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Estimates for the first eigenvalue of the p-Laplacian on Riemannian manifolds

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MATHEUS NUNES SOARES

ESTIMATES FOR THE FIRST EIGENVALUE OF p-LAPLACIAN ON RIEMANNIAN MANIFOLDS

Tese apresentada ao Programa de Pós-Graduação Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutor em Matemática.

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RESUMO

A seguinte tese teve por objetivo estudar estimativas do primeiro autovalor do operador p-Laplaciano em variedades Riemannianas compactas e completas não compactas. Estabelecemos um operador linearizado para o p-Laplaciano do tipo divergente, o que resultou em uma fórmula do tipo Bochner. A partir disso, obtivemos inicialmente estimativas inferiores do primeiro autovalor do p-Laplaciano através da norma da segunda forma fundamental para $p \geq 2$, com caracterização da igualdade. Em seguida, demonstramos um resultado similar para subvariedades com curvatura escalar prescrita e para subvariedades com curvatura média constante. Em cada caso anteriormente citado, apresentamos uma generalização para variedades com bordo não vazio através de uma fórmula do tipo Reilly para o operador linearizado. Além disso, apresentamos uma versão analítica dos resultados anteriores para o caso singular com $\frac{3}{2} < p < 2$. Por fim, desenvolvemos um teorema do tipo Liouville para variedades completas não compactas, com aplicações em produtos warped.

Palavras-Chave: Subvariedades compactas e fechadas; Subvariedades mínimas; Curvatura escalar prescrita; Formula de Bochner; Primeiro autovalor do p-Laplaciano; Produtos torcidos.

ABSTRACT

The following thesis aims to study estimates of the first eigenvalue of the p-Laplacian operator on compact Riemannian manifolds and complete non-compact manifolds. We established a linearized operator for the divergence-type p-Laplacian, which resulted in a Bochner-type formula. From this, we initially obtained lower bounds for the first eigenvalue of the p-Laplacian through the norm of the second fundamental form for $p \geq 2$, with characterization of equality. Next, we demonstrated a similar result for submanifolds with prescribed scalar curvature and for submanifolds with constant mean curvature. In each case reported above, we presented a generalization for manifolds with non-empty boundary through a Reilly-type formula for the linearized operator. Additionally, we presented an analytical version of the previous results for the singular case with $\frac{3}{2} < p < 2$. Finally, we developed a Liouville-type theorem for complete non-compact manifolds, with applications in warped products.

Keywords: Closed and compact submanifolds; Minimal submanifolds; Prescribed scalar curvature; Bochner formula; First eigenvalue p-Laplacian; Warped products.

CONTESTS

1 INTRODUCTION

The importance of elliptic operators in mathematics is widely acknowledged within the academic community. Over the past few decades, the study of the first nonzero eigenvalue of elliptic operators has garnered significant attention due to its profound implications in both physical and mathematical contexts. From a physical perspective, this eigenvalue plays a crucial role in determining the convergence rate of numerical schemes in numerical analysis, characterizing the ground-state energy of a particle in quantum mechanics, and describing the decay rate of heat flows in thermodynamics. Mathematically, this eigenvalue exhibits deep connections with various geometric objects and structures. For instance, the results of (LEUNG, 1983, 1992) study the first eigenvalue of the Laplacian in closed manifolds immersed in the Euclidean sphere. Specifically, (LEUNG, 1983) showed that if given M^n a closed (i.e., compact and without boundary) submanifold minimally immersed in the unit sphere $\mathbb{S}^m$ with a constant squared length of the second fundamental form $S = |A|^2$, then the quantity $n - \lambda_1(M)$ provides a lower bound for S. Here, $\lambda_1(M)$ denotes the first eigenvalue of the Laplace-Beltrami operator on M^n, which can also be interpreted as an upper bound for $\lambda_1(M)$.

Years later, (BARROS, 2002) refined Leung's result for minimal closed hypersurfaces in $\mathbb{S}^{n+1}$, proving that

$$S \geq c(n,k)(n-1)(n-\lambda_1(M)),$$

where $c(n,k) = \frac{n-k}{n(n-k-1)}$ and k is a constant depends on the dimension of the kernel of A. Subsequently, (BARBOSA; BARROS, 2003) further improved this estimate by showing that there exists a rational number $k \in [n/(n-1), n]$, depending either on A or on the first eigenfunction associated with $\lambda_1(M)$, such that

$$S \geq k\frac{(n-1)}{n}(n-\lambda_1(M)).$$

A natural extension of the Laplacian operator is the p-Laplacian operator Δ_p, defined for $1 < p < \infty$ as

$$\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

This operator arise naturally from the variational problem associated with the p-energy functional

$$E_p(u) = \int_M |\nabla u|^p dM, \tag{1.1}$$

with $u \in W_0^{1,p}(M)$, where $W_0^{1,p}(M)$ is the completion of the subspace of the functions of $\mathcal{C}^\infty(M)$ that has compact support on M^n under the Sobolev norm

$$\|u\|_{1,p} = \left(\int_M |u|^p dM + \int_M |\nabla u|^p dM\right)^{1/p},$$

where dM denotes the Riemannian volume element of M^n.

The p-Laplacian is of significant interest not only in mathematics but also in the theory of non-Newtonian fluids, including: dilatant fluids, for $p \geq 2$; pseudoplastic fluids, for $1 < p < 2$ (ASTARITA; MARRUCCI, 1974). Additionally, for $p \geq 2$, the p-Laplacian also holds geometric interest, with some aspects explored in (UHLENBECK, 1977). Due to its physical and mathematical relevance, numerous bounds have been established for the Dirichlet first eigenvalue of the Laplacian, and many of these results have been extended to the nonlinear p-Laplacian over the last two decades. In this context, (MATEI, 2000) extended Cheng's first Dirichlet eigenvalue comparison for balls to the p-Laplacian on complete manifolds with Ricci curvature bounded below by $(n-1)k$. For closed manifolds, Matei also provided lower bounds for the first nonzero eigenvalue of the p-Laplacian in the case $k > 0$. When $k = 0$, a sharp estimate was later achieved by (VALTORTA, 2012). For a general real k, further advancements were made by (NABER; VALTORTA, 2014) (see also (CAVALLETTI; MONDINO, 2017)). More recently, (WANG; LI, 2016) developed p-Bochner and p-Reilly formulas for the weighted p-Laplacian. As an application, they derived lower bounds for the first nonzero eigenvalue of the Dirichlet and Neumann problems for the weighted p-Laplacian on compact smooth metric measure spaces, both with and without boundaries. Additionally, (SETO; WEI, 2017) provided several estimates for the first eigenvalue of the p-Laplacian on closed Riemannian manifolds under integral curvature conditions. From the perspective of isometric immersions, upper bounds for the first nonzero eigenvalue of the p-Laplacian have also been extensively studied. In particular, Reilly-type inequalities for closed submanifolds in Riemannian space forms, involving the mean curvature vector and higher-order mean curvatures of the immersion, were obtained by (DU; MAO, 2015; CHEN; WEI, 2019).

For $\frac{3}{2} < p < 2$, several noteworthy applications emerge, highlighting the practical relevance of this range. For instance, in the specific case of $p = \frac{3}{2}$, a significant application is found in the study of unsaturated flow, as discussed by (DIAZ; DE THELIN, 1994). Similarly, for $p \in (1, \frac{4}{3}]$, this interval is closely linked to problems in glaciology, underscoring the diverse real-world scenarios where these mathematical frameworks are applied. Within the range $\frac{3}{2} < p < 2$, it is particularly worth mentioning Leibenson's work on the turbulent filtration of gas in porous media. This study offers valuable insights into fluid dynamics in complex systems, providing a compelling connection between theoretical models and physical phenomena (LEIBENSON, 1945). For those seeking a more comprehensive understanding of this area, Benedikt's work (BENEDIKT et al., 2018) serves as a rich resource, offering an in-depth exploration of the subject and its broader implications.

We begin our work by revisiting some classical definitions related to isometric immersions and differential operators to ensure the completeness of the discussion (Chapter 2). Following this, in Chapter 3, we introduce a new linearization of the p-Laplacian operator and establish several properties associated with it. Additionally, we present Bochner and Reilly-type formulas for our operator (cf. Propositions 3.3 and 3.4). Subsequently, we focus on minimal submanifolds isometrically immersed in the unit sphere. As an application of the Bochner-type formula, our first result extends the classical integral inequality of (LEUNG, 1983), which involves the first eigenvalue of the 2-Laplacian, the squared norm of the second fundamental form, and the dimension of the minimal submanifold in the Euclidean sphere. Specifically, we prove, for $p \geq 2$, that:

Theorem 1.1. *Let M^n be a closed minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the p-Laplacian of M^n associated to λ. Then*

$$\int_M (S + \alpha_{n,p}\lambda^{2/p} - n)|\nabla u|^{2p-2} dM \geq 0,$$

where

$$\alpha_{n,p} = \frac{n(p-1)^2 - 1}{(n-1)(p-1)^{2/p}}. \tag{1.2}$$

Moreover, if equality holds, then $p = 2$ and

(a) *either M^n is totally geodesic and λ is the first nonzero eigenvalue of the 2-Laplacian,*

(b) *or $n = 2$, $m = 2q$, M^n is isometric to $\mathbb{S}^2\left(\sqrt{q(q+1)/2}\right)$ and λ is the first nonzero eigenvalue of the 2-Laplacian.*

By denoting $\lambda_{1,p}(M)$ as the first eigenvalue of the p-Laplacian of M^n and assuming that the squared norm of the second fundamental form S is constant, we have a nice lower Leung-type estimate

Corollary 1.2. *Let M^n be a closed minimal submanifold in $\mathbb{S}^m$ with $S = const.$, then*

$$S \geq n - \alpha_{n,p}\lambda_{1,p}(M)^{2/p},$$

for $p \in [2, \infty)$.

Following this, the natural question arises: Can the hypothesis of an empty boundary be removed? To address this, we will apply the Reilly-type formula obtained in Chapter 3. Given that our submanifolds now include an additional structure (the boundary), it is natural to impose specific boundary conditions. To proceed, we focus on two types of eigenvalue problems for the p-Laplacian: the Dirichlet and Neumann problems. Under these conditions, we obtain our second result:

Theorem 1.3. *Let M^n be a compact minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the Neumann problem of the p-Laplacian of M^n associated to λ. Assume in addition that the second fundamental form of the boundary is negative semi-definite. Then*

$$\int_M \left(S - n + \alpha_{n,p}\lambda^{2/p}\right) |\nabla u|^{2p-2} dM \geq 0,$$

where $p \in [2, \infty)$, where $\alpha_{n,p}$ is defined in (1.2). Moreover, if equality holds, then $p = 2$ and M^n is isometric to the hemisphere $\mathbb{S}^n_+(1)$ and $\lambda = \lambda^N_{1,2}(\mathbb{S}^n_+)$, where $\lambda^N_{1,2}$ stands for the first eigenvalue of the 2-Laplacian related to the Neumann problem.

On the other hand, it is well known that submanifolds isometrically immersed in the unit sphere satisfy an equation that relates the length of the second fundamental form to their mean curvature H and scalar curvature R. Consequently, it is natural to replace the minimality hypothesis with a prescribed scalar curvature condition, specifically $R = 1$. With this perspective, (WEI, 2011) extended Leung's result by assuming that the scalar curvature of such a closed manifold immersed in the unit sphere satisfies $R = 1$, which is the same of the ambient. In this framework, our next result generalizes Wei's result to the context of the p-Laplacian. Specifically, we prove:

Theorem 1.4. *Let M^n be a compact submanifold of $\mathbb{S}^m$ with constant scalar curvature $R = 1$ and possibly with convex boundary. Let u be an eigenfunction of p-Laplacian (with the Neumann boundary condition if $\partial M \neq \emptyset$) of M^n associated to λ. Then*

$$\int_M \left(n - \frac{2(n-2)}{n} S - \alpha_{n,p} \lambda^{2/p} \right) |\nabla u|^{2p-2} dM \leq 0,$$

where $\alpha_{n,p}$ is defined in (1.2). *In particular, if the equality holds, then $p = 2$ and M^n is isometric to:*

(a) The sphere $\mathbb{S}^n(1)$, if $\partial M = \emptyset$;

(b) The closed hemisphere $\mathbb{S}^n_+(1)$, otherwise.

Finally, in terms of umbilicity tensor, we prove an similar integral inequality for arbitrary compact submanifolds immersed in the unit sphere without any additional assumptions in the geometry of the object. Explicitally,

Theorem 1.5. *Let M^n be a compact submanifold of $\mathbb{S}^m$ with possibly convex boundary. Let u be an eigenfunction of the p-Laplacian (with the Neumann boundary condition if $\partial M \neq \emptyset$) of M^n associated to λ. Then*

$$\int_M \left(n - \frac{n^2}{4(n-1)} |\phi|^2 - \alpha_{n,p} \lambda^{2/p} \right) |\nabla u|^{2p-2} dM \leq 0,$$

where $\alpha_{n,p}$ is a positive constant depending only given by n and p. In particular, if the equality holds, then $p = 2$ and M^n is isometric to

(a) The sphere $\mathbb{S}^n(1)$, if $\partial M = \emptyset$;

(b) The closed hemisphere $\mathbb{S}^n_+(1)$, otherwise.

It is important to note that all the results presented above hold exclusively for $p \geq 2$. The techniques employed are not sufficient for $1 < p < 2$, as they rely on Hölder's inequality and, at certain steps, explicitly use the condition $p - 2 \geq 0$. To address this limitation, we make appropriate modifications to the arguments, enabling us to establish the following lower bound for the first eigenvalue of the p-Laplacian in the case $1 < p < 2$.

Theorem 1.6. *Let M^n be a compact Riemannian manifold with mean convex boundary and consider M^n as a minimal submanifold of $\mathbb{S}^{n+q}$ with $S = const$. Then*

$$\lambda_{1,p}(M) \geq \sqrt{(n-S)C}, \qquad \frac{3}{2} < p < 2,$$

where C is a positive constant depending only n, p and M^n.

Unlike other inequalities derived for $p \geq 2$, the inequality in this result is never attained (cf. Remark 3.29).

In the final chapter (Chapter 4), we focus on complete and non-compact Riemannian manifolds. We begin by presenting some fundamental facts about the eigenvalue problem of the p-Laplacian operator on these manifolds. Following this, we establish a Palmer-type lower bound (see (PALMER, 1990)) for the first eigenvalue of the p-Laplacian, valid for $p \geq 2$. Specifically, we prove:

Lemma 1.7. *Let D be a relatively compact smoothly bounded domain on a Riemannian manifold M^n. Let $\lambda_1(D)$ denote the first eigenvalue of the problem*

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Suppose there exists a smooth function f on $\overline{D}$ that satisfies $\Delta_p f \geq 1$ in D. Then

$$\lambda_{1,p}(D) \geq \frac{1}{(\beta - \alpha)^{p-1}},$$

where α, β are any lower and upper bounds respectively of f on $\overline{D}$.

On the other hand, strongly p-subharmonic functions play an important role in the study of Riemannian manifolds. Let us recall that a smooth function $u : M^n \to \mathbb{R}$ is said to be *strongly p-subharmonic* if u satisfies the following differential inequality:

$$\Delta_p u \geq k > 0,$$

for some constant k. Concerning strongly p-subharmonic functions, we quote some well-known results as, for example, the ones due to (TAKEGOSHI, 2001) for relations lying between the existence of a certain strongly p-subharmonic function and the volume growth property of the case $p \geq 2$. Also, for $p = 2$, (COGHLAN L. ITOKAWA; KOSECKI, 1992) proved a Liouville-type result which said that every 2-subharmonic function on a complete non-compact Riemannian manifold must be unbounded provided that its sectional curvature is bounded. A few years later, (LEUNG, 1997) demonstrated that the same result holds when replacing the boundedness of the sectional curvature with the nullity of first eigenvalue of the 2-Laplacian. As an application, Leung obtained an estimate for the size of the image set of some types of maps between Riemannian manifolds. Proceeding with this picture, as an application of our Palmer-type lower estimate, we obtain the following extension of Leung's result for the context of the p-Laplacian, for all $p \geq 2$.

Theorem 1.8. *If M^n is a complete non-compact Riemannian manifold with $\lambda_{1,p}(M) = 0$, then every strongly p-subharmonic function on M^n is unbounded.*

As an application of our result, we study hypersurfaces immersed in a suitable warped product $I \times_f P^n$, where f stands for the warped function, $I \subset \mathbb{R}$ is the real line and P^n the Riemannian fiber. In this setting, our first application addresses the following nonexistence result.

Theorem 1.9. *There exists no complete non-compact immersed hypersurface M^n contained in a slab of $I \times_f P^n$ having $\lambda_{1,2}(M) = 0$ and mean curvature satisfying*

$$\sup_M |H| < \min_{[t_1,t_2]} \mathcal{H}(t),$$

where $\mathcal{H}(t)$ denotes the mean curvature function of the slices of $I \times_f P^n$.

We point out that (ALÍAS; DAJCZER, 2006) investigated complete surfaces properly immersed in a slab of $I \times_f P^2$ under suitable geometric assumptions on the Riemannian fiber P^2. Our result generalizes this one without any assumption on P^n.

In the next result, again in the context of warped product metric, we will deal with *helix-type* hypersurfaces or, those that have *constant angle*. In this setting, by considering that the warped product manifold $I \times_f M^n$ satisfies $\mathcal{H}(t) \geq 1$ for all $t \in I$, we obtain.

Theorem 1.10. *Let M^n be a complete non-compact helix-type hypersurface contained in a slab of $I \times_f P^n$ with $\lambda_{1,p}(M) = 0$, $p > 2$. If the mean curvature (not necessarily constant) satisfies $H^2 \leq 1$, then M^n is a slice.*

As a consequence, we end this with the following extension of (ALÍAS; DAJCZER, 2006, Theorem 4) and (AQUINO; LIMA, 2014, Theorem 3.3).

Corollary 1.11. *Let M^n be a complete non-compact helix-type hypersurface contained in a slab of pseudo-hyperbolic manifold $\mathbb{R} \times_{e^t} P^n$ with $\lambda_{1,p}(M) = 0$. If the mean curvature (not necessarily constant) satisfies $H^2 \leq 1$, then M^n is a slice.*

2 PRELIMINARIES

In this text, all arguments are presented in connected manifolds. We highlight that this does not imply the non-validity of arguments; rather, we repeat them in each connected component.

Let $\overline{M}^m$ be a Riemannian manifold endowed to the Riemannian metric $\langle\,,\rangle$, and M^n a submanifold isometrically immersed on $\overline{M}^m$. Eventually, we omit the dimension exponent of the manifolds when there is no reason for confusion. We denote by ∇ and $\overline{\nabla}$ the **Levi-Civita connection** of M^n and $\overline{M}^m$ respectively. Moreover, $\nabla^\perp$ denotes the **Normal Connection** of M^n in $\overline{M}^m$.

