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Magneto-Micropolar Equations: Decay Characterization of Solutions for the Homogeneous Case 2D and the Inviscid and Non-Resistive Limit Problem for the Nonhomogeneous Case 3D

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Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática

Área de Concentração: Análise

Orientador: Felipe Wergete Cruz

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RESUMO

Estudamos o fluxo magneto-micropolar tanto no caso de densidade constante (problema homogêneo), quanto no caso de densidade variável (problema não homogêneo). De fato, inicialmente caracterizamos as taxas de decaimento das soluções para o sistema magneto-micropolar homogêneo 2D em termos do caráter de decaimento dos dados iniciais. Além disso, obtivemos uma taxa de decaimento mais rápida para a velocidade micro-rotacional e estudamos o comportamento das soluções para tempo grande comparando-as com as soluções da parte linear. Por fim, no caso 3D, estabelecemos a convergência uniforme da solução do problema viscoso e resistivo com densidade variável para a solução do problema não viscoso e não resistivo, quando as viscosidades e a resistividade tendem a zero.

Palavras-chaves: Taxas de decaimento. Comportamento assintótico. Fluidos magneto-micropolares; limite invíscido e não resistivo. Fluxo magneto-micropolar não homogêneo.

ABSTRACT

We studied the magneto-micropolar flow in both the case of constant density (homogeneous problem) and variable density (nonhomogeneous problem). In fact, we initially characterized the decay rates of solutions for the 2D homogeneous magneto-micropolar system in terms of the decay character of the initial data. Furthermore, we obtained a faster decay rate for the microrotational velocity and studied the behavior of the solutions for large time by comparing them with solutions from the linear part. Finally, in the 3D case, we established the uniform convergence of the solution for the viscous and resistive problem with variable density to the solution of the inviscid and non-resistive problem when viscosities and resistivity tend to zero.

Keywords: Decay rates; asymptotic behavior; magneto-micropolar fluids; inviscid and non-resistive limit; nonhomogeneous magneto-micropolar flow.

SUMÁRIO

1 INTRODUCTION

The theory of micropolar fluids (also known as asymmetric fluids) was first studied by the Turkish-American engineer Ahmed C. Eringen in 1966 in his work titled "Theory of Micropolar Fluids". The governing equations for micropolar fluids extend the classical Navier-Stokes equations to include terms related to micro-rotational effects. These equations provide a more detailed description of fluid behavior, especially in situations where the microstructure is important.

Micropolar fluid theory has been applied in various scientific and engineering contexts, including the study of liquid crystals, polymer melts, ferrofluids, and other complex fluids where microstructure plays a crucial role. It allows researchers to better understand the intricate behavior of fluids in situations where classical fluid dynamics might not provide an accurate representation.

The magneto-micropolar (also known by MHD micropolar) fluids refers to a specialized class of fluids that combines the characteristics of magneto-fluids (fluids influenced by magnetic fields) and micropolar fluids. This combination suggests the consideration of both magnetic field interactions and microstructure effects in the description of fluid motion. If we consider the effect of the induced magnetic field on the fluid motion, we obtain the system of equations known as magneto-micropolar fluids, namely:

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\Big(\mathit{\Pi} + \dfrac{|\boldsymbol{b}|^2}{2}\Big) = (\mu+\chi)\Delta\boldsymbol{u} + \chi\nabla\times\boldsymbol{w} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} = \gamma\Delta\boldsymbol{w} + \kappa\nabla(\operatorname{div}\ \boldsymbol{w}) + \chi\nabla\times\boldsymbol{u} - 2\chi\boldsymbol{w}, \\ \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} = \nu\Delta\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \\ \operatorname{div}\ \boldsymbol{u} = 0, \quad \operatorname{div}\ \boldsymbol{b} = 0. \end{cases} \tag{1.1}$$

These equations describe the time evolution of the linear velocity $\mathbf{u}(x,t)\in\mathbb{R}^3$, of the hydrostatic pressure $\mathit{\Pi}(x,t)\in\mathbb{R}$, of the micro-rotational velocity $\mathbf{w}(x,t)\in\mathbb{R}^3$, and of the magnetic field $\mathbf{b}(x,t)\in\mathbb{R}^3$ of a electrically conducting micropolar fluid in the presence of a magnetic field. The function $|\mathbf{b}|^2/2$ is the magnetic pressure. In this way, we denote by $p \overset{\text{def}}{=} \mathit{\Pi} + |\mathbf{b}|^2/2$ the *total pressure* of the fluid. The positive constants μ, χ, γ, κ and ν are associated with specific properties of the fluid, where μ is the kinematic viscosity, χ is the vortex viscosity, γ and κ are spin viscosities, and ν is the resistivity constant acting as the magnetic diffusion coefficient of the magnetic field ($1/\nu$ is the magnetic Reynolds number). Note that the system

(1.1) reduces to the incompressible Navier–Stokes equations, when $\mathbf{b} \equiv \mathbf{w} \equiv \mathbf{0}$ and $\chi \equiv 0$; to the MHD system, when $\mathbf{w} \equiv \mathbf{0}$ and $\chi \equiv 0$; and to the micropolar system, when $\mathbf{b} \equiv \mathbf{0}$.

We supplement system (1.1) with given initial conditions $\mathbf{u}_0, \mathbf{b}_0$ and $\mathbf{w}_0$, i.e,

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{w}(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x}), \quad \mathbf{b}(\mathbf{x},0) = \mathbf{b}_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3.$$

The initial data for velocity and magnetic field satisfy $\operatorname{div}\ \mathbf{u}_0 = \operatorname{div}\ \mathbf{b}_0 = 0$ in distributional sense.

The symbols $\nabla, \Delta, \nabla\times$ and div denote, respectively, the gradient, Laplacian, curl and divergence operators; $\mathbf{u}_t, \mathbf{w}_t$ e $\mathbf{b}_t$ are the time derivatives of $\mathbf{u}, \mathbf{w}$ and $\mathbf{b}$. The i-th component of $(\mathbf{u}\cdot\nabla)\mathbf{v}$ and $(\mathbf{b}\cdot\nabla)\mathbf{v}$, respectively, are given by

$$[(\mathbf{u}\cdot\nabla)\mathbf{v}]_i = \sum_{j=1}^{n} u_j \frac{\partial v_i}{\partial x_j} \quad \text{and} \quad [(\mathbf{b}\cdot\nabla)\mathbf{v}]_i = \sum_{j=1}^{n} b_j \frac{\partial v_i}{\partial x_j}.$$

There are numerous results on the existence and uniqueness of solutions to the problem related to micropolar fluids (see, for example, (BOLDRINI; DURÁN; ROJAS-MEDAR, 2010), (ROJAS-MEDAR; BOLDRINI, 1998), (CHEN; MIAO, 2012), (ERINGEN, 1966), (GALDI, 2011), (LUKASZEWICZ, 2012), (ORTEGA-TORRES; ROJAS-MEDAR, 1999), (SERMANGE; TEMAM, 1983), (YUAN, 2008)). More precisely, in 1977, G.P. Galdi and S. Rionero (GALDI, 2011) demonstrated the existence and uniqueness of weak solutions for the initial value problem of the micropolar system. In 1997, M.A. Rojas-Medar showed the local existence and uniqueness of classical solutions. J.L. Boldrini and M.A. Rojas-Medar, in 1998, studied the existence of weak solutions in (ROJAS-MEDAR; BOLDRINI, 1998) for two and three spatial dimensions. In particular, they showed (in 2D) the uniqueness of such solutions. E.E. Ortega-Torres and M.A. Rojas-Medar in 1999, assuming small initial data, proved the global existence of a classical solution (see (ORTEGA-TORRES; ROJAS-MEDAR, 1999)). The results of these last three works were obtained through a spectral Galerkin method. In 2010, J.L. Boldrini, M. Durán, and M.A. Rojas-Medar (BOLDRINI; DURÁN; ROJAS-MEDAR, 2010) proved the existence and uniqueness of classical solutions in $L^p(\Omega)$, for $p > 3$. For initial data in $L^1 \cap L^2$ and with some additional hypotheses, there are some works where decay results for weak solutions are obtained via the (classical) Fourier Splitting techniques (see, e.g., (LI; SHANG, 2018)).

A weak solution (or *Leray–Hopf solution*) to the system (1.1) is any triple $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t) \in C_w([0,\infty), \boldsymbol{L}^2(\mathbb{R}^3)) \cap\ L^2_{loc}([0,\infty), \boldsymbol{H}^1(\mathbb{R}^3))$ with $(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,0) = (\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0)$ and which satisfies the equations weakly in $\mathbb{R}^3\times(0,\infty)$ and in addition is such that, for $s = 0$ and almost every $s > 0$, one has

$$
\begin{aligned}
&\| (\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^2 \\
&\quad + \; 2\int_s^t \Big\{ \mu \| \nabla \mathbf{u}(\cdot, \tau) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^2 + \gamma \| \nabla \mathbf{w}(\cdot, \tau) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^2 + \nu \| \nabla \mathbf{b}(\cdot, \tau) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^2 \Big\} \mathrm{d}\tau \\
&\quad + \; 2\chi \int_s^t \| \mathbf{w}(\cdot, \tau) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^2 \mathrm{d}\tau \; + \; 2\kappa \int_s^t \| \operatorname{div} \mathbf{w}(\cdot, \tau) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^2 \mathrm{d}\tau \; \leq \; \| (\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, s) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^2
\end{aligned} \tag{E}
$$

for all $t > s$. While the existence of Leray–Hopf solutions for arbitrary data $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)$ as above is well known, their uniqueness and exact regularity properties remain an open problem, except in the case of suitably small data, for instance if (see e.g. (SILVA; ZINGANO; ZINGANO, 2019; CRUZ; NOVAIS, 2020; LI; SHANG, 2018; ROJAS-MEDAR, 1997; SERMANGE; TEMAM, 1983; YUAN, 2008))

$$\| (\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^{\frac{1}{2}} \| (\nabla \mathbf{u}_0, \nabla \mathbf{w}_0, \nabla \mathbf{b}_0) \|_{\boldsymbol{L}^2(\mathbb{R}^3)}^{\frac{1}{2}} \leq \min\{\mu, \gamma, \nu\}.$$

In particular, it follows from the *strong* energy inequality (E) that Leray–Hopf solutions to (1.1) are smooth for $t \gg 1$. More precisely: given an arbitrary solution $(\mathbf{u}, \mathbf{w}, \mathbf{b})$, there exists regularity time $t_* \geq 0$ such that $(\mathbf{u}, \mathbf{w}, \mathbf{b}) \in C^\infty(\mathbb{R}^3 \times (t_*, \infty))$ and

$$(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t) \in L^\infty_{\text{loc}}((t_*, \infty), \boldsymbol{H}^m(\mathbb{R}^3)) \qquad \forall \; m \in \mathbb{N} \cup \{0\}. \tag{S3}$$

For initial datas $\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0 \in \boldsymbol{L}^2(\mathbb{R}^n)$, $n = 2$ or 3, Guterres and its collaborators (GUTERRES; NUNES; PERUSATO, 2018) proved that the norm of Leray–Hopf solutions tends to zero, i.e.,

$$\lim_{t \to \infty} \|(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)\|_{\boldsymbol{L}^2} = 0.$$

Furthermore, they obtained a sharper result for the micro-rotational velocity $\mathbf{w}$, namely

$$\lim_{t \to \infty} t^{\frac{1}{2}} \|\mathbf{w}(\cdot, t)\|_{\boldsymbol{L}^2} = 0.$$

In turn, when the initial datas $\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0$ is in $\boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$, Cruz and Novais (CRUZ; NOVAIS, 2020) (see also (LI; SHANG, 2018)) used the classical Fourier splitting technique to prove that the decay has algebraic rate, i.e.,

$$\|(\mathbf{u}, \mathbf{w}, \mathbf{b})(\cdot, t)\|_{\boldsymbol{L}^2} \leq C(t+1)^{-\frac{3}{4}}, \quad \forall \, t \geq 0.$$

Moreover, they proved that the decay rate for the micro-rotational field can be improved to

$$\|\mathbf{w}(\cdot, t)\|_{\boldsymbol{L}^2} \leq C(t+1)^{-\frac{5}{4}}, \quad \forall \, t \geq 0.$$

In (CRUZ et al., 2022), the authors proved that $\left\| (D^m\,\mathbf{u},\ D^m\,\mathbf{w},\ D^m\,\mathbf{b})(\cdot,t) \right\|_{\boldsymbol{L}^2(\mathbb{R}^n)} \leq C\,(t+1)^{-\frac{m}{2}-\frac{n}{4}}$ for $t>0$ sufficiently large (with $m \in \mathbb{N}\cup\{0\}$ and $n=2$ or 3). In addition, they also proved a faster decay rate for the micro-rotation, namely $\|D^m\,\mathbf{w}(\cdot,t)\|_{\boldsymbol{L}^2(\mathbb{R}^n)} \leq C\,(t+1)^{-\frac{m}{2}-\frac{n}{4}-\frac{1}{2}}$ for all $t \gg 1$. It was also shown that $\|(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t)-(\overline{\mathbf{u}},\overline{\mathbf{w}},\overline{\mathbf{b}})(\cdot,t)\|_{\boldsymbol{L}^2(\mathbb{R}^n)} \leq C\,(t+1)^{-\frac{n}{4}-\frac{1}{2}}$ and $\|\mathbf{w}(\cdot,t)-\overline{\mathbf{w}}(\cdot,t)\|_{\boldsymbol{L}^2(\mathbb{R}^n)} \leq C\,(t+1)^{-\frac{n}{4}-1}$ for every $t \geq 0$, where $(\overline{\mathbf{u}},\overline{\mathbf{w}},\overline{\mathbf{b}})$ is the solution to the related linear system with the same initial data. Finally, they also presented decay estimates for the total pressure of the fluid and space-time derivatives bounds.

Recently, Niche and Perusato (NICHE; PERUSATO, 2022) characterized the decay rate of solutions to the 3D MHD micropolar equations in terms of the decay character of the initial data and studied the long time behavior of its solutions by comparing them to solutions to the linear part.

In the following, we will describe how this work is structured.

In Chapter 1, we briefly present the functional spaces suitable for the mathematical analysis of the system (1.1), as well the most common results used throughout the text.

In Chapter 2, we prove a decay estimate for the two-dimensional MHD micropolar system for any initial data in $\boldsymbol{L}^2(\mathbb{R}^2)$. To achieve this, we associate to initial data $\mathbf{z}_0 \stackrel{\text{def}}{=} (\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0)$ in $\boldsymbol{L}^2(\mathbb{R}^2)$ a decay character $r^* = r^*(\mathbf{z}_0)$, and then use the "Fourier splitting method" (also called Schonbek's method) and some estimates for the linear part to characterize the decay of MHD micropolar equations.

In Chapter 3, we will study the inviscid and non-resistive limit for the solution to the nonhomogeneous magneto-micropolar with density variable flow in $\mathbb{R}^3 \times [0,T]$, $T>0$, given by:

$$\begin{cases} \rho\boldsymbol{u}_t + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = (\mu+\chi)\Delta\boldsymbol{u} + \chi\nabla\times\boldsymbol{w} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \rho\boldsymbol{f}, \\ \rho\boldsymbol{w}_t + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{w} = \gamma\Delta\boldsymbol{w} + \kappa\nabla(\operatorname{div}\ \boldsymbol{w}) + \chi\nabla\times\boldsymbol{u} - 2\chi\boldsymbol{w} + \rho\boldsymbol{g}, \\ \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} = \nu\Delta\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \\ \operatorname{div}\ \boldsymbol{u} = \operatorname{div}\ \boldsymbol{b} = 0, \\ \rho_t + \boldsymbol{u}\cdot\nabla\rho = 0, \end{cases} \tag{1.2}$$

with initial conditions

$$\begin{cases} \rho(\mathbf{x},0) = \rho_0(\mathbf{x}), \\ \boldsymbol{u}(\mathbf{x},0) = \boldsymbol{u}_0(\mathbf{x}), \quad \boldsymbol{w}(\mathbf{x},0) = \boldsymbol{w}_0(\mathbf{x}), \quad \boldsymbol{b}(\mathbf{x},0) = \boldsymbol{b}_0(\mathbf{x}). \end{cases} \tag{1.3}$$

We will establish the uniform convergence of the solution of this system to the solution of an ideal non-homogeneous asymmetric flow, i.e., when viscosities and resistivity tend to zero.

Finally, in Appendices A and B, we prove some estimates for the nonlinear term of the system's solution 2D MHD micropolar and for the matrix of symbols for the linear part used in Chapter 2.

2 PRELIMINARY CONSIDERATIONS

This chapter addresses the preliminary considerations essential for a comprehensive understanding of the subjets discussed in this thesis. It provides an overview of the main concepts, definitions, and contexts that sustain the presented research.

2.1 FUNCTIONAL SPACES

In results that it follows, we will always denote by Ω a domain (not necessarily bounded) in $\mathbb{R}^n$. Functions with values in $\mathbb{R}^n$, as their spaces and points, are represented with bold letters.

Definition 2.1 *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots \alpha_n)$ be a standard multi-index. We denote by D^α the differential operator of order*

$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

i.e.,

$$D^{\boldsymbol{\alpha}} = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}.$$

Definition 2.2 *Let $\phi : \Omega \to \mathbb{R}$ be a continuos function. The support of ϕ is the set*

$$\textit{supp } \phi = \overline{\{\boldsymbol{x} \in \Omega;\ \phi(\boldsymbol{x}) \neq 0\}}.$$

Definition 2.3 *Let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω.*

Definition 2.4 *A sequence $\{\phi_m\}_{m=1}^\infty \subset C_0^\infty(\Omega)$ tends to zero with respect to the topology τ if the following conditions are satisfied:*

i. There exists a compact set K such that supp $\phi_m \subset K$, for all $m \in \mathbb{N}$.

ii. For each multi-index $\alpha \in \mathbb{N}^n$ it holds $\sup_{x \in K} |D^\alpha \phi_m| \to 0$ as $m \to \infty$.

Remark 2.1 *Let $\phi \in C_0^\infty(\Omega)$, we say that the sequence $\{\phi_m\}_{m=1}^\infty$ tends to ϕ in $C_0^\infty(\Omega)$, with respect to the topology τ, when the sequence $\{\phi_m - \phi\}_{m=1}^\infty$ tends to zero in the sense of the previous definition.*

Definition 2.5 *The vector space $C_0^\infty(\Omega)$ equipped with the topology τ is denoted by $\mathcal{D}(\Omega)$ and called the space of test function on Ω.*

Definition 2.6 *A distribution $T : \mathcal{D}(\Omega) \to \mathbb{R}$ is a linear functional which is continuous with respect to the topology of $\mathcal{D}(\Omega)$.*

2.1.1 L^p spaces

Throughout this section $(\Omega, \mathcal{M}, \mu)$ denotes a σ-finite measure space, i.e., Ω is a σ-finite set, $\mathcal{M}$ is a σ-algebra of measurable sets, and μ is a measure.

Definition 2.7 *Let $p \in \mathbb{R}$ with $1 \le p < \infty$; we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \ ; \ f \text{ is mesurable and } \int_\Omega |f(x)|^p \, d\mu < \infty \right\}$$

with norm

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu \right)^{\frac{1}{p}}.$$

Remark 2.2 *The space $L^2(\Omega)$ equipped with the scalar product*

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x) \, d\mu$$

is a Hilbert space, i.e., L^2 is complet for the norm $\| \cdot \|_2$.

Definition 2.8 *Let $f : X \to \mathbb{R}$ be a measurable function on a measure space $(\Omega, \mathcal{M}, \mu)$. The essential supremum of f on Ω is defined by*

$$\text{ess sup }_\Omega f = \inf\{a \in \mathbb{R}; \ \mu\{x \in \Omega; \ f(x) > a\} = 0\}.$$

The function f is said to be essentially bounded on Ω if

$$\text{ess sup }_\Omega |f| < \infty.$$

Definition 2.9 *We set*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R}; \ f \text{ is measurable and essentially bounded on } \Omega\}$$

with norm

$$\|f\|_{L^\infty(\Omega)} = \|f\|_\infty = \text{ess sup }_\Omega |f|.$$

Theorem 2.1 (Hölder's inequality) *Let $1 \le p \le \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $p^{-1} + q^{-1} = 1$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

Theorem 2.2 (Young's inequality with ϵ) *Let a and b be nonnegative real numbers. If $p, q \in \mathbb{R}$ are such that $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then for all $\epsilon > 0$, we have that*

$$ab \le \epsilon a^p + C(\epsilon) b^q, \text{ where } C(\epsilon) = \frac{p-1}{p^q}\epsilon^{1-q}.$$

Definition 2.10 *Let $1 \le p \le \infty$. We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $f : K \to \mathbb{R}$ belongs to $L^p(\Omega)$ for every compact set K contained in Ω.*

2.1.2 Sobolev spaces

Let $1 < p < \infty$ and let k be a nonnegative integer. Now we define certain function spaces, whose members have weak derivatives of various orders lying in various L^p spaces.

We will consider the usual Sobolev spaces

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega);\ D^{\boldsymbol{\alpha}} f \in L^p(\Omega),\ \text{for all } \boldsymbol{\alpha} \text{ with } |\boldsymbol{\alpha}| \le k\}.$$

If $f \in W^{k,p}(\Omega)$, its norm is defined as

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\boldsymbol{\alpha}| \le k} \int_\Omega |D^{\boldsymbol{\alpha}} f|^p \, dx \right)^{1/p}, & 1 \le p < \infty; \\ \sum_{|\boldsymbol{\alpha}| \le k} \text{ess sup}_{\,\Omega} |D^{\boldsymbol{\alpha}} f|, & p = \infty. \end{cases}$$

We write $H^k(\Omega) := W^{k,2}(\Omega)$ and denote by $H^k_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$, where $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions $f : \Omega \to \mathbb{R}$, with compact support in Ω. As we know, $H^k(\Omega)$ is a separable Hilbert space and $H^1_0(\Omega)$ is characterized by

$$H^1_0(\Omega) = \{f \in H^1(\Omega);\ f\Big|_{\partial\Omega} = 0\}.$$

In the next inequality we obtain an important property for the spaces $H^1_0(\Omega)$. We say that an open set $\Omega \subset \mathbb{R}^n$ is bounded in some direction x_i if there exists a finite open interval (a, b) such that

$$pr_i\,\Omega \subset (a, b),$$

where $pr_i\,\Omega$ is the projection of $\mathbb{R}^n$ over the axis x_i.

Theorem 2.3 ((MEDEIROS; MIRANDA, 2000), Poincaré's inequality) *Let Ω be an open set of $\mathbb{R}^n$ bounded in some direction x_i such that $pr_i\,\Omega \subset (a, b)$. Then*

$$\int_\Omega |f|^2 \, dx \le (b-a)^2 \int_\Omega \left| \frac{\partial f}{\partial x_i} \right|^2 dx, \quad \forall f \in H^1_0(\Omega).$$

In $\mathbb{R}^n$, for $n = 2, 3$, we observe that pointwise values of functions can be estimated in terms of H^2 norms. One has

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq \|f\|_{L^2(\mathbb{R}^2)}^{1/2} \|D^2 f\|_{L^2(\mathbb{R}^2)}^{1/2}, \tag{2.1}$$

for arbitrary $f \in H^2(\mathbb{R}^2)$; likewise,

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)}^{1/4} \|D^2 f\|_{L^2(\mathbb{R}^3)}^{3/4}, \tag{2.2}$$

for $f \in H^2(\mathbb{R}^3)$. These are easily shown by Fourier transform and Parseval's identity (see e.g. (SCHUTZ et al., 2016)). By Fourier transform, we also get (for any n):

$$\|Df\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}^{1/2} \|D^2 f\|_{L^2(\mathbb{R}^n)}^{1/2} \tag{2.3}$$

or, more generally,

$$\|D^l f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}^{1-\theta} \|D^m f\|_{L^2(\mathbb{R}^n)}^{\theta}, \quad \theta = \frac{l}{m}, \tag{2.4}$$

for all $0 \leq l \leq m$.

Now, we will remember some inequalities that will be useful in the second chapter.

Lemma 2.1 ((ADAMS; FOURNIER, 1975), (SILVA; CRUZ; ROJAS-MEDAR, 2014a), (KATO; PONCE, *Let $s \in \mathbb{N}$, $\alpha \in \mathbb{R}^n$ be a standard multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and D^α be the usual differential operator $D^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$.*

(a) (Commutator estimate) If $f \in H^s$, $\nabla f \in L^\infty$ and $g \in H^{s-1} \cap L^\infty$, then

$$\sum_{|\alpha| \leq s} \|D^\alpha(fg) - f D^\alpha g\|_2 \leq C(\|\nabla f\|_{H^{s-1}} \|g\|_\infty + \|\nabla f\|_\infty \|g\|_{H^{s-1}}). \tag{2.5}$$

(b) (Product estimate) If $f, g \in H^s \cap L^\infty$, then

$$\sum_{|\alpha| \leq s} \|D^\alpha(fg)\|_2 \leq C(\|f\|_{H^s} \|g\|_\infty + \|f\|_\infty \|g\|_{H^s}). \tag{2.6}$$

(c) (Gagliardo-Nirenberg inequality) If $|\alpha| = k \leq s$, then

$$\sum_{|\alpha| \leq s} \|D^\alpha f\|_{\frac{2s}{k}} \leq C \|f\|_\infty^{1-k/s} \|f\|_{H^s}^{k/s}. \tag{2.7}$$

(d) (Sobolev inequalities) For any $f \in H^2$ and $q \in [2, 6]$, there holds that

$$\|f\|_q \leq C\|f\|_{H^1}, \quad \|f\|_6 \leq C\|\nabla f\|_2, \quad \|f\|_\infty \leq C\|\nabla f\|_{H^1} \leq C\|f\|_{H^2}. \tag{2.8}$$

(e) (An inequality in Sobolev spaces) For every $s \geq 1$, there exists a constant C_s such as $f, g \in H^s \cap L^\infty$, then

$$\|fg\|_{H^s} \leq C_s(\|f\|_{H^s}\|g\|_\infty + \|f\|_\infty\|g\|_{H^s}). \tag{2.9}$$

As observed in (**??**), if an inequality is satisfied for scalars functions u with a positive constant $C > 0$, then is also valid for vector functions $\mathbf{u}$ with the same constant C from the scalar case.

Theorem 2.4 (Green's formulas) *Let $f, g \in C^2(\bar{\Omega})$. Then*

(i) $\int_\Omega \Delta f \, dx = -\int_\Gamma \dfrac{\partial f}{\partial \eta} \, dS$

(ii) $\int_\Omega \nabla f \cdot \nabla g \, dx = -\int_\Omega f \Delta g \, dx + \int_\Gamma f \dfrac{\partial g}{\partial \eta} \, dS$

(iii) $\int_\Omega (f\Delta g - g\Delta f) \, dx = \int_\Gamma \left(f \dfrac{\partial g}{\partial \eta} - g \dfrac{\partial f}{\partial \eta} \right) dS$

2.2 TECHNICAL RESULTS

Definition 2.11 *If $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform by*

$$\mathcal{F}\{f\}(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x}) \, d\mathbf{x}, \quad \boldsymbol{\xi} \in \mathbb{R}^n,$$

where $i^2 = -1$.

Theorem 2.5 (Plancherel's Theorem) *If $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\widehat{u} \in L^2(\mathbb{R}^n)$ and*

$$\|\widehat{u}\| = (2\pi)^{\frac{n}{2}} \|u\|. \tag{2.10}$$

Proof: Firstly we will prove that if $v \in L^1(\mathbb{R}^n)$ then $\widehat{v} \in L^\infty(\mathbb{R}^n)$. Indeed,

$$|\widehat{v}(\boldsymbol{\xi})| = \left| \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} v(\mathbf{x}) \, d\mathbf{x} \right| \leq \int_{\mathbb{R}^n} |e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} v(\mathbf{x})| \, d\mathbf{x} = \int_{\mathbb{R}^n} |v(\mathbf{x})| \, d\mathbf{x} < \infty.$$

So, if $v, w \in L^1(\mathbb{R}^n)$ it follows that $\widehat{v}, \widehat{w} \in L^\infty(\mathbb{R}^n)$. Futhermore, we have that

$$\int_{\mathbb{R}^n} v(\mathbf{x}) \widehat{w}(\mathbf{x}) \, d\mathbf{x} = \int_{\mathbb{R}^n} \widehat{v}(\boldsymbol{\xi}) w(\boldsymbol{\xi}) \, d\boldsymbol{\xi}. \tag{2.11}$$

A straight calculation leads us to the next identity:

$$\int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}\cdot\mathbf{x} - t|\mathbf{x}|^2} \, d\mathbf{x} = \left(\frac{\pi}{t} \right)^{\frac{n}{2}} e^{\frac{-|\boldsymbol{\xi}|^2}{4t}}, \quad \forall t > 0.$$

Then by (2.11), for all $\epsilon > 0$, we get that

$$\int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi}) e^{-\epsilon|\boldsymbol{\xi}|^2}\, d\boldsymbol{\xi} = \left(\frac{\pi}{\epsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} w(\mathbf{x}) e^{\frac{-|\mathbf{x}|^2}{4\epsilon}}\, d\mathbf{x}. \tag{2.12}$$

Now, let $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and consider $v(\mathbf{x}) := \overline{u}(-\mathbf{x})$, where $\overline{u}$ is the complex conjugate of u. Define $w \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ as follows:

$$w(\mathbf{y}) = (2\pi)^n (u * v)(\mathbf{y}) = (2\pi)^n \int_{\mathbb{R}^n} u(\mathbf{x}) v(\mathbf{y} - \mathbf{x})\, d\mathbf{x}.$$

Then, we have that

$$\begin{aligned}
\widehat{w}(\boldsymbol{\xi}) &= (2\pi)^n \widehat{(u * v)}(\boldsymbol{\xi}) \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} \int_{\mathbb{R}^n} u(\mathbf{y}) v(\mathbf{x} - \mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{y}} u(\mathbf{y}) \left(\int_{\mathbb{R}^n} e^{-i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}}\ v(\mathbf{x} - \mathbf{y})\, d\mathbf{x} \right) d\mathbf{y} \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{y}} u(\mathbf{y}) \widehat{v}(\boldsymbol{\xi})\, d\mathbf{y} \\
&= (2\pi)^n \widehat{u}(\boldsymbol{\xi}) \widehat{v}(\boldsymbol{\xi}).
\end{aligned}$$

Thus, $\widehat{w} \in L^\infty(\mathbb{R}^n)$. Note that

$$\widehat{v}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\ \overline{u}(-\mathbf{x})\, d\mathbf{x} = \overline{\widehat{u}}(\boldsymbol{\xi}).$$

Then, $\widehat{w} = (2\pi)^n |\widehat{u}|^2$. Since w is continuos, we have that

$$\lim_{\epsilon \to 0} \left(\frac{\pi}{\epsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} w(\mathbf{x}) e^{-\frac{|\mathbf{x}|^2}{4\epsilon}}\, d\mathbf{x} = (2\pi)^n w(\mathbf{0}).$$

From the identity (2.12), we conclude that

$$\int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi})\, d\boldsymbol{\xi} = (2\pi)^n w(\mathbf{0}).$$

Therefore,

$$\int_{\mathbb{R}^n} |\widehat{u}|^2\, d\boldsymbol{\xi} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi})\, d\boldsymbol{\xi} = w(\mathbf{0}) = (2\pi)^n \int_{\mathbb{R}^n} u(\mathbf{x}) v(-\mathbf{x})\, d\mathbf{x} = (2\pi)^n \int_{\mathbb{R}^n} |u|^2\, d\mathbf{x},$$

what we wanted to prove.

The identity (2.10) is also known by *Parseval's identity*.

Lemma 2.2 *Let M be a Hermitian matrix with all eigenvalues $\{\lambda_i\}_{i=1}^n$ positives. Then*

$$\|e^{-Mt}\| \leq e^{-(\min_i \lambda_i)t}, \quad \forall t \geq 0.$$

Proof: Since M is a Hermitian matrix we get that M is diagonalizable. Consider then $\beta = \{v_1, \dots, v_n\}$ a basis of eigenvectors for M with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Thus, β is also a basis of eigenvectors for e^{-Mt} with eigenvalues $\{e^{-\lambda_1 t}, \dots, e^{-\lambda_n t}\}$. Let $w \in \mathbb{C}^n$ be such that

$$w = \alpha_1 v_1 + \cdots + \alpha_n v_n,$$

where $\alpha_i \in \mathbb{C}$ for all $i = 1, \dots, n$. As all norms in $\mathbb{C}^n$ are equivalent, consider the maximum norm with respect to basis β, i.e.,

$$\|w\| := \max_{1 \le i \le n} |\alpha_i|,$$

where $|\cdot|$ denotes the standard norm in $\mathbb{C}$. Thus, for all $t \geq 0$, we have that

$$\left\| e^{-Mt} w \right\| = \left\| \sum_{i=1}^{n} \alpha_i e^{-\lambda_i t} v_i \right\| \le \|w\| \left(\max_{1 \le i \le n} \left| e^{-\lambda_i t} \right| \right) = \|w\| e^{-(\min_i \lambda_i) t},$$

and this implies that

$$\left\| e^{-Mt} \right\| \le e^{-(\min_i \lambda_i) t}.$$

Definition 2.12 *The heat semigroup in $\mathbb{R}^n$ is the family of operators $(e^{\Delta t})_{t \ge 0}$ where*

$$e^{\Delta 0} v(x) = v(x), \quad x \in \mathbb{R}^n$$

and

$$e^{\Delta t} v(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} v(y)\, dy, \quad (x,t) \in \mathbb{R}^2 \times (0, \infty).$$

Theorem 2.6 *Let $\boldsymbol{u}(x,t) = e^{\Delta t} \boldsymbol{u}_0(x)$ be the solution of the initial value problem*

$$\begin{cases} \boldsymbol{u}_t = \Delta \boldsymbol{u}, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad x \in \mathbb{R}^n \end{cases} \tag{2.13}$$

where $\boldsymbol{u}_0 \in L^r(\mathbb{R}^n)$. Then

$$\|\nabla^m e^{\Delta t} \boldsymbol{u}_0\|_q = \|e^{\Delta t} \nabla^m \boldsymbol{u}_0\|_q \le C \|\boldsymbol{u}_0\|_r t^{-\frac{n}{2}\left(\frac{1}{r} - \frac{1}{q}\right) - \frac{m}{2}}, \quad \forall t > 0 \tag{2.14}$$

for any $1 \le r \le q \le \infty$, $m \in \mathbb{N}$ and where the positive constant C depends only of r, q, n and m.

