psi_j^T=B(\varphi^T,\psi^T),\ \forall\psi^T\in\mathbb{L}^2(\mathcal{T}).$$

Thus, for $\varphi(t,\cdot)=S(T-t)\varphi^T$, $p_j=-\partial_x^2\varphi_j(\cdot,l_j)$ and $g_j=\partial_x\varphi_j(\cdot,l_j)$ the solution u of (3.12) satisfies

$$\sum_{j=1}^{N}\int_0^{l_j} u_j^T\psi_j^T=B(\varphi^T,\psi^T)=\sum_{j=1}^{m}\int_0^T g_j(t)\psi_{jx}(t,l_j)-\sum_{j=m+1}^{N}\int_0^T p_j(t)\psi_{jxx}(t,l_j),$$

for every $\psi^T\in\mathbb{L}^2(\mathcal{T})$, which implies, by Lemma 3.1, that $u(T,\cdot)=u^T$, ensuring (ii).

3.5.1 Observability inequality

Note that the coercivity of B is equivalent to proving the following observability inequality, that is, the existence of a constant $C>0$ such that

$$\|\varphi^T\|^2_{\mathbb{L}^2(\mathcal{T})}\le C\left(\sum_{j=1}^{m}\|\partial_x\varphi_j(\cdot,l_j)\|^2_{L^2(0,T)}+\sum_{j=m+1}^{N}\|\partial_x^2\varphi_j(\cdot,l_j)\|^2_{L^2(0,T)}\right),\ \forall\varphi^T\in\mathbb{L}^2(\mathcal{T}). \tag{3.34}$$

But to check (3.34) it is sufficient to check

$$\|\varphi^T\|^2_{X_m}\le \tilde{C}\left(\sum_{j=1}^{m}\|\partial_x\varphi_j(\cdot,l_j)\|^2_{L^2(0,T)}+\sum_{j=m+1}^{N}\|\partial_x^2\varphi_j(\cdot,l_j)\|^2_{L^2(0,T)}\right),\ \forall\varphi^T\in X_m, \tag{3.35}$$

for some positive constant $\tilde{C}$.

Indeed, given $\varphi^T\in\mathbb{L}^2(\mathcal{T})$, since X_m is dense in $\mathbb{L}^2(\mathcal{T})$ there exists a sequence $(\varphi^{T,n})_{n\in\mathbb{N}}\subset X_m$ such that $\varphi^{T,n}\to\varphi^T$ in $\mathbb{L}^2(\mathcal{T})$. From Propositions 3.2 and 3.10 it follows that $\partial_x\varphi_j^n(\cdot,l_j)\to$

$\partial_x\varphi_j(\cdot,l_j)$ and $\partial_x^2\varphi_j^n(\cdot,l_j)\to\partial_x^2\varphi_j(\cdot,l_j)$ in $L^2(0,T)$. Furthermore, using the Poincaré inequality, we get, for some $c>0$,

$$\|\varphi^{T,n}\|_{\mathbb{L}^2(\mathcal{T})}\leq c\|\varphi^{T,n}\|_{X_m}.$$

If (3.35) holds, then

$$\|\varphi^{T,n}\|^2_{\mathbb{L}^2(\mathcal{T})}\leq c^2\tilde{C}\left(\sum_{j=1}^{m}\|\partial_x\varphi_j^n(\cdot,l_j)\|^2_{L^2(0,T)}+\sum_{j=m+1}^{N}\|\partial_x^2\varphi_j^n(\cdot,l_j)\|^2_{L^2(0,T)}\right)$$

and passing to the limit, it follows that

$$\|\varphi^{T}\|^2_{\mathbb{L}^2(\mathcal{T})}\leq c^2\tilde{C}\left(\sum_{j=1}^{m}\|\partial_x\varphi_j(\cdot,l_j)\|^2_{L^2(0,T)}+\sum_{j=m+1}^{N}\|\partial_x^2\varphi_j(\cdot,l_j)\|^2_{L^2(0,T)}\right)$$

so (3.34) holds with $C=c^2\tilde{C}>0$.

Remark 3.1 *The advantage of working with data in X_m for the adjoint system is that, in addition to the additional regularity for the solution, the estimate given in the Proposition 3.11 carries information related to the two types of controls used, while the corresponding estimate for data in $\mathbb{L}^2(\mathcal{T})$ carries only information related to the Neumann controls types.*

The following two results will be useful to investigate the inequality (3.35).

Lemma 3.2 *For any $L>0$, if $y\in H^1(0,L)$ then $-\partial_x y-\partial_x^3 y\in H^{-2}(0,L)$ with*

$$\|-\partial_x y-\partial_x^3 y\|_{H^{-2}(0,L)}\leq\|y\|_{H^1(0,L)}.$$

Proof: We know that $H^{-2}(0,L)$ has the following characterization:

$$H^{-2}(0,L)=\Big\{T\in\mathcal{D}'(0,L);\ T=g_0+\partial_x g_1+\partial_x^2 g_2,\ g_0,g_1,g_2\in L^2(0,L)\Big\}$$

where the derivatives are taken in the distributional sense. Moreover,

$$\|T\|_{H^{-2}(0,L)}=\inf\left\{\left(\sum_{j=1}^{2}\|g_i\|^2_{L^2(0,L)}\right)^{\frac{1}{2}};\ g_0,g_1,g_2\in L^2(0,L)\text{ and }T=g_0+\partial_x g_1+\partial_x^2 g_2\right\}.$$

Suppose that $y\in H^1(0,L)$. Defining $g_0=0$, $g_1=-y$ and $g_2=-\partial_x y$ we have $g_0,g_1,g_2\in L^2(0,L)$ and

$$-\partial_x y-\partial_x^3 y=g_0+\partial_x g_1+\partial_x^2 g_2$$

so $-\partial_x y - \partial_x^3 y \in H^{-2}(0,L)$. Furthermore,

$$\| -\partial_x y - \partial_x^3 y\|_{H^{-2}(0,L)} \leq \left(\sum_{j=1}^{2} \|g_i\|^2_{L^2(0,L)}\right)^{\frac{1}{2}} = \left(\|y\|^2_{L^2(0,L)} + \|\partial_x y\|^2_{L^2(0,L)}\right)^{\frac{1}{2}} = \|y\|_{H^1(0,L)},$$

giving the result. □

From now on, we will focus our efforts on describing the lengths $l_j = L > 0$ for which the observability (3.35) holds. We strictly follow the argument used in (ROSIER, 1997).

Lemma 3.3 *Let $T > 0$ be given. If the observability inequality* (3.35) *does not occur, then there exists $\varphi^T \in X_m$ with $\|\varphi^T\|_{X_m} = 1$ for which the corresponding solution φ of* (3.13) *satisfies*

$$\partial_x \varphi_j(\cdot, l_j) = 0,\ j = 1, \ldots, m \quad \textit{and} \quad \partial_x^2 \varphi_j(\cdot, l_j) = 0,\ j = m+1, \ldots, N. \tag{3.36}$$

Proof: Suppose that (3.35) is false. Then given $n \in \mathbb{N}$ there exists $\varphi^{T,n} \in X_m \backslash \{0\}$ such that

$$\sum_{j=1}^{m} \|\partial_x \varphi_j^n(\cdot, l_j)\|^2_{L^2(0,T)} + \sum_{j=m+1}^{n} \|\partial_x^2 \varphi_j^n(\cdot, l_j)\|^2_{L^2(0,T)} < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{3.37}$$

First note that thanks to Proposition 3.11 we have for $j = 1, \ldots, N$

$$\|\varphi_j^n\|_{L^2(0,T,H^1(0,l_j))} \leq \|\varphi^n\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))} \leq \|\varphi^n\|_{L^2(0,T;Y_m)} \leq C_9 \|\varphi^{T,n}\|_{X_m} = C_9,$$

that is, $(\varphi_j^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^1(0,l_j))$. On the other hand, using Lemma 3.2

$$\|\partial_t \varphi_j(t,\cdot)\|_{H^{-2}(0,l_j)} = \| -\partial_x \varphi_j^n(t,\cdot) - \partial_x^3 \varphi_j^n(t,\cdot)\|_{H^{-2}(0,l_j)} \leq \|\varphi_j^n(t,\cdot)\|_{H^1(0,l_j)}$$

from where do we obtain

$$\|\partial_t \varphi_j^n\|_{L^2(0,T;H^{-2}(0,l_j))} \leq \|\varphi_j^n\|_{L^2(0,T;H^1(0,l_j))} \leq C_9.$$

Thus $(\partial_t \varphi_j^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^{-2}(0,l_j))$. Since the first embedding in

$$H^1(0,l_j) \hookrightarrow L^2(0,l_j) \hookrightarrow H^{-2}(0,l_j)$$

is compact, from Aubin-Lions Lemma (see (AUBIN, 1963; SIMON, 1986)) it follows that $(\varphi_j^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;L^2(0,l_j))$ and therefore it admits a convergent subsequence in $L^2(0,T;L^2(0,l_j))$.

Analogously, the Proposition 3.11 gives us for $j = m+1, \ldots, N$

$$\|\varphi_j^n\|_{L^2(0,T;H^2(0,l_j))} \leq \|\varphi^n\|_{L^2(0,T;Y_m)} \leq C_9 \|\varphi^{T,n}\|_{X_m} = C_9,$$

that is, $(\varphi_j^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^2(0,l_j))$. As $(\partial_t\varphi_j^n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^{-2}(0,l_j))$ and the first embedding in

$$H^2(0,l_j)\hookrightarrow H^1(0,l_j)\hookrightarrow H^{-2}(0,l_j)$$

is compact, from Aubin-Lions Lemma (see (AUBIN, 1963; SIMON, 1986)) we have $(\varphi_j^n)_{n\in\mathbb{N}}$ relatively compact in $L^2(0,T;H^1(0,l_j))$, that is, it has a convergent subsequence in $L^2(0,T;H^1(0,l_j))$.

Now, due to Proposition 3.9, the sequence $(\varphi_1^n(\cdot,0))_{n\in\mathbb{N}}$ is bounded in $H^{\frac{1}{3}}(0,T)$ and thanks to compact embedding $H^{\frac{1}{3}}(0,T)\hookrightarrow L^2(0,T)$ we can extract from it a convergent subsequence in $L^2(0,T)$. In the same way, $(\varphi_j^{T,n})_{n\in\mathbb{N}}$ has a convergent subsequence in $L^2(0,l_j)$ due to compact embedding $H_0^1(0,l_j)\hookrightarrow L^2(0,l_j)$. In summary, there exists a subsequence $(\varphi^{T,n_k})_{k\in\mathbb{N}}$ of $(\varphi^{T,n})_{n\in\mathbb{N}}$ satisfying the following items:

$$\begin{cases}(\varphi_j^{n_k})_{k\in\mathbb{N}} \text{ converges in } L^2(0,T;L^2(0,l_j)),\ j=1,\dots,N,\\ (\varphi_j^{n_k})_{k\in\mathbb{N}} \text{ converges in } L^2(0,T;H^1(0,l_j)),\ j=m+1,\dots,N,\\ (\varphi_1^{n_k}(\cdot,0))_{k\in\mathbb{N}} \text{ converges in } L^2(0,T),\\ (\varphi_j^{T,n_k})_{k\in\mathbb{N}} \text{ converges in } L^2(0,l_j),\ j=m+1,\dots,N.\end{cases} \tag{3.38}$$

Now, let $(\varphi^{T,n_k})_{k\in\mathbb{N}}$ the subsequence of $(\varphi^{T,n})_{n\in\mathbb{N}}$ satisfying (3.38). From Proposition 3.11 we have

$$\begin{aligned}\|\varphi^{T,n_k}-\varphi^{T,n_r}\|_{X_m}^2 \leq &\frac{1}{T}\sum_{j=1}^{N}\|\varphi_j^{n_k}-\varphi_j^{n_r}\|_{L^2(0,T;L^2(0,l_j)}^2+\left(\frac{1}{T}+\frac{6}{\sigma}\right)\sum_{j=m+1}^{N}\|\varphi_j^{n_k}-\varphi_j^{n_r}\|_{L^2(0,T;H^1(0,l_j)}^2\\ &+2\left(\alpha-\frac{N}{2}\right)\|\varphi_1^{n_k}(\cdot,0)-\varphi_1^{n_r}(\cdot,0)\|_{L^2(0,T}^2+2\sum_{j=m+1}^{N}\|\varphi_j^{T,n_k}-\varphi_j^{T,n_r}\|_{L^2(0,l_j)}^2\\ &+\sum_{j=1}^{m}\|\partial_x\varphi_j^{n_k}(\cdot,l_j)-\partial_x\varphi_j^{n_r}(\cdot,l_j)\|_{L^2(0,T)}^2\\ &+\sum_{j=m+1}^{N}\|\partial_x^2\varphi_j^{n_k}(\cdot,l_j)-\partial_x^2\varphi_j^{n_r}(\cdot,l_j)\|_{L^2(0,T)}^2,\end{aligned}$$

which gives $(\varphi^{T,n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in X_m so, there exists $\varphi^T\in X_m$ such that $\varphi^{T,n_k}\to\varphi^T$ in X_m. Consider $\varphi=S(T-\cdot)\varphi^T$. Since $X_m\hookrightarrow\mathbb{L}^2(\mathcal{T})$ continuously, from Propositions 3.2 and 3.10 ensures that

$$\begin{cases}\partial_x\varphi_j^{n_k}(\cdot,l_j)\to\partial_x\varphi_j(\cdot,l_j) \text{ in } L^2(0,T),\quad j=1,\dots,m,\\ \partial_x^2\varphi_j^{n_k}(\cdot,l_j)\to\partial_x^2\varphi_j(\cdot,l_j) \text{ in } L^2(0,T),\quad j=m+1,\dots,N.\end{cases}$$

From (3.37) we get

$$\partial_x\varphi_j(\cdot,l_j)=0,\quad j=1,\dots,m \quad\text{and}\quad \partial_x^2\varphi_j(\cdot,l_2)=0,\quad j=m+1,\dots,N.$$

Since $\|\varphi^{T,n}\|_{X_m} = 1$ for every $n \in \mathbb{N}$, we also have, in the limit, that $\|\varphi^T\|_{X_m} = 1$. So, Lemma 3.3 holds. □

Lemma 3.4 *Let $T > 0$ and denote by N_T the space of data $\varphi^T \in X_m$ whose respective solutions $\varphi = S(T - \cdot)\varphi^T$ of* (3.13) *satisfy* (3.36). *If $N_T \neq \{0\}$ then there exists $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ such that*

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j),\ j = 1, \dots, N, \\ \varphi_j(0) = \varphi_1(0), & j = 1, \dots, N, \\ \sum_{j=1}^{N} \varphi_j''(0) = 0, & \\ \varphi_j(l_j) = \varphi_j'(0) = 0, & j = 1, \dots, N, \\ \varphi_j'(l_j) = 0, & j = 1, \dots, m, \\ \varphi_j''(l_j) = 0, & j = m+1, \dots, N. \end{cases} \tag{3.39}$$

Proof: Using the arguments as those given in (ROSIER, 1997, Lemma 3.4), follows that if $N_T \neq \emptyset$, the map $\varphi^T \in N_T \mapsto \tilde{A}\,(N_T) \subset \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at least) one eigenvalue; hence, there exists $\lambda \in \mathbb{C}$ and $\varphi^T \in N_T\backslash\{0\} \subset \mathcal{D}(A^*) \subset \mathbb{H}^3(\mathcal{T})$ such that (3.39) holds, which give us the lemma. □

From this point forward, we will organize our observability analysis into the following six cases:

- $N = 2$;
- $N \geq 3$ and $m = 1$;
- $N \geq 3$ and $m = N - 1$;
- $N > 3$ and $1 < m < N - 1$;
- $N \geq 2$ and $m = N$;
- $N \geq 2$ and $m = 0$,

(3.40)

remembering that $l_j = L$, for all j.

3.5.2 Case $N = 2$

In this case, we consider a slightly more general problem, namely

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j),\ j = 1, 2, \\ \varphi_2(0) = \varphi_1(0), & \\ \varsigma_{Ne}\varphi_1''(0) + \varsigma_{Di}\varphi_2''(0) = 0, & \\ \varphi_j(l_j) = \varphi_j'(0) = 0, & j = 1, 2, \\ \varphi_1'(l_1) = \varphi_2''(l_2) = 0, & \end{cases} \tag{3.41}$$

where $\varsigma_{Ne}, \varsigma_{Di} > 0$. Problem (3.39) corresponds to the case $\varsigma_{Ne} = \varsigma_{Di} = 1$. We can prove that the observability inequality (3.35) holds for any $L > 0$. This is a consequence of the following lemma.

Lemma 3.5 *Let $L > 0$ and suppose $l_1 = l_2 = L$. For any $\lambda \in \mathbb{C}$, does not exist $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ satisfying (3.41).*

Proof: For $\psi \in \mathbb{L}^2(\mathcal{T})$, we introduce the notation

$$\hat{\psi}_j(\xi) = \int_0^{l_j} \psi_j(x)e^{-ix\xi}dx.$$

Consider $\lambda \in \mathbb{C}$ and suppose that $\varphi \in \mathbb{H}^3(\mathcal{T})$ satisfies (3.41). We will show that the unique solution for (3.41) is the trivial one, that is, $\varphi = 0$.

Multiplying the system (3.41) by $e^{-ix\xi}$, integrating by parts in $(0, l_j)$ and using the boundary conditions we get, for every $\xi \in \mathbb{C}$,

$$\left[(i\xi)^3 + i\xi + \lambda\right] = \left[(1 - \xi^2)\varphi_j(0) - i\xi e^{-il_j\xi}\varphi_j'(l_j) - e^{-il_j\xi}\varphi_j''(l_j) + \varphi_j''(0)\right],\ j = 1, 2.$$

Writing $\lambda = ip$ with $p \in \mathbb{C}$ and multiplying this equation by i yields

$$\left(\xi^3 - \xi - p\right)\hat{\varphi}_j(\xi) = i\left[(1 - \xi^2)\varphi_j(0) - i\xi e^{-il_j\xi}\varphi_j'(l_j) - e^{-il_j\xi}\varphi_j''(l_j) + \varphi_j''(0)\right],\ j = 1, 2.$$

Setting

$$\kappa = \varphi_1(0) = \varphi_j(0), \quad \delta_j = -\varphi_j'(l_j), \quad \gamma_j = -\varphi_j''(l_j), \quad \beta_j = \varphi_j''(0)$$

one can write

$$\left(\xi^3 - \xi - p\right)\hat{\varphi}_j(\xi) = i\left[(1 - \xi^2)\kappa + i\delta_j\xi e^{-il_j\xi} + \gamma_j e^{-il_j\xi} + \beta_j\right], \quad j = 1, 2.$$

Since $\delta_1=\gamma_2=0$ and $l_1=l_2=L$ we have

$$\left(\xi^3-\xi-p\right)\hat{\varphi}_1(\xi)=i\left[(1-\xi^2)\kappa+\gamma_1 e^{-iL\xi}+\beta_1\right],\ \forall\ \xi\in\mathbb{C}, \tag{3.42}$$

$$\left(\xi^3-\xi-p\right)\hat{\varphi}_2(\xi)=i\left[(1-\xi^2)\kappa+i\delta_2\xi e^{-iL\xi}+\beta_2\right],\ \ \forall\ \xi\in\mathbb{C}. \tag{3.43}$$

Now, defining $f=\varphi_1-\varphi_2$ and $\beta=\beta_1-\beta_2$, from the above identities, we obtain

$$\left(\xi^3-\xi-p\right)\hat{f}(\xi)=i\left[(\gamma_1-i\delta_2\xi)e^{-iL\xi}+\beta\right],\ \forall\ \xi\in\mathbb{C}. \tag{3.44}$$

***Claim* 3.1** *If* $\gamma_1=0$ *or* $\beta=0$ *then* $\varphi=0$.

Proof: If $\gamma_1=0$ then φ_1 solves the problem

$$\begin{cases}\lambda\varphi_1+\varphi_1'+\varphi_1'''=0,\\ \varphi_1(L)=\varphi_1'(L)=\varphi_1''(L)=0,\end{cases}$$

so $\varphi_1=0$. Consequently, φ_2 satisfies

$$\begin{cases}\lambda\varphi_2+\varphi_2'+\varphi_2'''=0,\\ \varphi_2(0)=\varphi_2'(0)=\varphi_2''(0)=0,\end{cases}$$

then $\varphi_2=0$ and therefore $\varphi=0$.

Now, if $\beta=0$ then $\beta_1=\beta_2$ so φ_1 and φ_2 are solutions to the problem

$$\begin{cases}\lambda\varphi_i+\varphi_i'+\varphi_i'''=0,\quad i=1,2\\ \varphi_i(0)=\kappa,\quad \varphi_i'(0)=0,\quad \varphi_i''(0)=\beta_1.\end{cases}$$

By uniqueness of solution, it follows that $\varphi_1=\varphi_2$, which implies $f=0$. Hence $\hat{f}=0$ and since $\beta=0$, (3.44) becomes

$$(\gamma_1-i\delta_2\xi)e^{-iL\xi}=0,\ \forall\xi\in\mathbb{C}.$$

Evaluating this equality at $\xi=0$, we obtain $\gamma_1=0$, and, as seen before, this leads us to $\varphi=0$, showing Claim 3.1. □

Assuming

$$\gamma_1\neq 0\ \text{ and }\ \beta\neq 0, \tag{3.45}$$

with the following claim in hand, the Lemma 3.5 holds for the case $\lambda=0$.

***Claim* 3.2** *If* $\lambda=0$ *then does not exist* $\varphi=(\varphi_1,\varphi_2)\in\mathbb{H}^3(\mathcal{T})\backslash\{0\}$ *satisfying* (3.41).

Proof: We will show that for $\lambda = 0$, then we necessarily have $\varphi = 0$. To do this, let us multiply (3.42) and (3.43) by $-i$ to get

$$-i\left(\xi^3-\xi\right)\hat{\varphi}_1(\xi) = (1-\xi^2)\kappa + \gamma_1 e^{-iL\xi} + \beta_1, \ \forall\ \xi \in \mathbb{C}, \tag{3.46}$$

$$-i\left(\xi^3-\xi\right)\hat{\varphi}_2(\xi) = (1-\xi^2)\kappa + i\delta_2\xi e^{-iL\xi} + \beta_2, \quad \forall\ \xi \in \mathbb{C}. \tag{3.47}$$

Evaluating (3.46) at $\xi = 1$ and $\xi = -1$ we obtain, respectively,

$$\gamma_1 e^{-iL} = -\beta_1 \ \text{ and } \ \gamma_1 e^{iL} = -\beta_1$$

from where we have $\gamma_1 e^{-iL} = \gamma_1 e^{iL}$ and, since $\gamma_1 \neq 0$, it follows that

$$e^{-iL} = e^{iL}. \tag{3.48}$$

On the other hand, evaluating (3.47) at $\xi = 1$ and $\xi = -1$ we obtain, respectively,

$$i\delta_2 e^{-iL} = -\beta_2 \ \text{ and } \ -i\delta_2 e^{iL} = -\beta_2$$

which gives us

$$i\delta_2 e^{-iL} = -i\delta_2 e^{iL}.$$

If $\delta_2 \neq 0$, then the above equality provides

$$e^{-iL} = -e^{iL}. \tag{3.49}$$

Adding (3.48) and (3.49) we get $2e^{-iL} = 0$, which is not possible. Therefore, $\delta_2 = 0$ so φ_2 solves the problem

$$\begin{cases} \lambda\varphi_2 + \varphi_2' + \varphi_2''' = 0, \\ \varphi_2(L) = \varphi_2'(L) = \varphi_2''(L) = 0, \end{cases}$$

and consequently $\varphi_2 = 0$. Thanks to this fact and using the boundary conditions of (3.41), φ_1 is the solution of the following system

$$\begin{cases} \lambda\varphi_1 + \varphi_1' + \varphi_1''' = 0, \\ \varphi_1(0) = \varphi_1'(0) = \varphi_1''(0) = 0, \end{cases}$$

which tells us that $\varphi_1 = 0$ and, consequently $\varphi = 0$, which give the Claim 3.2. $\square$

Define $P_\lambda(\mu) = \mu^3 + \mu + \lambda$. Before studying the case $\lambda \neq 0$, we need to establish an important relationship between the multiplicity of the roots of P_λ and the solutions of (3.41).

***Claim* 3.3** *If P_λ admits multiples roots, then $\varphi = 0$.*

Proof: Let μ_0, μ_1 and μ_2 the roots of P_λ. By Girard's relations, we have

$$\mu_0 + \mu_1 + \mu_2 = 0, \tag{3.50}$$

$$\mu_0\mu_1 + \mu_1\mu_2 + \mu_0\mu_2 = 0, \tag{3.51}$$

$$\mu_0\mu_1\mu_2 = -\lambda. \tag{3.52}$$

If P_λ has a triple root thus, by (3.50), we have

$$\mu_0 = \mu_1 = \mu_2 = 0.$$

So, from (3.52) it follows that $\lambda = 0$. With this, the Claim 3.2 ensures that $\varphi = 0$.

Now, suppose that P_λ has a double root, let us say $\mu_1 = \mu_2 = \sigma$. Relation (3.50) gives $\mu_0 = -2\sigma$ and substituting it in (3.51) we have

$$-2\sigma \cdot \sigma + \sigma \cdot \sigma - 2\sigma \cdot \sigma = 1$$

which leads us to

$$\sigma = \pm\frac{i}{\sqrt{3}}. \tag{3.53}$$

By the theory of ordinary differential equations, the set

$$\left\{e^{-2\sigma x}, e^{\sigma x}, xe^{\sigma x}\right\}$$

is a generator of the solutions of the following ordinary differential equation

$$\lambda y + y' + y''' = 0.$$

Note that φ_2 solution of

$$\begin{cases} \lambda\varphi_2 + \varphi_2' + \varphi_2''' = 0, \\ \varphi_2(L) = \varphi_2'(0) = \varphi_2''(L) = 0, \end{cases}$$

ensures the existence of $d_0, d_1, d_2 \in \mathbb{C}$ such that

$$\varphi_2(x) = d_0 e^{-2\sigma x} + d_1 e^{\sigma x} + d_2 x e^{\sigma x}.$$

Consequently

$$\varphi_2'(x) = d_0(-2\sigma)e^{-2\sigma x} + d_1\sigma e^{\sigma x} + d_2(e^{\sigma x} + x\sigma e^{\sigma x}), \tag{3.54}$$

and

$$\varphi_2''(x) = d_0(-2\sigma)^2 e^{-2\sigma x} + d_1\sigma^2 e^{\sigma x} + d_2(\sigma e^{\sigma x} + \sigma e^{\sigma x} + x\sigma^2 e^{\sigma x}). \tag{3.55}$$

Observe that the boundary condition $\varphi_2(L) = 0$ gives us

$$d_0 e^{-2\sigma L} + d_1 e^{\sigma L} + d_2 L e^{\sigma L} = 0$$

which implies

$$d_1 + L d_2 = -d_0 e^{-3\sigma L}.$$

On the other hand, the condition $\varphi_2''(L) = 0$ provides, due the relation given by (3.55), that

$$4d_0\sigma^2 e^{-2\sigma L} + d_1\sigma^2 e^{\sigma L} + 2d_2\sigma e^{\sigma L} + d_2 L\sigma^2 e^{\sigma L} = 0,$$

or equivalently,

$$d_1\sigma + d_2(2 + L\sigma) = -4d_0\sigma e^{-3\sigma L}.$$

Thus, we have the system

$$\begin{cases} d_1 + L d_2 = -d_0 e^{-3\sigma L} \\ d_1\sigma + d_2(2 + L\sigma) = -4d_0\sigma e^{-3\sigma L} \end{cases}$$

Below, we solve this system for d_1 and d_2 using Gauss-Jordan elimination (we denote by r_k the k-th row of a matrix):

$$\begin{bmatrix} 1 & L & -d_0 e^{-3\sigma L} \\ \sigma & 2 + L\sigma & -4d_0\sigma e^{-3\sigma L} \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_2 - \sigma r_1} \begin{bmatrix} 1 & L & -d_0 e^{-3\sigma L} \\ 0 & 2 & -3d_0\sigma e^{-3\sigma L} \end{bmatrix} \xrightarrow{r_2 \leftrightarrow \frac{1}{2} r_2}$$

$$\begin{bmatrix} 1 & L & -d_0 e^{-3\sigma L} \\ 0 & 1 & -\frac{3d_0\sigma e^{-3\sigma L}}{2} \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_1 - L r_2} \begin{bmatrix} 1 & 0 & -d_0 e^{-3\sigma L} + \frac{3d_0\sigma L e^{-3\sigma L}}{2} \\ 0 & 1 & -\frac{3d_0\sigma e^{-3\sigma L}}{2} \end{bmatrix}.$$

Therefore

$$d_1 = -d_0 e^{-3\sigma L} + \frac{3d_0\sigma L e^{-3\sigma L}}{2} \quad \text{and} \quad d_2 = -\frac{3d_0\sigma e^{-3\sigma L}}{2}. \tag{3.56}$$

Using this values in (3.54) yields,

$$\varphi_2'(x) = d_0(-2\sigma)e^{-2\sigma x} + \left(-d_0 e^{-3\sigma L} + \frac{3d_0\sigma L e^{-3\sigma L}}{2}\right)\sigma e^{\sigma x} + \left(-\frac{3d_0\sigma e^{-3\sigma L}}{2}\right)(e^{\sigma x} + x\sigma e^{\sigma x}).$$

Since $\varphi_2'(0) = 0$ we get that

$$-2d_0\sigma - d_0\sigma e^{-3\sigma L} + \frac{3d_0\sigma^2 Le^{-3\sigma L}}{2} - \frac{3d_0\sigma e^{-3\sigma L}}{2} = 0.$$

Multiplying the previous equality by $2/\sigma$, we have

$$d_0\left(-4 - 2e^{-3\sigma L} + 3\sigma Le^{-3\sigma L} - 3e^{-3\sigma L}\right) = 0.$$

If $d_0 \neq 0$, then it follows that

$$-4 - 2e^{-3\sigma L} + 3\sigma Le^{-3\sigma L} - 3e^{-3\sigma L} = 0 \iff (-5 + 3\sigma L)\, e^{-3\sigma L} = 4.$$

Due to (3.53) this equality becomes

$$\left(-5 \pm \frac{3iL}{\sqrt{3}}\right) e^{\pm\frac{3iL}{\sqrt{3}}} = 4.$$

Taking the absolute value in the previous equality, we see that

$$\left|-5 \pm \frac{3iL}{\sqrt{3}}\right| = 4 \implies 25 + 3L^2 = 16 \implies 3L^2 < 0,$$

a contradiction. Therefore $d_0 = 0$ and from (3.56) it follows that $d_1 = d_2 = 0$, which results in $\varphi_2 = 0$. Hence, using the boundary conditions of (3.41), φ_1 solves the problem

$$\begin{cases} \lambda\varphi_1 + \varphi_1' + \varphi_1''' = 0, \\ \varphi_1(0) = \varphi_1'(0) = \varphi_1''(0) = 0, \end{cases}$$

that is, $\varphi_1 = 0$ and therefore $\varphi = 0$, giving the Claim 3.3. □

According to claims 3.2 and 3.3, it remains to study the case where $\lambda \in \mathbb{C}\backslash\{0\}$ and the roots of P_λ are all simple. In this case, defining $P(\xi) = \xi^3 - \xi - p$, due to identity

$$P_\lambda(i\xi) = -iP(\xi),\ \forall\xi \in \mathbb{C},$$

we have that the roots of P are all simple, too. Moreover, as $\lambda = ip$ we must have

$$p \in \mathbb{C}\backslash\{0\}.$$

With this in mind, the next claim completes the proof of Lemma 3.5.

***Claim* 3.4** *If $\lambda \in \mathbb{C}\backslash\{0\}$ and the roots of P_λ are all simple then $\delta_2 = 0$.*

Proof: Suppose, by contradiction, that $\delta_2 \neq 0$. Let ξ_0, ξ_1, ξ_2 the roots of P. As $p \neq 0$ we have $\xi_s \neq 0$ for $s = 0, 1, 2$. The relation (3.42) ensures that

$$(1 - \xi_s^2)\kappa + \gamma_1 e^{-iL\xi_s} + \beta_1 = 0.$$

Multiplying this equation by $1/\gamma_1$ yields

$$(1 - \xi_s^2)\kappa \frac{1}{\gamma_1} + e^{-iL\xi_s} + \frac{\beta_1}{\gamma_1} = 0, \ \ s = 0, 1, 2. \tag{3.57}$$

Now, from (3.43), note that

$$(1 - \xi_s^2)\kappa + i\delta_2 \xi_s e^{-iL\xi_s} + \beta_2 = 0.$$

Multiplying this equation by $1/i\delta_2\xi_s$, we obtain

$$(1 - \xi_s^2)\kappa \frac{1}{i\delta_2\xi_s} + e^{-iL\xi_s} + \frac{\beta_2}{i\delta_2\xi_s} = 0, \ \ s = 0, 1, 2. \tag{3.58}$$

Taking the difference between (3.58) and (3.57) yields that

$$(1 - \xi_s^2)\kappa \left(\frac{1}{\gamma_1} - \frac{1}{i\delta_2\xi_s} \right) + \frac{\beta_1}{\gamma_1} - \frac{\beta_2}{i\delta_2\xi_s} = 0.$$

Now, we multiply this equation by $\gamma_1 i\delta_2\xi_s$ to get

$$(1 - \xi_s^2)\kappa \left(i\delta_2\xi_s - \gamma_1 \right) + \beta_1 i\delta_2\xi_s - \beta_2\gamma_1 = 0,$$

that is,

$$\kappa i\delta_2\xi_s - \kappa i\delta_2\xi_s^3 - \kappa\gamma_1 + \kappa\gamma_1\xi_s^2 + \beta_1 i\delta_2\xi_s - \beta_2\gamma_1 = 0.$$

Reorganizing the terms, we can write

$$-\kappa i\delta_2\xi_s^3 + \kappa\gamma_1\xi_s^2 + (\kappa i\delta_2 + \beta_1 i\delta_2)\xi_s - (\kappa\gamma_1 + \beta_2\gamma_1) = 0, \ \ s = 0, 1, 2.$$

Since ξ_0, ξ_1, ξ_2 are all distinct, we conclude that ξ_s is a simple root of the polynomial $\tilde{P}$ defined by

$$\tilde{P}(\xi) = -\kappa i\delta_2\xi^3 + \kappa\gamma_1\xi^2 + (\kappa i\delta_2 + \beta_1 i\delta_2)\xi - (\kappa\gamma_1 + \beta_2\gamma_1),$$

for $s = 0, 1, 2$. Thus, there exist $c \in \mathbb{C}\backslash\{0\}$ such that

$$P(\xi) = c\tilde{P}(\xi), \ \forall \xi \in \mathbb{C}.$$

Hence, the corresponding coefficients of P and $c\tilde{P}$ must be equal. In particular,

$$-c\kappa i\delta_2 = 1 \quad \text{and} \quad c\kappa\gamma_1 = 0.$$

The first equality tells us that $\kappa \neq 0$, since in our hypothesis $\delta_2 \neq 0$. Thus, using this fact in the second equality, we obtain $\gamma_1 = 0$, which is a contradiction with (3.45); consequently, Claim 3.4 is complete. □

So, the previous claims ensure the Lemma 3.5. □

3.5.3 Case $N \geq 3$ and $m = 1$

Now we will consider a network with 3 or more edges, placing a Neumman control on the first edge and Dirichlet controls on the others. In this case the problem (3.39) becomes

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j), \ j = 1, ..., N, \\ \varphi_j(0) = \varphi_1(0), & j = 1, ..., N, \\ \sum_{j=1}^{N} \varphi_j''(0) = 0, & \\ \varphi_j(l_j) = \varphi_j'(0) = 0, & j = 1, ..., N, \\ \varphi_1'(l_1) = 0, & \\ \varphi_j''(l_j) = 0, & j = 2, ..., N, \end{cases} \tag{3.59}$$

and we have the following result.

Lemma 3.6 *Let $L > 0$ and assume $l_j = L$, for any $j = 1, ..., N$. There exist $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ satisfying* (3.59) *if and only if $L \in \mathcal{N}^*$.*

Proof: If $L \in \mathcal{N}^*$ then, from (GLASS; GUERRERO, 2010, Propositions 1 and 2), there exist $\lambda \in \mathbb{C}$ and $z \in H^3(0, L)\backslash\{0\}$ such that

$$\begin{cases} \lambda z + z' + z''' = 0, \\ z(0) = z(L) = z'(0) = z''(L) = 0. \end{cases}$$

Defining $\varphi_1 = 0, \ \varphi_2 = z, \ \varphi_3 = -z, \ \varphi_j = 0$ for $j = 4, ..., N$ and $\varphi = (\varphi_1, \varphi_2, ..., \varphi_N)$ then $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ and (λ, φ) satisfies (3.59).