For each $x \in M^n$, we denote by T_xM the **Tangent Space** of M^n in x. We denote by A the **Second Fundamental Form** of M^n in $\overline{M}^m$ and for each η in the **normal space** $T_xM^\perp$ we define the symmetric endomorphism $A_\eta : T_xM \to T_xM$ known as **Weiengarten Operator**. We recall the following formula:

$$\langle A(X,Y),\eta\rangle = \langle A_\eta X, Y\rangle,$$

for all $X, Y \in T_xM$. Since is possible to split $\overline{\nabla}_X Y$ in tangent and normal components, we recall the **Gauss formula** given by

$$\overline{\nabla}_X Y = \nabla_X Y + A(X,Y),$$

for all X, Y. From a similar idea we recall the **Weingarten formula** which relate the ambient connection of a tangent and a normal vector fields with the same in object. Explicitly,

$$\overline{\nabla}_X \eta = -A_\eta X + \nabla^\perp_X \eta, \tag{2.1}$$

where $\nabla^\perp$ denote the **Normal connection** of $\overline{\nabla}$. In particular, for **Hypersurfaces**, i.e. submanifolds with codimension 1, the Weingarten formula presents in a more simple way. Note that, if M^n is a hypersurface, then $T_xM^\perp = span\{\eta\}$ for some unitary vector η.

Hence,

$$0 = X(1) = X(|\eta|^2) = 2\langle \overline{\nabla}_X \eta, \eta \rangle.$$

We conclude that $\nabla^{\perp}_X \eta = 0$ and finally, for hypersurfaces, $\overline{\nabla}_X \eta = -A_\eta X$.

Another important intrinsic quantity of M^n is the **Mean curvature vector field** and the **Mean curvature function** defined, respectively, as follow

$$h = \frac{1}{n}\,\mathrm{tr}\,A \quad \text{and} \quad H = |h|.$$

In particular, if $H = 0$, we say that M^n is a **Minimal submanifold** of $\overline{M}^m$. Besides this, we say that M^n has **Parallel mean curvature** if the mean curvature vector field h is parallel as a section of the normal bundle, that is, $\nabla^{\perp} h = 0$.

We define the **Curvature tensor** of a manifold M^n as

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

where $[,]$ denote the **Lie bracket**. Relating the $\overline{M}^m$ and M^n we have the **Gauss equation**

$$\langle R(X,Y)Z, W\rangle = \langle \overline{R}(X,Y)Z, W\rangle + \langle A(X,Z), A(Y,W)\rangle - \langle A(Y,Z), A(X,W)\rangle \tag{2.2}$$

for all $X, Y, Z, W \in T_xM$. Taking the trace of the curvature, we have the known **Ricci tensor**, which is presented as

$$\mathrm{Ric}(X,Z) = \sum_{i=1}^{n} \langle R(X,e_i)Z, e_i\rangle \tag{2.3}$$

Hence, by (2.3) and (2.2),

$$\begin{aligned} \mathrm{Ric}(X,X) &= \mathrm{tr}(\overline{R}_X) + \langle A(X,X), A(e_i,e_i)\rangle \\ &- \langle A(e_i,X), A(X,e_i)\rangle \\ &= \mathrm{tr}(\overline{R}_X) + n\langle A(X,X), h\rangle - \sum_{i=1}^{n} |A(X,e_i)|^2, \end{aligned} \tag{2.4}$$

where $\mathrm{tr}(\overline{R}_X) = \sum_i \langle \overline{R}(X,e_i)X, e_i\rangle$ for every $X \in \mathfrak{X}(M)$. Taking the trace of Ric, we obtain the formula of the **Normalized scalar curvature** R, which is give by

$$\begin{aligned} n(n-1)R &= \sum_{j=1}^{n}\sum_{i=1}^{n}\langle \overline{R}(e_j,e_i)e_j,e_i\rangle + \sum_{j=1}^{n} n\langle A(e_j,e_j),h\rangle - \sum_{j=1}^{n}\sum_{i=1}^{n}|A(e_j,e_i)|^2 \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n}\langle \overline{R}(e_j,e_i)e_j,e_i\rangle + n^2H^2 - \sum_{j=1}^{n}\sum_{i=1}^{n}|A(e_j,e_i)|^2, \end{aligned} \tag{2.5}$$

the last term is precisely the Frobenius norm of A, the square length of the second fundamental form S. In particular, $\mathrm{Scal}(M) = n(n-1)R$ will stand for the **Scalar curvature**.

Fix a point $x \in M^n$ and take a local orthonormal frame $\{e_1,\ldots,e_m\}$ of $\overline{M}^m$ around of x such that $\{e_1,\ldots,e_n\}$ are tangent fields and $\{e_{n+1},\ldots,e_m\}$ are normal fields on M^n. For each α, we will define a linear maps $\phi_\alpha : T_xM \to T_xM$ by

$$\langle \phi_\alpha(X),Y\rangle = \langle A_\alpha(X),Y\rangle - \langle h,e_\alpha\rangle\langle X,Y\rangle, \tag{2.6}$$

and a bilinear map $\phi : T_xM \times T_xM \to (T_xM)^\perp$ by

$$\phi(X,Y) = \sum_\alpha \langle \phi_\alpha(X),Y\rangle e_\alpha, \tag{2.7}$$

where X, Y are tangent fields on M^n. It is easy to check that each map ϕ_α a is traceless and that

$$|\phi|^2 = S - nH^2. \tag{2.8}$$

Using this notation, we get a version of a theorem from Leung (1992, Main Theorem) in terms of (2.8) and the ambient curvature tensor.

Proposition 2.1. *Let M^n be a submanifold of the Riemannian manifold $\overline{M}^m$. Then*

$$\mathrm{Ric}(X,X) \geq \mathrm{tr}(\overline{R}_X) - \frac{n-1}{n}\left(|\phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| - nH^2\right),$$

for all $X \in \mathfrak{X}(M)$.

Proof. Firstly, we observe that in terms of (2.7), (2.4) can be rewritten

$$\begin{aligned}\mathrm{Ric}(X,X) = &\sum_i \langle \overline{R}(X,e_i)X, e_i\rangle + (n-2)\langle \phi_h(X), X\rangle \\ &+ (n-1)H^2|X|^2 - \sum_\alpha |\phi_\alpha(X)|^2.\end{aligned}$$

Since ϕ is traceless, from Cauchy-Schwarz inequality, we have

$$\langle \phi_h(X), X\rangle \geq -|\langle \phi_h(X), X\rangle| \geq -\sqrt{\frac{n-1}{n}}|\phi_h||X|^2$$

and

$$\sum_\alpha \langle \phi_\alpha^2(X), X\rangle \leq \frac{n-1}{n}\sum_\alpha |\phi_\alpha|^2|X|^2 = \frac{n-1}{n}|\phi|^2|X|^2.$$

Besides this, we observe that

$$\phi_h = \sum_\alpha \langle h, e_\alpha\rangle \phi_\alpha.$$

From Cauchy Schwarz's inequality and Hilbert-Schmidt's norm definition, we have

$$\begin{aligned}|\phi_h|^2 &= \sum_{\alpha,i}\langle h, e_\alpha\rangle^2\langle \phi_\alpha(e_i), \phi_\alpha(e_i)\rangle \\ &\leq \sum_{\alpha,i}|h|^2|e_\alpha|^2\langle \phi_\alpha(e_i), \phi_\alpha(e_i)\rangle \\ &\leq H^2|\phi|^2.\end{aligned}$$

Hence

$$\langle \phi_h(X), X\rangle \geq -\sqrt{\frac{n-1}{n}}H|\phi||X|^2,$$

and consequently,

$$\begin{aligned}\mathrm{Ric}(X,X) \geq &\sum_i \langle \overline{R}(X,e_i)X, e_i\rangle - \frac{(n-1)(n-2)}{\sqrt{n(n-1)}}H|\phi||X|^2 \\ &+ (n-1)H^2|X|^2 - \frac{n-1}{n}|\phi|^2|X|^2,\end{aligned}$$

for all X is a tangent vector field. ■

When the ambient space has constant sectional curvature κ, the Gauss equation (2.2) becomes in the form:

$$\langle R(X,Y)Z,W\rangle = \kappa\left(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\right) + \langle A(X,Z),A(Y,W)\rangle$$
$$- \langle A(Y,Z),A(X,W)\rangle,$$

for all $X,Y,Z,W \in T_xM$. So, when $\overline{M}^m$ has constant sectional curvature, we obtain the following Leung result (LEUNG, 1992)

Corollary 2.2. *Let $\overline{M}^m(\kappa)$ be a constant sectional curvature manifold and let M^n be a submanifold of $\overline{M}^m$. Let* Ric *denote the function that assigns the minimum Ricci curvature to each point of M^n. Then*

$$\mathrm{Ric}(X,X) \geq -\frac{n-1}{n}\left(|\phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| - n(\kappa + H^2)\right)|X|^2, \tag{2.9}$$

for all $X \in \mathfrak{X}(M)$.

Given a $\mathcal{C}^2$ funcion u defined on the Riemannian manifold M^n, the **Laplace-Beltrami** operator is a linear elliptical operator given by

$$\Delta u = -\mathrm{div}(\nabla u), \tag{2.10}$$

where $\mathrm{div}(\nabla u) := \mathrm{tr}(Z \mapsto \nabla_Z \nabla u)$.

A natural extension of this operator is the quasilinear elliptical operator p-**Laplacian**, that we will denote by Δ_p, where p is a real number satisfying $1 < p < +\infty$, which is defined by

$$\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

The p-Laplacian operator appears naturally from the variational problem associated to the p-energy functional

$$E_p(u) = \int_M |\nabla u|^p dM,$$

where $W_0^{1,p}(M)$ is the completion of the subspace of smooth functions $C^\infty(M)$ that has compact support on M^n under the **Sobolev norm**

$$||u||_{1,p} = \left(\int_M |u|^p dM + \int_M |\nabla u|^p dM\right)^{\frac{1}{p}},$$

where dM denotes the Riemannian volume element of M^n. We also define the **Eigenvalue problem**, for closed manifolds, which consist to find real numbers λ such that

$$\Delta_p u = \lambda |u|^{p-2} u. \tag{2.11}$$

It is natural to include additional information about the function's behavior on the boundary for compact manifolds with non-empty boundaries. Let ∂M denote the boundary of M^n, seen as an orientable hypersurface of M^n equipped with a normal vector field η. Throughout this work, we analyze three eigenvalue problems:

$$\begin{array}{lcl} & \Delta_p u = \lambda |u|^{p-2} u, & \text{in } M \\ \textbf{Dirichlet Problem} & u = 0, & \text{on } \partial M \\ \textbf{Neumann Problem} & \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial M, \end{array} \tag{2.12}$$

where

$$\frac{\partial u}{\partial \eta} = \langle \nabla u, \eta \rangle = d\eta(u)$$

stands for the normal derivative. When the boundary is empty we will call **Closed Case**. The real number λ is then called an eigenvalue of Δ_p on M^n. The first nonzero eigenvalue of Δ_p, which we will denote by $\lambda_{1,p}(M)$, has a Rayleigh type variational characterization (cf.(VERON, 1991)):

$$\lambda_{1,p}(M) = \inf \left\{ \frac{\int_M |\nabla u|^p dM}{\int_M |u|^p dM}; u \in W^{1,p}(M) \backslash \{0\} \text{ and } \int_M |u|^{p-2} u dM = 0 \right\}. \tag{2.13}$$

which has the same characterization as the Neumann problem. For the Dirichlet problem, according to Lindqvist (LINDQVIST, 1990), the first eigenvalue is simple, positive, and has the following characterization:

$$\mu_{1,p}(M) = \inf \left\{ \frac{\int_M |\nabla u|^p dM}{\int_M |u|^p dM}; u \in W_0^{1,p}(M) \backslash \{0\} \right\}.$$

An example of the first eigenvalue in Dirichlet problem is $\mu_{1,p}(\mathbb{H}^n(-\kappa^2)) = \frac{(n-1)^p \kappa^p}{p^p}$, where $\mathbb{H}^n(-\kappa^2)$ denotes the usual hyperbolic space with constant sectional curvature $-\kappa^2$. On the other hand, for other domains as the upper hemisphere (see (2.16)), the explicit expression is not known, this show the importance of obtain estimates.

Is important to highlight that the positivity of the first eigenvalue in the Neumann problem and the closed case does not occur. In fact, the first eigenvalue for compact manifolds in both cases is zero. Indeed, by Stokes theorem

$$\begin{aligned}\int_M \Delta_p u dM &= \int_M \lambda |u|^{p-2} u dM \\ -\int_M \operatorname{div}(|\nabla u|^{p-2}\nabla u) dM &= \lambda \int_M |u|^{p-2} u dM \\ 0 &= \lambda \int_M |u|^{p-2} u dM,\end{aligned}$$

where $d\sigma$ is the volume element of ∂M. Since the eigenfunctions associated to the first eigenvalue in both cases do not change sign (cf. (LÊ, 2006)), hence $\int_M |u|^{p-2} u dM > 0$. It is follows that $\lambda = 0$. In other words, the first nonzero Neumann (and closed case) eigenvalue of the p-Laplacian is, in fact, the second eigenvalue. This argument also justifies the integral equal to zero in (2.13).

A direct calculation by (2.11), allows us to write the p-Laplacian in terms of the classical 2-Laplacian, that is, if U is a domain of M^n where $\nabla u \neq 0$,

$$\begin{aligned}\Delta_p u &= -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \\ &= |\nabla u|^{p-2}\Delta u - \nabla u\left(|\nabla u|^{p-2}\right) \\ &= |\nabla u|^{p-2}\Delta u - \nabla u\left(|\nabla u|^{p-2}\right) \\ &= |\nabla u|^{p-2}\Delta u - (p-2)|\nabla u|^{p-4}\langle \operatorname{Hess} u(\nabla u), \nabla u\rangle.\end{aligned} \tag{2.14}$$

Here, $\operatorname{Hess} u$ stands the symmetric $(0,2)$-tensor called the hessian of u and defined by $\operatorname{Hess} u(X,Y) = (\nabla_X \nabla u)(Y) = XY(u) - (\nabla_X Y)u$ for any vector fields X and Y. We also define the $\infty-$**Laplacian** operator as

$$\Delta_\infty u := \lim_{p\to+\infty} \frac{\Delta_p u}{(p-2)|\nabla u|^{p-2}} = -\frac{\langle \operatorname{Hess} u(\nabla u), \nabla u\rangle}{|\nabla u|^2}, \tag{2.15}$$

when singularities occur, it is typical in the literature to write the operator without denominator $|\nabla u|^2$. This operator is a particular case of the functional $A_u : \mathcal{C}^2(M) \to \mathbb{R}$

(cf. (NABER; VALTORTA, 2014; VALTORTA, 2012)) defined as follow:

$$A_u(f) = \frac{\langle \mathrm{Hess} f(\nabla u), \nabla u\rangle}{|\nabla u|^2}.$$

In particular, the p-Laplacian definition can be written in this way

$$\Delta_p u = |\nabla u|^{p-2}(\Delta u - (p-2)A_u(u)).$$

We conclude this section by presenting three essential characterization theorems that serve as the foundation for our results. The first is attributed to Obata (OBATA, 1962):

Lemma 2.3. *Let $c > 0$. For a $\mathcal{C}^\infty$ complete Riemannian manifold (M^n, g) of dimension $n \geq 2$, there is a $\mathcal{C}^\infty$ nontrivial function u on M^n satisfying,*

$$\mathrm{Hess}\, u + cug = 0$$

if and only if (M^n, g) is isometric to the Euclidean n-sphere $(\mathbb{S}^n(\sqrt{c}), g_0)$ of radius $1/\sqrt{c}$, where g_0 denotes the canonical metric on the sphere

$$\mathbb{S}^n(\sqrt{c}) = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}\,;\, |\mathbf{x}| = 1/\sqrt{c}\},$$

with constant curvature $1/\sqrt{c}$.

The next two results regard the characterization of manifolds with non-empty boundaries. The first is a version of the Lichnerowicz-Obata theorem, presented in (REILLY, 1980). To establish the notation, we recall that the boundary ∂M is said to be convex if the second fundamental form of ∂M in M^n is negative semi-definite concerning the outward-pointing unit normal vector η. In particular, ∂M is considered mean convex if its mean curvature function is non-positive (cf. (REILLY, 1980)).

Lemma 2.4. *Let M^n he a compact manifold with nonempty mean convex boundary ∂M. Assume that there is a constant $c^2 > 0$ such that on $\mathrm{Ric} \geq (n-1)c^2 g$ (where g is the metric on M^n). Then the first eigenvalue $\mu_{1,2}(M)$ of Dirichlet case satisfies the inequality $\mu_{1,2}(M) \geq nc^2$. Moreover, $\mu_{1,2}(M) = nc^2$ if and only if M^n is isometric to a closed* *hemisphere*

$$\mathbb{S}^n_+\left(\sqrt{c}\right) = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1};\, |x| = 1/\sqrt{c} \text{ and } x_{n+1} \geq 0\}. \tag{2.16}$$

of the Euclidean sphere $\mathbb{S}^n(\sqrt{c})$ *of radius* $1/\sqrt{c}$*.*

The second result concerning manifolds with boundary is due to (ESCOBAR, 1990).

Lemma 2.5. *Let* M^n *be a compact Riemannian manifold with boundary. Assume that there exists a nonconstant function* u *such that*

$$\operatorname{Hess} u = -cuI \quad \text{and} \quad \frac{\partial u}{\partial \eta} = 0$$

for some $c > 0$*. Then* M^n *is isometric to* $\mathbb{S}^n_+\left(\sqrt{c}\right)$ *the upper hemisphere of radius* $1/\sqrt{c}$*.*

3 ESTIMATES ON COMPACT RIEMANNIAN MANIFOLDS

In the previous chapter, we introduced the notion of eigenvalues and provided a characterization. As already noted, the exact formula for the first eigenvalue of the p-Laplacian is not known — not even for simple domains such as the upper hemisphere, the unit sphere, or the square. At first, it seems natural to expect a straightforward extension from Δ_2 (the classical Laplacian) to Δ_p, given that the structure of the p-Laplacian appears to be a simple modification of the classical case. However, this expectation turns out to be misleading. The difficulties arise from the fact that the p-Laplacian lacks fundamental properties such as linearity and self-adjointness, which are crucial for many classical techniques, including spectral theory and superposition principles.

Moreover, the generalized operator may exhibit singular behavior at points where $|\nabla u| = 0$, particularly when $p < 2$, further complicating the analysis. As a result, explicit formulas for the eigenvalues of the p-Laplacian are generally unavailable, and one must instead rely on variational characterizations, estimates, or numerical approximations. In this chapter, we deal with estimates for compact submanifolds in the unit sphere.

3.1 A LINEARIZED OPERATOR

The nonlinear nature of the p-Laplacian operator necessitates linearized version to facilitate its analysis. In this setting, a remarkable work of (KAWAI; NAKAUCHI, 2003) introduces a linear operator that provides an effective framework for examining the properties and behavior of the p-Laplacian in a more manageable way. For every function $u \in \mathcal{C}^1(M)$, they defined the linearized operator P_u as follows: for a function $f \in \mathcal{C}^2(M)$, let

$$P_u(f) = |\nabla u|^{p-2}\Delta f - (p-2)|\nabla u|^{p-4}\langle \text{Hess} f(\nabla u), \nabla u\rangle.$$

By applying a direct computation

$$\begin{aligned} P_u(f) &= |\nabla u|^{p-2}\Delta f - (p-2)|\nabla u|^{p-4}\langle \text{Hess}u(\nabla u), \nabla f\rangle \\ &+ (p-2)|\nabla u|^{p-4}\langle \text{Hess}u(\nabla u), \nabla f\rangle \\ &- (p-2)|\nabla u|^{p-4}\langle \text{Hess}f(\nabla u), \nabla u\rangle \\ &= -\text{div}(|\nabla u|^{p-2}\nabla f) + (p-2)|\nabla u|^{p-2}\left(\langle \nabla \log|\nabla u|, \nabla f\rangle - A_u(f)\right). \end{aligned}$$

That is, the operator P_u can be split as follows:

$$P_u(f) = L_u(f) + (p-2)|\nabla u|^{p-2}R_u(f).$$

where $R_u(f) = \langle \nabla\log|\nabla u|, \nabla f\rangle - A_u(f)$. For our interest, we take only the divergent part of the operator P_u to define the new linearized p-Laplacian L_u as follows:

$$L_u(f) = -\text{div}(|\nabla u|^{p-2}\nabla f), \tag{3.1}$$

where $u \in \mathcal{C}^2(M)$. Furthermore, we should note that $L_u(u) = \Delta_p u$.