Proof: In this thesis we will only use this theorem when $m = 0$ or $m = 1$, so we will limit ourselfs to prove just these cases, the general case can be found in (GUADAGNIN, 2005). Firstly, we will prove the case $m = 0$, i.e,

$$\|e^{\Delta t} \boldsymbol{u}_0\|_q \le C \|\boldsymbol{u}_0\|_r t^{-\frac{n}{2}\left(\frac{1}{r} - \frac{1}{q}\right)}, \quad \forall t > 0. \tag{2.15}$$

Set

$$\phi(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, t > 0.$$

Note that $\boldsymbol{u}(x,t) = \int_{\mathbb{R}^n} \phi(x-y,t)\boldsymbol{u}_0(y)\, dy$ is a solution for the problem (2.13) and

$$\int_{\mathbb{R}^n} \phi(x,t)\, dx = 1 \tag{2.16}$$

for all $t > 0$. Consequently,

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx = (4\pi t)^{\frac{n}{2}}, \quad \forall t > 0.$$

Thus, by identity (2.16) and Young's inequality for convolutions, for $1 \leq r \leq \infty$ we get that

$$\|\boldsymbol{u}(\cdot,t)\|_r = \|\phi(\cdot,t) * \boldsymbol{u}_0\|_r \leq \|\phi(\cdot,t)\|_1 \|\boldsymbol{u}_0\|_r = \|\boldsymbol{u}_0\|_r. \tag{2.17}$$

Assumption 2.1 *For $1 \leq r < \infty$ we have that*

$$\|u(\cdot,t)\|_\infty \leq (4\pi t)^{-\frac{n}{2r}} \|\boldsymbol{u}_0\|_r. \tag{2.18}$$

Proof: Indeed, for all $t > 0$

$$|\boldsymbol{u}(x,t)| \leq \int_{\mathbb{R}^n} \phi(x-y,t)|\boldsymbol{u}_0(y)|\, dy \leq (4\pi t)^{-\frac{n}{2}} \|\boldsymbol{u}_0\|_1$$

since $\phi(x,t) \leq (4\pi t)^{-\frac{n}{2}}$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$. This proves (2.18) *for $r = 1$.*

If $1 < r < \infty$, let $p > 0$ be such as $p^{-1} + r^{-1} = 1$. Then by Hölder's inequality and (2.16), *it follows that*

$$\begin{aligned}
|\boldsymbol{u}(x,t)| \leq \int_{\mathbb{R}^n} \phi(x-y,t)|\boldsymbol{u}_0(y)|\, dy &= \int_{\mathbb{R}^n} \phi(x-y,t)^{\frac{1}{p}} \phi(x-y,t)^{\frac{1}{r}} |\boldsymbol{u}_0(y)|\, dy \\
&\leq (4\pi t)^{-\frac{n}{2r}} \int_{\mathbb{R}^n} \phi(x-y,t)^{\frac{1}{p}} |\boldsymbol{u}_0(y)|\, dy \\
&\leq (4\pi t)^{-\frac{n}{2r}} \left(\int_{\mathbb{R}^n} \phi(x-y,t)\, dy \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |\boldsymbol{u}_0(y)|^r\, dy \right)^{\frac{1}{r}} \\
&= (4\pi t)^{-\frac{n}{2r}} \|\boldsymbol{u}_0\|_r
\end{aligned}$$

which proves the assumption.

Using (2.17) and (2.18), for $1 \leq r \leq q < \infty$, we find that

$$\begin{aligned}
\|\boldsymbol{u}(\cdot,t)\|_q^q = \int_{\mathbb{R}^n} |\boldsymbol{u}(x,t)|^q\, dx = \int_{\mathbb{R}^n} |\boldsymbol{u}(x,t)|^{q-r} |\boldsymbol{u}(x,t)|^r\, dx &\leq \|\boldsymbol{u}(\cdot,t)\|_\infty^{q-r} \|\boldsymbol{u}(\cdot,t)\|_r^r \\
&\leq (4\pi t)^{-\frac{n}{2r}(q-r)} \|\boldsymbol{u}_0\|_r^{q-r} \|\boldsymbol{u}_0\|_r^r \\
&= (4\pi t)^{-\frac{n}{2r}(q-r)} \|\boldsymbol{u}_0\|_r^q.
\end{aligned}$$

The case where $q=\infty$ it follows directly from (2.18). This proves (2.15).

Now we will prove the case $m=1$, i.e,

$$\|\nabla e^{\Delta t}\boldsymbol{u}_0\|_q \leq C\|\boldsymbol{u}_0\|_r t^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}}, \quad \forall t>0.$$

First, we want to show the identity

$$\int_{\mathbb{R}^n} |\nabla^m \phi(x,t)|\,dx = \tilde{C}t^{-\frac{m}{2}} \tag{2.19}$$

where $\tilde{C}=\tilde{C}(m,n)>0$. Consider the function $g:\mathbb{R}^n\to\mathbb{R}^+$ given by $g(x)=e^{-\frac{|x|^2}{4}}$. Note that $\nabla^m g(x)=p_m(x)g(x)$ where $p_m(x)$ is a polynomial of degree m. As $0<g(x)\leq 1$ for all $x\in\mathbb{R}^n$, it follows that $\nabla^m g(x)$ is a bounded function, then

$$\int_{\mathbb{R}^n} |\nabla^m g(x)|\,dx := K(m,n) < \infty.$$

Note that

$$\phi(x,t) = (4\pi t)^{-\frac{n}{2}} g(t^{-\frac{1}{2}}x),$$

thus from the chain rule, we find that

$$\nabla^m \phi(x,t) = (4\pi t)^{-\frac{n}{2}} t^{-\frac{m}{2}} \nabla^m g(t^{-\frac{1}{2}}x).$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla^m \phi(x,t)|\,dx = (4\pi t)^{-\frac{n}{2}} t^{-\frac{m}{2}} \int_{\mathbb{R}^n} |\nabla^m g(t^{-\frac{1}{2}}x)|\,dx &= (4\pi t)^{-\frac{n}{2}} t^{-\frac{m}{2}} t^{\frac{n}{2}} \int_{\mathbb{R}^n} |\nabla^m g(y)|\,dy \\ &= (4\pi)^{-\frac{n}{2}} t^{-\frac{m}{2}} K(m,n) \\ &= \tilde{C}(m,n) t^{-\frac{m}{2}}\end{aligned}$$

where $\tilde{C}(m,n)=(4\pi)^{-\frac{n}{2}}K(m,n)$. This proves (2.19).

Using the identity (2.19) and the Young's inequality for convolutions, for $1\leq r\leq\infty$, we get that

$$\|\nabla^m \boldsymbol{u}(\cdot,t)\|_r = \|\nabla^m \phi(\cdot,t) * \boldsymbol{u}_0\|_r \leq \|\nabla^m \phi(\cdot,t)\|_1 \|\boldsymbol{u}_0\|_r = \tilde{C}(m,n) t^{-\frac{m}{2}} \|\boldsymbol{u}_0\|_r \tag{2.20}$$

for all $t>0$. Since

$$\boldsymbol{u}(x,t) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \boldsymbol{u}_0(y)\,dy, \tag{2.21}$$

for all $(x,t)\in\mathbb{R}^n\times(0,\infty)$, taking D_l in (2.21), for $l=1,\dots,n$, it follows that

$$\begin{aligned} D_l \boldsymbol{u}(x,t) &= -(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{1}{4t}\frac{\partial}{\partial x_l}(|x-y|^2) e^{-\frac{|x-y|^2}{4t}} \boldsymbol{u}_0(y)\,dy \\ &= -(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{1}{4t} 2(x_l-y_l) e^{-\frac{|x-y|^2}{4t}} \boldsymbol{u}_0(y)\,dy \\ &= -(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{1}{2t} \langle x-y, e_l\rangle e^{-\frac{|x-y|^2}{4t}} \boldsymbol{u}_0(y)\,dy. \end{aligned}$$

Then for the Cauchy-Schwarz inequality we have

$$\begin{aligned}|D_l\boldsymbol{u}(x,t)| &\le (4\pi t)^{-\frac{n}{2}}t^{-\frac{1}{2}}\int_{\mathbb{R}^n}\frac{1}{2t^{\frac{1}{2}}}|x-y||e_l|e^{-\frac{|x-y|^2}{4t}}|\boldsymbol{u}_0(y)|\,dy\\ &= 2^{-1}(4\pi t)^{-\frac{n}{2}}t^{-\frac{1}{2}}\int_{\mathbb{R}^n}\frac{|x-y|^2}{t^{\frac{1}{2}}}e^{-\frac{|x-y|^2}{4t}}|\boldsymbol{u}_0(y)|\,dy\\ &\le \tilde{C}(4\pi t)^{-\frac{n}{2}}t^{-\frac{1}{2}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4t}}|\boldsymbol{u}_0(y)|\,dy\end{aligned}$$

where $\tilde{C} := \max\{2^{-1}\lambda e^{\frac{-\lambda^2}{8}};\ \lambda>0\} = e^{-\frac{1}{2}}$. Therefore, if $p^{-1}+r^{-1}=1$, then

$$\begin{aligned}|\nabla\boldsymbol{u}(x,t)| &\le \tilde{C}(4\pi t)^{-\frac{n}{2}}t^{-\frac{1}{2}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{8tp}}e^{-\frac{|x-y|^2}{8tr}}|\boldsymbol{u}_0(y)|\,dy\\ &\le \tilde{C}(4\pi t)^{-\frac{n}{2}}t^{-\frac{1}{2}}\left(\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{8t}}\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{8t}}|\boldsymbol{u}_0(y)|^r\,dy\right)^{\frac{1}{r}}\\ &= \tilde{C}(4\pi t)^{-\frac{n}{2}}t^{-\frac{1}{2}}(8\pi t)^{\frac{n}{2p}}\left(\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{8t}}|\boldsymbol{u}_0(y)|^r\,dy\right)^{\frac{1}{r}}\\ &\le \tilde{C}(4\pi t)^{-\frac{n}{2r}}2^{\frac{n}{2p}}t^{-\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\boldsymbol{u}_0(y)|\,dy\right)^{\frac{1}{r}}\\ &\le \tilde{C}(4\pi t)^{-\frac{n}{2r}}2^{\frac{n}{2}}t^{-\frac{1}{2}}\|\boldsymbol{u}_0\|_r\end{aligned}$$

for all $(x,t)\in\mathbb{R}^n\times(0,\infty)$. Thus

$$\|\nabla\boldsymbol{u}(\cdot,t)\|_\infty \le \frac{\tilde{C}2^{\frac{n}{2}}}{(4\pi)^{\frac{n}{2r}}}\|\boldsymbol{u}_0\|_r t^{-\frac{n}{2r}-\frac{1}{2}},\quad \forall t>0. \tag{2.22}$$

Moreover

$$|\nabla\boldsymbol{u}(x,t)|^r \le \frac{\tilde{C}^r2^{\frac{nr}{2}}t^{-\frac{r}{2}}}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{8t}}|\boldsymbol{u}_0(y)|^r\,dy,\quad \forall(x,t)\in\mathbb{R}^n\times(0,\infty).$$

Then from Fubini's theorem, we have that

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla\boldsymbol{u}(x,t)|^r\,dx &\le \frac{\tilde{C}^r2^{\frac{nr}{2}}t^{-\frac{r}{2}}}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{8t}}|\boldsymbol{u}_0(y)|^r\,dy\right)dx\\ &= \frac{\tilde{C}^r2^{\frac{nr}{2}}t^{-\frac{r}{2}}}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{8t}}\,dx\right)|\boldsymbol{u}_0(y)|^r\,dy\\ &= \frac{\tilde{C}^r2^{\frac{nr}{2}}t^{-\frac{r}{2}}}{(4\pi t)^{\frac{n}{2}}}(8\pi t)^{\frac{n}{2}}\|\boldsymbol{u}_0\|_r^r = \tilde{C}^r2^{\frac{n}{2}}2^{\frac{nr}{2}}t^{-\frac{r}{2}}\|\boldsymbol{u}_0\|_r^r,\quad \forall t>0.\end{aligned}$$

Thus

$$\|\nabla\boldsymbol{u}(\cdot,t)\|_r \le \tilde{C}2^nt^{-\frac{1}{2}}\|\boldsymbol{u}_0\|_r. \tag{2.23}$$

Therefore, from (2.22), (2.23) and interpolation inequality, for $1 \leq r \leq q \leq \infty$, it follows that

$$
\begin{aligned}
\|\nabla \boldsymbol{u}(\cdot,t)\|_q &\leq \|\nabla \boldsymbol{u}(\cdot,t)\|_r^{r/q} \|\nabla \boldsymbol{u}(\cdot,t)\|_\infty^{1-r/q} \\
&\leq (\tilde{C}^{r/q} 2^{nr/q} \|\boldsymbol{u}_0\|_r^{r/q} t^{-r/2q}) \frac{2^{\frac{n}{2}(1-\frac{r}{q})} \tilde{C}^{1-\frac{r}{q}}}{(4\pi)^{\frac{n}{2}(1-\frac{r}{q})}} \|\boldsymbol{u}_0\|_r^{1-\frac{r}{q}} t^{-\left(\frac{n}{2r}+\frac{1}{2}\right)\left(1-\frac{r}{q}\right)} \\
&= \frac{\tilde{C} 2^{\frac{n}{2}\left(1+\frac{r}{q}\right)}}{(4\pi)^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}} \|\boldsymbol{u}_0\|_r^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}}.
\end{aligned}
$$

Therefore, we conclude the proof for cases $m = 0$ and $m = 1$.

3 DECAY CHARACTERIZATION OF WEAK SOLUTIONS TO THE 2D EQUATIONS OF THE MAGNETO-MICROPOLAR FLUIDS

3.1 INTRODUCTION

The 3D model for magneto-micropolar fluids given by (1.1) is a generalization of the classical Navier-Stokes equations, where the microstructure of the fluid particles and the effect of the induced magned field on the motion are considered.

The two-dimensional model is a special case of the 3D system when we consider $\boldsymbol{u}(x,t) = (u_1(x,t), u_2(x,t), 0)$, $\boldsymbol{b}(x,t) = (b_1(x,t), b_2(x,t), 0)$ and $\boldsymbol{w}(x,t) = (0, 0, w(x,t))$. In this way, substituting $\boldsymbol{u}$, $\boldsymbol{w}$ and $\boldsymbol{b}$ written in the above form in the system (1.1), we obtain the following 2D system of equations:

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = (\mu+\chi)\Delta\boldsymbol{u} + \chi\nabla\times\boldsymbol{w} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} = \gamma\Delta\boldsymbol{w} + \chi\nabla\times\boldsymbol{u} - 2\chi\boldsymbol{w}, \\ \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} = \nu\Delta\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \\ \operatorname{div}\ \boldsymbol{u} = 0, \quad \operatorname{div}\ \boldsymbol{b} = 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x), \quad \boldsymbol{b}(x,0) = \boldsymbol{b}_0(x). \end{cases} \tag{3.1}$$

Leray–Hopf solutions to (3.1) are defined similarly to the 3D case, for arbitrary $(\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0) \in \mathbf{L}^2(\mathbb{R}^2)\times\mathbf{L}^2(\mathbb{R}^2)\times\mathbf{L}^2(\mathbb{R}^2)$ with $\operatorname{div}\mathbf{u}_0 = \operatorname{div}\mathbf{b}_0 = 0$ as distributions. But this time they are known to be uniquely defined by the initial data and in $C^\infty(\mathbb{R}^2\times(0,\infty))$, with

$$(\mathbf{u},\mathbf{w},\mathbf{b})(\cdot,t) \in L^\infty_{\text{loc}}((0,\infty), \mathbf{H}^m(\mathbb{R}^2)) \qquad \forall\ m \in \mathbb{N}\cup\{0\}, \tag{S2}$$

so that $t_* = 0$ in dimension $n = 2$ (see (LUKASZEWICZ, 2012; REN et al., 2014; ROJAS-MEDAR, 1997; SERMANGE; TEMAM, 1983)). An important property of both 2D and 3D flows is that $\|\,(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\,\|_{\boldsymbol{L}^2(\mathbb{R}^n)}$ is eventually monotonic: there exists $t_{**} \geq t_*$ such that (see (SILVA; ZINGANO; ZINGANO, 2019; KREISS et al., 2003))

$$\|\,(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,t)\,\|_{\boldsymbol{L}^2(\mathbb{R}^n)} \leq \|\,(\nabla\mathbf{u},\nabla\mathbf{w},\nabla\mathbf{b})(\cdot,s)\,\|_{\mathbf{L}^2(\mathbb{R}^n)} \qquad \forall\ t_{**} < s < t. \tag{M}$$

The main goal of this chapter is to prove a decay estimate for the 2D MHD micropolar system for any initial data in $\mathbf{L}^2(\mathbb{R}^2)$. To achieve this, we associate to data $\mathbf{z}_0 \overset{\text{def}}{=} (\mathbf{u}_0, \mathbf{w}_0, \mathbf{b}_0)$ in $\mathbf{L}^2(\mathbb{R}^2)$ a decay character $r^* = r^*(\mathbf{z}_0)$, which measures the "order" of $\widehat{\mathbf{z}_0}(\xi)$ at $\xi = 0$ in frequency space and then find sharp decay estimates for solutions of the linear part in terms of r^* (see (BJORLAND; SCHONBEK, 2009)). We then use the "Fourier splitting method" (also called Schonbek's method) and these estimates for the linear part to characterize the decay of MHD micropolar equations. This same strategy was followed by some authors to obtain similar results for other fluid mechanics equations (see e.g. (NICHE; PERUSATO, 2022; NICHE; SCHONBEK, 2015; NICHE; SCHONBEK, 2016; NICHE, 2016)).

From now on, we denote by $\mathbf{z}(\cdot,t) \equiv \mathbf{z}(t) \stackrel{\text{def}}{=} (\mathbf{u},\mathbf{w},\mathbf{b})(t)$ a weak solution (or Leray-Hopf solution) of problem (3.1), with initial data $\mathbf{z}(0) \equiv \mathbf{z}_0 \stackrel{\text{def}}{=} (\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0) \in \mathbf{L}^2_\sigma(\mathbb{R}^2) \times \mathbf{L}^2(\mathbb{R}^2) \times \mathbf{L}^2_\sigma(\mathbb{R}^2)$, where the subscript σ means that the vector field is free divergence.

3.2 DECAY CHARACTER AND CHARACTERIZATION OF DECAY OF SOLUTIONS FOR THE LINEAR PART

3.2.1 Decay character

In order to prove sharp estimates for the decay of the linear part in (3.1), C. Bjorland and M. E. Schonbek (BJORLAND; SCHONBEK, 2009) introduced the idea of decay character $r^* = r^*(v_0)$ of a function v_0 in $L^2(\mathbb{R}^n)$. Roughly speaking, the decay character is the "order" of $|\widehat{v_0}(\xi)|$ in frequency space at $\xi = 0$. We recall now some of these definitions and results.

Definition 3.1 *Let $v_0 \in L^2(\mathbb{R}^n)$. For $r \in (-n/2,\infty)$, we define the decay indicator $P_r(v_0)$ corresponding to v_0 as*

$$P_r(v_0) = \lim_{\rho\to 0} \rho^{-2r-n} \int_{B(\rho)} |\widehat{v_0}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}, \quad B(\rho) = \{\boldsymbol{\xi} \in \mathbb{R}^n;\ |\boldsymbol{\xi}| \leq \rho\}$$

provided this limit exists.

Remark 3.1 *Observe that the decay indicator is always zero when $r \leq -n/2$. Moreover, note that the decay indicator compares $|\widehat{v}_0(\boldsymbol{\xi})|^2$ to $|\boldsymbol{\xi}|^{2r}$ near $\boldsymbol{\xi} = \mathbf{0}$.*

Definition 3.2 *The decay character of v_0, denoted by $r^* = r^*(v_0)$ is the unique $r \in (-n/2,\infty)$ such that $0 < P_r(v_0) < \infty$, provided that this number exists. If such $P_r(v_0)$ does not exist, we set $r^* = -\frac{n}{2}$, when $P_r(v_0) = \infty$ for all $r \in (-n/2,\infty)$ or $r^* = \infty$, if $P_r(v_0) = 0$ for all $r \in (-n/2,\infty)$.*

We use the decay character for establishing upper and lower bounds for decay rates of energy for solutions to a large family of dissipative linear operators. Let X be a Hilbert space on $\mathbb{R}^n$, we consider a pseudodifferential operator $\mathcal{L} : X^n \to (L^2(\mathbb{R}^n))^n$, with symbol $M(\boldsymbol{\xi})$ such that

$$M(\boldsymbol{\xi}) = P^{-1}(\boldsymbol{\xi})D(\boldsymbol{\xi})P(\boldsymbol{\xi}), \quad \boldsymbol{\xi} - \text{a.e}, \tag{3.2}$$

where $P(\boldsymbol{\xi}) \in O(n)$ and $D(\boldsymbol{\xi}) = -c_i|\boldsymbol{\xi}|^{2\alpha}\delta_{ij}$, for $c_i > c > 0$ and $0 < \alpha \leq 1$.

Given the linear equation

$$v_t = \mathcal{L}v \tag{3.3}$$

applying the Fourier transform in the previous equation we obtain, after multiplying for the Fourier Transform of v and integrating over $\mathbb{R}^n$, that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 = \frac{1}{2}\frac{1}{(2\pi)^n}\frac{d}{dt}\|\widehat{v}(t)\|_2^2 = \frac{1}{(2\pi)^n}\langle \widehat{v}, M\widehat{v}\rangle_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n}\langle \widehat{v}, P^{-1}DP\widehat{v}\rangle_{L^2(\mathbb{R}^n)} \\ &= -\frac{1}{(2\pi)^n}\langle (-D)^{\frac{1}{2}}P\widehat{v}, (-D)^{\frac{1}{2}}P\widehat{v}\rangle_{L^2(\mathbb{R}^n)} \\ &= -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |(-D)^{\frac{1}{2}}P\widehat{v}|^2 d\boldsymbol{\xi} \\ &\leq -C\int_{\mathbb{R}^n} |\boldsymbol{\xi}|^{2\alpha}|\widehat{v}|^2 d\boldsymbol{\xi} \end{aligned} \tag{3.4}$$

where in the first equality we used the Plancherel identity. Therefore

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 \leq -C\int_{\mathbb{R}^n} |\boldsymbol{\xi}|^{2\alpha}|\widehat{v}|^2 d\boldsymbol{\xi}$$

which is the very heart of Schonbek's *Fourier splitting* technique.

We now state the theorem that describes decay in terms of the decay character for linear operators as in (3.2).

Theorem 3.1 (Theorem 2.10, Niche and M.E. Schonbek (NICHE; SCHONBEK, 2015)) *Let $v_0 \in L^2(\mathbb{R}^n)$ with character decay $r^* = r^*(v_0)$. Let $v(t)$ be the solution of* (3.3) *with inicial data v_0. Then:*

(1) if $-\frac{n}{2} < r^* < \infty$, then there exist constants $C_1, C_2 > 0$ such that

$$C_1(1+t)^{-\frac{1}{\alpha}(\frac{n}{2}+r^*)} \leq \|v(t)\|_2^2 \leq C_2(1+t)^{-\frac{1}{\alpha}(\frac{n}{2}+r^*)},$$

(2) if $r^* = -\frac{n}{2}$, then there exists $C = C(\epsilon) > 0$ such that

$$\|v(t)\|_2^2 \geq C(1+t)^{-\epsilon}, \quad \forall \epsilon > 0,$$

that is, the decay of $\|v(t)\|_2^2$ is slower than any uniform algebric rate;

(3) if $r^* = \infty$, then there exists $C > 0$ such that

$$\|v(t)\|_2^2 \le C(1+t)^{-m}, \quad \forall m > 0,$$

that is, the decay of $\|v(t)\|_2^2$ is faster than any algebric rate.

3.2.2 Linear part

We now study the linear system associate to (3.1), namely

$$\begin{cases} \bar{\boldsymbol{u}}_t = (\mu+\chi)\Delta\bar{\boldsymbol{u}} + \chi\nabla\times\bar{\boldsymbol{w}}, \\ \bar{\boldsymbol{w}}_t = \gamma\Delta\bar{\boldsymbol{w}} + \chi\nabla\times\bar{\boldsymbol{u}} - 2\chi\bar{\boldsymbol{w}}, \\ \bar{\boldsymbol{b}}_t = \nu\Delta\bar{\boldsymbol{b}} \end{cases} \tag{3.5}$$

with inicial data $\bar{\boldsymbol{z}}_0 = \boldsymbol{z}_0 = (\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0) \in L^2_\sigma(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2_\sigma(\mathbb{R}^2)$, where we set $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{u}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{b}})$, for simplicity. Here $\operatorname{div}\ \bar{\mathbf{u}} = \operatorname{div}\ \bar{\mathbf{b}} = 0.$

We first address the relation between $r^*(\boldsymbol{z}_0), r^*(\boldsymbol{u}_0), r^*(\boldsymbol{w}_0)$ and $r^*(\boldsymbol{b}_0)$.

Lemma 3.1 *Let $r^*(\boldsymbol{u}_0)$, $r^*(\boldsymbol{w}_0)$, $r^*(\boldsymbol{b}_0) \in (-1, \infty)$. Then,*

$$r^*(\boldsymbol{z}_0) = \min\{r^*(\boldsymbol{u}_0), r^*(\boldsymbol{w}_0), r^*(\mathbf{b}_0)\}.$$

Proof: Let $\lambda = \min\{r^*(\boldsymbol{u}_0), r^*(\boldsymbol{w}_0), r^*(\boldsymbol{b}_0)\}$. Without loss of generality, assume $\lambda = r^*(\boldsymbol{u}_0)$ with $\lambda < r^*(\boldsymbol{w}_0) = \lambda_\star$ and $\lambda < r^*(\boldsymbol{b}_0) = \lambda_{\star\star}$. Observe that

$$P_\lambda(\boldsymbol{z}_0) = P_\lambda(\boldsymbol{u}_0) + P_\lambda(\boldsymbol{w}_0) + P_\lambda(\boldsymbol{b}_0),$$

and that $P_\lambda(\boldsymbol{u}_0) > 0$, $P_{\lambda_\star}(\boldsymbol{w}_0) > 0$ and $P_{\lambda_{\star\star}}(\boldsymbol{b}_0) > 0$. Hence,

$$\begin{aligned} P_\lambda(\mathbf{b}_0) &= \lim_{\rho\to 0^+} \rho^{-2\lambda-2} \int_{B(\rho)} |\widehat{\mathbf{b}_0}(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &= \lim_{\rho\to 0^+} \rho^{2(\lambda_{\star\star}-\lambda)} \cdot \rho^{-2\lambda_{\star\star}-2} \int_{B(\rho)} |\widehat{\mathbf{b}_0}(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi} \\ &= 0 \cdot P_{\lambda_{\star\star}}(\boldsymbol{b}_0) = 0, \end{aligned}$$

since $\lambda < \lambda_{\star\star}$ and $P_{\lambda_{\star\star}}(\boldsymbol{b}_0) > 0$. The same argument proves that $P_\lambda(\boldsymbol{w}_0) = 0$, and thus, $P_\lambda(\boldsymbol{z}_0) = P_\lambda(\mathbf{u}_0)$, from which we infer that $\lambda = r^*(\boldsymbol{z}_0)$.

Taking the Fourier transform of (3.5), we obtain

$$\begin{cases} \widehat{\bar{\mathbf{u}}}_t = -(\mu+\chi)|\boldsymbol{\xi}|^2\widehat{\bar{\mathbf{u}}} + \chi\mathcal{F}(\nabla\times\bar{\mathbf{w}}), \\ \widehat{\bar{\mathbf{w}}}_t = -\gamma|\boldsymbol{\xi}|^2\widehat{\bar{\mathbf{w}}} + \chi\mathcal{F}(\nabla\times\bar{\mathbf{u}}) - 2\chi\widehat{\bar{\mathbf{w}}}, \\ \widehat{\bar{\mathbf{b}}}_t = -\nu|\boldsymbol{\xi}|^2\widehat{\bar{\mathbf{b}}} \end{cases} \tag{3.6}$$

where $\mathcal{F}(v) = \widehat{v}$ denotes the Fourier transform of v and $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$. Note that

$$\nabla\times\bar{\mathbf{w}} = (D_2\bar{w}, -D_1\bar{w}, 0) \quad \text{and} \quad \nabla\times\bar{\mathbf{u}} = (0, 0, D_1\bar{u}_2 - D_2\bar{u}_1)$$

then

$$\mathcal{F}(\nabla\times\bar{\mathbf{w}}) = i(\xi_2\widehat{\bar{w}}, -\xi_1\widehat{\bar{w}}, 0) \quad \text{and} \quad \mathcal{F}(\nabla\times\bar{\mathbf{u}}) = i(0, 0, \xi_1\widehat{\bar{u}}_2 - i\xi_2\widehat{\bar{u}}_1) \tag{3.7}$$

where $i^2 = -1$. With this we can rewrite (3.6) as follows

$$\widehat{\bar{\mathbf{z}}}_t(\boldsymbol{\xi}, t) = M(\boldsymbol{\xi})\widehat{\bar{\mathbf{z}}}(\boldsymbol{\xi}, t), \tag{3.8}$$

where $M(\boldsymbol{\xi})$ is the matrix of symbols given by

$$M(\boldsymbol{\xi}) = \begin{pmatrix} -(\mu+\chi)|\boldsymbol{\xi}|^2 & 0 & i\chi\xi_2 & 0 & 0 \\ 0 & -(\mu+\chi)|\boldsymbol{\xi}|^2 & -i\chi\xi_1 & 0 & 0 \\ -i\chi\xi_2 & i\chi\xi_1 & -\gamma|\boldsymbol{\xi}|^2 - 2\chi & 0 & 0 \\ 0 & 0 & 0 & -\nu|\boldsymbol{\xi}|^2 & 0 \\ 0 & 0 & 0 & 0 & -\nu|\boldsymbol{\xi}|^2 \end{pmatrix}. \tag{3.9}$$

Since $M(\boldsymbol{\xi})$ is Hermitian, it is diagonalizable and $M(\boldsymbol{\xi}) = P^{-1}(\boldsymbol{\xi})D(\boldsymbol{\xi})P(\boldsymbol{\xi})$ where $P(\boldsymbol{\xi}) \in U(5)$ is the matrix whose columns are the vectors $e_i = v_i/|v_i|$ and $D(\boldsymbol{\xi}) = -\lambda_i|\boldsymbol{\xi}|^2\delta_{ij}$ is a diagonal matrix. To use Theorem 3.1 we need to calculate the eigenvalues of $M(\boldsymbol{\xi})$. Indeed

$$\lambda I_{5\times5} - M(\boldsymbol{\xi}) = \begin{pmatrix} \lambda+(\mu+\chi)|\boldsymbol{\xi}|^2 & 0 & -i\chi\xi_2 & 0 & 0 \\ 0 & \lambda+(\mu+\chi)|\boldsymbol{\xi}|^2 & i\chi\xi_1 & 0 & 0 \\ i\chi\xi_2 & -i\chi\xi_1 & \lambda+\gamma|\boldsymbol{\xi}|^2+2\chi & 0 & 0 \\ 0 & 0 & 0 & \lambda+\nu|\boldsymbol{\xi}|^2 & 0 \\ 0 & 0 & 0 & 0 & \lambda+\nu|\boldsymbol{\xi}|^2 \end{pmatrix},$$

where $I_{5\times 5}$ is the identity matrix of order 5. The characteristic polynomial $p(\lambda)$ is given by

$$p(\lambda) = (\lambda+\nu|\boldsymbol{\xi}|^2)^2 \begin{vmatrix} \lambda+(\mu+\chi)|\boldsymbol{\xi}|^2 & 0 & -i\chi\xi_2 \\ 0 & \lambda+(\mu+\chi)|\boldsymbol{\xi}|^2 & i\chi\xi_1 \\ i\chi\xi_2 & -i\chi\xi_1 & \lambda+\gamma|\boldsymbol{\xi}|^2+2\chi \end{vmatrix}$$

$$\begin{aligned}
&= (\lambda+\nu|\boldsymbol{\xi}|^2)^2\left[(\lambda+(\mu+\chi)|\boldsymbol{\xi}|^2)^2(\lambda+\gamma|\boldsymbol{\xi}|^2+2\chi)+(\lambda+(\mu+\chi)|\boldsymbol{\xi}|^2)((i\chi\xi_2)^2+(i\chi\xi_1)^2)\right] \\
&= (\lambda+\nu|\boldsymbol{\xi}|^2)^2(\lambda+(\mu+\chi)|\boldsymbol{\xi}|^2)\left[(\lambda+(\mu+\chi)|\boldsymbol{\xi}|^2)(\lambda+\gamma|\boldsymbol{\xi}|^2+2\chi)-\chi^2(\xi_1^2+\xi_2^2)\right] \\
&= (\lambda+\nu|\boldsymbol{\xi}|^2)^2(\lambda+(\mu+\chi)|\boldsymbol{\xi}|^2)\left[(\lambda+(\mu+\chi)|\boldsymbol{\xi}|^2)(\lambda+\gamma|\boldsymbol{\xi}|^2+2\chi)-\chi^2|\boldsymbol{\xi}|^2\right] \\
&= (\lambda+\nu|\boldsymbol{\xi}|^2)^2(\lambda+(\mu+\chi)|\boldsymbol{\xi}|^2)q(\lambda),
\end{aligned}$$

where $q(\lambda)$ is a second degree polynomial given by

$$q(\lambda) = \lambda^2+\lambda(\gamma|\boldsymbol{\xi}|^2+2\chi+(\mu+\chi)|\boldsymbol{\xi}|^2)+\gamma(\mu+\chi)|\boldsymbol{\xi}|^4+2\chi\mu|\boldsymbol{\xi}|^2+\chi^2|\boldsymbol{\xi}|^2.$$

From the Bhaskara's formula we have

$$\lambda = \frac{-(2\chi+(\mu+\gamma+\chi)|\boldsymbol{\xi}|^2)\pm\sqrt{[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2]^2+4\chi^2|\boldsymbol{\xi}|^2}}{2},$$

then the eigenvalues of $M(\boldsymbol{\xi})$ are

$$\lambda_1=\lambda_2=-\nu|\boldsymbol{\xi}|^2, \quad \lambda_3=-(\mu+\chi)|\boldsymbol{\xi}|^2,$$

$$\begin{aligned}
\lambda_4 &= -\frac{1}{2}(2\chi+(\mu+\gamma+\chi)|\boldsymbol{\xi}|^2)-\frac{1}{2}\sqrt{[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2]^2+4\chi^2|\boldsymbol{\xi}|^2} \\
\lambda_5 &= -\frac{1}{2}(2\chi+(\mu+\gamma+\chi)|\boldsymbol{\xi}|^2)+\frac{1}{2}\sqrt{[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2]^2+4\chi^2|\boldsymbol{\xi}|^2},
\end{aligned}$$

and the corresponding eigenvectors are

$$v_1=(0,0,0,0,1), \quad v_2=(0,0,0,1,0), \quad v_3=(\xi_1/\xi_2,1,0,0,0),$$

$$v_4=\left(-\frac{2i\chi\xi_2}{\eta_+},\frac{2i\chi\xi_1}{\eta_+},1,0,0\right) \quad \text{and} \quad v_5=\left(-\frac{2i\chi\xi_2}{\eta_-},\frac{2i\chi\xi_1}{\eta_-},1,0,0\right),$$

where $\eta_+ = 2\chi-(\mu+\chi+\gamma)|\boldsymbol{\xi}|^2+\sqrt{[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2]^2+4\chi^2|\boldsymbol{\xi}|^2}$ and $\eta_- = 2\chi-(\mu+\chi+\gamma)|\boldsymbol{\xi}|^2-\sqrt{[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2]^2+4\chi^2|\boldsymbol{\xi}|^2}$.