Now, assuming that there exist $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ satisfying (3.59), we will show by contradiction that this leads to $L \in \mathcal{N}^*$. Suppose that this does not occur, that is, $L \notin \mathcal{N}^*$. Define the following functions

$$\psi = \sum_{j=2}^{N} \varphi_j \quad \text{and} \quad \psi_j = \psi - (N-1)\varphi_j, \; j = 2, ..., N.$$

Note that, for each $j \in \{2, ..., N\}$, ψ_j solves the problem

$$\begin{cases} \lambda\psi_j + \psi_j' + \psi_j'' = 0, \\ \psi_j(0) = \psi_j(L) = \psi_j'(0) = \psi_j''(L) = 0. \end{cases} \tag{3.60}$$

Since $L \notin \mathcal{N}^*$, from (GLASS; GUERRERO, 2010, Propositions 1 and 2) it follows that

$$\psi_j = 0, \text{ for } j = 2, ..., N,$$

so that

$$\varphi_j = \frac{1}{N-1}\psi, \text{ for } j = 2, ..., N.$$

Consequently,

$$\varphi_j = \varphi_2, \text{ for } j = 2, ..., N. \tag{3.61}$$

This implies that, φ_1 and φ_N satisfy the spectral problem

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j), \; j = 1, N, \\ \varphi_N(0) = \varphi_1(0), \\ \varphi_1''(0) + (N-1)\varphi_N''(0) = 0, \\ \varphi_j(L) = \varphi_j'(0) = 0, & j = 1, N, \\ \varphi_1'(L) = \varphi_N''(L) = 0, \end{cases}$$

which corresponds to (3.41) in the case $\varsigma_{Ne} = 1$ and $\varsigma_{Di} = N - 1$. Therefore, the unique solution is $\varphi_1 = \varphi_N = 0$. Finally, as $0 = \varphi_N = \varphi_j$ for $j = 2, \ldots, N$, we obtain $\varphi = 0$. □

3.5.4 Case $N \geq 3$ and $m = N - 1$

Here, we will consider a star graph with 3 or more edges, placing Neumann controls on the first $N - 1$ of them and a Dirichlet control on the last one. Hence, the problem (3.39) becomes

$$
\begin{cases}
\lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j),\ j = 1, ..., N, \\
\varphi_j(0) = \varphi_1(0), & j = 1, ..., N, \\
\sum_{j=1}^{N} \varphi_j''(0) = 0, & \\
\varphi_j(l_j) = \varphi_j'(0) = 0, & j = 1, ..., N, \\
\varphi_j'(l_j) = 0, & j = 1, ..., N-1 \\
\varphi_N''(l_N) = 0, &
\end{cases}
\tag{3.62}
$$

and we have the following result.

Lemma 3.7 *Let $L > 0$ and assume $l_j = L$, for $j = 1, ..., N$. There exist $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ satisfying* (3.62) *if and only if $L \in \mathcal{N}$.*

Proof: If $L \in \mathcal{N}$ it follows direct by (ROSIER, 1997, Lemma 3.5) that there exist $\lambda \in \mathbb{C}$ and $z \in H^3(0, L)\backslash\{0\}$ such that

$$
\begin{cases}
\lambda z + z' + z''' = 0, \\
z(0) = z(L) = z'(0) = z'(L) = 0.
\end{cases}
$$

Defining $\varphi_1 = z,\ \varphi_2 = -z,\ \varphi_j = 0$ for $j = 3, ..., N$ and $\varphi = (\varphi_1, \varphi_2, ..., \varphi_N)$ then $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ with (λ, φ) satisfying (3.62).

Conversely, assume that there exist $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ satisfying (3.62). We will show that $L \in \mathcal{N}$. Suppose, by contradiction, that $L \notin \mathcal{N}$ and define

$$
\psi = \sum_{j=1}^{N-1} \varphi_j \quad \text{and} \quad \psi_j = \psi - (N-1)\varphi_j,\ j = 1, ..., N-1.
$$

For each $j \in \{1, ..., N-1\}$, ψ_j solves the problem

$$
\begin{cases}
\lambda\psi_j + \psi_j' + \psi_j''' = 0, \\
\psi_j(0) = \psi_j(L) = \psi_j'(0) = \psi_j'(L) = 0.
\end{cases}
$$

Since $L \notin \mathcal{N}$, from (ROSIER, 1997, Lemma 3.5) it follows that

$$
\psi_j = 0,\ j = 1, ..., N-1
$$

so that

$$
\varphi_j = \frac{1}{N-1}\psi,\ j = 1, ..., N-1.
$$

Consequently,

$$\varphi_j = \varphi_1, \; j = 1, ..., N-1. \tag{3.63}$$

This implies that, φ_1 and φ_N satisfy the spectral problem

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j), \; j = 1, N, \\ \varphi_N(0) = \varphi_1(0), & \\ (N-1)\varphi_1''(0) + 1\varphi_N''(0) = 0, & \\ \varphi_j(L) = \varphi_j'(0) = 0, & j = 1, N, \\ \varphi_1'(L) = \varphi_N''(L) = 0, & \end{cases}$$

which corresponds to (3.41) in the case $\varsigma_{Ne} = N-1$ and $\varsigma_{Di} = 1$. Therefore, the unique solution is $\varphi_1 = \varphi_N = 0$. Finally, as $0 = \varphi_1 = \varphi_j$ for $j = 1, \ldots, N-1$, we obtain $\varphi = 0$. □

3.5.5 Case $N > 3$ and $1 < m < N-1$

First, note that the case $N = 3$ is fully encompassed in the situations already analyzed. Therefore, here we consider $N > 3$ and place both types of control, Neumann and Dirichlet, on more than one edge. We recall the problem (3.39) given by

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j'' = 0, & x \in (0, l_j), \; j = 1, ..., N, \\ \varphi_j(0) = \varphi_1(0), & j = 1, ..., N, \\ \sum_{j=1}^{N} \varphi_j''(0) = 0, & \\ \varphi_j(l_j) = \varphi_j''(0) = 0, & j = 1, ..., N, \\ \varphi_j'(l_j) = 0, & j = 1, ..., m, \\ \varphi_j''(l_j) = 0, & j = m+1, ..., N. \end{cases} \tag{3.64}$$

For this case, we have the following result.

Lemma 3.8 *Let $L > 0$ and assume $l_j = L$ for $j = 1, ..., N$. There exist $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ satisfying* (3.64) *if and only if $L \in \mathcal{N} \cup \mathcal{N}^*$.*

Proof: If $L \in \mathcal{N} \cup \mathcal{N}^*$ the result is a direct consequence of (ROSIER, 1997, Lemma 3.5) and (GLASS; GUERRERO, 2010, Propositions 1 and 2) (see Lemmas 3.7 and 3.5). Conversely, suppose there exist $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T}) \setminus \{0\}$ satisfying (3.64). We will demonstrate, by contradiction, that this implies $L \in \mathcal{N} \cup \mathcal{N}^*$. Assume, for the sake of contradiction, that $L \notin \mathcal{N} \cup \mathcal{N}^*$, and define

$$\psi = \sum_{j=1}^{m} \varphi_j \quad \text{and} \quad \psi_j = \psi - m\varphi_j,\ j = 1, ..., m.$$

For each $j \in \{1, ..., m\}$, ψ_j solves the problem

$$\begin{cases} \lambda\psi_j + \psi_j' + \psi_j''' = 0, \\ \psi_j(0) = \psi_j(L) = \psi_j'(0) = \psi_j'(L) = 0. \end{cases}$$

Since $L \notin \mathcal{N}$, from (ROSIER, 1997, Lemma 3.5) it follows that

$$\psi_j = 0,\ j = 1, ..., m$$

so

$$\varphi_j = \frac{1}{m}\psi,\ j = 1, ..., m.$$

Consequently,

$$\varphi_j = \varphi_1,\ j = 1, ..., m. \tag{3.65}$$

Now define

$$\theta = \sum_{j=m+1}^{N} \varphi_j \quad \text{and} \quad \theta_j = \theta - (N-m)\varphi_j,\ j = m+1, ..., N.$$

Note that, for each $j \in \{m+1, ..., N\}$, θ_j solves the problem

$$\begin{cases} \lambda\theta_j + \theta_j' + \theta_j''' = 0, \\ \theta_j(0) = \theta_j(L) = \theta_j'(0) = \theta_j''(L) = 0. \end{cases}$$

Additionally, $L \notin \mathcal{N}^*$, from (GLASS; GUERRERO, 2010, Propositions 1 and 2) it follows that

$$\theta_j = 0,\ j = m+1, ..., N$$

so that

$$\varphi_j = \frac{1}{N-m}\theta,\ j = m+1, ..., N.$$

Consequently,

$$\varphi_j = \varphi_N,\ j = m+1, ..., N. \tag{3.66}$$

This implies that, φ_1 and φ_N satisfy the spectral problem

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j),\ j = 1, N, \\ \varphi_N(0) = \varphi_1(0), & \\ m\varphi_1''(0) + (N-m)\varphi_N''(0) = 0, & \\ \varphi_j(L) = \varphi_j'(0) = 0, & j = 1, N, \\ \varphi_1'(L) = \varphi_N''(L) = 0, & \end{cases}$$

which corresponds to (3.41) in the case $\varsigma_{Ne} = m$ and $\varsigma_{Di} = N - m$. Therefore, the unique solution is $\varphi_1 = \varphi_N = 0$. Finally, as $0 = \varphi_1 = \varphi_j$ for $j = 1, \ldots, m$ and $0 = \varphi_N = \varphi_j$ for $j = m+1, \ldots, N$, we obtain $\varphi = 0$. □

3.5.6 Case Full Neumann ($N \geq 2$ and $m = N$)

This case consists by of considering Neumann controls on all edges.The associate spectral problem arising for N Neumann controls is

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j),\ j = 1, ..., N, \\ \varphi_j(0) = \varphi_1(0), & j = 1, ..., N, \\ \sum_{j=1}^{N} \varphi_j''(0) = 0, & \\ \varphi_j(l_j) = \varphi_j'(0) = 0, & j = 1, ..., N, \\ \varphi_j'(l_j) = 0, & j = 1, ..., N. \end{cases} \tag{3.67}$$

For this case, we have the following result.

Lemma 3.9 *Let $L > 0$ and assume $l_j = L$ for $j = 1, ..., N$. There exist $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ satisfying* (3.67) *if and only if $L \in \mathcal{N} \cup \mathcal{N}^*$.*

Proof: Consider the function $\psi = \sum_{j=1}^{N} \varphi_j$. We will see that if the length is not critical, it is enough to analyze the spectral problem satisfied by the sum function.

***Claim* 3.5** *If $L \notin \mathcal{N}$, then $\psi \equiv 0$ if and only if $\varphi_j \equiv 0$ for all $j = 1, \ldots, N$.*

Proof: One direction is direct. In the other hand, if $\psi \equiv 0$, we have $0 = \psi(0) = N\varphi_j(0)$, thus φ_j solves

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j' = 0, \\ \varphi_j(0) = \varphi_j(L) = \varphi_j'(0) = \varphi_j'(L) = 0. \end{cases}$$

Since $L \notin \mathcal{N}$, from (ROSIER, 1997, Lemma 3.5) it follows that $\varphi_j \equiv 0$. □

The condition $L \notin \mathcal{N}$ is necessary. In fact, if not, we can not control $\psi_j = \varphi_j - \frac{1}{N}\psi$. By the previous claim, it is enough to study when we have a non-trivial function ψ. We can immediately see that ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(0) = \psi'(L) = \psi''(0) = 0. \end{cases}$$

Considering $\theta(x) = \psi(L - x)$ and $\tilde{\lambda} = -\lambda$, we get

$$\begin{cases} \tilde{\lambda}\theta + \theta' + \theta''' = 0, \\ \theta(0) = \theta'(L) = \theta'(0) = \theta''(L) = 0. \end{cases} \tag{3.68}$$

Define $P_{\tilde{\lambda}}(\mu) = \mu^3 + \mu + \tilde{\lambda}$.

***Claim* 3.6** *If $P_{\tilde{\lambda}}$ admits multiples roots, then $\theta = 0$.*

Proof: We follow the proof of Claim 3.3. Multiplying the system (3.68) by $e^{-ix\xi}$, integrating by parts in $(0, L)$ and using the boundary conditions we get, for every $\xi \in \mathbb{C}$

$$(\xi^3 - \xi - p)\hat{\theta}(\xi) = i\kappa(1 - \xi^2)e^{-iL\xi} + \beta,$$

where $p = i\tilde{\lambda}$, $\kappa = -\theta(L)$ and $\beta = \theta''(0)$. If $P_{\tilde{\lambda}}$ has a triple root, then by Girard's relations $\tilde{\lambda} = 0$, thus $p = 0$ and $\hat{\theta}$ satisfies

$$(\xi^3 - \xi)\hat{\theta}(\xi) = i\kappa(1 - \xi^2)e^{-iL\xi} + \beta, \qquad \forall \xi \in \mathbb{C}.$$

Evaluating in $\xi = 1$, we get $\theta''(0) = \beta = 0$. Therefore, θ satisfies $\theta(0) = \theta'(0) = \theta''(0) = 0$, which implies $\theta = 0$.

If $P_{\tilde{\lambda}}$ has a double root $\mu_1 = \mu_2 = \sigma$, by Girard's relations $\sigma = \pm\frac{i}{\sqrt{3}}$ and the solution θ of (3.68) can be written as

$$\theta(x) = d_0e^{-2\sigma x} + d_1e^{\sigma x} + d_2xe^{\sigma x}.$$

The boundary condition $\theta(0) = 0$ gives us $d_0+d_1 = 0$. While, $\theta'(0) = \theta'(L)$ give us respectively

$$-2\sigma d_0 + d_1\sigma + d_2 = 0$$

and

$$-2\sigma d_0 e^{-2\sigma L} + d_1\sigma e^{\sigma L} + d_2(e^{\sigma L} + L\sigma e^{\sigma L}) = 0.$$

Using $d_0 = -d_1$, we get

$$\begin{cases} 3\sigma d_1 + d_2 = 0 \\ \sigma d_1(2e^{-3\sigma L} + 1) + d_2(1 + \sigma L) = 0. \end{cases}$$

The previous system has a non-trivial solution if and only if $\sigma L = 2\sigma e^{-3\sigma L}$. In that case $d_2 = -3\sigma d_1$ On the other hand, the condition $\theta''(L) = 0$ provides that

$$4d_0\sigma^2 e^{-2\sigma L} + d_1\sigma^2 e^{\sigma L} + 2d_2\sigma e^{\sigma L} + d_2 L\sigma^2 e^{\sigma L} = 0,$$

or equivalently,

$$d_1\sigma + d_2(2 + L\sigma) = -4d_0\sigma e^{-3\sigma L}.$$

Using $d_0 = -d_1$, $\sigma L = 2\sigma e^{-3\sigma L}$ and $d_2 = -3\sigma d_1$ we obtain $-4 = 5\sigma L$, which is not possible since $L > 0$ and $\sigma \in i\mathbb{R}$. □

By the previous claim, $\theta(x) = d_0 e^{\mu_0 x} + d_1 e^{\mu_1 x} + d_2 e^{\mu_2 x}$, where μ_0, μ_1 and μ_2 are the simple roots of the characteristic polynomial $\mu^3 + \mu + \tilde{\lambda} = 0$. By imposing the boundary conditions, we deduce the following overdetermined system

$$\begin{cases} d_0 + d_1 + d_2 = 0 \\ \mu_0 d_0 + \mu_1 d_1 + \mu_2 d_2 = 0 \\ \mu_0 d_0 e^{\mu_0 L} + \mu_1 d_1 e^{\mu_1 L} + \mu_2 d_2 e^{\mu_2 L} = 0 \\ \mu_0^2 d_0 e^{\mu_0 L} + \mu_1^2 d_1 e^{\mu_1 L} + \mu_2^2 d_2 e^{\mu_2 L} = 0. \end{cases}$$

By calling $a = \mu_0 L$, $b = \mu_1 L$ and $c = \mu_2 L$, we obtain the following matrix system

$$\begin{bmatrix} 1 & 1 & 1 \\ ae^a & be^b & ce^c \\ a & b & c \\ a^2e^a & b^2e^b & c^2e^c \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 1 & 1 \\ ae^a & be^b & ce^c \\ a & b & c \\ a^2e^a & b^2e^b & c^2e^c \end{bmatrix} \xrightarrow{r_4 \leftrightarrow r_4 - ar_2} \begin{bmatrix} 1 & 1 & 1 \\ ae^a & be^b & ce^c \\ a & b & c \\ 0 & b(b-a)e^b & c(c-a)e^c \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_3 - ar_1}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ ae^a & be^b & ce^c \\ 0 & b-a & c-a \\ 0 & b(b-a)e^b & c(c-a)e^c \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_4 - be^b r_3}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ ae^a & be^b & ce^c \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(ce^c - be^b) \end{bmatrix} = \begin{bmatrix} & A_{a,b,c} & \\ 0 & 0 & B_{a,b,c} \end{bmatrix}.$$

We have a non-trivial function θ if and only if

- The rank of $A_{a,b,c}$ is one.
- The rank of $A_{a,b,c}$ is two and $B_{a,b,c} = 0$.

Easy computations give us

- $rank(A_{a,b,c}) = 1$, if and only if $a = b = c$. This case is not possible because the roots are simple.
- $rank(A_{a,b,c}) = 2$
  1. $a = b \neq c$, $be^b = ce^c$. Not possible.
  2. $a = b$, $be^b \neq ce^a$. Not possible.
  3. $ae^a \neq be^b$, $ce^c(a-b) + be^b(c-a) + ae^a(b-c) = 0$.
  4. $a \neq b$, $ae^a = be^b = ce^c$.

On the other hand, again as the roots are simple, $B_{a,b,c} = 0$ if and only if $be^b = ce^c$. Using this in the third case, we get

$$0 = ce^c(a-b) + be^b(c-a) + ae^a(b-c) = (b-c)(ae^a - ce^c),$$

which implies $ae^a = ce^c = be^b$, that contradicts the third case. Finally, in the fourth case, we have a non-trivial solution if $ae^a = be^b = ce^c$. Recall that Girard's relations $a+b+c=0$ and $L^2 = -(a^2+ab+b^2)$, which correspond to the expression (3.9) of the critical lengths $\mathcal{N}^*$. □

Remark 3.2 *In the previous analysis, we have shown that different spectral problems could have the same set of critical lengths. $\mathcal{N}^*$ is the set of critical lengths of the spectral problem* (3.68) *which is a different one than* (3.60) *studied in (GLASS; GUERRERO, 2010).*

3.5.7 Case Full Dirichlet ($N \geq 2$ and $m = 0$)

Finally we take a configuration with Dirichlet controls on all edges. By Lemma 3.4, it is enough to focus on the spectral problem (3.39).

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j''' = 0, & x \in (0, l_j),\ j = 1, ..., N, \\ \varphi_j(0) = \varphi_1(0), & j = 1, ..., N, \\ \sum_{j=1}^{N} \varphi_j''(0) = 0, & \\ \varphi_j(l_j) = \varphi_j'(0) = 0, & j = 1, ..., N, \\ \varphi_j''(l_j) = 0, & j = 1, ..., N. \end{cases} \tag{3.69}$$

For this case, we have the following result.

Lemma 3.10 *Let $L > 0$ and assume $l_j = L$ for $j = 1, ..., N$. There exist $\lambda \in \mathbb{C}$ and $\varphi \in \mathbb{H}^3(\mathcal{T})\backslash\{0\}$ satisfying* (3.69) *if and only if $L \in \mathcal{N}^* \cup \mathcal{N}^\dagger$.*

As in the full Neumann case, it is enough to analyze the spectral problem associated with the sum function $\psi = \sum_{j=1}^{N} \varphi_j$.

***Claim* 3.7** *If $L \notin \mathcal{N}^*$, then $\psi \equiv 0$ if and only if $\varphi_j \equiv 0$ for all $j = 1, \ldots, N$.*

Proof: One direction is direct. In the other hand, if $\psi \equiv 0$, we have $0 = \psi(0) = N\varphi_j(0)$, thus φ_j solves

$$\begin{cases} \lambda\varphi_j + \varphi_j' + \varphi_j'' = 0, \\ \varphi_j(0) = \varphi_j(L) = \varphi_j'(0) = \varphi_j'(L) = 0. \end{cases}$$

Since $L \notin \mathcal{N}^*$, from (GLASS; GUERRERO, 2010, Proposition 1 and 2) it follows that $\varphi_j \equiv 0$. □

We can immediately see that ψ solves

$$\begin{cases} \lambda\psi + \psi' + \psi''' = 0, \\ \psi(L) = \psi'(0) = \psi''(L) = \psi''(0) = 0. \end{cases}$$

Considering $\theta(x) = \psi(L - x)$ and $\tilde{\lambda} = -\lambda$, we get

$$\begin{cases} \tilde{\lambda}\theta + \theta' + \theta'' = 0, \\ \theta(0) = \theta'(L) = \theta''(0) = \theta''(L) = 0. \end{cases} \tag{3.70}$$

Define $P_{\tilde{\lambda}}(\mu) = \mu^3 + \mu + \tilde{\lambda}$.

***Claim* 3.8** *If $P_{\tilde{\lambda}}$ admits multiples roots, then $\theta = 0$.*

Proof: We follow the proof of Claim 3.3. Multiplying the system (3.70) by $e^{-ix\xi}$, integrating by parts in $(0, L)$ and using the boundary conditions we get, for every $\xi \in \mathbb{C}$

$$(\xi^3 - \xi - p)\hat{\theta}(\xi) = i\kappa(1 - \xi^2)e^{-iL\xi} + \iota\xi,$$

where $p = i\tilde{\lambda}$, $\kappa = -\theta(L)$ and $\iota = i\theta'(0)$. If $P_{\tilde{\lambda}}$ has a triple root, then by Girard's relations $\tilde{\lambda} = 0$, thus $p = 0$ and $\hat{\theta}$ satisfies

$$(\xi^3 - \xi)\hat{\theta}(\xi) = i\kappa(1 - \xi^2)e^{-iL\xi} + \iota\xi, \qquad \forall \xi \in \mathbb{C}.$$

Evaluating in $\xi = 1$, we get $\theta'(0) = \iota = 0$. Therefore, θ satisfies $\theta(0) = \theta'(0) = \theta''(0) = 0$, which implies $\theta = 0$.

If $P_{\tilde{\lambda}}$ has a double root $\mu_1 = \mu_2 = \sigma$, by Girard's relations $\sigma = \pm\dfrac{i}{\sqrt{3}}$ and the solution θ of (3.70) can be written as

$$\theta(x) = d_0 e^{-2\sigma x} + d_1 e^{\sigma x} + d_2 x e^{\sigma x}.$$

The boundary conditions $\theta(0) = \theta'(L) = 0$ gives us respectively

$$d_0 + d_1 = 0,$$

and

$$-2\sigma d_0 + d_1\sigma + d_2 = 0,$$

from where we get $d_0 = -d_1$ and $d_2 = -3\sigma d_1$. On the other hand, the boundary condition $\theta''(0)$ provides

$$4d_0\sigma^2 + d_1\sigma^2 + 2d_2\sigma = 0.$$

Replacing $d_0 = -d_1$ and $d_2 = -3\sigma d_1$ in the above expression we get $d_1 = 0$, and finally $\theta = 0$. □

By the previous claim, $\theta(x) = d_0e^{\mu_0 x} + d_1e^{\mu_1 x} + d_2e^{\mu_2 x}$, where μ_0, μ_1 and μ_2 are the simple roots of the characteristic polynomial $\mu^3 + \mu + \tilde{\lambda} = 0$. By imposing the boundary conditions and calling $a = \mu_0 L$, $b = \mu_1 L$ and $c = \mu_2 L$, we obtain the following matrix system

$$\begin{bmatrix} 1 & 1 & 1 \\ ae^a & be^b & ce^c \\ a^2 & b^2 & c^2 \\ a^2e^a & b^2e^b & c^2e^c \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By Gauss-Jordan elimination, $c \neq 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ ae^a & be^b & ce^c \\ a & b & c \\ a^2e^a & b^2e^b & c^2e^c \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ ae^a & be^b & ce^c \\ 0 & b^2 - a^2 & c^2 - a^2 \\ 0 & 0 & \dfrac{(c-a)}{c}(c^2e^c - b^2e^b) \end{bmatrix} = \begin{bmatrix} & \tilde{A}_{a,b,c} & \\ 0 & 0 & \tilde{B}_{a,b,c} \end{bmatrix}.$$

Easy computations give us

- $rank(\tilde{A}_{a,b,c}) = 1$, if and only if

  $$ce^c - ae^a = 0,\ a^2 - c^2 = 0, ce^c - be^b = 0,\ b^2 - c^2 = 0.$$

  Not possible.

- $rank(\tilde{A}_{a,b,c}) = 2$

  1. $ce^c - ae^a = 0$, $a^2 - b^2 = 0$, $ce^c - be^b = 0$, $b^2 - c^2 \neq 0$. Not possible.
  2. $be^b - ae^a = 0$, $a^2 - b^2 = 0$, $ce^c - be^b \neq 0$.
  3. $ce^c - ae^a = 0$, $a^2 - b^2 \neq 0$, $ce^c - be^b = 0$. Not possible, because $L \notin \mathcal{N}^*$.
  4. $a^2e^a(b - c) + b^2e^b(c - a) + c^2e^c(a - b) = 0$.

On the other hand, by symmetry

$$\tilde{B}_{a,b,c} = \frac{(c-a)(c^2e^c - b^2e^b)}{c} = 0, \quad \Longrightarrow c^2e^c = b^2e^b = a^2e^a.$$

Then we have a non-trivial solution if $c^2e^c = b^2e^b = a^2e^a$. Recall that Girard's relations $a+b+c=0$ and $L^2 = -(a^2+ab+b^2)$, which corresponds to the expression (3.10) of the critical lengths $\mathcal{N}^{\dagger}$.

3.5.8 Proof of Theorem 3.1

The main result in this chapter, Theorem 3.1, is a consequence of Lemmas 3.3 and 3.4, combined with the results obtained in the study of each case in (3.40), through Lemmas 3.5, 3.6, 3.7, 3.8, 3.9 and 3.10.

3.6 CONCLUSION

In this chapter, we present a detailed study of the controllability of the Kdv equation in star graphs, under appropriate boundary conditions. From a detailed analysis of the spectral problems associated with some control configurations, we were able to give a quite satisfactory framework on the phenomenon of critical lengths for the linear problem associated with KdV in this type of domain, which can be summarized in Table 1. Consequently, the controllability of system (3.1) could be achieved by using N controls in different situations of controlled boundary conditions, and without using control at the central node of the network. In chapter 5 we will present some open problems regarding the systems studied in this chapter.

4 CONTROL OF KAWAHARA EQUATION USING FLAT OUTPUTS

In this chapter, we focused on the linear Kawahara equation in a bounded domain. Using two boundary controls, we show the null controllability for this equation in the space $L^2(0, L)$. Moreover, employing the flatness approach, which is a new approach for higher-order dispersive systems, we achieve the exact controllability within a space of analytic functions.

4.1 BACKGROUND AND LITERATURE REVIEW

It is known that, in an appropriate non-dimensional form, water waves can be modeled as a free boundary problem of the incompressible, irrotational Euler equation. This involves two non-dimensional parameters: $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength, and the amplitude of the free surface are respectively denoted by h, λ and a. Additionally, the parameter μ, known as the Bond number, measures the relative importance of gravitational forces compared to surface tension forces. Long waves, also known as shallow water waves, are characterized by the condition $\delta \ll 1$. There are various long-wave approximations depending on the relationship between ε and δ.

The discussion above suggests that, instead of relying on models with poor asymptotic properties, one can rescale the mentioned parameters to find systems that reveal asymptotic models for surface and internal waves, such as the Kawahara model. Specifically, by setting $\varepsilon = \delta^4 \ll 1$, $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, considering the critical Bond numbe $\mu = \frac{1}{3}$, the Kawahara equation is derived. This equation, first introduced by Hasimoto and Kawahara (HASIMOTO, 1970; KAWAHARA, 1972), takes the form

$$\pm 2v_t + 3vv_x - \nu v_{xxx} + \frac{1}{45}v_{xxxxx} = 0,$$

or, after re-scaling,

$$v_t + \alpha v_x + \beta v_{xxx} - v_{xxxxx} + vv_x = 0.$$

This equation is also known as the fifth-order Korteweg-de Vries (KdV) equation (BOYD, 1991) or the singularly perturbed KdV equation (POMEAU; RAMANI; GRAMMATICOS, 1988). It describes a dispersive partial differential equation that encompasses various wave phenomena, such as magneto-acoustic waves in a cold plasma (KAKUTANI, 1960), the propagation of long waves in a shallow liquid beneath an ice sheet (IGUCHI, 2007), and gravity waves on the surface of a heavy liquid (CUI; DENG; TAO, 2006), among others.

Significant efforts in recent decades have aimed to understand this model within various research frameworks. For instance, numerous studies have focused on analytical and numerical methods for solving the equation. These methods include the tanh-function method (BERLOFF; HOWARD, 1997), extended tanh-function method (BISWAS, 2009), sine-cosine method (YUSUFOğLU; BEKIR; ALP, 2008), Jacobi elliptic functions method (HUNTER; SCHEURLE, 1988), direct algebraic method and numerical simulations (POLAT; KAYA; TUTALAR, 2006), decomposition methods (KAYA; AL-KHALED, 2007), as well as variational iteration and homotopy perturbation methods (JIN, 2009). Another important research direction is the study of the Kawahara equation from the perspective of control theory, specifically addressing the boundary controllability problem (GLASS; GUERRERO, 2009), which is our motivation.

So, in this context, we are interested in the boundary controllability issue of the Kawahara equation in a bounded domain. Precisely, we investigate the linear control problem

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = 0, & (x,t) \in (-1,0) \times (0,T), \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t \in (0,T), \\ u(-1,t) = h_1(t), \quad u_x(-1,t) = h_2(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (-1,0), \end{cases} \tag{4.1}$$

where h_1, h_2 are the controls input and u is the state function. The adoption of the interval $(-1,0)$ as the spatial domain is just a convention to simplify calculations. Performing a scale in the time and space together with a translation in the space, we can see that the results are valid for arbitrary domains $(0,L)$, with $L > 0$.

This chapter addresses two main issues:

Question $\mathcal{A}_4$: Null controllability. *Given an initial data u_0 in a suitable space, is it possible to find control functions h_1, h_2 such that the state solution u of the system* (4.1) $u(x,T) = 0$?

Question $\mathcal{B}_4$: Reachable functions. *Can we find a space $\mathcal{R}$ with the property that, if the final data $u_1 \in \mathcal{R}$ then one can get control functions h_1, h_2 such that the solution u of the system* (4.1) *with $u_0 = 0$ satisfies $u(\cdot,T) = u_1$?*

It is important to highlight several results related to control problems associated with the system (4.1). Here are some of them. Regarding the analysis of the Kawahara equation in a bounded interval, pioneering work was done by Silva and Vasconcellos (VASCONCELLOS; SILVA, 2008), (VASCONCELLOS; SILVA, 2011). They studied the stabilization of global solutions of the linear Kawahara equation in a bounded interval under the influence of a localized damping mechanism. The second contribution in this area was made by Capistrano-Filho *et al.* (ARARUNA; CAPISTRANO-FILHO; DORONIN, 2012), who considered a generalized Kawahara equation in a bounded domain.

Glass and Guerrero (GLASS; GUERRERO, 2009) considered the problem

$$\begin{cases} y_t + \alpha y_{5x} = \sum_{k=0}^{3} a_k(x,t)\partial_x^k y + h(x,t), & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = v_1(t), \quad y(1,t) = v_2(t), \quad y_x(0,t) = v_3(t), & t \in \times(0,T) \\ y_x(1,t) = v_4(t), \quad y_{xx}(0,t) = v_5(t), & t \in \times(0,T), \\ y(x,0) = y_0, & x \in (0,1), \end{cases} \tag{4.2}$$

where $\alpha > 0$ and $y_0, h, v_1, ..., v_5$ are given functions. Using a Carleman estimate, they demonstrated that the system (4.2) is null controllable in the energy space $L^2(0,1)$ using only the controls on the right side of the boundary, v_2 and v_4, meaning $v_1 = v_3 = v_5 = 0$. However, the authors noted that the controllability properties might fail if the set of controls is altered. For example, if $v_1 = v_2 = v_3 = v_4 = 0$, meaning only v_5 is used as a control input, controllability does not occur because the adjoint system associated with (4.2) may have unobservable solutions.

The internal controllability problem for the Kawahara equation with homogeneous boundary conditions has been addressed by Chen (CHEN, 2019). Using Carleman estimates associated with the linear operator of the Kawahara equation with internal observation, a null controllability result was demonstrated when the internal control is effective in a subdomain $\omega \subset (0, L)$. In (CAPISTRANO-FILHO; GOMES, 2021), the authors consider the Kawahara equation with an internal control $f(t, x)$ and homogeneous boundary conditions, the equation is shown to be exactly controllable in L^2-weighted Sobolev spaces. Additionally, it is shown to be controllable by regions in the L^2-Sobolev space.

4.2 FLATNESS APPROACH AND MAIN RESULTS

The flatness approach (FLIESS et al., 1995), (ARMENTANO; CORON; GONZÁLEZ-BURGOS, 2011), (FLIESS; JOIN, 2013), also known as differential flatness, represents a powerful concept in control theory and nonlinear system analysis. It characterizes certain dynamical systems where all states and inputs can be described as algebraic functions of a finite set of independent variables, referred to as flat outputs, and their derivatives. This approach simplifies the control and trajectory planning of complex systems. It has found extensive application across various partial differential equations, including the heat equation (MARTIN; ROSIER; ROUCHON, 2014), (MARTIN; ROSIER; ROUCHON, 2016b), (CHEN; ROSIER, 2022), 1-dimensional parabolic equations (MARTIN; ROSIER; ROUCHON, 2016a), the 1-dimensional Schrödinger equation (MARTIN; ROSIER; ROUCHON, 2018), the linear KdV equation (MARTIN et al., 2019), and more recently, the linear Zakharov-Kuznetsov equation (CHEN; ROSIER, 2020). Here are the key aspects of the flatness approach:

1. **Flat outputs:** In a flat system, outputs (flat outputs) describe the system's entire state and input trajectories using algebraic relationships involving a finite number of their derivatives.

2. **Simplified control design:** By utilizing flat outputs, complex nonlinear control problems can be transformed into simpler linear problems. This transformation simplifies the design of control laws and facilitates the generation of desired trajectories.

3. **Trajectory planning:** The flatness approach enables systematic trajectory planning. Desired trajectories for flat outputs can be planned first, and then corresponding state and input trajectories can be computed using the flatness property.

4. **Real-world applications:** The flatness approach has been successfully applied across various fields, including robotics, aerospace, process control, and automotive systems. For instance, in robotics, it aids in trajectory design for manipulators and mobile robots.

It is crucial to recognize the advantages of the flatness approach. Typically, designing controls for nonlinear systems is complex; however, this approach mitigates this complexity. Furthermore, flatness offers a systematic approach to generating trajectories and control inputs and, as previously noted, is applicable across a broad spectrum of dynamical systems.

Nevertheless, it is important to acknowledge the limitations and challenges associated with this method. Identifying flat outputs for a specific system can be demanding and necessitates a profound understanding of the system's dynamics. Additionally, not all systems exhibit flatness, which restricts the universal applicability of this approach. In summary, the flatness approach provides a robust framework for controlling and analyzing nonlinear dynamical systems through the concept of flat outputs. It simplifies the design of control laws and trajectory planning, making it a valuable tool across various engineering applications.

Regarding the primary contribution of this chapter, we advance the study of the control problem for the fifth-order dispersive system (4.1). Unlike recent works that utilize boundary controls and use the Hilbert uniqueness method, introduced by Lions (LIONS, 1988b), to show control properties, this work achieves the problems $\mathcal{A}_4$ and $\mathcal{B}_4$ using two control inputs via the flatness approach. Let us now present the main results of this work.

Motivated by the smoothing effect exhibited by the equation (4.1) with free evolution ($h_1 = h_2 = 0$), we explore the reachable sets within smooth function spaces, specifically, *Gevrey spaces*. First, let us introduce the definition of these spaces.

Definition 4.1 (Gevrey spaces) *Given $s_1, s_2 \geq 0$, a function $u : [a,b] \times [t_1,t_2] \to \mathbb{R}$ is said to be Gevrey of order s_1 on $[a,b]$ and s_2 on $[t_1,t_2]$ if $u \in C^\infty([a,b] \times [t_1,t_2])$ and there exist positive constants C, R_1 and R_2 such that*

$$|\partial_x^n \partial_t^m y(x,t)| \leq C \frac{n!^{s_1}}{R_1^n} \frac{m!^{s_2}}{R_2^m}, \ \forall n, m \geq 0, \ \forall (x,t) \in [a,b] \times [t_1,t_2].$$

The vectorial space of all functions on $[a,b] \times [t_1,t_2]$ which are Gevrey of order s_1 in x and s_2 in t is denoted by $G^{s_1,s_2}([a,b] \times [t_1,t_2])$.