It should be noted that, in the compact case, the operator L_u (and P_u as well) must have singular points due to Weierstrass theorem's. To address this problem, we use an ε-approximation argument, which is presented in detail in Appendix 4.2.

Concerning to the L_u operator, we have the following properties:

Proposition 3.1. *For any $u \in \mathcal{C}^2$, the operator L_u satisfies the following proprieties*

a) L_u is a linear operator.

b) L_u is an elliptic operator

c) For any $f_1, f_2 \in \mathcal{C}^2$,

$$L_u(f_1 f_2) = f_1 L_u(f_2) + f_2 L_u(f_1) + 2|\nabla u|^{p-2}\langle \nabla f_2, \nabla f_1\rangle$$

Proof. The items *a)* and *b)* are trivial by the definition of the operator. Moreover, from (2.10)

$$\begin{aligned} L_u(f_1f_2) &= \operatorname{div}(|\nabla u|^{p-2}\nabla(f_1f_2)) \\ &= |\nabla u|^{p-2}\Delta(f_1f_2) - (p-2)|\nabla u|^{p-4}\langle \mathrm{Hess}u(\nabla u), \nabla(f_1f_2)\rangle \end{aligned}$$

Now, by using the Hessian properties,

$$\begin{aligned} L_u(f_1f_2) &= |\nabla u|^{p-2}\left(f_1\Delta f_2 + f_2\Delta f_1 + 2\langle \nabla f_2, \nabla f_1\rangle\right) \\ &- (p-2)|\nabla u|^{p-4}\langle \mathrm{Hess}u(\nabla u), f_1\nabla(f_2) + f_2\nabla(f_1)\rangle \\ &= |\nabla u|^{p-2}f_1\Delta f_2 + |\nabla u|^{p-2}f_2\Delta f_1 + 2|\nabla u|^{p-2}\langle \nabla f_2, \nabla f_1\rangle \\ &- (p-2)|\nabla u|^{p-4}f_2\langle \mathrm{Hess}u(\nabla u), \nabla(f_1)\rangle \\ &- (p-2)|\nabla u|^{p-4}f_1\langle \mathrm{Hess}u(\nabla u), \nabla(f_2)\rangle. \end{aligned}$$

Hence,

$$L_u(f_1f_2) = f_1L_u(f_2) + f_2L_u(f_1) + 2|\nabla u|^{p-2}\langle \nabla f_2, \nabla f_1\rangle.$$

■

To establish a connection between geometric quantities and the eigenvalues of the p-Laplacian, we begin by presenting a generalization of the well-known Bochner formula. We include the formal statement and detailed proof below for completeness.

Lemma 3.2 (Bochner's Formula)**.** *Let M^n be a Riemannian manifold. If $u \in \mathcal{C}^2(M)$, then*

$$\frac{1}{2}\Delta|\nabla u|^2 = \langle \nabla u, \nabla\Delta u\rangle - |\mathrm{Hess}u|^2 - \mathrm{Ric}(\nabla u, \nabla u).$$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame on $\mathfrak{X}(M)$ and write

$$\nabla|\nabla u|^2 = \sum_i e_i(|\nabla u|^2)e_i.$$

Then,

$$\frac{1}{2}\mathrm{div}\left(\nabla|\nabla u|^2\right) = \frac{1}{2}\sum_i e_i\left(e_i(|\nabla u|^2)\right) + e_i(|\nabla u|^2)\mathrm{div}(e_i)$$
$$= \sum_i e_i\left(\langle \nabla_{e_i}\nabla u, \nabla u\rangle\right) + \langle \nabla_{e_i}\nabla u, \nabla u\rangle \mathrm{div}(e_i)$$
$$= \sum_i \langle \nabla_{e_i}\nabla_{\nabla u}\nabla u, e_i\rangle + \langle \nabla_{e_i}\nabla u, \nabla_{e_i}\nabla u\rangle - 2\langle \nabla_{\nabla u}\nabla u, \nabla_{e_i}e_i\rangle.$$

On the other hand, direct computation gives

$$\langle \nabla u, \nabla \Delta u\rangle = -\sum_i \nabla u\left(\langle \nabla_{e_i}\nabla u, e_i\rangle\right)$$
$$= -\sum_i \langle \nabla_{\nabla u}\nabla_{e_i}\nabla u, e_i\rangle - \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u}e_i\rangle$$

and

$$\mathrm{Ric}(\nabla u, \nabla u) = \sum_i \left(\langle \nabla_{\nabla u}\nabla_{e_i}\nabla u, e_i\rangle - \langle \nabla_{e_i}\nabla_{\nabla u}\nabla u, e_i\rangle - \langle \nabla_{e_i}\nabla u, \nabla_{e_i}\nabla u\rangle\right)$$
$$+ \sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u}e_i\rangle.$$

To end this proof, we will show that $\sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u}e_i\rangle = 0$. In fact, since $\langle e_i, e_j\rangle = \delta_{ij}$, we have

$$\langle \nabla_{e_j}e_i, e_j\rangle = -\langle e_i, \nabla_{e_j}e_j\rangle \quad \text{and} \quad \langle \nabla_{\nabla u}e_i, e_j\rangle = -\langle e_i, \nabla_{\nabla u}e_j\rangle.$$

Hence,

$$\sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u}e_i\rangle = \sum_i \langle \nabla_{\nabla_{\nabla u}e_i}\nabla u, e_i\rangle = \sum_{i,j}\langle \nabla_{\nabla u}e_i, e_j\rangle\langle \nabla_{e_j}\nabla u, e_i\rangle$$
$$= \sum_{i,j}\langle \nabla_{\nabla u}e_i, e_j\rangle\langle \nabla_{e_j}\nabla u, e_i\rangle = -\sum_{i,j}\langle \nabla_{\nabla u}e_j, e_i\rangle\langle \nabla_{e_j}\nabla u, e_i\rangle$$
$$= -\sum_j \langle \nabla_{\nabla u}e_j, \nabla_{e_j}\nabla u\rangle = -\sum_i \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u}e_i\rangle,$$

as desired. ■

We point out that the reason for using the following formula, rather than others in the literature, is that our operator is of divergence type. This property enables the application of the Divergence Theorem in computations involving integrals.

Proposition 3.3. *Let M^n be a Riemannian manifold. If $u \in \mathcal{C}^3(M)$ and $|\nabla u| \neq 0$, then*

$$\begin{aligned}\frac{1}{p}L_u\left(|\nabla u|^p\right) =&|\nabla u|^{p-2}\left(\langle \nabla u, \nabla \Delta_p u\rangle - (p-2)\left(\Delta_\infty u \Delta_p u + 2|\nabla u|^{p-4}|\mathrm{Hess}u(\nabla u)|^2\right)\right)\\ &+ |\nabla u|^{2p-4}\left((p-2)\langle \nabla u, \nabla \Delta_\infty u\rangle - |\mathrm{Hess}u|^2 - \mathrm{Ric}(\nabla u, \nabla u)\right)\end{aligned} \tag{3.2}$$

for every $p \in (1, \infty)$.

Proof. By using (3.1), a direct computation shows

$$\begin{aligned}-L_u(|\nabla u|^p) &= \mathrm{div}(|\nabla u|^{p-2}\nabla|\nabla u|^p)\\ &= \frac{p}{2}\mathrm{div}(|\nabla u|^{2p-4}\nabla|\nabla u|^2)\\ &= \frac{p}{2}|\nabla u|^{2p-4}\mathrm{div}(\nabla|\nabla u|^2) + p\nabla|\nabla u|^2\left(|\nabla u|^{2p-4}\right).\end{aligned}$$

Since,

$$\nabla|\nabla u|^2\left(|\nabla u|^{2p-4}\right) = 2\frac{(p-2)}{2}|\nabla u|^{2p-6}\langle\nabla|\nabla u|^2, \nabla|\nabla u|^2\rangle,$$

and

$$\langle\nabla|\nabla u|^2, X\rangle = X(|\nabla u|^2) = 2\langle\nabla_X \nabla u, \nabla u\rangle = 2\langle \mathrm{Hess}u(\nabla u), X\rangle$$

for all $X \in \mathfrak{X}(M)$. We have that

$$\frac{1}{p}L_u(|\nabla u|^p) = \frac{1}{2}|\nabla u|^{2p-4}\Delta|\nabla u|^2 - 2(p-2)|\nabla u|^{2p-6}|\mathrm{Hess}\, u\,(\nabla u)|^2.$$

By Proposition 3.2, we have

$$\begin{aligned}\frac{1}{p}L_u(|\nabla u|^p) = |\nabla u|^{2p-4}&\left(\langle\nabla u, \nabla\Delta u\rangle - |\mathrm{Hess}\, u|^2 - \mathrm{Ric}(\nabla u, \nabla u)\right)\\ &- 2(p-2)|\nabla u|^{2p-6}|\mathrm{Hess}\, u(\nabla u)|^2.\end{aligned} \tag{3.3}$$

Finally, from (2.14) and (2.15), we see that the first term of (3.3) can be write as follow

$$\begin{aligned}\langle \nabla u, \nabla \Delta_p u\rangle &= \langle \nabla u, \nabla \left(|\nabla u|^{p-2}\Delta u - (p-2)|\nabla u|^{p-2}\Delta_\infty u\right)\rangle \\ &= |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta u\rangle + \frac{(p-2)}{2}|\nabla u|^{p-4}\Delta u\langle \nabla u, \nabla|\nabla u|^2\rangle \\ &- (p-2)|\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_\infty u\rangle - \frac{(p-2)}{2}\Delta_\infty u|\nabla u|^{p-4}\langle \nabla u, \nabla |\nabla u|^2\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}|\nabla u|^{2p-4}\langle \nabla u, \nabla \Delta u\rangle = |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_p u\rangle + (p-2)|\nabla u|^{2p-4}\langle \nabla u, \nabla \Delta_\infty u\rangle \\ - (p-2)|\nabla u|^{p-2}\Delta_\infty u\,\Delta_p u.\end{aligned} \tag{3.4}$$

Therefore, by putting (3.4) in (3.3) we finish this proof. ■

To deal with manifolds that have a boundary, some additional information is needed. Let M^n be a submanifold of $\mathbb{S}^m$ and let ∂M be the boundary of M^n viewed as a smooth embedded submanifold of M^n. Let $\mathcal{A}$ be the second fundamental form of ∂M in the normal direction $\eta \in \mathfrak{X}^\perp(M)$ which is related with the Weingarten operator $\mathcal{A}_\eta$ of ∂M in M^n by

$$\nabla_X Y = \nabla^\partial_X Y + \mathcal{A}(X,Y), \qquad \mathcal{A}_\eta(X) = -\nabla_X \eta,$$

for every $X, Y \in \mathfrak{X}(\partial M)$, while the second one derivate immediately from the equation (2.1) and the fact that ∂M is a hypersurface of M^n. Here, ∇^∂ denotes the Levi-Civita connection of ∂M. The mean curvature function at $x \in \partial M$ is defined by

$$H^\partial(x) = \frac{1}{n-1}\mathrm{tr}(\mathcal{A}_\eta(x)).$$

As a by-product of the previous digression, denoting Δ^∂ the Laplacian on ∂M regarding the induced metric, from Proposition 3.3 we have the following Reilly-type formula:

Proposition 3.4. *Let M^n be a compact manifold with boundary ∂M. Then, for any* *function* $u \in \mathcal{C}^2(M)$ *with* $\nabla u \neq 0$ *on* M^n*, we have*

$$\begin{aligned}\int_M |\nabla u|^{2p-4}\left(|\mathrm{Hess}\, u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + (p-2)^2(\Delta_\infty u)^2\right) dM \\ = \int_M (\Delta_p u)^2 dM - 2(p-2)\int_M |\nabla u|^{2p-6}|\mathrm{Hess}\, u(\nabla u)|^2 \\ + \int_{\partial M} |\nabla u|^{2p-4} Q(u) d\sigma,\end{aligned} \tag{3.5}$$

where

$$Q(u) = du(\eta)\left(\Delta^\partial u + (n-1)H^\partial\, du(\eta)\right) + \langle \mathcal{A}_\eta(\nabla^\partial u), \nabla^\partial u\rangle + \langle \nabla^\partial u, \nabla^\partial du(\eta)\rangle,$$

and $d\sigma$ *denotes the Riemannian volume element on* ∂M*.*

Proof. By integrating (3.2) over M^n, we have

$$\begin{aligned}-\int_M |\nabla u|^{2p-4}\left(|\mathrm{Hess}\, u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + 2(p-2)|\mathrm{Hess}\, u(\nabla u)|^2|\nabla u|^{-2}\right) dM \\ = \frac{1}{p}\int_M L_u(|\nabla u|^p) dM - \int_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_p u\rangle dM \\ - (p-2)\int_M |\nabla u|^{p-2}\left(|\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_\infty u\rangle - \Delta_\infty u \Delta_p u\right) dM.\end{aligned} \tag{3.6}$$

Integration by parts immediately yields

$$\begin{aligned}\int_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_p u\rangle dM &= \int_M \mathrm{div}(\Delta_p u |\nabla u|^{p-2}\nabla u) dM + \int_M (\Delta_p u)^2 dM \\ &= \int_{\partial M} |\nabla u|^{p-2} du(\eta)\Delta_p u\, d\sigma + \int_M (\Delta_p u)^2 dM,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\int_M |\nabla u|^{p-2}\left(|\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_\infty u\rangle - \Delta_\infty u \Delta_p u\right) dM \\ = \int_M \mathrm{div}(\Delta_\infty u |\nabla u|^{2p-4}\nabla u) dM + (p-2)\int_M |\nabla u|^{2p-4}(\Delta_\infty u)^2 dM \\ = \int_{\partial M} |\nabla u|^{2p-4} du(\eta)\Delta_\infty u\, d\sigma + (p-2)\int_M |\nabla u|^{2p-4}(\Delta_\infty u)^2 dM.\end{aligned} \tag{3.8}$$

On the other hand, the divergence theorem implies

$$
\begin{aligned}
\frac{1}{p}\int_M L_u(|\nabla u|^p)dM &= -\frac{1}{p}\int_M \operatorname{div}(|\nabla u|^{p-2}\nabla|\nabla u|^p)dM \\
&= -\frac{1}{p}\int_{\partial M} |\nabla u|^{p-2}\langle\nabla|\nabla u|^p,\eta\rangle d\sigma.
\end{aligned}
\tag{3.9}
$$

Hence, by replacing (3.7), (3.8) and (3.9) in (3.6),

$$
\begin{aligned}
&\int_M |\nabla u|^{2p-4}\left(|\operatorname{Hess} u|^2 + \operatorname{Ric}(\nabla u,\nabla u)\right) dM \\
&\qquad = -2(p-2)\int_M |\operatorname{Hess} u(\nabla u)|^2|\nabla u|^{2p-6}dM + \int_M (\Delta_p u)^2 dM \\
&\qquad - (p-2)^2\int_M |\nabla u|^{2p-4}(\Delta_\infty u)^2 + \int_{\partial M} |\nabla u|^{p-2}\Gamma(u)d\sigma,
\end{aligned}
\tag{3.10}
$$

where,

$$
\Gamma(u) = \frac{1}{p}\langle\nabla|\nabla u|^p,\eta\rangle + |\nabla u|^{p-2}du(\eta)\Delta u. \tag{3.11}
$$

Now, in order to get (3.11), let us consider $\{e_1,\dots,e_n\}$ an orthonormal frame on M^n adapted to ∂M, that is, $e_1,\dots,e_{n-1}$ are tangent to ∂M and $e_n=\eta$. From (2.1),

$$
\begin{aligned}
\Delta u &= -\sum_{i=1}^{n-1}\langle\nabla_{e_i}\nabla u, e_i\rangle - \langle\nabla_\eta\nabla u,\eta\rangle \\
&= -\sum_{i=1}^{n-1}\langle\nabla^\partial_{e_i}\nabla^\partial u, e_i\rangle + du(\eta)\sum_{i=1}^{n-1}\langle\mathcal{A}_\eta(e_i),e_i\rangle - \langle\nabla_\eta\nabla u,\eta\rangle \\
&= \Delta^\partial u + (n-1)H^\partial\, du(\eta) - \langle\nabla_\eta\nabla u,\eta\rangle.
\end{aligned}
\tag{3.12}
$$

On the other hand, by using one more time (2.1), a direct computation gives

$$
\begin{aligned}
\langle\nabla|\nabla u|^p,\eta\rangle &= p|\nabla u|^{p-2}\langle\nabla_\eta\nabla u,\nabla u\rangle \\
&= p|\nabla u|^{p-2}\left(\langle\nabla_{\nabla^\partial u}\nabla^\partial u,\eta\rangle + \langle\nabla^\partial u,\nabla^\partial du(\eta)\rangle + du(\eta)\langle\nabla_\eta\nabla u,\eta\rangle\right) \\
&= p|\nabla u|^{p-2}\left(\langle\nabla^\partial u,\nabla^\partial du(\eta)\rangle + \langle\mathcal{A}_\eta(\nabla^\partial u),\nabla^\partial u\rangle + du(\eta)\langle\nabla_\eta\nabla u,\eta\rangle\right).
\end{aligned}
\tag{3.13}
$$

Thus, from (3.12) and (3.13), (3.11) reads

$$
\begin{aligned}
\Gamma(u) &= |\nabla u|^{p-2}\left(du(\eta)\left(\Delta^\partial u + (n-1)H^\partial\, du(\eta)\right)\right) \\
&\quad + |\nabla u|^{p-2}\left(\langle\mathcal{A}_\eta(\nabla^\partial u),\nabla^\partial u\rangle + \langle\nabla^\partial u,\nabla^\partial du(\eta)\rangle\right).
\end{aligned}
$$

Therefore, by inserting this in (3.10), we have

$$\begin{aligned}\int_M |\nabla u|^{2p-4}\left(|\mathrm{Hess}\, u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + 2(p-2)|\mathrm{Hess}\, u(\nabla u)|^2|\nabla u|^{-2}\right) dM \\ = \int_M (\Delta_p u)^2 dM - (p-2)^2 \int_M |\nabla u|^{2p-4}(\Delta_\infty u)^2 \\ + \int_{\partial M} |\nabla u|^{2p-4} Q(u) d\sigma,\end{aligned} \tag{3.14}$$

where $Q(u)$ is defined in (3.5). ∎

The following inequalities will be useful in the proof of our results.