Lemma 3.2 *For $M(\boldsymbol{\xi})$ given in (3.9), we have that*

$$\lambda_{\max}(M(\boldsymbol{\xi})) \le -C|\boldsymbol{\xi}|^2, \quad C=C(\mu,\gamma,\nu)>0,$$

where $\lambda_{\max}(M(\boldsymbol{\xi}))$ is the largest eigenvalue of $M(\boldsymbol{\xi})$.

Proof: Note that

$$\lambda_j \le -C_1|\boldsymbol{\xi}|^2, \quad j = 1,2,3,$$

where $C_1 \le \min\{\mu, \nu\}$. Futhermore

$$\begin{aligned}\lambda_4 &= -\frac{1}{2}(2\chi + (\mu+\gamma+\chi)|\boldsymbol{\xi}|^2) - \frac{1}{2}\sqrt{[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2]^2 + 4\chi^2|\boldsymbol{\xi}|^2}\\ &\le -\frac{1}{2}(2\chi + (\mu+\gamma+\chi)|\boldsymbol{\xi}|^2)\\ &\le -\frac{1}{2}(\mu+\gamma)|\boldsymbol{\xi}|^2\\ &\le -\min\{\mu,\gamma\}|\boldsymbol{\xi}|^2.\end{aligned}$$

Now, we will show that

$$\lambda_5 \le -\frac{1}{4}\min\{\mu,\gamma\}|\boldsymbol{\xi}|^2. \tag{3.10}$$

For this, we set

$$\varrho_- \overset{\text{def}}{=} \min\{\mu,\gamma\} \quad \text{and} \quad \varrho_+ \overset{\text{def}}{=} \max\{\mu,\gamma\}.$$

Since $\gamma - \mu \le \frac{\varrho_-}{2} + \chi + \varrho_+$ and $|\gamma-\mu-\chi| \le \frac{\varrho_-}{2} + \chi + \varrho_+$, we have

$$4\chi(\gamma-\mu)|\boldsymbol{\xi}|^2 \le 4\chi\Big(\frac{\varrho_-}{2} + \chi + \varrho_+\Big)|\boldsymbol{\xi}|^2$$

and

$$(\gamma-\mu-\chi)^2|\boldsymbol{\xi}|^4 \le \Big(\frac{\varrho_-}{2} + \chi + \varrho_+\Big)^2|\boldsymbol{\xi}|^4.$$

Thus,

$$4\chi^2 + 4\chi(\gamma-\mu)|\boldsymbol{\xi}|^2 + (\gamma-\mu-\chi)^2|\boldsymbol{\xi}|^4 \le 4\chi^2 + 4\chi\Big(\frac{\varrho_-}{2} + \chi + \varrho_+\Big)|\boldsymbol{\xi}|^2 + \Big(\frac{\varrho_-}{2} + \chi + \varrho_+\Big)^2|\boldsymbol{\xi}|^4,$$

or yet

$$\Big[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2\Big]^2 + 4\chi^2|\boldsymbol{\xi}|^2 \le \Big(2\chi + \Big(\frac{\varrho_-}{2} + \chi + \varrho_+\Big)|\boldsymbol{\xi}|^2\Big)^2.$$

This implies that

$$\begin{aligned}\sqrt{[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2]^2 + 4\chi^2|\boldsymbol{\xi}|^2} &\le 2\chi + \Big(\frac{\varrho_-}{2} + \chi + \varrho_+\Big)|\boldsymbol{\xi}|^2\\ &= 2\chi + (\mu+\chi+\gamma)|\boldsymbol{\xi}|^2 - \frac{\varrho_-}{2}|\boldsymbol{\xi}|^2,\end{aligned}$$

from which, we infer

$$-\Big(2\chi + (\mu+\chi+\gamma)|\boldsymbol{\xi}|^2\Big) + \sqrt{[(2\chi+\gamma|\boldsymbol{\xi}|^2)-(\mu+\chi)|\boldsymbol{\xi}|^2]^2 + 4\chi^2|\boldsymbol{\xi}|^2} \le -\frac{\varrho_-}{2}|\boldsymbol{\xi}|^2.$$

This proves inequality (3.10) and finishes the proof of lemma.

The estimate of Lemma 3.2 leads us to (3.4). Now we can use the Theorem 3.1 to obtain the following result that characterize the decay of solutions for the linear system (3.5) in terms of the decay character $r^* = r^*(\mathbf{z}_0)$.

Theorem 3.2 *Let $\bar{\mathbf{z}}_0 \in \mathbf{L}^2(\mathbb{R}^2)$ with character decay $r^* = r^*(\bar{\mathbf{z}}_0)$. Let $\bar{\mathbf{z}}(t)$ be the solution of* (3.5) *with inicial data $\bar{\mathbf{z}}_0$. Then:*

(1) if $-1 < r^* < \infty$, then there exists constants $C_1, C_2 > 0$ such that

$$C_1(1+t)^{-(1+r^*)} \leq \|\bar{\mathbf{z}}(t)\|_2^2 \leq C_2(1+t)^{-(1+r^*)}$$

(2) if $r^* = -1$, then there exists $C = C(\epsilon) > 0$ such that

$$\|\bar{\mathbf{z}}(t)\|_2^2 \geq C(1+t)^{-\epsilon}, \quad \forall \epsilon > 0,$$

that is, the decay of $\|\bar{\mathbf{z}}(t)\|_2^2$ is slower than any uniform algebric rate;

(3) if $r^* = \infty$, then there exists $C > 0$ such that

$$\|\bar{\mathbf{z}}(t)\|_2^2 \leq C(1+t)^{-m}, \quad \forall m > 0,$$

that is, the decay of $\|\bar{\mathbf{z}}(t)\|_2^2$ is faster than any algebric rate.

Remark 3.2 *Since the decay character of $\mathbf{z}_0$ is the minimum of the decay characters of $\mathbf{u}_0$, $\mathbf{w}_0$ and $\mathbf{b}_0$ (see Lemma 3.1), it follows that decay of solution $\bar{\mathbf{z}}(t)$ to* (3.5) *is the slowest of the decays of $\bar{\mathbf{u}}(t)$, $\bar{\mathbf{w}}(t)$ and $\bar{\mathbf{b}}(t)$.*

3.3 L^2-DECAY OF WEAK SOLUTIONS

Theorem 3.3 *Let $\mathbf{z}$ be a weak solution to* (3.1). *Let $r^*(\mathbf{z}_0) = r^*$ be the decay character of $\mathbf{z}_0$, with $-1 < r^* < \infty$. Then, for all $t \geq 0$*

$$\|\mathbf{z}(t)\|_2^2 \leq C(1+t)^{-\min\{1+r^*,2\}}. \tag{3.11}$$

Proof: Multiplying $(3.1)_1$ by $2\mathbf{u}$ in $L^2(\mathbb{R}^2)$, we obtain

$$\begin{aligned} 2(\mathbf{u}_t, \mathbf{u})_{L^2} + 2((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{u})_{L^2} + 2(\nabla p, \mathbf{u})_{L^2} &= 2(\mu + \chi)(\Delta \mathbf{u}, \mathbf{u})_{L^2} + 2\chi(\nabla \times \mathbf{w}, \mathbf{u})_{L^2} \\ &+ 2((\mathbf{b} \cdot \nabla \mathbf{b}), \mathbf{u})_{L^2} \end{aligned} \tag{3.12}$$

Firstly we will estimate each term of (3.12):

$$2(\mathbf{u}_t, \mathbf{u})_{L^2} = \frac{d}{dt}(\mathbf{u}, \mathbf{u})_{L^2} = \frac{d}{dt}\|\mathbf{u}(t)\|_2^2. \tag{3.13}$$

Using integration by parts, we get

$$2((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{u})_{L^2} = 2\sum_{j,i=1}^{2}\int_{\mathbb{R}^2} u_j(\mathbf{x},t)D_j u_i(\mathbf{x},t)u_i(\mathbf{x},t)\,d\mathbf{x} = -2\sum_{i,j}^{2}\int_{\mathbb{R}^2} D_j u_j(\mathbf{x},t)[u_i(\mathbf{x},t)]^2\,d\mathbf{x} - 2\sum_{i,j}^{2}\int_{\mathbb{R}^2} u_j(\mathbf{x},t)u_i(\mathbf{x},t)D_j u_i(\mathbf{x},t)\,d\mathbf{x}. \tag{3.14}$$

Since $\operatorname{div}\ \mathbf{u}=0$, the first term on the right side become:

$$-2\sum_{i,j}^{2}\int_{\mathbb{R}^2} D_j u_j(\mathbf{x},t)[u_i(\mathbf{x},t)]^2\,d\mathbf{x} = -2\int_{\mathbb{R}^2}\operatorname{div}\ \mathbf{u}\ |\mathbf{u}|^2\,d\mathbf{x} = 0.$$

Replacing this equality in (3.14), we obtain

$$2((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{u})_{L^2} = -2\sum_{i,j}^{2}\int_{\mathbb{R}^2} u_j(\mathbf{x},t)D_j u_i(\mathbf{x},t)u_i(\mathbf{x},t)\,d\mathbf{x} = -2((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{u})_{L^2}$$

hence $((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{u})_{L^2}=0$. In the same way, using the incompressible condition $\operatorname{div}\ \mathbf{u}=0$ we find that $(\nabla p,\mathbf{u})_{L^2}=0$. At last, using integration by parts, we get

$$\begin{aligned} 2(\mu+\chi)(\Delta\mathbf{u},\mathbf{u})_{L^2} &= 2(\mu+\chi)\sum_{i,j=1}^{2}\int_{\mathbb{R}^2} D_j^2 u_i(\mathbf{x},t)u_i(\mathbf{x},t)\,d\mathbf{x} \\ &= -2(\mu+\chi)\sum_{i,j=1}^{2}\int_{\mathbb{R}^2}[D_j u_i(\mathbf{x},t)]^2\,d\mathbf{x} \\ &= -2(\mu+\chi)\|\nabla\mathbf{u}(t)\|_2^2. \end{aligned} \tag{3.15}$$

Replacing the previous estimates in (3.12), we obtain

$$\frac{d}{dt}\|\mathbf{u}(t)\|_2^2 + 2(\mu+\chi)\|\nabla\mathbf{u}(t)\|_2^2 = 2\chi(\nabla\times\mathbf{w},\mathbf{u})_{L^2} + 2((\mathbf{b}\cdot\nabla)\mathbf{b},\mathbf{u})_{L^2}. \tag{3.16}$$

Similarly, multiplying by $2\mathbf{w}$ and $2\mathbf{b}$ the equations $(3.1)_2$ and $(3.1)_3$, respectively, and integrating over $\mathbb{R}^2$, we get

$$\frac{d}{dt}\|\mathbf{w}(t)\|_2^2 + 2\gamma\|\nabla\mathbf{w}(t)\|_2^2 + 4\chi\|\mathbf{w}(t)\|_2^2 = 2\chi(\nabla\times\mathbf{u},\mathbf{w})_{L^2} \tag{3.17}$$

and

$$\frac{d}{dt}\|\mathbf{b}(t)\|_2^2 + 2\nu\|\nabla\mathbf{b}(t)\|_2^2 = 2((\mathbf{b}\cdot\nabla)\mathbf{u},\mathbf{b})_{L^2}. \tag{3.18}$$

Adding (3.16), (3.17) and (3.18), we find

$$\begin{aligned} &\frac{d}{dt}(\|\mathbf{u}(t)\|_2^2+\|\mathbf{w}(t)\|_2^2+\|\mathbf{b}(t)\|_2^2) + 2(\mu+\chi)\|\nabla\mathbf{u}(t)\|_2^2 + 2\gamma\|\nabla\mathbf{w}(t)\|_2^2 + 2\nu\|\nabla\mathbf{b}(t)\|_2^2 \\ &+4\chi\|\mathbf{w}(t)\|_2^2 = 2\chi(\nabla\times\mathbf{w},\mathbf{u})_{L^2} + 2((\mathbf{b}\cdot\nabla)\mathbf{b},\mathbf{u})_{L^2} + 2\chi(\nabla\times\mathbf{u},\mathbf{w})_{L^2} + 2((\mathbf{b}\cdot\nabla)\mathbf{u},\mathbf{b})_{L^2}. \end{aligned} \tag{3.19}$$

Note that $(\nabla \times \mathbf{w}, \mathbf{u})_{L^2} = (\nabla \times \mathbf{u}, \mathbf{w})_{L^2}$. Indeed, since $\nabla \times \mathbf{w} = (D_2 w, -D_1 w, 0)$ and $\nabla \times \mathbf{u} = (0, 0, D_1 u_2 - D_2 u_1)$, it follows that

$$\begin{aligned}(\nabla \times \mathbf{w}, \mathbf{u})_{L^2} &= \int_{\mathbb{R}^2} D_2 w(\mathbf{x},t) u_1(\mathbf{x},t)\, d\mathbf{x} - \int_{\mathbb{R}^2} D_1 w(\mathbf{x},t) u_2(\mathbf{x},t)\, d\mathbf{x} \\ &= -\int_{\mathbb{R}^2} w(\mathbf{x},t) D_2 u_1(\mathbf{x},t)\, d\mathbf{x} + \int_{\mathbb{R}^2} w(\mathbf{x},t) D_1 u_2(\mathbf{x},t)\, d\mathbf{x} \\ &= (\nabla \times \mathbf{u}, \mathbf{w})_{L^2}.\end{aligned}$$

Futhermore, as $\mathrm{div}\ \mathbf{u} = 0$ and $\mathrm{div}\ \mathbf{b} = 0$, we obtain

$$\begin{aligned}((\mathbf{b}\cdot\nabla)\mathbf{b}, \mathbf{u})_{L^2} = \sum_{i,j=1}^{2} \int_{\mathbb{R}^2} b_j(\mathbf{x},t) D_j b_i(\mathbf{x},t) u_i(\mathbf{x},t)\, d\mathbf{x} &= -\int_{\mathbb{R}^2} b_j(\mathbf{x},t) D_j u_i(\mathbf{x},t) b_i(\mathbf{x},t)\, d\mathbf{x} \\ &= -((\mathbf{b}\cdot\nabla)\mathbf{u}, \mathbf{b})_{L^2}.\end{aligned}$$

Thus, from (3.19)

$$\begin{aligned}\frac{d}{dt}(\|\mathbf{u}(t)\|_2^2 + \|\mathbf{w}(t)\|_2^2 + \|\mathbf{b}(t)\|_2^2) + 2(\mu+\chi)\|\nabla\mathbf{u}(t)\|_2^2 + 2\gamma\|\nabla\mathbf{w}(t)\|_2^2 \\ +2\nu\|\nabla\mathbf{b}(t)\|_2^2 + 4\chi\|\mathbf{w}(t)\|_2^2 = 4\chi(\nabla\times\mathbf{u}, \mathbf{w})_{L^2}.\end{aligned} \tag{3.20}$$

From Cauchy-Schwarz and Young's inequalities, we have

$$4\chi(\nabla\times\mathbf{u},\mathbf{w})_{L^2} \le 4\chi\|\nabla\times\mathbf{u}\|_2\|\mathbf{w}\|_2 \le 2\chi\|\nabla\times\mathbf{u}\|_2^2 + 2\chi\|\mathbf{w}\|_2^2. \tag{3.21}$$

Note that

$$\nabla\times(\nabla\times\mathbf{u}) = \nabla(\mathrm{div}\ \mathbf{u}) - \Delta\mathbf{u} = -\Delta\mathbf{u}$$

since $\mathrm{div}\ \mathbf{u} = 0$, which yields

$$\begin{aligned}\|\nabla\times\mathbf{u}\|_2^2 &= (\nabla\times\mathbf{u}, \nabla\times\mathbf{u})_{L^2} = (\mathbf{u}, \nabla\times(\nabla\times\mathbf{u}))_{L^2} \\ &= -(\mathbf{u}, \Delta\mathbf{u})_{L^2} = \|\nabla\mathbf{u}\|_2^2.\end{aligned} \tag{3.22}$$

Then from (3.20), (3.21) and (3.22) we obtain the energy estimate:

$$\frac{d}{dt}\|\mathbf{z}(t)\|_2^2 \le -2\beta\|\nabla\mathbf{z}(t)\|_2^2. \tag{3.23}$$

where $\beta = \min\{\mu, \gamma, \nu\}$. In particular

$$\|\mathbf{z}(t)\|_2^2 + 2\beta\int_0^t \|\nabla\mathbf{z}(s)\|_2^2\, ds \le \|\mathbf{z}_0\|_2^2, \quad t \ge 0. \tag{3.24}$$

Next, let $f(t)$ be a differentiable function of one real variable satisfying

$$f(0) = 1, \quad f(t) \ge 1, \quad f'(t) > 0, \quad \frac{f'(t)}{f(t)} \le \text{const}, \quad \forall\, t \ge 0, \quad \text{and} \quad \frac{f'(t)}{f(t)} \xrightarrow[t\to\infty]{} 0$$

which will be chosen later. Moreover, we define the ball at the origin with radius $\rho(t)$

$$B(t) \stackrel{\text{def}}{=} \left\{\boldsymbol{\xi} \in \mathbb{R}^2 : \ |\boldsymbol{\xi}| \le \rho(t)\right\}, \quad \text{where} \quad \rho(t) \stackrel{\text{def}}{=} \left[\frac{f'(t)}{2\beta f(t)}\right]^{\frac{1}{2}}, \quad \forall\, t \ge 0.$$

From Plancherel theorem

$$\begin{aligned} -2\beta\|\nabla \mathbf{z}(t)\|_2^2 = -2\beta(2\pi)^{-2}\|\widehat{\nabla \mathbf{z}}(\boldsymbol{\xi},t)\|_2^2 = -\frac{\beta}{2\pi^2}\int_{\mathbb{R}^2} |\widehat{\nabla \mathbf{z}}(\boldsymbol{\xi},t)|^2\, d\boldsymbol{\xi} &= -\frac{\beta}{2\pi^2}\int_{\mathbb{R}^2} |\boldsymbol{\xi}|^2|\widehat{\mathbf{z}}(\boldsymbol{\xi},t)|^2\, d\boldsymbol{\xi} \\ &\le \frac{\beta}{2\pi^2}\int_{\mathbb{R}^2\setminus B(t)} |\boldsymbol{\xi}|^2|\widehat{\mathbf{z}}(\boldsymbol{\xi},t)|^2\, d\boldsymbol{\xi} \\ &\le -\frac{1}{4\pi^2}\frac{f'(t)}{f(t)}\int_{\mathbb{R}^2\setminus B(t)} |\widehat{\mathbf{z}}(\boldsymbol{\xi},t)|^2\, d\boldsymbol{\xi}. \end{aligned}$$

Thus, from (3.23)

$$\frac{d}{dt}\|\mathbf{z}(t)\|_2^2 + \frac{f'(t)}{4\pi^2 f(t)}\int_{\mathbb{R}^2\setminus B(t)} |\widehat{\mathbf{z}}(\boldsymbol{\xi},t)|^2\, d\boldsymbol{\xi} \le 0.$$

Consequently

$$\frac{d}{dt}\|\mathbf{z}(t)\|_2^2 + \frac{f'(t)}{f(t)}\|\mathbf{z}(t)\|_2^2 \le \frac{f'(t)}{4\pi^2 f(t)}\int_{B(t)} |\widehat{\mathbf{z}}(\boldsymbol{\xi},t)|^2\, d\boldsymbol{\xi}. \tag{3.25}$$

Multiplying (3.25) by the integration factor $f(t)$, we find

$$\frac{d}{dt}\Big[\|\mathbf{z}(t)\|_2^2 f(t)\Big] \le \frac{f'(t)}{4\pi^2}\int_{B(t)} |\widehat{\mathbf{z}}(\boldsymbol{\xi},t)|^2\, d\boldsymbol{\xi}. \tag{3.26}$$

Applying the Fourier transform in (3.1) we get the following initial value problem:

$$\begin{cases} \widehat{\mathbf{u}}_t + \mathcal{F}\{(\mathbf{u}\cdot\nabla)\mathbf{u}\} + (\mu+\chi)|\boldsymbol{\xi}|^2\widehat{\mathbf{u}} + i\boldsymbol{\xi}\widehat{p} = \mathcal{F}\{(\mathbf{b}\cdot\nabla)\mathbf{b}\} + \chi\mathcal{F}(\nabla\times\mathbf{w}), \quad \boldsymbol{\xi}\in\mathbb{R}^2,\ t>0, \\ \widehat{\mathbf{w}}_t + \mathcal{F}\{(\mathbf{u}\cdot\nabla)\mathbf{w}\} + \gamma|\boldsymbol{\xi}|^2\widehat{\mathbf{w}} + 2\chi\widehat{\mathbf{w}} = \chi\mathcal{F}(\nabla\times\mathbf{u}), \quad \boldsymbol{\xi}\in\mathbb{R}^2,\ t>0, \\ \widehat{\mathbf{b}}_t + \mathcal{F}\{(\mathbf{u}\cdot\nabla)\mathbf{b}\} + \nu|\boldsymbol{\xi}|^2\widehat{\mathbf{b}} = \mathcal{F}\{(\mathbf{b}\cdot\nabla)\mathbf{u}\}, \quad \boldsymbol{\xi}\in\mathbb{R}^2,\ t>0, \\ i\boldsymbol{\xi}\cdot\widehat{\mathbf{u}} = i\boldsymbol{\xi}\cdot\widehat{\mathbf{b}} = 0, \quad \boldsymbol{\xi}\in\mathbb{R}^2,\ t>0, \\ \widehat{\mathbf{u}}(\boldsymbol{\xi},0) = \widehat{\mathbf{u}_0}(\boldsymbol{\xi}),\ \widehat{\mathbf{w}}(\boldsymbol{\xi},0) = \widehat{\mathbf{w}_0}(\boldsymbol{\xi}),\ \widehat{\mathbf{b}}(\boldsymbol{\xi},0) = \widehat{\mathbf{b}_0}(\boldsymbol{\xi}), \quad \boldsymbol{\xi}\in\mathbb{R}^2. \end{cases} \tag{3.27}$$

Multiplying $(3.27)_1$, $(3.27)_2$ and $(3.27)_3$ by $\widehat{\mathbf{u}}$, $\widehat{\mathbf{w}}$ and $\widehat{\mathbf{b}}$, respectively, and summing up the resulting equations, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(|\widehat{\mathbf{u}}|^2 + |\widehat{\mathbf{w}}|^2 + |\widehat{\mathbf{b}}|^2\Big) + (\mu+\chi)|\boldsymbol{\xi}|^2|\widehat{\mathbf{u}}|^2 + \gamma|\boldsymbol{\xi}|^2|\widehat{\mathbf{w}}|^2 + \nu|\boldsymbol{\xi}|^2|\widehat{\mathbf{b}}|^2 + 2\chi|\widehat{\mathbf{w}}|^2 + \mathcal{F}(\nabla p)\cdot\widehat{\mathbf{u}} \\ = \mathcal{F}((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot\widehat{\mathbf{u}} - \mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\widehat{\mathbf{u}} + \chi\mathcal{F}(\nabla\times\mathbf{w})\cdot\widehat{\mathbf{u}} - \mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{w})\cdot\widehat{\mathbf{w}} \\ +\chi\mathcal{F}(\nabla\times\mathbf{u})\cdot\widehat{\mathbf{w}} - \mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{b})\cdot\widehat{\mathbf{b}} + \mathcal{F}\{(\mathbf{b}\cdot\nabla)\mathbf{u}\}\cdot\widehat{\mathbf{b}}. \end{aligned} \tag{3.28}$$

Observe that, from definition of $\mathcal{F}(\nabla\times\mathbf{w})$ and $\mathcal{F}(\nabla\times\mathbf{u})$ given in (3.7), the sum $\mathcal{F}(\nabla\times\mathbf{w})\cdot\widehat{\mathbf{u}}+\mathcal{F}(\nabla\times\mathbf{u})\cdot\widehat{\mathbf{w}}$ is equal to zero. On other hand, we have that $\mathcal{F}(\nabla p)=i\boldsymbol{\xi}\widehat{p}$. Since $i\boldsymbol{\xi}\cdot\widehat{\mathbf{u}}=0$, $\mathcal{F}(\nabla p)\cdot\widehat{\mathbf{u}}=0$. Then

$$\begin{aligned}\frac{d}{dt}|\widehat{\mathbf{z}}|^2+2\beta|\boldsymbol{\xi}|^2|\widehat{\mathbf{z}}|^2+4\chi|\widehat{\mathbf{w}}|^2 &\leq 2|\mathcal{F}((\mathbf{b}\cdot\nabla)\mathbf{b})||\widehat{\mathbf{u}}|+2|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{u})||\widehat{\mathbf{u}}|+2|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{w})||\widehat{\mathbf{w}}|\\ &\quad+2|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{b})||\widehat{\mathbf{b}}|+2|\mathcal{F}\{(\mathbf{b}\cdot\nabla)\mathbf{u}\}||\widehat{\mathbf{b}}|.\end{aligned} \tag{3.29}$$

For $F_1=\mathbf{u},\mathbf{b}$ and $F_2=\mathbf{u},\mathbf{w},\mathbf{b}$, note that

$$\mathcal{F}(F_1\cdot\nabla F_2)=\mathcal{F}(\nabla\cdot(F_1\otimes F_2))=i\boldsymbol{\xi}\cdot(\widehat{F_1\otimes F_2}).$$

Thus,

$$|\mathcal{F}(F_1\cdot\nabla F_2)|=|\boldsymbol{\xi}||\widehat{F_1\otimes F_2}|\leq|\boldsymbol{\xi}|\|\widehat{F_1\otimes F_2}\|_\infty\leq|\boldsymbol{\xi}|\|F_1\otimes F_2\|_1\leq|\boldsymbol{\xi}|\|F_1\|_2\|F_2\|_2\leq|\boldsymbol{\xi}|\|\mathbf{z}\|_2^2. \tag{3.30}$$

Applying these estimates in (3.29), we obtain

$$\frac{d}{dt}|\widehat{\mathbf{z}}|^2+2\beta|\boldsymbol{\xi}|^2|\widehat{\mathbf{z}}|^2\leq 10|\boldsymbol{\xi}|\|\mathbf{z}\|_2^2|\widehat{\mathbf{z}}|.$$

Integrating the previous inequality in time and using a standard Gronwall type argument, we have that

$$|\widehat{\mathbf{z}}(\boldsymbol{\xi},t)|\leq|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})|+5\int_0^t|\boldsymbol{\xi}|\|\mathbf{z}(s)\|_2^2\,\mathrm{d}s,\qquad\forall\,t\geq 0, \tag{3.31}$$

where $\widehat{\mathbf{z}_0}=\widehat{\mathbf{z}}(\cdot,0)$. Replacing (3.31) in (3.26), we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[f(t)\|\mathbf{z}(t)\|_2^2\Big]\ \leq\ Cf'(t)\left[\int_{B(t)}|\widehat{\boldsymbol{z}_0}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}+\int_{B(t)}\left(\int_0^t|\boldsymbol{\xi}|\|\mathbf{z}(s)\|_2^2\,\mathrm{d}s\right)^2\mathrm{d}\boldsymbol{\xi}\right]. \tag{3.32}$$

On one hand, from the definition of decay indicator, we have that

$$\int_{B(t)}|\widehat{\mathbf{z}_0}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}=\rho^{2r^*+2}\rho^{-2r^*-2}\int_{B(t)}|\widehat{\mathbf{z}_0}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}\leq\rho^{2r^*+2}P_{r^*}(\mathbf{z}_0),$$

where $r^*=r^*(\mathbf{z}_0)$ is the decay character of $\mathbf{z}_0$. On the other hand, from Cauchy inequality, we get that

$$\int_{B(t)} \left(\int_0^t |\boldsymbol{\xi}| \|\mathbf{z}(s)\|_2^2 \, ds \right)^2 d\boldsymbol{\xi} \leq \int_{B(t)} \left(\int_0^t |\boldsymbol{\xi}|^2 \, ds \right) \left(\int_0^t \|\mathbf{z}(s)\|_2^4 \, ds \right) d\boldsymbol{\xi}$$
$$= \int_{B(t)} t|\boldsymbol{\xi}|^2 \left(\int_0^t \|\mathbf{z}(s)\|_2^4 \, ds \right) d\boldsymbol{\xi}$$
$$\leq t\rho(t)^2 \left(\int_0^t \|\mathbf{z}(s)\|_2^4 \, ds \right) \left(\int_{B(t)} d\boldsymbol{\xi} \right)$$
$$\leq Ct\rho(t)^4 \int_0^t \|\mathbf{z}(s)\|_2^4 \, ds$$

since $\int_{B(t)} d\boldsymbol{\xi} = \pi\rho(t)^2$. Then, usando these estimates in (3.32):

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[f(t)\|\mathbf{z}(t)\|_2^2 \Big] \leq Cf'(t)\Big[\rho(t)^{2(1+r^*)} + t\rho(t)^4 \int_0^t \|\mathbf{z}(s)\|_2^4 \, \mathrm{d}s \Big], \tag{3.33}$$

where $C > 0$ depends only on $P_r(\mathbf{z}_0)$.