In this context, we investigate the null controllability problem associated with (4.1) using the *flatness approach* to establish controllability properties. The goal is to identify a set of functions in a Gevrey space that are null controllable and to demonstrate that, at some intermediate time $\tau \in (0,T)$, the solutions to the free evolution problem fall into this set. Specifically, the first main result of this work can be stated as follows.

Theorem 4.1 (Null controllability) *Let $s \in [\frac{5}{2}, 5)$ and $T > 0$ be given. For any $u_0 \in L^2(-1,0)$ there exist control inputs $h_1, h_2 \in G^s([0,T])$ such that the solution of* (4.1) *belongs to the class $u \in C([0,T], L^2(-1,0)) \cap G^{\frac{s}{5},s}([-1,0] \times [\varepsilon,T]), \ \forall \varepsilon \in (0,T)$, and satisfies $u(\cdot,T) = 0$.*

The previous result confirms that two flat outputs can be used to achieve null controllability, thereby addressing Question $\mathcal{A}_4$ presented at the beginning of this work. Now, to present our second main result, we need to introduce some notations. Given $z_0 \in \mathbb{C}$ and $R > 0$, we denote by $D(z_0,R)$ the open disk given by

$$D(z_0,R) = \{z \in \mathbb{C}; \ |z - z_0| < R\}$$

and $H(D(z_0,R))$ denote the set of holomorphic functions on $D(z_0,R)$. Henceforth, we consider the operators

$$Pu = \partial_x u + \partial_x^3 u - \partial_x^5 u,$$

where $P^0 = I_d$ and $P^n = P \circ P^{n-1}$, when $n \geq 1$. In this way, the Kawahara equation can be expressed as

$$\partial_t u + Pu = 0. \tag{4.3}$$

Using induction and the fact that ∂_t and P commute we see that, if $u = u(x,t)$ satisfies (4.3) then

$$\partial_t^n u + (-1)^{n-1} P^n u = 0. \tag{4.4}$$

For every $R > 1$, we define the set

$$\mathcal{R}_R := \left\{ u \in C([-1,0]);\ \exists z \in H(D(0,R));\ u = z|_{[-1,0]} \text{ and } (\partial_x^j P^n u)(0) = 0,\ j = 0,1,2 \right\}, \tag{4.5}$$

now on called *a set of reachable states*. The following result is the second main result in this chapter, answering the Question $\mathcal{B}_4$.

Theorem 4.2 *Let $T > 0$, $R_0 := 2 \cdot 12^{5^{-1}} \cdot e^{(5e)^{-1}} > 1$ and $R > 2R_0$ be given. For every $u_1 \in \mathcal{R}_R$ there exist control inputs $h_1, h_2 \in G^5([0,T])$ for which the solution u of* (4.1) *with $u_0 = 0$ satisfies $u(\cdot, T) = u_1$ and $u \in G^{1,5}([-1,0] \times [0,T])$.*

4.2.1 Chapter outline

Let us conclude this introduction with an outline of our chapter. Section 4.3 addresses Question $\mathcal{A}_4$ by presenting the control result, which is a consequence of the flatness property and the smoothing effect of the Kawahara equation. In Section 4.4, we provide an example of a set that can be reached from 0 by the system (4.1). This result, stemming from the flatness property extended to the limit case $s = 5$, partially answers Question $\mathcal{B}_4$. Finally, in Section 4.5 we conclude the chapter with some additional considerations about our studies and their possible future developments.

4.3 CONTROLLABILITY RESULT

We want to prove the null controllability property for the system (4.1). Our initial objective is to show the flatness property. This property ensures that we can parameterize the solution of the system (4.1) using the "flat outputs" $\partial_x^3 u(0,t)$ and $\partial_x^4 u(0,t)$. To prove it, we will examine the ill-posed system

$$\begin{cases} u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (x,t) \text{ in } (-1,0)\times(0,T), \\ u(0,t) = \partial_x u(0,t) = \partial_x^2 u(0,t) = 0, & t \text{ in } (0,T), \\ \partial_x^3 u(0,t) = y(t), \quad \partial_x^4 u(0,t) = z(t), & t \text{ in } (0,T). \end{cases} \tag{4.6}$$

Remark 4.1 *In the control literature, the term ill-posed has been used to refer to systems whose control provided by the HUM presents difficulties in numerical simulations. In cases of systems for which the solution has a smoothing effect, the energy space for the adjoint system has very low regularity and, consequently, the control written in terms of the solution of the adjoint problem has irregularities that are difficult to capture numerically. In other words, the term ill-posed refers to the difficulty of numerically determining the minimum norm control on $L^2(0,T)$ provided by the HUM, that is, this is a " numerical ill-posedness" of the control problem. For a better explanation we suggest references (MUNCH; ZUAZUA, 2010), (MICU; ZUAZUA, 2011) and (BELGACEM; KABER, 2011).*

Precisely, we will show that the solution of (4.6) belongs to $G^{\frac{s}{5},s}([-1,0]\times[\varepsilon,T])$ for all $\varepsilon \in (0,T)$ and can be written in the form

$$u(x,t) = \sum_{j=0}^{\infty} f_j(x) y^{(j)}(t) + \sum_{j=0}^{\infty} g_j(x) z^{(j)}(t), \;\; (x,t) \in [-1,0]\times[\varepsilon,T] \tag{4.7}$$

where $y, z \in G^s([0,T])$ for some $s \geq 0$, with $y^{(j)}(T) = z^{(j)}(T) = 0$ for every $j \geq 0$. Here, f_j and g_j are called the generating functions and are constructed following the ideas introduced in (MARTIN; ROSIER; ROUCHON, 2016a). The smoothing effect is responsible for ensuring that, from time ε onward, the solution of the free evolution problem associated to (4.1) is Gevrey of order $\frac{1}{2}$ in x and $\frac{5}{2}$ in t. Finally, a result of unique continuation is used to ensure that this solution coincides with the one associated with (4.6), described in (4.7).

4.3.1 Flatness property

We need to prove that there exists a one-to-one correspondence between solutions of (4.6) and a certain space of smooth functions, which we will call the flatness property. We will often use Stirling's formula:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}, \;\; \forall n,$$

and the inequality

$$(n+m)! \leq 2^{n+m} n! m!. \tag{4.8}$$

For $n \in \mathbb{N}$, $p \in [1, \infty]$ and $f \in W^{n,p}(-1,0)$ we denote

$$\|f\|_p = \|f\|_{L^p(-1,0)} \quad \text{and} \quad \|f\|_{n,p} = \sum_{i=0}^{n} \|\partial_x^i f\|_p.$$

Let us, in this part, find a solution u for (4.6) in the form

$$u(x,t) = \sum_{j\geq 0} f_j(x) y^{(j)}(t) + \sum_{j\geq 0} g_j(x) z^{(j)}(t). \tag{4.9}$$

Here, y, z, f_j, g_j satisfies the following conditions:

1) $y, z \in G^s([0,T])$ with $s \in (1,5)$;

2) $f_j, g_j \in L^\infty([-1,0])$ with polynomial growths in the form

$$|f_j(x)| \leq 2^j \frac{|x|^{5j+r}}{(5j+r)!} \quad \text{and} \quad |g_j(x)| \leq 2^j \frac{|x|^{5j+r}}{(5j+r)!}, \ \forall j \geq 0, \ \forall x \in [-1,0],$$

for some $r \in \{0,1,2,3,4\}$;

3) $\partial_x^3 u(0,t) = y(t)$ and $\partial_x^4 u(0,t) = z(t)$.

Considering u as in (4.9) and assuming that we can derive term by term, we get that

$$\begin{aligned} u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u =& \sum_{j\geq 0} f_j(x) y^{(j+1)}(t) + \sum_{j\geq 0} \left(f_{jx} + f_{j3x} - f_{j5x}\right)(x) y^{(j)}(t) \\ &+ \sum_{j\geq 0} g_j(x) z^{(j+1)}(t) + \sum_{j\geq 0} \left(g_{jx} + g_{j3x} - g_{j5x}\right)(x) z^{(j)}(t). \end{aligned}$$

Considering a new index $l = j+1$ gives us

$$\sum_{j\geq 0} f_j(x) y^{(j+1)}(t) = \sum_{l\geq 1} f_{l-1}(x) y^{(l)}(t) \quad \text{and} \quad \sum_{j\geq 0} g_j(x) z^{(j+1)}(t) = \sum_{l\geq 1} g_{l-1}(x) z^{(l)}(t).$$

Thus

$$\begin{aligned} u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u =& \sum_{j\geq 1} f_{j-1}(x) y^{(j)}(t) + \sum_{j\geq 1} \left(f_{jx} + f_{j3x} - f_{j5x}\right)(x) y^{(j)}(t) \\ &+ \left(f_{0x} + f_{03x} - f_{05x}\right)(x) y(t) + \left(g_{0x} + g_{03x} - g_{05x}\right)(x) z(t) \\ &+ \sum_{j\geq 1} g_{j-1}(x) z^{(j)}(t) + \sum_{j\geq 1} \left(g_{jx} + g_{j3x} - g_{j5x}\right)(x) z^{(j)}(t). \end{aligned}$$

Hence, if we have

$$\begin{cases} f_{0x} + f_{03x} - f_{05x} = g_{0x} + g_{03x} - g_{05x} = 0, \\ f_{jx} + f_{j3x} - f_{j5x} = -f_{j-1}, \quad \forall j \geq 1, \\ g_{jx} + g_{j3x} - g_{j5x} = -g_{j-1}, \quad \forall\, j \geq 1, \end{cases}$$

then

$$u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0.$$

Furthermore, imposing the conditions

$$\begin{cases} f_0(0) = f_{0x}(0) = f_{02x}(0) = f_{04x}(0) = 0, \quad f_{03x}(0) = 1, \\ f_j(0) = f_{jx}(0) = f_{j2x}(0) = f_{j3x}(0) = f_{j4x}(0) = 0, \quad \forall j \geq 1 \end{cases}$$

and

$$\begin{cases} g_0(0) = g_{0x}(0) = g_{02x}(0) = g_{03x}(0) = 0, \quad g_{04x}(0) = 1, \\ g_j(0) = g_{jx}(0) = g_{j2x}(0) = g_{j3x}(0) = g_{j4x}(0) = 0, \quad \forall j \geq 1, \end{cases}$$

we obtain,

$$\begin{cases} u(0,t) = \partial_x u(0,t) = \partial_x^2 u(0,t) = 0, \\ \partial_x^3 u(0,t) = y(t), \quad \partial_x^4 u(0,t) = z(t), \end{cases}$$

for $t \in (0,T)$. This leads us to define inductively the functions $\{f_j\}_{j\geq 0}$ and $\{g_j\}_{j\geq 0}$ as follows:

$1)$ f_0 is the solution of the IBVP

$$\begin{cases} f_{0x} + f_{03x} - f_{05x} = 0, \\ f_0(0) = f_{0x}(0) = f_{02x}(0) = f_{04x}(0) = 0, \quad f_{03x}(0) = 1. \end{cases} \tag{4.10}$$

and, for $j \geq 1$, f_j is the solution for the IBVP

$$\begin{cases} f_{jx} + f_{j3x} - f_{j5x} = -f_{j-1}, \\ f_j(0) = f_{jx}(0) = f_{j2x}(0) = f_{j3x}(0) = f_{j4x}(0) = 0. \end{cases} \tag{4.11}$$

$2)$ g_0 is the solution of the IBVP

$$\begin{cases} g_{0x} + g_{03x} - g_{05x} = 0, \\ g_0(0) = g_{0x}(0) = g_{02x}(0) = g_{03x}(0) = 0, \quad g_{04x}(0) = 1. \end{cases} \tag{4.12}$$

and, for $j \geq 1$, g_j is the solution for the IVP

$$\begin{cases} g_{jx} + g_{j3x} - g_{j5x} = -g_{j-1}, \\ g_j(0) = g_{jx}(0) = g_{j2x}(0) = g_{j3x}(0) = g_{j4x}(0) = 0. \end{cases} \tag{4.13}$$

We will prove that, given $s \in (1,5)$, there exists a one to one correspondence between solutions of (4.6) and pairs of functions $(y,z) \in G^s([0,T]) \times G^s([0,T])$, namely,

$$u \mapsto \left(\partial_x^3 y(0,\cdot), \partial_x^4 z(0,\cdot)\right).$$

In the sense of this bijection, we shall say that the system (4.6) is flat. Observe that, an expression in terms of the families $\{f_j\}_{j\geq 0}$ and $\{g_j\}_{j\geq 0}$ for a solution u of (4.6) as in (4.9) must be unique, that is, if

$$u(x,t) = \sum_{j\geq 0} f_j(x) y^{(j)}(t) + \sum_{j\geq 0} g_j(x) z^{(j)}(t)$$

and

$$u(x,t) = \sum_{j\geq 0} f_j(x) \tilde{y}^{(j)}(t) + \sum_{j\geq 0} g_j(x) \tilde{z}^{(j)}(t),$$

with $y, \tilde{y}, z, \tilde{z} \in G^s([0,T])$, then $y = \tilde{y}$ and $z = \tilde{z}$.

Remark 4.2 1) *Note that considering a toy model, that is,*

$$\begin{cases} u_t - \partial_x^5 u = 0, & (x,t) \in (-1,0) \times (0,T), \\ u(0,t) = \partial_x u(0,t) = \partial_x^2(0,t) = 0, & t \in (0,T), \\ u(-1,t) = h_1(t) \quad \partial_x u(-1,t) = h_2(t), & t \in (0,T), \\ u(x,0) = u_0(x), \end{cases}$$

the first equation in the systems (4.10)-(4.12) *do not have the first and third derivatives terms. Then, direct computation gives*

$$f_j(x) = \frac{x^{5j+3}}{(5j+3)!} \quad \textit{and} \quad g_j(x) = \frac{x^{5j+4}}{(5j+4)!}, \quad \forall\, j \geq 0,\ x \in [-1,0].$$

2) *Returning to the full system, with terms of the first and third derivatives, we have that the solutions f_0 and g_0 of* (4.10) *and* (4.12) *are given by*

$$f_0(x) = \frac{1}{\sqrt{a}(a+b)} \sinh\left(\sqrt{a}x\right) - \frac{1}{\sqrt{b}(a+b)} \sin\left(\sqrt{b}x\right) \tag{4.14}$$

and

$$g_0(x) = \frac{1}{a(a+b)}\cosh\left(\sqrt{a}x\right) + \frac{1}{b(a+b)}\cos\left(\sqrt{b}x\right) - \frac{1}{a(a+b)} - \frac{1}{b(a+b)}, \quad (4.15)$$

respectively, where $a = \frac{\sqrt{5}+1}{2}$ *and* $b = \frac{\sqrt{5}-1}{2}$.

To conclude that the system (4.6) is flat, it is enough to show that the solutions of (4.6) can be expressed as in (4.9) with $\{f_j\}_{j\geq 0}$ and $\{g_j\}_{j\geq 0}$ given by (4.10)-(4.13). Precisely, we will see that given $y, z \in G^s([0,T])$, then u given by (4.9) is well defined, belongs to $G^{\frac{s}{5},s}([-1,0]\times[0,T])$ and it solves the problem

$$\begin{cases} u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0 & (x,t) \text{ in } (-1,0)\times(0,T) \\ u(0,t) = \partial_x u(0,t) = \partial_x^2 u(0,t) = 0, & t \text{ in } (0,T) \\ \partial_x^3 u(0,t) = y(t), \quad \partial_x^4 u(0,t) = z(t) & t \in (0,T). \end{cases} \quad (4.16)$$

To do this, first, we need to establish estimates for the norms $\|f_j\|_{L^\infty(-1,0)}$ and $\|g_j\|_{L^\infty(-1,0)}$, as suggested before. At this point, it is very useful to note that, for $j \geq 1$, f_j (respectively g_j) can be written in terms of f_0 and f_{j-1} (respectively g_0 and g_{j-1}).

Lemma 4.1 *For any* $j \geq 1$ *and* $x \in [-1,0]$ *we have*

$$f_j(x) = \int_0^x \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi dy \quad (4.17)$$

and

$$g_j(x) = \int_0^x g_0(x-\xi) g_{j-1}(\xi) d\xi. \quad (4.18)$$

Proof: Let $x \in [-1,0]$, $y \in [x,0]$ and $\xi \in [y,0]$ be. From (4.11) we have

$$f_{j\xi}(\xi)f_0(y-\xi) + f_{j3\xi}(\xi)f_0(y-\xi) - f_{j5\xi}(\xi)f_0(y-\xi) = -f_0(y-\xi)f_{j-1}(\xi).$$

Integrating with respect to ξ we obtain

$$\int_0^y f_j(\xi)\left(f_{0\xi}(y-\xi) + f_{03\xi}(y-\xi) - f_{05\xi}(y-\xi)\right)d\xi - f_{j\xi}(y) = -\int_0^y f_0(y-\xi)f_{j-1}(\xi)d\xi$$

and by (4.10), after some integration by parts, it follows that

$$f_j'(y) = \int_0^y f_0(y-\xi)f_{j-1}(\xi)d\xi.$$

Integrating from 0 to x with respect to y we get

$$f_j(x) = \int_0^x \int_0^y f_0(y-\xi)f_{j-1}(\xi)d\xi dy, \ \forall x \in [-1,0],$$

showing (4.17). Similarly, (4.18) is verified. □

The next lemma will ensure that the series in (4.9) are convergent.

Lemma 4.2 *For every $j \geq 0$ we have*

$$|f_j(x)| \leq 2^j \frac{|x|^{5j+1}}{(5j+1)!} \tag{4.19}$$

and

$$|g_j(x)| \leq 2^j \frac{|x|^{5j+1}}{(5j+1)!}, \tag{4.20}$$

for all $j \geq 0$ and $x \in [-1, 0]$.

Proof: Let us start to proving (4.19) by induction on j. First, we must verify the case $j = 0$:

$$|f_0(x)| \leq |x|, \quad \forall x \in [-1, 0].$$

Note that $f_0(x) \leq 0$ in $[-1, 0]$. Indeed, from (4.14), since $\cosh > 1$ in $\mathbb{R}\backslash\{0\}$ and $\cos \leq 1$ in $\mathbb{R}$,

$$f_{0x} = \frac{1}{a+b}\cosh\left(\sqrt{a}x\right) - \frac{1}{a+b}\cos\left(\sqrt{b}x\right) > \frac{1}{a+b} - \frac{1}{a+b} = 0,$$

for all $x \in (-\infty, 0)$. This implies that f_0 is increasing, thus $f_0(x) \leq f_0(0) = 0$. Therefore, it is sufficient to show that

$$f_0(x) \geq x, \quad \forall x \in [-1, 0]. \tag{4.21}$$

Defining $\varphi(x) = f_0(x) - x$ we have $\varphi'(x) = f_0'(x) - 1$. On the other hand, we have

$$f_{03x}(x) = \frac{a}{a+b}\cosh\left(\sqrt{a}x\right) + \frac{b}{a+b}\cos\left(\sqrt{b}x\right) > \frac{a-b}{a+b} = \frac{1}{\sqrt{5}} > 0,$$

for $x \in (-\infty, 0)$, and so f_{02x} is increasing in $(-\infty, 0]$, implying that $f_{02x}(x) < f_{02x}(0) = 0$ for $x \in (-\infty, 0]$. Consequently f_{0x} is decreasing in $(-\infty, 0]$ which implies that

$$f_{0x}(x) < f_{0x}(-1) \approxeq 0,54 < 1, \quad \forall\, x \in (-1, 0),$$

so $\varphi' < 0$ in $(-1, 0)$. Hence φ is decreasing on $[-1, 0]$ and therefore $\varphi(x) \geq \varphi(0) = 0$, for $x \in [-1, 0]$, which implies (4.21).

For the next step, we need to verify the estimate

$$|f_{03x}(x)| \leq 2, \tag{4.22}$$

when $x \in [-1, 0]$. Note that we have

$$f_{05x}(x) = \frac{a^2}{a+b}\cosh\left(\sqrt{a}x\right) - \frac{b^2}{a+b}\cos\left(\sqrt{b}x\right) > \frac{a^2-b^2}{a+b} = a - b = 1,$$

for $x \in (-\infty, 0)$. This gives us that f_{04x} is increasing in $(-\infty, 0]$, in particular, $f_{04x}(x) < f_{04x}(0) = 0$. Hence f_{03x} is decreasing in $(-\infty, 0]$, and so $1 = f_{03x}(0) \leq f_{03x}(x) \leq f_{03x}(-1) \approxeq 1,59 < 2$, for $x \in [-1, 0]$, getting (4.22).

Now, suppose that for some $j \geq 1$

$$|f_j(x)| \leq 2^{j-1} \frac{|x|^{5(j-1)+1}}{[5(j-1)+1]!}, \ \forall x \in [-1, 0],$$

holds. By Lemma 4.1 we have

$$f_j(x) = \int_0^x \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi dy, \quad \forall x \in [-1, 0].$$

Using integration by parts (with respect to ξ) together with the boundary conditions in (4.10) we obtain

$$\begin{aligned} \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi &= \int_0^y f_0(y-\xi) \frac{d}{d\xi} \int_0^\xi f_{j-1}(\sigma) d\sigma d\xi \\ &= \left[f_0(y-\xi) \int_0^\xi f_{j-1}(\sigma) d\sigma \right]_0^y + \int_0^y \int_0^\xi f_{j-1}(\sigma) d\sigma f_{0\xi}(y-\xi) d\xi \\ &= \int_0^y f_{0\xi}(y-\xi) \int_0^\xi f_{j-1}(\sigma) d\sigma d\xi. \end{aligned}$$

Define $F(\xi) = \int_0^\xi f_{j-1}(\sigma) d\sigma$, with this, we can write

$$\int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi = \int_0^y f_{0\xi}(y-\xi) F(\xi) d\xi = \int_0^y f_{0\xi}(y-\xi) \frac{d}{d\xi} \int_0^\xi F(\tau) d\tau d\xi.$$

Integration by parts, together with the boundary conditions given in (4.10), we get

$$\begin{aligned} \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi &= \left[f_{0\xi}(y-\xi) \int_0^\xi F(\tau) d\tau \right]_0^y + \int_0^y \int_0^\xi F(\tau) d\tau f_{02\xi}(y-\xi) d\xi \\ &= \int_0^y f_{02\xi}(y-\xi) \int_0^\xi F(\tau) d\tau d\xi. \end{aligned}$$

Setting also $G(\xi) = \int_0^\xi F(\tau) d\tau$, and we have

$$\int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi = \int_0^y f_{02\xi}(y-\xi) G(\xi) d\xi = \int_0^y f_{02\xi}(y-\xi) \frac{d}{d\xi} \int_0^\xi G(\rho) d\rho d\xi.$$

Using again integration by parts together with (4.10), it follows that

$$\begin{aligned} \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi &= \left[f_{02\xi}(y-\xi) \int_0^\xi G(\rho) d\rho \right]_0^y + \int_0^y \int_0^\xi G(\rho) d\rho f_{03\xi}(y-\xi) d\xi \\ &= \int_0^y f_{03\xi}(y-\xi) \int_0^\xi G(\rho) d\rho d\xi. \end{aligned}$$

Therefore, the previous equality ensures that

$$\begin{aligned}\int_0^y f_0(y-\xi)f_{j-1}(\xi)d\xi &= \int_0^y f_{03\xi}(y-\xi)\int_0^\xi G(\rho)d\rho d\xi \\ &= \int_0^y f_{03\xi}(y-\xi)\int_0^\xi\int_0^\rho F(\tau)d\tau d\rho d\xi \\ &= \int_0^y f_{03\xi}(y-\xi)\int_0^\xi\int_0^\rho\int_0^\tau f_{j-1}(\sigma)d\sigma d\tau d\rho d\xi \\ &= \int_0^y\int_0^\xi\int_0^\rho\int_0^\tau f_{03\xi}(y-\xi)f_{j-1}(\sigma)d\sigma d\tau d\rho d\xi,\end{aligned}$$

so

$$f_j(x) = \int_0^x\int_0^y\int_0^\xi\int_0^\rho\int_0^\tau f_{03\xi}(y-\xi)f_{j-1}(\sigma)d\sigma d\tau d\rho d\xi dy, \quad \forall x \in [-1,0].$$

Then, by the induction hypothesis, we get that

$$\begin{aligned}|f_j(x)| &\le \int_0^x\int_0^y\int_0^\xi\int_0^\rho\int_0^\tau 2\cdot 2^{j-1}\frac{|\sigma|^{5(j-1)+1}}{[5(j-1)+1]!}d\sigma d\tau d\rho d\xi dy \\ &\le 2^j\left|\int_0^x\int_0^y\int_0^\xi\int_0^\rho\int_0^\tau \frac{|\sigma|^{5j-4}}{(5j-4)!}d\sigma d\tau d\rho d\xi dy\right| \\ &= 2^j\left|\int_0^x\int_0^y\int_0^\xi\int_0^\rho\int_0^\tau \frac{(-\sigma)^{5j-4}}{(5j-4)!}d\sigma d\tau d\rho d\xi dy\right|,\end{aligned}$$

for $x \in [-1,0]$. Pick $r = -\sigma$, thus $dr = -d\sigma$, that is,

$$\int_0^\tau \frac{(-\sigma)^{5j-4}}{(5j-4)!}d\sigma = -\int_0^{-\tau}\frac{r^{5j-4}}{(5j-4)!}dr = \left[-\frac{r^{5j-3}}{(5j-3)(5j-4)!}\right]_0^{-\tau} = -\frac{(-\tau)^{5j-3}}{(5j-3)!}$$

thus

$$|f_j(x)| \le 2^j\left|\int_0^x\int_0^y\int_0^\xi\int_0^\rho \frac{(-\tau)^{5j-3}}{(5j-3)!}d\tau d\rho d\xi dy\right|.$$

Now set $r = -\tau$ to get

$$\int_0^\rho \frac{(-\tau)^{5j-3}}{(5j-3)!}d\tau = -\int_0^{-\rho}\frac{r^{5j-3}}{(5j-3)!}dr = \left[-\frac{r^{5j-2}}{(5j-2)(5j-3)!}\right]_0^{-\rho} = -\frac{(-\rho)^{5j-2}}{(5j-2)!},$$

and, consequently,

$$|f_j(x)| \le 2^j\left|\int_0^x\int_0^y\int_0^\xi \frac{(-\rho)^{5j-2}}{(5j-2)!}d\rho d\xi dy\right|.$$

Proceeding with three more integrations, we obtain

$$|f_j(x)| \le 2^j\left|\int_0^x\int_0^y \frac{(-\xi)^{5j-1}}{(5j-1)!}d\xi dy\right| \le 2^j\left|\int_0^x \frac{(-y)^{5j}}{(5j)!}dy\right| \le 2^j\left|\frac{(-x)^{5j+1}}{(5j+1)!}\right|,$$

for all $x \in [-1,0]$, showing (4.19).

Now, for (4.20), arguing similarly, we start by checking that $|g_0(x)| \leq |x|$, for all $x \in [-1,0]$. Since $\cosh \geq 1$ and $\cos \geq 1$, from (4.15) we have $g_0 \geq 0$. Then it is enough to show that $g_0(x) \leq -x$, for $x \in [-1,0]$. Defining $\psi(x) = g_0(x) + x$ we have $\psi'(x) = g_0'(x) + 1$. On the other hand,

$$g_{02x}(x) = \frac{1}{a+b}\cosh\left(\sqrt{a}x\right) - \frac{1}{a+b}\cos\left(\sqrt{b}x\right) > 0, \quad \forall x \in (-\infty, 0),$$

so g_{0x} is increasing in $(-\infty, 0]$. Thus $g_{0x}(x) > g_{0x}(-1) \approxeq -0,18 > -1$, for all $x \in (-1,0)$, which implies that $\psi' > 0$ in $(-1,0)$ and therefore ψ is increasing in $[-1,0]$. Consequently

$$\psi(x) \leq \psi(0) = 0 \iff g_0(x) \leq -x, \quad \forall x \in [-1,0].$$

In the next part, we need the estimate $|g_{04x}(x)| < 2$, for all $x \in [-1,0]$. This immediately follows from (4.22) and the fact that $g_{04x} \equiv f_{03x}$.

Now, assume that

$$|g_{j-1}(x)| \leq 2^{j-1}\frac{|x|^{5(j-1)+1}}{[5(j-1)+1]!} \quad \forall x \in [-1,0],$$

holds for some $j \geq 1$. By Lemma 4.1 we have

$$g_j(x) = \int_0^x g_0(x-\xi)g_{j-1}(\xi)d\xi.$$

Proceeding with integration by parts as in the case of f_j and using the boundary conditions in (4.13), we obtain

$$g_j(x) = \int_0^x\int_0^\xi\int_0^\lambda\int_0^\rho\int_0^\tau g_{04x}(x-\xi)g_{j-1}(\sigma)d\sigma d\tau d\rho d\lambda d\xi.$$

Then, by the induction hypothesis, we have

$$\begin{aligned}
|g_j(x)| &\leq \int_0^x\int_0^\xi\int_0^\lambda\int_0^\rho\int_0^\tau 2\cdot 2^{j-1}\frac{|\sigma|^{5(j-1)+1}}{[5(j-1)+1]!}d\sigma d\tau d\rho d\lambda d\xi \\
&\leq 2^j\left|\int_0^x\int_0^\xi\int_0^\lambda\int_0^\rho\int_0^\tau \frac{|\sigma|^{5j-4}}{(5j-4)!}d\sigma d\tau d\rho d\lambda d\xi\right| \\
&= 2^j\left|\int_0^x\int_0^\xi\int_0^\lambda\int_0^\rho\int_0^\tau \frac{(-\sigma)^{5j-4}}{(5j-4)!}d\sigma d\tau d\rho d\lambda d\xi\right|,
\end{aligned}$$

for $x \in [-1,0]$. Using the same process of repeated integration by substitution as in the case of f_j, we obtain (4.20), which concludes the proof. □

We are now in a position to solve the system (4.16).

Proposition 4.1 *Let $s \in (1,5)$, $T > 0$ and $y, z \in G^s([0,T])$. The function $u(x,t)$ defined in* (4.9) *belongs to $G^{\frac{s}{5},s}([-1,0]\times[0,T])$ and it solves* (4.16)*. In particular, the corresponding controls $h_1(t) := u(-1,t)$ and $h_2(t) := \partial_x u(-1,t)$ are Gevrey of order s on $[0,T]$.*

Proof: Let us briefly explain our strategy to prove this result. Consider $m, n \geq 0$ and $(x,t) \in [-1,0] \times [0,T]$ be given. Our task is first to derivate the series in (4.9) as follows

$$\partial_x^n \partial_t^m u(x,t) = \sum_{j\geq 0} \partial_x^n \partial_t^m \left(f_j(x) y^{(j)}(t) \right) + \sum_{j\geq 0} \partial_x^n \partial_t^m \left(g_j(x) z^{(j)}(t) \right). \tag{4.23}$$

With this, since m, n are arbitrary, we will prove that the series in (4.23) is uniformly convergent in $[-1,0] \times [0,T]$, which will ensure that $u \in C^\infty([-1,0] \times [0,T])$. Next, to conclude that $u \in G^{\frac{s}{5},s}([-1,0] \times [0,T])$ we must prove that

$$|\partial_x^n \partial_t^m u(x,t)| \leq C \frac{n!^{\frac{s}{5}}}{R_1^n} \frac{m!^s}{R_2^m}$$

for some constants $C, R_1, R_2 > 0$.

Let us star, choose $s \in (1,5)$ and $y, z \in G^s([0,T])$. Then there exists $M, R > 0$ such that

$$\left| y^{(j)}(t) \right| \leq M \frac{j!^s}{R^j} \quad \text{and} \quad \left| z^{(j)}(t) \right| \leq M \frac{j!^s}{R^j} \ \ \forall j \geq 0, \ \forall t \in [0,T]. \tag{4.24}$$

For $k \geq 0$, since ∂_t and P commute, we have that

$$\partial_t^m P^k \left(f_j(x) y^{(j)}(t) \right) = P^k \left(f_j(x) \right) y^{(j+m)}(t).$$

From (4.10) and (4.11) follows that $Pf_0 = 0$ and $Pf_j = -f_{j-1}$, for $j \geq 1$. We will split our analysis of $P^k(f_j)$ into two cases, namely, $j - k \geq 0$ and $j - k < 0$. First, suppose $j - k \geq 0$. In this case,

$$P^k(f_j) = (-1)^1 P^{k-1}(f_{j-1}) = (-1)^2 P^{k-2}(f_{j-2}) = \ \cdots = (-1)^{k-1} P^{k-(k-1)} \left(f_{j-(k-1)} \right),$$

or equivalently,

$$P^k(f_j) = (-1)^{k-1} P(f_{j-k+1}) = (-1)^{k-1}(-f_{j-k}) = (-1)^k f_{j-k}.$$

Secondly, assuming $j - k < 0$ we have $k - j > 0$ so $k - j \geq 1$. Hence

$$P^k(f_j) = (-1)^1 P^{k-1}(f_{j-1}) = (-1)^2 P^{k-2}(f_{j-2}) = \ \cdots = (-1)^j P^{k-j-1}(0) = 0.$$

Putting both information together, we get that

$$P^k(f_j) = \begin{cases} (-1)^k f_{j-k}, & \text{if } j - k \geq 0. \\ 0, & \text{if } j - k < 0, \end{cases} \tag{4.25}$$

and consequently

$$\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)=\begin{cases}(-1)^k f_{j-k}(x)y^{(j+m)}(t), & \text{if } j-k\geq 0,\\ 0, & \text{if } j-k<0.\end{cases} \tag{4.26}$$

Assume $j\geq k$, that is, $j-k\geq 0$. Then thanks to the relation (4.26) and Lemma 4.2, the following estimate holds

$$\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right|\leq 2^{j-k}\frac{|x|^{5(j-k)+1}}{[5(j-k)+1]!}M\frac{(j+m)!^s}{R^{j+m}}\leq M2^{j-k}\frac{(j+m)!^s}{R^{j+m}}\frac{1}{(5(j-k)+1)!}.$$

Setting $l=j-k$, we can write the previous inequality as follows

$$\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right|\leq M2^l\frac{(l+k+m)!^s}{R^{l+k+m}}\frac{1}{(5l+1)!}.$$

Furthermore, writing $N=k+m$ yields that

$$\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right|\leq M2^l\frac{(l+N)!^s}{R^{l+N}}\frac{1}{(5l+1)!}.$$

Then, using the inequality (4.8), we obtain that

$$\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right|\leq M2^l\frac{(2^l 2^N l!N!)^s}{R^l R^N}\frac{1}{(5l+1)!}=M\frac{2^{(1+s)l}2^{Ns}l!^sN!^s}{R^lR^N}\frac{1}{(5l+1)!}. \tag{4.27}$$

Stirling's formula gives that

$$l!\sim\left(\frac{l}{e}\right)^l\sqrt{2\pi l}\quad\text{and}\quad(5l)!\sim\left(\frac{5l}{e}\right)^{5l}\sqrt{2\pi 5l}.$$

Observer that

$$\left(\frac{5l}{e}\right)^{5l}\sqrt{2\pi 5l}=5^{5l+\frac{1}{2}}\left[\left(\frac{l}{e}\right)^l\left(\sqrt{2\pi l}\right)\right]^5\left(\sqrt{2\pi l}\right)^{-4}\sim 5^{5l+\frac{1}{2}}\left(\sqrt{2\pi l}\right)^{-4}l!^5$$

and, consequently, $(5l)!\sim 5^{5l+\frac{1}{2}}\left(\sqrt{2\pi l}\right)^{-4}l!^5$. Thus, from (4.27), we get the following

$$\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right|\leq\tilde{M}\frac{l!^s}{\left[2^{-(1+s)}R\right]^l}\frac{N!^s}{\left(2^{-s}R\right)^N}\frac{1}{(5l+1)5^{5l+\frac{1}{2}}\left(\sqrt{2\pi l,}\right)^{-4}l!^5},$$

for a suitable constant $\tilde{M}>0$. Since $\frac{1}{(5l+1)5^{5l+\frac{1}{2}}}\leq 1$, for $l\geq 0$, it follows that

$$\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right|\leq\tilde{M}\frac{N!^s}{\left(2^{-s}R\right)^N}\frac{(2\pi l)^2}{[2^{-(1+s)}R]^l l!^{5-s}}.$$

Using the inequality (4.8), we get that

$$\frac{N!^s}{\left(2^{-s}R\right)^N}\leq\frac{k!^s}{\left(\frac{R}{4^s}\right)^k}\frac{m!^s}{\left(\frac{R}{4^s}\right)^m}$$

and, consequently, the following holds

$$\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right| \leq \tilde{M}4\pi^2 \frac{k!^s}{\left(4^{-s}R\right)^k}\frac{m!^s}{\left(4^{-s}R\right)^m}\frac{l^2}{\tilde{R}^l l!^{5-s}},$$

where $\tilde{R} = 2^{-(1+s)}R$. Therefore,

$$\sum_{j\geq k}\left\|\partial_t^m P^k\left(f_j y^{(j)}(t)\right)\right\|_\infty \leq 4\tilde{M}\pi^2 \frac{k!^s}{\left(4^{-s}R\right)^k}\frac{m!^s}{\left(4^{-s}R\right)^m}\sum_{l\geq 0}\frac{l^2}{\tilde{R}^l l!^{5-s}},$$

for all $m, k \geq 0$. To finish this case, observe that the following series $\sum_{l\geq 0}\frac{l^2}{\tilde{R}^l l!^{5-s}}$ is convergent, so by the Weierstrass M-test, the following series

$$\sum_{j\geq k}\left\|\partial_t^m P^k\left(f_j y^{(j)}(t)\right)\right\|_\infty$$

is uniformly convergent on $[-1,0]\times[0,T]$, for all $m, k \geq 0$. Furthermore

$$\sum_{j\geq k}\left\|\partial_t^m P^k\left(f_j y^{(j)}(t)\right)\right\|_\infty \leq \overline{M}\frac{k!^s}{\left(4^{-s}R\right)^k}\frac{m!^s}{\left(4^{-s}R\right)^m}, \tag{4.28}$$

for $m, k \geq 0$, where $\overline{M} = 4\tilde{M}\pi^2\sum_{l\geq 0}\frac{l^2}{\tilde{R}^l l!^{5-s}}$.