Lemma 3.5. *Let M^n be a Riemannian manifold and u a C^2 function such that $\nabla u \neq 0$. Then,*

(a) $(\Delta_p u)^2 \leq n(p-1)^2|\nabla u|^{2p-4}|\mathrm{Hess}\, u|^2$,

(b) $(\Delta_\infty u)^2 \leq \dfrac{|\mathrm{Hess}\, u(\nabla u)|^2}{|\nabla u|^2}$,

for all $p \geq 2$.

Proof. Let us prove (a). For this, we will consider the following $(1,1)$-tensor $T : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$T(X) = \nabla_X(|\nabla u|^{p-2}\nabla u).$$

Direct computation allows that

$$T(X) = (p-2)|\nabla u|^{p-4}\langle \mathrm{Hess}\, u(\nabla u), X\rangle \nabla u + |\nabla u|^{p-2}\nabla_X \nabla u, \tag{3.15}$$

for all $X \in \mathfrak{X}(M)$. In particular, $\mathrm{tr}(T) = -\Delta_p u$ and

$$\begin{aligned}\langle T(X), T(X)\rangle &= (p-2)^2|\nabla u|^{2p-6}\langle \mathrm{Hess} u(\nabla u), X\rangle^2 \\ &+ 2(p-2)|\nabla u|^{2p-6}\langle \mathrm{Hess} u(\nabla u), X\rangle\langle \nabla u, \nabla_X \nabla u\rangle \\ &+ |\nabla u|^{2p-4}\langle \nabla_X \nabla u, \nabla_X \nabla u\rangle\end{aligned}$$

If $\{e_i\}_i$ is an orthonormal frame of M^n, then

$$\begin{aligned} |T|^2 &= (p-2)^2 \sum_i |\nabla u|^{2p-6} \langle \mathrm{Hess} u(\nabla u), e_i\rangle^2 \\ &\quad + 2(p-2) \sum_i |\nabla u|^{2p-6} \langle \mathrm{Hess} u(\nabla u), e_i\rangle \langle \nabla u, \nabla_{e_i} \nabla u\rangle \\ &\quad + \sum_i |\nabla u|^{2p-4} \langle \nabla_{e_i} \nabla u, \nabla_{e_i} \nabla u\rangle \\ &= p(p-2)|\nabla u|^{2p-6}|\mathrm{Hess} u(\nabla u)|^2 + |\nabla u|^{2p-4}|\mathrm{Hess} u|^2 \\ &\leq (p-1)^2 |\nabla u|^{2p-4} |\mathrm{Hess} u|^2 \end{aligned} \tag{3.16}$$

We use $p \geq 2$ and the Cauchy-Schwarz inequality in the last line. Thence, from the well-known operator inequality $n|T|^2 \geq \mathrm{tr}(T)^2$, we have

$$|\nabla u|^{2p-4} |\mathrm{Hess}\, u|^2 \geq \frac{(\Delta_p u)^2}{n(p-1)^2}. \tag{3.17}$$

The item (b) follows from a suitable application of the Cauchy-Schwarz inequality. ■

3.2 MINIMAL SUBMANIFOLDS

In this section, we will use the machinery developed in the previous section to extend the integral inequality obtained in (LEUNG, 1983, Theorem 3) for the p-Laplacian in the case where $p \in [2, \infty)$.

Theorem 3.6. *Let M^n be a closed minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the p-Laplacian of M^n associated to λ. Then*

$$\int_M (S + \alpha_{n,p} \lambda^{2/p} - n) |\nabla u|^{2p-2} dM \geq 0,$$

where $p \in [2, \infty)$ and $\alpha_{n,p}$ is defined in (1.2). *Moreover, if equality holds, then $p = 2$ and*

(a) either M^n is totally geodesic and λ is the first nonzero eigenvalue of the 2-Laplacian,

(b) or $n = 2$ and $m = 2q$ and M^n is isometric to $\mathbb{S}^2\left(\sqrt{q(q+1)/2}\right)$ and λ is the first nonzero eigenvalue of the 2-Laplacian.

Proof. Taking into account the integral in Proposition 3.3, from the divergence theorem, since the boundary is empty, we have

$$\begin{aligned}\int_M |\nabla u|^{2p-4}\left(|\operatorname{Hess} u|^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) dM &= \int_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_p u\rangle dM \\ &- 2(p-2)\int_M |\nabla u|^{2p-6}|\operatorname{Hess} u(\nabla u)|^2 dM \\ &+ (p-2)\int_M |\nabla u|^{p-2}\left(|\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_\infty u\rangle - \Delta_\infty u \Delta_p u\right) dM,\end{aligned} \tag{3.18}$$

By using the divergence theorem again,

$$\begin{aligned}0 &= \int_M \operatorname{div}(|\nabla u|^{p-2}\Delta_\infty u |\nabla u|^{p-2}\nabla u) dM \\ &= -\int_M |\nabla u|^{p-2}\Delta_\infty u \Delta_p u dM + \int_M |\nabla u|^{p-2}\nabla u\left(|\nabla u|^{p-2}\Delta_\infty u\right) dM \\ &= -\int_M |\nabla u|^{p-2}\Delta_\infty u \Delta_p u dM + \int_M |\nabla u|^{p-2}\Delta_\infty u \nabla u\left(|\nabla u|^{p-2}\right) dM \\ &+ \int_M |\nabla u|^{2p-4}\langle \nabla u, \nabla \Delta_\infty u\rangle dM \\ &= -\int_M |\nabla u|^{p-2}\Delta_\infty u \Delta_p u dM + (p-2)\int_M |\nabla u|^{2p-4}(\Delta_\infty u)^2 dM \\ &+ \int_M |\nabla u|^{2p-4}\langle \nabla u, \nabla \Delta_\infty u\rangle dM.\end{aligned}$$

So, by replacing this in (3.18), we get

$$\begin{aligned}&\int_M |\nabla u|^{2p-4}\left(|\operatorname{Hess} u|^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) dM \\ &\quad = \int_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_p u\rangle dM - (p-2)^2\int_M |\nabla u|^{2p-4}(\Delta_\infty u)^2 dM \\ &\quad - 2(p-2)\int_M |\nabla u|^{2p-6}|\operatorname{Hess} u(\nabla u)|^2 dM.\end{aligned} \tag{3.19}$$

Applying Lemma 3.5, (3.19) reads

$$\begin{aligned}\int_M |\nabla u|^{2p-4}\left(|\operatorname{Hess} u|^2 + \operatorname{Ric}(\nabla u, \nabla u)\right) dM &\leq \int_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_p u\rangle dM \\ &- p(p-2)\int_M |\nabla u|^{2p-4}(\Delta_\infty u)^2 dM.\end{aligned} \tag{3.20}$$

Since $p \geq 2$, we get

$$\int_M |\nabla u|^{2p-4} \left(|\mathrm{Hess}\, u|^2 + \mathrm{Ric}(\nabla u, \nabla u)\right) dM \leq \int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \Delta_p u\rangle dM, \tag{3.21}$$

with equality occurring if and only if either u is ∞-harmonic or $p = 2$.

Inserting (3.17) in (3.21), we obtain

$$\begin{aligned}\int_M |\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u) dM \leq& \int_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_p u\rangle dM \\ &- \frac{1}{n(p-1)^2}\int_M (\Delta_p u)^2 dM.\end{aligned}$$

By using the divergence theorem,

$$0 = \int_M \mathrm{div}(\Delta_p u |\nabla u|^{p-2}\nabla u) dM = -\int_M (\Delta_p u)^2 dM + \int_M |\nabla u|^{p-2}\langle \nabla u, \nabla \Delta_p u\rangle dM. \tag{3.22}$$

Thence,

$$\int_M |\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u) dM \leq \left(\frac{n(p-1)^2 - 1}{n(p-1)^2}\right)\int_M (\Delta_p u)^2 dM.$$

Since u be an eigenfunction of the p-Laplacian of M^n associated to λ, from (2.11)

$$\int_M |\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u) dM \leq \lambda^2 \left(\frac{n(p-1)^2 - 1}{n(p-1)^2}\right)\int_M |u|^{2p-2} dM. \tag{3.23}$$

On the one hand, a direct computation in (3.22) gives,

$$\begin{aligned}\mathrm{div}(|u|^{p-2}u|\nabla u|^{p-2}\nabla u) &= -|u|^{p-2}u\Delta_p u + |\nabla u|^{p-2}\langle \nabla u, \nabla(|u|^{p-2}u)\rangle \\ &= -\lambda |u|^{2p-2} + (p-1)|\nabla u|^p |u|^{p-2}\end{aligned} \tag{3.24}$$

and so,

$$\lambda \int_M |u|^{2p-2} dM = (p-1)\int_M |\nabla u|^p |u|^{p-2} dM. \tag{3.25}$$

Since $p > 2$,

$$\frac{p}{2p-2} + \frac{p-2}{2p-2} = 1.$$

Note that

$$\frac{2p-2}{p} = 2 - \frac{2}{p} > 1 \qquad \frac{2p-2}{p-2} = \frac{p}{p-2} + 1 > 1.$$

By using Hölder's inequality, for any $p > 2$, we have

$$\begin{aligned}\int_M |\nabla u|^p |u|^{p-2} dM &\leq \left(\int_M (|\nabla u|^p)^{\frac{2p-2}{p}} dM\right)^{\frac{p}{2p-2}} \left(\int_M (|u|^{p-2})^{\frac{2p-2}{p-2}} dM\right)^{\frac{p-2}{2p-2}} \\ &= \left(\int_M |\nabla u|^{2p-2} dM\right)^{\frac{p}{2p-2}} \left(\int_M |u|^{2p-2} dM\right)^{\frac{p-2}{2p-2}}.\end{aligned} \tag{3.26}$$

Hence, from (3.25) and (3.26),

$$\int_M |u|^{2p-2} dM \leq \left(\frac{\lambda}{p-1}\right)^{\frac{2-2p}{p}} \int_M |\nabla u|^{2p-2} dM. \tag{3.27}$$

Therefore, by replacing this inequality in (3.23),

$$\int_M |\nabla u|^{2p-4} \mathrm{Ric}(\nabla u, \nabla u) dM \leq \lambda^{2/p} \left(\frac{n(p-1)^2 - 1}{n(p-1)^{2/p}}\right) \int_M |\nabla u|^{2p-2} dM. \tag{3.28}$$

Now, let us suppose that M^n is a minimal submanifold in $\mathbb{S}^m$. By Corollary 2.2, we have that,

$$\begin{aligned}\mathrm{Ric}(\nabla u, \nabla u) &\geq -\frac{n-1}{n}\left(|\phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| - n(\kappa + H^2)\right)|\nabla u|^2 \\ &= \frac{n-1}{n}(n - S)|\nabla u|^2\end{aligned}$$

By inserting this in (3.28),

$$\int_M (n - S)|\nabla u|^{2p-2} dM \leq \alpha_{n,p} \lambda^{2/p} \int_M |\nabla u|^{2p-2} dM,$$

where $\alpha_{n,p}$ is a positive constant given by

$$\alpha_{n,p} = \frac{n(p-1)^2 - 1}{(n-1)(p-1)^{2/p}}.$$

Therefore,

$$\int_M \left(n - S - \alpha_{n,p}\lambda^{2/p}\right)|\nabla u|^{2p-2} dM \leq 0, \tag{3.29}$$

which concludes the first part of the result.

If the equality holds in (3.29), all the inequalities along the proof become equal. In the next steps, we analyze each of them.

The first inequality is the one provided by Lemma 3.5, which is an equality case of Cauchy-Schwarz or $p = 2$.[1] Hence, $\nabla u = \gamma \mathrm{Hess}(\nabla u)$ for some constant γ. From now on, we assume that $p \neq 2$. The next inequality[2] is (3.21). When the equality happens, $0 = \gamma\langle \mathrm{Hess}u(\nabla u), \nabla u\rangle = \gamma|\nabla u|^2$. Hence, $\nabla u = 0$, and consequently, the eigenvalue associated with u is zero, which contradicts the initial assumptions. Therefore, if equality in (3.29) holds, we must have $p = 2$. Hence, from Lemma 3.5,

$$\frac{\lambda}{n} uI(X) = -\mathrm{Hess}\, u(X)$$

for all $X \in \mathfrak{X}(M)$. Thus, Lemma 2.3 assures that M^n must be isometric to $\mathbb{S}^n\left(\sqrt{n/\lambda}\right)$. Consequently, M^n has a second fundamental form of constant length. Moreover, by (3.29),

$$n - S - \lambda = 0. \tag{3.30}$$

Therefore, by (2.5)

$$n(n-1)\frac{\lambda}{n} = n(n-1) - (n - \lambda).$$

By direct computation, $(n-\lambda)(n-2) = 0$. So, if $n \geq 3$, $n = \lambda$ and then by (3.30) we have that $S = 0$. For $n = 2$, we have, by (CHERN, 1970), that $m = 2q$ and M^n is isometric to $\mathbb{S}^2\left(\sqrt{q(q+1)/2}\right)$. ■

It is important to highlight that the existence of a minimal immersion of $\mathbb{S}^2\left(\sqrt{q(q+1)/2}\right)$ into $\mathbb{S}^{2q}(1)$ is guaranteed by (WALLACH; CARMO, 1971, Theorem 1.2).

Remark 3.7. *Another way to see the conclusion of the equality, if $p \neq 2$, is through the equality* (3.21)*. In this case, we have* $\mathrm{Hess}\, u(\nabla u) = 0$*. On the other hand, the inequality in* (3.16) *turn into an equality, hence* $|\mathrm{Hess}(\nabla u)| = \gamma|\mathrm{Hess}u| = 0$*. Besides this, from* (3.17)*,* $\nabla(|\nabla u|^{p-2}\nabla u) = -(1/n)\Delta_p uI$*, where I is the identity in the algebra of smooth vector*

[1]The second case happens because using Cauchy-Schwarz inequality is unnecessary if $p = 2$. The terms that are compared will be zero.

[2]The inequality (3.20) is not analyzed because is a direct computation of (3.19) and Lemma 3.5.

fields on M^n. From this and (3.15),

$$0 = (p-2)|\nabla u|^{p-4}\langle \mathrm{Hess}\, u(\nabla u), \cdot\rangle \nabla u + |\nabla u|^{p-2}\mathrm{Hess}\, u(\cdot) = -\frac{1}{n}\Delta_p u I,$$

that is,

$$0 = \frac{\lambda}{n}|u|^{p-2}uI.$$

Thus, $\lambda = 0$, which is a contraction.

As an immediate consequence of Theorem 3.6, we have the following Leung's type estimate for the length of the second fundamental form in terms of the first eigenvalue of the p-Laplacian (cf. (LEUNG, 1983, Corollary 2)).

Corollary 3.8. *Let M^n be a closed minimal submanifold in $\mathbb{S}^m$ with $S = const.$, then*

$$S \geq n - \alpha_{n,p}\lambda_{1,p}(M)^{2/p},$$

for $p \in [2, \infty)$.

By using the identity (2.5), it is easy to see that any minimal submanifolds with constant scalar curvature immersed in $\mathbb{S}^m$ must have constant squared norm of the second fundamental form. In other words, we have the following result:

Corollary 3.9. *Let M^n be a closed minimal submanifold in $\mathbb{S}^m$ with constant scalar curvature. We have*

$$Scal_M \leq n(n-2) + \alpha_{n,p}\lambda_{1,p}(M)^{2/p},$$

for $p \in [2, \infty)$.

Proof. By equation (2.5),

$$\begin{aligned} n(n-1)R = n(n-1) - S &\leq n(n-1) - n + \alpha_{n,p}\lambda_{1,p}(M)^{2/p} \\ &\leq n(n-2) + \alpha_{n,p}\lambda_{1,p}(M)^{2/p}. \end{aligned}$$

■

Now we will focus our attention on studying compact manifolds M^n (with nonempty boundary ∂M and $n \geq 3$) to obtain estimates similar to those obtained in Theorem 3.6.

For the study of the first eigenvalue of compact manifolds with boundary, it is more appropriate to work with certain eigenvalue equations. The first eigenvalue equation that we will consider is the Neumann eigenvalue problem:

$$\begin{aligned} \Delta_p u &= \lambda |u|^{p-2}u, \qquad \text{in} \quad M^n \\ du(\eta) &= 0, \qquad\qquad \text{on} \quad \partial M, \end{aligned} \tag{3.31}$$

where η denotes the outward-pointing normal unit vector field to ∂M. Let us denote by $\lambda_{1,p}^N(M)$ the infimum of the set of nonzero eigenvalues for this problem. According to (VERON, 1991), a similar argument to the closed case, assures that $\lambda_{1,p}^N(M)$ is itself a positive eigenvalue with the variational characterization

$$\lambda_{1,p}^N(M) = \inf \left\{ \frac{\int_M |\nabla u|^p dM}{\int_M |u|^p dM} \; ; \; u \in W^{1,p}(M)\backslash\{0\} \text{ and } \int_M |u|^{p-2}u dM = 0 \right\}.$$

As an application of Proposition 3.4, we present our first version of Theorem 3.6 concerning the Neumann eigenvalue problem. It is important to highlight that, in the next two theorems (and their generalizations for non-minimal submanifolds), the case $p = 2$ was not known in the literature (unlike the closed case).

Theorem 3.10. *Let M^n be a compact minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the Neumann problem of the p-Laplacian of M^n associated to λ. Assume in addition that the boundary ∂M is convex. Then*

$$\int_M \left(S - n + \alpha_{n,p}\lambda^{2/p}\right) |\nabla u|^{2p-2} dM \geq 0,$$

where $p \in [2, \infty)$ and $\alpha_{n,p}$ is defined in (1.2). *Moreover, if equality holds, then $p = 2$ and M^n is isometric to the hemisphere $\mathbb{S}^n_+(1)$ and $\lambda = \lambda_{1,2}^N(\mathbb{S}^n_+)$*

Proof. By using Lemma 3.5, we can estimate (3.5) as follows

$$\begin{aligned} \int_M |\nabla u|^{2p-4} \mathrm{Ric}(\nabla u, \nabla u) dM \leq &\left(\frac{n(p-1)^2 - 1}{n(p-1)^{2/p}} \right) \int_M (\Delta_p u)^2 dM \\ &+ \int_{\partial M} |\nabla u|^{2p-4} Q(u) d\sigma, \end{aligned} \tag{3.32}$$

with equality if and only if $\mathrm{Hess}\, u(\nabla u) = 0$ or $p = 2$.

From the Neumann boundary conditions (3.31), and by using the assumption on the boundary ∂M, we have $Q(u) \leq 0$. Hence, (3.32) becomes

$$\int_M |\nabla u|^{2p-4} \mathrm{Ric}(\nabla u, \nabla u) dM \leq \lambda^2 \left(\frac{n(p-1)^2 - 1}{n(p-1)^{2/p}} \right) \int_M |u|^{2p-2} dM.$$

By applying the divergence theorem in (3.24),

$$\begin{aligned} -\lambda \int_M |u|^{2p-2} dM + (p-1) \int_M |\nabla u|^p |u|^{p-2} dM &= \int_M \mathrm{div}(|u|^{p-2} u |\nabla u|^{p-2} \nabla u) dM \\ &= \int_{\partial M} |u|^{p-2} u |\nabla u|^{p-2} du(\eta) d\sigma, \end{aligned}$$

and using one more time (3.31), we have

$$\lambda \int_M |u|^{2p-2} dM = (p-1) \int_M |\nabla u|^p |u|^{p-2} dM.$$

Since $p \geq 2$, we can use (3.27) in order to obtain,

$$\int_M |\nabla u|^{2p-4} \mathrm{Ric}(\nabla u, \nabla u) dM \leq \lambda^{2/p} \left(\frac{n(p-1)^2 - 1}{n(p-1)^{2/p}} \right) \int_M |\nabla u|^{2p-2} dM.$$

As M^n is minimal, from Corollary 2.2,

$$\int_M \left(n - S - \alpha_{n,p} \lambda^{2/p} \right) |\nabla u|^{2p-2} dM \leq 0,$$

where $\alpha_{n,p}$ is defined in (1.2).