We will first obtain a preliminary decay, which will be later used to obtain the optimal decay rate. Now, choosing $f(t) = (\ln(e+t))^3$, for all $t \geq 0$, where $\ln(\cdot)$ denotes the natural logarithmic function, then

$$f'(t) = \frac{3\ln(e+t)^2}{(e+t)} \quad \text{and} \quad \frac{f'(t)}{f(t)} = \frac{3}{\ln(e+t)(e+t)}.$$

Making use of the energy estimative given in (3.24), we get

$$\int_0^t \|\mathbf{z}(s)\|_2^4 \, ds \leq \int_0^t \|\mathbf{z}_0\|_2^4 \, ds = t\|\mathbf{z}_0\|_2^4 \leq Ct, \quad C = C(\|\mathbf{z}_0\|_2)$$

thus

$$Cf'(t)t\rho(t)^4 \int_0^t \|\mathbf{z}(s)\|_2^4 \, ds = Cf'(t)t\left(\frac{f'(t)}{f(t)}\right)^2 \int_0^t \|\mathbf{z}(s)\|_2^4 \, ds$$
$$\leq C\ln(e+t)^2(e+t)^{-1}t^2\ln(e+t)^{-2}(e+t)^{-2}$$
$$\leq C\ln(e+t)^2(e+t)^{-1}(e+t)^2\ln(e+t)^{-2}(e+t)^{-2}$$
$$\leq C(e+t)^{-1}.$$

Replacing in (3.33), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\ln(e+t)^3\|\mathbf{z}(t)\|_2^2 \Big] \leq C\ln(e+t)^2(e+t)^{-1}\ln(e+t)^{-1-r^*}(e+t)^{-1-r^*} + C(e+t)^{-1}$$
$$\leq C[\ln(e+t)^{1-r^*}(e+t)^{-2-r^*} + (e+t)^{-1}] \tag{3.34}$$

where $C > 0$ depends only on $P_r(\boldsymbol{z}_0)$, β, r^*, and $\|\mathbf{z}_0\|_2$. Integrating with respect to t, we find

$$(\ln(e+t))^3 \|\mathbf{z}(t)\|_2^2 \leq \|\mathbf{z}_0\|_2^2 + C + C\ln(e+t) \leq C\ln(e+t), \quad \forall\, t \geq 0.$$

Therefore, we have deduced the preliminary decay

$$\|\mathbf{z}(t)\|_2^2 \leq C\left(\ln(t+e)\right)^{-2}, \quad \forall\, t \geq 0, \tag{3.35}$$

where $C \in \mathbb{R}^+$ depends only on $P_r(\boldsymbol{z}_0)$, β, r^*, and $\|\mathbf{z}_0\|_2$. Now, taking $f(t) = (1+t)^\lambda$ in (3.33) with $\lambda > \max\{1+r^*, 2\}$, we have that

$$\begin{aligned}\frac{d}{dt}\Big[(1+t)^\lambda\|\mathbf{z}(t)\|_2^2\Big] &\leq C(1+t)^{\lambda-1}(1+t)^{-1-r^*} + C(1+t)^{\lambda-1}(1+t)^{-2}t\int_0^t \|\mathbf{z}(s)\|_2^4\,ds\\ &\leq C(1+t)^{\lambda-2-r^*} + C(1+t)^{\lambda-3}t\int_0^t \|\mathbf{z}(s)\|_2^4\,ds.\end{aligned}$$

Integrating the previous inequality in time, it follows that

$$\begin{aligned}(1+t)^\lambda\|\mathbf{z}(t)\|_2^2 &\leq \|\mathbf{z}_0\|_2^2 + C\int_0^t (1+\tau)^{\lambda-2-r^*}\,d\tau + C\int_0^t (1+\tau)^{\lambda-3}\tau\left(\int_0^\tau \|\mathbf{z}(s)\|_2^4\,ds\right)d\tau\\ &\leq \|\mathbf{z}_0\|_2^2 + C(1+t)^{\lambda-1-r^*} + C\int_0^t \|\mathbf{z}(\tau)\|_2^4\,d\tau\int_0^t \tau(1+\tau)^{\lambda-3}\,d\tau.\end{aligned} \tag{3.36}$$

Let $\tau < \tilde{\tau} < (\tau+1)$, so

$$\begin{aligned}\int_0^t \|\mathbf{z}(\tau)\|_2^4\,d\tau\int_0^t \tau(1+\tau)^{\lambda-3}\,d\tau &< \int_0^t \|\mathbf{z}(\tau)\|_2^4\,d\tau\int_0^t \tilde{\tau}(1+\tau)^{\lambda-3}\,d\tau\\ &= \int_0^t \tilde{\tau}\|\mathbf{z}(\tau)\|_2^4\,d\tau\int_0^t (1+\tau)^{\lambda-3}\,d\tau\\ &< \int_0^t (1+\tau)\|\mathbf{z}(\tau)\|_2^4\,d\tau\int_0^t (1+\tau)^{\lambda-3}\,d\tau\\ &\leq C(1+t)^{\lambda-2}\int_0^t (1+\tau)\|\mathbf{z}(\tau)\|_2^4\,d\tau.\end{aligned} \tag{3.37}$$

From (3.36) and (3.37), we get that

$$(1+t)^\lambda\|\mathbf{z}(t)\|_2^2 \leq \|\mathbf{z}_0\|_2^2 + C(1+t)^{\lambda-1-r^*} + C(1+t)^{\lambda-2}\int_0^t (1+\tau)\|\mathbf{z}(\tau)\|_2^4\,d\tau.$$

Dividing by $(1+t)^{\lambda-2}$ and using (3.35), we conclude that

$$\begin{aligned}(1+t)^2\|\mathbf{z}(t)\|_2^2 &\leq (1+t)^{-\lambda+2}\|\mathbf{z}_0\|_2^2 + C(1+t)^{1-r^*}\\ &\quad + C\int_0^t \ln(e+\tau)^{-2}(1+\tau)^{-1}(1+\tau)^2\|\mathbf{z}(\tau)\|_2^2\,d\tau.\end{aligned} \tag{3.38}$$

Let

$$\psi(t) \overset{\text{def}}{=} (1+t)^2\|\mathbf{z}(t)\|_2^2, \quad a(t) \overset{\text{def}}{=} (1+t)^{-\lambda+2}\|\mathbf{z}_0\|_2^2 + C(1+t)^{1-r*},$$

$$b(t) \overset{\text{def}}{=} \ln(e+t)^{-2}(1+t)^{-1}.$$

Then the inequality (3.38) becomes

$$\psi(t) \le a(t) + C \int_0^t b(\tau)\psi(\tau)\, ds.$$

Observe that

$$\int_0^t b(\tau)\, d\tau = \int_0^t \ln(e+\tau)^{-2}(1+\tau)^{-1}\, d\tau \le \int_0^\infty \ln(e+\tau)^{-2}(1+s)^{-1}\, ds < \infty, \quad \forall t \ge 0.$$

We need to consider two cases:

Case 1: $r^* \ge 1$**.** In this case $1 - r^* \le 0$. Hence, $a(t) \le C$. This gives, by a standard Gronwall type argument, that

$$\psi(t) \le C \exp\left(\int_0^t b(s)\, ds \right)$$

i.e,

$$(1+t)^2 \|\mathbf{z}(t)\|_2^2 \le C \exp\left(\int_0^t b(s)\, ds \right) \le C$$

wich yields

$$\|\mathbf{z}(t)\|_2^2 \le C(1+t)^{-2}. \tag{3.39}$$

Case 2: $-1 < r^* \le 1$. In this case, $a(t)$ is a nondecreasing function. Thus, by a corollary of Gronwall's Lemma (see (BAINOV; SIMEONOV, 2013), p. 4, Corollary 1.2), we find that

$$\psi(t) \le a(t) \exp\left(\int_0^t b(s)\, ds \right)$$

i.e,

$$(1+t)^2 \|\mathbf{z}(t)\|_2^2 \le a(t) \exp\left(\int_0^t b(s)\, ds \right) \le C a(t). \tag{3.40}$$

Since $\lambda > \max\{1 + r^*, 2\}$, we have that

$$\begin{aligned} (1+t)^{-2} a(t) &= \|\mathbf{z}_0\|_2^2 (1+t)^{-\lambda} + C(1+t)^{-(1+r^*)} \\ &\le C(1+t)^{-(1+r^*)}. \end{aligned}$$

From (3.40), it follows that

$$\|\mathbf{z}(t)\|_2^2 \le C(1+t)^{-(1+r^*)}. \tag{3.41}$$

Therefore, from (3.39) and (3.41), we conclude that

$$\|\mathbf{z}(t)\|_2^2 \le C(1+t)^{-\min\{1+r^*,2\}}.$$

Noticed that, due to the damping term $2\chi\mathbf{w}$ in the Eq. $(3.1)_2$, it would be possible to improve the decay rate for the angular velocity. Our proof follows that of Theorem 3.3 in Cruz *et al.* (CRUZ et al., 2022), where a analogous result is proved for the MHD micropolar fluids with $\mathbf{z}_0 \in \boldsymbol{L}^1(\mathbb{R}^2) \cap \boldsymbol{L}^2(\mathbb{R}^2)$.

We first prove

Lemma 3.3 *Let* $\mathbf{z}$ *be a weak solution to* (3.1). *Let* $r^*(\mathbf{z}_0) = r^*$ *be the decay character of* $\mathbf{z}_0$, *with* $-1 < r^* < \infty$. *Then*

$$\|\nabla \mathbf{z}(t)\|_2^2 \le C(1+t)^{-\min\{2+r^*,3\}}, \quad \forall t > 0.$$

Proof: Let $\alpha(r^*) \stackrel{\text{def}}{=} \min\{1+r^*, 2\}$. Taking $\delta > 0$ and using the enegy estimate given in (3.23), we have that

$$\begin{aligned}\frac{d}{dt}\Big[(1+t)^{\alpha(r^*)+\delta}\|\mathbf{z}(t)\|_2^2\Big] &= (\alpha(r^*)+\delta)(1+t)^{\alpha(r^*)+\delta-1}\|\mathbf{z}(t)\|_2^2 + (1+t)^{\alpha(r^*)+\delta}\frac{d}{dt}\|\mathbf{z}(t)\|_2^2 \\ &\le \Big(\alpha(r^*)+\delta\Big)(1+t)^{\alpha(r^*)+\delta-1}\|\mathbf{z}(t)\|_2^2 - 2\beta(1+t)^{\alpha(r^*)+\delta}\|\nabla\mathbf{z}(t)\|_2^2.\end{aligned}$$

Choosing some $t_1 \gg 1$ (fixed, but otherwise arbitrary), and integrating on (t_1, t), we get

$$\begin{aligned}&(t+1)^{\alpha(r^*)+\delta}\|\mathbf{z}(t)\|_2^2 + 2\beta\int_{t_1}^{t}(s+1)^{\alpha(r^*)+\delta}\|\nabla\mathbf{z}(s)\|_2^2\,\mathrm{d}s \\ &\le (t_1+1)^{\alpha(r^*)+\delta}\|\mathbf{z}(t_1)\|_2^2 + (\alpha(r^*)+\delta)\int_{t_1}^{t}(s+1)^{\alpha(r^*)+\delta-1}\|\mathbf{z}(s)\|_2^2\,\mathrm{d}s \\ &\le (t_1+1)^{\alpha(r^*)+\delta}\|\boldsymbol{z}_0\|_2^2 + C\int_{t_1}^{t}(s+1)^{\delta-1}\,\mathrm{d}s \\ &\le C(t+1)^{\delta},\end{aligned} \tag{3.42}$$

where we used the Theorem 3.3 in the second inequality. Here, $C > 0$ depends only on $\|\boldsymbol{z}_0\|_2$, t_1, δ, $\alpha(r^*)$, $P_r(\boldsymbol{z}_0)$, and β. Now, taking D_k in $(3.1)_1$, we obtain

$$\begin{aligned}\frac{d}{dt}D_k\mathbf{u} + (D_k\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)D_k\mathbf{u} + \nabla(D_k p) &= (\mu+\chi)\Delta(D_k\mathbf{u}) + \chi\nabla\times D_k\mathbf{w} \\ &\quad + (D_k\mathbf{b}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)D_k\mathbf{b}\end{aligned} \tag{3.43}$$

multiplying (3.43) by $2D_k\mathbf{u}$ and integrating over $\mathbb{R}^2$:

$$\begin{aligned}\frac{d}{dt}\|D_k\mathbf{u}(t)\|_2^2 + 2(\mu+\chi)\|\nabla D_k\mathbf{u}(t)\|_2^2 &= 2\chi(\nabla\times D_k\mathbf{w}, D_k\mathbf{u})_{L^2} + 2((D_k\mathbf{b}\cdot\nabla)\mathbf{b}, D_k\mathbf{u})_{L^2} \\ &\quad + 2((\mathbf{b}\cdot\nabla)D_k\mathbf{b}, D_k\mathbf{u})_{L^2} - 2((D_k\mathbf{u}\cdot\nabla)\mathbf{u}, D_k\mathbf{u})_{L^2}.\end{aligned} \tag{3.44}$$

Analogously, taking D_k in the second and third equation from (3.1) and multiplying by $2D_k\mathbf{w}$ and $2D_k\mathbf{b}$ in $L^2(\mathbb{R}^2)$, respectively, we obtain

$$\begin{aligned}\frac{d}{dt}\|D_k\mathbf{w}(t)\|_2^2 + 2\gamma\|\nabla D_k\mathbf{w}(t)\|_2^2 + 4\chi\|D_k\mathbf{w}(t)\|_2^2 &= 2\chi(\nabla\times D_k\mathbf{u}, D_k\mathbf{w})_{L^2} \\ &\quad - 2(D_k\mathbf{u}\cdot\nabla\mathbf{w}, D_k\mathbf{w})_{L^2}\end{aligned} \tag{3.45}$$

and

$$\frac{d}{dt}\|D_k\mathbf{b}(t)\|_2^2 + 2\nu\|\nabla D_k\mathbf{b}(t)\|_2^2 = 2(D_k\mathbf{b}\cdot\nabla\mathbf{u}, D_k\mathbf{b})_{L^2} + 2(\mathbf{b}\cdot\nabla D_k\mathbf{u}, D_k\mathbf{b})_{L^2} \\ - 2(D_k\mathbf{u}\cdot\nabla\mathbf{b}, D_k\mathbf{b})_{L^2}. \tag{3.46}$$

Since $\operatorname{div}\ \mathbf{b}=0$ then $(\mathbf{b}\cdot D_k\mathbf{b}, D_k\mathbf{u})_{L^2} + ((\mathbf{b}\cdot\nabla)\mathbf{u}, D_k\mathbf{b})_{L^2} = 0$. Therefore, summing the Eq. (3.44), (3.45) and (3.46) over $k=1,2$, we find

$$\begin{aligned}&\frac{d}{dt}\|\nabla\mathbf{z}(t)\|_2^2 + 2(\mu+\chi)\|D^2\mathbf{u}(t)\|_2^2 + 2\gamma\|D^2\mathbf{w}(t)\|_2^2 + 2\nu\|D^2\mathbf{u}(t)\|_2^2 + 4\chi\|\nabla\mathbf{w}(t)\|_2^2\\ &= 4\chi\sum_{k=1}^{2}(\nabla\times D_k\mathbf{u}, D_k\mathbf{w})_{L^2} + 2\sum_{k=1}^{2}((D_k\mathbf{b}\cdot\nabla)\mathbf{b}, D_k\mathbf{u})_{L^2} - 2\sum_{k=1}^{2}((D_k\mathbf{u}\cdot\nabla)\mathbf{u}, D_k\mathbf{u})_{L^2}\\ &-2\sum_{k=1}^{2}((D_k\mathbf{u}\cdot\nabla)\mathbf{w}, D_k\mathbf{w})_{L^2} + 2\sum_{k=1}^{2}((D_k\mathbf{b}\cdot\nabla)\mathbf{u}, D_k\mathbf{b})_{L^2} - 2\sum_{k=1}^{2}((D_k\mathbf{u}\cdot\nabla)\mathbf{b}, D_k\mathbf{b})_{L^2}.\end{aligned} \tag{3.47}$$

From Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned}4\chi\sum_{k=1}^{2}(\nabla\times D_k\mathbf{u}, D_k\mathbf{w})_{L^2} &\le 4\chi\sum_{k=1}^{2}\|\nabla\times D_k\mathbf{u}(t)\|_2\|D_k\mathbf{w}(t)\|_2\\ &\le 2\chi\sum_{k=1}^{2}\|\nabla\times D_k\mathbf{u}(t)\|_2^2 + 2\chi\sum_{k=1}^{2}\|D_k\mathbf{w}(t)\|_2^2\\ &\le 2\chi\|D^2\mathbf{u}(t)\|_2^2 + 2\chi\|\nabla\mathbf{w}(t)\|_2^2.\end{aligned} \tag{3.48}$$

Also, for $\mathbf{F}=\mathbf{u},\mathbf{b}$ and $\mathbf{G}=\mathbf{u},\mathbf{b},\mathbf{w}$, we get

$$\begin{aligned}\pm2\sum_{k=1}^{2}((D_k\mathbf{F}\cdot\nabla)\mathbf{G}, D_k\mathbf{G})_{L^2} &= \pm2\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^2}(D_kF_jD_jG_i)(D_kG_i)\,d\mathbf{x}\\ &= \pm2\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^2}(D_kF_jG_i)(D_jD_kG_i)\,d\mathbf{x}\\ &\le 2\sum_{i,j,k=1}^{2}\|G_i\|_\infty\int_{\mathbb{R}^2}|D_kF_j||D_jD_kG_i|\,d\mathbf{x}\\ &\le 2\sum_{i,j,k=1}^{2}\|G_i\|_\infty\|D_kF_j\|_2\|D_jD_kG_i\|_2\\ &\le 2\|\mathbf{G}\|_\infty\|\nabla\mathbf{F}\|_2\|D^2\mathbf{G}\|_2 \le 2\|\mathbf{z}\|_\infty\|\nabla\mathbf{z}\|_2\|D^2\mathbf{z}\|_2,\end{aligned} \tag{3.49}$$

where in the second equality we used that $\operatorname{div}\ \mathbf{F}=0$. Then from (3.47), (3.48) and (3.49) we obtain

$$\frac{d}{dt}\|\nabla\mathbf{z}(t)\|_2^2 + 2\beta\|D^2\mathbf{z}(t)\|_2^2 \le 10\|\mathbf{z}(t)\|_\infty\|\nabla\mathbf{z}(t)\|_2\|D^2\mathbf{z}(t)\|_2 \tag{3.50}$$

where $\beta = \min\{\mu, \gamma, \nu\}$. Multiplying (3.50) by $(1+t)^{\alpha(r^*)+\delta+1}$ and integrating on (t_1, t) we have

$$\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta+1}\frac{d}{dt}\|\nabla\mathbf{z}(s)\|_2^2\,ds + 2\beta\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(s)\|_2^2\,ds$$
$$\leq 10\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta+1}\|\mathbf{z}(s)\|_\infty\|\nabla\mathbf{z}(s)\|_2\|D^2\mathbf{z}(s)\|_2\,ds.$$

Integrating by parts the first integral, we obtain

$$\begin{aligned}&(1+t)^{\alpha(r^*)+\delta+1}\|\nabla\mathbf{z}(t)\|_2^2 + 2\beta\textstyle\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(s)\|_2^2\,ds\\ &\leq (1+t_1)^{\alpha(r^*)+\delta+1}\|\nabla\mathbf{z}(t_1)\|_2^2 + (\alpha(r^*)+\delta+1)\textstyle\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta}\|\nabla\mathbf{z}(s)\|_2^2\,ds\\ &+10\textstyle\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta+1}\|\mathbf{z}(s)\|_\infty\|\nabla\mathbf{z}(s)\|_2\|D^2\mathbf{z}(s)\|_2\,ds.\end{aligned} \tag{3.51}$$

Using the inequalities (2.1) and (2.3) in (3.51), we find

$$\begin{aligned}(1+t)^{\alpha(r^*)+\delta+1}\|\nabla\mathbf{z}(t)\|_2^2 + 2\beta\int_{t_1}^{t}&(1+s)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(s)\|_2^2\,ds\\ &\leq (1+t_1)^{\alpha(r^*)+\delta+1}\|\nabla\mathbf{z}(t_1)\|_2^2\\ &+C\int_{t_1}^{t}(1+t)^{\alpha(r^*)+\delta}\|\nabla\mathbf{z}(s)\|_2^2\,ds\\ &+C\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta+1}\|\mathbf{z}(s)\|_2\|D^2\mathbf{z}(s)\|_2^2\,ds.\end{aligned} \tag{3.52}$$

By Theorem 3.3, we infer that $\|\mathbf{z}(t)\|_2 \ll 1$, for each $t \gg 1$. Consequently, increasing t_1, if necessary, we have that $\|\mathbf{z}(t)\|_2 \leq \beta/5$, for all $t \geq t_1$. Therefore, for any $t \geq t_1$, we obtain

$$\begin{aligned}(1+t)^{\alpha(r^*)+\delta+1}\|\nabla\mathbf{z}(t)\|_2^2 + C\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(s)\|_2^2\,ds &\leq (1+t_1)^{\alpha(r^*)+\delta+1}\|\nabla\mathbf{z}(t_1)\|_2^2\\ &+C\int_{t_1}^{t}(1+t)^{\alpha(r^*)+\delta}\|\nabla\mathbf{z}(s)\|_2^2\,ds\\ &\leq C(1+t)^{\delta}\end{aligned} \tag{3.53}$$

where the last inequality is a consequence of (S2) and (3.42). Therefore, for any $t \geq t_1$, we get

$$\|\nabla\mathbf{z}(t)\|_2^2 \leq C(t+1)^{-\min\{2+r^*,3\}}.$$

Moreover, for every $0 < t \leq t_1$, it follows from (S2) that

$$(t+1)^{\min\{2+r^*,3\}}\|\nabla\mathbf{z}(t)\|_2^2 \leq (t_1+1)^{\min\{2+r^*,3\}}\|\nabla\mathbf{z}(t)\|_2^2 \leq C.$$

This proves the claim.

Now, we can proof the decay estimate for angular velocity:

Theorem 3.4 *Let* $\mathbf{z}$ *be a weak solution to* (3.1)*. Let* $r^*(\mathbf{z}_0) = r^*$ *be the decay character of* $\mathbf{z}_0$*, with* $-1 < r^* < \infty$*. Then, we have the improved decay estimate*

$$\|\mathbf{w}(t)\|_2^2 \leq C(1+t)^{-\min\{2+r^*,3\}}, \quad \forall t \geq 0.$$

Proof: Firstly, note that, by the integral representation of Eq. $(3.1)_2$, we have

$$\mathbf{w}(t) = e^{-2\chi t} e^{\gamma\Delta t}\mathbf{w}(0) + \int_0^t e^{-2\chi(t-s)} e^{\gamma\Delta(t-s)} \{\chi\nabla\times\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\mathbf{w}\}(s)\,\mathrm{d}s, \quad \forall\, t \geq 0, \tag{3.54}$$

where $\left(e^{\gamma\Delta\tau}\right)_{\tau\geq 0}$ is the heat semigroup. Thus,

$$\begin{aligned}\|\mathbf{w}(t)\|_2 \leq e^{-2\chi t}\|e^{\gamma\Delta t}\mathbf{w}(0)\|_2 + \int_0^t e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}(\nabla\times\mathbf{u})(s)\|_2\,ds \\ + \int_0^t e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}(\mathbf{u}\cdot\nabla\mathbf{w})(s)\|_2\,ds.\end{aligned} \tag{3.55}$$

On one hand, from the estimate (2.14), we have

$$e^{-2\chi t}\|e^{\gamma\Delta t}\mathbf{w}(0)\|_2 = e^{-2\chi t}\|e^{\gamma\Delta t}\mathbf{w}_0\|_2 \leq Ce^{-2\chi t}, \quad C = C(\|\mathbf{z}_0\|_2) \tag{3.56}$$

and

$$\|e^{\gamma\Delta(t-s)}(\nabla\times\mathbf{u})(s)\|_2 \leq C\|(\nabla\times\mathbf{u})(s)\|_2 = C\|\nabla\mathbf{u}(s)\|_2 \leq C(1+s)^{-\frac{1}{2}\min\{2+r^*,3\}}, \quad \forall s > 0 \tag{3.57}$$

where the last inequality it follows from Lemma 3.3. On the other hand, using the energy estimate given in (3.24) and Lemma 3.3, we obtain

$$\begin{aligned}\|e^{\gamma\Delta(t-s)}[(\mathbf{u}\cdot\nabla)\mathbf{w}](s)\|_2 &\leq C\|[(\mathbf{u}\cdot\nabla)\mathbf{w}](s)\|_1(t-s)^{-\frac{1}{2}} \\ &\leq C\|\mathbf{u}(s)\|_2\|\nabla\mathbf{w}(s)\|_2(t-s)^{-\frac{1}{2}} \\ &\leq C(t-s)^{-\frac{1}{2}}\|\mathbf{z}(s)\|_2\|\nabla\mathbf{z}(s)\| \\ &\leq C(t-s)^{-\frac{1}{2}}\|\mathbf{z}_0\|_2\|\nabla\mathbf{z}(s)\| \\ &\leq C(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}\min\{2+r^*,3\}}, \quad \forall s > 0\end{aligned} \tag{3.58}$$

Using the estimates (3.56), (3.57) and (3.58) in (3.55), we get

$$\begin{aligned}\|\mathbf{w}(t)\|_2 \leq Ce^{-2\chi t} + \underbrace{\int_0^t e^{-2\chi(t-s)}(1+s)^{-\frac{1}{2}\min\{2+r^*,3\}}\,ds}_{I_1(t)} \\ + \underbrace{\int_0^t e^{-2\chi(t-s)}(1+s)^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}\min\{2+r^*,3\}}\,ds}_{I_2(t)}.\end{aligned} \tag{3.59}$$

Let $\lambda(r^*) \stackrel{\text{def}}{=} \min\{2+r^*, 3\}$ and $h(t) = e^{-2\chi t}(1+t)^{\frac{\lambda(r^*)}{2}}$, for all $t > -1$. Then $h(t) \le h(t^*)$ for all $t > -1$ where $t^* = -1 + \lambda(r^*)/4\chi$. Then

$$e^{-2\chi t} = h(t)(1+t)^{-\frac{\lambda(r^*)}{2}} \le h(t^*)(1+t)^{-\frac{\lambda(r^*)}{2}} \le C(1+t)^{-\frac{\lambda(r^*)}{2}}. \tag{3.60}$$

Now, we will find estimates from $I_1(t)$ and $I_2(t)$.

Estimate for $I_1(t)$: We can rewrite $I_1(t)$ as follows

$$\begin{aligned} I_1(t) &= \int_0^{\frac{t}{2}} e^{-2\chi(t-s)}(1+s)^{-\frac{\lambda(r^*)}{2}}\, ds + \int_{\frac{t}{2}}^t e^{-2\chi(t-s)}(1+s)^{-\frac{\lambda(r^*)}{2}}\, ds \\ &= I_{11}(t) + I_{12}(t) \end{aligned}$$

Since $s \le t/2$ in $I_{11}(t)$, we have that $e^{-2\chi(t-s)} \le e^{-\chi t}$. Hence

$$I_{11}(t) \le e^{-\chi t}\int_0^{\frac{t}{2}} (1+s)^{-\frac{\lambda(r^*)}{2}}\, ds \le e^{-\chi t}\int_0^t (1+s)^{-\frac{\lambda(r^*)}{2}}\, ds.$$

If $-1 < r^* < 0$, i.e, $\lambda(r^*)/2 < 1$, we find

$$I_{11}(t) \le Ce^{-\chi t}(1+t)^{1-\frac{\lambda(r^*)}{2}}.$$

Let $h_1(t) = e^{-\chi t}(1+t)$, for all $t > -1$. Then $h_1(t) \le h_1(t_1^*)$ where $t_1^* = -1 + 1/\chi$. Thus

$$I_{11}(t) \le C(1+t)^{-\frac{\lambda(r^*)}{2}} h_1(t) \le C(1+t)^{-\frac{\lambda(r^*)}{2}}.$$

If $r^* = 0$, i.e, $\lambda(r^*)/2 = 1$, we have

$$\begin{aligned} I_{11}(t) \le Ce^{-\chi t}\ln(1+t) \le Ce^{-\chi t}(1+t) &\le Ce^{-\chi t}(1+t)^{\frac{\lambda(r^*)}{2}+1}(1+t)^{-\frac{\lambda(r^*)}{2}} \\ &\le C(1+t)^{-\frac{\lambda(r^*)}{2}} h_2(t_2^*) \\ &\le C(1+t)^{-\frac{\lambda(r^*)}{2}} \end{aligned}$$

where $h_2(t) = e^{-\chi t}(1+t)^{1+\frac{\lambda(r^*)}{2}}$ and $t_2^* = -1 + [\lambda(r^*)/2 + 1]/\chi$.

Similarly, if $r^* > 0$, i.e, $\lambda(r^*)/2 > 1$ we have

$$I_{11}(t) \le Ce^{-\chi t} \le C(1+t)^{-\frac{\lambda(r^*)}{2}} h_3(t_3^*) \le C(1+t)^{-\frac{\lambda(r^*)}{2}}$$

where $h_3(t) = e^{-\chi t}(1+t)^{\frac{\lambda(r^*)}{2}}$, for all $t > -1$, and $t_3^* = -1 + \lambda(r^*)/2\chi$.

Then, we conclude that

$$I_{11}(t) \le C(1+t)^{-\frac{\lambda(r^*)}{2}}, \quad t > 0. \tag{3.61}$$

Now, since $t/2 \le s$ in $I_{12}(t)$, it follows that

$$(1+s)^{-\frac{\lambda(r^*)}{2}} \le 2^{\frac{\lambda(r^*)}{2}}(2+t)^{-\frac{\lambda(r^*)}{2}} \le C(1+t)^{-\frac{\lambda(r^*)}{2}}.$$

Thus

$$
\begin{aligned}
I_{12}(t) \leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}\, ds &\leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \int_{0}^{t} e^{-2\chi(t-s)}\, ds \\
&\leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \int_{0}^{2\chi t} e^{-\tau}\, d\tau \\
&\leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \int_{0}^{\infty} e^{-\tau}\, d\tau \\
&= C(1+t)^{-\frac{\lambda(r^*)}{2}}.
\end{aligned}
\tag{3.62}
$$

Therefore, from (3.61) and (3.62) we have that

$$
I_1(t) \leq C(1+t)^{-\frac{\lambda(r^*)}{2}}, \quad t > 0. \tag{3.63}
$$

Estimate for $I_2(t)$***:*** Note that

$$
\begin{aligned}
I_2(t) &= \int_0^t e^{-2\chi(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{\lambda(r^*)}{2}}\, ds \\
&= \int_0^{\frac{t}{2}} e^{-2\chi(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{\lambda(r^*)}{2}}\, ds + \int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{\lambda(r^*)}{2}}\, ds \\
&= I_{21}(t) + I_{22}(t).
\end{aligned}
$$

Since $s \leq t/2$ in I_{21}, it follows that, $e^{-2\chi(t-s)} \leq e^{-\chi t}$ and $(t-s)^{-\frac{1}{2}} \leq 2^{\frac{1}{2}} t^{-\frac{1}{2}}$. Hence, on one hand,

$$
I_{21}(t) \leq Ce^{-\chi t} t^{-\frac{1}{2}} \int_0^{\frac{t}{2}} (1+s)^{-\frac{\lambda(r^*)}{2}}\, ds \leq Ce^{-\chi t} t^{-\frac{1}{2}} \int_0^{t} (1+s)^{-\frac{\lambda(r^*)}{2}}\, ds.
$$

From the estimate of $I_{11}(t)$, we get that

$$
e^{-\chi t} \int_0^t (1+s)^{-\frac{\lambda(r^*)}{2}}\, ds \leq C(1+t)^{-\frac{\lambda(r^*)}{2}}.
$$

On the other hand, taking $t > t_0$ for some $t_0 > 0$, we have that $t^{-1/2} < t_0^{-1/2}$, then

$$
I_{21}(t) \leq C(1+t)^{-\frac{\lambda(r^*)}{2}}, \quad t > t_0. \tag{3.64}
$$

Since $t/2 \leq s$ in $I_{22}(t)$, it follows that, $(1+s)^{-\frac{\lambda(r^*)}{2}} \leq C(1+t)^{-\frac{\lambda(r^*)}{2}}$. Hence

$$
\begin{aligned}
I_{22}(t) \leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}(t-s)^{-\frac{1}{2}}\, ds &\leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \int_0^t e^{-2\chi(t-s)}(t-s)^{-\frac{1}{2}}\, ds \\
&\leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \int_0^{2\chi t} e^{-\tau}\tau^{-\frac{1}{2}}\, d\tau \\
&\leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \int_0^{\infty} e^{-\tau}\tau^{-\frac{1}{2}}\, d\tau \\
&\leq C(1+t)^{-\frac{\lambda(r^*)}{2}} \Gamma(1/2) \\
&\leq C(1+t)^{-\frac{\lambda(r^*)}{2}}
\end{aligned}
\tag{3.65}
$$

where Γ is the Euler's Gamma function and $\Gamma(1/2) = \sqrt{\pi}$. From (3.64) and (3.65) we have

$$I_2(t) \leq C(1+t)^{-\frac{\lambda(r^*)}{2}}, \quad t > t_0. \tag{3.66}$$

Then from (3.59), (3.63) and (3.66) we have

$$\|\mathbf{w}(t)\|_2^2 \leq \tilde{C}(1+t)^{-\min\{2+r^*,3\}}, \quad t > t_0.$$

Moreover, for every $0 \leq t \leq t_0$, it follows from energy estimate that

$$(t+1)^{\min\{2+r^*,3\}}\|\mathbf{w}(t)\|_2^2 \leq (t_0+1)^{\min\{2+r^*,3\}}\|\mathbf{z}(t)\|_2^2 \leq C.$$

This ends the proof.