Analogously, it turns out that

$$P^k(g_j) = \begin{cases} (-1)^k g_{j-k}, & \text{if } j-k\geq 0,\\ 0, & \text{if } j-k<0,\end{cases} \tag{4.29}$$

and

$$\partial_t^m P^k\left(g_j(x)z^{(j)}(t)\right) = \begin{cases} (-1)^k g_{j-k}(x)z^{(j+m)}(t), & \text{if } \ j-k\geq 0,\\ 0, & \text{if } \ j-k<0,\end{cases} \tag{4.30}$$

so that, using inequalities (4.8), (4.24), Stirling's formula and Lemma 4.1 we get

$$\sum_{j\geq k}\left\|\partial_t^m P^k\left(g_j z^{(j)}(t)\right)\right\|_\infty \leq 4\tilde{M}\pi^2 \frac{k!^s}{\left(4^{-s}R\right)^k}\frac{m!^s}{\left(4^{-s}R\right)^m}\sum_{l\geq 0}\frac{l^2}{\tilde{R}^l l!^{5-s}},$$

for all $m, k \geq 0$. Consequently the series $\sum\left\|\partial_t^m P^k\left(g_j z^{(j)}(t)\right)\right\|_\infty$ is uniformly convergent in $[-1,0]\times[0,T]$, for all $m, k\geq 0$, with

$$\sum_{j\geq k}\left\|\partial_t^m P^k\left(g_j z^{(j)}(t)\right)\right\|_\infty \leq \overline{M}\frac{k!^s}{\left(4^{-s}R\right)^k}\frac{m!^s}{\left(4^{-s}R\right)^m}. \tag{4.31}$$

Let $K_3 > 0$ be as in Lemma 4.5 for $p=\infty$. From (4.26) and (4.28) we have

$$\sum_{j=0}^{\infty}\left\|\partial_t^m\left(f_j y^{(j)}(t)\right)\right\|_{5i,\infty} = K_3^i\sum_{k=0}^{i}\sum_{j\geq k}\left\|\partial_t^m P^k\left(f_j y^{(j)}(t)\right)\right\|_\infty \leq K_3^i\sum_{k=0}^{i}\overline{M}\frac{k!^s}{\left(4^{-s}R\right)^k}\frac{m!^s}{\left(4^{-s}R\right)^m}.$$

Thus

$$\sum_{j=0}^{\infty}\left\|\partial_t^m\left(f_jy^{(j)}(t)\right)\right\|_{5i,\infty}\leq K_3^i\overline{M}i!^s\left(\sum_{k=0}^{i}\frac{1}{(4^{-s}R)^k}\right)\frac{m!^s}{(4^{-s}R)^m}.$$

Defining $b=1+\frac{1}{4^{-s}R}$ and noting that

$$\sum_{j=0}^{i}\frac{1}{(4^{-s}R)^k}\leq\sum_{k=0}^{i}b^k=1\cdot\frac{b^{i+1}-1}{b-1}\leq\frac{b^{i+1}}{b-1}=\frac{b}{b-1}b^i,$$

yields

$$\sum_{j=0}^{\infty}\left\|\partial_t^m\left(f_jy^{(j)}(t)\right)\right\|_{5i,\infty}\leq K_3^i\overline{M}i!^s\frac{b}{b-1}b^i\frac{m!^s}{(4^{-s}R)^m}.$$

Hence, defining $\hat{M}=\overline{M}\frac{b}{b-1}$ it follows that

$$\sum_{j=0}^{\infty}\left\|\partial_t^m\left(f_jy^{(j)}(t)\right)\right\|_{5i,\infty}\leq\hat{M}\frac{i!^s}{(K_3^{-1}b^{-1})^i}\frac{m!^s}{(4^{-s}R)^m},\ \forall m,i\geq0,\ t\in[0,T]. \tag{4.32}$$

Similarly, from (4.30), (4.31) and Lemma 4.5 we obtain

$$\sum_{j=0}^{\infty}\left\|\partial_t^m\left(g_jz^{(j)}(t)\right)\right\|_{5i,\infty}\leq\hat{M}\frac{i!^s}{(K_3^{-1}b^{-1})^i}\frac{m!^s}{(4^{-s}R)^m},\ \forall m,i\geq0,\ t\in[0,T]. \tag{4.33}$$

Given $m,n\geq0$ consider $i\geq0$ such that $n\in\{5i-r,\ r=0,1,2,3,4\}$. Then from (4.32) we get, for $(x,t)\in[-1,0]\times[0,T]$, that

$$\sum_{j=0}^{\infty}\left|\partial_x^n\partial_t^m\left(f_j(x)y^{(j)}(t)\right)\right|\leq\hat{M}\frac{i!^s}{(K_3^{-1}b^{-1})^i}\frac{m!^s}{(4^{-s}R)^m},$$

and from (4.33)

$$\sum_{j=0}^{\infty}\left|\partial_x^n\partial_t^m\left(g_j(x)z^{(j)}(t)\right)\right|\leq\hat{M}\frac{i!^s}{(K_3^{-1}b^{-1})^i}\frac{m!^s}{(4^{-s}R)^m}.$$

By the Weierstrass M-test, it follows that these series are uniformly convergent in $[-1,0]\times[0,T]$ for all $m,n\geq0$. Thus, defining $u:[-1,0]\times[0,T]\to\mathbb{R}$ as in (4.9) we have that $\partial_x^n\partial_t^mu$ is continuous. Consequently $u\in C^\infty([-1,0]\times[0,T])$ and satisfies

$$|\partial_x^n\partial_t^mu(x,t)|\leq2\hat{M}\frac{i!^s}{(K_3^{-1}b^{-1})^i}\frac{m!^s}{(4^{-s}R)^m}\quad\forall(x,t)\in[-1,0]\times[0,T]. \tag{4.34}$$

As seen before, Stirling's formula gives us $(5i)!\sim5^{5i+\frac{1}{2}}\left(\sqrt{2\pi i}\right)^{-4}i!^5=5^{5i+\frac{1}{2}}(2\pi i)^{-2}i!^5$, and so that $i!^5\sim\frac{(2\pi i)^2(5i)!}{5^{5i+\frac{1}{2}}}$. Therefore $i!^s\sim\left[\frac{(2\pi i)^2(5i)!}{5^{5i+\frac{1}{2}}}\right]^{\frac{s}{5}}$. Once we have that $\frac{i^2}{5^{5i+\frac{1}{2}}}\leq1\ \ \forall i\geq0$, we can infer that $i!^s\leq M^*(4\pi^2)^{\frac{s}{5}}(5i)!^{\frac{s}{5}}$, for some constant $M^*>0$. Furthermore, $n=5i-r$, that is, $5i=n+r$ with $r\in\{0,1,2,3,4\}$. Then, by (4.8),

$$i!^s\leq M^*\left(4\pi^2\right)^{\frac{s}{5}}(2^n2^rn!r!)^{\frac{s}{5}}=M^*\left(4\pi^2\cdot2^r\cdot r!\right)^{\frac{s}{5}}(2^n)^{\frac{s}{5}}n!^{\frac{s}{5}}\leq M'\left(2^{\frac{s}{5}}\right)^nn!^{\frac{s}{5}},$$

where $M' = M'(r) = M^* \left(4\pi^2 \cdot 2^r \cdot r!\right)^{\frac{s}{5}}$. With this, returning to inequality (4.34) we obtain

$$|\partial_x^n \partial_t^m u(x,t)| \leq 2\hat{M}M' \left(2^{\frac{s}{5}}\right)^n \frac{n!^{\frac{s}{5}}}{\left(K_3^{-1}b^{-1}\right)^{\frac{n+r}{5}}} \frac{m!^s}{(4^{-s}R)^m} \leq \frac{2\hat{M}M'}{\left(K_3^{-1}b^{-1}\right)^{\frac{r}{5}}} \frac{n!^{\frac{s}{5}}}{\left(K_3^{-\frac{1}{5}}b^{-\frac{1}{5}}2^{-\frac{s}{5}}\right)^n} \frac{m!^s}{(4^{-s}R)^m}.$$

Finally, defining

$$M'' := \max\left\{\frac{2\hat{M}M'}{\left(K_3^{-1}b^{-1}\right)^{\frac{r}{5}}};\ r = 0,1,2,3,4\right\}, \quad R_1 := K_3^{-\frac{1}{5}}b^{-\frac{1}{5}}2^{-\frac{s}{5}}, \text{ and } R_2 := 4^{-s}R,$$

we have

$$|\partial_x^n \partial_t^m u(x,t)| \leq M'' \frac{n!^{\frac{s}{5}}}{R_1^n} \frac{m!^s}{R_2^m}, \quad \forall n,m \geq 0,\ \forall (x,t) \in [-1,0] \times [0,T].$$

Moreover, u solves (4.16) by construction, the result follows. □

4.3.2 Estimates in $W^{5n,p}(-1,0)$-norm

Let us start by remembering that the map

$$\begin{aligned} \|\cdot\|_* : W^{5,p}(-1,0) &\rightarrow \mathbb{R}_+ \\ f &\mapsto \|f\|_* := \|f\|_p + \|Pf\|_p \end{aligned}$$

is a norm called the *graph* norm associated with the operator $P : W^{5,p}(-1,0) \to L^p(-1,0)$.

Lemma 4.3 *Let $p \in [1,\infty]$ be. For all $n \geq 0$ we have*

$$\|P^n f\|_p \leq 3^n \|f\|_{5n,p},\ \forall\ f \in W^{5n,p}(-1,0). \tag{4.35}$$

Proof: For $n = 0$, the inequality is obvious. For $n = 1$, given $f \in W^{5n,p}(-1,0)$ follows that

$$\|Pf\|_p = \|\partial_x f + \partial_x^3 f - \partial_x^5 f\|_p \leq \|\partial_x f\|_p + \|\partial_x^3 f\|_p + \|\partial_x^5 f\|_p \leq 3\|f\|_{5,p}.$$

Suppose that (4.35) for $0, 1, ..., n-1$. Then for $f \in W^{5n,p}(-1,0)$ we get

$$\|P^n f\|_p = \|P^{n-1}Pf\|_p \leq 3^{n-1}\|Pf\|_{5n-5,p} \leq 3^{n-1}\left(3\sum_{j=0}^{5n}\|\partial_x^j f\|_p\right) = 3^n\|f\|_{5n,p},$$

so (4.35) is true for n, which concludes the proof. □

Lemma 4.4 *Let $p \in [1,\infty]$ be. There exists a constant $K_1 = K_1(p) > 0$ such that*

$$\|f\|_{5,p} \leq K_1\left(\|f\|_p + \|Pf\|_p\right),\ \forall\ f \in W^{5,p}(-1,0).$$

Proof: We know that $\|\cdot\|_*$ is a norm in $W^{5,p}(-1,0)$. We will show that $(W^{5,p}(-1,0), \|\cdot\|_*)$ is a Banach space. To do this, consider a Cauchy sequence in $(W^{5,p}(-1,0), \|\cdot\|_*)$, $(f_n)_{n\geq 0}$. Then,

$$\|f_m - f_n\|_* = \|f_m - f_n\|_p + \|Pf_m - Pf_n\|_p \to 0 \text{ as } m,n \to \infty.$$

Since $L^p(-1,0)$ is a Banach space, there exist $f, g \in L^p(-1,0)$ such that

$$\|f_n - f\|_p \to 0, \quad \|Pf_n - g\|_p \to 0 \text{ as } n \to \infty.$$

Given $\varphi \in C_0^\infty(-1,0)$ we have

$$\left|\int_{-1}^0 (f_n - f)\varphi\right| \leq \int_{-1}^0 |f_n - f||\varphi| \leq \begin{cases} \|f_n - f\|_p \|\varphi\|_q, & 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \|f_n - f\|_1 \|\varphi\|_\infty, & p = 1 \\ \|f_n - f\|_\infty \|\varphi\|_1, & p = \infty. \end{cases}$$

Therefore

$$\int_{-1}^0 (f_n - f)\varphi \to 0, \quad \forall \varphi \in C_0^\infty(-1,0)$$

which implies that

$$f_n \to f \text{ in } \mathcal{D}'(-1,0). \tag{4.36}$$

Analogously we infer that $Pf_n \to g$ in $\mathcal{D}'(-1,0)$. But (4.36) implies that

$$\partial_x^i f_n \to \partial_x^i f \text{ in } \mathcal{D}'(-1,0),\ \forall i \in \mathbb{N} \cup \{0\}$$

and consequently $Pf_n \to Pf$ in $\mathcal{D}'(-1,0)$. By the uniqueness of the limit, it follows that

$$Pf = g \text{ in } \mathcal{D}'(-1,0). \tag{4.37}$$

Consider $T_1, T_2 \in D'(-1,0)$ given by $T_1 = f + \partial_x^2 f - \partial_x^4 f$ and $T_2 = h_1$, where

$$h_1(x) = \int_{-1}^x g(t)dt.$$

Note that (4.37) implies that $\partial_x T_1 = g$ in $\mathcal{D}'(-1,0)$. On the other hand, $h_1 \in L^p(-1,0)$ and

$$\langle \partial_x T_2, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = -\int_{-1}^0 h_1(x)\varphi'(x)dx = -\int_{-1}^0 \left(\int_{-1}^x g(t)dt\right)\varphi'(x)dx = -\int_{-1}^0 \int_{-1}^x g(t)\varphi'(x)dtdx.$$

For $-1 \leq t \leq x \leq 0$ the Fubini's theorem gives us

$$\langle \partial_x T_2, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = -\int_{-1}^0 g(t)(\varphi(0) - \varphi(t))dt = \int_{-1}^0 g(t)\varphi(t)dt = \langle g, \varphi \rangle_{\mathcal{D}',\mathcal{D}}$$

and therefore $\partial_x T_2 = g = \partial_x T_1$ in $\mathcal{D}'(-1,0)$. Consequently, there exists a constant $k_1 \in \mathbb{R}$ such that $T_1 = T_2 + k_1$ in $\mathcal{D}'(-1,0)$, that is,

$$f + \partial_x^2 f - \partial_x^4 f = h_1 + k_1 \ \text{ in } \ \mathcal{D}'(-1,0).$$

Defining $\tilde{h}_1 = h_1 + k_1 - f$ we have $\tilde{h}_1 \in L^p(-1,0)$ and

$$\partial_x^2 f - \partial_x^4 f = \tilde{h}_1 \ \text{ in } \ \mathcal{D}'(-1,0). \tag{4.38}$$

Now, consider $T_3, T_4 \in \mathcal{D}'(-1,0)$ given by $T_3 = \partial_x f - \partial_x^3 f$ and $T_4 = h_2$, where $h_2 \in L^p(-1,0)$ is given by

$$h_2(x) = \int_1^x \tilde{h}_1(t)dt.$$

Proceeding as done before we obtain $\partial_x T_3 = \tilde{h}_1 = \partial_x T_4$ in $\mathcal{D}'(-1,0)$, thus there exists a constant $k_2 \in \mathbb{R}$ such that $T_3 = T_4 + k_2$ in $\mathcal{D}'(-1,0)$, or equivalently,

$$\partial_x f - \partial_x^3 f = h_2 + k_2 \ \text{ in } \ \mathcal{D}'(-1,0).$$

Defining $\tilde{h}_2 := h_2 + k_2$ we have $\tilde{h}_2 \in L^p(-1,0)$ and

$$\partial_x f - \partial_x^3 f = \tilde{h}_2 \ \text{ in } \ \mathcal{D}'(-1,0). \tag{4.39}$$

Set $T_5, T_6 \in \mathcal{D}'(-1,0)$ by $T_5 = f - \partial_x^2 f$ and $T_6 = h_3$, where $h_3 \in L^p(-1,0)$ is given by

$$h_3(x) = \int_{-1}^x \tilde{h}_2(t)dt.$$

By the same argument as done before,

$$f - \partial_x^2 f = h_3 + k_3 \ \text{ in } \ \mathcal{D}'(-1,0)$$

with $k_3 \in \mathbb{R}$. Defining $\tilde{h}_3 = f - h_3 - k_3$ we have $\tilde{h}_3 \in L^p(-1,0)$ and

$$\partial_x^2 f = \tilde{h}_3 \ \text{ in } \ \mathcal{D}'(-1,0). \tag{4.40}$$

Now define $T_7, T_8 \in \mathcal{D}'(-1,0)$ by $T_7 = \partial_x f$ and $T_8 = h_4$, where $h_4 \in L^p(-1,0)$ is given by

$$h_4(x) = \int_{-1}^t \tilde{h}(t)dt.$$

Again, the same argument ensures that there exists $k_4 \in \mathbb{R}$ such that, defining $\tilde{h}_4 := h_4 + k_4$ we have $\tilde{h}_4 \in L^p(-1,0)$ and

$$\partial_x f = \tilde{h}_4 \ \text{ in } \ \mathcal{D}'(-1,0). \tag{4.41}$$

Since $\tilde{h}_1\tilde{h}_2, \tilde{h}_3, \tilde{h}_4 \in L^p(-1,0)$, the equalities (4.37)-(4.41) gives us $\partial_x f, \partial_x^2 f, \partial_x^3 f, \partial_x^4 f$ and $\partial_x^5 f$ belong to $L^p(-1,0)$, so that $f \in W^{5,p}(-1,0)$. Furthermore,

$$\|f_n - f\|_* = \|f_n - f\|_p + \|Pf_n - Pf\|_p = \|f_n - f\|_p + \|Pf_n - g\|_p$$

which implies that

$$\|f_n - f\|_* \to 0 \text{ as } n \to \infty,$$

that is, $f_n \to f$ in $(W^{5,p}(-1,0), \|\cdot\|_*)$ showing that $(W^{5,p}(-1,0), \|\cdot\|_*)$ is a Banach space.

Consider the map

$$\begin{aligned} I : (W^{5,p}(-1,0), \|\cdot\|_{5,p}) &\to (W^{5,p}(-1,0), \|\cdot\|_*) \\ f &\mapsto I(f) = f. \end{aligned}$$

Note that I is linear and bijective and $\|I(f)\|_* \le \|f\|_{5,p}$, so that I is continuous. Thus, I^{-1} is also continuous. Therefore, there exists $K_1 > 0$ such that

$$\|f\|_{5,p} \le K_1 \left(\|f\|_p + \|Pf\|_p\right) \quad \forall f \in W^{5,p}(-1,0),$$

showing the result. □

Remark 4.3 *As a consequence of the Lemma 4.4, we see that the norms* $\|\cdot\|_*$ *and* $\|\cdot\|_{5,p}$ *are equivalents in the space* $W^{5,p}(-1,0)$.

Lemma 4.5 *Let* $p \in [1,\infty]$ *be. There exists a constant* $K_2 = K_2(p) > 0$ *such that, for every* $n \ge 1$,

$$2 \cdot 3^{-(n+1)} \sum_{i=0}^{n} \|P^i f\|_p \le \|f\|_{5n,p} \le (1+2K_2)^{n-1} K_2 \sum_{i=0}^{n} \|P^i f\|_p, \ \forall\ f \in W^{5n,p}(-1,0).$$

Consequently,

$$2 \cdot 3^{-(n+1)} \sum_{i=0}^{n} \|P^i f\|_p \le \|f\|_{5n,p} \le K_3^n \sum_{i=0}^{n} \|P^i f\|_p, \ \forall\ f \in W^{5n,p}(-1,0),$$

for every $n \ge 0$, *where* $K_3 = K_3(p) = 1 + 2K_2$.

Proof: Given $n \ge 0$ and $f \in W^{5n,p}(-1,0)$ we have $f \in W^{5i,p}(-1,0)$ and $\|f\|_{5i,p} \le \|f\|_{5n,p}$, $0 \le i \le n$. Then by Lemma 4.3 we get

$$\sum_{i=0}^{n} \|P^i f\|_p \le \sum_{i=0}^{n} 3^i \|f\|_{5i,p} \le \left(\sum_{i=0}^{n} 3^i\right) \|f\|_{5n,p} = 1 \cdot \frac{3^{n+1}-1}{3-1} \|f\|_{5n,p} \le \frac{3^{n+1}}{2} \|f\|_{5n,p}$$

and therefore

$$2 \cdot 3^{-(n+1)} \sum_{i=0}^{n} \|P^i f\|_p \leq \|f\|_{5n,p}, \quad \forall f \in W^{5n,p}(-1,0), \quad \forall n \geq 0.$$

To prove

$$\|f\|_{5n,p} \leq (1+2K_1)^{n-1} K_1 \sum_{i=0}^{n} \|P^i f\|_p \quad \forall f \in W^{5n,p}(-1,0), \quad \forall n \geq 1$$

we use induction on n. For $n = 1$ this is true since, by Lemma 4.4,

$$\|f\|_{5,p} \leq K_1 \left(\|f\|_p + \|Pf\|_p\right) = K_1 \sum_{i=0}^{1} \|P^i f\|_p = (1+2K_1)^0 K_1 \sum_{i=0}^{1} \|P^i f\|_p.$$

Assume that, for $n \geq 2$, the inequality is true up to the rank $n-1$. Then, for any $f \in W^{5n,p}(-1,0)$ we have (setting $j = i - 5n + 5$)

$$\|f\|_{5n,p} = \|f\|_{5n-5,p} + \sum_{j=1}^{5} \|\partial_x^j \partial_x^{5n-5} f\|_p \leq \|f\|_{5n-5,p} + \|\partial_x^{5n-5} f\|_{5,p}.$$

Using Lemma 4.4 and the induction hypothesis, we get

$$\begin{aligned} \|f\|_{5n,p} &\leq (1+2K_1)^{(n-1)-1} K_1 \sum_{i=0}^{n-1} \|P^i f\|_p + K_1 \left(\|\partial_x^{5n-5} f\|_p + \|P\partial_x^{5n-5} f\|_p\right) \\ &\leq (1+2K_1)^{n-2} K_1 \sum_{i=0}^{n-1} \|P^i f\|_p + K_1 \left(\|f\|_{5n-5,p} + \|Pf\|_{5n-5,p}\right). \end{aligned}$$

Once again induction hypothesis yields

$$\begin{aligned} \|f\|_{5n,p} \leq &(1+2K_1)^{n-2} K_1 \sum_{i=0}^{n-1} \|P^i f\|_p \\ &+ K_1 \left((1+2K_1)^{n-2} K_1 \sum_{i=0}^{n-1} \|P^i f\|_p + (1+2K_1)^{n-2} K_1 \sum_{i=0}^{n-1} \|P^i P f\|_p\right). \end{aligned}$$

Then

$$\begin{aligned} \|f\|_{5n,p} &\leq (1+2K_1)^{n-2} K_1 \sum_{i=0}^{n} \|P^i f\|_p + 2K_1 (1+2K_1)^{n-2} K_1 \sum_{i=0}^{n} \|P^i f\|_p \\ &= (1+2K_1)^{n-1} K_1 \sum_{i=0}^{n} \|P^i f\|_p, \end{aligned}$$

from where the desired follows with $K_2 = K_1$.

Finally, consider $K_3 = 1 + 2K_2$ and $f \in W^{5n,p}(-1,0)$. For $n = 0$, we see that $\|f\|_{5n,p} = K_3^n \sum_{i=0}^{n} \|P^i f\|_p$. Now, For $n \geq 1$, we have that

$$\|f\|_{5n,p} \leq (1+2K_2)^{n-1} K_2 \sum_{i=0}^{n} \|P^i f\|_p \leq (1+2K_2)^{n-1}(1+2K_2) \sum_{i=0}^{n} \|P^i f\|_p = K_3^n \sum_{i=0}^{n} \|P^i f\|_p.$$

□

4.3.3 Smoothing property

With the previous section in hand, let us show that for any $u_0 \in L^2(-1,0)$, the solution of (4.1) with $h_1 = h_2 = 0$ is a Gevrey unction of order $\frac{1}{2}$ in the variable x and $\frac{5}{2}$ in the variable t. Precisely, we will prove the existence of positive constants M, R_1, and R_2 such that

$$|\partial_x^p \partial_t^n u(x,t)| \leq \frac{M}{t^{\frac{5n+p+5}{2}}} \frac{p!^{\frac{1}{2}}}{R_1^p} \frac{n!^{\frac{5}{2}}}{R_2^n}, \tag{4.42}$$

for all $t \in (0,T]$ and for all $x \in [-1,0]$.

To do this, using the inequality (4.42) on intervals of length one, we can assume, without loss of generality, $T = 1$. Consider the operator given by $Au = -Pu = -\partial_x u - \partial_x^3 u + \partial_x^5 u$, with $D(A) = \{u \in H^5(-1,0);\ u(-1) = u(0) = u_x(-1) = u_x(0) = u_{xx}(0) = 0\}$. It has been proven in (ARARUNA; CAPISTRANO-FILHO; DORONIN, 2012) that A generates a semigroup of contractions $\{S(t)\}_{t\geq 0}$ in $L^2(-1,0)$, moreover, that smoothing effect is verified, that is, for $u_0 \in L^2(-1,0)$, the mild solution $u(\cdot,t) = S(t)u_0$ of (4.1) with $h_1 = h_2 = 0$ satisfies $u \in C([0,1], L^2(-1,0)) \cap L^2(0,1,H^2(-1,0))$ and

$$\|u\|_{L^2(0,T,H^2(-1,0))} \leq \sqrt{3}\|u_0\|_{L^2(-1,0)}. \tag{4.43}$$

Now on, we will denote the norm $\|\cdot\|_{L^2(-1,0)}$ for simplicity for $\|\cdot\|_{L^2}$ and the spaces

$$\begin{aligned}
&X_0 = L^2(-1,0) \quad X_1 = H_0^1(-1,0), \quad X_2 = H_0^2(-1,0),\\
&X_3 = \left\{u \in H^3(-1,0);\ u(-1) = u(0) = u_x(-1) = u_x(0) = u_{xx}(0) = 0\right\},\\
&X_4 = \left\{u \in H^4(-1,0);\ u(-1) = u(0) = u_x(-1) = u_x(0) = u_{xx}(0) = 0\right\},\\
&X_5 = D(A).
\end{aligned}$$

For any $m \in \{1,2,3,4,5\}, (X_m, \|\cdot\|_{H^m})$ is a Banach space. The next propositions are paramount in our analysis and given several estimates in the X_s-spaces.

Proposition 4.2 *There exists some constant $C_1 > 0$ such that for all $u_0 \in L^2(-1,0)$ and all $t \in (0,1]$ it holds $\|S(t)u_0\|_{H^2} \leq (C_1/\sqrt{t})\|u_0\|_{L^2}$.*

Proof: Given $u_0 \in D(A)$, the semigroup theory ensures that $S(t)u_0 \in D(A)$ with

$$\frac{d}{dt}S(t)u_0 = AS(t)u_0 = S(t)Au_0 \tag{4.44}$$

and $u = S(\cdot)u_0 \in C([0,\infty), D(A))$. From (4.44) we obtain $\|AS(t)u_0\|_{L^2} = \|S(t)Au_0\|_{L^2}$ and, therefore, $\|S(t)u_0\|_{D(A)} \leq \|u_0\|_{D(A)}$.

Using the Lemma 4.4 (see also Remark 4.3) we obtain a positive constant $C' > 0$ (which does not depend on t) such that $\|S(t)u_0\|_{H^5} \leq C_1'\|u_0\|_{H^5}$. Then we see that the map $S(t) : (D(A), \|\cdot\|_{H^5}) \to (D(A), \|\cdot\|_{H^5})$ is continuous. Since $S(t) : L^2(-1,0) \to L^2(-1,0)$ is also continuous and, by interpolation argument, $[X_0, X_5]_{\frac{2}{5}} = X_2$, we have that $S(t) : (X_2, \|\cdot\|_{H^2}) \to (X_2, \|\cdot\|_{H^2})$ is continuous with $\|u(\cdot,t)\|_{H^2} = \|S(t)u_0\|_{H^2} \leq C_1''\|u_0\|_{H^2}$, for $u_0 \in X_2$ and $t \in (0,1]$, where $C_1'' = \max\{C_1', 1\}$.

In this way, given $u_0 \in X_2$, for any $t \in (0,1]$ we have

$$\|u(\cdot,t)\|_{H^2}^2 \leq (C_1'')^2\|u(\cdot,s)\|_{H^2}^2,$$

for all $s \in (0,t]$. Integrating with respect to s from 0 to t we get

$$t\|u(\cdot,t)\|_{H^2}^2 \leq (C_1'')^2 \int_0^t \|u(\cdot,s)\|_{H^2}^2 ds \leq (C_1'')^2\|u\|_{L^2(0,1,H^2(-1,0))}^2.$$

Using (4.43) we obtain

$$\|u(\cdot,t)\|_{H^2} \leq \frac{C_1''\sqrt{3}}{\sqrt{t}}\|u_0\|_{L^2},$$

and the result is achieved. □

Proposition 4.3 *There exists a positive constant C_2 such that, for all $u_0 \in D(A)$ and all $t \in (0,1]$ it holds $\|S(t)u_0\|_{H^7} \leq (C_2/\sqrt{t})\|u_0\|_{H^5}$.*

Proof: First, we need to check that $S(t)u_0 \in H^7(-1,0)$ whenever $u_0 \in D(A)$. Indeed, given $u_0 \in D(A)$ we have $u(\cdot,t) = S(t)u_0 \in D(A)$ and (4.44) holds. Then $Au(\cdot,t) = AS(t)u_0 = S(t)Au_0 \in H^2(-1,0)$ which implies that $\partial_x^5 u(\cdot,t) \in H^2(-1,0)$ and consequently $u(\cdot,t) \in H^7(-1,0)$. Furthermore, Proposition 4.2 gives us

$$\|u(\cdot,t)\|_{H^2} + \|Au(\cdot,t)\|_{H^2} \leq \frac{C_1}{\sqrt{t}}\left(\|u_0\|_{L^2} + \|Au_0\|_{L^2}\right). \tag{4.45}$$

Now, let $v \in H^7(-1,0)$. Observe that $\partial_x^6 v = \partial_x^2 v + \partial_x^4 v - \partial_x P v$ and $\partial_x^7 v = \partial_x^3 v + \partial_x^5 v - \partial_x^2 P v$. Then,

$$\|v\|_{H^7} \leq C_2'\left(\|v\|_{H^5} + \|Pv\|_{H^2}\right),$$

for some positive constant C_2'. Thus, using the Lemma 4.4, we obtain

$$\|v\|_{H_7} \leq C_2'\left(\|v\|_{H^2} + \|Av\|_{H^2}\right).$$

With this and (4.45), we get that

$$\|u(\cdot,t)\|_{H^7} \leq \frac{C_2'C_1}{\sqrt{t}}\left(\|u_0\|_{L^2} + \|Au_0\|_{L^2}\right) = \frac{C_2'C_1}{\sqrt{t}}\|u_0\|_{D(A)} \leq \frac{C_2}{\sqrt{t}}\|u_0\|_{H^5},$$

showing the proposition. □

Proposition 4.4 *There exists a constant* $C_3 > 1$ *such that, for all* $u_0 \in X_m$ *and all* $t \in (0,1]$ *it holds* $\|S(t)u_0\|_{H^{m+2}} \leq (C_3/\sqrt{t})\|u_0\|_{H^m}$.

Proof: By Propositions 4.2 and 4.3, the linear maps $S(t) : L^2(-1,0) \to H^2(-1,0)$ and $S(t) : D(A) \to H^7(-1,0)$ are continuous. Moreover, there exists $C_3 > 1$ such that

$$\begin{cases} \|S(t)u_0\|_{H^2} \leq \dfrac{C_3}{\sqrt{t}}\|u_0\|_{L^2}, \ \forall\ u_0 \in L^2(-1,0), \\ \|S(t)u_0\|_{H^7} \leq \dfrac{C_3}{\sqrt{t}}\|u_0\|_{H^5}, \ \forall\ u_0 \in D(A). \end{cases} \tag{4.46}$$

For $m = 1,2,3,4$, thanks to the interpolation arguments, it follows that $S(t)\left([X_0, X_5]_{\frac{m}{5}}\right) \subset [H^2, H^7]_{\frac{m}{5}}$, and the maps $S(t) : [X_0, X_5]_{\frac{m}{5}} \to [H^2, H^7]_{\frac{m}{5}}$ are continuous with

$$\|S(t)u_0\|_{[H^2,H^7]_{\frac{m}{5}}} \leq \frac{C_3}{\sqrt{t}}\|u_0\|_{[X_0,X_5]_{\frac{m}{5}}}, \ \forall\ u_0 \in [X_0, X_5]_{\frac{m}{5}}.$$

Since $[H^2, H^7]_{\frac{m}{5}} = H^{(1-\frac{m}{5})2+\frac{m}{2}7}$ and $[X_0, X_5]_{\frac{m}{5}} = X_m$, it follows that the maps $S(t) : X_m \to H^{m+2}$ for $m = 1,2,3,4$, are continuous with

$$\|S(t)u_0\|_{H^{m+2}} \leq \frac{C_3}{\sqrt{t}}\|u_0\|_{H^m}, \ \forall\ u_0 \in H^m. \tag{4.47}$$

From (4.46) and (4.47) we obtain the desired. $\square$

It is convenient to remember that for $n \in \mathbb{N}$, we define inductively by

$$D(A^n) = \left\{v \in L^2(-1,0);\ v \in D(A^{n-1}) \text{ and } Av \in D(A^{n-1})\right\}, \quad A^n v = A^{n-1}(Av).$$

So, with this in hand, we have the following result.

Proposition 4.5 *Let* $u_0 \in L^2(-1,0)$ *and* $t \in (0,1]$. *We have for* $u = S(\cdot)u_0$ *that:*

(i) $u(\cdot,t) \in D(A)$ *and*

$$\|Au(\cdot,t)\|_{L^2} \leq \frac{C_4}{t^{\frac{3}{2}}}\|u_0\|_{L^2}$$

for some positive constant $C_4 > 1$ *(which does not depend on* t*).*

(ii) $u(\cdot,t) \in D(A^n)$ *for every* $n \in \mathbb{N}$ *and*

$$\|A^n u(\cdot,t)\|_{L^2} \leq \frac{C_5^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|u_0\|_{L^2},$$

where $C_5 > \max\{1, C_4\}$ *is a constant which does not depend on* t.

(iii) $u \in C((0,1], D(A^n))$, *for every* $n \in \mathbb{N} \cup \{0\}$.

Proof: ***(i)*** Assume $u_0 \in D(A)$. Splitting $[0,t]$ into $[0,t/3]\cup[t/3,2t/3]\cup[2t/3,t]$, using Proposition 4.4 and take in mind that $Au(\cdot,t) = -\partial_x u(\cdot,t) - \partial_x^3 u(\cdot,t) + \partial_x^5 u(\cdot,t)$, we get that

$$\|Au(\cdot,t)\|_{L^2} \leq \frac{\tilde{C}_4}{t^{\frac{3}{2}}}\|u_0\|_{L^2},\ \forall\ u_0 \in D(A), \tag{4.48}$$

where $\tilde{C}_4 = \frac{3C_3^3}{\left(\sqrt{\frac{1}{3}}\right)^3}$ and $\tilde{C}_4 > 1$.