In the case of equality, we just see that equality (3.32) implies that $\mathrm{Hess}\, u(\nabla u) = 0$. Consequently, we can reason as in the last part of the proof of Theorem 3.6 to conclude that $p = 2$. Similar to what happened in the minimal case,

$$\mathrm{Hess}\, u = -\frac{\lambda}{n} u I \quad \text{and} \quad du(\eta) = 0.$$

From this, we can apply Lemma 2.5 to obtain that M^n is isometric to $\mathbb{S}^n_+(\sqrt{\lambda/n})$, the upper hemisphere of radius $\sqrt{n/\lambda}$. Following the last part of the proof of Theorem 3.6, we have that S is constant and in fact by (1.2),

$$S = n(n-1) - n(n-1)\frac{\lambda}{n}.$$

Hence, equality holds

$$\lambda + n(n-1) - (n-1)\lambda - n = 0 \quad \Longrightarrow \quad (n-2)(n-\lambda) = 0.$$

Therefore, $\lambda = n$ and M^n is isometric to the totally geodesic hemisphere $\mathbb{S}^n_+(1)$ and λ is the first nonzero eigenvalue of the 2-Laplacian. ■

As a consequence,

Corollary 3.11. *In the situation of Theorem 3.10, if the square length of the second fundamental form S is constant, then*

$$S \geq n - \alpha_{n,p}\lambda^N_{1,p}(M)^{2/p}.$$

The second eigenvalue problem associated with the p-Laplacian that we will consider is the Dirichlet problem

$$\begin{aligned} \Delta_p u &= \mu|u|^{p-2}u, && \text{in} \quad M^n \\ u &= 0, && \text{on} \quad \partial M. \end{aligned} \tag{3.33}$$

We will denote by $\mu_{1,p}(M)$ the infimum of the set of eigenvalues of the problem (3.33). According to Lindqvist (LINDQVIST, 1990), $\mu_{1,p}(M)$ is simple, strictly positive and characterized by

$$\mu_{1,p}(M) = \inf\left\{\frac{\int_M |\nabla u|^p dM}{\int_M |u|^p dM}\,;\, u \in W^{1,p}_0(M)\backslash\{0\}\right\}.$$

In this setting, concerning the Dirichlet eigenvalue problem, we have.

Theorem 3.12. *Let M^n be a compact minimal submanifold in $\mathbb{S}^m$ and let u be an eigenfunction of the Dirichlet problem of the p-Laplacian of M^n associated to μ. Assume in* *addition that ∂M is mean convex. Then*

$$\int_M \left(S - n + \alpha_{n,p}\mu^{2/p}\right) |\nabla u|^{2p-2} dM \geq 0,$$

where $p \in [2, \infty)$ and $\alpha_{n,p}$ is defined in (1.2). Moreover, if equality holds and $S = const.$, then $p = 2$ and M^n is isometric to a closed hemisphere of the Euclidean sphere $\mathbb{S}^n(1)$ and $\mu = \mu_{1,2}(\mathbb{S}^n)$.

Proof. The inequality part follows as the proof of Theorem 3.10 by changing the convex boundary hypothesis to a mean convex boundary. Let us consider then the case where equality holds. By using the equality in Theorem 3.10, we have $p = 2$ and, as S is constant, we write

$$S = n - \mu_{1,2}(M).$$

From Ricci curvature estimate (Corollary 2.2),

$$\mathrm{Ric}(X, X) \geq \frac{n-1}{n}(n - S)|X|^2 = (n-1)c^2|X|^2, \quad \text{with} \quad c^2 = \frac{\mu_{1,2}(M)}{n},$$

for every $X \in \mathfrak{X}(M)$. Hence, using Reilly's result 2.4, we conclude that M^n is isometric to a closed hemisphere of the Euclidean sphere $\mathbb{S}^n(c^2)$. At this point, we reason as in the last part of the proof of Theorem 3.10 to conclude the result. ■

By thinking as in Corollary 3.9, we also have the following result:

Corollary 3.13. *Let M^n be a compact minimal submanifold in $\mathbb{S}^m$ with constant scalar curvature. For a real number $p \in [2, \infty)$, we have*

(i) if ∂M is mean convex, then

$$Scal_M \leq n(n-2) + \alpha_{n,p}\mu_{1,p}(M)^{2/p};$$

(ii) if ∂M is convex boundary, then

$$Scal_M \leq n(n-2) + \alpha_{n,p}\lambda^N_{1,p}(M)^{2/p}.$$

3.2.1 Hypersurfaces with constant relative nullity index

Our aim in this subsection is to extend part of the results of (BARBOSA; BARROS, 2003) to the first eigenvalue of the p-Laplacian by using the analytical tools developed in the previous sections. For this, let us consider M^n a connected isometrically immersed hypersurface of $\mathbb{S}^{n+1}$ which we assume to be orientable and oriented by a globally defined unit normal vector field N. Let $A = A_N$ be the Weingarten operator of the immersion concerning the normal direction N. We define the **Relative nullity space** for every $x \in M^n$ by the set (cf. (DAJCZER, 1990, Chapter 3))

$$\mathfrak{L}(x) = \{v \in T_xM\,;\, v \in \mathrm{Ker}(A_x)\}.$$

The dimension $\nu(x)$ of $\mathfrak{L}(x)$ is called the **Index of relative nullity** of M^n at x. In this setting, we say that M^n has constant index of relative nullity k if $\nu(x) = k$ for all $x \in M^n$. By using this notation, (BARBOSA; BARROS, 2003, Lemma 2) proved the following result:

Lemma 3.14. *Let V be a finite-dimensional vector space of dimension n, and let $T : V \to V$ be a traceless, symmetric, nontrivial linear operator. Consider an orthonormal basis $\{e_1, \dots, e_n\}$ such that $Te_i = \mu_i e_i$ for $i = 1, \dots, n$. If $k = \dim(\ker T)$, then for a nonzero vector $v = \sum_{i=1}^n v_i e_i$, the following inequality holds:*

$$\frac{1}{n-k}|T|^2|v|^2 \leq \sum_{i=1}^{n} \mu_i^2 v_i^2.$$

In our context, to assume that M^n has a constant index of relative nullity k if $\nu(x) = k$ for all $x \in M^n$ is equivalent to assuming that $k = \dim(\ker A)$. So, besides this, if we consider that M^n is minimal in $\mathbb{S}^{n+1}$, from Corollary 2.2, we get the following Ricci low estimate:

$$\mathrm{Ric}(X,X) \geq \frac{1}{n-k}\left((n-1)(n-k) - S\right)|X|^2,$$

for all $X \in \mathfrak{X}(M)$. By this, we present our first result concerning to constant relative nullity index, which extends (BARBOSA; BARROS, 2003, Theorem 2) to the p-Laplacian first eigenvalue.

Theorem 3.15. *Let M^n be a closed minimal hypersurface in $\mathbb{S}^{n+1}$ having constant relative nullity index k and let u be an eigenfunction of the p-Laplacian of M^n associated to λ, then*

$$\int_M \left(S-(n-1)\left(1-\frac{k}{n}\right)\left(n-\alpha_{n,p}\lambda_{2/p}\right)\right)|\nabla u|^{2p-2}dM \geq 0,$$

where $\alpha_{n,p}$ is defined in (1.2)*. Moreover, if the equality holds, then $p=2$. In particular, if $S = const.$, we have*

$$S \geq \frac{(n-k)(n-1)}{n}\left(n-\alpha_{n,p}\lambda_{1,p}(M)^{2/p}\right).$$

Proof. The proof follows as the one of Theorem 3.6 until we reach the inequality given in Corollary (2.2). Now, since we are supposing that M^n is minimal and has a constant relative nullity index k, hence

$$\mathrm{Ric}(\nabla u, \nabla u) \geq \frac{1}{n-k}\left((n-1)(n-k)-S\right)|\nabla u|^2. \tag{3.34}$$

By inserting (3.34) in (3.23), we get

$$\int_M |\nabla u|^{2p-2}\left((n-1)(n-k)-S\right)dM \leq \frac{(n(p-1)^2-1)(n-k)}{n(p-1)^{2/p}}\lambda^{2/p}\int_M |\nabla u|^{2p-2}dM.$$

From (1.2), we notice that

$$(n-1)(n-k)-\frac{(n(p-1)^2-1)(n-k)}{n(p-1)^{2/p}}\lambda^{2/p} = (n-1)(n-k)\left(1-\frac{\alpha_{n,p}\lambda^{2/p}}{n}\right).$$

Therefore, we obtain

$$\int_M |\nabla u|^{2p-2}\left((n-1)(n-k)\left(1-\frac{\alpha_{n,p}\lambda^{2/p}}{n}\right)-S\right)dM \leq 0, \tag{3.35}$$

which concludes the integral inequality part. As in Theorem 3.6, if the equality in (3.35) holds, then $p=2$. ■

Following the same ideas of the proof of the Theorems 3.10 and 3.12, we have a p-Laplacian version of (BARBOSA; BARROS, 2003, Theorem 2) for compact manifolds.

Theorem 3.16. *Let M^n be a compact minimal hypersurface in $\mathbb{S}^{n+1}$ having constant* *relative nullity index k and let u be an eigenfunction of the Dirichlet (resp. Neumann) problem of the p-Laplacian of M^n associated to λ. Assume in addition that,*

i. for the Dirichlet problem the boundary ∂M is mean convex,

ii. for the Neumann problem the boundary ∂M is convex.

Then

$$\int_M \left(S - (n-1)(n-k)\left(1 - n^{-1}\alpha_{n,p}\lambda^{2/p}\right)\right)|\nabla u|^{2p-2} dM \geq 0,$$

where $\alpha_{n,p}$ is defined in (1.2). Moreover, if the equality holds, then $p = 2$. In particular, if $S = const.$, then

$$S \geq \frac{(n-k)(n-1)}{n} \cdot \begin{cases} n - \alpha_{n,p}\mu_{1,p}(M)^{2/p} & (Dirichlet) \\ n - \alpha_{n,p}\lambda_{1,p}^N(M)^{2/p} & (Neumann). \end{cases}$$

We finish this subsection with the following version of Corollaries 3.9 and 3.13 for hypersurfaces with constant relative nullity index:

Corollary 3.17. *Let M^n be a compact minimal hypersurface in $\mathbb{S}^{n+1}$ having constant relative nullity index k and constant scalar curvature. For a real number $p \in [2, \infty)$, we have that*

(a) for $\partial M = \emptyset$,

$$Scal_M \leq \frac{n-1}{n}\left(nk + (n-k)\alpha_{n,p}\lambda_{1,p}(M)^{2/p}\right),$$

(b) for $\partial M \neq \emptyset$,

(i) if ∂M is mean convex, then

$$Scal_M \leq \frac{n-1}{n}\left(nk + (n-k)\alpha_{n,p}\mu_{1,p}(M)^{2/p}\right),$$

(ii) if ∂M is convex, then

$$Scal_M \leq \frac{n-1}{n}\left(nk + (n-k)\alpha_{n,p}\lambda_{1,p}^N(M)^{2/p}\right).$$

3.3 SUBMANIFOLDS WITH CONSTANT SCALAR CURVATURE

In this section, we do not assume that the manifold is minimal. Since in the equality case we obtained great spheres and these submanifolds in the unit sphere has constant scalar curvature, hence it is natural to do the opposite direction: if we assume that $M^n \subset \mathbb{S}^m$ is a submanifold with prescribed scalar curvature, can we conclude a similar inequality and again obtain the great spheres (and hemispheres for non-empty boundary)? The answer to this question is positive and is the main result of this section.

Note that one of the most relevant parts of the proof in the minimal case is the Ricci estimates. We then enunciate the following rereading of the lower estimate of the Ricci curvature 2.1 in terms of the normalized scalar curvature and the square norm of the umbilicity tensor.

Lemma 3.18. *Let M^n be a submanifold of the Riemannian manifold $\mathbb{S}^m$. Let* Ric *denote the function that assigns the minimum Ricci curvature to each point of M^n. Then*

$$\mathrm{Ric} \geq \frac{1}{n}\left(-(n-2)|\phi|^2 - (n-2)|\phi|\sqrt{|\phi|^2 + n(n-1)(R-1)} + n(n-1)R\right). \tag{3.36}$$

Proof. Now we fix the ambient to be the unit Euclidean sphere. By using (2.5) and (2.8), we write

$$H^2 = \frac{1}{n(n-1)}|\phi|^2 + R - 1,$$

and as $H \geq 0$,

$$H = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\phi|^2 + n(n-1)(R-1)}.$$

Thus, from (2.9), we obtain

$$\begin{aligned} -\frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(1+H^2) = &-\frac{n-2}{n-1}|\phi|\sqrt{|\phi|^2 + n(n-1)(R-1)} \\ &+ \frac{1}{n-1}|\phi|^2 + nR. \end{aligned} \tag{3.37}$$

Therefore, by inserting (3.37) in (2.9) we conclude the proof. ■

We present now a generalization of Theorem 3.6

Theorem 3.19. *Let M^n be a compact submanifold of $\mathbb{S}^m$ with constant scalar curvature $R = 1$ and possibly with convex boundary. Let u be an eigenfunction of the p-Laplacian (with the Neumann boundary condition if $\partial M \neq \emptyset$) of M^n associated to λ. Then*

$$\int_M \left(n - \frac{2(n-2)}{n}S - \alpha_{n,p}\lambda^{2/p}\right)|\nabla u|^{2p-2}dM \leq 0,$$

where $\alpha_{n,p}$ is a positive constant depending only n and p.

In particular, if the equality holds, then $p = 2$ and M^n is isometric to:

(a) The sphere $\mathbb{S}^n(1)$, if $\partial M = \emptyset$;

(b) The closed hemisphere $\mathbb{S}^n_+(1)$, otherwise.

Proof. We start the proof from equation (3.28) since until this part, the minimality of the manifold is not used. Since $R = 1$, equation (2.5) give us that $n^2H^2 = S$. On the other hand, the Ricci estimates (3.36) with equation (2.6) can be rewrite as

$$\begin{aligned}\mathrm{Ric}(\nabla u, \nabla u) &\geq \frac{n-1}{n}\left(n - 2n(n-2)H^2\right)|\nabla u|^2\\ &= \frac{n-1}{n}\left(n - \frac{2(n-2)}{n}S\right)|\nabla u|^2\end{aligned}$$

Hence, the result follows. ∎

3.4 SUBMANIFOLDS WITH CONSTANT MEAN CURVATURE

In this section, we present a result in a similar argument that was done in the previous section. A natural and intriguing question emerges in this context: is it possible to relax or entirely remove the hypothesis that the submanifold in question is minimal, while simultaneously not imposing any restrictions on its scalar curvature? Addressing this question requires a careful examination of the underlying geometric properties; moreover, we can maintain the characterization of equality. The principal objective of this section is to present a significant result in the form of an integral inequality, specifically for submanifolds characterized by having constant mean curvature. To achieve this, we set out to generalize the following well-known theorem for the p-Laplacian operator (cf. (LIU; ZHANG, 2007)): If M^n is a compact submanifold immersed in the standard Euclidean unit sphere $\mathbb{S}^N$ and u is an eigenfunction of the Laplacian, then we have an integral inequality involving the square length of the second fundamental form and the first eigenvalue of the Laplacian. We emphasize that, unlike the estimate obtained by Leung, the theorem above does not describe which geometric object achieves equality in the integral inequality. In this section, our main result is the following:

Theorem 3.20. *Let M^n be a compact with possibly convex boundary submanifold immersed in $\mathbb{S}^m$. Let u be an eigenfunction of the p-Laplacian of M^n associated to λ given by equation* (2.12). *Then*

$$\int_M \left(\frac{\sqrt{n-1}}{2} S - \frac{(n-1)(n-\alpha_{n,p}\lambda^{\frac{2}{p}})}{n} \right) |\nabla u|^{2p-2} dM \geq 0,$$

for $p \geq 2$. If the equality happens, then $p = 2$. In this case, by assuming in addition that M^n has parallel mean curvature vector field in $\mathbb{S}^m$, then M^n is isometric to

i. the sphere $\mathbb{S}^2(\sqrt{2/q(q+1)})$, with λ being the first eigenvalue and $m = 2q$, for the closed case if $n = 2$;

ii. the great sphere $\mathbb{S}^n(1)$, with λ being the first eigenvalue, for the closed case if $n \geq 3$;

iii. the upper hemisphere $\mathbb{S}^n_+(1)$, with λ being the first eigenvalue, corresponding to Dirichlet and Neumann problems.

From Proposition 2.1, we have that the Ricci curvature of M^n satisfies:

$$\text{Ric} \geq n - 1 + \frac{n-1}{n} \left(nH^2 - \frac{n-2}{\sqrt{n-1}} \sqrt{n} H \sqrt{S - nH^2} - (S - nH^2) \right). \tag{3.38}$$

To estimate (3.38) from below, let us consider the following quadratic form with eigenvalues $\pm \dfrac{n}{2\sqrt{n-1}}$:

$$F(x,y) = x^2 - \frac{n-2}{\sqrt{n-1}} xy - y^2.$$

We note that the orthogonal transformation

$$\begin{cases} w &= \frac{1}{2n} \left[(1+\sqrt{n-1})x + (1-\sqrt{n-1})y \right] \\ v &= \frac{1}{2n} \left[(\sqrt{n-1}-1)x + (1+\sqrt{n-1})y \right] \end{cases}$$

is such that $x^2+y^2=w^2+v^2$. Thus, by taking $x=\sqrt{n}H$ and $y=\sqrt{S-nH^2}$ we have $x^2+y^2=S$. Hence,

$$\begin{aligned}F(x,y)&=x^2-\frac{n-2}{\sqrt{n-1}}xy-y^2=\frac{n}{2\sqrt{n-1}}(w^2-v^2)\\&=\frac{n}{2\sqrt{n-1}}(w^2-v^2+w^2-w^2)\\&\geq-\frac{n}{2\sqrt{n-1}}(w^2+v^2)\\&=-\frac{n}{2\sqrt{n-1}}S,\end{aligned}\tag{3.39}$$

with equality holding if and only if $w=0$. So, by inserting (3.39) in (3.38),

$$\mathrm{Ric}\geq\left(n-1-\frac{\sqrt{n-1}}{2}S\right).\tag{3.40}$$

Moreover, if the equality

$$\int_M\left(\frac{(n-1)(n-\alpha_{n,p}\lambda^{\frac{2}{p}})}{n}-\frac{\sqrt{n-1}}{2}S\right)|\nabla u|^{2p-2}dM=0.\tag{3.41}$$

For the same argument done for the minimal submanifold, we must have that $p=2$. Thus, $\alpha_{n,2}=1$ and (3.41) turns in

$$\int_M\left(\frac{(n-1)(n-\lambda)}{n}-\frac{\sqrt{n-1}}{2}S\right)|\nabla u|^2dM=0.$$

Hence, if $\partial M=\emptyset$, by applying Lemma 2.3, it follows that M^n is isometric to the sphere $\mathbb{S}^n(\lambda/n)$. In the case where the boundary ∂M is nonempty and convex, we can apply Lemma 2.5 to obtain that M^n is isometric to a hemisphere $\mathbb{S}^n_+(\lambda/n)$. As (3.40) holds equality, then (3.39) also holds equality, which implies $w=0$. Thence,

$$nH^2=\left(\frac{\sqrt{n-1}-1}{\sqrt{n-1}+1}\right)^2(S-nH^2).\tag{3.42}$$

First we observe that if $n=2$, then $H=0$ and M^n is a minimal submanifold of $\mathbb{S}^N$. Hence, by using a similar argument to made in (LEUNG, 1983, Theorem 3) we conclude that $N=2q$ and M^n is isometric to $\mathbb{S}^2\left(\sqrt{2/q(q+1)}\right)$.