Lemma 3.4 *Let* $\mathbf{z}$ *be a weak solution to* (3.1). *Let* $r^*(\mathbf{z}_0) = r^*$ *be the decay character of* $\mathbf{z}_0$. *Then,*

$$\|\nabla \mathbf{w}(t)\|_2^2 + \|D^2\mathbf{z}(t)\|_2^2 + (1+t)\|D^3\mathbf{z}(t)\|_2^2 \leq C(1+t)^{-\min\{3+r^*,4\}}, \quad \forall t > 0.$$

Proof: First of all, we deduce from (S2) that

$$\| D^2\mathbf{z}(t)\|_2 \,\leq\, C, \quad\; \forall\, t > 0. \tag{3.67}$$

Now, differentiating Eqs. $(3.1)_1$, $(3.1)_2$ and $(3.1)_3$ with respect to x_{ℓ_1} and x_{ℓ_2}, multiplying by $2D_{\ell_1}D_{\ell_2}\mathbf{u}$, $2D_{\ell_1}D_{\ell_2}\mathbf{w}$, and $2D_{\ell_1}D_{\ell_2}\mathbf{b}$ in $\mathbf{L}^2(\mathbb{R}^2)$, respectively, and summing over $\ell_1, \ell_2 = 1, 2$, we get

$$
\begin{aligned}
&\frac{d}{dt}\|D^2\mathbf{z}(t)\|_2^2 + 2(\mu+\chi)\|D^3\mathbf{u}(t)\|_2^2 + 2\gamma\|D^3\mathbf{w}(t)\|_2^2 + 2\nu\|D^3\mathbf{b}(t)\|_2^2 + 4\chi\|D^2\mathbf{w}(t)\|_2^2 \\
&= 4\chi\sum_{l_1,l_2=1}^{2}(\nabla\times(D_{l_1}D_{l_2}\mathbf{u}), D_{l_1}D_{l_2}\mathbf{w})_{L^2} + 2\sum_{l_1,l_2=1}^{2}((D_{l_1}D_{l_2}\mathbf{b}\cdot\nabla)\mathbf{b}, D_{l_1}D_{l_2}\mathbf{u})_{L^2} \\
&+2\sum_{l_1,l_2=1}^{2}((D_{l_1}\mathbf{b}\cdot\nabla)D_{l_2}\mathbf{b}, D_{l_1}D_{l_2}\mathbf{u})_{L^2} + 2\sum_{l_1,l_2=1}^{2}((D_{l_2}\mathbf{b}\cdot\nabla)D_{l_1}\mathbf{b}, D_{l_1}D_{l_2}\mathbf{u})_{L^2} \\
&-2\sum_{l_1,l_2=1}^{2}((D_{l_1}D_{l_2}\mathbf{u}\cdot\nabla)\mathbf{u}, D_{l_1}D_{l_2}\mathbf{u})_{L^2} - 2\sum_{l_1,l_2=1}^{2}((D_{l_1}\mathbf{u}\cdot\nabla)D_{l_2}\mathbf{u}, D_{l_1}D_{l_2}\mathbf{u})_{L^2}
\end{aligned} \tag{3.68}
$$

$$
\begin{aligned}
&-2\sum_{l_1,l_2=1}^{2}((D_{l_2}\mathbf{u}\cdot\nabla)D_{l_1}\mathbf{u}, D_{l_1}D_{l_2}\mathbf{u})_{L^2} - 2\sum_{l_1,l_2=1}^{2}((D_{l_1}D_{l_2}\mathbf{u}\cdot\nabla)\mathbf{w}, D_{l_1}D_{l_2}\mathbf{w})_{L^2} \\
&-2\sum_{l_1,l_2=1}^{2}((D_{l_1}\mathbf{u}\cdot\nabla)D_{l_2}\mathbf{w}, D_{l_1}D_{l_2}\mathbf{w})_{L^2} - 2\sum_{l_1,l_2=1}^{2}((D_{l_2}\mathbf{u}\cdot\nabla)D_{l_1}\mathbf{w}, D_{l_1}D_{l_2}\mathbf{w})_{L^2} \\
&+2\sum_{l_1,l_2=1}^{2}((D_{l_1}D_{l_2}\mathbf{b}\cdot\nabla)\mathbf{u}, D_{l_1}D_{l_2}\mathbf{b})_{L^2} + 2\sum_{l_1,l_2=1}^{2}((D_{l_1}\mathbf{b}\cdot\nabla)D_{l_2}\mathbf{u}, D_{l_1}D_{l_2}\mathbf{b})_{L^2} \\
&+2\sum_{l_1,l_2=1}^{2}((D_{l_2}\mathbf{b}\cdot\nabla)D_{l_1}\mathbf{u}, D_{l_1}D_{l_2}\mathbf{b})_{L^2} - 2\sum_{l_1,l_2=1}^{2}((D_{l_1}D_{l_2}\mathbf{u}\cdot\nabla)\mathbf{b}, D_{l_1}D_{l_2}\mathbf{b})_{L^2} \\
&-2\sum_{l_1,l_2=1}^{2}((D_{l_1}\mathbf{u}\cdot\nabla)D_{l_2}\mathbf{b}, D_{l_1}D_{l_2}\mathbf{b})_{L^2} - 2\sum_{l_1,l_2=1}^{2}((D_{l_2}\mathbf{u}\cdot\nabla)D_{l_1}\mathbf{b}, D_{l_1}D_{l_2}\mathbf{b})_{L^2}.
\end{aligned}
$$

Now, we will estimate each term in the right hand of (3.68). From Hölder and Young's inequalities, we have

$$
\begin{aligned}
4\chi\sum_{l_1,l_2=1}^{2}(\nabla\times(D_{l_1}D_{l_2}\mathbf{u}), D_{l_1}D_{l_2}\mathbf{w})_{L^2} &\le 4\chi\sum_{l_1,l_2=1}^{2}\|\nabla\times(D_{l_1}D_{l_2}\mathbf{u})\|_2\|D_{l_1}D_{l_2}\mathbf{w}\|_2 \\
&\le 2\chi\sum_{l_1,l_2=1}^{2}\left(\|\nabla\times(D_{l_1}D_{l_2}\mathbf{u})\|_2^2 + \|D_{l_1}D_{l_2}\mathbf{w}\|_2^2\right) \\
&= 2\chi\sum_{l_1,l_2=1}^{2}\left(\|\nabla(D_{l_1}D_{l_2}\mathbf{u})\|_2^2 + \|D_{l_1}D_{l_2}\mathbf{w}\|_2^2\right) \\
&= 2\chi\|D^3\mathbf{u}\|_2^2 + 2\chi\|D^2\mathbf{w}\|_2^2.
\end{aligned} \tag{3.69}
$$

Let $\mathbf{F}=\mathbf{u},\mathbf{b}$ and $\mathbf{G}=\mathbf{u},\mathbf{w},\mathbf{b}$. Then, integrating by parts and using that $\operatorname{div}\ \mathbf{F}=0$, we have

$$\pm 2\sum_{l_1,l_2=1}^{2}((D_{l_1}D_{l_2}\mathbf{F}\cdot\nabla)D_{l_1}D_{l_2}\mathbf{G},D_{l_1}D_{l_2}\mathbf{G})_{L^2} = \pm 2\sum_{i,j,l_1,l_2=1}^{2}\int_{\mathbb{R}^2}(D_{l_1}D_{l_2}F_jD_jG_i)(D_{l_1}D_{l_2}G_i)\,d\mathbf{x}$$
$$= \pm 2\sum_{i,j,l_1,l_2=1}^{2}\int_{\mathbb{R}^2}(D_{l_1}D_{l_2}F_j)G_i(D_jD_{l_1}D_{l_2}G_i)\,d\mathbf{x}$$
$$\leq 2\sum_{i,j,l_1,l_2=1}^{2}\|G_i\|_\infty\int_{\mathbb{R}^2}|D_{l_1}D_{l_2}F_j||D_jD_{l_1}D_{l_2}G_i|\,d\mathbf{x}$$
$$\leq 2\sum_{i,j,l_1,l_2=1}^{2}\|G_i\|_\infty\|D_{l_1}D_{l_2}F_j\|_2\|D_jD_{l_1}D_{l_2}G_i\|_2$$
$$= 2\|\mathbf{G}\|_\infty\|D^2\mathbf{F}\|_2\|D^3\mathbf{G}\|_2 \leq 2\|\mathbf{z}\|_\infty\|D^2\mathbf{z}\|_2\|D^3\mathbf{z}\|_2$$

and

$$\pm 2\sum_{l_1,l_2=1}^{2}((D_{l_1}\mathbf{F}\cdot\nabla)D_{l_2}\mathbf{G},D_{l_1}D_{l_2}\mathbf{G})_{L^2} = \pm 2\sum_{i,j,l_1,l_2=1}^{2}\int_{\mathbb{R}^2}(D_{l_1}F_jD_jG_i)(D_{l_1}D_{l_2}G_i)\,d\mathbf{x}$$
$$= \pm 2\sum_{i,j,l_1,l_2=1}^{2}\int_{\mathbb{R}^2}(D_{l_1}F_j)G_i(D_jD_{l_1}D_{l_2}G_i)\,d\mathbf{x}$$
$$\leq 2\sum_{i,j,l_1,l_2=1}^{2}\|G_i\|_\infty\int_{\mathbb{R}^2}|D_{l_1}F_j||D_jD_{l_1}D_{l_2}G_i|\,d\mathbf{x}$$
$$\leq 2\|\mathbf{G}\|_\infty\|\nabla\mathbf{F}\|_2\|D^3\mathbf{G}\|_2 \leq 2\|\mathbf{z}\|_\infty\|\nabla\mathbf{z}\|_2\|D^3\mathbf{z}\|_2. \tag{3.70}$$

Analogously, we have that

$$\pm 2\sum_{l_1,l_2=1}^{2}((D_{l_2}\mathbf{F}\cdot\nabla)D_{l_1}\mathbf{G},D_{l_1}D_{l_2}\mathbf{G})_{L^2} \leq 2\|\mathbf{z}\|_\infty\|\nabla\mathbf{z}\|_2\|D^3\mathbf{z}\|_2. \tag{3.71}$$

Hence, from (3.68)-(3.71) we find

$$\frac{d}{dt}\|D^2\mathbf{z}(t)\|_2^2+2\beta\|D^3\mathbf{z}(t)\|_2^2 \leq 10\|\mathbf{z}(t)\|_\infty\|D^2\mathbf{z}(t)\|_2\|D^3\mathbf{z}(t)\|_2+20\|\mathbf{z}(t)\|_\infty\|\nabla\mathbf{z}(t)\|_2\|D^3\mathbf{z}(t)\|_2 \tag{3.72}$$

where $\beta=\min\{\mu,\gamma,\nu\}$. As before, let $\alpha(r^*)\stackrel{\text{def}}{=}\min\{1+r^*,2\}$. Taking $\delta>0$, we have

$$\begin{aligned}
&\frac{d}{dt}\Big[(1+t)^{\alpha(r^*)+\delta+2}\|D^2\mathbf{z}(t)\|_2^2\Big]+(1+t)^{\alpha(r^*)+\delta+2}2\beta\|D^3\mathbf{z}(t)\|_2^2\\
&\leq C(1+t)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(t)\|_2^2+(1+t)^{\alpha(r^*)+\delta+2}\left[\frac{d}{dt}\|D^2\mathbf{z}(t)\|_2^2+\beta\|D^3\mathbf{z}(t)\|_2^2\right]\\
&\leq C(1+t)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(t)\|_2^2+10(1+t)^{\alpha(r^*)+\delta+2}\|\mathbf{z}(t)\|_\infty\|D^2\mathbf{z}(t)\|_2\|D^3\mathbf{z}(t)\|_2\\
&\quad+20(1+t)^{\alpha(r^*)+\delta+2}\|\mathbf{z}(t)\|_\infty\|\nabla\mathbf{z}(t)\|_2\|D^3\mathbf{z}(t)\|_2
\end{aligned} \tag{3.73}$$

where in the last inequality we used (3.72). Take $t_2 \gg 1$ (fixed, but otherwise arbitrary), integrating on (t_2, t) and using the inequalities (see e.g. (GUTERRES et al., 2019, Lemma 2))

$$\|\mathbf{v}\|_\infty \|D^2\mathbf{v}\|_2 \leq C\|\mathbf{v}\|_2\|D^3\mathbf{v}\|_2 \quad \text{and} \quad \|\nabla\mathbf{v}\|_\infty\|\nabla\mathbf{v}\|_2 \leq C\|\mathbf{v}\|_2\|D^3\mathbf{v}\|_2, \quad \forall\, \mathbf{v} \in \boldsymbol{H}^3(\mathbb{R}^2),$$

we obtain

$$\begin{aligned}
&(t+1)^{\alpha(r^*)+\delta+2}\|D^2\mathbf{z}(t)\|_2^2 + 2\beta\int_{t_2}^{t}(s+1)^{\alpha(r^*)+\delta+2}\|D^3\mathbf{z}(s)\|_2^2\,\mathrm{d}s \\
&\leq (t_2+1)^{\alpha(r^*)+\delta+2}\|D^2\mathbf{z}(t_2)\|_2^2 + C\int_{t_2}^{t}(s+1)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(s)\|_2^2\,\mathrm{d}s \\
&+ 10\int_{t_2}^{t}(s+1)^{\alpha(r^*)+\delta+2}\|\mathbf{z}(s)\|_\infty\|D^2\mathbf{z}(s)\|_2\|D^3\mathbf{z}(s)\|_2\,\mathrm{d}s \qquad (3.74)\\
&+ 20\int_{t_2}^{t}(s+1)^{\alpha(r^*)+\delta+2}\|\nabla\mathbf{z}(s)\|_\infty\|\nabla\mathbf{z}(s)\|_2\|D^3\mathbf{z}(s)\|_2\,\mathrm{d}s \\
&\leq (t_2+1)^{\alpha(r^*)+\delta+2}\|D^2\mathbf{z}(t_2)\|_2^2 + C\int_{t_2}^{t}(s+1)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(s)\|_2^2\,\mathrm{d}s \\
&+ \widetilde{C}\int_{t_2}^{t}(s+1)^{\alpha(r^*)+\delta+2}\|\mathbf{z}(s)\|_2\|D^3\mathbf{z}(s)\|_2^2\,\mathrm{d}s\ .
\end{aligned}$$

From (3.53), we have that

$$\int_{t_1}^{t}(1+s)^{\alpha(r^*)+\delta+1}\|D^2\mathbf{z}(s)\|_2^2\,ds \leq C(1+t)^\delta$$

and from Theorem 3.3 increasing t_2, if necessary, we have $\|\mathbf{z}(t)\|_2 \leq \beta/\widetilde{C}$, for any $t \geq t_2$. Then, from (3.74), it follows that

$$(1+t)^{\alpha(r^*)+\delta+2}\|D^2\mathbf{z}(t)\|_2^2 + C\int_{t_2}^{t}(1+s)^{\alpha(r^*)+\delta+2}\|D^3\mathbf{z}(s)\|_2^2\,ds \leq C(1+t)^\delta \qquad (3.75)$$

and thus

$$\|D^2\mathbf{z}(t)\|_2^2 \leq C(t+1)^{-\alpha(r^*)-2}, \quad \forall\, t \geq t_2. \qquad (3.76)$$

In a similar way, for some $t_3 \gg 1$ (fixed, but otherwise arbitrary), we have that

$$(1+t)^{\alpha(r^*)+\delta+3}\|D^3\mathbf{z}(t)\|_2^2 + C\int_{t_3}^{t}(1+s)^{\alpha(r^*)+\delta+3}\|D^4\mathbf{z}(s)\|_2^2\,ds \leq C(1+t)^\delta, \quad t \geq t_3. \qquad (3.77)$$

Therefore, by (3.76) and (3.77), and having in mind (S2), we conclude that

$$\|D^2\mathbf{z}(t)\|_2^2 + (1+t)\|D^3\mathbf{z}(t)\|_2^2 \leq C(1+t)^{-\alpha(r^*)-2}, \quad t > 0. \qquad (3.78)$$

Furthermore, by standard Sobolev embeddings, Theorem 3.3, Lemma 3.3 and (3.77) we obtain, for all $t > 0$, that

$$\|\mathbf{z}\|_4^2 \le C\|\mathbf{z}\|_2\|\nabla\mathbf{z}\|_2 \le C(1+t)^{-\frac{\alpha(r^*)}{2}}(1+t)^{-\frac{\alpha(r^*)}{2}-\frac{1}{2}} \le C(1+t)^{-\alpha(r^*)-\frac{1}{2}}, \tag{3.79}$$

$$\|\nabla\mathbf{z}\|_4^2 \le C\|\nabla\mathbf{z}\|_2\|D^2\mathbf{z}\|_2 \le C(1+t)^{-\frac{\alpha(r^*)}{2}-\frac{1}{2}}(1+t)^{-\frac{\alpha(r^*)}{2}-1} \le C(1+t)^{-\alpha(r^*)-1}, \tag{3.80}$$

$$\|D^2\mathbf{z}\|_4^2 \le C\|D^2\mathbf{z}\|_2\|D^3\mathbf{z}\|_2 \le C(1+t)^{-\frac{\alpha(r^*)}{2}-1}(1+t)^{-\frac{\alpha(r^*)}{2}-\frac{3}{2}} \le C(1+t)^{-\alpha(r^*)-\frac{5}{2}}. \tag{3.81}$$

Thus, from (3.79)-(3.81), we find

$$\|\nabla^j\mathbf{z}(t)\|_4^2 \le C(1+t)^{-\alpha(r^*)-\frac{1}{2}-j}, \quad \forall t > 0. \tag{3.82}$$

This particular bound allows us to estimate the gradient of the micro-rotational field. Indeed, taking the gradient in Eq. $(3.1)_2$ we have

$$\frac{d}{dt}\nabla\mathbf{w} + 2\chi\nabla\mathbf{w} = \gamma\Delta(\nabla\mathbf{w}) + \chi\nabla(\nabla\times\mathbf{u}) - \nabla(\mathbf{u}\cdot\nabla\mathbf{w}).$$

For $t \ge t_0 > 0$, the integral representation of $\nabla\mathbf{w}$ is given by

$$\nabla\mathbf{w}(\mathbf{x},t) = e^{-2\chi(t-t_0)}e^{\gamma\Delta(t-t_0)}\mathbf{w}(\mathbf{x},t_0) + \int_{t_0}^t e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}[\chi\nabla(\nabla\times\mathbf{u}) - \nabla(\mathbf{u}\cdot\nabla\mathbf{w})](\mathbf{x},s)\,ds.$$

Thus

$$\begin{aligned}\|\nabla\mathbf{w}(t)\|_2 \le{}& e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\nabla\mathbf{w}(t_0)\|_2 + \chi\int_{t_0}^t e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\times\mathbf{u})(s)\|_2\,ds \\ &+ \int_{t_0}^t e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\mathbf{u}\cdot\nabla\mathbf{w})(s)\|_2\,ds.\end{aligned} \tag{3.83}$$

From (2.14) and (S2), we have that

$$e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\nabla\mathbf{w}(t_0)\|_2 \le Ce^{-2\chi(t-t_0)}\|\nabla\mathbf{z}(t_0)\|_2 \le Ce^{-2\chi t}. \tag{3.84}$$

Let $\tilde{\lambda}(r^*) \overset{\mathsf{def}}{=} \min\{3+r^*, 4\}$ and $\tilde{h}(t) = e^{-2\chi t}(1+t)^{\frac{\tilde{\lambda}(r^*)}{2}}$, for all $t > -1$. Then, $\tilde{h}(t) \le \tilde{h}(\tilde{t}^*)$ where $\tilde{t}^* = -1 + \tilde{\lambda}(r^*)/4\chi$. Hence, on one hand, we get

$$e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\nabla\mathbf{w}(t_0)\|_2 \le C\tilde{h}(t)(1+t)^{-\frac{\tilde{\lambda}(r^*)}{2}} \le C(1+t)^{-\frac{\tilde{\lambda}(r^*)}{2}}. \tag{3.85}$$

On the other hand, from (2.14) we have

$$\|e^{\gamma\Delta(t-s)}\nabla(\nabla\times\mathbf{u})(t)\|_2 \le C\|\nabla(\nabla\times\mathbf{u})(t)\|_2 \le C\|D^2\mathbf{u}(t)\|_2 \tag{3.86}$$

and

$$\|e^{\gamma\Delta(t-s)}\nabla((\mathbf{u}\cdot\nabla)\mathbf{w})(t)\|_2 \le C\|\nabla((\mathbf{u}\cdot\nabla)\mathbf{w})(t)\|_2 \\ \le C(\|\mathbf{u}(t)\|_4\|D^2\mathbf{w}(t)\|_4 + \|\nabla\mathbf{u}(t)\|_4\|\nabla\mathbf{w}(t)\|_4). \tag{3.87}$$

Thus, from (3.83), (3.85), (3.86) and (3.87), we find

$$\|\nabla\mathbf{w}(t)\|_2 \le C(1+t)^{-\frac{\tilde\lambda(r^*)}{2}} + C\int_{t_0}^t e^{-2\chi(t-s)}\|D^2\mathbf{u}(s)\|_2\,ds \\ + \int_{t_0}^t e^{-2\chi(t-s)}(\|\mathbf{u}(s)\|_4\|D^2\mathbf{w}(s)\|_4 + \|\nabla\mathbf{u}(s)\|_4\|\nabla\mathbf{w}(s)\|_4)\,ds.$$

It follows, from (3.78) and (3.82), that

$$\|\nabla\mathbf{w}(t)\|_2 \le C(1+t)^{-\frac{\tilde\lambda(r^*)}{2}} + C\int_{t_0}^t e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds \\ + C\int_{t_0}^t e^{-2\chi(t-s)}(1+s)^{-\alpha(r^*)-\frac{3}{2}}\,ds \\ \le C(1+t)^{-\frac{\tilde\lambda(r^*)}{2}} + C\int_{t_0}^t e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds. \tag{3.88}$$

Note that

$$\int_{t_0}^t e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds \le \int_0^t e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds \\ = \int_0^{\frac{t}{2}} e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds + \int_{\frac{t}{2}}^t e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds. \tag{3.89}$$

If $s \le t/2$ then $e^{-2\chi(t-s)} \le e^{-\chi t}$, so

$$\int_0^{\frac{t}{2}} e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds \le e^{-\chi t}\int_0^t (1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds \\ = e^{-\chi t}\Big[-(\alpha(r^*)/2)^{-1}(1+t)^{-\frac{\alpha(r^*)}{2}} + (\alpha(r^*)/2)^{-1}\Big] \\ \le Ce^{-\chi t} = C\tilde h_1(t)(1+t)^{-\frac{\alpha(r^*)}{2}-1} \le C\tilde h_1(\tilde t_1^*)(1+t)^{-\frac{\alpha(r^*)}{2}-1} \tag{3.90}$$

where $\tilde h_1(t) = e^{-\chi t}(1+t)^{\frac{\alpha(r^*)}{2}+1}$ and $\tilde t_1^* = -1 + (\alpha(r^*)+2)/2\chi$.

If $t/2 \le s$, then $(1+s)^{-\frac{\alpha(r^*)}{2}-1} \le C(1+t)^{-\frac{\alpha(r^*)}{2}-1}$, hence

$$\int_{\frac{t}{2}}^t e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds \le C(1+t)^{-\frac{\alpha(r^*)}{2}-1}\int_0^t e^{-2\chi(t-s)}\,ds \\ \le C(1+t)^{-\frac{\alpha(r^*)}{2}-1}\int_0^\infty e^{-\tau}\,d\tau \\ \le C(1+t)^{-\frac{\alpha(r^*)}{2}-1}. \tag{3.91}$$

Then, from (3.89), (3.90) and (3.91), we have

$$\int_{t_0}^t e^{-2\chi(t-s)}(1+s)^{-\frac{\alpha(r^*)}{2}-1}\,ds \le C(1+t)^{-\frac{\alpha(r^*)}{2}-1} = C(1+t)^{-\frac{\tilde\lambda(r^*)}{2}}. \tag{3.92}$$

since $\tilde{\lambda}(r^*) = \alpha(r^*) + 2$. Therefore, from (3.88) and (3.92), we get that

$$\|\nabla \mathbf{w}(t)\|_2^2 \leq C(1+t)^{-\min\{3+r^*,4\}}, \quad t > t_0.$$

Due to (S2), we conclude the proof.

Now, we will compare the evolution of the solution $\mathbf{z}(t)$ with the solution $\overline{\mathbf{z}}(t) \stackrel{\text{def}}{=} (\overline{\mathbf{u}}, \overline{\mathbf{w}}, \overline{\mathbf{b}})(t)$ of the linear system associated with the same initial data. Moreover, we improve the decay rate for the micro-rotational field.

Theorem 3.5 *Let $\mathbf{z}$ be a weak solution to* (3.1)*. Let $r^*(\mathbf{z}_0) = r^*$ be the decay character of $\mathbf{z}_0$, with $-1 < r^* < \infty$. Then*

$$\|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_2^2 \leq C(1+t)^{-\min\{2+2r^*,2\}}, \quad t \geq 0 \tag{3.93}$$

and

$$\|\mathbf{w}(t) - \bar{\mathbf{w}}(t)\|_2^2 \leq C(1+t)^{-\min\{3+2r^*,3\}}, \quad t \geq 0. \tag{3.94}$$

Proof: We shall first prove (3.93). We can write the the linear system (3.5) as

$$\frac{d}{dt}\bar{\mathbf{z}}(\mathbf{x},t) = \mathbb{A}(\mathbf{x},t)\bar{\mathbf{z}}(\mathbf{x},t) \tag{3.95}$$

where $\mathbb{A}$ is the matrix of symbols given by

$$\mathbb{A} \stackrel{\text{def}}{=} \begin{pmatrix} (\mu+\chi)\Delta & \chi\nabla\times & 0 \\ \chi\nabla\times & \gamma\Delta - 2\chi & 0 \\ 0 & 0 & \nu\Delta \end{pmatrix}. \tag{3.96}$$

It is easy to check that solution of (3.5) that of (3.95) is given by

$$\bar{\mathbf{z}}(\mathbf{x},t) = e^{\mathbb{A}t}\mathbf{z}_0(\mathbf{x})$$

where $\left(e^{\mathbb{A}\tau}\right)_{\tau\geq 0}$ is the semigroup associated to (3.5). Then we infer from integral representation of $\mathbf{z}(\mathbf{x},t)$ that

$$\mathbf{z}(\mathbf{x},t) - \bar{\mathbf{z}}(\mathbf{x},t) = \int_0^t e^{\mathbb{A}(t-s)}\widetilde{Q}(s)\,ds \tag{3.97}$$

where

$$\widetilde{Q}(t) \stackrel{\text{def}}{=} \begin{bmatrix} \mathbb{P}[(\mathbf{b}\cdot\nabla)\mathbf{b} - (\mathbf{u}\cdot\nabla)\mathbf{u}](t) \\ -(\mathbf{u}\cdot\nabla)\mathbf{w}(t) \\ (\mathbf{b}\cdot\nabla)\mathbf{u}(t) - (\mathbf{u}\cdot\nabla)\mathbf{b}(t) \end{bmatrix}, \tag{3.98}$$

and $\mathbb{P}$ is the Helmholtz projector. Thus,

$$\|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_2 \leq \int_0^t \|e^{\mathbb{A}(t-s)}\widetilde{Q}(s)\|_2 \, ds.$$

By Plancherel's theorem, Lemma 3.2 gives, for all $\tau \geq 0$,

$$\|e^{\mathbb{A}\tau}\mathbf{v}\|_2 \leq \|e^{C\Delta\tau}\mathbf{v}\|_2, \quad \forall\, \mathbf{v} \in \boldsymbol{L}^2(\mathbb{R}^2), \tag{3.99}$$

where $C = C(\mu, \gamma, \nu) \in \mathbb{R}^+$ (a proof of (3.99) is given in the Appendix). Thus, using (3.99) and the fundamental property that the Helmholtz projector is bounded in L^q if $1 < q < \infty$ (see (FOIAS et al., 2001; GALDI, 2011; FILHO; LOPES, 1999)), we obtain

$$\int_0^t \|e^{\mathbb{A}(t-s)}\widetilde{Q}(s)\|_2 \, ds \leq \int_0^t \|e^{C\Delta(t-s)}Q(s)\|_2 \, ds$$

where

$$Q(t) \overset{\text{def}}{=} \begin{bmatrix} (\mathbf{b}\cdot\nabla)\mathbf{b}(t) - (\mathbf{b}\cdot\nabla)\mathbf{u}(t) \\ -(\mathbf{u}\cdot\nabla)\mathbf{w}(t) \\ (\mathbf{b}\cdot\nabla)\mathbf{u}(t) - (\mathbf{u}\cdot\nabla)\mathbf{b}(t) \end{bmatrix}. \tag{3.100}$$

Then

$$\begin{aligned} \|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_2 \leq \int_0^t \|e^{C\Delta(t-s)}Q(s)\|_2 \, ds &= \int_0^{\frac{t}{2}} \|e^{C\Delta(t-s)}Q(s)\|_2 \, ds + \int_{\frac{t}{2}}^t \|e^{C\Delta(t-s)}Q(s)\|_2 \, ds \\ &\leq C\int_0^{\frac{t}{2}} (t-s)^{-1}\|\mathbf{z}(s)\|_2^2 \, ds \\ &\quad + C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}\|\mathbf{z}(s)\|_2\|\nabla\mathbf{z}(s)\|_2 \, ds \end{aligned} \tag{3.101}$$

where the last inequality it follows from (A.7) and (A.8) in Appendix. From the Theorem 3.3, we have

$$\int_0^{\frac{t}{2}} (t-s)^{-1}\|\mathbf{z}(s)\|_2^2 \, ds \leq C\int_0^{\frac{t}{2}} (t-s)^{-1}(1+s)^{-\alpha(r^*)} \, ds$$

where $\alpha(r^*) \overset{\text{def}}{=} \min\{1 + r^*, 2\}$. Since $s \leq t/2$, implies that, $(t-s)^{-1} \leq 2t^{-1}$. Then

$$\int_0^{\frac{t}{2}} (t-s)^{-1}\|\mathbf{z}(s)\|_2^2 \, ds \leq Ct^{-1}\int_0^{\frac{t}{2}} (1+s)^{-\alpha(r^*)} \, ds. \tag{3.102}$$

If $r^* < 0$, i.e, $\alpha(r^*) = 1 + r^* < 1$, we get

$$\begin{aligned} \int_0^{\frac{t}{2}} (t-s)^{-1}\|\mathbf{z}(s)\|_2^2 \, ds \leq Ct^{-1}\int_0^{\frac{t}{2}} (1+s)^{-\alpha(r^*)} \, ds &\leq Ct^{-1}\int_0^t (1+s)^{-1-r^*} \, ds \\ &\leq Ct^{-1}(1+t)^{-r^*}, \end{aligned}$$

then, for $t > t_1 >> 1$, we find

$$\int_0^{\frac{t}{2}} (t-s)^{-1} \|\mathbf{z}(s)\|_2^2 \, ds \le K(1+t)^{-(1+r^*)}. \tag{3.103}$$

If $r^* = 0$, i.e, $\alpha(r^*) = 1$, it follows that

$$\begin{aligned} \int_0^{\frac{t}{2}} (t-s)^{-1} \|\mathbf{z}(s)\|_2^2 \, ds \le Ct^{-1} \int_0^t (1+s)^{-1} \, ds &= Ct^{-1} \ln(1+t) \\ &\le Ct^{-1} \ln(e+t) \\ &\le K(1+t)^{-1} \end{aligned} \tag{3.104}$$

for all $t > t_1$. Finally, if $r^* > 0$, i.e, $\alpha(r^*) > 1$ we have

$$\int_0^{\frac{t}{2}} (t-s)^{-1} \|\mathbf{z}(s)\|_2^2 \, ds \le Ct^{-1} \int_0^t (1+s)^{-\alpha(r^*)} \, ds \le Ct^{-1} \le K(1+t)^{-1} \tag{3.105}$$

for all $t > t_1$. Then from (3.102)-(3.105) we obtain

$$\int_0^{\frac{t}{2}} (t-s)^{-1} \|\mathbf{z}(s)\|_2^2 \, ds \le C(1+t)^{-\min\{1+r^*,1\}}, \quad t > t_1. \tag{3.106}$$

Now, from the Theorem 3.3 and Lemma 3.3 we have

$$\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \|\mathbf{z}(s)\|_2 \|\nabla \mathbf{z}(s)\|_2 \, ds \le C \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} (1+s)^{-\alpha(r^*)-\frac{1}{2}} \, ds, \quad t > 0. \tag{3.107}$$

Since $t/2 \le s$, implies that, $(1+s)^{-\alpha(r^*)-\frac{1}{2}} \le C(1+t)^{-\alpha(r^*)-\frac{1}{2}}$. Hence

$$\begin{aligned} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \|\mathbf{z}(s)\|_2 \|\nabla \mathbf{z}(s)\|_2 \, ds &\le C(1+t)^{-\alpha(r^*)-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} \, ds \\ &\le C(1+t)^{-\alpha(r^*)-\frac{1}{2}} \int_0^t \tau^{-\frac{1}{2}} \, d\tau \\ &\le Ct^{\frac{1}{2}} (1+t)^{-\alpha(r^*)-\frac{1}{2}} \\ &\le K(1+t)^{-\alpha(r^*)}, \end{aligned} \tag{3.108}$$

for all $t > t_1$. Note that for all $r^* > -1$ we have $\alpha(r^*) \ge \min\{1+r^*, 1\}$, then

$$\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \|\mathbf{z}(s)\|_2 \|\nabla \mathbf{z}(s)\|_2 \, ds \le K(1+t)^{-\min\{1+r^*,1\}}, \quad t > t_1. \tag{3.109}$$

From (3.101), (3.119) and (3.109) we have

$$\|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_2^2 \le K(1+t)^{-\min\{2+2r^*,2\}}, \quad t > t_1. \tag{3.110}$$

Moreover, we will show that the inequality (3.110) holds for all $t \ge 0$. Indeed, from the energy estimate, we set

$$\tilde{M} = \max_{0 \le t \le t_1} \{(1+t)^{\min\{2+2r^*,2\}} \|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_2^2\} < \infty.$$

Then, we conclude that

$$\|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_2^2 \leq C(1+t)^{-\min\{2+2r^*,2\}}, \quad t \geq 0$$

where $C > \max\{K, \tilde{M}\}$.