Note that, thanks to the inequality (4.48), the linear operator $S(t) : (D(A), \|\cdot\|_{L^2}) \to (D(A), \|\cdot\|_{D(A)})$ is a bounded linear operator. Moreover, we have

$$\|S(t)u_0\|_{D(A)} \leq \frac{2\tilde{C}_4}{t^{\frac{3}{2}}}\|u_0\|_{L^2},$$

for $u_0 \in D(A)$. Since $D(A)$ is dense in $L^2(-1,0)$, there exists a bounded linear operator $\Lambda_t : (L^2(-1,0), \|\cdot\|_{L^2}) \to (D(A), \|\cdot\|_{D(A)})$ such that

$$\Lambda_t\Big|_{D(A)} = S(t) \quad \text{and} \quad \|\Lambda_t v\|_{D(A)} \leq \frac{2\tilde{C}_4}{t^{\frac{3}{2}}}\|v\|_{L^2},\ \ \forall\ v \in L^2(-1,0). \tag{4.49}$$

Now, given $u_0 \in L^2(-1,0)$, there exists $(u_k) \subset D(A)$ such that $u_k \to u_0$ in $L^2(-1,0)$. Then from (4.49),

$$\|\Lambda_t u_0 - S(t)u_0\|_{L^2} \leq \frac{2\tilde{C}_4}{t^{\frac{3}{2}}}\|u_0 - u_k\|_{L^2} + \|u_0 - u_k\|_{L^2}.$$

Making $k \to \infty$ we obtain $\Lambda_t u_0 = S(t)u_0$. Therefore, $S(t)u_0 \in D(A)$, for $u_0 \in L^2(-1,0)$, and by (4.49)

$$\|Au(\cdot,t)\|_{L^2} = \|AS(t)u_0\|_{L^2} \leq \|S(t)u_0\|_{D(A)} = \|\Lambda_t u_0\|_{D(A)} \leq \frac{C_4}{t^{\frac{3}{2}}}\|u_0\|_{L^2},$$

where $C_4 = 2\tilde{C}_4$, giving the iten (i).

(ii) Assume $u_0 \in D(A^n)$. From the demigroup theory we have $S(t)u_0 \in D(A^n)$ and $A^n u(\cdot,t) = AS(t)A^{n-1}u_0$. Using the item (i) we get

$$\|A^n u(\cdot,t)\|_{L^2} = \|AS(t)A^{n-1}u_0\|_{L^2} \leq \frac{C_4}{t^{\frac{3}{2}}}\left\|A^{n-1}u_0\right\|_{L^2}. \tag{4.50}$$

Splitting $[0,t]$ into $[0,t] = [0,t/n]\cup[t/n,2t/n]\cup\cdots\cup[(n-1)t/n,t]$ and using (4.50) several times we obtain

$$\begin{aligned}\|A^n u(\cdot,t)\|_{L^2} \leq \frac{C_4}{t^{\frac{3}{2}}}\left\|A^{n-1}u(\cdot,(n-1)t/n)\right\|_{L^2} &\leq \cdots \leq \frac{C_4}{t^{\frac{3}{2}}}\frac{C_4}{\left(\frac{n-1}{n}t\right)^{\frac{3}{2}}}\frac{C_4}{\left(\frac{n-2}{n}t\right)^{\frac{3}{2}}}\cdots\frac{C_4}{\left(\frac{2t}{n}\right)^{\frac{3}{2}}}\frac{C_4}{\left(\frac{t}{n}\right)^{\frac{3}{2}}}\|u_0\|_{L^2}\\ &\leq \left(\frac{C_4}{\left(\frac{t}{n}\right)^{\frac{3}{2}}}\right)^n \|u_0\|_{L^2},\end{aligned}$$

thus, for $u_0 \in D(A^n)$, holds that

$$\|A^n u(\cdot,t)\|_{L^2} \leq \frac{C_4^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|u_0\|_{L^2}.$$

Now, remark that $S(t) : (D(A^n), \|\cdot\|_{L^2}) \to \left(D(A^n), \|\cdot\|_{D(A^n)}\right)$ is a bounded linear operator, since, for $u_0 \in D(A^n)$, the previous estimate ensures that

$$\|S(t)u_0\|_{D(A^n)} = \|S(t)u_0\|_{L^2} + \|A^n S(t)u_0\|_{L^2} \leq \left(1 + \frac{C_4^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}}\right) \|u_0\|_{L^2}.$$

Since $D(A^n)$ is dense in $L^2(-1,0)$, there exists a bounded linear operator $\Lambda_{t,n} : (L^2(-1,0), \|\cdot\|_{L^2}) \to \left(D(A^n), \|\cdot\|_{D(A^n)}\right)$, such that

$$\Lambda_{t,n}\Big|_{D(A^n)} = S(t) \quad \text{and} \quad \|\Lambda_{t,n} v\|_{D(A^n)} \leq \frac{C_5^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|v\|_{L^2},\ \forall v \in L^2(-1,0) \tag{4.51}$$

for some constant $C_5 > \max\{1, C_4\}$ which does not depend on t.

Given $u_0 \in L^2(-1,0)$, there exists $(u_k) \subset D(A^n)$ such that $u_k \to u_0$ in $L^2(-1,0)$. Thus, (4.51) gives that

$$\begin{aligned}\|\Lambda_{t,n} u_0 - S(t)u_0\|_{L^2} &\leq \|\Lambda_{t,n} u_0 - \Lambda_{t,n} u_k\|_{D(A^n)} + \|S(t)u_k - S(t)u_0\|_{L^2} \\ &\leq \frac{C_5^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|u_0 - u_k\|_{L^2} + \|u_k - u_0\|_{L^2}.\end{aligned}$$

Making $k \to \infty$ we obtain $S(t)u_0 = \Lambda_{t,n} u_0$. Therefore, $S(t)u_0 \in D(A)$, for $u_0 \in L^2(-1,0)$, and, due to (4.51),

$$\|A^n u(\cdot,t)\|_{L^2} \leq \|S(t)u_0\|_{D(A^n)} = \|\Lambda_{t,n} u_0\|_{D(A^n)} \leq \frac{C_5^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|u_0\|_{L^2},$$

and item (ii) holds.

(iii) Let $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$ be. By the item (ii) we have $S(t)u_0 \in D(A^n)$. In particular, $S(\varepsilon)u_0 \in D(A^n)$ so, from the semigroup theory, we obtain $S(\cdot)u_0 \in \bigcap_{j=0}^n C^{n-j}([\varepsilon,1]; D(A^j))$. Taking $j = n$ we get $S(\cdot)u_0 \in C([\varepsilon,1]; D(A^n))$, and as $\varepsilon \in (0,1)$ is arbitrary it follows that $S(\cdot)u_0 \in C((0,1]; D(A^n))$, showing the item (iii), and the proof is finished. □

The last lemma will be useful to prove the main result of this section.

Lemma 4.6 *For every $n \in \mathbb{N} \cap \{0\}$ we have $D(A^n) \subset H^{5n}(-1,0)$.*

Proof: The result is obvious for $n = 0$ and $n = 1$. For $n = 2$, given $v \in D(A^2)$ there exists $g \in D(A) \subset H^5$ such that $Av = g$, that is, $\partial_x^5 v = g + \partial_x v + \partial_x^3 v \in H^2$ and therefore $v \in H^7$. Thus, $\partial_x^7 v = \partial_x^2 g + \partial_x^3 v + \partial_x^5 v \in H^2$, so $v \in H^9$. Deriving, again, we get $\partial_x^9 v = \partial_x^4 g + \partial_x^5 v + \partial_x^7 v \in H^1$, and so $v \in H^{10}$. Therefore $D(A^2) \subset H^{10}$.

To conclude the result for any $n \in \mathbb{N}$, we proceed by induction. Suppose that for some $n \geq 1$ we have $D(A^n) \subset H^{5n}(-1,0)$. Let $v \in D(A^{n+1})$, by induction hypothesis, and using the same procedure, we have $v, Av \in H^{5n}$ which implies that there exists $f \in H^{5n}$ with $Av = f$, that is, $\partial_x^5 v \in H^{5n-3}$ and so $v \in H^{5n+2}$. By deriving $Av = f$ twice we obtain $v \in H^{5n+4}$. Deriving again twice it follows that $v \in H^{5n+5}$ and therefore $D(A^{n+1}) \subset H^{5(n+1)}$, which concludes the proof. □

The previous results ensure the following ones.

Proposition 4.6 *For any $u_0 \in L^2(-1,0)$ the solution $u(\cdot,t) = S(t)u_0$ of* (4.1) *with $h_1 = h_2 = 0$ satisfies $u \in C^\infty([-1,0] \times (0,1])$.*

Proof: Consider $n \in \mathbb{N}$, $t \in (0,1]$ and $u_0 \in L^2(-1,0)$. From Proposition 4.5 we have $u(\cdot,t) \in D(A^n)$. Using Lemma 4.6 we obtain $u(\cdot,t) \in H^{5n}$. The Sobolev embedding $H^{5n} \hookrightarrow C^{5n-1}([-1,0])$ provides $u(\cdot,t) \in C^{5n-1}([-1,0])$. Since $n \in \mathbb{N}$ is arbitrary, it follows that $u(\cdot,t) \in C^\infty([-1,0])$. On the other hand, given $\varepsilon > 0$ and $n \in \mathbb{N}$, Proposition 4.5 yields that $u(\cdot,\varepsilon) \in D(A^{n+1})$ so, the semigroup theory, follows that $u \in \bigcap_{j=0}^{n+1} C^{n+1-j}\left([\varepsilon,1], D(A^j)\right)$. In particular, taking $j = 1$, we have that $u \in C^n([\varepsilon,1], D(A))$. Using the Lemma 4.4 and the embedding $H^5 \hookrightarrow C^4([-1,0])$ we obtain that $\partial_t^i u(x,\cdot)$ is continuous at t_0 and, as $t_0 \in [\varepsilon,1]$ is arbitrary we conclude that $\partial_t^i u(x,\cdot)$ is continuous for $i = 0,1,...,n$. Since $n \in \mathbb{N}$ and $x \in [-1,0]$ are arbitrary we get $u(x,\cdot) \in C^\infty([\varepsilon,1]$. Furthermore, $u(x,\cdot) \in C^\infty((0,1])$, and the result holds. □

We are now in a position to prove a smooth property for the solutions of (4.1). With these auxiliary results in hand, the main result of this section is the following one.

Proposition 4.7 *Let $u_0 \in L^2(-1,0)$ and $h_1(t) = h_2(t) = 0$ for $t \in [0,1]$. Then the corresponding solution u of* (4.1) *satisfies $u \in G^{\frac{1}{2},\frac{5}{2}}([-1,0] \times [\varepsilon,1])$, for all $\varepsilon \in (0,1)$, that is, we can find positive constants M, R_1, R_2 such that*

$$|\partial_x^p \partial_t^n u(x,t)| \leq \frac{M}{t^{\frac{5n+p+5}{2}}} \frac{p!^{\frac{1}{2}}}{R_1^p} \frac{n!^{\frac{5}{2}}}{R_2^n}.$$

Proof: Consider $p \in \mathbb{N} \cup \{0\}$ and choose $n \in \mathbb{N}$ such that $5n - 5 \leq p \leq 5n - 1$. Then, the

Sobolev embedding, Lemma 4.5 and Proposition 4.5 ensures that

$$\begin{aligned}\|\partial_x^p u(\cdot,t)\|_\infty &\le C_6K_3^n\sum_{i=0}^n \left\|P^i u(\cdot,t)\right\|_{L^2}\\ &= C_6K_3^n\left(\|u(\cdot,t)\|_{L^2}+\sum_{i=1}^n\left\|A^iu(\cdot,t)\right\|_{L^2}\right)\\ &\le C_6K_3^n\left(1+\sum_{i=1}^n\frac{C_5^i}{t^{\frac{3i}{2}}}i^{\frac{3i}{2}}\right)\|u_0\|_{L^2}\\ &\le C_6K_3^n(n+1)\frac{C_5^n}{t^{\frac{3n}{2}}}n^{\frac{3n}{2}}\|u_0\|_{L^2}\\ &= C_6K_3^n(n+1)\frac{C_5^n}{t^{\frac{5n}{2}}}\frac{(5n)^{\frac{5n}{2}}}{5^{\frac{5n}{2}}}\|u_0\|_{L^2}.\end{aligned} \tag{4.52}$$

Using Stirling's formula we get

$$(5n)^{5n}\sim\frac{e^{5n}(5n)!}{(2\pi)^{\frac12}(5n)^{\frac12}} \iff (5n)^{\frac{5n}{2}}\sim\frac{e^{\frac{5n}{2}}(5n)!^{\frac12}}{(2\pi)^{\frac14}(5n)^{\frac14}} \iff \frac{(5n)^{\frac{5n}{2}}}{5^{\frac{5n}{2}}}\sim\frac{1}{(2\pi)^{\frac14}(5n)^{\frac14}}\left(\frac{e}{5}\right)^{\frac{5n}{2}}(5n)!^{\frac12},$$

which implies that

$$\frac{(5n)^{\frac{5n}{2}}}{5^{\frac{5n}{2}}}\le C_7\frac{1}{(2\pi)^{\frac14}(5n)^{\frac14}}\left(\frac{e}{5}\right)^{\frac{5n}{2}}(5n)!^{\frac12}\le C_7\left(\frac{e}{5}\right)^{\frac{5n}{2}}(5n)!^{\frac12}$$

for some constant $C_7>0$. Since $(n+1)\le e^{\frac{5n}{2}}$, for $n\in\mathbb{N}$, the inequality (4.52) gives us

$$\|\partial_x^p u(\cdot,t)\|_\infty\le C_6K_3^ne^{\frac{5n}{2}}\frac{C_5^n}{t^{\frac{5n}{2}}}C_7\left(\frac{e}{5}\right)^{\frac{5n}{2}}(5n)!^{\frac12}\|u_0\|_{L^2}=C_6C_7\frac{C_5^n}{t^{\frac{5n}{2}}}K_3^n\left(\frac{e^2}{5}\right)^{\frac{5n}{2}}(5n)!^{\frac12}\|u_0\|_{L^2}.$$

Remember that $5n-5\le p\le 5n-1$ which allow us write $p=5n-r$ with $r\in\{1,2,3,4,5\}$, that is, $5n=p+r$, with $r\in\{1,2,3,4,5\}$. Using (4.8) it follows that

$$\begin{aligned}\|\partial_x^p u(\cdot,t)\|_\infty &\le C_6C_7\frac{C_5^{\frac{p+r}{5}}}{t^{\frac{p+r}{2}}}K_3^{\frac{p+r}{5}}\left(\frac{e^2}{5}\right)^{\frac{p+r}{2}}(p+r)!^{\frac12}\|u_0\|_{L^2}\\ &\le C_6C_7C_5^{\frac r5}K_3^{\frac r5}\left(\frac{e^2}{5}\right)^{\frac r2}\frac{C_5^{\frac p5}}{t^{\frac{p+r}{2}}}K_3^{\frac p5}\left(\frac{e^2}{5}\right)^{\frac p2}\left(2^p2^rp!r!\right)^{\frac12}\|u_0\|_{L^2}\\ &= C_6C_7C_5^{\frac r5}K_3^{\frac r5}\left(\frac{e^2}{5}\right)^{\frac r2}2^{\frac r2}r!^{\frac12}\|u_0\|_{L^2}\frac{1}{t^{\frac{p+r}{2}}}\left(\frac{C_5^{\frac15}K_3^{\frac15}e2^{\frac12}}{5^{\frac12}}\right)^p p!^{\frac12}.\end{aligned}$$

Consequently

$$\|\partial_x^p u(\cdot,t)\|_\infty\le\frac{C_8}{t^{\frac{p+r}{2}}}\left(\frac{C_5^{\frac15}K_0^{\frac15}e2^{\frac12}}{5^{\frac12}}\right)^p p!^{\frac12}=\frac{C_8}{t^{\frac{p+r}{2}}}\frac{p!^{\frac12}}{\left(5^{\frac12}C_5^{-\frac15}K_0^{-\frac15}e^{-1}2^{-\frac12}\right)^p},$$

with

$$C_8=C_8(r):=C_6C_7C_5^{\frac r5}K_3^{\frac r5}\left(\frac{e^2}{5}\right)^{\frac r2}2^{\frac r2}r!^{\frac12}\|u_0\|_{L^2}\quad\text{and}\quad K_0=K_3+1.$$

Defining $R = 5^{\frac{1}{2}} C_5^{-\frac{1}{5}} K_0^{-\frac{1}{5}} e^{-1} 2^{-\frac{1}{2}}$, follows that $R \in (0,1)$ and

$$\|\partial_x^p u(\cdot,t)\|_\infty \leq \frac{C_8}{t^{\frac{p+r}{2}}} \frac{p!^{\frac{1}{2}}}{R^p}.$$

Here, $p \geq 0$, $r \in \{1,2,3,4,5\}$ and $t \in (0,1]$. Observe that, as $t \in (0,1]$ and $0 \leq r \leq 5$ holds that

$$\|\partial_x^p u(\cdot,t)\|_\infty \leq \frac{C_8}{t^{\frac{p+5}{2}}} \frac{p!^{\frac{1}{2}}}{R^p}, \ \forall\, p \geq 0, \ \forall t \in (0,1]. \tag{4.53}$$

Finally, for every $n, p \geq 0$ from (4.4) we have

$$\partial_t^n \partial_x^p u = \partial_x^p \partial_t^n u = \partial_x^p \left(-(-1)^{n-1} P^n u\right) = \partial_x^p \left((-1)^n P^n u\right) = (-1)^n P^n \partial_x^p u.$$

Using Newton's Binomial theorem, we have that

$$\begin{aligned} P^n \partial_x^p u &= \sum_{q=0}^{n} \binom{n}{q} \sum_{j=0}^{q} \binom{q}{j} (-\partial_x^5)^j (\partial_x^3)^{q-j} \partial_x^{n-q} \partial_x^p u \\ &= \sum_{q=0}^{n} \binom{n}{q} \sum_{j=0}^{q} \binom{q}{j} (-1)^j \partial_x^{5j} \partial_x^{3q-3j} \partial_x^{n-q} \partial_x^p u. \end{aligned}$$

Thus, from (4.53), for $(x,t) \in [-1,0] \times (0,1]$,

$$\begin{aligned} |\partial_t^n \partial_x^p u(x,t)| &\leq \sum_{q=0}^{n} \binom{n}{q} \sum_{j=0}^{q} \binom{q}{j} \left|\partial_x^{n+2q+2j+p} u(x,t)\right| \\ &\leq \sum_{q=0}^{n} \binom{n}{q} \sum_{j=0}^{q} \binom{q}{j} \frac{C_8}{t^{\frac{n+2q+2j+p+5}{2}}} \frac{(n+2q+2j+p)!^{\frac{1}{2}}}{R^{n+2q+2j+p}}. \end{aligned}$$

Once that $t \in (0,1]$, $R < 1$ and $n + 2q + 2j + p \leq 5n + p$ it follows that

$$|\partial_t^n \partial_x^p u(x,t)| \leq \frac{C_8}{t^{\frac{5n+p+5}{2}}} \frac{(5n+p)!^{\frac{1}{2}}}{R^{5n+p}} \sum_{q=0}^{n} \binom{n}{q} \sum_{j=0}^{q} \binom{q}{j}.$$

Noting that

$$2^q = (1+1)^q = \sum_{j=0}^{q} \binom{q}{j} 1^j \cdot 1^{q-j} = \sum_{j=0}^{q} \binom{q}{j}$$

and

$$\sum_{q=0}^{n} \binom{n}{q} \sum_{j=0}^{q} \binom{q}{j} = \sum_{q=0}^{n} \binom{n}{q} 2^q = \sum_{q=0}^{n} \binom{n}{q} 2^q \cdot 1^{n-q} = (2+1)^n = 3^n,$$

using (4.8) one more time, yields

$$|\partial_t^n \partial_x^p u(x,t)| \leq \frac{C_8}{t^{\frac{5n+p+5}{2}}} \frac{3^n \cdot 2^{\frac{5n}{2}} 2^{\frac{p}{2}} (5n)!^{\frac{1}{2}} p!^{\frac{1}{2}}}{R^{5n} R^p}.$$

Stirling's formula gives us

$$(5n)! \sim \left(\frac{5n}{e}\right)^{5n} \sqrt{2\pi 5n} = (10\pi)^{\frac{1}{2}} \left(\frac{5n}{e}\right)^{5n} n^{\frac{1}{2}}. \tag{4.54}$$

Moreover, we also have $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} = (2\pi)^{\frac{1}{2}} \left(\frac{n}{e}\right)^n n^{\frac{1}{2}}$, then

$$\frac{5^{5n} n!^5}{(2\pi)^{\frac{5}{2}} n^{\frac{5}{2}}} \sim \left(\frac{5n}{e}\right)^{5n}.$$

This previous relation together with (4.54) leads us

$$(5n)! \sim (10\pi)^{\frac{1}{2}} \frac{5^{5n} n!^5}{(2\pi)^{\frac{5}{2}} n^{\frac{5}{2}}} n^{\frac{1}{2}} = \frac{(10\pi)^{\frac{1}{2}}}{(2\pi)^{\frac{5}{2}}} \frac{5^{5n} n!^5}{n^2} \sim \frac{(10\pi)^{\frac{1}{4}}}{(2\pi)^{\frac{5}{4}}} \frac{5^{\frac{5n}{2}} n!^{\frac{5}{2}}}{n}.$$

Thus, there exists a constant $C_9 > 0$ such that $(5n)!^{\frac{1}{2}} \leq C_9 5^{\frac{5n}{2}} n!^{\frac{5}{2}}$. Hence

$$|\partial_t^n \partial_x^p u(x,t)| \leq \frac{C_8 C_9}{t^{\frac{5n+p+5}{2}}} \left(\frac{3 \cdot 2^{\frac{5}{2}} \cdot 5^{\frac{5}{2}}}{R^5}\right)^n n!^{\frac{5}{2}} \left(\frac{2^{\frac{1}{2}}}{R}\right)^p p!^{\frac{1}{2}} = \frac{C_8 C_9}{t^{\frac{5n+p+5}{2}}} \frac{n!^{\frac{5}{2}}}{\left(3^{-1} \cdot 10^{-\frac{5}{2}} \cdot R^5\right)^n} \frac{p!^{\frac{1}{2}}}{\left(2^{-\frac{1}{2}} R\right)^p}$$

and the result is achieved with $M = C_8 C_9$, $R_1 = 2^{-\frac{1}{2}} R$ and $R_2 = 3^{-1} \cdot 10^{-\frac{5}{2}} \cdot R^5$. □

4.3.4 Null controllability results

Let us now prove the null controllability result. Precisely, employing two flat output controls, the solution of (4.1) satisfies $u(\cdot, T) = 0$.

Proof of Theorem 4.1: Consider $u_0 \in L^2(-1,0)$ and denote by $\overline{u}$ the solution of (4.1) for $h_1 = h_2 = 0$. From Proposition 4.7 we have, for $\varepsilon \in (0,T)$, that $\overline{u} \in G^{\frac{1}{2},\frac{5}{2}}([-1,0] \times [\varepsilon, T])$. In particular, $\partial_x^3 \overline{u}(0,t), \partial_x^4 \overline{u}(0,t) \in G^{\frac{5}{2}}([\varepsilon, T])$ for any $\varepsilon \in (0,T)$. Choose $\tau \in (0,T)$ and define

$$y(t) = \phi_s \left(\frac{t-\tau}{T-\tau}\right) \partial_x^3 \overline{u}(0,t) \quad \text{and} \quad z(t) = \phi_s \left(\frac{t-\tau}{T-\tau}\right) \partial_x^4 \overline{u}(0,t),$$

where ϕ_s is the step function given by

$$\phi_s(r) = \begin{cases} 1, & \text{if } r \leq 0, \\ 0, & \text{if } r \geq 1, \\ \dfrac{e^{-\frac{K}{(1-r)^\sigma}}}{e^{-\frac{K}{r^\sigma}} + e^{-\frac{K}{(1-r)^\sigma}}}, & \text{if } r \in (0,1), \end{cases}$$

with $K > 0$ and $\sigma := (s-1)^{-1}$. As ϕ_s is Gevrey of order s (see, for example, (MARTIN; ROSIER; ROUCHON, 2018)) and $s \geq \frac{5}{2}$ we infer that $y, z \in G^s([\varepsilon, T])$, $\forall\, \varepsilon \in (0,T)$. Then, defining $u : [-1,0] \times (0,T] \to \mathbb{R}$ by

$$u(x,t) = \begin{cases} u_0(x), & \text{if } t = 0, \\ \sum_{i \geq 0} f_i(x) y^{(i)}(t) + \sum_{i \geq 0} g_i(x) z^{(i)}(t), & \text{if } t \in (0,T], \end{cases}$$

the Proposition 4.1 gives us that u satisfies (4.16) with $u \in G^{\frac{s}{5},s}([-1,0]\times[\varepsilon,T])$ for all $\varepsilon \in (0,T)$. In particular

$$\partial_x^m u(0,t) = 0,\ m = 0,1,2,\quad \partial_x^3 u(0,t) = y(t) \quad \text{and} \quad \partial_x^4 u(0,t) = z(t).$$

Furthermore, by construction

$$\partial_x^m \overline{u}(0,t) = 0,\ m = 0,1,2$$

and, for $t \in (0,\tau)$

$$y(t) = \underbrace{\phi_s\left(\frac{t-\tau}{T-\tau}\right)}_{=1} \partial_x^3 \overline{u}(0,t) = \partial_x^3 \overline{u}(0,t),$$

and

$$z(t) = \underbrace{\phi_s\left(\frac{t-\tau}{T-\tau}\right)}_{=1} \partial_x^4 \overline{u}(0,t) = \partial_x^4 \overline{u}(0,t).$$

Therefore $\partial_x^m u(0,t) = \partial_x^m \overline{u}(0,t)$, for $m = 0,1,2,3,4$ and $t \in (0,\tau)$. Thanks to the Holmgren theorem, we conclude that $u(x,t) = \overline{u}(x,t)$, for all $(x,t) \in [-1,0]\times(0,\tau)$.

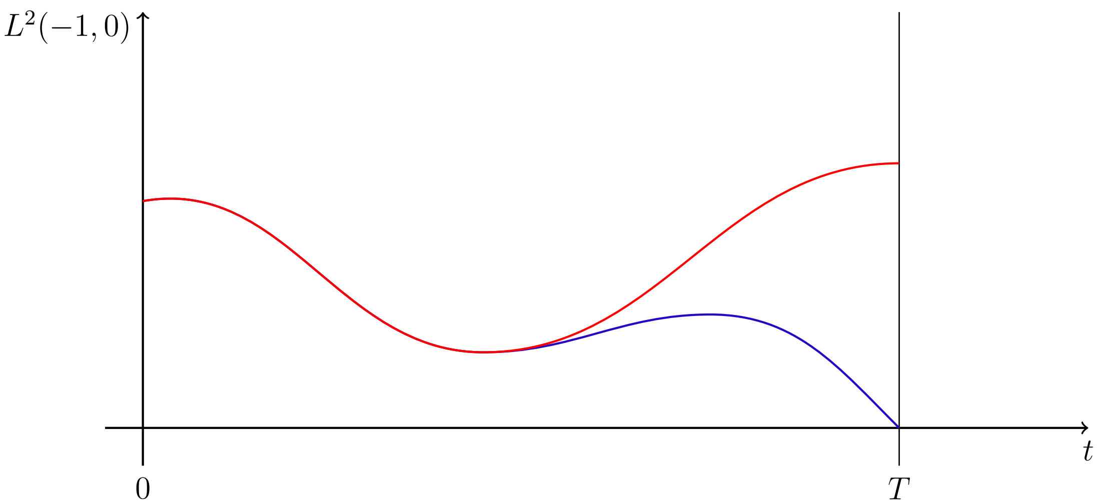

Figure 9 – Trajectories $\overline{u}$ (in red) and u (in blue)

Hence, $u \in C([0,T], L^2(-1,0))$ and it solves the system (4.1) with

$$h_1(t) = \sum_{i\geq 0} f_i(-1) y^{(i)}(t) + \sum_{i\geq 0} g_i(-1) z^{(i)}(t)$$

and

$$h_2(t) = \sum_{i\geq 0} f_{ix}(-1) y^{(i)}(t) + \sum_{i\geq 0} g_{ix}(-1) z^{(i)}(t).$$

Observe that $h_1, h_2 \in G^s([0,T])$ and $h_1(t) = h_2(t) = 0$ for $0 < t < \tau$ since

$$\begin{cases} h_1(t) = u(-1,t) = \overline{u}(-1,t) = 0, \\ h_2(t) = u_x(-1,t) = \overline{u}_x(-1,t) = 0, \end{cases} \quad \forall\, t \in (0,\tau).$$

Finally, once we have $\operatorname{supp} y^{(i)} \subset \operatorname{supp} y \subset (-\infty, T)$ and $\operatorname{supp} z^{(i)} \subset \operatorname{supp} z \subset (-\infty, T)$, follows that $y^{(i)}(T) = 0$ and $z^{(i)}(T) = 0$, for every $i \geq 0$, so that $u(\cdot, T) = 0$, and the first main result is showed. □

4.4 A CLASS OF REACHABLE FUNCTIONS

In this section, we will establish a class of sets that can be reachable from 0 by the system (4.1). Our goal is to prove that, given $u_1 \in \mathcal{R}_R$ (see the definition in (4.5)), one can find control inputs h_1 and h_2 for which the solution of (4.1), with $u_0 = 0$, satisfies $u(x,T) = u_1(x)$.

4.4.1 Auxiliary results

To prove what we mentioned before, we need auxiliary results. The first establishes the flatness property for the limit case $s = 5$.

Proposition 4.8 *Let $R > 2^{\frac{1}{5}}$ and $y, z \in G^5([0,T])$ with*

$$|y^{(j)}(t)|, |z^{(j)}(t)| \leq M\frac{(5j)!}{R^{5j}}, \ \forall j \geq 0, \ \forall t \in [0,T]. \tag{4.55}$$

Then, defining $u(x,t)$ as in (4.9) *we have $u \in G^{1,5}([-1,0]\times[0,T])$ and it solves* (4.16).

Proof: Let $m, k \geq 0$. For $j \geq k$ we have, from (4.55) and Lemma 4.2, that

$$\sum_{j=0}^{\infty} \left| f_{j-k}(x) y^{(j+m)}(t) \right| \leq \sum_{j \geq k} 2^{j-k} \frac{|x|^{5(j-k)+1}}{[5(j-k)+1]!} M \frac{[5(j+m)]!}{R^{5(j+m)}}.$$

Pick $l = 5(j-k)$ and $N = 5(k+m)$ so $l + N = 5(j+m)$, gives that

$$\sum_{j=0}^{\infty} \left| f_{j-k}(x) y^{(j+m)}(t) \right| \leq M \sum_{l \geq 0} 2^{\frac{l}{5}} \frac{1}{(l+1)!} \frac{(l+N)!}{R^{l+N}}.$$

If $N \leq 1$ then

$$\sum_{j=0}^{\infty} \left| f_{j-k}(x) y^{(j+m)}(t) \right| \leq \frac{M}{R^N} \sum_{l \geq 0} \left(\frac{1}{R \cdot 2^{-\frac{1}{5}}} \right)^l < \infty.$$

Assume $N > 1$, note that $(l+N)! = (l+N)(l+N-1)\cdots(l+2)(l+1)!$, so that

$$\begin{aligned}\sum_{j=0}^{\infty}\left|f_{j-k}(x)y^{(j+m)}(t)\right| &\leq M\sum_{l\geq 0}2^{\frac{l}{5}}\frac{(l+N)(l+N-1)\cdots(l+2)}{R^{l+N}}\\ &\leq M\sum_{q\geq 0}\sum_{qN\leq l<(q+1)N}\frac{2^{\frac{l+N}{5}}(l+N)^{N-1}}{R^{l+N}}\\ &= M\sum_{q\geq 0}\sum_{qN\leq l<(q+1)N}\frac{[(q+2)N]^{N-1}}{\left(2^{-\frac{1}{5}}R\right)^{l+N}}.\end{aligned}$$

For $l > qN$ we have $l+N \geq (q+1)N$ and, since $2^{-\frac{1}{5}}R > 1$, follows that

$$\frac{1}{\left(2^{-\frac{1}{5}}R\right)^{l+N}} \leq \frac{1}{\left(2^{-\frac{1}{5}}R\right)^{(q+1)N}}.$$

Thus, thanks to the relation (4.26), the following estimate

$$\begin{aligned}\sum_{j=0}^{\infty}\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right| &\leq M\sum_{q\geq 0}\sum_{qN\leq l<(q+1)N}\frac{(q+2)^{N-1}N^{N-1}}{\left(2^{-\frac{1}{5}}R\right)^{(q+1)N}}\\ &\leq MN^N\sum_{q\geq 0}\left(\frac{q+2}{\tilde{R}^{q+1}}\right)^N,\end{aligned}$$

is verified with $\tilde{R} = 2^{-\frac{1}{5}}R$. Pick any $\sigma \in (0,1)$ and define $f_\sigma(x) = \frac{x+2}{\left(\tilde{R}^{1-\sigma}\right)^{x+1}}$. Note that $\lim_{x\to\infty}\left(\tilde{R}^{1-\sigma}\right)^{x+1} = \infty$, since $\tilde{R} > 1$. The L'Hospital rule ensures that

$$\lim_{x\to\infty} f_\sigma(x) = \lim_{x\to\infty}\frac{1}{\ln(\tilde{R}^{1-\sigma})(\tilde{R}^{1-\sigma})^{x+1}} = 0,$$

and so,

$$\frac{q+2}{\left(\tilde{R}^{1-\sigma}\right)^{q+1}} \to 0, \quad \text{as} \quad q\to\infty.$$

Defining $a := \sup_{q\geq 0}\frac{q+2}{\left(\tilde{R}^{1-\sigma}\right)^{q+1}}$ holds that

$$\sum_{q\geq 0}\left(\frac{q+2}{\tilde{R}^{q+1}}\right)^N \leq \frac{a^N}{\tilde{R}^{N\sigma}}\sum_{q\geq 0}\frac{1}{\tilde{R}^{N\sigma q}} = \frac{a^N}{\tilde{R}^{N\sigma}}\left(\frac{1}{1-\frac{1}{\tilde{R}^{N\sigma}}}\right).$$

Once we have the following convergence

$$\alpha_N := \frac{1}{1-\frac{1}{\tilde{R}^{N\sigma}}} \to 1, \quad \text{as} \quad N\to\infty,$$

we can define $\tilde{M} := \sup_{N>1}\alpha_N$ to get

$$\sum_{q\geq 0}\left(\frac{q+2}{\tilde{R}^{q+1}}\right)^N \leq \tilde{M}\frac{a^N}{\tilde{R}^{N\sigma}}.$$

Consequently

$$\sum_{j=0}^{\infty}\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right| \leq MN^N\tilde{M}\frac{a^N}{\tilde{R}^{N\sigma}},$$

and using Stirling's formula, we get that

$$\frac{a^N N^N}{\tilde{R}^{N\sigma}} \sim \frac{1}{\sqrt{2\pi}}\frac{a^N e^N}{\tilde{R}^{N\sigma}}\frac{N!}{N^{\frac{1}{2}}},$$

which ensures the following estimate

$$\sum_{j=0}^{\infty}\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right| \leq M'\left(\frac{ae}{\tilde{R}^{\sigma}}\right)^N\frac{N!}{N^{\frac{1}{2}}},$$

for some constant $M'>0$. Moreover, noting that $N!=(5k+5m)!\leq 2^{5k}2^{5m}(5k)!(5m)!$ and using Stirling's formula again, namely, $(5m)!\sim 5^{5m+\frac{1}{2}}\left(\sqrt{2\pi m}\right)^{-4}m!^5$, follows that

$$\sum_{j=0}^{\infty}\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right| \leq \sqrt{5}M''\left(\frac{2ae}{\tilde{R}^{\sigma}}\right)^{5k}(5k)!\left(\frac{10ae}{\tilde{R}^{\sigma}}\right)^{5m}m!^5\frac{1}{(k+1)^{\frac{1}{2}}}.$$

Now, define $R_1=(2ae)^{-1}\tilde{R}^{\sigma}$, $R_2=\left[(10ae)^{-1}\tilde{R}\right]^5$ and $M'''=\sqrt{5}M''$, it follows that

$$\sum_{j=0}^{\infty}\left|\partial_t^m P^k\left(f_j(x)y^{(j)}(t)\right)\right| \leq M'''\frac{(5k)!}{R_1^{5k}}\frac{m!^5}{R_2^m}\frac{1}{(k+1)^{\frac{1}{2}}}.$$

Observe that we can assume $R_1<1$. Let $K_3>0$ as in Lemma 4.5 for $p=\infty$. Then, the previous inequality yields that

$$\sum_{j=0}^{\infty}\left\|\partial_t^m\left(f_j y^{(j)}(t)\right)\right\|_{5i,\infty} \leq M'''K_3^i\frac{m!^5}{R_2^m}\frac{(5i)!}{R_1^{5i}}\sum_{k=0}^{i}\frac{1}{(k+1)^{\frac{1}{2}}},$$

for all $i\geq 0$. Given $m,n\geq 0$ consider $i\geq 0$ such that $n\in\{5i-r,\ r=0,1,2,3,4\}$. Thus,

$$\sum_{j=0}^{\infty}\left|\partial_x^n\partial_t^m\left(f_j(x)y^{(j)}(t)\right)\right| \leq M'''K_3^i\frac{m!^5}{R_2^m}\frac{(5i)!}{R_1^{5i}}\sum_{k=0}^{i}\frac{1}{(k+1)^{\frac{1}{2}}},$$

for $(x,t)\in[-1,0]\times[0,T]$. Analogously, one can see that

$$\sum_{j=0}^{\infty}\left|\partial_x^n\partial_t^m\left(g_j(x)z^{(j)}(t)\right)\right| \leq M'''K_3^i\frac{m!^5}{R_2^m}\frac{(5i)!}{R_1^{5i}}\sum_{k=0}^{i}\frac{1}{(k+1)^{\frac{1}{2}}}.$$

Therefore these series are uniformly convergent on $[-1,0]\times[0,T]$, for all $m,n\geq 0$ so that, the function u defined by (4.9) satisfies $u\in C^{\infty}([-1,0]\times[0,T])$. Furthermore,

$$\sum_{j=0}^{\infty}\left|\partial_x^n\partial_t^m\left(f_j(x)y^{(j)}(t)\right)\right|+\sum_{j=0}^{\infty}\left|\partial_x^n\partial_t^m\left(g_j(x)z^{(j)}(t)\right)\right| \leq M'''K_3^i\frac{m!^5}{R_2^m}\frac{(5i)!}{R_1^{5i}}\sum_{k=0}^{i}\frac{1}{(k+1)^{\frac{1}{2}}}.$$

On the other hand, given $\rho > 1$, one can get a constant $c > 0$ such that

$$\sum_{k=0}^{i} \frac{1}{(k+1)^{\frac{1}{2}}} \leq i+1 \leq c\rho^i.$$

Hence, adjusting the constants M''' and K_3 we can write

$$|\partial_x^n \partial_t^m u(x,t)| \leq M''' K_3^i \frac{m!^5}{R_2^m} \frac{(5i)!}{R_1^{5i}}.$$

Since $n = 5i - r$ with $r \in \{0,1,2,3,4\}$ we have

$$\frac{K_3^i}{R_1^{5i}}(5i)! \leq \frac{K_3^{\frac{r}{5}} \cdot 2^r \cdot r!}{R_1^r} \cdot \frac{n!}{\left(K_3^{-\frac{1}{5}} \cdot 2^{-1} \cdot R_1\right)^n}.$$

Defining

$$\hat{M} = 2M''' \left(\max_{0 \leq r \leq 4} \frac{K_3^{\frac{r}{5}} \cdot 2^r \cdot r!}{R_1^r} \right) \quad \text{and} \quad R_1' = K_3^{-\frac{1}{5}} \cdot 2^{-1} \cdot R_1$$

it follows that

$$|\partial_x^n \partial_t^m u(x,t)| \leq \hat{M} \frac{n!}{(R_1')^n} \frac{m!^5}{R_2^m} \quad \forall\, n,m \geq 0, \quad \forall (x,t) \in [-1,0] \times [0,T],$$

which concludes the proof. □

The next result is a particular case of (MARTIN; ROSIER; ROUCHON, 2016b, Proposition 3.6) with $a_0 = 1$ and $a_p = [5p(5p-1)(5p-2)(5p-3)(5p-4)]^{-1}$, for $p \geq 1$.