From now on, we will assume that $n \neq 2$. In this case, (3.42) becomes

$$S = \frac{2n^2}{(\sqrt{n-1}-1)^2} H^2. \tag{3.43}$$

Being M^n isometric to $\mathbb{S}^n(\lambda/n)$ (similarly for $\mathbb{S}^n_+(\lambda/n)$), by putting (3.43) in (2.5), we have

$$\lambda = n + \frac{n^2}{n-1}\left(1 - \frac{2}{(\sqrt{n-1}-1)^2}\right) H^2. \tag{3.44}$$

Since H is constant, from identity (3.42) we must have that S is constant. So, (3.44) can be written as follows

$$\lambda - n = -\frac{n}{2\sqrt{n-1}} S. \tag{3.45}$$

Hence, by inserting (3.43) and (3.45) in (3.44) we get

$$\frac{n^2}{n-1}\left(1 - \frac{2}{(\sqrt{n-1}-1)^2}\right) H^2 = -\frac{n}{2\sqrt{n-1}} \frac{2n^2}{(\sqrt{n-1}-1)^2} H^2.$$

Thus,

$$\left(\frac{1}{n-1}\left((\sqrt{n-1}-1)^2 - 2\right) + \frac{n}{\sqrt{n-1}}\right) H^2 = (n-2)(\sqrt{n-1}+1)H^2 = 0.$$

Once that $n \neq 2$, it follows $H = 0$ and hence, $S = 0$, from (3.43). Therefore, by returning to (3.45), we conclude that $n = \lambda$ and consequently M^n is isometric to $\mathbb{S}^n(1)$ if $\partial M = \emptyset$ and isometric to $\mathbb{S}^n_+(1)$ otherwise.

To end this proof, we will assume that ∂M is nonempty, convex, and satisfies the Dirichlet boundary condition. Since $w = 0$, from (3.38)

$$\mathrm{Ric} \geq n - 1 + \frac{(n-1)(\lambda - n)}{n} = \frac{(n-1)\lambda}{n} > 0.$$

Being ∂M convex, follows that H^∂ is nonpositive. Hence, we can apply the classical result (REILLY, 1977, Theorem 4) in order to obtain that M^n is isometric to $\mathbb{S}^n_+(\lambda/n)$. By thinking as before, we conclude that M^n is isometric to $\mathbb{S}^n_+(1)$.

Remark 3.21. *A natural question arises as to why the integral inequality for minimal submanifolds was addressed earlier rather than first using the one presented above. In ad-*

dition to the order in which the work is done, it is important to highlight that Theorem 3.6 provides a better estimate than the one presented in 3.20. In fact, by direct computation, we see that

$$\frac{\sqrt{n-1}}{2}S - \frac{(n-1)(n-\alpha_{n,p}\lambda^{\frac{2}{p}})}{n} = \left(\frac{n-1}{n}\right)\left(\frac{n}{2\sqrt{n-1}}S - n + \alpha_{n,p}\lambda^{\frac{2}{p}}\right)$$

and

$$\frac{n}{2\sqrt{n-1}} \geq 1.$$

3.5 AN INTEGRAL INEQUALITY VIA THE UMBILICITY TENSOR

The next result gives an integral inequality involving the squared norm of the total umbilicity tensor and the nonzero eigenvalue of the p-Laplacian without assuming that $R = 1$ or constant mean curvature.

Theorem 3.22. *Let M^n be a compact submanifold of $\mathbb{S}^m$ with possibly convex boundary. Let u be an eigenfunction of the p-Laplacian (with the Neumann boundary condition if $\partial M \neq \emptyset$) of M^n associated to λ. Then*

$$\int_M \left(n - \frac{n^2}{4(n-1)}|\phi|^2 - \alpha_{n,p}\lambda^{2/p}\right)|\nabla u|^{2p-2}dM \leq 0,$$

where $\alpha_{n,p}$ is a positive constant depending only given by n and p. In particular, if the equality holds, then $p = 2$ and M^n is isometric to

(a) The sphere $\mathbb{S}^n(1)$, if $\partial M = \emptyset$;

(b) The closed hemisphere $\mathbb{S}^n_+(1)$, otherwise.

Proof. By using ε-Young's inequality $xy \leq \varepsilon x^2 + \varepsilon^{-1}y^2$ for

$$x = \sqrt{|\phi|^2 + n(n-1)(R-1)} \quad \text{and} \quad y = |\phi|$$

we have

$$2|\phi|\sqrt{|\phi|^2 + n(n-1)(R-1)} \leq \varepsilon(|\phi|^2 + n(n-1)(R-1)) + \varepsilon^{-1}|\phi|^2. \tag{3.46}$$

Consequently, by taking $\varepsilon = \frac{2}{n-2}$, (3.46) reads

$$(n-2)|\phi|\sqrt{|\phi|^2 + n(n-1)(R-1)} \leq |\phi|^2 + n(n-1)(R-1) + \frac{(n-2)^2}{4}|\phi|^2.$$

Hence, inserting this in Lemma 2.2 we reach at the following inequality

$$\begin{aligned} \mathrm{Ric}(\nabla u, \nabla u) &\geq \frac{1}{n}\left(-(n-2)|\phi|^2 - |\phi|^2 + n(n-1) - \frac{(n-2)^2}{4}|\phi|^2\right)|\nabla u|^2 \\ &= \frac{n-1}{n}\left(n - \frac{n^2}{4(n-1)}|\phi|^2\right)|\nabla u|^2. \end{aligned} \tag{3.47}$$

Therefore, by replacing (3.47) in (3.32), we obtain

$$\int_M \left(n - \frac{n^2}{4(n-1)}|\phi|^2\right)|\nabla u|^{2p-2} dM \leq \lambda^{2/p}\alpha_{n,p}\int_M |\nabla u|^{2p-2} dM, \tag{3.48}$$

where $\alpha_{n,p}$ is defined in (1.2).

Now, we analyze the equality in (3.48) for the closed case. The case not closed will follow the same ideas. In fact, if the equality holds in (3.48), then we can reason as in the proof of Theorem 3.19 to conclude that $p = 2$. Hence, according Lemma 2.3, M^n is isometric to $\mathbb{S}^n(\sqrt{\lambda/n})$ and therefore $|\phi|$ is constant. Furthermore, the equality implies

$$|\phi|^2 = \frac{4(n-\lambda)(n-1)}{n^2}. \tag{3.49}$$

On the other hand, the equality in ε-Young's inequality (3.46), reads

$$|\phi|^2 + n(n-1)(R-1) = \left(\frac{n-2}{2}\right)^2 |\phi|^2. \tag{3.50}$$

Since $R = \lambda/n$, (3.50) can rewritten as follows

$$\lambda = \frac{n(n-4)}{4(n-1)}|\phi|^2 + n. \tag{3.51}$$

Hence, (3.49) and (3.51) implies $(n-\lambda)(n-2) = 0$. Provide that $n \geq 3$, we get $\lambda = n$ and $M^n = \mathbb{S}^n(1)$. ∎

Remark 3.23. *Note that in the case that M^n is minimal, this implies that $|\phi|^2 = S$.*

3.6 THE CASE $\frac{3}{2} < p < 2$

In order to give completeness to the work, we present a weak version of our theorem for $\frac{3}{2} \leq p \leq 2$. Despite the geometry results in the literature for the p-Laplacian being restricted most of the time to $p \geq 2$, for $p < 2$ the operator still have applications. From the perspective of Physics, various applications arise, such as those concerning pseudo-plastic fluids. For instance, one notable application pertains to the study of unsaturated flow for $p = 3/2$ (cf. (DIAZ; DE THELIN, 1994)) and glaciology for $p \in (1, 4/3]$. In particular, within $3/2 < p < 2$, Leibenson investigated turbulent gas filtration in porous media (for further details, see (LEIBENSON, 1945)). Additionally, comprehensive insights into this subject can be found in the work of Benedikt (cf. (BENEDIKT et al., 2018)).

Lemma 3.24. *Let M^n be a compact manifold with mean-convex boundary ∂M. If u is a solution of the Dirichlet eigenvalue problem and $3/2 < p < 2$, then*

$$\int_M \mathrm{Ric}(\nabla u, \nabla u)|\nabla u|^{2p-4} dM < \lambda^2 \widetilde{C} \int_M |\nabla u|^{2p-2} dM,$$

where $\widetilde{C} := \widetilde{C}(n, p, C_1)$ is a positive constant depending only n, p and the Sobolev constant C_1.

Proof. We start the proof from the equation (3.14). Since $p < 2$, from Cauchy-Schwarz's inequality,

$$\begin{aligned} \int_M |\nabla u|^{2p-4} \mathrm{Ric}(\nabla u, \nabla u) dM \leq& -(2p-3) \int_M |\nabla u|^{2p-4} |\mathrm{Hess}\, u|^2 dM \\ &+ \int_M (\Delta_p u)^2 dM - (p-2)^2 \int_M |\nabla u|^{2p-4} (\Delta_\infty u)^2 dM \\ &+ \int_{\partial M} |\nabla u|^{2p-4} \mathcal{Q}(u) d\sigma. \end{aligned} \tag{3.52}$$

By using that $p > 3/2$, we can use again the Cauchy-Schwarz inequality on the operator T (defined in equation (3.15)) in order to obtain

$$-(2p-3)|\nabla u|^{2p-4} |\mathrm{Hess} u|^2 \leq -\frac{(2p-3)}{n} (\Delta_p u)^2 \tag{3.53}$$

with equality holding if and only if

$$(p-2)|\nabla u|^{p-4}\langle \operatorname{Hess} u(\nabla u), X\rangle \nabla u + |\nabla u|^{p-2}\operatorname{Hess} u(X) = -\frac{1}{n}(\Delta_p u)I(X),$$

for all $X \in \mathfrak{X}(M)$. Thus, by inserting (3.53) in (3.52), we get

$$\begin{aligned}\int_M |\nabla u|^{2p-4}\operatorname{Ric}(\nabla u, \nabla u)dM \leq{}& \frac{n-2p+3}{n}\int_M (\Delta_p u)^2 dM \\ &- (p-2)^2\int_M |\nabla u|^{2p-4}(\Delta_\infty u)^2 dM + \int_{\partial M} |\nabla u|^{2p-4}\mathcal{Q}(u)d\sigma \\ \leq{}& \frac{n-2p+3}{n}\int_M (\Delta_p u)^2 dM + \int_{\partial M} |\nabla u|^{2p-4}\mathcal{Q}(u)d\sigma\end{aligned}$$

Therefore,

$$\int_M |\nabla u|^{2p-4}\operatorname{Ric}(\nabla u, \nabla u)dM \leq \left(\frac{n-2p+3}{n}\right)\lambda^2\int_M |u|^{2p-2}dM + \int_{\partial M} |\nabla u|^{2p-4}\mathcal{Q}(u)d\sigma,$$

with equality if and only if $\Delta_\infty u = 0$.

On the other hand, since $u = 0$ on ∂M and the boundary is mean convex, we can estimate the $\mathcal{Q}$ as follows

$$\int_{\partial M} |\nabla u|^{2p-4}\mathcal{Q}(u)d\sigma \leq 0,$$

and then

$$\int_M |\nabla u|^{2p-4}\operatorname{Ric}(\nabla u, \nabla u)dM \leq \lambda^2\left(\frac{n-2p+3}{n}\right)\int_M |u|^{2p-2}dM. \tag{3.54}$$

Since $p \in (3/2, 2)$, we have $1 < 2p-2 < p$. Consequently, by Sobolev's embedding theory, $W_0^{1,p}(M) \subset W_0^{1,2p-2}(M)$. As $2p-2 < (2p-2)^*$, from Hölder inequality,

$$\int_M |u|^{2p-2}dM \leq \operatorname{vol}(M)^{\frac{2p-2}{n}}\left(\int_M |u|^{(2p-2)^*}dM\right)^{\frac{2p-2}{(2p-2)^*}} \tag{3.55}$$

Hence, by using Gagliardo-Nirenberg-Sobolev's inequality for compact Riemannian manifolds with boundary (AUBIN, 2012) (see also (HEBEY, 2000)), we have

$$\int_M |u|^{2p-2}dM \leq \operatorname{vol}(M)^{\frac{2p-2}{n}}C_1\int_M |\nabla u|^{2p-2}dM, \tag{3.56}$$

where C_1 is a constant positive depending on M^n. By inserting this in (3.54),

$$\int_M |\nabla u|^{2p-4}\mathrm{Ric}(\nabla u, \nabla u)dM \leq \lambda^2 \widetilde{C}(n,p,C_1)\int_M |\nabla u|^{2p-2}dM, \tag{3.57}$$

where $\widetilde{C}$ is a positive constant depending only n, p and the Sobolev constant C_1.

By assuming the equality case, then all inequalities become equalities, in particular the equality (3.55) implies the equality in Hölder inequality. Hence, $|u| = 1$ almost everywhere, and then the p-Laplacian of u is identically zero, which cannot happen since the first eigenvalue is nonzero. Therefore, the inequality (3.57) is not sharp. ■

As a first application, we get the following lower estimate:

Theorem 3.25. *Let M^n be a compact Riemannian manifold with mean-convex boundary and consider M^n as a minimal submanifold of $\mathbb{S}^{n+q}$ with $S = const$. Then*

$$\lambda_{1,p}(M) \geq \sqrt{(n-S)C}, \qquad p \in (3/2, 2),$$

where C is a positive constant depending only n, p and M^n.

Proof. From Lemma 3.24,

$$\int_M \mathrm{Ric}(\nabla u, \nabla u)|\nabla u|^{2p-4}dM \leq \widetilde{C}\int_M |\nabla u|^{2p-4}dM. \tag{3.58}$$

On the other hand, since M^n is a minimal submanifold of $\mathbb{S}^{n+q}$, by taking $H = 0$ in Corollary 2.2,

$$\mathrm{Ric}(\nabla u, \nabla u) \geq -\frac{n-1}{n}(S-n)|\nabla u|^2.$$

By replacing this in (3.58),

$$-\frac{n-1}{n}(S-n)\int_M |\nabla u|^{2p-2}dM \leq \lambda^2\widetilde{C}\int_M |\nabla u|^{2p-2}dM$$

where was that $S = const.$. Hence

$$0 \leq \left(\lambda^2\widetilde{C} + \frac{n-1}{n}(S-n)\right)\int_M |\nabla u|^{2p-2}dM$$

and since $\int_M |\nabla u|^{2p-2} dM > 0$, we have

$$\lambda^2 \geq \frac{n}{(n-1)\widetilde{C}} (n - S).$$

In particular,

$$\lambda_{1,p}(M)^2 \geq (n - S)C,$$

and we have the desired result. ■

By assuming that the submanifold has a prescribed constant scalar curvature, we get

Theorem 3.26. *Let M^n be a compact Riemannian manifold with constant scalar curvature $R = 1$ and mean convex boundary. Consider M^n as a submanifold of $\mathbb{S}^{n+q}$ with $S = const.$. Then*

$$\lambda_{1,p}(M) \geq \sqrt{(n^2 - 2(n-2)S)C}, \qquad p \in (3/2, 2),$$

where C is a positive constant depending only n, p and M^n.

Proof. Again, since $R = 1$, from (2.5) we have $S = n^2 H^2$. By Corollary 2.2 and Lemma 3.24,

$$0 \leq \left(\lambda^2 \widetilde{C} + \frac{n-1}{n^2} \left(2(n-2)S - n^2 \right) \right) \int_M |\nabla u|^{2p-2} dM.$$

So

$$\lambda^2 \geq \frac{n-1}{n^2 \widetilde{C}} (n^2 - 2(n-2)S),$$

and therefore,

$$\lambda_{1,p}(M) \geq \sqrt{(n^2 - 2(n-2)S)C}.$$

■

We end this section with the following low estimate to $\lambda_{1,p}(M)$ involving the squared norm of the total umbilicity tensor and without the necessity of prescribing the scalar curvature.

Theorem 3.27. *Let M^n be a compact Riemannian manifold having mean-convex boundary and consider M^n as a submanifold of $\mathbb{S}^{n+q}$ with $S - nH^2 = const.$. Then*

$$\lambda_{1,p}(M) \geq \sqrt{(4(n-1) - n(S - nH^2))\, C}, \qquad p \in (3/2, 2),$$

where C is a positive constant depending only n, p and M^n.

Remark 3.28. *It should be noted that, by applying a similar argument, the Theorems 3.26 and 3.27 also hold for closed manifolds. In fact, the Hölder inequality used in* (3.55) *does not depend on the boundary. Moreover, the Sobolev inequality* (3.56) *is still true for closed manifolds (see (HEBEY, 2000, Theorem 4.5)).*

Remark 3.29. *Another important point to highlight is the veracity of the inequalities presented before in the case of the p-Laplacian with $1 < p < \frac{3}{2}$. The tools presented here are not sufficient to show, but we conjecture that the inequalities do not hold since $2p - 2 < 1$, and then the Hölder inequalities and Sobolev Immersions are not true (in the classical sense).*

4 ESTIMATES ON COMPLETE non-compact RIEMANNIAN MANIFOLDS

In this chapter, we present a Lioville-type result for strongly subharmonic functions (which in defined in the next steps) in terms of some hypotheses on the first eigenvalue of the p-Laplacian. We present a generalization of part of the results presented by Leung (LEUNG, 1997). To give completeness to the work, we enunciate some classical definitions and results about the p-Laplacian in complete manifolds, similarly to what was done in chapter 2.