To prove (3.94) we will need the following gradient estimate:

Lemma 3.5 *Sejam* $\mathbf{z}$ *a weak solution of* (3.1) *and* $r^*(\mathbf{z}_0) = r^*$ *be the decay character of* $\mathbf{z}_0$*. Then*

$$\|\nabla\mathbf{z}(t) - \nabla\bar{\mathbf{z}}(t)\|_2^2 \leq C(1+t)^{-\min\{3+2r^*,3\}}, \quad \forall t > 0.$$

Proof: Note that, by (3.97), we have

$$\|\nabla\mathbf{z}(t) - \nabla\overline{\mathbf{z}}(t)\|_2 \leq \int_0^t \|\nabla e^{\mathbb{A}(t-s)}\widetilde{Q}(s)\|_2 \, \mathrm{d}s, \qquad \forall\, t \geq 0,$$

where $\widetilde{Q}(t)$ is defined in (3.100). Once again using inequality (3.99) and the fact that the Helmholtz projector is bounded in L^2, we get

$$\begin{aligned} \|\nabla\mathbf{z}(t) - \nabla\bar{\mathbf{z}}(t)\|_2 &\leq C\int_0^t \|\nabla e^{C\Delta(t-s)}Q(s)\|_2\, ds \\ &= C\int_0^{t/2} \|\nabla e^{C\Delta(t-s)}Q(s)\|_2\, ds + C\int_{t/2}^t \|\nabla e^{C\Delta(t-s)}Q(s)\|_2\, ds. \end{aligned} \tag{3.111}$$

On one hand, from estimate (A.10) in Appendix, it follows that

$$\|\nabla e^{C\Delta(t-s)}Q(s)\|_2 \leq C(t-s)^{-\frac{3}{2}}\|\mathbf{z}(s)\|_2^2. \tag{3.112}$$

On other hand, from (2.14) we also have that

$$\|\nabla e^{C\Delta(t-s)}Q(s)\|_2 \leq C(t-s)^{-\frac{3}{4}}\|Q(s)\|_{\frac{4}{3}}. \tag{3.113}$$

Note that from Hölder's inequality, we have

$$\begin{aligned} \|Q(s)\|_{4/3}^{4/3} &= \|\mathbf{b}\cdot\nabla\mathbf{b} - \mathbf{u}\cdot\nabla\mathbf{u}\|_{4/3}^{4/3} + \|\mathbf{u}\cdot\nabla\mathbf{w}\|_{4/3}^{4/3} + \|\mathbf{b}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{b}\|_{4/3}^{4/3} \\ &\leq \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{4/3}^{4/3} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{4/3}^{4/3} + \|\mathbf{u}\cdot\nabla\mathbf{w}\|_{4/3}^{4/3} + \|\mathbf{b}\cdot\nabla\mathbf{u}\|_{4/3}^{4/3} + \|\mathbf{u}\cdot\nabla\mathbf{b}\|_{4/3}^{4/3} \\ &\leq \|\mathbf{b}\|_4^{4/3}\|\nabla\mathbf{b}\|_2^{4/3} + \|\mathbf{u}\|_4^{4/3}\|\nabla\mathbf{u}\|_2^{4/3} + \|\mathbf{u}\|_4^{4/3}\|\nabla\mathbf{w}\|_2^{4/3} \\ &+ \|\mathbf{b}\|_4^{4/3}\|\nabla\mathbf{u}\|_2^{4/3} + \|\mathbf{u}\|_4^{4/3}\|\nabla\mathbf{b}\|_2^{4/3} \\ &\leq C\|\mathbf{z}\|_4^{4/3}\|\nabla\mathbf{z}\|_2^{4/3}. \end{aligned} \tag{3.114}$$

and by Gagliardo-niremberg's inequality, we have that

$$\|\mathbf{z}(s)\|_4 \leq C\|\mathbf{z}(s)\|_2^{1/2}\|\nabla\mathbf{z}(s)\|_2^{1/2}. \tag{3.115}$$

Then, from (3.113), (3.114) and (3.115), we obtain

$$\|\nabla e^{C\Delta(t-s)}Q(s)\|_2 \leq C(t-s)^{-\frac{3}{4}}\|\mathbf{z}(s)\|_2^{1/2}\|\nabla\mathbf{z}(s)\|_2^{3/2}. \tag{3.116}$$

Hence, from (3.111)-(3.116), it follows that

$$\begin{aligned}\|\nabla\mathbf{z}(t)-\nabla\bar{\mathbf{z}}(t)\|_2 &\leq C\int_0^{\frac{t}{2}}(t-s)^{-\frac{3}{2}}\|\mathbf{z}(s)\|_2^2\,ds + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4}}\|\mathbf{z}(s)\|_2^{1/2}\|\nabla\mathbf{z}(s)\|_2^{3/2}\,ds\\ &= \mathcal{I}_1(t)+\mathcal{I}_2(t).\end{aligned} \tag{3.117}$$

If $s\leq t/2$ then $(t-s)^{-\frac{3}{2}}\leq 2^{\frac{3}{2}}t^{-\frac{3}{2}}$. Thus from Theorem 3.3, we have

$$\mathcal{I}_1(t)\leq Ct^{-\frac{3}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\alpha(r^*)}\,ds \leq Ct^{-\frac{3}{2}}\int_0^{t}(1+s)^{-\alpha(r^*)}\,ds. \tag{3.118}$$

where $\alpha(r^*)\overset{\text{def}}{=}\min\{1+r^*,2\}$. A straightforward calculation gives, for all $t>t_2\gg 1$,

$$\mathcal{I}_1(t)\leq C(t+1)^{-\left(\frac{3}{2}+r^*\right)},\quad \text{when } -1<r^*<0,$$

$$\mathcal{I}_1(t)\leq C(t+1)^{-\frac{3}{2}},\quad \text{when } r^*>0,$$

$$\mathcal{I}_1(t)\leq Ct^{-\frac{3}{2}}\ln(t+e)\leq C(t+1)^{-\frac{3}{2}},\quad \text{when } r^*=0.$$

for some appropriate constant $C\in\mathbb{R}^+$. Then, from (3.102), we obtain

$$\mathcal{I}_1(t)\leq C(t+1)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}},\quad \forall\, t>t_2. \tag{3.119}$$

From Theorem 3.3 and Lemma 3.3 we have

$$\mathcal{I}_2(t)\leq C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4}}(1+s)^{-\alpha(r^*)-\frac{3}{4}},\quad t>0.$$

If $t/2\leq s$ then $(1+s)^{-\alpha(r^*)-\frac{3}{4}}\leq C(1+t)^{-\alpha(r^*)-\frac{3}{4}}$. Hence

$$\begin{aligned}\mathcal{I}_2(t)\leq C(1+t)^{-\alpha(r^*)-\frac{3}{4}}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4}}\,ds &\leq C(1+t)^{-\alpha(r^*)-\frac{3}{4}}\int_0^{t}(t-s)^{-\frac{3}{4}}\,ds\\ &\leq C(1+t)^{-\alpha(r^*)-\frac{3}{4}}\int_0^{t}\tau^{-\frac{3}{4}}\,d\tau\\ &\leq Ct^{\frac{1}{4}}(1+t)^{-\alpha(r^*)-\frac{3}{4}}\\ &\leq K(1+t)^{-\min\{\frac{3}{2}+r^*,\frac{5}{2}\}},\end{aligned}$$

for $t > 0$. Note that $\min\{3/2 + r^*, 3/2\} \leq \min\{3/2 + r^*, 5/2\}$ for all $r^* > -1$, then

$$\mathcal{I}_2(t) \leq C(1+t)^{-\min\{\frac{3}{2}+r^*,\frac{3}{2}\}}, \quad t > t_2. \tag{3.120}$$

From (3.117), (3.118) and (3.120) we conclude that

$$\|\nabla \mathbf{z}(t) - \nabla \bar{\mathbf{z}}(t)\|_2^2 \leq C(1+t)^{-\min\{3+2r^*,3\}}, \quad t > t_2.$$

Lastily, having in mind (S2), we can extend the result to all $t > 0$

We return to the proof of (3.94). Let $\mathcal{E}_{\mathbf{u}}(t) \stackrel{\text{def}}{=} \mathbf{u}(t) - \bar{\mathbf{u}}(t)$, $\mathcal{E}_{\mathbf{w}}(t) \stackrel{\text{def}}{=} \mathbf{w}(t) - \bar{\mathbf{w}}(t)$ and $\mathcal{E}_{\mathbf{z}}(t) \stackrel{\text{def}}{=} \mathbf{z}(t) - \bar{\mathbf{z}}(t)$. Thus from $(3.1)_2$ and $(3.5)_2$, it follows that

$$\frac{d}{dt}\mathcal{E}_{\mathbf{w}}(t) + 2\chi \mathcal{E}_{\mathbf{w}}(t) = \gamma \Delta \mathcal{E}_{\mathbf{w}}(t) + \chi(\nabla \times \mathcal{E}_{\mathbf{u}})(t) - ((\mathbf{u}\cdot\nabla)\mathbf{w})(t). \tag{3.121}$$

Hence, the integral representation of (3.121) is given by

$$\mathcal{E}_{\mathbf{w}}(t) = e^{-2\chi t} e^{\gamma \Delta t} \mathcal{E}_{\mathbf{w}}(0) + \int_0^t e^{-2\chi(t-s)} e^{\gamma\Delta(t-s)}[\chi(\nabla \times \mathcal{E}_{\mathbf{u}})(s) - ((\mathbf{u}\cdot\nabla)\mathbf{w})(s)]\, ds,$$

where $\mathcal{E}_{\mathbf{w}}(0) = \mathbf{w}(0) - \bar{\mathbf{w}}(0) = \mathbf{w}_0 - \mathbf{w}_0 = 0$. Then, we find that

$$\|\mathcal{E}_{\mathbf{w}}(t)\|_2 \leq \chi \int_0^t e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} \nabla \times \mathcal{E}_{\mathbf{u}}(s)\|_2\, ds + \int_0^t e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)}(\mathbf{u}\cdot\nabla)\mathbf{w}(s)\|_2\, ds. \tag{3.122}$$

Firstly, note that, from (2.14) and Lemma 3.5, it follows that

$$\begin{aligned}\|e^{\gamma\Delta(t-s)}(\nabla \times \mathcal{E}_{\mathbf{u}})(t)\|_2 \leq C\|(\nabla \times \mathcal{E}_{\mathbf{u}})(t)\|_2 \leq C\|\nabla \mathcal{E}_{\mathbf{u}}(s)\|_2 \leq C\|\nabla \mathcal{E}_{\mathbf{z}}(t)\|_2 \\ \leq C(1+t)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}},\end{aligned} \tag{3.123}$$

for all $t > 0$. Next, using (2.14), Hölder's inequality, Theorem 3.3 and Lemma 3.4, we infer

$$\begin{aligned}\|e^{\gamma\Delta(t-s)}((\mathbf{u}\cdot\nabla)\mathbf{w})(t)\|_2 \leq C(t-s)^{-\frac{1}{2}}\|((\mathbf{u}\cdot\nabla)\mathbf{w})(t)\|_1 &\leq C(t-s)^{-\frac{1}{2}}\|\mathbf{u}(t)\|_2\|\nabla \mathbf{w}(t)\|_2 \\ &\leq C(t-s)^{-\frac{1}{2}}\|\mathbf{z}(t)\|_2\|\nabla \mathbf{w}(t)\|_2 \\ &\leq C(t-s)^{-\frac{1}{2}}(1+t)^{-\min\{2+r^*,3\}}\end{aligned} \tag{3.124}$$

for all $t > 0$. Therefore, by (3.122)-(3.124), we obtain that

$$\begin{aligned}\|\mathcal{E}_{\mathbf{w}}(t)\|_2 &\leq C\int_0^t e^{-2\chi(t-s)}(1+s)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}}\, ds + C\int_0^t e^{-2\chi(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\min\{2+r^*,3\}}\, ds \\ &= \mathbb{I}_1(t) + \mathbb{I}_2(t).\end{aligned} \tag{3.125}$$

Note that we can write

$$\begin{aligned}\mathbb{I}_1(t) &\leq C\int_0^{\frac{t}{2}} e^{-2\chi(t-s)}(1+s)^{-\frac{\tilde{\lambda}(r^*)}{2}}\,ds + C\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}(1+s)^{-\frac{\tilde{\lambda}(r^*)}{2}}\,ds \\ &= \mathbb{I}_{1,1}(t) + \mathbb{I}_{1.2}(t)\end{aligned} \tag{3.126}$$

where $\tilde{\lambda} = \min\{3+2r^*, 3\}$ and

$$\begin{aligned}\mathbb{I}_2(t) &\leq C\int_0^{\frac{t}{2}} e^{-2\chi(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\min\{2+r^*,3\}}\,ds \\ &+ C\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}(t-s)^{-\frac{1}{2}}(1+s)^{-\min\{2+r^*,3\}}\,ds \\ &= \mathbb{I}_{2,1}(t) + \mathbb{I}_{2,2}(t).\end{aligned} \tag{3.127}$$

As before, a direct calculation gives us

$$\mathbb{I}_{1,1}(t) \leq Ce^{-\chi t}(t+1)^{-\frac{1}{2}-r^*} \leq C(t+1)^{-\left(\frac{3}{2}+r^*\right)}, \quad \text{for } -1 < r^* < -\tfrac{1}{2},$$

$$\mathbb{I}_{1,1}(t) \leq Ce^{-\chi t}\ln(t+e) \leq C(t+1)^{-1}, \quad \text{for } r^* = -\tfrac{1}{2},$$

$$\mathbb{I}_{1,1}(t) \leq Ce^{-\chi t} \leq C(t+1)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}}, \quad \text{for } r^* > -\tfrac{1}{2},$$

and

$$\mathbb{I}_{1,2}(t) \leq C(t+1)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}}, \quad -1 < r^* < \infty.$$

Thus,

$$\mathbb{I}_1(t) \leq C(t+1)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}}, \quad \forall\, t > 0. \tag{3.128}$$

Similarly, we have, for $t > 1$ that

$$\mathbb{I}_{2,1}(t) \leq \frac{Ce^{-\chi t}}{\sqrt{t}} \leq Ce^{-\chi t} \leq C(t+1)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}}, \quad \text{for } -1 < r^* < \infty,$$

and, for $t > 0$,

$$\mathbb{I}_{2,2}(t) \leq (2\chi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)2^{\min\{2+r^*,3\}}(t+1)^{-\min\{2+r^*,3\}} \leq C(t+1)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}}, \quad \forall\, r^* > -1,$$

where Γ is the Euler's Gamma function and $\Gamma(1/2) = \sqrt{\pi}$. Therefore,

$$\mathbb{I}_2(t) \leq C(t+1)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}}, \tag{3.129}$$

for all $t > 1$. In view of estimates (3.125)–(3.129), we get

$$\|\mathcal{E}_{\mathbf{w}}(t)\|_2 \leq C(1+t)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}} \quad t > 1. \tag{3.130}$$

For $t \in [0,1]$, the result follows from (3.24). This ends the proof of theorem.

In the next theorem, we analyze the lower bounds for rates of decay. These bounds are obtained using the reverse triangle inequality and the decays proved in Theorems 3.3 and 3.5.

Theorem 3.6 *Let* $\mathbf{z}$ *be a weak solution of* (3.1) *and* $r^*(\mathbf{z}_0) = r^*$ *be the decay character of* $\mathbf{z}_0$. *Then for* $-1 < r^* \leq 1$ *we have that*

$$\|\mathbf{z}(t)\|_2^2 \geq C(1+t)^{-(1+r^*)}. \tag{3.131}$$

Proof: Note that, by the reverse triangle inequality, we infer

$$\|\mathbf{z}(t)\|_2 = \|\mathbf{z}(t) - \bar{\mathbf{z}}(t) + \bar{\mathbf{z}}(t)\|_2 \geq \|\bar{\mathbf{z}}(t)\|_2 - \|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_2$$

where $\overline{\mathbf{z}}$ is the solution of the linear problem (3.5). As, by Theorem 2.15, we solely have upper bounds for the decay of $\|\mathbf{z}(t) - \overline{\mathbf{z}}(t)\|_2$, then the lower bound (3.131) holds only when the decay of linear part is slower than that of the difference $\mathbf{z} - \overline{\mathbf{z}}$. For this to happen, we must have $-1 < r^* \leq 1$. Therefore, the lower bound (3.131) follows from the estimates in Theorems 2.15 and 3.2. This completes the proof of corollary.

4 ON THE INVISCID AND NON-RESISTIVE LIMIT TO THE EQUATIONS OF THE NONHOMOGENEOUS MAGNETO-MICROPOLAR FLOW

We will prove that when viscosities and resistivity tend to zero, the solution of the system of equations modeling the motion of an incompressible and non-homogeneous magneto-micropolar flow converges to the solution of the ideal magneto-micropolar system, i.e., where these constants are zero (in the L^2 context). A similar result was proven by S. Itoh in (ITOH, 1996) in the L^2 context and by S. Itoh and A. Tani in the L^p context $(p > 3)$ in (ITOH; TANI, 1999), both for the non-homogeneous Navier-Stokes equations. In (SILVA; FERNÁNDEZ-CARA; ROJAS-MEDAR, 2007), the authors extend the results of S. Itoh and A. Tani to the system of a non-homogeneous, viscous, incompressible, and asymmetric fluid (also in the L^p context, $p > 3$) and in (SILVA; CRUZ; ROJAS-MEDAR, 2014b) the authors P. Braz e Silva, F. W. Cruz, and M. Rojas-Medar in the L^2 context.

4.1 INTRODUCTION AND MAIN RESULT

The magneto-micropolar system is a set of partial differential equations that describe the behavior of a flow with both magnetic and micropolar characteristics. The specific form of these equations may vary depending on the assumptions and simplifications made in the modeling process. Below is a generalized form of the magneto-micropolar flow equations.

Let ρ be the density, $\mathbf{u}$ the velocity vector, $\mathbf{w}$ the microrotation vector, $\mathbf{b}$ the magnetic induction vector, p the pression and $\mathbf{f}$ and $\mathbf{g}$ the external forces. The equations can be expressed as:

$$\begin{cases} \rho\mathbf{u}_t + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = (\mu+\chi)\Delta\mathbf{u} + \chi\nabla\times\mathbf{w} + (\mathbf{b}\cdot\nabla)\mathbf{b} + \rho\mathbf{f} \\ \rho\mathbf{w}_t + \rho(\mathbf{u}\cdot\nabla)\mathbf{w} = \gamma\Delta\mathbf{w} + \kappa\nabla(\operatorname{div}\ \mathbf{w}) + \chi\nabla\times\mathbf{u} - 2\chi\mathbf{w} + \rho\mathbf{g} \\ \mathbf{b}_t + (\mathbf{u}\cdot\nabla)\mathbf{b} = \nu\Delta\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u} \\ \operatorname{div}\ \mathbf{u} = \operatorname{div}\ \mathbf{b} = 0 \\ \rho_t + \mathbf{u}\cdot\nabla\rho = 0. \end{cases} \tag{4.1}$$

Note that the system (1.1) is a particular case of the system (4.1) where the density and the external forces are taken to be identically equal to one and zero, respectively.

In this chapter the system (4.1) will be considered in the set $\mathcal{Q}_T = \mathbb{R}^3 \times [0,T]$, $T > 0$, with the initial conditions

$$\begin{cases} \rho(\mathbf{x},0)=\rho_0(\mathbf{x}), \\ \mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \quad \mathbf{w}(\mathbf{x},0)=\mathbf{w}_0(\mathbf{x}), \quad \mathbf{b}(\mathbf{x},0)=\mathbf{b}_0(\mathbf{x}). \end{cases} \tag{4.2}$$

We are interested in studying the inviscid and non-resistive limit for the system (4.1)-(4.2), that is, the transition from a viscous and resistive system to a non-viscous and non-resistive system.

To simplify the notation, let us introduce

$$\bar{\mu}=\mu+\chi.$$

When $\bar{\mu}=\gamma=\kappa=\nu=0$, the system (4.1) becomes

$$\begin{cases} \rho\mathbf{u}_t+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=(\mathbf{b}\cdot\nabla)\mathbf{b}+\rho\mathbf{f} \\ \rho\mathbf{w}_t+\rho(\mathbf{u}\cdot\nabla)\mathbf{w}=\rho\mathbf{g} \\ \mathbf{b}_t+(\mathbf{u}\cdot\nabla)\mathbf{b}=(\mathbf{b}\cdot\nabla)\mathbf{u} \\ \operatorname{div}\ \mathbf{u}=\operatorname{div}\ \mathbf{b}=0 \\ \rho_t+\mathbf{u}\cdot\nabla\rho=0 \end{cases} \tag{4.3}$$

in $Q_T=\mathbb{R}^3\times[0,T]$, $T>0$. We will denote the solution of problem (4.3)-(4.2) by $(\rho^0,\mathbf{u}^0,\mathbf{w}^0,\mathbf{b}^0)$.

Remark 4.1 *Note that we are considering the problems* (4.1)-(4.2) *and* (4.3)-(4.2) *with the same inicial data* $(\rho_0,\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0)$ *and the same external forces* $\mathbf{f}$ *and* $\mathbf{g}$.

The main result of this chapter is to establish the uniform convergence of the solution to the problem (4.1)-(4.2) to the solution of the problem (4.3)-(4.2) as viscosities and resistivity tend to zero. We will prove the following result:

Theorem 4.1 (Main Theorem) *Let* $0<\bar{\mu},\gamma,\nu\le 1$, $\gamma\ge\max\{\kappa,\chi\}$, *and assume that*

$$\rho_0\in C^0(\mathbb{R}^3),\quad \nabla\rho_0\in\mathbf{H}^2(\mathbb{R}^3),\quad 0<m\le\rho_0(x,t)\le M<\infty$$

$$\mathbf{u}_0,\mathbf{w}_0,\mathbf{b}_0\in\mathbf{H}^3(\mathbb{R}^3),\quad \operatorname{div}\ \mathbf{u}_0=\operatorname{div}\ \mathbf{b}_0=0$$

$$\mathbf{f},\mathbf{g}\in L^2([0,T];\mathbf{H}^3(\mathbb{R}^3)).$$

Then there exists $T_0\in(0,T]$, *independent of* $\bar{\mu},\gamma$ *and* ν *such that problem* (4.1)-(4.2) *has a unique solution* $(\rho,\mathbf{u},\mathbf{w},\mathbf{b})$ *which satisfies*

$$\rho\in C^0(\mathbb{R}^3\times[0,T_0]),\quad \nabla\rho\in C^0([0,T_0];\mathbf{H}^2(\mathbb{R}^3)),\quad 0<m\le\rho(x,t)\le M<\infty$$

$$\mathbf{u},\mathbf{w},\mathbf{b}\in C^0([0,T_0];\mathbf{H}^3(\mathbb{R}^3)).$$

Analogous results also holds for the solution $(\rho^0, \mathbf{u}^0, \mathbf{w}^0, \mathbf{b}^0)$ *of the problem* (4.3)-(4.2). *Futhermore, we have*

$$\sup_{0\le t\le T_0} [\|\rho-\rho^0\|_{H^2} + \|\mathbf{u}-\mathbf{u}^0\|_{H^2} + \|\mathbf{w}-\mathbf{w}^0\|_{H^2} + \|\mathbf{b}-\mathbf{b}^0\|_{H^2}] \to 0 \tag{4.4}$$

as $\bar{\mu}, \gamma, \nu \to 0.$

4.2 A PRIORI ESTIMATES

In this section we will obtain estimates for the solutions for the initial value problems (4.1)-(4.2) and (4.3)-(4.2). We will denote by C the many positives constants that may depend from the initial datas, external forces and from the immersion theorems, but not from $\bar{\mu}, \gamma$ and ν. Futhermore, we will focus only on proving estimates for the solution of the problem (4.1)-(4.2), since the proof for the solutions of (4.3)-(4.2) follows analogously.

Let $(\rho, \mathbf{u}, \mathbf{w}, \mathbf{b})$ be a sufficiently smooth solution of the problem (4.1)-(4.2). Firstly, we will prove a estimate for the density:

Lemma 4.1 *Let*

$$\psi(t) = \int_0^t \left(1 + \|\nabla\rho(\cdot, s))\|^2_{H^2} + \|\mathbf{u}(\cdot, s)\|^2_{H^3} + \|\mathbf{w}(\cdot, s)\|^2_{H^3} + \|\mathbf{b}(\cdot, s)\|^2_{H^3}\right)^2 ds. \tag{4.5}$$

For all $(x, t) \in \mathcal{Q}_T = \mathbb{R}^3 \times [0, T]$, *we have*

$$m \le \rho(\mathbf{x}, t) \le M \tag{4.6}$$

and futhermore, we have that

$$\|\nabla\rho(\cdot, t)\|^2_{H^2} \le \|\nabla\rho_0\|^2_{H^2} + C\psi(t). \tag{4.7}$$

Proof: It is well-known that, according to the classical method of characteristics, the solution of problem $(4.1)_5$-$(4.2)_1$ is given by

$$\rho(\mathbf{x}, t) = \rho(\mathbf{y}(s; \mathbf{x}, t), s)\Big|_{s=t} = \rho_0(\mathbf{y}(s; \mathbf{x}, t))\Big|_{s=0},$$

where the particle trajectory $\mathbf{y}(s; \mathbf{x}, t)$ is defined by the solution of the Cauchy problem

$$\frac{d\mathbf{y}}{ds} = \mathbf{u}(\mathbf{y}, s), \quad \mathbf{y}\Big|_{s=t} = \mathbf{x}.$$

Then, from the assumption that $m \le \rho_0(\mathbf{x}) \le M$, it follows the estimate (4.6).

Now, we will prove the estimate (4.7). Taking the operator D^α on each side of the equation $(4.1)_5$, multiplying the resulting equation by $D^\alpha\rho$ in L^2 and summing over $|\alpha| = 1, 2, 3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho(\cdot,t)\|_{H^2}^2 = -\sum_{|\alpha|=1}^{3}(D^\alpha(\mathbf{u}\cdot\nabla\rho), D^\alpha\rho)_{L^2}.$$

Integrating by parts and using the incompressibility condition $\operatorname{div}\ \mathbf{u} = 0$, we find that

$$\sum_{|\alpha|=1}^{3}(\mathbf{u}\cdot D^\alpha(\nabla\rho), D^\alpha\rho)_{L^2} = -\frac{1}{2}\sum_{|\alpha|=1}^{3}\int_{\mathbb{R}^3}(\operatorname{div}\ \mathbf{u})|D^\alpha\rho|^2\,d\mathbf{x} = 0$$

then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\rho(\cdot,t)\|_{H^2}^2 &= -\sum_{|\alpha|=1}^{3}(D^\alpha(\mathbf{u}\cdot\nabla\rho) - \mathbf{u}\cdot D^\alpha(\nabla\rho), D^\alpha\rho)_{L^2}\\ &\le \sum_{|\alpha|=1}^{3}|(D^\alpha(\mathbf{u}\cdot\nabla\rho) - \mathbf{u}\cdot D^\alpha(\nabla\rho), D^\alpha\rho)_{L^2}|\\ &\le \sum_{|\alpha|=1}^{3}\|D^\alpha(\mathbf{u}\cdot\nabla\rho) - \mathbf{u}\cdot D^\alpha(\nabla\rho)\|_2\|D^\alpha\rho\|_2.\end{aligned}$$

Using the inequalities (2.5) and (2.8), we have

$$\begin{aligned}\sum_{|\alpha|=1}^{3}\|D^\alpha(\mathbf{u}\cdot\nabla\rho) - \mathbf{u}\cdot D^\alpha(\nabla\rho)\|_2\|D^\alpha\rho\|_2 &\le C(\|\nabla\mathbf{u}\|_{H^2}\|\nabla\rho\|_\infty + \|\nabla\mathbf{u}\|_\infty\|\nabla\rho\|_{H^2})\|\nabla\rho\|_{H^2}\\ &\le C\|\nabla\mathbf{u}\|_{H^2}\|\nabla\rho\|_{H^2}^2.\end{aligned}$$

Hence,

$$\frac{d}{dt}\|\nabla\rho(\cdot,t)\|_{H^2}^2 \le C\|\nabla\mathbf{u}\|_{H^2}\|\nabla\rho\|_{H^2}^2. \tag{4.8}$$

Therefore, using Young's inequality and integrating with respect to t, it is easy to see that inequality (4.7) holds.

In the next three lemmas we will get uniform estimates for $\|\mathbf{u}(\cdot,t)\|_{H^3}$, $\|\mathbf{w}(\cdot,t)\|_{H^3}$ and $\|\mathbf{b}(\cdot,t)\|_{H^3}$.

Lemma 4.2 *For all $t \in [0,T]$, we have*

$$\begin{aligned}\frac{d}{dt}\sum_{|\alpha|\le 3}\|\sqrt{\rho}D^\alpha\mathbf{u}(\cdot,t)\|_2^2 + \bar{\mu}\|\nabla\mathbf{u}(\cdot,t)\|_{H^3}^2 &\le C[(1+\|\nabla\rho\|_{H^2})\|\mathbf{u}(\cdot,t)\|_{H^3}^3 + \|\mathbf{w}(\cdot,t)\|_{H^3}^2\\ &+\|\nabla\rho(\cdot,t)\|_{H^2}\|\mathbf{u}_t(\cdot,t)\|_{H^2}\|\mathbf{u}(\cdot,t)\|_{H^3} + (1+\|\nabla\rho\|_{H^2})\|\mathbf{f}(\cdot,t)\|_{H^3}\|\mathbf{u}(\cdot,t)\|_{H^3}\\ &+\|\mathbf{b}(\cdot,t)\|_{H^3}^2\|\mathbf{u}(\cdot,t)\|_{H^3}] + 2\sum_{|\alpha|\le 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2}.\end{aligned} \tag{4.9}$$

Proof: Taking the operator D^α on each side of the equation $(4.1)_1$, multiplying the resulting equation by $D^\alpha \mathbf{u}$ in $\mathbf{L}^2$, we deduce, after summing over $|\alpha| \leq 3$, that

$$\begin{aligned}
L_1 + L_2 + L_3 &:= \sum_{|\alpha|\leq 3} (\rho D^\alpha \mathbf{u}_t, D^\alpha \mathbf{u})_{L^2} - \bar{\mu} \sum_{|\alpha|\leq 3} (\Delta D^\alpha \mathbf{u}, D^\alpha \mathbf{u})_{L^2} + \sum_{|\alpha|\leq 3} (D^\alpha \nabla p, D^\alpha \mathbf{u})_{L^2} \\
&= - \sum_{|\alpha|\leq 3} (D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}), D^\alpha \mathbf{u})_{L^2} + \chi \sum_{|\alpha|\leq 3} (\nabla \times D^\alpha \mathbf{w}, D^\alpha \mathbf{u})_{L^2} \\
&+ \sum_{|\alpha|\leq 3} (D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{b}), D^\alpha \mathbf{u})_{L^2} - \sum_{|\alpha|\leq 3} (D^\alpha(\rho \mathbf{u}_t) - \rho D^\alpha \mathbf{u}_t, D^\alpha \mathbf{u})_{L^2} \\
&+ \sum_{|\alpha|\leq 3} (D^\alpha(\rho \mathbf{f}), D^\alpha \mathbf{u})_{L^2}.
\end{aligned} \tag{4.10}$$

It is easy to deduce through integration by parts and from the identity $(4.1)_5$ that

$$\begin{aligned}
L_1 &= \frac{1}{2}\frac{d}{dt} \sum_{|\alpha|\leq 3} \|\sqrt{\rho} D^\alpha \mathbf{u}\|_2^2 - \frac{1}{2} \sum_{|\alpha|\leq 3} (\rho_t D^\alpha \mathbf{u}, D^\alpha \mathbf{u})_{L^2} \\
&= \frac{1}{2}\frac{d}{dt} \sum_{|\alpha|\leq 3} \|\sqrt{\rho} D^\alpha \mathbf{u}\|_2^2 + \frac{1}{2} \sum_{|\alpha|\leq 3} ((\mathbf{u}\cdot\nabla\rho) D^\alpha \mathbf{u}, D^\alpha \mathbf{u})_{L^2} \\
&= \frac{1}{2}\frac{d}{dt} \sum_{|\alpha|\leq 3} \|\sqrt{\rho} D^\alpha \mathbf{u}\|_2^2 - \sum_{|\alpha|\leq 3} (\rho \mathbf{u}\cdot\nabla D^\alpha \mathbf{u}, D^\alpha \mathbf{u})_{L^2}.
\end{aligned}$$

Through integration by parts it is easy to see that $L_2 = \bar{\mu}\|\nabla \mathbf{u}\|_{H^3}^2$. Using again the incompressibility condition $\operatorname{div}\ \mathbf{u} = 0$, we find that

$$L_3 = \sum_{|\alpha|\leq 3} \int_{\mathbb{R}^3} D^\alpha \nabla p \cdot D^\alpha \mathbf{u}\, d\mathbf{x} = - \sum_{|\alpha|\leq 3} \int_{\mathbb{R}^3} D^\alpha p D^\alpha (\nabla \times \mathbf{u})\, d\mathbf{x} = 0.$$

Thus, identity (4.10) can be rewritten as

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \sum_{|\alpha|\leq 3} \|\sqrt{\rho} D^\alpha \mathbf{u}\|_2^2 + \bar{\mu}\|\nabla \mathbf{u}\|_{H^3}^2 &= - \sum_{|\alpha|\leq 3} (D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}) - \rho \mathbf{u}\cdot\nabla D^\alpha \mathbf{u}, D^\alpha \mathbf{u})_{L^2} \\
&+ \chi \sum_{|\alpha|\leq 3} (\nabla \times D^\alpha \mathbf{w}, D^\alpha \mathbf{u})_{L^2} + \sum_{|\alpha|\leq 3} (D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{b}), D^\alpha \mathbf{u})_{L^2} \\
&- \sum_{|\alpha|\leq 3} (D^\alpha(\rho \mathbf{u}_t) - \rho D^\alpha \mathbf{u}_t, D^\alpha \mathbf{u})_{L^2} + \sum_{|\alpha|\leq 3} (D^\alpha(\rho \mathbf{f}), D^\alpha \mathbf{u})_{L^2} \\
&:= R_1 + R_2 + R_3 + R_4 + R_5.
\end{aligned} \tag{4.11}$$