Proposition 4.9 *Let $(d_q)_{q\geq 0}$ be a sequence of real numbers satisfying $|d_q| \leq CH^q(5q)!$, for all $q \geq 0$ and for some constants $H > 0$ and $C > 0$. Then, for each $\tilde{H} > e^{e^{-1}}H$, one can find a function $f \in C^\infty(\mathbb{R})$ such that $f^{(q)}(0) = d_q$, for all $q \geq 0$, and*

$$\left| f^{(q)}(x) \right| \leq C\tilde{H}^q (5q)!, \quad \forall q \geq 0,\ \forall x \in \mathbb{R}.$$

The next lemma is a consequence of the theory of analytic functions.

Lemma 4.7 *Let $\psi \in G^1([-1,0])$ be such that $\partial_x^j P^n \psi(0) = 0,\ \forall n \geq 0,\ j = 0,1,2,3,4$. Then $\psi \equiv 0$.*

Proof: First, remember (from the proof of Proposition 4.7) that

$$P^n = \sum_{q=0}^{n} \binom{n}{q} \sum_{k=0}^{q} \binom{q}{k} (-1)^k \partial_x^{n+2q+2k} \quad \forall n \geq 0$$

so

$$\partial_x^j P^n = \sum_{q=0}^{n} \binom{n}{q} \sum_{k=0}^{q} \binom{q}{k} (-1)^k \partial_x^{n+2q+2k+j} \quad \forall n, j \geq 0.$$

We claim that for any $n \geq 0$

$$\partial_x^j \psi(0) = 0 \ \ \forall j \in \{0, 1, ..., 5n+4\}. \tag{4.56}$$

To prove this, we use induction on n. For $n = 0$, this immediately follows from the hypothesis, since

$$\partial_x^j \psi(0) = \partial_x^j P^0 \psi(0) = 0, \quad j = 0, 1, 2, 3, 4. \tag{4.57}$$

Let us also analyze the case $n = 1$. Using the hypothesis and (4.57) we obtain

$$\begin{cases} P\psi(0) = 0 \Rightarrow \partial_x^5 \psi(0) = 0, \quad \partial_x P\psi(0) = 0 \partial_x^6 \psi(0) = 0, \quad \partial_x^2 P\psi(0) = 0 \Rightarrow \partial_x^7 \psi(0) = 0, \\ \partial_x^3 P\psi(0) = 0 \Rightarrow \partial_x^8 \psi(0) = 0, \quad \partial_x^4 P\psi(0) = 0 \Rightarrow \partial_x^9 \psi(0) = 0. \end{cases}$$

Combining this with (4.57) we get (4.56), for $n = 1$. Now, suppose that (4.56) holds for some $n \geq 1$ and let us show that

$$\partial_x^j \psi(0) = 0 \ \ \forall j \in \{0, 1, ..., 5(n+1)+4\}.$$

By the induction hypothesis, it is sufficient to show that $\partial_x^j \psi(0) = 0$ for $j = 5(n+1) + r$ with $r = 0, 1, 2, 3, 4$. From hypothesis $P^{n+1}\psi(0) = 0$, that is,

$$\sum_{q=0}^{n+1} \binom{n+1}{q} \sum_{k=0}^{q} \binom{q}{k} (-1)^k \partial_x^{n+1+2q+2k} \psi(0) = 0.$$

Thus,

$$\sum_{q=0}^{n} \binom{n+1}{q} \sum_{k=0}^{q} \binom{q}{k} (-1)^k \partial_x^{n+1+2q+2k} \psi(0) + \binom{n+1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \partial_x^{n+1+2(n+1)+2k} \psi(0) = 0.$$

Note that $n + 1 + 2q + 2k \leq 5n + 1$, for $0 \leq k \leq q \leq n$. So, from induction hypothesis it follows that $\partial_x^{n+1+2q+2k} \psi(0) = 0$ and the last equality becomes

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \partial_x^{3(n+1)+2k} \psi(0) = 0,$$

or, equivalently,

$$\sum_{k=0}^{n} \binom{n+1}{k} (-1)^k \partial_x^{3(n+1)+2k} \psi(0) + \binom{n+1}{n+1} (-1)^{n+1} \partial_x^{5(n+1)} \psi(0) = 0.$$

But, for $0 \leq k \leq n$ we have, $3(n+1)+2k \leq 5n+3$, and the induction hypothesis gives us $\partial_x^{3(n+1)+2k}\psi(0) = 0$ and therefore, from the last equality we concludes $\partial_x^{5(n+1)}\psi(0) = 0$. In a similar way,

$$\partial_x^r P^{n+1}\psi(0) = 0 \implies \partial_x^{5(n+1)+r}\psi(0) = 0, \quad r = 1, 2, 3, 4,$$

concluding the proof of (4.56), and we conclude $\partial_x^j\psi(0) = 0$, for all $j \geq 0$. Since ψ is analytic in $[-1, 0]$, it follows that $\psi \equiv 0$. □

4.4.2 Reachable states

We are now in a position to prove the second main result of the article.

Proof of Theorem 4.2: Let $R > 2R_0$ and $u_1 \in \mathcal{R}_R$. Later on, it will be shown that u_1 can be written in the form

$$u_1(x) = \sum_{i\geq 0} c_i f_i(x) + \sum_{i\geq 0} b_i g_i(x), \ \forall x \in [-1, 0]. \tag{4.58}$$

Assume for a moment that (4.58) holds with a convergence in $W^{n,\infty}(-1, 0)$ for all $n \geq 0$. Then, using (4.25) and (4.29) we obtain

$$P^n u_1(x) = (-1)^n \sum_{i\geq n} c_i f_{i-n}(x) + (-1)^n \sum_{i\geq n} b_i g_{i-n}(x)$$

and

$$\partial_x^j P^n u_1(x) = (-1)^n \sum_{i\geq n} c_i \partial_x^j f_{i-n}(x) + (-1)^n \sum_{i\geq n} b_i \partial_x^j g_{i-n}(x) \ \ \forall j \geq 0.$$

From (4.10)-(4.13) it follows that

$$\partial_x^3 P^n u_1(0) = (-1)^n c_n \partial_x^3 f_0(0) + (-1)^n \sum_{i>n} c_i \partial_x^3 f_{i-n}(0) + (-1)^n \sum_{i\geq n} b_i \partial_x^3 g_{i-n}(0) = (-1)^n c_n$$

and

$$\partial_x^4 P^n u_1(0) = (-1)^n \sum_{i\geq n} c_i \partial_x^4 f_{i-n}(0) + (-1)^n b_n \partial_x^4 g_0(0) + (-1)^n \sum_{i>n} b_i \partial_x^4 g_{i-n}(0) = (-1)^n b_n.$$

This leads us to define

$$c_n = (-1)^n \partial_x^3 P^n u_1(0) \quad \text{and} \quad b_n = (-1)^n \partial_x^4 P^n u_1(0), \ \forall \ n \geq 0. \tag{4.59}$$

Claim. There exist $r \in (R_0, R)$ and a constant $K = K(r) > 0$ such that

$$|\partial_x^n u_1(x)| \leq K \frac{n!}{r^n}, \ \forall \ x \in [-1, 0].$$

Indeed, since $R > 2R_0$ we can write $R = 2R_0+\alpha$ with $\alpha > 0$. Then $R > R_0+R_0+\frac{\alpha}{2}$, taking $r = R_0+\frac{\alpha}{2}$ we have $r \in (R_0, R)$ and $R_0+r < R$. Consequently $\overline{D(w,r)} \subset \overline{D(0,R_0+r)} \subset D(0,R)$, for all $w \in \overline{D(0,R_0)}$. Define $K := \max\left\{|z(w)|;\ w \in \overline{D(0,R_0+r)}\right\}$, where $z \in H(D(0,R))$ is such that $z|_{[-1,0]} = u_1$. So, under the hypothesis of $u_1 \in \mathcal{R}$, follows that

$$|\partial_x^n u_1(x)| \leq K\frac{n!}{r^n},\ \forall\ x \in [-1,0],$$

as desired, showing the claim.

Using Lemma 4.5, with $p = \infty$, the claim, (4.8), the fact that $x \mapsto \frac{(x+3)!}{r^{x+3}}$ is increasing and that $5n+1 \leq 6n \leq 6\cdot 2^n$, we get

$$|c_n| \leq \sum_{i=0}^{n} \|P^i\partial_x^3 u_1\|_\infty \leq \frac{3^{n+1}}{2}\|\partial_x^3 u_1\|_{5n,\infty} \leq \frac{9\cdot 2^3\cdot 3!}{r^3}K\cdot\frac{6^n\cdot 2^{5n}}{r^{5n}}\cdot(5n)!$$

and analogously

$$|b_n| \leq \frac{9\cdot 2^4\cdot 4!}{r^4}K\cdot\frac{6^n\cdot 2^{5n}}{r^{5n}}\cdot(5n)!.$$

Therefore

$$|c_n|, |b_n| \leq K'\left(\frac{6\cdot 2^5}{r^5}\right)^n (5n)!,\ \forall\ n \geq 0,$$

for some positive constant $K' > 0$.

Define $H = \frac{6\cdot 2^5}{r^5}$ and observe that $r > R_0$ implies that $He^{e^{-1}} < \frac{1}{2}$. Choose $\tilde{H} \in \left(He^{e^{-1}}, \frac{1}{2}\right)$. Then, from Proposition 4.9, there exist functions $\tilde{f}, \tilde{g} \in C^\infty(\mathbb{R})$ such that

$$\tilde{f}^{(n)}(0) = c_n, \quad \tilde{g}^{(n)}(0) = b_n, \quad \text{and} \quad \left|\tilde{f}^{(n)}(t)\right|, \left|\tilde{g}^{(n)}(t)\right| \leq K'\tilde{H}^n(5n)!\ \ \forall t \in \mathbb{R},$$

for every $n \geq 0$. Define $f(t) = \tilde{f}(t-T)$ and $g(t) = \tilde{g}(t-T)$. Then $f, g \in C^\infty(\mathbb{R})$ with

$$f^{(n)}(t) = \tilde{f}^{(n)}(t-T) \quad \text{and} \quad g^{(n)}(t) = \tilde{g}^{(n)}(t-T)\ \ \forall n \geq 0,\ \forall t \in \mathbb{R}.$$

Moreover, $f^{(n)}(T) = c_n$, $g^{(n)}(T) = b_n$, and

$$\left|f^{(n)}(t)\right|, \left|g^{(n)}(t)\right| \leq K'\tilde{H}^n(5n)! \quad \forall n \geq 0,\ \forall t \in \mathbb{R}.$$

From the last inequality, we have

$$\left|f^{(n)}(t)\right|, \left|g^{(n)}(t)\right| \leq K'\frac{(5n)!}{R_3^{5n}} \quad \forall n \geq 0,\ \forall t \in \mathbb{R}, \tag{4.60}$$

where $R_3 = \tilde{H}^{-\frac{1}{5}} > 2^{\frac{1}{5}}$. Note that (4.60) implies that $f, g \in G^5([0,T])$. Indeed, from Stirling's formula we have

$$(5n)! \sim 5^{5n+\frac{1}{2}} \left(\sqrt{2\pi n}\right)^{-4} n!^5 = \frac{5^{\frac{1}{2}}(2\pi)^{-2}}{n^2} 5^{5n} n!^5.$$

Hence, for some positive constant $K'' > 0$ we have

$$\left|f^{(n)}(t)\right|, \left|g^{(n)}(t)\right| \leq K' \frac{(5n)!}{R_3^{5n}} \leq K'' \frac{5^{5n}}{R_3^{5n}} n!^5 = K'' \frac{n!^5}{(5^{-5}R_3^5)^n}. \tag{4.61}$$

Pick any $\tau \in (0,T)$ and let

$$\beta(t) = 1 - \phi_2\left(\frac{t-\tau}{T-\tau}\right), \quad t \in [0,T].$$

Observe that

$$\beta^{(i)}(t) = -(T-\tau)^{-i}\phi_2^{(i)}\left(\frac{t-\tau}{T-\tau}\right),\ i \geq 1.$$

By definition we have $\operatorname{supp}\,\phi_2 \subset (-\infty, 1)$ so $\beta(T) = 1$ and $\beta^{(i)}(T) = 0,$ for all $i \geq 1$. Define

$$y(t) = f(t)\beta(t) \quad \text{and} \quad z(t) = g(t)\beta(t), \quad t \in [0,T]. \tag{4.62}$$

From the Leibniz rule, we have

$$y^{(n)}(t) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(t)\beta^{(n-j)}(t) \quad \text{and} \quad z^{(n)}(t) = \sum_{j=0}^{n} \binom{n}{j} g^{(j)}(t)\beta^{(n-j)}(t).$$

Then

$$y^{(n)}(T) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(T)\beta^{(n-j)}(T) = f^{(n)}(T)\beta(T) = c_n \tag{4.63}$$

and

$$z^{(n)}(T) = \sum_{j=0}^{n} \binom{n}{j} g^{(j)}(T)\beta^{(n-j)}(T) = g^{(n)}(T)\beta(T) = b_n. \tag{4.64}$$

On the other hand, by definition $\phi_2 \equiv 1$ in $(-\infty, 0]$ so that $\beta \equiv 0$ in $(-\infty, \tau)$. Consequently $y^{(n)} \equiv z^{(n)} \equiv 0$ in $(-\infty, \tau)$ for any $n \geq 0$. In particular

$$y^{(n)}(0) = z^{(n)}(0) = 0, \quad \forall n \geq 0. \tag{4.65}$$

Since $\beta \in G^2([0,T])$ and $f, g \in G^5([0,T])$ we have $y, z \in G^5([0,T])$. Moreover, from (MARTIN; ROSIER; ROUCHON, 2016b, Lemma 3.7), the same constant "R " of f and g in the definition of $G^5([0,T])$ works for y and z. Hence, from (4.61), there exists $K''' > 0$ such that

$$\left|y^{(n)}(t)\right|, \left|z^{(n)}(t)\right| \leq K''' \frac{n!^5}{(5^{-5}R_3^5)^n}, \quad \forall n \geq 0, \ \forall t \in [0,T].$$

Using Stirling's formula we obtain $(5n)! \sim 5^{5n+\frac{1}{2}} \left(\sqrt{2\pi n}\right)^{-4} n!^5$, and so

$$n!^5 \sim 5^{-\left(5n+\frac{1}{2}\right)} \left(\sqrt{2\pi n}\right)^4 (5n)! = \frac{(2\pi)^2}{5^{\frac{1}{2}}} n^2 \frac{(5n)!}{5^{5n}}.$$

Then, there exists a constant $\overline{K} > 0$ such that

$$\left|y^{(n)}(t)\right|, \left|z^{(n)}(t)\right| \leq K''' \frac{1}{(5^{-5}R_3^5)^n} \overline{K} n^2 \frac{(5n)!}{5^{5n}} = K''' \overline{K} \cdot \frac{n^2 (5n)!}{R_3^{5n}}.$$

Note that, since $R_3 > 2^{\frac{1}{5}}$ we can pick $\rho \in \left(1, R_3 2^{-\frac{1}{5}}\right)$, so that the sequence $\left(\frac{n^2}{\rho^{5n}}\right)_{n\in\mathbb{N}}$ is bounded. Indeed, using the L'Hospital rule, we see that

$$\lim_{x\to\infty} \frac{x^2}{\rho^{5x}} = \lim_{x\to\infty} \frac{2x}{(5\ln\rho)\rho^{5x}} = \lim_{x\to\infty} \frac{2}{(5\ln\rho)^2\rho^{5x}} = 0.$$

Hence, there exists $\overline{K}' > 0$ such that $n^2 \leq \overline{K}' \rho^{5n}$, and therefore

$$\left|y^{(n)}(t)\right|, \left|z^{(n)}(t)\right| \leq K''' \overline{K} \cdot \overline{K}' \cdot \frac{\rho^{5n}(5n)!}{R_3^{5n}} = K''' \overline{K} \cdot \overline{K}' \cdot \frac{(5n)!}{(R_3\rho^{-1})^{5n}}.$$

Defining $K'''' = K''' \overline{K} \cdot \overline{K}'$ and $R_3' = \frac{R_3}{\rho}$ we have $R_3' > 2^{\frac{1}{5}}$ and

$$\left|y^{(n)}(t)\right|, \left|z^{(n)}(t)\right| \leq K'''' \frac{(5n)!}{(R_3')^{5n}}, \quad \forall n \geq 0, \forall t \in [0,T]. \tag{4.66}$$

Let u be as in (4.9) corresponding to y and z given in (4.62). From (4.66) and by Proposition 4.8 we have that $y \in G^{1,5}([-1,0]\times[0,T])$ and it solves (4.16). Furthermore, (4.65) gives us

$$u(x,0) = \sum_{j\geq 0} f_j(x) y^{(j)}(0) + \sum_{j\geq 0} g_j(x) z^{(j)}(0) = 0.$$

Setting $h_1 = u(-1,t)$ and $h_2 = u_x(-1,t)$, we get $h_1, h_2 \in G^5([0,T])$ and therefore u solves (4.1) with h_1 and h_2 as control inputs and $y_0 = 0$ as initial data. From the proof of Proposition 4.8 we know that for all $n, m \in \mathbb{N}$ the sequence of the series

$$\sum_{j\geq 0} \partial_t^m \partial_x^n \left(f_j(x) y^{(j)}(t) + g_j(x) z^{(j)}(t) \right)$$

converges uniformly on $[-1,0]\times[0,T]$ to $\partial_t^m\partial_x^n u$ and consequently, for all $n,i\geq 0$,

$$\begin{aligned}P^n u(x,t) &= \sum_{j\geq 0} P^n f_j(x)y^{(j)}(t) + \sum_{j\geq 0} P^n g_j(x)z^{(j)}(t)\\ &= (-1)^n\sum_{j\geq n} f_{j-n}(x)y^{(j)}(t) + (-1)^n\sum_{j\geq n} g_{j-n}(x)z^{(j)}(t)\end{aligned}$$

and

$$\partial_x^i P^n u(x,t) = (-1)^n\sum_{j\geq n}\partial_x^i f_{j-n}(x)y^{(j)}(t) + (-1)^n\sum_{j\geq n}\partial_x^i g_{j-n}(x)z^{(j)}(t),$$

for every $(x,t)\in[-1,0]\times[0,T]$. In particular, (4.63) and (4.64) gives us that

$$\partial_x^i P^n u(x,T) = (-1)^n\sum_{j\geq n} c_j\partial_x^i f_{j-n}(x) + (-1)^n\sum_{j\geq n} b_j\partial_x^i g_{j-n}(x)\ \ \forall i,n\geq 0,\ \forall x\in[-1,0].$$

Thus, (4.10)-(4.13) provide us

$$\begin{cases} P^n u(0,T) = \partial_x P^n u(0,T) = \partial_x^2 P^n u(0,T) = 0,\\ \partial_x^3 P^n u(0,T) = (-1)^n\sum_{j\geq n} c_j\partial_x^3 f_{j-n}(0) + (-1)^n\sum_{j\geq n} b_j\partial_x^3 g_{j-n}(0) = (-1)^n c_n,\\ \partial_x^4 P^n u(0,T) = (-1)^n\sum_{j\geq n} c_j\partial_x^4 f_{j-n}(0) + (-1)^n\sum_{j\geq n} b_j\partial_x^4 g_{j-n}(0) = (-1)^n b_n,\end{cases}$$

and therefore, using (4.59),

$$\begin{cases}\partial_x^j P^n u(0,T) = 0,\ j=0,1,2,\\ \partial_x^3 P^n u(0,T) = \partial_x^3 P^n u_1(0), \quad \partial_x^4 P^n u(0,T) = \partial_x^4 P^n u_1(0).\end{cases} \tag{4.67}$$

Define $\psi\in G^1([-1,0])$ by $\psi(x) = u(x,T) - u_1(x)$, for all $x\in[-1,0]$, and using the fact that $u_1\in\mathcal{R}_R$ together with (4.67), holds that $\partial_x^j P^n\psi(0) = 0$, for $j = 0,1,2,3,4$. From Lemma 4.7 it follows that $\psi\equiv 0$, that is, $u(x,T) = u_1(x)$, which concludes the proof. □

4.5 FURTHER COMMENTS

In this chapter, we study the controllability of the Kawahara equation. Using the flatiness approach, it was possible to show that the system (4.1) is controllable to zero by using two controls at the boundary. Furthermore, we showed that a given space of analytic functions is reachable from zero and, consequently, system (4.1) is exactly controllable in this space. Let us present some comments about our work.

i. Observe that Theorem 4.2 ensures that for the linear Kawahara equation, any reachable state can likely be extended as a holomorphic function on some open set in $\mathbb{C}$. Moreover, the reachable states corresponding to controllability of the trajectories are in $G^{\frac{1}{2}}([-1,0])$, allowing them to be extended as functions in $H(\mathbb{C})$. In contrast, the reachable functions in Theorem 4.2 do not need to be holomorphic over the entire set $\mathbb{C}$; they can have poles outside $D(0,R)$.

ii. As previously mentioned, several authors have applied the strategy of this article to different systems. In our case, additional difficulties arise when dealing with a fifth-order operator in space, specifically $Pu = \partial_x u + \partial_x^3 u - \partial_x^5 u$. Consequently, addressing the smoothing properties is challenging because we are working with five sets of different regularities, and additional terms appear. Furthermore, the set (4.5), and consequently, Theorem 4.2, can be obtained using the strategies outlined in (MARTIN et al., 2019). However, the Gevrey space level needs to be adjusted, and the operator's order must be carefully adapted to our specific case.

iii. Finally, observe that our results are verified to the Benney–Lin type equation:

$$\partial_t u + \partial_x u + \partial_x^3 u + \mu_0 \partial_x^4 u - \partial_x^5 u = 0, \quad (x,t) \in (-1,0) \times (0,T),$$

with the same boundary conditions as in equation (4.1) and with $\mu_0 > 0$. This equation describes the evolution of one-dimensional small but finite amplitude long waves in various physical systems in fluid dynamics (see (BENNEY, 1966) and (LIN, 1974)). The coefficient $\mu_0 > 0$ introduces nonconservative dissipative effects to the dispersive Kawahara equation (4.1) (where $\mu_0 = 0$), and thus it is sometimes referred to as the strongly dissipative Kawahara equation (ZHOU, 2019), the fifth-order Korteweg-de Vries equation (ZHAO; ZHANG; FENG, 2018), or the generalized Kawahara equation (CHEN et al., 2022). For more details about the Benney-Lin type equation, we encourage the reader to see the reference (COCLITE; RUVO, 2022).

5 PERSPECTIVES ON RESEARCH

In this chapter, we present some open questions concerning the systems studied throughout this thesis. These questions outline possible directions for the continuation of this research, with the expectation that new results may be obtained in the near future.

5.1 KDV EQUATION WITH NEUMANN BOUNDARY CONDITIONS

In the Chapter 2 we prove the local controllability to the nonlinear system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & \text{in } (0,L)\times(0,T), \\ y_{xx}(0,t) = y_{xx}(L,t) = 0, & \text{in } (0,T), \\ y_x(L,t) = h(t), & \text{in } (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,L), \end{cases} \tag{5.1}$$

when L belongs to the critical lengths set associated to the corresponding linear system. We believe that the results presented here constitute an important step toward the understanding of the KdV equation posed on the bounded domain $[0,L]$ under Neumann boundary conditions, especially when L belongs to the critical lengths set $\mathcal{R}_c$. We hope that this broader understanding may contribute, shortly, to the investigation of other properties of (5.1), such as results related to stabilization. For instance, the following question is of particular interest to us.

Question $\mathcal{A}_5$: Given $\lambda > 0$ is it possible to find $C > 0$, $r > 0$ and a feedback control law $h(t) = h(y(\cdot,t))$ such that the corresponding solution y of (5.1) satisfies

$$\|y(\cdot,t)\|_{L^2(0,L} \leq Ce^{-\lambda t}\|y_0\|_{L^2(0,L)}, \quad \forall t \geq 0,$$

whenever $\|y_0\|_{L^2(0,L)} \leq r$?

5.2 KDV EQUATION ON STAR GRAPHS

As mentioned before, the study of KdV on graphs is relatively new and still under development. In the particular case studied here, it could not be different. For example, regarding well-posedness, we believe that the notion of solution presented here for the system

$$\begin{cases} \partial_t u_j(t,x) + \partial_x u_j(t,x) + \partial_x^3 u_j(t,x) = 0, & t \in (0,T),\ x \in (0,\ell_j),\ j = 1,\dots,N, \\ u_j(t,0) = u_1(t,0), & t \in (0,T),\ \forall j = 2,\dots,N, \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) + g_0(t), & t \in (0,T), \\ u_j(t,\ell_j) = p_j(t), \quad \partial_x u_j(t,\ell_j) = g_j(t), & t \in (0,T),\ j = 1,\dots,N, \\ u_j(0,x) = u_j^0(x), & x \in I_j := (0,l_j), \end{cases} \tag{5.2}$$

can be improved to give solutions with more regularities, say in the $C([0,T];\mathbb{H}^s(\mathcal{T}) \cap L^2(0,T;\mathbb{H}^{s+1}(\mathcal{T}))$ space for data in $H^s(0,L)$. Precisely, the following question has caught our attention.

Question $\mathcal{B}_5$: Let $s \geq 0$. For data $u^0 \in \mathbb{H}^s(\mathcal{T})$, $g \in \left[H^{\frac{s}{3}}(0,T)\right]^{N+1}$ and $p \in \left[H^{\frac{s+1}{3}}(0,T)\right]^{N}$, does problem (5.2) admit a unique solution in $C([0,T];\mathbb{H}^s(\mathcal{T}))\cap L^2(0,T;\mathbb{H}^{s+1}(\mathcal{T}))$ depending continuously on the data?

If Question $\mathcal{B}_5$ has an affirmative answer, the well-posedness could be extended to the nonlinear problem, as well as the controllability outside critical lengths. Thus, the question arises concerning the controllability of the nonlinear system when L is a critical length.

Question $\mathcal{C}_5$: Is the nonlinear system

$$\begin{cases} \partial_t u_j(t,x) + \partial_x u_j(t,x) + \partial_x^3 u_j(t,x) + u_j(t,x)\partial_x u_j(t,x) = 0, \\ u_j(t,0) = u_1(t,0), \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) + g_0(t), \\ u_j(t,\ell_j) = p_j(t), \quad \partial_x u_j(t,\ell_j) = g_j(t), \\ u_j(0,x) = u_j^0(x), \end{cases} \tag{5.3}$$

exactly controllable when L is a critical lenght?

This problem could be attacked using some method already known in the literature for these situations, for example, power series or the return method. Even in the scenario with solutions by transposition, the linear system still has open questions, such as the case where $\alpha \neq N$ or the edges are considered to have different lengths, giving rise to the following two questions.

Question $\mathcal{D}_5$: Is the system (5.1) exactly controllable when $\alpha \neq N$ by using N controls?

Question $\mathcal{E}_5$: Is the system (5.1) exactly controllable by using N controls when there exist $j,k \in \{1,...,N\}$ such that $\ell_j \neq \ell_k$?

In addition, other boundary and coupling conditions can be considered. With all this in mind, we hope to soon have new results on this topic.

5.3 KAWAHARA EQUATION AND FLATNESS APPROACH

In the Chapter 4 we prove the null controllability to the system

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = 0, & (x,t) \in (-1,0) \times (0,T), \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t \in (0,T), \\ u(-1,t) = h_1(t), \quad u_x(-1,t) = h_2(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (-1,0), \end{cases} \tag{5.4}$$

Moreover, we investigate the controllability of this system in analytic function spaces. Below, we present some significant observations concerning this study, which motivate the formulation of new questions.

i. Our result is entirely linear and applies only to the linear system (5.4). Therefore, a natural extension is to consider the nonlinear problem, which includes the term uu_x. However, it is required to modify the method used here. We believe that the strategy used in (LAURENT; ROSIER, 2020) could be adapted to our work, though this remains an open problem.

ii. Note that the set defined by (4.5) is an example of reachable functions, though they are not completely understood. We believe there are other sets for which Theorem 4.2 remains valid. For instance, in (CHEN; ROSIER, 2020), the authors presented another example for an extension of the KdV equation in a two-dimensional case. Thus, it remains an open question to verify other sets where Theorem 4.2 holds.

In summary, these observations lead to the following problems:

Question $\mathcal{F}_5$: Can we use the ideas contained in the flatness approach to study the controllability of the nonlinear system

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = 0, & (x,t) \in (-1,0) \times (0,T), \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t \in (0,T), \\ u(-1,t) = h_1(t), \quad u_x(-1,t) = h_2(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (-1,0) \quad ? \end{cases}$$

Question $\mathcal{G}_5$**:** Are there other spaces in which the controllability of (5.4) still holds?

REFERENCES

ADAMI, R.; CACCIAPUOTI, C.; FINCO, D.; NOJA, D. Variational properties and orbital stability of standing waves for nls equation on a star graph. *Journal of Differential Equations*, v. 257, n. 10, p. 3738–3777, 2014.

ADAMI, R.; CACCIAPUOTI, C.; FINCO, D.; NOJA, D. Stable standing waves for a nls on star graphs as local minimizers of the constrained energy. *Journal of Differential Equations*, v. 260, n. 10, p. 7397–7415, 2016.

ADAMS, R. A. *Sobolev Spaces*. London: Academic Press, 1975.

AHAMED, S.; MONDAL, D. Global controllability of the kawahara equation at any time. *Nonlinear Analysis: Real World Applications*, v. 85, p. 104374, 2025. ISSN 1468-1218.

AIRY, G. B. Tides and waves. In: SMEDLEY, E.; ROSE, H. J.; ROSE, H. J. (Ed.). *Encyclopaedia Metropolitana*. London: William Clowes and Sons, 1845. v. 5, p. 241–396.

AMMARI, K.; CRÉPEAU, E. Feedback stabilization and boundary controllability of the korteweg–de vries equation on a star-shaped network. *SIAM Journal on Control and Optimization*, v. 56, n. 3, p. 1620–1639, 2018.

AMMARI, K.; CRÉPEAU, E. Well-posedness and stabilization of the Benjamin-Bona-Mahony equation on star-shaped networks. *Systems & Control Letters*, v. 127, p. 39–43, 2019.

ANGULO, J.; CAVALCANTE, M. *Nonlinear Dispersive Equations on Star Graphs, 32º Colóquio Brasileiro de Matemática*. Rio de janeiro: IMPA, 2013.

ANGULO, J.; GOLOSHCHAPOVA, N. Extension theory approach in the stability of the standing waves for the nls equation with point interactions on a star graph. *Advances in Differential Equations*, Khayyam Publishing, Inc., v. 23, n. 11/12, p. 793 – 846, 2018.

ANGULO, J.; GOLOSHCHAPOVA, N. *On the orbital instability of excited states for the NLS equation with the <i></i>-interaction on a star graph*. 2018. 5039-5066 p.

ARARUNA, F.; CAPISTRANO-FILHO, R.; DORONIN, G. Energy decay for the modified kawahara equation posed in a bounded domain. *Journal of Mathematical Analysis and Applications*, v. 385, n. 2, p. 743–756, 2012.

ARMENTANO, F.; CORON, J.; GONZÁLEZ-BURGOS, M. Flatness-based control of distributed parameter systems: Some examples. In: *18th IFAC World Congress*. Milan, Italy: [s.n.], 2011.

AUBIN, J.-P. Un théorème de compacité. *C. R. Acad. Sci.*, Paris, n. 256, p. 5042–5044, 1963.

BAUDOUIN, L.; YAMAMOTO, M. Inverse problem on a tree-shaped network: unified approach for uniqueness. *Applicable Analysis*, Taylor & Francis, v. 94, n. 11, p. 2370–2395, 2015.

BELGACEM, F. B.; KABER, S. M. On the dirichlet boundary controllability of the one-dimensional heat equation: semi-analytical calculations and ill-posedness degree. *Inverse Problems*, v. 27, n. 5, p. 055012, apr 2011. Available at: <https://dx.doi.org/10.1088/0266-5611/27/5/055012>.

BENJAMIN, T. B.; BONA, J. L.; MAHONY, J. J. Model equations for long waves in nonlinear dispersive systems. *Philosophical Transactions of the Royal Society A: Mathematical*, Physical and Engineering Sciences, v. 272, n. 1220, p. 47–78, 1972.

BENNEY, D. J. Long waves on liquid films. *Journal of Mathematics and Physics*, v. 45, n. 1-4, p. 150–155, 1966.

BERKOLAIKO, G.; KUCHMENT, P. *Introduction to Quantum Graphs, Math. Surveys Monogr. 186*. Providence: American Mathematical Society, 2013.

BERLOFF, N. G.; HOWARD, L. N. Solitary and periodic solutions of nonlinear nonintegrable equations. *Studies in Applied Mathematics*, v. 99, 1997.

BISWAS, A. Solitary wave solution for the generalized kawahara equation. *Applied Mathematics Letters*, v. 22, n. 2, p. 208–210, 2009.

BLANK, J.; EXNER, P.; HAVLICEK, M. *Hilbert Space Operators in Quantum Physics*. 2. ed. New York: Springer, 2008. (Theoretical and Mathematical Physics).

BONA, J.; CASCAVAL, R. Nonlinear dispersive waves on trees. *Can. Appl. Math. Q.*, v. 16, p. 1–18, 2008.

BONA, J. L.; SUN, S. M.; ZHANG, B.-Y. A nonhomogeneous boundary-value problem for the korteweg–de vries equation posed on a finite domain. *Communications in Partial Differential Equations*, Taylor & Francis, v. 28, n. 7-8, p. 1391–1436, 2003.