Let us take M^n an n-dimensional complete non-compact manifold and, let $\{\Omega_k\}_{k\in\mathbb{N}}$ be an exhaustion of M^n by compact domains, that is, $\{\Omega_k\}_{k\in\mathbb{N}}$ are compact domains such that $M^n = \cup_{k=1}^{\infty}\Omega_k$ and $\Omega_k \subset \Omega_{k+1}$, for all $k \in \mathbb{N}$. We will consider the first eigenvalue $\mu_{1,p}(\Omega_k)$ of the following Dirichlet boundary value problem:

$$\begin{cases} \Delta_p u &= -\lambda |u|^{p-2}u, \qquad \text{in} \quad \Omega_k \\ u &= 0, \qquad \text{on} \quad \partial\Omega_k. \end{cases} \tag{4.1}$$

The existence of the eigenvalue problem (4.1) and the variational characterization as in (1.1) were proved by Veron (VERON, 1991). On the other hand, in (LINDQVIST, 1990), Lindqvist proved that $\mu_{1,p}(\Omega_k)$ is simple for each compact domain Ω_k, $k \in \mathbb{N}$. By definition, we see that $\mu_{1,p}(M) \leq \mu_{1,p}(\Omega_k)$ for each compact domain Ω_k, $k \in \mathbb{N}$. Using the domain monotonicity of $\mu_{1,p}(\Omega_k)$, we deduce that $\mu_{1,p}(\Omega_k)$ is non-increasing in $k \in \mathbb{N}$ and has a limit which is independent of the choice of the exhaustion of M^n. Therefore

$$\mu_{1,p}(M) = \lim_{k\to\infty} \mu_{1,p}(\Omega_k).$$

On the other hand, strongly p-subharmonic functions play an important role in studying Riemannian manifolds. Let us recall that, a smooth function $u : M^n \to \mathbb{R}$ is said to be *strongly p-subharmonic* if u satisfies the following differential inequality

$$\Delta_p u \geq k > 0. \tag{4.2}$$

Concerning strongly p-subharmonic functions, we quote the results due to (TAKEGOSHI, 2001) for the relations that lie between the existence of a certain strongly p-subharmonic function and the volume growth property for the case $p \geq 2$. Also, for $p = 2$, Coghlan, Itokawa, and Kosecki (COGHLAN L. ITOKAWA; KOSECKI, 1992) proved a nice Liouville-type result, which said that every 2-subharmonic function on a complete non-compact Riemannian manifold must be unbounded provided that its sectional curvature is bounded. A few years later, Leung (LEUNG, 1997) proved that the same result is valid by replacing the bounded sections' curvature with the vanishing first eigenvalue of the 2-Laplacian. As an application, Leung obtained an estimate for the size of the image set of some types of maps between Riemannian manifolds.

Note that all arguments previously presented in the other chapters are done for a compact manifold. Here, we are interested in studying the first eigenvalue of the p-Laplacian of complete non-compact Riemannian manifolds. More precisely, we obtain a characterization of p-subharmonic functions on a complete non-compact Riemannian manifold by assuming that the first eigenvalue of the p-Laplacian is zero. Proceeding with this picture, we obtain the following extension of the main result of (LEUNG, 1997, Theorem 1) for the context of the p-Laplacian, for all $p \geq 2$.

By using this previous machinery, the next result is an extension of (PALMER, 1990, Lemma 1) due to Palmer to the p-Laplacian for $p \geq 2$ (see also (LEUNG, 1992)).

Lemma 4.1. *Let D be a relatively compact smoothly bounded domain on a Riemannian manifold M^n. Suppose that exists a smooth function f on $\overline{D}$ that satisfies $\Delta_p f \geq 1$ in D. Then*

$$\mu_{1,p}(D) \geq \frac{1}{(\beta - \alpha)^{p-1}},$$

where α, β are any lower and upper bounds respectively of f on $\overline{D}$.

Proof. First of all, let us consider u the first eigenfunction associated to $\mu_{1,p}(D)$ and assume, without loss of generality, that $u > 0$. We consider a any constant such that $\beta < a$. Let us define then the function $v = \frac{u}{a-f}$. We observe that v is attains the maximum at some point $x_0 \in \operatorname{int}(D)$ since $v \equiv 0$ in ∂D and $a - f > 0$. Hence in x_0, we have that $\nabla v(x_0) = 0$ and

$$\begin{aligned} L_u(v)(x_0) &= |\nabla u|^{p-2}\Delta v + (p-2)|\nabla u|^{p-4}\langle \mathrm{Hess} u(\nabla u), \nabla v\rangle \\ &= |\nabla u|^{p-2}\Delta v \leq 0. \end{aligned}$$

A direct computation gives that the gradient of v is

$$\nabla v = \frac{1}{a-f}\nabla u + \frac{u}{(a-f)^2}\nabla f \tag{4.3}$$

and that, for all $X \in \mathfrak{X}(M)$, the hessian of v is

$$\begin{aligned} \nabla_X \nabla v = &\frac{1}{(a-f)^2}\langle \nabla f, X\rangle \nabla u + \frac{1}{a-f}\nabla_X \nabla u + \frac{u}{(a-f)^2}\nabla_X \nabla f \\ &+ \frac{1}{(a-f)^3}\left((a-f)\langle \nabla u, X\rangle + 2u\langle X, \nabla f\rangle\right)\nabla f. \end{aligned} \tag{4.4}$$

By tracing (4.4), we have

$$\Delta v = \frac{1}{a-f}\Delta u + \frac{u}{(a-f)^2}\Delta f + \frac{2u}{(a-f)^3}|\nabla f|^2 + \frac{2}{(a-f)^2}\langle \nabla u, \nabla f\rangle.$$

Moreover, from (4.3),

$$\langle \mathrm{Hess}\, u(\nabla u), \nabla v\rangle = \frac{1}{a-f}A_u(u) + \frac{u}{(a-f)^2}\langle \mathrm{Hess}\, u(\nabla u), \nabla f\rangle. \tag{4.5}$$

Hence, by inserting (4.3) and (4.5) in the definition of L_u, we get

$$\begin{aligned} L_u(v) = &\frac{1}{a-f}\Delta_p u + \frac{u}{(a-f)^2}L_u(f) + \frac{2u}{(a-f)^3}|\nabla f|^2|\nabla u|^{p-2} \\ &+ \frac{2}{(a-f)^2}\langle \nabla u, \nabla f\rangle |\nabla u|^{p-2}. \end{aligned} \tag{4.6}$$

On the other hand, once $\nabla v(x_0) = 0$, (4.3) gives, $\nabla u = -\dfrac{u}{a-f}\nabla f$ in x_0. Hence, in x_0,

$$\mathrm{Hess}\, u(\nabla u) = \frac{u^2}{(a-f)^2}\mathrm{Hess}\, f(\nabla f),$$

and consequently

$$\begin{aligned} L_u(f) &= |\nabla u|^{p-2}\Delta f + (p-2)|\nabla u|^{p-4}\langle \operatorname{Hess} u(\nabla u), \nabla f\rangle \\ &= \left(\frac{u}{a-f}\right)^{p-2} |\nabla f|^{p-2}\Delta f + (p-2)\left(\frac{u}{a-f}\right)^{p-2} |\nabla f|^{p-2} A_f(f) \\ &= \left(\frac{u}{a-f}\right)^{p-2} \Delta_p f. \end{aligned} \tag{4.7}$$

Besides this,

$$\frac{2u}{(a-f)^3}|\nabla f|^2 + \frac{2}{(a-f)^2}\langle \nabla u, \nabla f\rangle = 0. \tag{4.8}$$

Therefore, from (4.6), (4.7) and (4.8) we obtain in x_0,

$$0 \geq L_u(v) = \frac{1}{a-f}\Delta_p u + \frac{u^{p-1}}{(a-f)^p}\Delta_p f. \tag{4.9}$$

Since $\Delta_p u = -\mu_{1,p}(D)|u|^{p-2}u$, $u(x_0) > 0$, $\alpha < f(x_0) < a$, and $\Delta_p f \geq 1$, (4.9) implies

$$\mu_{1,p}(D) \geq \frac{1}{(a-\alpha)^{p-1}}.$$

By taking $a \to \beta$ we obtain the desired. ■

4.1 A LIOUVILLE-TYPE RESULT

Before presenting the proof of our main result, we will give some examples and classical results concerning complete Riemannian manifolds that have vanished the first p-Laplacian eigenvalue. The first is (cf. (CARMO; ZHOU, 1999))

Example 4.2. *Let $M^2 = (\mathbb{R}^2, ds^2)$, with $ds^2 = dr^2 + g^2(r)d\theta^2$, where $g(r)$ is a nonnegative $\mathcal{C}^2([0, +\infty))$ function with $g(0) = 0$, $g(r) > 0$ for $0 < r \leq 1$ and $g(r) = e^{-r}$ for $r > 1$. Note that* vol$(M) < +\infty$. *Hence, the constant functions are in $L^p(M)$ and, therefore, $\mu_{1,p}(M) = 0$.*

We say that the volume of M^n has polynomial growth if there exist positive numbers a and c such that $\operatorname{vol}(B_r(q)) \leq cr^a$. In this setting, the next two results are due to Batista et al. (BATISTA M. CAVALCANTE; SANTOS, 2014), extending the classical result to do

Carmo (CARMO; ZHOU, 1999) for all $p \geq 2$. The first one assumes that the Riemannian manifold has infinite volume:

Lemma 4.3. *Let M^n be an open manifold with infinite volume and Ω an arbitrary compact set of M^n. If M^n has polynomial volume growth, then $\mu_{1,p}(M\backslash\Omega) = 0$.*

The second assumption is that the Riemannian manifold is complete and non-compact :

Lemma 4.4. *Let M^n be a complete non-compact Riemannian manifold with polynomial volume growth, then $\mu_{1,p}(M) = 0$.*

Now, we can prove our main result.

Theorem 4.5. *If M^n is a complete non-compact Riemannian manifold with $\mu_{1,p}(M) = 0$, then every strongly p-subharmonic function on M^n is unbounded.*

Proof. Let us suppose that $u \in \mathcal{C}^2(M)$ is bounded. Since u is a strongly p-subharmonic function, from (4.2)

$$\Delta_p u \geq k > 0, \tag{4.10}$$

for some positive constant k. Now, we consider the function $f : M^n \to \mathbb{R}$ given by

$$f = \frac{u}{k^{\frac{1}{p-1}}}. \tag{4.11}$$

By a direct computation,

$$\nabla f = \frac{1}{k^{\frac{1}{p-1}}} \nabla u$$

Hence, it follows from (4.10) and the definition of the p-Laplacian operator,

$$\begin{aligned}
\Delta_p f &= |\nabla f|^{p-2} \Delta f + (p-2)|\nabla u|^{p-4} \langle \nabla_{\nabla f} \nabla f, \nabla f \rangle \\
&= \left(k^{-\frac{p-2}{p-1}} |\nabla u|^{p-2}\right) \left(k^{-\frac{1}{p-1}} \Delta u\right) + (p-2) \left(k^{-\frac{p-4}{p-1}} |\nabla u|^{p-4}\right) \left(k^{-\frac{3}{p-1}}\right) \langle \nabla_{\nabla u} \nabla u, \nabla u \rangle \\
&= k^{-1} \Delta_p u \geq 1.
\end{aligned}$$

On the other hand, from (4.11) we see that f is bounded. Thus, there exists positive constants α and β such that $\alpha < f < \beta$. Therefore, by applying Lemma 4.1,

$$\mu_{1,p}(M) \geq \frac{1}{(\beta - \alpha)^{p-1}} > 0,$$

which contradicts the fact that the first eigenvalue is zero. ■

As an immediate application, it follows from Lemma 4.1,

Corollary 4.6. *If M^n is a complete non-compact Riemannian manifold with polynomial volume growth, then every strongly p-subharmonic function on M^n is unbounded.*

4.2 HYPERSURFACES IN WARPED PRODUCT MANIFOLDS

Let $(P^n, \langle\, , \rangle_P)$ be a connected n-dimensional ($n \geq 2$) oriented Riemannian manifold and $f : I \subset \mathbb{R} \to \mathbb{R}_+$ a positive smooth function. In the product differentiable manifold $I \times P^n$. A particular class of Riemannian manifolds is the one obtained by endowing $I \times P^n$ with the metric

$$\langle v, w\rangle = \langle (\pi_I)_* v, (\pi_I)_* w\rangle_I + (f \circ \pi_I)^2 \langle (\pi_P)_* v, (\pi_P)_* w\rangle_P,$$

with $(t, q) \in I \times P^n$ and $v, w \in T_{(t,q)}(I \times P)$, where π_I and π_P denote the projections onto the corresponding factor. This space is called a *warped product* and f the *warped function*, and in what follows, we write $I \times_f P^n$ to denote it. In particular the family of hypersurfaces $P_t^n = \{t\} \times P^n$ (called here a slices) form a foliation $t \in I \to P_t^n$ of $I \times_f P^n$ by totally umbilical leaves of constant mean curvature

$$\mathcal{H}(t) = \sum_i \langle -\overline{\nabla}_{e_i} \partial_t, e_i\rangle = \frac{f'(t)}{f(t)} = (\ln f)'(t),$$

with respect $-\partial_t$, where

$$\partial_t := (\partial/\partial_t)\big|_{(p,t)}, \quad (t, q) \in I \times P^n$$

is a conformal and unitary vector field, that is,

$$\overline{\nabla}_X \partial_t = \mathcal{H}(t)(X - \langle X, \partial_t\rangle \partial_t) \quad \text{and} \quad \langle \partial_t, \partial_t\rangle = 1,$$

for all $X \in \mathfrak{X}(P)$.

Let $x : M^n \to I \times_f P^n$ be an isometrically immersed connected hypersurface in the warped product manifold $I \times_f M^n$ with second fundamental A concerning the normal direction N. We define the **Height function** $h : M^n \to \mathbb{R}$ by setting $h(q) = \langle x(q), \partial_t \rangle$. A direct computation shows that the gradient of h on M^n is

$$\nabla h = \partial_t^\top = \partial_t - \langle N, \partial_t \rangle N,$$

where $(\,\cdot)^\top$ denotes the tangential component of a vector field on $I \times_f P^n$ along M^n and $\langle N, \partial_t \rangle$ is the **Angle function**. Thus, we get

$$|\nabla h|^2 = |\partial_t^\top|^2 = 1 - \langle N, \partial_t \rangle^2, \tag{4.12}$$

where $|\cdot|$ denotes the norm of a vector field on M^n.

As a sub-product of the digression above, we have the p-Laplacian version of the result presented in (ALÍAS L.J. IMPERA; RIGOLI, 2013, Proposition 3).

Lemma 4.7. *Let $x : M^n \to I \times_f P^n$ be an isometric immersion into a warped product manifold. Define*

$$\sigma(h) = \int_{t_0}^{h(\cdot)} f(u)du,$$

where h is the height function. If $q \in M^n$ is a point contained in a domain U and $\nabla h \neq 0$ on U, then

$$\begin{aligned} \Delta_p h = \langle N, \partial_t \rangle \left(nH + (p-2)|\nabla h|^{-2} \langle A(\nabla h), \nabla h \rangle \right) |\nabla h|^{p-2} \\ + \mathcal{H}(h) \left(n + p - 2 - (p-1)|\nabla h|^2 \right) |\nabla h|^{p-2}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \Delta_p \sigma(h) = f(h)^{p-1} \langle N, \partial_t \rangle \left(nH + (p-2)|\nabla h|^{-2} \langle A(\nabla h), \nabla h \rangle \right) |\nabla h|^{p-2} \\ + (n+p-2)\mathcal{H}(h) f(h)^{p-1} |\nabla h|^{p-2}, \end{aligned} \tag{4.14}$$

for all $p \in (1, +\infty)$.

Proof. From Gauss and Weingarten formulas, the Hessian of the height function and the gradient of the angle function satisfy:

$$\begin{aligned}
\overline{\nabla}_X \nabla h &= \overline{\nabla} \left(\partial_t - \langle N, \partial_t \rangle N \right) \\
&= \overline{\nabla}_X \left(\partial_t - \langle N, \partial_t \rangle N \right) \\
&= \mathcal{H}(t)(X - \langle X, \partial_t \rangle \partial_t) - \langle \overline{\nabla}_X N, \partial_t \rangle N - \langle N, \overline{\nabla}_X \partial_t \rangle N - \langle N, \partial_t \rangle \overline{\nabla}_X N \\
&= \mathcal{H}(t)(X - \langle X, \partial_t \rangle \partial_t) + \langle A(X), \partial_t \rangle N - \langle N, \overline{\nabla}_X \partial_t \rangle N + \langle N, \partial_t \rangle A(X) \\
&= \mathcal{H}(t)(X - \langle X, \nabla h \rangle \nabla h) - \mathcal{H}(t) \langle X, \nabla h \rangle \langle \partial_t, N \rangle N + \langle A(\nabla u), X \rangle N \\
&- \langle N, \overline{\nabla}_X \partial_t \rangle N + \langle N, \partial_t \rangle A(X)
\end{aligned}$$

Hence, the tangent component must be equal to

$$\nabla_X \nabla h = \langle N, \partial_t \rangle A(X) + \mathcal{H}(h) \left(X - \langle X, \nabla h \rangle \nabla h \right),$$

and

$$X \langle N, \partial_t \rangle = -\langle A(\nabla h), X \rangle - \mathcal{H}(h) \langle N, \partial_t \rangle \langle \nabla h, X \rangle, \tag{4.15}$$

for every $X \in \mathfrak{X}(M)$. Consequently,

$$\Delta h = n \langle N, \partial_t \rangle H + \mathcal{H}(h)(n - |\nabla h|^2), \tag{4.16}$$

and

$$A_h(h) = |\nabla h|^{-2} \langle N, \partial_t \rangle \langle A(\nabla h), \nabla h \rangle + \mathcal{H}(h) \left(1 - |\nabla h|^2 \right). \tag{4.17}$$

Hence, by inserting (4.16) and (4.17) in the equation (2.14), we get

$$\begin{aligned}
\Delta_p h = \langle N, \partial_t \rangle & \left(nH + (p-2)|\nabla h|^{-2} \langle A(\nabla h), \nabla h \rangle \right) |\nabla h|^{p-2} \\
& + \mathcal{H}(h) \left(n + p - 2 - (p-1)|\nabla h|^2 \right) |\nabla h|^{p-2}.
\end{aligned} \tag{4.18}$$

On the other hand, since $\nabla \sigma(h) = f(h) \nabla h$, we have

$$\nabla_X \nabla \sigma(h)(X) = f(h) \nabla_X \nabla h + f'(h) \langle X, \nabla h \rangle \nabla h.$$

Consequentially,

$$\begin{aligned}
A_{\sigma(h)}(\sigma(h)) &= |\nabla\sigma(h)|^{-2}\langle\nabla_{\nabla\sigma(h)}\nabla\sigma(h),\nabla\sigma(h)\rangle\\
&= |\nabla h|^{-2}\langle\nabla_{\nabla h}\nabla\sigma(h),\nabla h\rangle\\
&= f(h)|\nabla h|^{-2}\langle\nabla_{\nabla h}\nabla h,\nabla h\rangle + |\nabla h|^{-2}\langle\nabla h(f(h))\nabla h,\nabla h\rangle\\
&= f(h)|\nabla h|^{-2}\langle\nabla_{\nabla h}\nabla h,\nabla h\rangle + f'(h)|\nabla h|^{2}\\
&= f(h)\left(A_h(h)+\mathcal{H}(h)|\nabla h|^2\right).
\end{aligned}$$

and

$$\Delta\sigma(h) = f(h)\left(\Delta h + \mathcal{H}(h)|\nabla h|^2\right)$$

Hence

$$\begin{aligned}
\Delta_p\sigma(h) &= |\nabla\sigma(h)|^{p-2}\left(\Delta\sigma(h)+(p-2)A_{\sigma(h)}\sigma(h)\right)\\
&= f(h)^{p-1}|\nabla h|^{p-2}\left(\Delta h+\mathcal{H}(h)|\nabla h|^2+(p-2)\left(A_h(h)+\mathcal{H}(h)|\nabla h|^2\right)\right)\\
&= f(h)^{p-1}\left(\Delta_p h+(p-1)\mathcal{H}(h)|\nabla h|^p\right).
\end{aligned} \tag{4.19}$$

So, by inserting (4.18) in (4.19), we conclude that

$$\begin{aligned}
\Delta_p\sigma(h) = {}& f(h)^{p-1}\langle N,\partial_t\rangle\left(nH+(p-2)|\nabla h|^{-2}\langle A(\nabla h),\nabla h\rangle\right)|\nabla h|^{p-2}\\
&+(n+p-2)\mathcal{H}(h)f(h)^{p-1}|\nabla h|^{p-2},
\end{aligned}$$

for all $p\in(1,+\infty)$. ■

From now on, we will deal with hypersurface contained in a *slab* of $I\times_f P^n$ which means that M^n lies between two leaves $P^n_{t_1}$, $P^n_{t_2}$ with $t_1<t_2$ of the foliation P^n_t; in other words, M^n is contained in a bounded region of the type

$$[t_1,t_2]\times P^n = \{(t,q)\in I\times_f P^n\,;\, t_1\le t\le t_2 \text{ and } q\in P^n\}. \tag{4.20}$$

It is clear that if the M^n is contained in the region (4.20), the height function satisfies: $t_1\le h(q)\le t_2$ for all $q\in M^n$. Using this notation, Alías and Dajczer (ALÍAS; DAJCZER, 2006) investigated complete surfaces properly immersed in a slab of $I\times_f P^n$ under suitable geometric assumptions on the Riemannian fiber P^n. Our first result generalizes this one without any assumption on P^n.