Now, we will estimate each term on the right-hand side of identity (4.11). Due to the estimates

(2.5), (2.8), (4.6) and the Cauchy-Schwarz's inequality, one finds

$$
\begin{aligned}
|R_1| &\leq \sum_{|\alpha|\leq 3} \|D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}) - \rho\mathbf{u}\cdot\nabla D^\alpha\mathbf{u}\|_2\|D^\alpha\mathbf{u}\|_2 \\
&\leq C(\|\nabla(\rho\mathbf{u})\|_\infty\|\nabla\mathbf{u}\|_{H^2} + \|\nabla(\rho\mathbf{u})\|_{H^2}\|\nabla\mathbf{u}\|_\infty)\|\mathbf{u}\|_{H^3} \\
&\leq C\|\nabla(\rho\mathbf{u})\|_{H^2}\|\nabla\mathbf{u}\|_{H^2}\|\mathbf{u}\|_{H^3} \\
&\leq C[\|\mathbf{u}\|_{H^2}\|\nabla\rho\|_{H^2} + \|\mathbf{u}\|_{H^3}(1+\|\nabla\rho\|_{H^2})]\|\mathbf{u}\|_{H^3}^2 \\
&\leq C(1+\|\nabla\rho\|_{H^2})\|\mathbf{u}\|_{H^3}^3.
\end{aligned}
$$

Now using also Young's inequality, we bound

$$
\begin{aligned}
|R_2| \leq \chi\sum_{|\alpha|\leq 3}|(\nabla\times D^\alpha\mathbf{w}, D^\alpha\mathbf{u})_{L^2}| &= \chi\sum_{|\alpha|\leq 3}|(\nabla\times D^\alpha\mathbf{u}, D^\alpha\mathbf{w})_{L^2}| \\
&\leq \chi\sum_{|\alpha|\leq 3}\|\nabla\times D^\alpha\mathbf{u}\|_2\|D^\alpha\mathbf{w}\|_2 \\
&\leq \chi\sum_{|\alpha|\leq 3}\|\nabla D^\alpha\mathbf{u}\|_2\|D^\alpha\mathbf{w}\|_2 \\
&\leq C\chi\|\nabla\mathbf{u}\|_{H^3}\|\mathbf{w}\|_{H^3} \\
&\leq \frac{\chi^2}{2}\|\nabla\mathbf{u}\|_{H^3}^2 + C\|\mathbf{w}\|_{H^3}^2 \leq \frac{\bar{\mu}}{2}\|\nabla\mathbf{u}\|_{H^3}^2 + C\|\mathbf{w}\|_{H^3}^2
\end{aligned}
$$

since $\chi^2 \leq \chi \leq \bar{\mu}$. Similarly, one gets

$$
\begin{aligned}
R_3 &= \sum_{|\alpha|\leq 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2} + \sum_{|\alpha|\leq 3}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{b}) - (\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2} \\
&\leq \sum_{|\alpha|\leq 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2} + \sum_{|\alpha|\leq 3}|(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{b}) - (\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2}| \\
&\leq \sum_{|\alpha|\leq 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2} + \sum_{|\alpha|\leq 3}\|D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{b}) - (\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}\|_2\|D^\alpha\mathbf{u}\|_2 \\
&\leq \sum_{|\alpha|\leq 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2} + C\|\nabla\mathbf{b}\|_\infty\|\nabla\mathbf{b}\|_{H^2}\|\mathbf{u}\|_{H^3} \\
&\leq \sum_{|\alpha|\leq 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2} + C\|\mathbf{b}\|_{H^3}^2\|\mathbf{u}\|_{H^3},
\end{aligned}
$$

$$
\begin{aligned}
|R_4| &\leq \sum_{|\alpha|\leq 3}\|D^\alpha(\rho\mathbf{u}_t) - \rho D^\alpha\mathbf{u}_t\|_2\|D^\alpha\mathbf{u}\|_2 \\
&\leq C(\|\nabla\rho\|_\infty\|\mathbf{u}_t\|_{H^2} + \|\nabla\rho\|_{H^2}\|\mathbf{u}_t\|_\infty)\|\mathbf{u}\|_{H^3} \\
&\leq C\|\nabla\rho\|_{H^2}\|\mathbf{u}_t\|_{H^2}\|\mathbf{u}\|_{H^3}.
\end{aligned}
$$

Finally, we estimate R_5. Due to inequalities (2.5), (2.8) and (4.6), we find

$$
\begin{aligned}
|R_5| &\le \sum_{|\alpha|\le 3} |(D^\alpha(\rho\mathbf{f}) - \rho D^\alpha\mathbf{f}, D^\alpha\mathbf{u})_{L^2}| + \sum_{|\alpha|\le 3} |(\rho D^\alpha\mathbf{f}, D^\alpha\mathbf{u})_{L^2}| \\
&\le \sum_{|\alpha|\le 3} \|D^\alpha(\rho\mathbf{f}) - \rho D^\alpha\mathbf{f}\|_2 \|D^\alpha\mathbf{u}\|_2 + \sum_{|\alpha|\le 3} \|\rho D^\alpha\mathbf{f}\|_2 \|D^\alpha\mathbf{u}\|_2 \\
&\le C(\|\nabla\rho\|_\infty \|\mathbf{f}\|_{H^2} + \|\nabla\rho\|_{H^2}\|\mathbf{f}\|_\infty)\|\mathbf{u}\|_{H^3} + C\|\mathbf{f}\|_{H^3}\|\mathbf{u}\|_{H^3} \\
&\le C(1 + \|\nabla\rho\|_{H^2})\|\mathbf{f}\|_{H^3}\|\mathbf{u}\|_{H^3}.
\end{aligned}
$$

Then, using the estimates for R_1, R_2, R_3 and R_4 , we obtain

$$
\begin{aligned}
\frac{d}{dt}\sum_{|\alpha|\le 3} \|\sqrt{\rho}D^\alpha\mathbf{u}(\cdot,t)\|_2^2 + \bar{\mu}\|\nabla\mathbf{u}(\cdot,t)\|_{H^3}^2 &\le C[(1+\|\nabla\rho\|_{H^2})\|\mathbf{u}(\cdot,t)\|_{H^3}^3 + \|\mathbf{w}(\cdot,t)\|_{H^3}^2 \\
&+ \|\nabla\rho(\cdot,t)\|_{H^2}\|\mathbf{u}_t(\cdot,t)\|_{H^2}\|\mathbf{u}(\cdot,t)\|_{H^3} + (1+\|\nabla\rho\|_{H^2})\|\mathbf{f}(\cdot,t)\|_{H^3}\|\mathbf{u}(\cdot,t)\|_{H^3} \\
&+ \|\mathbf{b}(\cdot,t)\|_{H^3}^2\|\mathbf{u}(\cdot,t)\|_{H^3} + \sum_{|\alpha|\le 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{u})_{L^2}]
\end{aligned}
$$

which proves the Lemma.

Lemma 4.3 *For all $t \in [0,T]$, we have*

$$
\begin{aligned}
\frac{d}{dt}\sum_{|\alpha|\le 3} \|\sqrt{\rho}D^\alpha\mathbf{w}(\cdot,t)\|_2^2 + \gamma\|\nabla\mathbf{w}(\cdot,t)\|_{H^3}^2 &\le C[(1+\|\nabla\rho\|_{H^2})\|\mathbf{u}(\cdot,t)\|_{H^3}\|\mathbf{w}(\cdot,t)\|_{H^3}^2 \\
&+ \|\mathbf{u}(\cdot,t)\|_{H^3}^2 + \|\nabla\rho(\cdot,t)\|_{H^2}\|\mathbf{w}_t(\cdot,t)\|_{H^2}\|\mathbf{w}(\cdot,t)\|_{H^3} \\
&+ (1+\|\nabla\rho\|_{H^2})\|\mathbf{g}(\cdot,t)\|_{H^3}\|\mathbf{w}(\cdot,t)\|_{H^3}].
\end{aligned}
\tag{4.12}
$$

Proof: Taking the operator D^α on each side of the equation $(4.1)_2$, multiplying the resulting equation by $D^\alpha\mathbf{w}$ in $\mathbf{L}^2$, we deduce, after summing over $|\alpha| \le 3$, that

$$
\begin{aligned}
E_1 + E_2 + E_3 &:= \sum_{|\alpha|\le 3}(\rho D^\alpha\mathbf{w}_t, D^\alpha\mathbf{w})_{L^2} - \sum_{|\alpha|\le 3}(\gamma\Delta D^\alpha\mathbf{w}, D^\alpha\mathbf{w})_{L^2} + 2\chi\sum_{|\alpha|\le 3}(D^\alpha\mathbf{w}, D^\alpha\mathbf{w})_{L^2} \\
&= -\sum_{|\alpha|\le 3}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{w}), D^\alpha\mathbf{w})_{L^2} + \kappa\sum_{|\alpha|\le 3}(\nabla(\operatorname{div}\ D^\alpha\mathbf{w}), D^\alpha\mathbf{w})_{L^2} \\
&+ \chi\sum_{|\alpha|\le 3}(\nabla\times D^\alpha\mathbf{u}, D^\alpha\mathbf{w})_{L^2} + \sum_{|\alpha|\le 3}(D^\alpha(\rho\mathbf{g}), D^\alpha\mathbf{w})_{L^2} \\
&- \sum_{|\alpha|\le 3}(D^\alpha(\rho\mathbf{w}_t) - \rho D^\alpha\mathbf{w}_t, D^\alpha\mathbf{w})_{L^2}
\end{aligned}
\tag{4.13}
$$

A straightforward calculation gives

$$\begin{aligned} E_1 &= \frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 3}\|\sqrt{\rho}D^\alpha \mathbf{w}\|_2^2 - \frac{1}{2}\sum_{|\alpha|\le 3}(\rho_t D^\alpha \mathbf{w}, D^\alpha \mathbf{w})_{L^2} \\ &= \frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 3}\|\sqrt{\rho}D^\alpha \mathbf{w}\|_2^2 + \frac{1}{2}\sum_{|\alpha|\le 3}((\mathbf{u}\cdot\nabla\rho) D^\alpha \mathbf{w}, D^\alpha \mathbf{w})_{L^2} \\ &= \frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 3}\|\sqrt{\rho}D^\alpha \mathbf{w}\|_2^2 - \sum_{|\alpha|\le 3}(\rho\mathbf{u}\cdot\nabla D^\alpha \mathbf{w}, D^\alpha \mathbf{w})_{L^2} \end{aligned}$$

$$E_2 = \gamma\|\nabla\mathbf{w}\|_{H^3}^2, \quad E_3 = 2\chi\|\mathbf{w}\|_{H^3}^2.$$

Thus, we can rewrite the identity (4.13) as follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 3}\|\sqrt{\rho}D^\alpha \mathbf{w}\|_2^2 + \gamma\|\nabla\mathbf{w}\|_{H^3}^2 + 2\chi\|\mathbf{w}\|_{H^3}^2 &= -\sum_{|\alpha|\le 3}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{w}) - \rho\mathbf{u}\cdot\nabla D^\alpha\mathbf{w}, D^\alpha\mathbf{w})_{L^2} \\ &\quad - \sum_{|\alpha|\le 3}(D^\alpha(\rho\mathbf{w}_t) - \rho D^\alpha\mathbf{w}_t, D^\alpha\mathbf{w})_{L^2} \\ &\quad + \chi\sum_{|\alpha|\le 3}(\nabla\times D^\alpha\mathbf{u}, D^\alpha\mathbf{w})_{L^2} + \sum_{|\alpha|\le 3}(D^\alpha(\rho\mathbf{g}), D^\alpha\mathbf{w})_{L^2} \\ &\quad + \kappa\sum_{|\alpha|\le 3}(\nabla(\operatorname{div}\ D^\alpha\mathbf{w}), D^\alpha\mathbf{w})_{L^2} \\ &:= D_1 + D_2 + D_3 + D_4 + D_5. \end{aligned} \tag{4.14}$$

Let us estimate D_i for $i = 1, \cdots, 5$. Similary how we estimate the terms R_i for $i = 1, \cdots, 4$ in Lemma 4.2, we find that

$$\begin{aligned} |D_1| &\le C(1 + \|\nabla\rho\|_{H^2})\|\mathbf{u}\|_{H^3}\|\mathbf{w}\|_{H^3}^2, \\ |D_2| &\le C\|\nabla\rho\|_{H^2}\|\mathbf{w}_t\|_{H^2}\|\mathbf{w}\|_{H^3}, \\ |D_3| &\le \frac{\gamma}{2}\|\nabla\mathbf{w}\|_{H^3}^2 + C\|\mathbf{u}\|_{H^3}^2 \text{ (since } \chi^2 \le \chi \le \gamma\text{)}, \\ |D_4| &\le C(1 + \|\nabla\rho\|_{H^2})\|\mathbf{g}\|_{H^3}\|\mathbf{w}\|_{H^3}. \end{aligned}$$

Futhermore, from integration by parts, note that

$$D_5 = -\kappa\|\operatorname{div}\ \mathbf{w}\|_{H^3}^2.$$

Then, using the estimates for D_1, D_2, D_3, D_4 and D_5 into identity (4.14), we obtain

$$\begin{aligned} \frac{d}{dt}\sum_{|\alpha|\le 3}\|\sqrt{\rho}D^\alpha\mathbf{w}(\cdot,t)\|_2^2 + \gamma\|\nabla\mathbf{w}(\cdot,t)\|_{H^3}^2 &\le C[(1 + \|\nabla\rho\|_{H^2})\|\mathbf{u}(\cdot,t)\|_{H^3}\|\mathbf{w}(\cdot,t)\|_{H^3}^2 \\ &\quad + \|\mathbf{u}(\cdot,t)\|_{H^3}^2 + \|\nabla\rho(\cdot,t)\|_{H^2}\|\mathbf{w}_t(\cdot,t)\|_{H^2}\|\mathbf{w}(\cdot,t)\|_{H^3} \\ &\quad + (1 + \|\nabla\rho\|_{H^2})\|\mathbf{g}(\cdot,t)\|_{H^3}\|\mathbf{w}(\cdot,t)\|_{H^3}]. \end{aligned}$$

Lemma 4.4 *For all $t \in [0,T]$, we have*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{b}(\cdot,t)\|_{H^3}^2 + \nu\|\nabla\mathbf{b}(\cdot,t)\|_{H^3}^2 \le C\|\mathbf{u}(\cdot,t)\|_{H^3}\|\mathbf{b}(\cdot,t)\|_{H^3}^2 + \sum_{|\alpha|\le 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{u}, D^\alpha\mathbf{b})_{L^2}. \tag{4.15}$$

Proof: The proof of this lemma is similar to the previous ones. Taking the operator D^α on each side of the equation $(4.1)_3$, multiplying the resulting equation by $D^\alpha\mathbf{b}$ in $\mathbf{L}^2$, we deduce, after summing over $|\alpha| \le 3$, that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{b}\|_{H^3}^2 + \nu\|\nabla\mathbf{b}\|_{H^3}^2 = \sum_{|\alpha|\le 3}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{u}), D^\alpha\mathbf{b})_{L^2} - \sum_{|\alpha|\le 3}(D^\alpha((\mathbf{u}\cdot\nabla)\mathbf{b}), D^\alpha\mathbf{b})_{L^2}. \tag{4.16}$$

Due to the incompressible condition $\operatorname{div}\ \mathbf{u} = 0$, it follows that

$$\sum_{|\alpha|\le 3}((\mathbf{u}\cdot\nabla)D^\alpha\mathbf{b}, D^\alpha\mathbf{b})_{L^2} = 0.$$

Then, we can rewrite the identity (4.16) as follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{b}\|_{H^3}^2 + \nu\|\nabla\mathbf{b}\|_{H^3}^2 &= \sum_{|\alpha|\le 3}(\mathbf{b}\cdot\nabla D^\alpha\mathbf{u}, D^\alpha\mathbf{b})_{L^2} \\ &+ \sum_{|\alpha|\le 3}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{u}) - \mathbf{b}\cdot\nabla D^\alpha\mathbf{u}, D^\alpha\mathbf{b})_{L^2} \\ &- \sum_{|\alpha|\le 3}(D^\alpha((\mathbf{u}\cdot\nabla)\mathbf{b}) - \mathbf{u}\cdot\nabla D^\alpha\mathbf{b}, D^\alpha\mathbf{b})_{L^2} \\ &:= \sum_{|\alpha|\le 3}(\mathbf{b}\cdot\nabla D^\alpha\mathbf{u}, D^\alpha\mathbf{b})_{L^2} + I_1 + I_2.\end{aligned} \tag{4.17}$$

Analogously to the proof of Lemma 4.3, one can bounds

$$|I_1| \le C\|\mathbf{b}\|_{H^3}^2\|\mathbf{u}\|_{H^3}, \quad |I_2| \le C\|\mathbf{b}\|_{H^3}^2\|\mathbf{u}\|_{H^3}.$$

Thus, using the estimates for I_1 and I_2 into identity 4.17, we concludes that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{b}(\cdot,t)\|_{H^3}^2 + \nu\|\nabla\mathbf{b}(\cdot,t)\|_{H^3}^2 \le C\|\mathbf{u}(\cdot,t)\|_{H^3}\|\mathbf{b}(\cdot,t)\|_{H^3}^2 + \sum_{|\alpha|\le 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{u}, D^\alpha\mathbf{b})_{L^2}.$$

Lemma 4.5 *Let A be the constant defined as bellow:*

$$A := 1 + \|\nabla\rho_0\|_{H^2}^2 + \|\mathbf{u}_0\|_{H^3}^2 + \|\mathbf{w}_0\|_{H^3}^2 + \|\mathbf{b}_0\|_{H^3}^2 + \int_0^T (\|\mathbf{f}(\cdot,t)\|_{H^3}^2 + \|\mathbf{g}(\cdot,t)\|_{H^3}^2)\,dt. \tag{4.18}$$

For all $t \in [0,T]$, there holds

$$\begin{gathered}\|\nabla\rho(\cdot,t)\|_{H^2}^2 + \|\mathbf{u}(\cdot,t)\|_{H^3}^2 + \|\mathbf{w}(\cdot,t)\|_{H^3}^2 + \|\mathbf{b}(\cdot,t)\|_{H^3}^2 \\ +\bar{\mu}\textstyle\int_0^t \|\nabla\mathbf{u}(\cdot,s)\|_{H^3}^2\,ds + \gamma\int_0^t \|\nabla\mathbf{w}(\cdot,s)\|_{H^3}^2\,ds + \nu\int_0^t \|\nabla\mathbf{b}(\cdot,s)\|_{H^3}^2\,ds \\ \le C\textstyle\int_0^t(\|\mathbf{u}_t(\cdot,s)\|_{H^2}^2 + \|\mathbf{w}_t(\cdot,s)\|_{H^2}^2)\,ds + C[A + \psi(t)].\end{gathered} \tag{4.19}$$

Proof: Integrating by parts and using the condition $\operatorname{div}\ \mathbf{b}=0$, we find that

$$\sum_{|\alpha|\le 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{u},D^\alpha\mathbf{b})_{L^2}+\sum_{|\alpha|\le 3}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{b},D^\alpha\mathbf{u})_{L^2}=0.$$

Then, collecting inequalities (4.8), (4.9), (4.12) and (4.15) together, and using the Young's inequality we obtain directly , after integrating the resulting inequality with respect to t and using the estimate (4.6), the desired result.

Now we desired to estimate the integrals on the right hand of (4.19). For this purpose, we will prove the following lemma.

Lemma 4.6 *For all $t\in[0,T]$, we have*

$$\begin{array}{c}\bar\mu\|\nabla\mathbf{u}(\cdot,t)\|_{H^2}^2+\gamma\|\nabla\mathbf{w}(\cdot,t)\|_{H^2}^2+\nu\|\nabla\mathbf{b}(\cdot,t)\|_{H^2}^2\\ +m\int_0^t(\|\mathbf{u}_t(\cdot,s)\|_{H^2}^2+\|\mathbf{w}_t(\cdot,s)\|_{H^2}^2)\,ds+\int_0^t\|\mathbf{b}_t(\cdot,s)\|_{H^2}^2\,ds\le C[A+\psi(t)]^3.\end{array}\tag{4.20}$$

Proof: We will divide the proof into five steps.

Step 1. Multiplying $(4.1)_1$ by $\mathbf{u}_t$ in $\mathbf{L}^2$, we obtain

$$\|\sqrt{\rho}\mathbf{u}_t\|_2^2+\frac{\bar\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_2^2=\chi(\nabla\times\mathbf{w},\mathbf{u}_t)_{L^2}+((\mathbf{b}\cdot\nabla)\mathbf{b},\mathbf{u}_t)_{L^2}+(\rho\mathbf{f},\mathbf{u}_t)_{L^2}-(\rho(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{u}_t)_{L^2}\tag{4.21}$$

From inequalities (2.6), (2.8) and (4.6), we bound the terms in the right-hand side of identity (4.21) as follows

$$\begin{aligned}|\chi(\nabla\times\mathbf{w},\mathbf{u}_t)_{L^2}|&=\chi\Big|\Big(\frac{1}{\sqrt{\rho}}\nabla\times\mathbf{w},\sqrt{\rho}\mathbf{u}_t\Big)_{L^2}\Big|\\ &\le\frac{1}{\sqrt{m}}\|\nabla\times\mathbf{w}\|_2\|\sqrt{\rho}\mathbf{u}_t\|_2\ \text{(since } \chi\le\bar\mu\le 1)\\ &\le C\|\nabla\mathbf{w}\|_2\|\sqrt{\rho}\mathbf{u}_t\|_2\\ &\le C\|\mathbf{w}\|_{H^3}\|\sqrt{\rho}\mathbf{u}_t\|_2,\end{aligned}$$

$$\begin{aligned}|((\mathbf{b}\cdot\nabla)\mathbf{b},\mathbf{u}_t)_{L^2}|&\le C\|(\mathbf{b}\cdot\nabla)\mathbf{b}\|_2\|\sqrt{\rho}\mathbf{u}_t\|_2\\ &\le C(\|\mathbf{b}\|_\infty\|\nabla\mathbf{b}\|_2+\|\mathbf{b}\|_2\|\nabla\mathbf{b}\|_\infty)\|\sqrt{\rho}\mathbf{u}_t\|_2\\ &\le C\|\mathbf{b}\|_{H^3}^2\|\sqrt{\rho}\mathbf{u}_t\|_2,\end{aligned}$$

$$|(\rho\mathbf{f},\mathbf{u}_t)_{L^2}|\le\|\sqrt{\rho}\mathbf{f}\|_2\|\sqrt{\rho}\mathbf{u}_t\|_2\le C\|\mathbf{f}\|_{H^3}\|\sqrt{\rho}\mathbf{u}_t\|_2,$$

$$\begin{aligned}|(\rho(\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{u}_t)_{L^2}|&\le\|\sqrt{\rho}(\mathbf{u}\cdot\nabla)\mathbf{u}\|_2\|\sqrt{\rho}\mathbf{u}_t\|_2\\ &\le C(\|\mathbf{u}\|_\infty\|\nabla\mathbf{u}\|_2+\|\mathbf{u}\|_2\|\nabla\mathbf{u}\|_\infty)\|\sqrt{\rho}\mathbf{u}_t\|_2\\ &\le C\|\mathbf{u}\|_{H^3}^2\|\sqrt{\rho}\mathbf{u}_t\|_2.\end{aligned}$$

Using these bounds in identity (4.21), we deduce by the estimate (4.6) and the Young's inequality, that

$$m\|\mathbf{u}_t(\cdot,t)\|_2^2+\bar{\mu}\frac{d}{dt}\|\nabla\mathbf{u}(\cdot,t)\|_2^2\le C[\|\mathbf{u}(\cdot,t)\|_{H^3}^4+\|\mathbf{w}(\cdot,t)\|_{H^3}^2+\|\mathbf{b}(\cdot,t)\|_{H^3}^4+\|\mathbf{f}(\cdot,t)\|_{H^3}^2]. \tag{4.22}$$

Integrating the inequality (4.22) over $[0.T]$, we find that

$$m\int_0^t\|\mathbf{u}_t(\cdot,s)\|_2^2\,ds+\bar{\mu}\|\nabla\mathbf{u}(\cdot,t)\|_2^2\le C[A+\psi(t)] \tag{4.23}$$

where $\psi(t)$ and A are given by (4.5) and (4.18), respectively. Similarly how we obtained the estimate (4.23), one deduces from $(4.1)_2$ and $(4.1)_3$ that

$$\begin{aligned}m\|\mathbf{w}_t(\cdot,t)\|_2^2+\gamma\frac{d}{dt}\|\nabla\mathbf{w}(\cdot,t)\|_2^2+\kappa\frac{d}{dt}\|\operatorname{div}\ \mathbf{w}(\cdot,t)\|_2^2+2\chi\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|_2^2\\ \le C[\|\mathbf{u}(\cdot,t)\|_{H^3}^2\|\mathbf{w}(\cdot,t)\|_{H^3}^2+\|\mathbf{u}(\cdot,t)\|_{H^3}^2+\|\mathbf{g}(\cdot,t)\|_{H^3}^2],\end{aligned} \tag{4.24}$$

and

$$\|\mathbf{b}_t(\cdot,t)\|_2^2+\nu\frac{d}{dt}\|\nabla\mathbf{b}(\cdot,t)\|_2^2\le C\|\mathbf{u}(\cdot,t)\|_{H^3}^2\|\mathbf{b}(\cdot,t)\|_{H^3}^2. \tag{4.25}$$

Thus integrating (4.24) over $[0,T]$, we have that

$$m\int_0^t\|\mathbf{w}_t(\cdot,s)\|_2^2\,ds+\gamma\|\nabla\mathbf{w}(\cdot,t)\|_2^2\le C[A+\psi(t)]. \tag{4.26}$$

Step 2 Taking the operator D^α on each side of the equation $(4.1)_1$, multiplying the resulting equation by $D^\alpha\mathbf{u}_t$ in $\mathbf{L}^2$ we deduce, after summing over $|\alpha|=1$, that

$$\begin{aligned}S_1+S_2+S_3:=&\sum_{|\alpha|=1}(\rho D^\alpha\mathbf{u}_t,D^\alpha\mathbf{u}_t)_{L^2}-\bar{\mu}\sum_{|\alpha|=1}(\Delta D^\alpha\mathbf{u},D^\alpha\mathbf{u}_t)_{L^2}+\sum_{|\alpha|=1}(\nabla D^\alpha p,D^\alpha\mathbf{u}_t)_{L^2}\\ =&-\sum_{|\alpha|=1}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}),D^\alpha\mathbf{u}_t)_{L^2}+\chi\sum_{|\alpha|=1}(\nabla\times D^\alpha\mathbf{w},D^\alpha\mathbf{u}_t)_{L^2}\\ &+\sum_{|\alpha|=1}(D^\alpha(\rho\mathbf{f}),D^\alpha\mathbf{u}_t)_{L^2}+\sum_{|\alpha|=1}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{b}),D^\alpha\mathbf{u}_t)_{L^2}\\ &+\sum_{|\alpha|=1}(D^\alpha(\rho\mathbf{u}_t)-\rho D^\alpha\mathbf{u}_t,D^\alpha\mathbf{u}_t)_{L^2}.\end{aligned} \tag{4.27}$$

Note that $S_1=\|\sqrt{\rho}\nabla\mathbf{u}(\cdot,t)\|_2^2$. Integrating by parts the term S_2 it is easy to see that

$$S_2=\frac{\bar{\mu}}{2}\frac{d}{dt}\|D^2\mathbf{u}(\cdot,t)\|_2^2$$

and using the assumption $\operatorname{div}\ \mathbf{u}_t=0$, we conclude that

$$S_3=-\sum_{|\alpha|=1}\int_{\mathbb{R}^3}D^\alpha pD^\alpha(\operatorname{div}\ \mathbf{u}_t)\,d\mathbf{x}=0.$$

Then, we can rewrite the identity (4.27) as follows

$$
\begin{aligned}
\|\sqrt{\rho}\nabla \mathbf{u}_t\|_2^2 + \frac{\bar{\mu}}{2}\frac{d}{dt}\|D^2\mathbf{u}\|_2^2 = &-\sum_{|\alpha|=1}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}), D^\alpha \mathbf{u}_t)_{L^2} + \chi\sum_{|\alpha|=1}(\nabla\times D^\alpha \mathbf{w}, D^\alpha \mathbf{u}_t)_{L^2}\\
&+\sum_{|\alpha|=1}(D^\alpha(\rho \mathbf{f}), D^\alpha \mathbf{u}_t)_{L^2} + \sum_{|\alpha|=1}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{b}), D^\alpha \mathbf{u}_t)_{L^2}\\
&+\sum_{|\alpha|=1}(D^\alpha(\rho \mathbf{u}_t) - \rho D^\alpha \mathbf{u}_t, D^\alpha \mathbf{u}_t)_{L^2}\\
&:= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}
\tag{4.28}
$$

Using the estimates (2.5), (2.6), (2.8), (4.6) and the Cauchy-Schwarz's inequality, we obtain

$$
\begin{aligned}
|J_1| \le &\sum_{|\alpha|=1}\|D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}) - \rho D^\alpha((\mathbf{u}\cdot\nabla)\mathbf{u})\|_2\|D^\alpha \mathbf{u}_t\|_2 + \sum_{|\alpha|=1}\|\rho D^\alpha((\mathbf{u}\cdot\nabla)\mathbf{u})\|_2\|D^\alpha \mathbf{u}_t\|_2\\
&\le C(\|\nabla\rho\|_\infty\|\mathbf{u}\cdot\nabla\mathbf{u}\|_2 + \|\nabla\rho\|_2\|\mathbf{u}\cdot\nabla\mathbf{u}\|_\infty)\|\nabla\mathbf{u}_t\|_2\\
&+ C(\|\mathbf{u}\|_\infty\|\nabla\mathbf{u}\|_{H^1} + \|\mathbf{u}\|_{H^1}\|\nabla\mathbf{u}\|_\infty)\|\nabla\mathbf{u}_t\|_2\\
&\le C(\|\nabla\rho\|_{H^2}\|\mathbf{u}\|_{H^2}\|\nabla\mathbf{u}\|_{H^2} + \|\mathbf{u}\|_{H^2}\|\nabla\mathbf{u}\|_{H^2})\|\nabla\mathbf{u}_t\|_2\\
&\le C(1+\|\nabla\rho\|_{H^2})\|\mathbf{u}\|_{H^3}^2\|\nabla\mathbf{u}_t\|_2,\\
|J_2| \le &\;\chi\sum_{|\alpha|=1}\|\nabla\times D^\alpha\mathbf{w}\|_2\|D^\alpha\mathbf{u}_t\|_2\\
&\le \|\nabla D^\alpha\mathbf{w}\|_2\|D^\alpha\mathbf{u}_t\|_2\\
&\le C\|\mathbf{w}\|_{H^3}\|\nabla\mathbf{u}_t\|_2,\\
|J_3| \le &\sum_{|\alpha|=1}\|D^\alpha(\rho\mathbf{f}) - \rho D^\alpha\mathbf{f}\|_2\|D^\alpha\mathbf{u}_t\|_2 + \sum_{|\alpha|=1}\|\rho D^\alpha\mathbf{f}\|_2\|D^\alpha\mathbf{u}_t\|_2\\
&\le C(\|\nabla\rho\|_\infty\|\mathbf{f}\|_{H^2} + \|\nabla\rho\|_{H^2}\|\mathbf{f}\|_\infty)\|\nabla\mathbf{u}_t\|_2 + C\|\nabla\mathbf{f}\|_2\|\nabla\mathbf{u}_t\|_2\\
&\le C(1+\|\nabla\rho\|_{H^2})\|\mathbf{f}\|_{H^3}\|\nabla\mathbf{u}_t\|_2,
\end{aligned}
$$

$$
\begin{aligned}
|J_4| &\le \sum_{|\alpha|=1}\|D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{b})\|_2\|\nabla\mathbf{u}_t\|_2\\
&\le C(\|\mathbf{b}\|_\infty\|\nabla\mathbf{b}\|_{H^1} + \|\mathbf{b}\|_{H^1}\|\nabla\mathbf{b}\|_\infty)\|\nabla\mathbf{u}_t\|_2\\
&\le C\|\mathbf{b}\|_{H^3}^2\|\nabla\mathbf{u}_t\|_2,\\
|J_5| &\le \sum_{|\alpha|=1}\|D^\alpha(\rho\mathbf{u}_t) - \rho D^\alpha\mathbf{u}_t\|_2\|D^\alpha\mathbf{u}_t\|_2\\
&\le C\|\nabla\rho\|_\infty\|\mathbf{u}_t\|_2\|\nabla\mathbf{u}_t\|_2\\
&\le C\|\nabla\rho\|_{H^2}\|\mathbf{u}_t\|_2\|\nabla\mathbf{u}_t\|_2.
\end{aligned}
$$