BONA, J. L.; WINTHER, R. The korteweg–de vries equation, posed in a quarter-plane. *SIAM Journal on Mathematical Analysis*, Society for Industrial and Applied Mathematics, v. 14, n. 6, p. 1056–1106, 1983.

BOUSSINESQ, J. V. Théorie de l'intumescene liquide, applelée onde solitaire ou de translation, se propageant dans un canal rectangulaire. *C. R. Acad. Sci.*, v. 72, p. 755–759, 1871.

BOYD, J. P. Weakly non-local solitons for capillary-gravity waves: Fifth-degree korteweg-de vries equation. *Physica D: Nonlinear Phenomena*, v. 48, n. 1, p. 129–146, 1991.

BREZIS, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. [S.l.]: Springer, 2011.

BURIONI, R.; CASSI, D.; RASETTI, M.; SODANO, P.; VEZZANI, A. Bose-einstein condensation on inhomogeneous complex networks. *Journal of Physics B: Atomic, Molecular and Optical Physics*, v. 34, n. 23, p. 4697, 2001.

CAICEDO, M. A.; CAPISTRANO-FILHO, R.; ZHANG, B.-Y. Neumann boundary controllability of the korteweg–de vries equation on a bounded domain. *SIAM Journal on Control and Optimization*, v. 55, n. 6, p. 3503–3532, 2017.

CAPISTRANO-FILHO, R. A.; GALLEGO, F. A.; PAZOTO, A. F. On the well-posedness and large-time behavior of higher order boussinesq system. *Nonlinearity*, v. 32, n. 5, p. 1852, 2019.

CAPISTRANO-FILHO, R. A.; SOUSA, L. S. de; GALLEGO, F. A. Control of kawahara equation with overdetermination condition: The unbounded cases. *Mathematical Methods in the Applied Sciences*, v. 46, n. 14, p. 15162–15185, 2023.

CAPISTRANO-FILHO, R. d. A.; CHENTOUF, B.; SOUSA, L. S. de; MARTINEZ, V. H. G. Two stability results for the kawahara equation with a time-delayed boundary control. *Zeitschrift für angewandte Mathematik und Physik*, v. 74, n. 1, p. 1–26, 2023.

CAPISTRANO-FILHO, R. de A.; GOMES, M. M. de S. Well-posedness and controllability of kawahara equation in weighted sobolev spaces. *Nonlinear Analysis*, v. 207, 2021.

CAPISTRANO–FILHO, R. A.; PAZOTO, A. F.; ROSIER, L. Control of a boussinesq system of kdv–kdv type on a bounded interval. *ESAIM: COCV*, v. 25, n. 58, p. 1–55, 2019.

CARREÑO, N.; LOYOLA, C. An explicit time for the uniform null controllability of a linear korteweg-de vries equation. *Journal of Evolution Equations*, v. 23, 2023.

CAVALCANTE, M. The korteweg–de vries equation on a metric star graph. *Zeitschrift für angewandte Mathematik und Physik*, v. 69, n. 124, 2018.

CERPA, E. Exact controllability of a nonlinear korteweg–de vries equation on a critical spatial domain. *SIAM Journal on Control and Optimization*, v. 46, n. 3, p. 877–899, 2007.

CERPA, E.; CRÉPEAU, E. Boundary controllability for the nonlinear korteweg–de vries equation on any critical domain. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, v. 26, n. 2, p. 457–475, 2009.

CERPA, E.; CRÉPEAU, E.; MORENO, C. On the boundary controllability of the korteweg–de vries equation on a star-shaped network. *IMA Journal of Mathematical Control and Information*, v. 37, n. 1, p. 226–240, 2020.

CERPA, E.; CRÉPEAU, E.; VALEIN, J. Boundary controllability of the korteweg-de vries equation on a tree-shaped network. *Evolution Equations and Control Theory*, v. 9, n. 3, p. 673–692, 2020.

CERPA, E.; RIVAS, I.; ZHANG, B.-Y. Boundary controllability of the korteweg–de vries equation on a bounded domain. *SIAM Journal on Control and Optimization*, v. 51, n. 4, p. 2976–3010, 2013.

CHEN, M. Internal controllability of the kawahara equation on a bounded domain. *Nonlinear Analysis*, v. 185, p. 356–373, 2019.

CHEN, M.; ROSIER, L. Exact controllability of the linear zakharov-kuznetsov equation. *Discrete and Continuous Dynamical Systems - B*, v. 25, n. 10, p. 3889–3916, 2020.

CHEN, M.; ROSIER, L. Reachable states for the distributed control of the heat equation. *Comptes Rendus. Mathématique*, v. 360, p. 627–639, 2022.

CHEN, S.; DU, Z.; LIU, J.; WANG, K. The dynamic properties of a generalized kawahara equation with kuramoto-sivashinsky perturbation. *Discrete and Continuous Dynamical Systems - B*, v. 27, n. 3, p. 1471–1496, 2022.

CHUIKO, G.; DVORNIK, O.; SHYIAN, S.; BAGANOV, Y. A new age-related model for blood stroke volume. *Computers in Biology and Medicine*, v. 79, p. 144–148, 2016.

COCLITE, G. M.; RUVO, L. di. On the solutions for a benney-lin type equation. *Discrete and Continuous Dynamical Systems - B*, v. 27, n. 11, p. 6865–6883, 2022.

CORON, J.-M. Global asymptotic stabilization for controllable systems without drift. *Math. Control Signal Systems*, v. 5, p. 295–312, 1992.

CORON, J.-M. Contrôlabilité exacte frontiére de l'équation d'euler des fluids parfaits incompressibles bidimensionnels. *C. R. Acad. Sci. Paris*, p. 271–276, 1993.

CORON, J.-M. *Control and Nonlinearity, Math. Surveys Monogr. 136*. Providence: American Mathematical Society, 2020.

CORON, J.-M.; CRÉPEAU, E. Exact boundary controllability of a nonlinear kdv equation with critical lengths. *Journal of the European Mathematical Society*, European Mathematical Society Publishing House, v. 6, n. 3, p. 367–398, 2004.

CORON, J.-M.; KOENIG, A.; NGUYEN, H.-M. On the small-time local controllability of a kdv system for critical lengths. *J. Eur. Math. Soc.*, v. 26, n. 4, p. 1193–1253, 2024.

CRÉPEAU, E. Exact boundary controllability of the korteweg?de vries equation around a non-trivial stationary solution. *International Journal of Control*, Taylor & Francis, v. 74, n. 11, p. 1096–1106, 2001.

CRÉPEAU, E.; SORINE, M. A reduced model of pulsatile flow in an arterial compartment. *Chaos, Solitons Fractals*, v. 34, n. 2, p. 594–605, 2007.

CUI, S. B.; DENG, D. gao; TAO, S. Global existence of solutions for the cauchy problem of the kawahara equation with l2 initial data. *Acta Mathematica Sinica*, v. 22, p. 1457–1466, 2006.

FLIESS, M.; JOIN, C. Model-free control. *International Journal of Control*, v. 86, p. 2228 – 2252, 2013.

FLIESS, M.; LÉVINE, J.; MARTIN, P.; AND, P. R. Flatness and defect of non-linear systems: introductory theory and examples. *International Journal of Control*, Taylor & Francis, v. 61, n. 6, p. 1327–1361, 1995.

GLASS, O. Controllability and asymptotic stabilization of the camassa–holm equation. *Journal of Differential Equations*, v. 245, n. 6, p. 1584–1615, 2008.

GLASS, O.; GUERRERO, S. On the controllability of the fifth-order korteweg–de vries equation. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, v. 26, n. 6, p. 2181–2209, 2009.

GLASS, O.; GUERRERO, S. Controllability of the korteweg–de vries equation from the right dirichlet boundary condition. *Systems Control Letters*, v. 59, n. 7, p. 390–395, 2010.

GUILLERON, J.-P. Null controllability of a linear kdv equation on an interval with special boundary conditions. *Mathematics of Control, Signals, and Systems*, v. 26, p. 375–401, 2014.

HASIMOTO, H. Water waves. *Kagaku*, v. 40, p. 401–408, 1970.

HUNTER, J. K.; SCHEURLE, J. Existence of perturbed solitary wave solutions to a model equation for water waves. *Physica D: Nonlinear Phenomena*, v. 32, n. 2, p. 253–268, 1988.

IGNAT, L. I.; PAZOTO, A. F.; ROSIER, L. Inverse problem for the heat equation and the schrödinger equation on a tree. *Inverse Problems*, v. 28, n. 1, 2011.

IGUCHI, T. A long wave approximation for capillary-gravity waves and the kawahara equations. *Bull. Inst. Math. Acad. Sinica (New Series)*, v. 2, n. 2, p. 179–220, 2007.

JIN, L. Application of variational iteration method and homotopy perturbation method to the modified kawahara equation. *Mathematical and Computer Modelling*, v. 49, n. 3, p. 573–578, 2009.

KAKUTANI, T. Axially symmetric stagnation-point flow of an electrically conducting fluid under transverse magnetic field. *Journal of the Physical Society of Japan*, v. 15, n. 4, p. 688–695, 1960.

KAKUTANI, T.; ONO, H. Weak non-linear hydromagnetic waves in a cold collision-free plasma. *Journal of the Physical Society of Japan*, v. 26, n. 5, p. 1305–1318, 1969.

KAWAHARA, T. Oscillatory solitary waves in dispersive media. *Journal of the Physical Society of Japan*, v. 33, n. 1, p. 260–264, 1972.

KAYA, D.; AL-KHALED, K. A numerical comparison of a kawahara equation. *Physics Letters A*, v. 363, n. 5, p. 433–439, 2007.

KORTEWEG, D. J.; VRIES, G. de. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.*, v. 39, n. 240, p. 422–443, 1895.

KRAMER, E.; RIVAS, I.; ZHANG, B.-Y. Well-posedness of a class of non-homogeneous boundary value problems of the korteweg-de vries equation on a finite domain. *ESAIM: COCV*, v. 19, n. 2, p. 358–384, 2013.

KRAMER, E. F.; ZHANG, B.-Y. Nonhomogeneous boundary value problems for the korteweg-de vries equation on a bounded domain. *Journal of Systems Science and Complexity*, v. 23, n. 2, p. 499–526, 2010.

KUCHMENT, P. Quantum graphs: I. some basic structures. *Waves in Random Media*, v. 14, p. 107–128, 2004.

LAROCHE, B.; MARTIN, P.; ROUCHON, P. Motion planning for the heat equation. *International Journal of Robust and Nonlinear Control*, v. 10, n. 8, p. 629–643, 2000.

LAURENT, C.; ROSIER, L. Exact controllability of semilinear heat equations in spaces of analytic functions. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, v. 37, n. 4, p. 1047–1073, 2020.

LIN, S. P. Finite amplitude side-band stability of a viscous film. *Journal of Fluid Mechanics*, v. 63, n. 3, p. 417–429, 1974.

LINARES, F.; PONCE, G. *Introduction to Nonlinear Dispersive Equation*. 2. ed. New York: Springer, 2015. ISBN 978-1-4939-2180-5.

LIONS, J.-L. Contrôlabillité exacte, pertubations et stabilisation de systèmes distribuès. *SIAM Review*, 1988.

LIONS, J. L. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Review*, v. 30, n. 1, p. 1–68, 1988.

LIONS, J.-L. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Review*, v. 30, n. 1, p. 1–68, 1988.

LIONS, J.-L.; MAGENES, E. *Problèmes aux Limites Non Homogènes et Applications*. Paris: Dunod, 1968.

MARTIN, P.; RIVAS, I.; ROSIER, L.; ROUCHON, P. Exact controllability of a linear korteweg–de vries equation by the flatness approach. *SIAM Journal on Control and Optimization*, v. 57, n. 4, p. 2467–2486, 2019.

MARTIN, P.; ROSIER, L.; ROUCHON, P. Null controllability of the heat equation using flatness. *Automatica*, v. 50, n. 12, p. 3067–3076, 2014.

MARTIN, P.; ROSIER, L.; ROUCHON, P. Null controllability of one-dimensional parabolic equations by the flatness approach. *SIAM Journal on Control and Optimization*, v. 54, n. 1, p. 198–220, 2016.

MARTIN, P.; ROSIER, L.; ROUCHON, P. On the reachable states for the boundary control of the heat equation. *Applied Mathematics Research eXpress*, v. 2016, n. 2, p. 181–216, 2016.

MARTIN, P.; ROSIER, L.; ROUCHON, P. Controllability of the 1d schrödinger equation using flatness. *Automatica*, v. 91, p. 208–216, 2018. ISSN 0005-1098.

MICU, S.; ZUAZUA, E. Regularity issues for the null-controllability of the linear 1-d heat equation. *Systems Control Letters*, v. 60, n. 6, p. 406–413, 2011. ISSN 0167-6911. Available at: <https://www.sciencedirect.com/science/article/pii/S0167691111000624>.

MUGNOLO, D. *Mathematical Technology of Networks*. [S.l.]: Springer, 2015. (Bielefeld, Springer Proceedings in Mathematics & Statistics, v. 128).

MUGNOLO, D.; RAULT, J.-F. Construction of exact travelling waves for the benjamin-bona-mahony equation on networks. *Bull. Belg. Mathe. Soc. Simon Stevin*, The Belgian Mathematical Society, v. 21, n. 3, p. 415–436, 2014.

MUNCH, A.; ZUAZUA, E. Numerical approximation of null controls for the heat equation: Ill-posedness and remedies. *Inverse Problems*, v. 26, n. 8, p. 085018, jul 2010. Available at: <https://dx.doi.org/10.1088/0266-5611/26/8/085018>.

PARADA, H. Null controllability of kdv equation in a star-shaped network. *Evolution Equations and Control Theory*, v. 13, n. 3, p. 719–750, 2024.

PARADA, H.; CRÉPEAU, E.; PRIEUR, C. Delayed stabilization of the korteweg–de vries equation on a star-shaped network. *Mathematics of Control, Signals, and Systems*, v. 34, n. 3, p. 559–605, 2022.

PARADA, H.; CRÉPEAU, E.; PRIEUR, C. Global well-posedness of the kdv equation on a star-shaped network and stabilization by saturated controllers. *SIAM Journal on Control and Optimization*, v. 60, n. 4, p. 2268–2296, 2022.

PAZOTO, A. F.; SOTO, M. *Biorthogonal functions for complex exponentials and an application to the controllability of the Kawahara equation via a moment approach*. 2023. Available at: <https://arxiv.org/abs/2212.00158>.

PAZY, A. *Semigroups of linear operators and applications to the partial differential equations*. New York: Springer-Verlag, 1983. (Applied Mathematical Sciences, v. 44).

POLAT, N.; KAYA, D.; TUTALAR, H. I. A analytic and numerical solution to a modified kawahara equation and a convergence analysis of the method. *Applied Mathematics and Computation*, v. 181, n. 1, p. 193–199, 2006.

POMEAU, Y.; RAMANI, A.; GRAMMATICOS, B. Structural stability of the korteweg-de vries solitons under a singular perturbation. *Physica D: Nonlinear Phenomena*, v. 31, n. 1, p. 127–134, 1988.

RAYLEIGH, L. On waves. *Philosophical Magazine*, v. 1, p. 257–279, 1876.

ROSIER, L. Exact boundary controllability for the korteweg-de vries equation on a bounded domain. *ESAIM: COCV*, v. 2, p. 33–55, 1997.

RUSSELL, J. S. Report on waves. *Fourteenth meeting of the British Association for the Advancement of Science*, 1844.

SIMON, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.*, v. 146, p. 65–96, 1986.

STOKES, G. G. On the theroy of oscillatory waves. *Transactions of the Cambridge Philosophical Society*, v. 1, n. 8, p. 441–455, 1847.

VASCONCELLOS, C. F.; SILVA, P. N. da. Stabilization of the linear kawahara equation with localized damping. *Asymptotic Analysis*, v. 58, n. 4, p. 229–252, 2008.

VASCONCELLOS, C. F.; SILVA, P. N. da. Stabilization of the kawahara equation with localized damping. *ESAIM: COCV*, v. 17, n. 1, p. 102–116, 2011.

YOSIDA, K. *Functional Analysis*. 6. ed. New York: Springer, 1980.

YUSUFOğLU, E.; BEKIR, A.; ALP, M. Periodic and solitary wave solutions of kawahara and modified kawahara equations by using sine–cosine method. *Chaos, Solitons Fractals*, v. 37, n. 4, p. 1193–1197, 2008.

ZHANG, B.-Y.; ZHAO, X. Control and stabilization of the kawahara equation on a periodic domain. *Communications in Information and Systems*, v. 12, n. 1, p. 77–96, 2012.

ZHAO, X.; ZHANG, B.-Y. Global controllability and stabilizability of kawahara equation on a periodic domain. *Mathematical Control and Related Fields*, v. 5, n. 2, p. 335–358, 2015.

ZHAO, X.; ZHANG, X.; FENG, H. Global well-posedness and exponential decay for fifth-order korteweg–de vries equation posed on the finite domain. *Journal of Mathematical Analysis and Applications*, v. 468, n. 2, p. 976–997, 2018.

ZHOU, D. On the cauchy problem for the strong dissipative fifth-order kdv equations. *Nonlinear Analysis: Real World Applications*, v. 48, p. 466–484, 2019. ISSN 1468-1218.

ZUAZUA, E. Controllability of partial differential equations. 2006.

APPENDIX A – BASIC THEORY

In this chapter, we bring together important elements of the mathematical concepts that support this thesis.

A.1 FUNCTIONAL SPACES

This section is mainly based on (ADAMS, 1975) and (BREZIS, 2011).

A.1.1 Distributions and Sobolev spaces

We refer to a domain, denoted by Ω, for a nonempty open set in n-dimensional real space $\mathbb{R}^n$. We will focus on the differentiability and integrability of functions defined on the set Ω. Given $n \in \mathbb{N}$, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha^n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Moreover, if $D_j = \frac{\partial}{\partial x_j}$, then

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha^n}$$

denotes a differential operator of order $|\alpha|$. Notice that, $D^{(0,\ldots,0)}u = u$. If α and β are two multi-indices, we say that $\beta \leq \alpha$ provided $\beta_j \leq \alpha_j$ for $1 \leq j \leq n$. Then $\alpha - \beta$ is also a multi-index, and $|\alpha - \beta| + |\beta| = |\alpha|$. Moreover, we also denote $\alpha! = \alpha_1! \ldots \alpha_n!$ and if $\beta \leq \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

With this, for u, v regular enough functions, we state the Leibniz rule given by

$$D^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} v(x).$$

Let $\Omega \subset \mathbb{R}^n$, we denote by $\overline{\Omega}$ the closure of Ω in $\mathbb{R}^n$. Let u a function defined on Ω, we describe the support of u to be the set

$$\operatorname{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We say that u has compact support in Ω if $\operatorname{supp}(u)$ is compact.

For any $m \in \mathbb{N}$, let $C^m(\Omega)$ denote the vector spaces

$$C^m(\Omega) = \{\phi : D^\alpha \phi, |\alpha| \leq m \text{ is continuous on } \Omega\}.$$

We denote $C^0(\Omega) \equiv C(\Omega)$. Let $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$. The subspaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consists of all those functions in $C(\Omega)$ and $C^\infty(\Omega)$, respectively, that have compact support in Ω.

Example A.1 *Let $\theta : \mathbb{R} \to \mathbb{R}$ defined by*

$$\theta(x) = \begin{cases} e^{-x^{-2}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

then $\theta \in C^\infty(\mathbb{R})$.

The example above in $\mathbb{R}$ is a motivation to build classical examples of C_0^∞ functions in $\mathbb{R}^n$.

Example A.2 *Let $\Omega \subset \mathbb{R}^n$ be an open such that $B_1(0) = \{x \in \mathbb{R}^n; \|x\| < 1\}$ is compactly contained in Ω. Let us consider $f : \Omega \to \mathbb{R}$ such that*

$$f(x) = \begin{cases} e^{\frac{1}{\|x\|^2 - 1}}, & \|x\| < 1 \\ 0, & \|x\| \geq 1 \end{cases}$$

where $x = (x_1, \dots, x_n)$ and $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$. We have $f \in C^\infty(\Omega)$ and $\operatorname{supp}(f) = \overline{B_1(0)}$ is compact, that is, $f \in C_0^\infty(\Omega)$.

Definition A.1 *We say that $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ converges to $\varphi \in C_0^\infty(\Omega)$, denoted by $\varphi_n \to \varphi$, if*

i. There exists a compact K of Ω such that $\operatorname{supp}(\varphi) \subset K$ and $\operatorname{supp}(\varphi_n) \subset K, \forall n \in \mathbb{N}$;

ii. $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly in K, for all multi-index α.

By $D(\Omega)$ we represent the space $C_0^\infty(\Omega)$, equipped with the convergence defined above, and will be called the space of test functions on Ω. We define a distribution over Ω, as defined by Schwartz, to any linear form T over $\mathcal{D}(\Omega)$ that is continuous in the sense of convergence defined above, that is, for every sequence $(\varphi_n)_n \subset \mathcal{D}(\Omega)$ that converges to $\varphi \in \mathcal{D}(\Omega)$, then $(\langle T, \varphi_n \rangle)_n \subset \mathbb{K}$ converges to $\langle T, \varphi \rangle$ in $\mathbb{K}$[1].

Remark A.1 *The dual space $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ is called the space of distributions on Ω, it is endowed with the weak-star topology as the dual of $\mathcal{D}(\Omega)$, and is a locally convex topological vector space (TVS) with that topology.*

[1] Observe that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\langle T, \varphi \rangle$ is the evaluation of T in φ, i.e. $T(\varphi)$.

The following example of scalar distributions plays a key role in the theory. First, recall that a function u defined a.e on Ω is said to be locally integrable on Ω provided $u \in L^1(\omega)$ for every open ω compactly contained in Ω and we write $u \in L^1_{\text{loc}}(\Omega)$.

Example A.3 *For every $u \in L^1_{loc}(\Omega)$ we can associate a distribution $T_u : \mathcal{D}(\Omega) \to \mathbb{R}$, defined by*

$$\langle T_u, \varphi \rangle = \int_\Omega u(x)\varphi(x)dx.$$

Remark A.2 *Not every distribution $T \in \mathcal{D}'(\Omega)$ is on the form T_u for some $u \in L^1_{loc}(\Omega)$. Indeed, if $0 \in \Omega$, there cannot exist any locally integrable function δ over Ω such that for every $\varphi \in \mathcal{D}(\Omega)$*

$$\int_\Omega \delta(x)\varphi(x)dx = \varphi(0).$$

However, the linear functional δ defined on $\mathcal{D}(\Omega)$ by $\langle \delta, \varphi \rangle = \varphi(0)$ can be shown that is continuous, and hence a distribution on Ω. It is called the Dirac distribution.

Lemma A.1 (Du Bois Raymond) *Let $u \in L^1_{loc}(\Omega)$. Then*

$$\int_\Omega u(x)\varphi(x)dx = 0, \forall \varphi \in \mathcal{D}(\Omega)$$

if and only if $u = 0$ almost everywhere in Ω.

Let α a multi-index and $\varphi \in \mathcal{D}(\Omega)$, if $u \in C^{|\alpha|}(\Omega)$, then integrating by parts $|\alpha|$ times leads to

$$\int_\Omega \left(D^\alpha u(x)\right)\varphi(x)dx = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\varphi(x)dx$$

This motivates the definition of the derivative $D^\alpha T$ of a distribution $T \in \mathcal{D}'(\Omega)$ as being

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|}\langle T, D^\alpha \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

It is notable that:

- Each distribution T over Ω has derivatives of all orders.
- $D^\alpha T$ is a distribution over Ω, where $T \in \mathcal{D}'(\Omega)$. It is easily seen that $D^\alpha T$ is linear. Now, we show that it is continuous, consider $(\varphi_n)_n \subset \mathcal{D}(\Omega)$ converging to $\varphi \in \mathcal{D}(\Omega)$. Thus,

  $$|\langle D^\alpha T, \varphi_n \rangle - \langle D^\alpha T, \varphi \rangle| \leq |\langle T, D^\alpha \varphi_n - D^\alpha \varphi \rangle| \Rightarrow 0$$

  when $n \to \infty$.

- The map $D^\alpha : \mathcal{D}'(\Omega) \Rightarrow \mathcal{D}'(\Omega)$, such that $T \mapsto D^\alpha T$, is linear and continuous in the sense of convergence defined in $\mathcal{D}'(\Omega)$.

Example A.4 *Let $u : \mathbb{R} \to \mathbb{R}$ the Heaviside function defined by*

$$u(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

Notice that $u \in L^1_{loc}(\mathbb{R})$ but $u' = \delta \notin L^1_{loc}(\mathbb{R})$. Indeed,

$$\langle u', \varphi \rangle = -\langle u, \varphi' \rangle = -\int_0^\infty \varphi'(x)dx = \varphi(0) = \langle \delta, \varphi \rangle$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$.

The Example A.4 shows that the derivative of a $L^1_{\text{loc}}(\Omega)$ function is not, in general, besides to $L^1_{\text{loc}}(\Omega)$. This motivates the well-recognized definition of Sobolev spaces that will be introduced later. First, for $1 \le p < \infty$, we denote by $L^p(\Omega)$ the space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ such that $|u|^p$ is Lebesgue integrable in Ω. This is a Banach space with the norm

$$\|u\|^p_{L^p(\Omega)} = \int_\Omega |u(x)|^p dx.$$

The space $L^\infty(\Omega)$ consists of all essentially bounded functions in Ω equipped with the norm

$$\|u\|_{L^\infty(\Omega)} = \text{esssup}_{x\in\Omega} |u(x)| = \inf\{C : |v(x)| \le C \text{ a.e. in } \Omega\}.$$

which is a Banach space. When $p = 2$ we have the Hilbert space $L^2(\Omega)$ with the inner product

$$\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx$$

and induced norm

$$\|u\|^2_{L^2(\Omega)} = \int_\Omega |u(x)|^2 dx.$$

Given an integer $m > 0$ and $1 \le p \le \infty$, the Sobolev space $W^{m,p}(\Omega)$ consists of (classes of) functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$, for every multi-index α, with $|\alpha| \le m$. Of course $W^{m,p}(\Omega)$ is a vector space. Considering the following norm

$$\|u\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^p \, dx$$

when $1 \le p < \infty$ and

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\le m} \text{supp}_{x\in\Omega} |D^\alpha u(x)|$$

when $p = \infty$, then Sobolev spaces $W^{m,p}(\Omega)$ are Banach spaces. When $p = 2$, the space $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, which equipped with the inner product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx$$

is a Hilbert space.

Let us denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ relative to the norm of the space $W^{m,p}(\Omega)$, i.e.

$$\overline{C_0^\infty(\Omega)}^{W^{m,F}(\Omega)} = W_0^{m,p}(\Omega).$$

Whenever Ω is bounded at least in one direction x_i of $\mathbb{R}^n$, the norm of $W_0^{m,p}(\Omega)$ is given by

$$\|u\|^p_{W_0^{m,p}(\Omega)} = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^p \, dx.$$

We denote by $W^{-m,q}(\Omega)$ the topological dual of $W_0^{m,p}(\Omega)$, where $1 \leq p < \infty$ and q is the Hölder conjugated index of p[2]. We write $H^{-m}(\Omega)$ to denote the topological dual of $H_0^m(\Omega)$.

Let X and Y be two normed vector spaces such that $X \subseteq Y$. If the inclusion map i: $x \in X \mapsto x \in Y$ is continuous for every $x \in X$, then X is said to be continuously embedded in Y and will be denoted $X \hookrightarrow Y$.

Theorem A.1 (Sobolev embeddings) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary, and consider an integer $m \geq 1$ and $1 \leq p < \infty$. Then,*

i. If $\frac{1}{p} - \frac{m}{n} > 0$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$;

ii. If $\frac{1}{p} - \frac{m}{n} = 0$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, for all $p \leq q < +\infty$;

iii. If $\frac{1}{p} - \frac{mn}{n} < 0$, then $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Theorem A.2 (Rellich-Kondrachov) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary and consider $n \geq 2$. Then,*

i. If $p < n$, then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, for all $1 \leq q < \frac{np}{n-p}$;

ii. If $p = n$, then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, for all $p \leq q < \infty$;

iii. If $p > n$ then $W^{1,p}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$.

[2] q is said to be the Hölder conjugated index of $1 \leq p \leq \infty$ if $\frac{1}{p} + \frac{1}{q} = 1$.

We will denote by $L^p(0,T;X), 1 \leq p < \infty$, the space of Banach of (classes of) functions u, defined in $(0,T)$ with values in X, that are strongly measurable and $\|u(t)\|_X^p$ is Lebesgue integrable in $(0,T)$, with the norm

$$\|u(t)\|^p_{L_F(0,T;X)} = \int_0^T \|u(t)\|^p_X dt$$

Furthermore, if $p = \infty$, $L^\infty(0,T;X)$ represents the Banach space of (classes of) functions u, defined in $(0,T)$ with values in X, that are strongly measurable and $\|u(t)\|_X$ has supreme essential finite in $(0,T)$, with the norm

$$\|u(t)\|_{L^\infty(0,T;X)} = \operatorname{ess}\sup_{t\in(0,T)} \|u(t)\|_X.$$

Remark A.3 *When $p = 2$ and X is a Hilbert space, the space $L^2(0,T;X)$ is a Hilbert space, whose inner product is given by*

$$\langle u, v\rangle_{L^2(0,T;X)} = \int_0^T \langle u(t), v(t)\rangle_X dt.$$

Consider the space $L^p(0,T;X), 1 < p < \infty$, with X being Hilbert separable space. With respect to topological dual spaces, we have the following relationship.

$$[L^p(0,T;X)]' \simeq L^q\left(0,T;X'\right)$$

where p and q are Hölder conjugated index. When $p = 1$, we will associate

$$\left[L^1(0,T;X)\right]' \simeq L^\infty\left(0,T;X'\right).$$

Given a Banach space X. The vector space of linear and continuous maps of $\mathcal{D}(0,T)$ on X is called the Space of Vector Distributions on $(0,T)$ with values in X and denoted by $\mathcal{D}'(0,T;X)$.

Example A.5 *Given $u \in L^p(0,T;X),\ 1 \leq p < \infty$, and $\varphi \in \mathcal{D}(0,T)$ the application $T_u : \mathcal{D}(0,T) \to X$, defined by*

$$T_u(\varphi) = \int_0^T u(t)\varphi(t)dt$$

is a X-valued distribution on $(0,T)$.

The integral in the example above is considered in the sense of Bochner (See Yosida 1980 for more details). The map

$$\begin{aligned} L^p(0,T;X) &\to \mathcal{D}(0,T) \\ u &\mapsto T_u \end{aligned}$$

is injective, so we can identify u with T_u and, in this sense, we have

$$L^p(0,T;X) \subset \mathcal{D}'(0,T;X).$$

Given $S \in \mathcal{D}(0,T;X)$, inspired by the previous derivative of distribution, we define the derivative of order m of S as the vector distribution over $(0,T)$ with values in X given for

$$\left\langle \frac{\mathrm{d}^m S}{\mathrm{d}t^n}, \varphi \right\rangle = (-1)^m \left\langle S, \frac{\mathrm{d}^n \varphi}{\mathrm{d}t^n} \right\rangle, \text{ for all } \varphi \in \mathcal{D}(0,T)$$

Let us consider the Banach space

$$W^{m,p}(0,T;X) = \left\{ u \in L^p(0,T;X) : u^{(j)} \in L^p(0,T,X), j = 1, \dots, m \right\}$$

where $u^{(j)}$ represents the j-th derivative of u in the sense of distributions, and the space is endowed with the norm

$$\|u\|^p_{W^{m,p}(0,T;X)} = \sum_{j=0}^{m} \left\| u^{(j)} \right\|^p_{L^p(0,T;X)}.$$

When $p = 2$ and X is a Hilbert space, the space $W^{m,2}(0,T;X)$ is denoted by $H^m(0,T;X)$ which, equipped with the inner product

$$\langle u, v \rangle_{H^m(0,T;X)} = \sum_{j=0}^{m} \left\langle u^{(j)}, v^{(j)} \right\rangle_{L^2(0,T;X)},$$

is a Hilbert space. It is denoted by $H^m_0(0,T;X)$ the closure, in $H^m(0,T;X)$, of $\mathcal{D}(0,T;X)$ and by $H^{-m}(0,T;X)$ the topological dual of $H^m_0(0,T;X)$.

A.1.2 Interpolation of Sobolev spaces

Most of the results that we will enunciate in this subsection, as well as their demonstrations, can be found in (LIONS; MAGENES, 1968).

Let X and Y be two separable Hilbert spaces, with continuous and dense embedding, $X \hookrightarrow Y$. Let $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ be the inner products of X and Y, respectively. We will denote by $D(S)$ the set of all functions u defined in X, such that the application $v \mapsto \langle u, v \rangle_X, v \in X$, is continuous in the topology induced by Y. Then, $\langle u, v \rangle_X = \langle Su, v \rangle_Y$ defines S, as an (unbounded) operator on Y with domain $D(S)$, dense in Y. Since S is a self-adjoint and strictly positive operator, by using the spectral decomposition of self-adjoint operators, we can define $S^\theta, \theta \in \mathbb{R}$. In particular we will use $A = S^{\frac{1}{2}}$. The operator A, is self-adjoint, positive defined on Y, with domain X and

$$\langle u, v \rangle_X = \langle Au, Av \rangle_Y, \quad \text{for all } u, v \in X.$$

Definition A.2 *Under the previous assumptions, we define the intermediate space*

$$[X,Y]_\theta = D\left(A^{1-\theta}\right), 0 \le \theta \le 1$$

equipped with the norm

$$\|u\|^2_{[X,Y]_\theta} = \|u\|^2_Y + \left\|A^{1-\theta}u\right\|^2_Y$$

Note that:

i. $X \hookrightarrow [X,Y]_\theta \hookrightarrow Y$.

ii. $\|u\|_{[X,Y]_\theta} \le \|u\|^{1-\theta}_X \|u\|^\theta_Y$.

iii. If $0 < \theta_0 < \theta_1 < 1$, then $[X,Y]_{\theta_0} \hookrightarrow [X,Y]_{\theta_1}$.

iv. $[[X,Y]_{\theta_0}, [X,Y]_{\theta_1}]_\theta = [X,Y]_{(1-\theta)\theta_0+\theta\theta_1}$.

Throughout this thesis, the following result is quite useful.

Theorem A.3 *Let $\Omega \subset \mathbb{R}^n$ and $\theta_1 \ge \theta_2 \ge 0, \theta_1, \theta_2 \ne k + \frac{1}{2}$, for any integer k. If $s = (1-\theta)\theta_1 + \theta\theta_2 \ne k + \frac{1}{2}$, then*

$$\left[H^{\theta_1}_0(\Omega), H^{\theta_2}_0(\Omega)\right] = H^s_0(\Omega)$$

and

$$\left[H^m_0(\Omega), L^2(\Omega)\right]_\theta = H^s_0(\Omega), \quad s = (1-\theta)m \ne k + \frac{1}{2}$$

with equivalent norms.

A.1.3 Classical remarkable results

Now, let us present a series of classical results that will be used throughout this thesis. The results are classical, and the proofs will be omitted (see (ADAMS, 1975; BREZIS, 2011) and references therein).

Lemma A.2 (Young's Inequality) *Let a and b be positive constants, $1 \le p, q \le \infty$, such that p and q are Hölder conjugated index. Then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Moreover, for all $\varepsilon > 0$,

$$ab \le \varepsilon a^p + C(\varepsilon) b^q.$$

Lemma A.3 (Cauchy-Schwarz's Inequality) *Let $(E, \langle\cdot,\cdot\rangle)$ be a vector space with an inner product and $\|\cdot\|$ the induced norm of the inner product, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \quad \forall x, y \in E.$$

Furthermore, equality holds if and only if x and y are linearly independent.

Lemma A.4 (Hölder's Inequality) *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, consider $1 \leq p, q \leq \infty$ such that p and q are Hölder conjugated. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} = \int_\Omega |fg| \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

Lemma A.5 (Poincaré-Friedrichs inequality) *Let Ω be a bounded open subset of $\mathbb{R}^n$, then for every $1 \leq p < \infty$ there exists a constant $C = C(\Omega, p) > 0$, such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

Remark A.4 *Poincaré's inequality remains true if Ω has a finite measure and also if Ω has a bounded projection on some axis.*

The following result can be found in (AUBIN, 1963) and is a fundamental compactness criterion in the study of the observability inequalities presented in this thesis.

Theorem A.4 (Aubin-Lions) *Let X_0, X and X_1 be Banach spaces such that $X_0 \subset X \subset X_1$ with X_0 compactly embedded in X and $X \hookrightarrow X_1$. Suppose that $1 < p, q \leq \infty$ and*

$$W = \{u \in L^p([0,T]; X_0) : u_t \in L^q([0,T]; X_1)\}.$$

i. If $p < \infty$ then W is compactly embedded into $L^p([0,T], X)$.

ii. If $p = \infty$ and $q > 1$ then $W \hookrightarrow C([0,T]; X)$ is compact.