Theorem 4.8. *There exists no complete non-compact immersed hypersurface P^n contained in a slab of $I \times_f P^n$ having $\mu_{1,2}(M) = 0$ and mean curvature satisfying*

$$\sup_M |H| < \min_{[t_1,t_2]} \mathcal{H}(t). \tag{4.21}$$

Proof. Suppose there exists a complete non-compact immersed hypersurface M^n into the slab of $I \times_f P^n$, such that $\mu_{1,2}(M) = 0$ and that the mean curvature satisfy (4.21). By taking $p = 2$ in (4.14),

$$\Delta\sigma(h) = nf(h)\left(\langle N, \partial_t\rangle H + \mathcal{H}(h)\right).$$

From Cauchy-Schwarz' inequality and (4.21),

$$\langle N, \partial_t\rangle H + \mathcal{H}(h) \geq -\sup_M |H| + \inf_M \mathcal{H}(h) > 0.$$

Hence

$$\Delta\sigma(h) \geq n \inf_M f(h)\left(-\sup_M |H| + \inf_M \mathcal{H}(h)\right) > 0.$$

Thus, $\sigma(h)$ is a strongly 2-subharmonic function on M^n. Therefore, from Theorem 4.5, $\sigma(h)$ must be unbounded, which contradicts the fact that M^n is contained in a slab. ■

As application of Corollary 4.6, we get the following consequence of Theorem 4.8:

Corollary 4.9. *There exists no complete non-compact immersed hypersurface M^n contained in a slab of $I \times_f P^n$ with polynomial volume growth and mean curvature satisfying*

$$\sup_M |H| < \min_{[t_1,t_2]} \mathcal{H}(t).$$

Now, by assuming that the warped product manifold $I \times_f M^n$ satisfies $\mathcal{H}(t) \geq 1$ for all $t \in I$, we obtain

Theorem 4.10. *There exists no complete non-compact immersed hypersurface M^n contained in a slab of $I \times_f P^n$ having $\mu_{1,2}(M) = 0$ and mean curvature satisfying* $\sup_M H < 1$.

Proof. Let us suppose, by contradiction, the existence of such a hypersurface. From (4.12) we can easily see that $n - |\nabla h|^2 > 0$. Then, (4.16) can be write as follows,

$$\Delta h \geq n\langle N, \partial_t\rangle H + n - |\nabla h|^2,$$

provided that $\mathcal{H} \geq 1$.

By using ε-Young's inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$, $\varepsilon > 0$, for

$$a = \sqrt{n}|\langle N, \partial_t\rangle| \quad \text{and} \quad b = \sqrt{n}|H|$$

we have

$$\begin{aligned}\Delta h &\geq -n|\langle N, \partial_t\rangle||H| + n - |\nabla h|^2 \\ &\geq \frac{n}{2}\left(2 - \varepsilon + \left(\varepsilon - \frac{2}{n}\right)|\nabla h|^2 - \frac{1}{\varepsilon}H^2\right).\end{aligned}$$

By considering $\varepsilon = 1$, we get

$$\Delta h \geq \frac{n}{2}\left(1 + \frac{n-2}{n}|\nabla h|^2 - H^2\right) \geq \frac{n}{2}\left(1 - \sup_M H^2\right) > 0. \tag{4.22}$$

Which means that h is strongly 2-subharmonic. Therefore, we are in a position to apply Theorem 4.5 to infer that h is unbounded, a contradiction. ■

Corollary 4.11. *There exists no complete non-compact immersed hypersurface M^n contained in a slab of $I \times_f P^n$ having polynomial volume growth and mean curvature satisfying* $\sup_M H < 1$.

Following (TASHIRO, 1965), when the warped function f is the exponential, the product warped manifold $\mathbb{R} \times_{e^t} P^n$ belongs to a class of manifolds known as *pseudo-hyperbolic space.* This terminology comes from the observation that the hyperbolic space, $\mathbb{H}^{n+1}$, can be described as a warped product, $\mathbb{R} \times_{e^t} \mathbb{R}^n$. In this structure, the slices represent the set of all horospheres that share the same fixed point on the asymptotic boundary, $\partial\mathbb{H}^{n+1}$. Together, these horospheres provide a complete foliation of $\mathbb{H}^{n+1}$ (see also (ALÍAS; DAJCZER, 2006; MONTIEL, 1999)). In this setting, we have the following extension of (ALÍAS; DAJCZER, 2006, Corollary 3).

Corollary 4.12. *There exists no complete non-compact immersed hypersurface M^n contained in a slab of pseudo-hyperbolic manifold $\mathbb{R} \times_{e^t} P^n$ having $\mu_{1,2}(M) = 0$ and mean curvature satisfying $\sup_M H < 1$.*

For the next result, we will deal with *helix-type* hypersurfaces or having *constant angle.* We say that a hypersurface M^n of $I \times_f P^n$ is a helix-type hypersurface if the angle function $\langle N, \partial_t \rangle$ is constant on M^n. In this setting, by considering that the warped product manifold $I \times_f P^n$ satisfies $\mathcal{H}(t) \geq 1$ for all $t \in I$, we obtain.

Theorem 4.13. *Let M^n be a complete non-compact helix-type hypersurface contained in a slab of $I \times_f P^n$ with $\mu_{1,p}(M) = 0$, $p > 2$. If the mean curvature (not necessarily constant) satisfies $H^2 \leq 1$, then M^n is a slice.*

Proof. We will assume for contradiction, that M^n is not a slice of $I \times_f P^n$ Since M^n is a helix-type hypersurface which is not a slice, we must have $\langle N, \partial_t \rangle \neq \pm 1$, and hence $|\nabla h| = const. > 0$. From (4.12) and (4.15), we have

$$|\nabla h|^{-2} \langle N, \partial_t \rangle \langle A(\nabla h), \nabla h \rangle = -\mathcal{H}(h) \langle N, \partial_t \rangle^2 = -\mathcal{H}(h)(1 - |\nabla h|^2).$$

By replacing this in (4.13),

$$\Delta_p h = \big(n \langle N, \partial_t \rangle H + \mathcal{H}(h)(n - |\nabla h|^2) \big) |\nabla h|^{p-2},$$

that is,

$$\Delta_p h = |\nabla h|^{p-2} \Delta h.$$

Hence, from (4.22),

$$\Delta_p h \geq \left(\frac{n-2}{2} \right) |\nabla h|^p = const. > 0.$$

Therefore, h is a strongly p-subharmonic function. Since $\mu_{1,p}(M) = 0$, it follows from Theorem 4.5 that h is unbounded, a contradiction. ∎

Corollary 4.14. *Let M^n be a complete non-compact helix-type hypersurface contained in a slab of $I \times_f P^n$ with polynomial volume growth. If the mean curvature (not necessarily constant) satisfies $H^2 \leq 1$, then M^n is a slice.*

In the direction of Corollary 4.11, we close this thesis with the following extension of (ALÍAS; DAJCZER, 2006, Theorem 4) and (AQUINO; LIMA, 2014, Theorem 3.3).

Corollary 4.15. *Let M^n be a complete non-compact helix-type hypersurface contained in a slab of pseudo-hyperbolic manifold $\mathbb{R} \times_{e^t} P^n$ with $\mu_{1,p}(M) = 0$. If the mean curvature (not necessarily constant) satisfies $H^2 \leq 1$, then M^n is a slice.*

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APPENDIX A. AN ε-APPROXIMATION APPROACH

To avoid possible issues with points x in M^n such that $\nabla u = 0$ (which exists by the compactness of M^n in the first part of the thesis), we make a ε-approximation argument (KOTSCHWAR; NI, 2009). Another problem that this technique fixes is that we calculate second derivatives along most calculations, and we cannot expect a good regularity of eigenfunctions of the p-Laplacian due to the singularities. To complete the work, we recall the following definition:

Definition A. 1. *The Hölder space $C^{k,\alpha}(M)$ is defined for $0 < \alpha < 1$ as the set of $u \in C^k(M)$ such that the norm*

$$||u||_{C^{k,\alpha}} := ||u||_{C^k} + \sup_{p_1,p_2} \frac{|\nabla^k u(p_1) - \nabla^k u(p_2)|}{d_g(p_1,p_2)^\alpha}$$

is finte, where d_g is the distance function defined in M^n.

The regularity of the eigenfunctions of the p-Laplacian in general cannot be any better than $C^{1,\alpha}(M)$ (LÊ, 2006, Theorem 4.5). The main idea to solve this is consider the perturbation by a positive constant $|\nabla u(\cdot)|^2 + \varepsilon$, repeat the same arguments as used in the proof of theorem 3.6 (and its generalizations) to the modified function, and, at the end, take $\varepsilon \to 0^+$.

Suppose that λ is an eigenvalue of the p-Laplacian operator. Let $W_\epsilon = |\nabla u_\varepsilon|^2 + \varepsilon$ for $\epsilon > 0$ where u_ε is the solution of the problem

$$\Delta_{p,\epsilon} u_\varepsilon := -\mathrm{div}(W^{\frac{p-2}{2}} \nabla u_\varepsilon) = \lambda |u_\epsilon|^{p-2} u_\varepsilon. \tag{23}$$

Note that, when $\epsilon \to 0$, u_ϵ must coincide with the eigenfunction of the p-Laplacian associated to λ.

To ensure the regularity, we recall a theorem due to Ural'tseva (1960)

Theorem A.1. *Let F be a solution of the following variational problem:*

$$I(u) = \int_\Omega F(x, u, \nabla u) d\Omega.$$

If F has the first derivative bounded (in the Lipschitz sense) and F satisfies

$$F_{u_iu_j}\xi_i\xi_j \geq M|\xi|^2,$$

for some $M > 0$ and for all $\xi \in \mathbb{R}^n$, then $u \in C^{2,\alpha}(\Omega)$. If we assume that $u(x) = \varphi(x)$ for all $x \in \partial\Omega$, $\varphi \in C^{2,\alpha}(\partial\Omega)$ and the boundary is at least $C^{2,\alpha}$, then $u \in C^{2,\alpha}(\overline{\Omega})$.

The notation $F_{u_iu_j}$ means the derivatives of F with respect to $\langle\nabla u, e_i\rangle$. In other words, if we rewrite the variational problem as a second-order quasilinear PDE of the form

$$a_{ij}(x, u, \nabla u)\langle \mathrm{Hess}u(e_i), e_j\rangle + b(x, u, \nabla u) = 0,$$

then $F_{u_iu_j} = a_{ij}$. For the regularized p-Laplacian, we have the following

$$F(u) = (\varepsilon + |\nabla u|^2)^{p/2} - \lambda|u|^p.$$

Hence, by direct computation,

$$a_{ij} = (\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}}\delta_{ij} + (p-2)(\varepsilon + |\nabla u|^2)^{\frac{p-4}{2}}\langle\nabla u, e_i\rangle\langle\nabla u, e_j\rangle.$$

Hence, for any ξ,

$$a_{ij}\langle\xi, e_i\rangle\langle\xi, e_j\rangle \geq (\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}}|\xi|^2$$

If $p \geq 2$, then $a_{ij} \geq \varepsilon^{\frac{p-2}{2}}|\xi|^2$. Otherwise,

$$\begin{aligned} a_{ij}\langle\xi, e_i\rangle\langle\xi, e_j\rangle &\geq (\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}}|\xi|^2 + (p-2)(\varepsilon + |\nabla u|^2)^{\frac{p-4}{2}}|\nabla u|^2|\xi|^2 \\ &\geq (1 + (p-2))(\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}}|\xi|^2. \end{aligned}$$

Taking the maximum of $|\nabla u|$ on $\overline{\Omega}$, we get the desired inequality. The other hypotheses are guaranteed by the smoothness of our submanifold. For the Neumann boundary condition, see Ladyzhenskaya and Ural'tseva (1961).

Theorem A.2. *Let M^n be a closed minimal submanifold in $\mathbb{S}^m$ and let u_ε be a solution* *of the problem* (23)*. Then*

$$0 \leq \alpha_{n,p} \lambda^{2/p} \int_M \left(W_\varepsilon^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 \right)^{\frac{2p-2}{p}} dM - \int_M (n-S) W_\varepsilon^{\frac{2p-4}{2}} |\nabla u_\varepsilon|^2 dM$$

where $\alpha_{n,p}$ *is defined in* (1.2).

Proof. Note that

$$\frac{2}{p} \nabla W_\varepsilon^{\frac{p}{2}} = W_\varepsilon^{\frac{p-2}{2}} \nabla W_\varepsilon = \left(|\nabla u_\varepsilon|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla |\nabla u_\varepsilon|^2.$$

Hence, if we denote $\Delta_{\infty,\varepsilon} u = W_\varepsilon^{-1} \langle \mathrm{Hess}(u_\varepsilon)(\nabla u_\varepsilon), \nabla u_\varepsilon \rangle$, then

$$\begin{aligned} \frac{1}{p} L_u \left(W_\varepsilon^{\frac{p}{2}} \right) =& W_\varepsilon^{\frac{p-2}{2}} \Bigg(\langle \nabla u_\epsilon, \nabla \Delta_{p,\epsilon} u_\epsilon \rangle \\ & - (p-2) \left(\Delta_\infty u_\epsilon \Delta_{p,\epsilon} u_\epsilon + 2 W_\varepsilon^{\frac{p-4}{2}} |\mathrm{Hess}(u_\epsilon)(\nabla u_\epsilon)|^2 \right) \Bigg) dM \\ & + W_\varepsilon^{\frac{2p-4}{2}} \left((p-2) \langle \nabla u_\epsilon, \nabla \Delta_{\infty,\varepsilon} u_\epsilon \rangle - |\mathrm{Hess}(u_\epsilon)|^2 - \mathrm{Ric}(\nabla u_\epsilon, \nabla u_\epsilon) \right) dM \end{aligned}$$

Repeating the integration part arguments, we have that

$$\begin{aligned} i)\ \ 0 =& - \int_M \mathrm{div} \left(\Delta_{p,\varepsilon} u_\varepsilon W_\varepsilon^{\frac{p-2}{2}} \nabla u_\epsilon \right) dM = \int_M (\Delta_{p,\varepsilon} u_\varepsilon)^2 dM - \int_M \langle \nabla u_\varepsilon, \nabla \Delta_{p,\varepsilon} u_\varepsilon \rangle dM \\ ii)\ 0 =& - \int_M \mathrm{div} \left(W_\varepsilon^{\frac{p-2}{2}} \Delta_{\infty,\varepsilon} u_\varepsilon W_\varepsilon^{\frac{p-2}{2}} \nabla u_\epsilon \right) dM = \int_M W_\varepsilon^{\frac{p-2}{2}} \Delta_{\infty,\varepsilon} u_\varepsilon \Delta_{p,\varepsilon} u_\varepsilon dM \\ & - \int_M W_\varepsilon^{\frac{2p-4}{2}} \langle \nabla u_\varepsilon, \nabla \Delta_{\infty,\varepsilon} u_\varepsilon \rangle dM - (p-2) \int_M W_\varepsilon^{\frac{2p-6}{2}} \left(\Delta_{\infty,\varepsilon} u_\varepsilon \right)^2 dM \end{aligned}$$

Hence, by the Cauchy-Schwarz inequality,

$$0 \leq \int_M \lambda^2 |u_\varepsilon|^{2p-2} dM - \int_M W_\varepsilon^{\frac{2p-4}{2}} |\mathrm{Hess}(u_\varepsilon)|^2 dM - \int_M W_\varepsilon^{\frac{2p-4}{2}} \mathrm{Ric}(\nabla u_\varepsilon, \nabla u_\varepsilon) dM$$

We also define the approximation tensor T_ε as $T_\varepsilon(X) = \nabla_X (W_\varepsilon^{\frac{p-2}{2}} \nabla u_\varepsilon)$. Note that $\Delta_{p,\varepsilon} u_\varepsilon$ for $X \in \mathfrak{X}(M)$. Note that $\mathrm{tr}(T_\varepsilon) = -\Delta_{p,\varepsilon} u_\varepsilon$ and

$$|T_\varepsilon(X)|^2 = W_\varepsilon^{\frac{2p-4}{2}} \langle \nabla_X \nabla u_\varepsilon, \nabla_X \nabla u_\varepsilon \rangle + p(p-2) W_\varepsilon^{\frac{2p-6}{2}} \langle \nabla_X \nabla u_\varepsilon, \nabla u_\varepsilon \rangle \langle X, \nabla_{\nabla u_\varepsilon} \nabla u_\varepsilon \rangle$$

Hence,

$$n(p-1)^2 W_\varepsilon^{\frac{2p-4}{2}} |\mathrm{Hess}(u_\varepsilon)|^2 \geq (\Delta_{p,\varepsilon} u_\varepsilon)^2$$

By direct computation,

$$0 \leq \left(1 - \frac{1}{n(p-1)^2}\right) \int_M \lambda^2 |u_\varepsilon|^{2p-2} dM - \frac{n-1}{n} \int_M (n-S) W_\varepsilon^{\frac{2p-4}{2}} |\nabla u_\varepsilon|^2 dM$$

Moreover, by the divergence theorem,

$$\begin{aligned} 0 &= -\int_M \mathrm{div}\left(\lambda |u_\varepsilon|^{p-2} u_\varepsilon W_\varepsilon^{\frac{p-2}{2}} \nabla u_\varepsilon\right) dM \\ &= \int_M \lambda |u_\varepsilon|^{2p-2} dM - (p-1) \int_M W_\varepsilon^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 |u_\varepsilon|^{p-2} dM \end{aligned}$$

By Hölder's inequality,

$$\frac{\lambda}{p-1} \int_M |u_\varepsilon|^{2p-2} dM \leq \left(\int_M \left(W_\varepsilon^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2\right)^{\frac{2p-2}{p}} dM\right)^{\frac{p}{2p-2}} \left(\int_M |u_\varepsilon|^{2p-2} dM\right)^{\frac{p-2}{2p-2}}$$

Hence,

$$0 \leq \alpha_{n,p} \lambda^{2/p} \int_M \left(W_\varepsilon^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2\right)^{\frac{2p-2}{p}} dM - \int_M (n-S) W_\varepsilon^{\frac{2p-4}{2}} |\nabla u_\varepsilon|^2 dM$$

Taking $\varepsilon \to 0$ we obtain the desired inequality. ■