A.2 SEMIGROUP THEORY

In this section, we present the main elements of semigroup theory, which provides a framework for analyzing the systems described by evolution equations. For an in-depth study, we recommend (PAZY, 1983). In the sequel, we will denote by $(X, \|\cdot\|_X)$ a Banach space and by $\mathcal{L}(X)$ the space of bounded linear operators defined from X into itself.

A.2.1 Basic properties and infinitesimal generator

Definition A.3 *A map $S : [0,\infty) \longrightarrow \mathcal{L}(X)$ is a semigroup of bounded linear operators on X if*

(i) $S(0) = I$, *where I is the identity operator on X.*

(ii) $S(t+s) = S(t)S(s), \ \forall t, s \in [0,\infty)$.

We say that the semigroup $\{S(t)\}_{t\geq 0}$ is uniformly continuous if

$$\lim_{t\to 0^+} \|S(t) - I\| = 0.$$

Finally, $\{S(t)\}_{t\geq 0}$ is said to be strongly continuous, or of class C^0 when

$$\lim_{t\to 0^+} \|(S(t) - I)x\| = 0, \quad \forall x \in X.$$

In this case we can also say that $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup.

Example A.6 *If $A \in \mathcal{L}(X)$ then $S(t) = e^{tA}$ defines a C^0-semigroup in X.*

Example A.7 *Let Y a Banach space and X the space of the bounded and uniformly continuous functions $f : \mathbb{R} \to Y$. Considering in X the norm of the supremum, it follows that X is a Banach space. Defining*

$$S(t)f(s) = f(t+s),$$

we can see that the family $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup on X.

Lemma A.6 *There exist $M \geq 1$ and $\delta > 0$ such that, for $0 \leq t \leq \delta$,*

$$\|S(t)\| \leq M.$$

Theorem A.5 *There exist $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0.$$

Corollary A.1 *For each $u \in X$, the map*

$$f : [0,\infty) \longrightarrow X$$
$$t \longmapsto f(t) = S(t)u$$

is continuous. More precisely, for each $u \in X$

$$S(\cdot)u \in C([0,\infty); X).$$

Definition A.4 *A C^0-semigroup $\{S(t)\}_{t\geq 0}$ is said to be uniformly bounded if there exists $M > 0$ such that $\|S(t)\| \leq M$ for every $t \geq 0$. In the particular case where $M = 1$, we say that $\{S(t)\}_{t\geq 0}$ is a semigroup of contractions.*

Lemma A.7 *For any $u \in X$ and $h \geq 0$,*

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} S(\tau)u \, \mathrm{d}\tau = S(t)u.$$

Definition A.5 *The infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\geq 0}$ in X, is the linear operator $A : D(A) \subset X \to X$ defined by*

$$D(A) = \left\{ u \in X; \lim_{t\to 0^+} \frac{S(t)u - u}{t} \quad \text{exist} \right\}$$

and

$$Au = \lim_{t\to 0^+} \frac{S(t)u - u}{t}, \quad u \in D(A)$$

Remark A.5 (The graph norm) *If $A : D(A) \subset X \to X$ is a closed operator, then $D(A)$ endowed with the graph norm*

$$\|u\|_{D(A)} = \|u\|_X + \|Au\|_X$$

is a Banach space. As we will see later, if A is an infinitesimal generator of a C^0-semigroup in X, then A is closed. Henceforth, whenever we refer to $D(A)$ will consider it in this space. When X is a Hilbert space, then $D(A)$ is also a Hilbert space with inner product given by

$$(u, v)_{D(A)} = (u, v)_X + (Au, Av)_X.$$

Theorem A.6 *Let A the infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\geq 0}$ and $u \in D(A)$. Then $S(t)u \in D(A)$ for every $t \geq 0$,*

$$S(\cdot)u \in C^1([0,\infty); X) \cap C([0,\infty); D(A))$$

and

$$\frac{d}{dt}(S(t)u) = AS(t)u = S(t)Au.$$

Theorem A.7 *Let A the infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\geq 0}$ on X. Then, for any $u \in X$,*

$$\int_0^t S(\tau)u \, \mathrm{d}\tau \in D(A)$$

and

$$A\left(\int_0^t S(\tau)u \, \mathrm{d}\tau\right) = S(t)u - u.$$

Corollary A.2 *If A is the infinitesimal generator of the C^0-semigroup $\{S(t)\}_{t\geq 0}$ in X, then A is closed and densely defined.*

Theorem A.8 *Let $\{S_1(t)\}_{t\geq 0}$ and $\{S_2(t)\}_{t\geq 0}$ two C^0-semigroup with the same infinitesimal generator A. Then $\{S_1(t)\}$ and $\{S_2(t)\}$ are identical.*

Corollary A.3 *If $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup whose infinitesimal generator A is bounded, then*

$$S(t) = e^{tA}.$$

A.2.2 The Hille-Yosida theorem

In this section, A will always be the infinitesimal generator of a C^0-semigroup of contractions. For each $\lambda > 0$ we define the operator

$$\begin{aligned} R(\lambda) : X &\longrightarrow X \\ u &\longmapsto R(\lambda)u = \int_0^\infty e^{-\lambda\tau} S(\tau)u \, \mathrm{d}\tau. \end{aligned}$$

One can see that $R(\lambda)$ is well-defined, bounded and satisfies

$$\|R(\lambda)\| \leq \frac{1}{\lambda}, \quad \forall \lambda > 0. \tag{A.1}$$

Theorem A.9 *If A is an infinitesimal generator of semigroup of contractions $\{S(t)\}_{t\geq 0}$, then $(\lambda I - A)$ is invertible for every $\lambda > 0$ and*

$$(\lambda I - A)^{-1} = R(\lambda).$$

Furthermore, by (A.1) *we have, for each $\lambda > 0$,*

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}.$$

Theorem A.10 *Let A a densely defined operator such that, for any $\lambda > 0$, the operator $(\lambda I - A)^{-1}$ there exists, is linear and bounded with*

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}.$$

Then A is the infinitesimal generator of a semigroup of contractions.

Theorem A.11 (Hille-Yosida) *A linear operator A in a Banach space X is an infinitesimal generator of a semigroup of contractions if, and only if,*

1. A is closed;

2. A is densely defined;

3. For each $\lambda > 0$, $(\lambda I - A)^{-1}$ there exists and is a bounded linear operator with

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}.$$

Theorem A.12 *The operator A is an infinitesimal generator of a C^0-semigroup $\{S(t)\}$, with*

$$\|S(t)\| \leq e^{\omega t}, \quad \omega \geq 0, \quad \forall t \geq 0,$$

if, and only if, it is closed, densely defined, and for any $\lambda > \omega$, there exists the inverse $(\lambda I - A)^{-1}$ and

$$\left\|(\lambda I - A)^{-1}\right\| \leq (\lambda - \omega)^{-1}.$$

A.2.3 The Lumer Phillips theorem

We will now present another characterization of infinitesimal generators of semigroups. Denote by X' the topological dual of X. Given $x' \in X'$ we use the notation

$$x'(x) = \langle x', x \rangle = \langle x, x' \rangle, \quad \forall x \in X.$$

For every $x \in X$ we define the duality set $F(x) \subseteq X'$ by

$$F(x) = \left\{x' : x' \in X' \quad \text{and } \langle x', x \rangle = \|x\|^2 = \|x'\|^2\right\}.$$

From the Hahn-Banach theorem, it follows that $F(x) \neq \varnothing$ for any $x \in X$.

Definition A.6 *We say that the linear operator $A : D(A) \subset X \longrightarrow X$ is maximal if,*

$$R(I + A) = X,$$

that is, for every $f \in X$ there exist $u \in D(A)$ such that $(I + A)u = f$.

Definition A.7 *A linear operator A is dissipative if, for every $x \in D(A)$, there exists $x' \in F(x)$ such that $\operatorname{Re}\langle Ax, x' \rangle \leq 0$. If in addition there exists $\lambda > 0$ such that $R(\lambda I - A) = X$, then we say that A is maximal dissipative or simply m-dissipative*

Theorem A.13 *A linear operator A is dissipative if and only if,*

$$\|(\lambda I - A)x\| \geq \lambda\|x\|,$$

for every $x \in D(A)$ and $\lambda > 0$.

Proposition A.1 *If A is m-dissipative and $R(\lambda_0 I - A) = X$ for some $\lambda_0 > 0$, then:*

1. *$\lambda_0 \in \rho(A)$ and A is closed.*

2. *$(0, \infty) \subset \rho(A)$.*

3. *$R(\lambda I - A) = X$, for every $\lambda > 0$.*

Theorem A.14 (Lumer-Phillips) *Let A be a linear operator with domain $D(A)$ dense in X.*

(a) *If A is dissipative and there exists $\lambda_0 > 0$such that $R(\lambda_0 I - A) = X$, then A is a infinitesimal generator of a C^0-semigroup of contractions X.*

(b) *If A is the infinitesimal generator of a C^0-semigroup of contractions on X, then $R(\lambda I - A) = X$ for every $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x' \in F(x)$, we have $\mathrm{Re}\langle Ax, x'\rangle \leq 0$.*

Remark A.6 *In summary, the Lumer-Phillips theorem says us that, a linear operator $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a C^0-semigroup of contractions on X if, and only if, A is maximal, dissipative and densely defined.*

A.2.4 Regularity for semigroups of contractions

If A is the infinitesimal generator of a C^0-semigroup, then we define

$$D\left(A^2\right) = \{u \in D(A); Au \in D(A)\}.$$

In general, for $k \geq 2$

$$D\left(A^k\right) = \left\{u \in D\left(A^{k-1}\right); Au \in D\left(A^{k-1}\right)\right\}.$$

Lemma A.8 *Let A be the infinitesimal generator of a semigroup of contractions $\{S(t)\}$. Then* *i.* $D(A^2)$ *is dense in* $D(A)$*, with respect to the graph norm (and consequently with respect to the norm* $\|\cdot\|_X$*).*

ii. $\bigcap_{k=0}^{\infty} D(A^k)$ *is dense in* X

Theorem A.15 *Let* $u_0 \in D(A^2)$*, where* A *is the infinitesimal generator of a semigroup of contractions* $\{S(t)\}$*. Then*

$$S(\cdot)u_0 \in C^2([0,\infty);X) \cap C^1([0,\infty);D(A)) \cap C\left([0,\infty);D\left(A^2\right)\right).$$

In general, if $u_0 \in D\left(A^k\right), k \geq 2$*, then*

$$S(\cdot)u_0 \in \bigcap_{j=0}^{k} C^{k-j}\left([0,\infty);D\left(A^j\right)\right),$$

where $A^0 = I$*.*

Corollary A.4 *Let* $u_0 \in D(A)$*, where* A *is the infinitesimal generator of a semigroup of contractions* $\{S(t)\}$*. If* $u(t) = S(t)u_0$ *then we have*

$$\left\|\frac{du}{dt}(t)\right\| \leq \|Au_0\|.$$

A.2.5 Semigroups of contractions in Hilbert spaces

Semigroups have more formidable properties in a Hilbert space X, since we can identify X with its topological dual. In this subsection, we consider X a Hilbert space with inner product $(\cdot,\cdot)_X$.

Definition A.8 *An operator* $F : D(F) \subset X \to X$ *is called monotone when* $(Fu,u)_X \geq 0$*, for every* $u \in D(F)$*. If, in addition,* $R(I-F) = X$*, then* F *is called maximal monotone.*

Definition A.9 *We say that an operator* $F : D(F) \subset X \to X$ *is dissipative when* $(Fu,u)_X \leq 0$*, for every* $u \in D(F)$*. If, in addition,* $R(I-F) = X$*, then we say that* F *is maximal dissipative. In other words,* F *is dissipative when* $-F$ *is monotone, and* F *is maximal dissipative when* $-F$ *is maximal monotone.*

Proposition A.2 *If* $A : D(A) \subset X \to X$ *is the infinitesimal generator of a* C^0*-semigroup of contractions, then* A *is maximal dissipative.*

The next result provides the converse of this proposition above, showing that only the infinitesimal generators of semigroups in a Hilbert space are maximal dissipative.

Theorem A.16 *Let $A : D(A) \subset X \to X$ be a maximal dissipative operator. Then A is closed, densely defined and for each $\lambda > 0, (\lambda I - A)$ is invertible with*

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}.$$

Remark A.7 (Lumer-Phillips in Hilbert Spaces) *Proposition A.2 and Theorem A.16 (together with the Hille-Yosida theorem) deal with the Lumer-Phillips theorem in the Hilbert space context. With this in mind, to verify that A generates a C_0-semigroup, it is enough to verify that*

(i) *A is dissipative;*

(ii) *$R(\lambda_0 I - A) = X$, for some $\lambda_0 > 0$.*

Let A the infinitesimal generator of a C^0-semigroup of contractions $\{S(t)\}$ in X. In the previous sections, we present results concerning the regularity of $S(\cdot)u_0$ when $u_0 \in D(A)$. The next two theorems give us conditions to get similar results for $u_0 \in X$.

Theorem A.17 *Let A be a maximal dissipative self-adjoint operator. If $u_0 \in X$, then*

$$S(\cdot)u_0 \in C([0,\infty); X) \cap C^1((0,\infty); X) \cap C((0,\infty); D(A))$$

Moreover

$$\begin{cases} \|S(t)u_0\| \leq \|u_0\|, \quad t \geq 0 \\ \left\|\dfrac{d}{dt}S(t)u_0\right\| = \|Au(t)\| \leq \frac{1}{t}\|u_0\|, \quad t > 0. \end{cases}$$

Theorem A.18 *Let A be a maximal dissipative self-adjoint operator in X. Then, for each $u_0 \in X$,*

$$S(\cdot)u_0 \in C^k\left((0,\infty); D\left(A^l\right)\right)$$

for any non-negative integers k and l.

A.2.6 Relationships between infinitesimal generator and its adjoint

The definitions and results in this subsection can be found in (CORON, 2020). Here, X will always be a Hilbert space and $A : D(A) \subset X \to X$ a linear operator.

Theorem A.19 *Assume that A is densely defined and closed. If A and A^* are both dissipative, then A is the infinitesimal generator of a C^0-semigroup.*

Theorem A.20 *Assume that A is densely defined and closed. If A and A^* are both dissipative, then for each $u_0 \in D(A)$ we have*

$$S(\cdot)u_0 \in C^1([0, \infty); X) \cap C([0, \infty); D(A)).$$

Furthermore

$$\begin{cases} \|S(t)u_0\| \leq \|u_0\| \quad t \geq 0 \\ \left\| \dfrac{d}{dt} S(t)u_0 \right\| = \|Au(t)\| \leq \|Au_0\| \quad t \geq 0. \end{cases}$$

Theorem A.21 *Let $\{S(t)\}_{t\geq 0}$ a C^0-semigroup on X with infinitesimal generator A. Then $\{S(t)^*\}_{t\geq 0}$ is a C^0-semigroup on X with infinitesimal generator A^*.*

Definition A.10 (Group of bounded linear operators) *A family of bounded linear operators $S(t) : X \to X$, $t \in \mathbb{R}$, is a group on X if*

$$S(0) = \mathrm{Id},$$
$$S(t_1 + t_2) = S(t_1) \circ S(t_2), \forall (t_1, t_2) \in \mathbb{R}.$$

Definition A.11 (C^0-group of bounded linear operators) *Let $\{S(t)\}_{t\in\mathbb{R}}$ a group of bounded linear operators in X. we say that $\{S(t)\}_{t\in\mathbb{R}}$ is a group strongly continuous (or a C^0-group) of bounded linear operators in X if*

$$\lim_{t\to 0} S(t)x = x, \forall x \in X.$$

Definition A.12 *Let $\{S(t)\}_{t\in\mathbb{R}}$ a group of bounded linear operators in X. The infinitesimal generator of $\{S(t)\}_{t\in\mathbb{R}}$ is the linear operator $A : D(A) \subset X \to X$ defined by*

$$D(A) = \left\{ x \in X; \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists} \right\},$$
$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t}, \forall x \in D(A).$$

The main properties of a C^0-group and its infinitesimal generator are analogous to those dealing with semigroups, with the necessary adaptations. We highlight the following.

Theorem A.22 *Let $\{S(t)\}_{t\in\mathbb{R}}$ um C_0-group on X with infinitesimal generator A. Then $\{S(t)^*\}_{t\in\mathbb{R}}$ is a C_0-group on X whose infinitesimal generator is A^*.*

Theorem A.23 *Assume that A is densely defined and $A = -A^*$. Then A is the infinitesimal generator of a C_0-group of isometries on X.*

Corollary A.5 *Assume that A is densely defined and that $A = -A^*$. Then, for each $u_0 \in D(A)$, we have $S(\cdot)u_0 \in C^1(\mathbb{R}; X) \cap C(\mathbb{R}; D(A))$.*

A.2.7 The abstract Cauchy problem

With the results presented above, we have in our hands powerful tools to study the abstract problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + f(t, u(t)), \quad t > 0 \\ u(0) = u_0 \end{cases}$$

Let us start with the linear case, that is, when $f = 0$.

A.2.7.1 The linear case

As a consequence of Theorem A.6, we have the existence and uniqueness of the classical solution to the problem

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t), \ t \geq 0 \\ u(0) = u_0 \end{cases} \tag{A.2}$$

when A is a infinitesimal generator of C^0-semigroup $\{S(t)\}_{t\geq 0}$ in X and $u_0 \in D(A)$.

Definition A.13 *We say that a function $u : [0, \infty) \to X$ is a classical solution of* (A.2) *when $u \in C^1([0, \infty); X) \cap C([0, \infty); D(A))$ and it verifies* (A.2).

Proposition A.3 *If A is the infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\geq 0}$, then for every $u_0 \in D(A)$ we have that $u(t) = S(t)u_0$ is the unique solution to the problem* (A.2).

Theorem A.24 (Regularity) *Let* $u_0 \in D(A^2)$*, where* A *is the infinitesimal generator of a semigroup of contractions* $\{S(t)\}$*.Then the solution* $u(t) = S(t)u_0$ *of the initial value problem*

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) \quad t \geq 0 \\ u(0) = u_0, \end{cases}$$

satisfies

$$u \in C^2([0,\infty); X) \cap C^1([0,\infty); D(A)) \cap C\left([0,\infty); D\left(A^2\right)\right).$$

In general, if $u_0 \in D\left(A^k\right), k \geq 2$*, then* u *satisfies*

$$u \in \bigcap_{j=0}^{k} C^{k-j}\left([0,\infty); D\left(A^j\right)\right),$$

where $A^0 = I$*.*

Theorem A.25 *Let* X *a Hilbert space and* A *a self-adjoint maximal dissipative operator in* X*. If* $u_0 \in X$*, then there exist a unique* u *such that*

$$u \in C([0,\infty); X) \cap C^1((0,\infty); X) \cap C((0,\infty); D(A))$$

and

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t), \quad t > 0 \\ u(0) = u_0 \end{cases}$$

Furthermore

$$\begin{cases} \|u(t)\| \leq \|u_0\| \quad t \geq 0 \\ \left\|\dfrac{du}{dt}(t)\right\| = \|Au(t)\| \leq \frac{1}{t}\|u_0\| \quad t > 0 \end{cases}.$$

Theorem A.26 *Let* X *be a Hilbert space and* A *a self-adjoint maximal dissipative operator in* X*. Let* u *be the solution of the following initial value problem*

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) \quad t > 0 \\ u(0) = u_0, \end{cases}$$

$\mathrm{com}\, u_0 \in X$*. Then*

$$u \in C^k\left((0,\infty); D\left(A^l\right)\right)$$

for any nonnegative integers k *and* l*.*

Theorem A.27 *Let X a Hilbert space and A a closed operator densely defined in X. If A and A^* are both dissipative then for any $u_0 \in D(A)$ there exists a unique u such that*

$$u \in C^1([0,\infty);X) \cap C([0,\infty);D(A))$$

and

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t), \quad t > 0 \\ u(0) = u_0 \end{cases}$$

Furthermore

$$\begin{cases} \|u(t)\| \leq \|u_0\| \quad t \geq 0 \\ \left\|\dfrac{du}{dt}(t)\right\| = \|Au(t)\| \leq \|Au_0\| \quad t \geq 0. \end{cases}$$

Corollary A.6 *Let X be a Hilbert space and A an operator densely defined in X such that $A = -A^*$. Then, for any $u_0 \in D(A)$, then the Cauchy problem*

$$\begin{cases} \dfrac{du}{dt} = Au \\ u(0) = u_0 \end{cases}$$

possesses a unique solution $u \in C^1(\mathbb{R};X) \cap C(\mathbb{R};D(A))$ which is given by

$$u(t) = S(t)u_0, \ \forall t \in \mathbb{R}.$$

A.2.7.2 The semi-linear case

From now on unless otherwise stated, A is the infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space X. Now we will study the initial value problem

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) + f(t), \quad 0 < t < T \\ u(0) = u_0, \end{cases} \tag{A.3}$$

where $f : [0,T] \to X$ is a given function.

Definition A.14 *A function $u : [0,T] \to X$ is a **classical solution** to the problem* (A.3) *if u is continuous in $[0,T]$, continuously differentiable in $(0,T), u(t) \in D(A)$ for $0 < t < T$ and satisfies* (A.3) *in $(0,T)$.*

Definition A.15 *Let $u_0 \in X$ and $f \in L^1([0,T];X)$. We say that a function $u \in C([0,T];X)$ is a **mild solution** (or generalized solution) to the problem* (A.3) *when*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\mathrm{d}s. \tag{A.4}$$

Theorem A.28 *Let $f : [0,T] \to X$ be continuous and*

$$v(t) = \int_0^t S(t-s)f(s)\mathrm{d}s, \quad 0 \le t \le T.$$

If (A.3) *has a (unique) classical solution for $u_0 \in D(A)$, then v possesses the following properties:*

(i) *$v(t)$ is continuously differentiable in $(0,T)$;*

(ii) *$v(t) \in D(A)$ for every $t \in (0,T)$ and $Av(t)$ is continuous in $(0,T)$.*

Remark A.8 *The conditions (i) and (ii) of the theorem A.28 are equivalent.*

Theorem A.29 *If condition (i) (or equivalently (ii)) of Theorem A.28 holds, then the mild solution to the problem* (A.3) *is a classical solution, for $u_0 \in D(A)$.*

Corollary A.7 *If $f \in C^1([0,T];X)$, then the problem* (A.3) *has a unique classical solution, for every $u_0 \in D(A)$.*

Corollary A.8 *Let $f \in C([0,T];D(A))$ be, with $D(A)$ endowed by the norm graph. Then the problem* (A.3) *possesses a unique classical solution, for every $u_0 \in D(A)$.*

Corollary A.9 *Let X be a reflexive Banach space, A the generator of a C^0-semigroup $\{S(t)\}_{t\ge 0}$ and $f : [0,T] \to X$ a Lipschitz continuous function, that is, there exists a constant $L \ge 0$ such that*

$$\|f(t_1) - f(t_2)\| \le L\,|t_1 - t_2| \quad \textit{para quaisquer} \quad t_1, t_2 \in [0,T].$$

Then for every $u_0 \in D(A)$, the problem (A.3) *has a unique classical solution.*

A.2.7.3 The nonlinear case

For $T > 0$, consider the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + f(t,u(t)), \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{A.5}$$

where A is the infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\ge 0}$ in a Banach space X, and $f : [0,T] \times X \to X$ is continuous in t and satisfies a Lipschitz condition in u.

Definition A.16 *A function $u : [0,T] \to X$ is a **classical solution** of* (A.5) *in $[0,T]$ if u satisfies* (A.5) *in $[0,T]$ and if $u \in C([0,T]; D(A)) \cap C^1((0,T]; X)$.*

Definition A.17 *A function $u \in C([0,T]; X)$ satisfying*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s,u(s))ds, \ \forall \ t \in [0,T]$$

*is called a **mild solution** of* (A.5) *in $[0,T]$.*

Proposition A.4 *Every classical solution of* (A.5) *is a mild solution.*

Theorem A.30 *Let $f : [0,T] \times X \to X$ be continuous in $[0,T]$ and uniformly Lipschitz continuous (with constant L) on X. If A is the infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\geq 0}$ in X then, for every $u_0 \in X$, the initial value problem* (A.5) *possesses a unique mild solution $u \in C([0,T]; X)$. Moreover, the map $u_0 \to u$ is Lipschitz continuous from X into $C([0,T]; X)$.*

Corollary A.10 *If A and f satisfy the conditions of Theorem A.30, then, for each $g \in C([0,T]; X)$, the integral equation*

$$w(t) = g(t) + \int_0^t S(t-s)f(s,w(s))ds$$

has a unique solution $w \in C([0,T]; X)$.

Theorem A.31 *Let $f : [0,\infty) \times X \to X$ continuous in t and locally Lipschitz continuous in u, uniformly in t on bounded intervals. If A is the infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\geq 0}$ in X, then for each $u_0 \in X$ there exists $t_{\max} \leq \infty$ such that, then initial value problem*

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = f(t,u(t)), & t \geq 0 \\ u(0) = u_0 \end{cases}$$

has a unique mild solution u defined in $[0, t_{\max})$. Moreover, if$t_{\max} < \infty$ then

$$\lim_{t \to t_{\max}^-} \|u(t)\| = \infty.$$

Theorem A.32 (Regularity) *Let A the infinitesimal generator of a C^0-semigroup $\{S(t)\}_{t\geq 0}$ on X. If $f : [0,T] \times X \to X$ is continuously differentiable from $[0,T] \times X$ into X, then the mild solution of* (A.5) *with $u_0 \in D(A)$ is actually a classical solution.*

A.3 SOME CLASSICAL CONCEPTS ABOUT CONTROL AND STABILIZATION

Here, we present some definitions, tools, as well as techniques, that will be useful throughout this manuscript and are inspired by (LIONS, 1988a), (LIONS, 1988c), (ZUAZUA, 2006), and (CORON, 2020).

A.3.1 Control for finite-dimensional linear systems

Some essential concepts of control and stabilization come from finite-dimensional systems (ODE) and, after generalization in some sense, to infinite-dimensional systems (PDE). Therefore, let us consider $m, n \in \mathbb{N}^*, T > 0$ and the finite-dimensional system

$$\begin{cases} x'(t) = Ax(t) + Bv(t), \quad 0 < t < T \\ x(0) = x^0 \end{cases} \tag{A.6}$$

where $m \leq n, A$ is a real $n \times n$ matrix, B is a real $n \times m$ matrix and $x^0 \in \mathbb{R}^n$. The function $x : [0, T] \to \mathbb{R}^n$ represents the state and $u : [0, T] \to \mathbb{R}^m$ are called the control. The most desirable goal is, of course, controlling the system using a minimum number of m controls.

Note that, by the variations of constants formula, if $u \in L^2(0, T; \mathbb{R}^m)$ (A.6) has a unique solution $x \in H^1(0, T; \mathbb{R}^n)$ given by

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds, \quad \forall t \in [0, T] \tag{A.7}$$

Definition A.18 *We say* (A.6) *is exactly controllable in time $T > 0$ if given any initial and final data $x^0, x^1 \in \mathbb{R}^n$ there exists $u \in L^2(0, T; \mathbb{R}^m)$ such that the solution* (A.7) *of* (A.6) *satisfies $x(T) = x^1$.*

- The aim of the control consists in driving the solution from the initial data x^0 to the final one x^1 in time T by acting on the system through the control u.
- It is desirable to make the number of controls m as small as possible. However, this may affect the control properties of the system.

Definition A.19 *We say* (A.6) *is null controllable in time $T > 0$ if given any initial and final data $x^0 \in \mathbb{R}^n$ there exists $u \in L^2(0, T; \mathbb{R}^m)$ such that the solution* (A.7) *of* (A.6) *satisfies $x(T) = 0$.*

Remark A.9 *Exact and null controllability are equivalent properties in the case of finite-dimensional linear systems. But this is not necessarily the case for nonlinear systems, or for strongly time-irreversible infinite-dimensional systems.*

A.3.2 Control as a minimization problem

Let us introduce the homogeneous adjoint system of (A.6)

$$\begin{cases} -\varphi' = A^*\varphi, \quad 0 < t < T, \\ \varphi(T) = \varphi_T, \end{cases} \tag{A.8}$$

where A^* denotes the adjoint matrix of A. Next, by the adjoint properties, we have a characterization of the exact controllability property.

Lemma A.9 *An initial data $x^0 \in \mathbb{R}^n$ of* (A.6) *is driven to zero in time T by using a control $u \in L^2(0,T)$ if and only if*

$$\int_0^T \langle u, B^*\varphi \rangle \, dt + \left\langle x^0, \varphi(0) \right\rangle = 0 \tag{A.9}$$

for any $\varphi_T \in \mathbb{R}^n, \varphi$ being the solution of the adjoint system (A.8).

The equality (A.9) is an optimality condition for the critical points of the functional $J : \mathbb{R}^n \to \mathbb{R}^n$,

$$J(\varphi_T) = \frac{1}{2} \int_0^T |B^*\varphi|^2 \, dt + \left\langle x^0, \varphi(0) \right\rangle$$

with φ the solution of the adjoint system (A.8) with initial data φ_T at time $t = T$. More precisely,

Lemma A.10 *Suppose that J has a minimizer $\hat{\varphi}_T \in \mathbb{R}^n$ and let $\hat{\varphi}$ be the solution of the adjoint system* (A.8) *with initial data $\hat{\varphi}_T$. Then*

$$u = B^*\hat{\varphi}$$

is a control of the system (A.6) *with initial data x^0.*

The Lemma (A.10) gives a variational method to obtain the control as a minimum of the functional J. Remark that J is continuous. Therefore, the existence of a minimum is ensured if J is coercive too, that is,

$$\lim_{|\varphi_T| \to \infty} J(\varphi_T) = \infty. \tag{A.10}$$

The coercivity of J follows from the next concept, named observability.

Definition A.20 *We say that* (A.8) *is observable in time* $T > 0$ *if there exists* $C > 0$ *such that*

$$\int_0^T |B^*\varphi|^2\, dt \geq C|\varphi(0)|^2, \quad \forall \varphi_T \in \mathbb{R}^n \tag{A.11}$$

where φ *being the solution of* (A.8).

Remark A.10 *The observability inequality* (A.11) *is equivalent to the following assertion: there exists* $C > 0$ *such that*

$$\int_0^T |B^*\varphi|^2\, dt \geq C\, |\varphi_T|^2, \quad \forall \varphi_T \in \mathbb{R}^n \tag{A.12}$$

where φ *being the solution of* (A.8).

Finally, the next Theorem ensures that the exact controllability can be reduced to the study of observability.

Theorem A.33 *The system* (A.6) *is exactly controllable in time* T *if and only if* (A.8) *is observable in time* T.

A.3.3 Control and stabilization extended to infinite-dimensional systems

All of the concepts and results mentioned above can be generalized (in some sense) to infinite-dimensional systems. Let $T > 0, H$, and V be real Hilbert spaces, and consider the following control system

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = Au + Bv, \quad 0 < t < T \\ u(0) = u_0 \end{cases} \tag{A.13}$$

where u denotes the states and $v \in L^2(0, T; V)$ is the control. The operator $A \colon D(A) \to H$ is a linear operator and $B \in \mathcal{L}(V, D(A^*)')$[3], where $D(A^*)'$ denotes the dual space of $D(A^*)$ and A^* is the adjoint of the operator A. Additionally, A^* is associated with the homogeneous adjoint system

$$\begin{cases} \dfrac{d\varphi}{dt} = -A^*\varphi, \quad 0 < t < T, \\ \varphi(T) = \varphi_T, \end{cases} \tag{A.14}$$

Now, we state the most classical notions of controllability for the abstract system (A.13).

[3] This functional setting gives the possibility to consider boundary control operators (instead of the stronger one $B \in \mathcal{L}(V, H)$)

Definition A.21 *The system* (A.13) *is exactly controllable in time* $T > 0$ *if, for every initial and final data* $u_0, u_T \in H$*, there exists* $v \in L^2(0, T; V)$ *such that the solution of* (A.13) *satisfies* $u(T) = u_T$*.*

Definition A.22 *The system* (A.13) *is null controllable in time* $T > 0$ *if, for every initial data* $u_0 \in H$*, there exists* $v \in L^2(0, T; V)$ *such that the solution of* (A.13) *satisfies* $u(T) = 0$*.*

Definition A.23 *The system* (A.13) *is approximately controllable in time* $T > 0$ *if, for every initial and final data* $u_0, u_T \in H$*, and* $\varepsilon > 0$*, there exists* $v \in L^2(0, T; V)$ *such that the solution of* (A.13) *satisfies*

$$\|u(T) - u_T\|_H \leq \varepsilon.$$

Similar to the mentioned for finite-dimensional, a control may be obtained from the solution of the homogeneous system (A.14) with the initial data minimizing the functional $J \colon H \to \mathbb{R}$ given by

$$J(\varphi) = \frac{1}{2} \int_0^T \langle u, B^* \varphi \rangle \, dt + \langle u_0, \varphi(0) \rangle_H - \langle u_T, \varphi_T \rangle_H.$$

Hence, the controllability is reduced to a minimization problem. To guarantee that J has a unique minimizer, we use the next fundamental result in the calculus of variations, which can be found in (BREZIS, 2011).

Theorem A.34 *Let* H *be a reflexive Banach space,* K *a closed convex subset of* H *and* $J \colon K \to \mathbb{R}$ *a function with the following properties:*

(i) J *is convex,*

(ii) J *is lower semi-continuous,*

(iii) If K *is unbounded then* J *is coercive, i.e.*

$$\lim_{\|x\| \to \infty} J(x) = \infty.$$

Then J *reaches its minimum in* K*, i.e., there exists* $x_0 \in K$ *such that*

$$J(x_0) = \min_{x \in K} \varphi(x)$$

Note that J is continuous and convex. The existence of a minimum is ensured if J is also coercive, which is obtained with the *observability inequality*

$$\int_0^T \|B^* \varphi\|_H^2 \, dt \geq C \|\varphi(0)\|_H^2, \quad \forall \varphi(0) \in H. \tag{A.15}$$

APPENDIX B – THE SET $\mathcal{N}^\dagger$ IS NON EMPTY AND COUNTABLE

In this part, we prove that the new set of critical length, defined in (3.10), is non-empty and countable. Recall that $L \in \mathcal{N}^\dagger$ if it can be written as $L^2 = -\left(w_1^2 + w_1 w_2 + w_2^2\right)$ where

$$(w_1, w_2) \in \mathbb{C}^2, \quad \text{such that} \quad w_1^2 e^{w_1} = w_2^2 e^{w_2} = (w_1 + w_2)^2 e^{-(w_1+w_2)}. \tag{B.1}$$

In particular, we show a direct relation between this set and $\mathcal{N}^*$, defined in (3.9). Let (w_1, w_2) satisfying (B.1), take $c = w_i^2 e^{w_i} = (w_1 + w_2)^2 e^{-(w_1+w_2)}$, $i = 1, 2$ then we have that $z_i = \frac{w_i}{2}$ is solution of $z_i e^{z_i} = \pm(\frac{c}{4})^{1/2}$. Similarly, $(z_1 + z_2)^2 e^{-2(z_1+z_2)} = \frac{c}{4}$, which implies $(z_1 + z_2)e^{-(z_1+z_2)} = \pm(\frac{c}{4})^{1/2}$. Then, by symmetry, we have either one of the following cases

- $z_1 e^{z_1} = z_2 e^{z_2} = -(z_1 + z_2)e^{-(z_1+z_2)}$.
- $z_1 e^{z_1} = -z_2 e^{z_2} = (z_1 + z_2)e^{-(z_1+z_2)}$.
- $z_1 e^{z_1} = z_2 e^{z_2} = (z_1 + z_2)e^{-(z_1+z_2)}$.

The first case is the equation related to the critical set $\mathcal{N}^*$, therefore, it has a countable number of solutions. For the second and third cases, we can follow (GLASS; GUERRERO, 2010, Proposition 3 and 4) to ensure that it has a countable number of solutions.

Finally, observe that if $L \in \mathcal{N}^*$, then $2L \in \mathcal{N}^\dagger$. In fact, for $L \in \mathcal{N}^*$ we have $L^2 = -\left(z_1^2 + z_1 z_2 + z_2^2\right)$, for some $(z_1, z_2) \in \mathbb{C}^2$ such that

$$z_1 e^{z_1} = z_2 e^{z_2} = -(z_1 + z_2)e^{-(z_1+z_2)}.$$

It is easy to see that $w_i = 2z_i$, for $i = 1, 2$, satisfy (B.1). Moreover

$$-\left(w_1^2 + w_1 w_2 + w_2^2\right) = -4\left(z_1^2 + z_1 z_2 + z_2^2\right) = (2L)^2.$$