Due these estimates, from Young's inequality it follows that

$$\sum_{i=1}^{5}|J_i| \le C\|\nabla \mathbf{u}_t\|_2[\|\mathbf{w}\|_{H^3}+\|\mathbf{b}\|_{H^3}^2+\|\nabla\rho\|_{H^2}\|\mathbf{u}_t\|_2+(1+\|\nabla\rho\|_{H^2})(\|\mathbf{f}\|_{H^3}+\|\mathbf{u}\|_{H^3}^2)]$$
$$\le \frac{m}{2}\|\nabla \mathbf{u}_t\|_2^2+C[\|\mathbf{w}\|_{H^3}^2+\|\mathbf{b}\|_{H^3}^4+\|\nabla\rho\|_{H^2}^2\|\mathbf{u}_t\|_2^2+(1+\|\nabla\rho\|_{H^2})^2(\|\mathbf{f}\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^4)]. \tag{4.29}$$

Replacing the estimate (4.29) into the identity (4.28), we find that

$$m\|\nabla \mathbf{u}_t\|_2^2+\bar{\mu}\frac{d}{dt}\|D^2\mathbf{u}\|_2^2 \le C[\|\mathbf{w}\|_{H^3}^2+\|\mathbf{b}\|_{H^3}^4+\|\nabla\rho\|_{H^2}^2\|\mathbf{u}_t\|_2^2+(1+\|\nabla\rho\|_{H^2})^2(\|\mathbf{f}\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^4)]. \tag{4.30}$$

We can obtain a similary estimate for $\mathbf{w}$ and $\mathbf{b}$. Indeed, taking the operator D^α with $|\alpha|=1$ on each side of $(4.1)_2$ and $(4.1)_3$, multiplying the resulting equations by $D^\alpha\mathbf{w}_t$ and $D^\alpha\mathbf{b}_t$ in $\mathbf{L}^2$, respectively, and summing over $|\alpha|=1$, one gets

$$\begin{aligned} &m\|\nabla \mathbf{w}_t\|_2^2+\gamma\frac{d}{dt}\|D^2\mathbf{w}\|_2^2+\kappa\frac{d}{dt}\|\nabla(\operatorname{div}\ \mathbf{w})\|_2^2+2\chi\frac{d}{dt}\|\nabla\mathbf{w}\|_2^2\\ &\le C[\|\mathbf{u}\|_{H^3}^2+\|\nabla\rho\|_{H^2}^2\|\mathbf{w}_t\|_2^2+(1+\|\nabla\rho\|_{H^2})^2(\|\mathbf{g}\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^2\|\mathbf{w}\|_{H^3}^2)]\end{aligned} \tag{4.31}$$

and

$$\|\nabla \mathbf{b}_t\|_2^2+\nu\frac{d}{dt}\|D^2\mathbf{b}\|_2^2 \le C\|\mathbf{u}\|_{H^3}^2\|\mathbf{b}\|_{H^3}^2. \tag{4.32}$$

Step 3 Summing inequalities (4.22) and (4.30) and integrating the result over $[0,t]$, one finds

$$\begin{aligned}\bar{\mu}\|\nabla\mathbf{u}(\cdot,t)\|_{H^1}^2+m\int_0^t\|\mathbf{u}_t(\cdot,s)\|_{H^1}^2\,ds &\le \|\nabla\mathbf{u}_0(\cdot,t)\|_{H^1}^2+C\sup_{0\le s\le t}\|\nabla\rho(\cdot,s)\|_{H^2}^2\int_0^t\|\mathbf{u}_t(\cdot,s)\|_2^2\,ds\\ &+C\Big(1+\sup_{0\le s\le t}\|\nabla\rho(\cdot,s)\|_{H^2}^2\Big)\int_0^t(\|\mathbf{f}(\cdot,s)\|_{H^3}^2+\|\mathbf{u}(\cdot,s)\|_{H^3}^4)\,ds\\ &+C\int_0^t(\|\mathbf{w}(\cdot,s)\|_{H^3}^2+\|\mathbf{b}(\cdot,s)\|_{H^3}^4)\,ds\end{aligned}$$

which, together with inequalities (4.7) and (4.23), directly gives

$$\bar{\mu}\|\nabla\mathbf{u}(\cdot,t)\|_{H^1}^2+m\int_0^t\|\mathbf{u}_t(\cdot,s)\|_{H^1}^2\,ds \le C[A+\psi(t)]^2. \tag{4.33}$$

Similarly, summing (4.24) and (4.31) and integrating the result over $[0,t]$, one obtains

$$\begin{aligned}\gamma\|\nabla\mathbf{w}(\cdot,t)\|_{H^1}^2+m\int_0^t\|\mathbf{w}_t(\cdot,s)\|_{H^1}^2\,ds &\le C\|\nabla\mathbf{w}_0(\cdot)\|_{H^1}^2+\|\operatorname{div}\ \mathbf{w}_0(\cdot)\|_{H^1}^2+2\|\mathbf{w}_0(\cdot)\|_{H^1}^2\\ &+C\Big(1+\sup_{0\le s\le t}\|\nabla\rho(\cdot,s)\|_{H^2}^2\Big)\int_0^t(\|\mathbf{g}(\cdot,s)\|_{H^3}^2+\|\mathbf{u}(\cdot,s)\|_{H^3}^2\|\mathbf{w}(\cdot,s)\|_{H^3}^2)\,ds\\ &+C\sup_{0\le s\le t}\|\nabla\rho(\cdot,s)\|_{H^2}^2\int_0^t\|\mathbf{w}_t(\cdot,s)\|_2^2\,ds.\end{aligned}$$

Hence, due to estimates (4.7) and (4.26),

$$\gamma\|\nabla \mathbf{w}(\cdot,t)\|_{H^1}^2 + m\int_0^t \|\mathbf{w}_t(\cdot,s)\|_{H^1}^2\,ds \le C[A+\psi(t)]^2 \tag{4.34}$$

Step 4 Analogously to the development of Step 2, but with $|\alpha|=2$, we find that

$$\begin{aligned} m\|D^2\mathbf{u}_t(\cdot,t)\|_2^2 + \bar{\mu}\frac{d}{dt}\|D^3\mathbf{u}(\cdot,t)\|_2^2 \le C[\|\nabla\rho(\cdot,t)\|_{H^2}^2\|\mathbf{u}_t(\cdot,t)\|_{H^1}^2 + \|\mathbf{w}(\cdot,t)\|_{H^3}^2 \\ +\|\mathbf{b}(\cdot,t)\|_{H^3}^4 + (1+\|\nabla\rho(\cdot,t)\|_{H^2})^2(\|\mathbf{f}(\cdot,t)\|_{H^3}^2 + \|\mathbf{u}(\cdot,t)\|_{H^3}^4)], \end{aligned} \tag{4.35}$$

$$\begin{aligned} m\|D^2\mathbf{w}_t(\cdot,t)\|_2^2 + \gamma\frac{d}{dt}\|D^3\mathbf{w}(\cdot,t)\|_2^2 + \kappa\frac{d}{dt}\|D^2 \operatorname{div}\ \mathbf{w}(\cdot,t)\|_2^2 + 2\chi\frac{d}{dt}\|D^2\mathbf{w}(\cdot,t)\|_2^2 \\ \le C[(1+\|\nabla\rho(\cdot,t)\|_{H^2})^2(\|\mathbf{g}(\cdot,t)\|_{H^3}^2 + \|\mathbf{u}(\cdot,t)\|_{H^3}^2\|\mathbf{w}(\cdot,t)\|_{H^3}^2) \\ +\|\nabla\rho(\cdot,t)\|_{H^2}^2\|\mathbf{w}_t(\cdot,t)\|_{H^1}^2 + \|\mathbf{u}(\cdot,t)\|_{H^3}^2], \end{aligned} \tag{4.36}$$

$$\|D^2\mathbf{b}_t(\cdot,t)\|_2^2 + \nu\frac{d}{dt}\|D^3\mathbf{b}(\cdot,t)\|_2^2 \le C\|\mathbf{u}(\cdot,t)\|_{H^3}^2\|\mathbf{b}(\cdot,t)\|_{H^3}^2. \tag{4.37}$$

Step 5 Summing the inequalities (4.22), (4.30) and (4.35), and integrating the result over $[0,t]$, one obtains

$$\begin{aligned} \bar{\mu}\|\nabla\mathbf{u}(\cdot,t)\|_{H^2}^2 + m\int_0^t \|\mathbf{u}_t(\cdot,s)\|_{H^2}^2\,ds \le \|\nabla\mathbf{u}_0\|_{H^2}^2 + C\sup_{0\le s\le t}\|\nabla\rho(\cdot,s)\|_{H^2}^2\int_0^t \|\mathbf{u}_t(\cdot,s)\|_{H^1}^2\,ds \\ +C\Big(1+\sup_{0\le s\le t}\|\nabla\rho(\cdot,s)\|_{H^2}^2\Big)\int_0^t(\|\mathbf{f}(\cdot,s)\|_{H^3}^2 + \|\mathbf{u}(\cdot,s)\|_{H^3}^4)\,ds \\ +C\int_0^t(\|\mathbf{w}(\cdot,s)\|_{H^3}^2 + \|\mathbf{b}(\cdot,s)\|_{H^3}^4)\,ds \end{aligned}$$

which, together with inequalities (4.7) and (4.33), directly gives

$$\bar{\mu}\|\nabla\mathbf{u}(\cdot,t)\|_{H^2}^2 + m\int_0^t \|\mathbf{u}_t(\cdot,s)\|_{H^2}^2\,ds \le C[A+\psi(t)]^3. \tag{4.38}$$

Now, summing the estimates (4.24), (4.31) and (4.36). Then, due to bounds (4.7) and (4.34), we get

$$\gamma\|\nabla\mathbf{w}(\cdot,t)\|_{H^2}^2 + m\int_0^t \|\mathbf{w}_t(\cdot,s)\|_{H^2}^2\,ds \le C[A+\psi(t)]^3. \tag{4.39}$$

Finally, summing the inequalities (4.25), (4.32) and (4.37), and integrating the result over $[0,t]$, one obtains

$$\begin{aligned} \nu\|\nabla\mathbf{b}(\cdot,t)\|_{H^2}^2 + \int_0^t \|\mathbf{b}_t(\cdot,s)\|_{H^2}^2\,ds \le \|\nabla\mathbf{b}_0\|_{H^2}^2 + C\int_0^t \|\mathbf{u}(\cdot,s)\|_{H^3}^2\|\mathbf{b}(\cdot,s)\|_{H^3}^2\,ds \\ \le C[A+\psi(t)]^3. \end{aligned} \tag{4.40}$$

Therefore, the inequalities (4.38),(4.39) and (4.40) directly imply the desired bound (4.20). Summarizing the previous results, we found the following lemma:

Lemma 4.7 *For all $t \in [0,T]$, we have*

$$\|\nabla\rho(\cdot,t)\|_{H^2}^2 + \|\mathbf{u}(\cdot,t)\|_{H^3}^2 + \|\mathbf{w}(\cdot,t)\|_{H^3}^2 + \|\mathbf{b}(\cdot,t)\|_{H^3}^2$$
$$+ \int_0^t (\|\mathbf{u}_t(\cdot,s)\|_{H^2}^2 + \|\mathbf{w}_t(\cdot,s)\|_{H^2}^2 + \|\mathbf{b}_t(\cdot,s)\|_{H^2}^2)\,ds + \bar{\mu}\int_0^t \|\nabla\mathbf{u}(\cdot,s)\|_{H^3}^2\,ds$$
$$+\gamma\int_0^t \|\nabla\mathbf{w}(\cdot,s)\|_{H^3}^2\,ds + \nu\int_0^t \|\nabla\mathbf{b}(\cdot,s)\|_{H^3}^2\,ds \le C[A+\psi(t)]^3.$$

Lemma 4.8 *There exists $T_0 \in (0,T]$ independent of $\bar{\mu}, \gamma, \kappa$ and ν such that*

$$\psi(t) \le A \quad \textit{for} \;\; t \le T_0.$$

Proof: From Lemma 4.7, we have

$$\|\nabla\rho(\cdot,t)\|_{H^2}^2 + \|\mathbf{u}(\cdot,t)\|_{H^3}^2 + \|\mathbf{w}(\cdot,t)\|_{H^3}^2 + \|\mathbf{b}(\cdot,t)\|_{H^3}^2 \le C[A+\psi(t)]^3.$$

Due the definition of $\psi(t)$ given by (4.5), one gets

$$\begin{aligned}\left(\frac{d}{dt}\psi(t)\right)^{1/2} &= 1 + \|\nabla\rho(\cdot,t)\|_{H^2}^2 + \|\mathbf{u}(\cdot,t)\|_{H^3}^2 + \|\mathbf{w}(\cdot,t)\|_{H^3}^2 + \|\mathbf{b}(\cdot,t)\|_{H^3}^2 \\ &\le C[A+\psi(t)]^3.\end{aligned} \tag{4.41}$$

Set $Y(t) = \psi(t) + A$. From (4.41) we find the following differential inequation:

$$\frac{d}{dt}Y(t) \le CY^6(t),$$

or yet

$$Y^{-6}(t)\frac{d}{dt}Y(t) \le C. \tag{4.42}$$

Taking $I(t) = \int_0^t Y^{-6}(s)\frac{d}{ds}Y(s)\,ds$, after integration by parts one concludes

$$I(t) = \frac{1}{5A^5} - \frac{1}{5Y^5(t)}.$$

From inequality (4.42) it follows that

$$\frac{1}{5A^5} - \frac{1}{5Y^5(t)} \le Ct.$$

A straightforward calculation gives

$$Y(t) \le A(1 - 5CA^5 t)^{-1/5} \quad \text{with } 0 < t < (5CA^5)^{-1}.$$

Therefore

$$\psi(t) \le A \quad \text{for} \;\; t \le T_0 = \frac{31}{160CA^5}.$$

Due to the Lemmas 4.7 and 4.8, we obtain the following uniform estimates for the solutions of problems (4.1)-(4.2) and (4.3)-(4.2), respectively

Proposition 4.1 *There exists a positive constant C such that*

$$\sup_{0\le t\le T_0}[\|\nabla\rho(\cdot,t)\|_{H^2}^2+\|\mathbf{u}(\cdot,t)\|_{H^3}^2+\|\mathbf{w}(\cdot,t)\|_{H^3}^2+\|\mathbf{b}(\cdot,t)\|_{H^3}^2]$$
$$+\int_0^{T_0}(\|\mathbf{u}_t(\cdot,t)\|_{H^2}^2+\|\mathbf{w}_t(\cdot,t)\|_{H^2}^2+\|\mathbf{b}_t(\cdot,t)\|_{H^2}^2)\,dt+\bar{\mu}\int_0^{T_0}\|\nabla\mathbf{u}(\cdot,t)\|_{H^3}^2\,dt$$
$$+\gamma\int_0^{T_0}\|\nabla\mathbf{w}(\cdot,t)\|_{H^3}^2\,dt+\nu\int_0^{T_0}\|\nabla\mathbf{b}(\cdot,t)\|_{H^3}^2\,dt\le C.$$

and

$$\sup_{0\le t\le T_0}[\|\nabla\rho^0(\cdot,t)\|_{H^2}^2+\|\mathbf{u}^0(\cdot,t)\|_{H^3}^2+\|\mathbf{w}^0(\cdot,t)\|_{H^3}^2+\|\mathbf{b}^0(\cdot,t)\|_{H^3}^2]$$
$$+\int_0^{T_0}(\|\mathbf{u}_t^0(\cdot,t)\|_{H^2}^2+\|\mathbf{w}_t^0(\cdot,t)\|_{H^2}^2+\|\mathbf{b}_t^0(\cdot,t)\|_{H^2}^2)\,dt\le C.$$

4.3 PROOF OF MAIN THEOREM

The local existence of solutions for problems (4.1)-(4.2) and (4.3)-(4.2) is proved through a semi-Galerkin method, since the estimates established in Proposition 4.1 holds for semi-Galerkin aproximations, which are enough for taking the limit to obtain the local existence. This process is similar to that one proved in (KAZHIKHOV; MONAKHOV, 1990), here we will omit the details of the proof and focus only on establishing our mail result (4.4).

Set

$$\sigma=\rho-\rho^0,\quad \mathbf{v}^{\mathbf{u}}=\mathbf{u}-\mathbf{u}^0,\quad \mathbf{v}^{\mathbf{w}}=\mathbf{w}-\mathbf{w}^0,\quad \mathbf{v}^{\mathbf{b}}=\mathbf{b}-\mathbf{b}^0,\quad q=p-p^0.$$

Subtracting (4.3) from (4.1), we get the following system of equations:

$$\begin{cases}\rho\mathbf{v}_t^{\mathbf{u}}+\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{u}}-\bar{\mu}\Delta\mathbf{v}^{\mathbf{u}}-\bar{\mu}\Delta\mathbf{u}^0-(\mathbf{b}\cdot\nabla)\mathbf{v}^{\mathbf{b}}+\nabla q=-\rho(\mathbf{v}^{\mathbf{u}}\cdot\nabla)\mathbf{u}^0+\dfrac{\sigma}{\rho^0}\nabla(p^0)\\ +\chi\nabla\times\mathbf{w}+(\mathbf{v}^{\mathbf{b}}\cdot\nabla)\mathbf{b}^0-\dfrac{\sigma}{\rho^0}(\mathbf{b}^{\mathbf{0}}\cdot\nabla)\mathbf{b}^0:=\mathbf{F}\\ \rho\mathbf{v}_t^{\mathbf{w}}+\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{w}}+L\mathbf{v}^{\mathbf{w}}+L\mathbf{w}^0+2\chi\mathbf{v}^{\mathbf{w}}+2\chi\mathbf{w}^0=-\rho(\mathbf{v}^{\mathbf{u}}\cdot\nabla)\mathbf{w}^0+\chi\nabla\times\mathbf{u}:=\mathbf{G}\\ \mathbf{v}_t^{\mathbf{b}}+(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{b}}-\nu\Delta\mathbf{v}^{\mathbf{b}}-\nu\Delta\mathbf{b}^0-(\mathbf{b}\cdot\nabla)\mathbf{v}^{\mathbf{u}}=(\mathbf{v}^{\mathbf{b}}\cdot\nabla)\mathbf{u}^0-(\mathbf{v}^{\mathbf{u}}\cdot\nabla)\mathbf{b}^0:=\mathbf{H}\\ \operatorname{div}\ \mathbf{v}^{\mathbf{u}}=\operatorname{div}\ \mathbf{v}^{\mathbf{b}}=0\\ \sigma_t+\mathbf{u}\cdot\nabla\sigma=-\mathbf{v}^{\mathbf{u}}\cdot\nabla\rho^0\end{cases}\tag{4.43}$$

in $\mathcal{Q}_T=\mathbb{R}^3\times[0,T], T>0$, with initial conditions

$$\sigma(\mathbf{x},0)=0,\quad \mathbf{v}^{\mathbf{u}}(\mathbf{x},0)=\mathbf{v}^{\mathbf{w}}(\mathbf{x},0)=\mathbf{v}^{\mathbf{b}}(\mathbf{x},0)=0,\quad \mathbf{x}\in\mathbb{R}^3.\tag{4.44}$$

Where in the previous system L is the linear operator defined by:

$$L\mathbf{v}^{\mathbf{w}}=-\gamma\Delta\mathbf{v}^{\mathbf{w}}-\kappa\nabla(\operatorname{div}\ \mathbf{v}^{\mathbf{w}}).$$

We will divide the proof into three steps:

Step 1**.** Taking the operator D^α on each side of $(4.43)_5$, multiplying the resulting equation by $D^\alpha\sigma$ in L^2, we obtain, after summing over $|\alpha|\le 2$, that

$$\frac{1}{2}\frac{d}{dt}\|\sigma(\cdot,t)\|_{H^2}^2 = -\sum_{|\alpha|\le 2}(D^\alpha(\mathbf{u}\cdot\nabla\sigma), D^\alpha\sigma)_{L^2} - \sum_{|\alpha|\le 2}(D^\alpha(\mathbf{v}^{\mathbf{u}}\cdot\nabla\rho^0), D^\alpha\sigma)_{L^2}. \tag{4.45}$$

Using integration by parts, we have

$$\sum_{|\alpha|\le 2}(\mathbf{u}\cdot\nabla D^\alpha\sigma, D^\alpha\sigma)_{L^2} = -\frac{1}{2}\sum_{|\alpha|\le 2}\int_{\mathbb{R}^3}|D^\alpha\sigma|^2(\operatorname{div}\ \mathbf{u})\,d\mathbf{x} = 0$$

since $\operatorname{div}\ \mathbf{u} = 0$. Then, the identity (4.45) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\|\sigma(\cdot,t)\|_{H^2}^2 = -\sum_{|\alpha|\le 2}(D^\alpha(\mathbf{u}\cdot\nabla\sigma) - \mathbf{u}\cdot\nabla D^\alpha\sigma, D^\alpha\sigma)_{L^2} - \sum_{|\alpha|\le 2}(D^\alpha(\mathbf{v}^{\mathbf{u}}\cdot\nabla\rho^0), D^\alpha\sigma)_{L^2}. \tag{4.46}$$

To estimate the first term on the right side of (4.46), note that

$$\begin{aligned}\sum_{|\alpha|\le 2}(D^\alpha(\mathbf{u}\cdot\nabla\sigma) - \mathbf{u}\cdot\nabla D^\alpha\sigma, D^\alpha\sigma)_{L^2} &= \Big(\sum_{|\alpha|=1} + \sum_{|\alpha|=2}\Big)(D^\alpha(\mathbf{u}\cdot\nabla\sigma) - \mathbf{u}\cdot\nabla D^\alpha\sigma, D^\alpha\sigma)_{L^2}\\ &:= B_1 + B_2.\end{aligned}$$

Then, using Cauchy-Schwarz and Hölder's inequalities, we find that

$$\begin{aligned}|B_1| &\le \|\nabla\mathbf{u}\|_\infty\|\nabla\sigma\|_2\|\nabla\sigma\|_2 \le C\|\nabla\mathbf{u}\|_{H^2}\|\nabla\sigma\|_2^2 \le C\|\mathbf{u}\|_{H^3}\|\sigma\|_{H^2}^2\\ |B_2| &\le C(\|D^2\mathbf{u}\|_4\|\nabla\sigma\|_4 + \|\nabla\mathbf{u}\|_\infty\|D^2\sigma\|_2)\|D^2\sigma\|_2 \le C\|\mathbf{u}\|_{H^3}\|\sigma\|_{H^2}^2.\end{aligned}$$

Due these estimates and Proposition 4.1, one concludes

$$\sum_{|\alpha|\le 2}(D^\alpha(\mathbf{u}\cdot\nabla\sigma) - \mathbf{u}\cdot\nabla D^\alpha\sigma, D^\alpha\sigma)_{L^2} \le C\|\mathbf{u}\|_{H^3}\|\sigma\|_{H^2}^2 \le C\|\sigma\|_{H^2}^2. \tag{4.47}$$

For bound the second term on the right side of (4.46), we use the estimates (2.6), (2.8), Cauchy-Schwarz and Young's inequalities and the Proposition 4.1, to get

$$\begin{aligned}\Big|\sum_{|\alpha|\le 2}(D^\alpha(\mathbf{v}^{\mathbf{u}}\cdot\nabla\rho^0), D^\alpha\sigma)_{L^2}\Big| &\le C(\|\mathbf{v}^{\mathbf{u}}\|_{H^2}\|\nabla\rho^0\|_\infty + \|\mathbf{v}^{\mathbf{u}}\|_\infty\|\nabla\rho^0\|_{H^2})\|\sigma\|_{H^2}\\ &\le C\|\mathbf{v}^{\mathbf{u}}\|_{H^2}\|\nabla\rho^0\|_{H^2}\|\sigma\|_{H^2} \le C(\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2 + \|\sigma\|_{H^2}^2).\end{aligned} \tag{4.48}$$

Replacing the inequalities (4.47) and (4.48) into identity (4.46) and integrating the result over $[0,t]$, we find that

$$\|\sigma(\cdot,t)\|_{H^2}^2 \le C\int_0^t(\|\mathbf{v}^{\mathbf{u}}(\cdot,s)\|_{H^2}^2 + \|\sigma(\cdot,s)\|_{H^2}^2)\,ds, \tag{4.49}$$

since $\sigma(\mathbf{x},0)=0$ for all $\mathbf{x}\in\mathbb{R}^3$.

Step 2***.*** Taking the operator D^α on each side of the equation $(4.43)_1$, multiplying the resulting equation by $D^\alpha\mathbf{v}^{\mathbf{u}}$ in $\mathbf{L}^2$ and summing over $|\alpha|\le 2$, one obtains

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 2}\|\sqrt{\rho}D^\alpha\mathbf{v}^{\mathbf{u}}\|_2^2+\bar{\mu}\|\nabla\mathbf{v}^{\mathbf{u}}\|_{H^2}^2=-\sum_{|\alpha|\le 2}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{u}}),D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}\\ +\bar{\mu}\sum_{|\alpha|\le 2}(\Delta D^\alpha\mathbf{u}^0,D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}+\sum_{|\alpha|\le 2}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{v}^{\mathbf{b}}),D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}+\sum_{|\alpha|\le 2}(D^\alpha\mathbf{F},D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}\\ -\frac{1}{2}\sum_{|\alpha|\le 2}((\mathbf{u}\cdot\nabla\rho)D^\alpha\mathbf{v}^{\mathbf{u}},D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}-\sum_{|\alpha|\le 2}(D^\alpha(\rho\mathbf{v}_t^{\mathbf{u}})-\rho D^\alpha\mathbf{v}_t^{\mathbf{u}},D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}.\end{gathered}\tag{4.50}$$

In the same way, taking the operator D^α on each side of the equations $(4.43)_2$ and $(4.43)_3$, multiplying the resulting equations by $D^\alpha\mathbf{v}^{\mathbf{w}}$ and $D^\alpha\mathbf{v}^{\mathbf{b}}$ in $\mathbf{L}^2$, respectively, and summing over $|\alpha|\le 2$, we obtain that

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le 2}\|\sqrt{\rho}D^\alpha\mathbf{v}^{\mathbf{w}}\|_2^2+\gamma\|\nabla\mathbf{v}^{\mathbf{w}}\|_{H^2}^2+\kappa\|\operatorname{div}\ \mathbf{v}^{\mathbf{w}}\|_{H^2}^2+2\chi\|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2=\\ -2\chi\sum_{|\alpha|\le 2}(D^\alpha\mathbf{w}^0,D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}-\sum_{|\alpha|\le 2}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{w}}),D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}+\gamma\sum_{|\alpha|\le 2}(\Delta D^\alpha\mathbf{w}^0,D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}\\ +\sum_{|\alpha|\le 2}(D^\alpha\mathbf{G},D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}+\kappa\sum_{|\alpha|\le 2}(\nabla(\operatorname{div}\ D^\alpha\mathbf{w}^0),D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}-\frac{1}{2}\sum_{|\alpha|\le 2}((\mathbf{u}\cdot\nabla\rho)D^\alpha\mathbf{v}^{\mathbf{w}},D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}\\ -\sum_{|\alpha|\le 2}(D^\alpha(\rho\mathbf{v}_t^{\mathbf{w}})-\rho D^\alpha\mathbf{v}_t^{\mathbf{w}},D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2},\end{gathered}\tag{4.51}$$

and

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}^{\mathbf{b}}\|_{H^2}^2+\nu\|\nabla\mathbf{v}^{\mathbf{b}}\|_{H^2}^2=\nu\sum_{|\alpha|\le 2}(\Delta D^\alpha\mathbf{b}^0,D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2}+\sum_{|\alpha|\le 2}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{v}^{\mathbf{u}}),D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2}\\ -\sum_{|\alpha|\le 2}(D^\alpha((\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{b}}),D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2}+\sum_{|\alpha|\le 2}(D^\alpha\mathbf{H},D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2}.\end{gathered}\tag{4.52}$$

Summing (4.50), (4.51) and (4.52), we find

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{|\alpha|\leq 2}\|\sqrt{\rho}D^\alpha\mathbf{v}^{\mathbf{u}}\|_2^2+\sum_{|\alpha|\leq 2}\|\sqrt{\rho}D^\alpha\mathbf{v}^{\mathbf{w}}\|_2^2+\|\mathbf{v}^{\mathbf{b}}\|_{H^2}^2\right)+\bar{\mu}\|\nabla\mathbf{v}^{\mathbf{u}}\|_{H^2}^2+\gamma\|\nabla\mathbf{v}^{\mathbf{w}}\|_{H^2}^2$$
$$+\nu\|\nabla\mathbf{v}^{\mathbf{b}}\|_{H^2}^2+\kappa\|\text{ div }\mathbf{v}^{\mathbf{w}}\|_{H^2}^2+2\chi\|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2=\bar{\mu}\sum_{|\alpha|\leq 2}(\Delta D^\alpha\mathbf{u}^0,D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}$$
$$+\gamma\sum_{|\alpha|\leq 2}(\Delta D^\alpha\mathbf{w}^0,D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}+\nu\sum_{|\alpha|\leq 2}(\Delta D^\alpha\mathbf{b}^0,D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2}+\kappa\sum_{|\alpha|\leq 2}(\nabla(\text{ div }D^\alpha\mathbf{w}^0),D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}$$
$$-\sum_{|\alpha|\leq 2}(D^\alpha(\rho\mathbf{v}_t^{\mathbf{u}})-\rho D^\alpha\mathbf{v}_t^{\mathbf{w}},D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}-\frac{1}{2}\sum_{|\alpha|\leq 2}((\mathbf{u}\cdot\nabla\rho)D^\alpha\mathbf{v}^{\mathbf{u}},D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}$$
$$-\sum_{|\alpha|\leq 2}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{u}}),D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}+\sum_{|\alpha|\leq 2}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{v}^{\mathbf{b}}),D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}$$
$$-\sum_{|\alpha|\leq 2}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{w}}),D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}-\sum_{|\alpha|\leq 2}(D^\alpha(\rho\mathbf{v}_t^{\mathbf{w}})-\rho D^\alpha\mathbf{v}_t^{\mathbf{w}},D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}$$
$$-2\chi\sum_{|\alpha|\leq 2}(D^\alpha\mathbf{w}^0,D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}-\frac{1}{2}\sum_{|\alpha|\leq 2}((\mathbf{u}\cdot\nabla\rho)D^\alpha\mathbf{v}^{\mathbf{w}},D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}$$
$$+\sum_{|\alpha|\leq 2}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{v}^{\mathbf{u}}),D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2}-\sum_{|\alpha|\leq 2}(D^\alpha((\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{b}}),D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2}$$
$$+\sum_{|\alpha|\leq 2}(D^\alpha\mathbf{F},D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2}+\sum_{|\alpha|\leq 2}(D^\alpha\mathbf{G},D^\alpha\mathbf{v}^{\mathbf{w}})_{L^2}+\sum_{|\alpha|\leq 2}(D^\alpha\mathbf{H},D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2}:=\sum_{i=1}^{17}R_i \tag{4.53}$$

We will bound each term on the right side of identity (4.53). On one hand, using integration by parts, Cauchy-Schwarz and Young's inequalities and Proposition 4.1, we have that

$$\begin{aligned}|R_1|+|R_2|+|R_3|+|R_4|&\leq\frac{1}{2}(\bar{\mu}\|\nabla\mathbf{v}^{\mathbf{u}}\|_{H^2}^2+\gamma\|\nabla\mathbf{v}^{\mathbf{w}}\|_{H^2}^2+\nu\|\nabla\mathbf{v}^{\mathbf{b}}\|_{H^2}^2+\kappa\|\text{ div }\mathbf{v}^{\mathbf{w}}\|_{H^2}^2)\\&+C(\bar{\mu}\|\mathbf{u}^0\|_{H^3}^2+\gamma\|\mathbf{w}^0\|_{H^3}^2+\nu\|\mathbf{b}^0\|_{H^3}^2)\\&\leq\frac{1}{2}(\bar{\mu}\|\nabla\mathbf{v}^{\mathbf{u}}\|_{H^2}^2+\gamma\|\nabla\mathbf{v}^{\mathbf{w}}\|_{H^2}^2+\nu\|\nabla\mathbf{v}^{\mathbf{b}}\|_{H^2}^2+\kappa\|\text{ div }\mathbf{v}^{\mathbf{w}}\|_{H^2}^2)\\&+C(\bar{\mu}+\gamma+\nu).\end{aligned}$$

On the other hand, using estimate (2.8), Young's inequality and Proposition 4.1, we obtain

$$\begin{aligned}|R_5|&\leq C(\|\nabla\rho\|_\infty\|\mathbf{v}_t^{\mathbf{u}}\|_2+\|D^2\rho\|_4\|\mathbf{v}_t^{\mathbf{u}}\|_4+\|D\rho\|_\infty\|\nabla\mathbf{v}_t^{\mathbf{u}}\|_2)\|\mathbf{v}^{\mathbf{u}}\|_{H^2}\\&\leq C\|\nabla\rho\|_{H^2}\|\mathbf{v}_t^{\mathbf{u}}\|_{H^1}\|\mathbf{v}^{\mathbf{u}}\|_{H^2}\leq C[\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2+\|\mathbf{v}_t^{\mathbf{u}}\|_{H^1}^2].\end{aligned}$$

For bound the term R_6 we use the estimate (2.6), Cauchy-Schwarz's inequality and the Proposition 4.1 to get

$$\begin{aligned}|R_6|&\leq C(\|\mathbf{u}\|_\infty\|\nabla\rho\|_2+\|\mathbf{u}\|_2\|\nabla\rho\|_\infty)\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2\\&\leq C\|\mathbf{u}\|_{H^3}\|\nabla\rho\|_{H^2}\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2\leq C\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2.\end{aligned}$$

Note that we can rewrite the term R_7 as

$$R_7 = -\sum_{|\alpha|\le 2}(\rho(\mathbf{u}\cdot\nabla)D^\alpha \mathbf{v}^{\mathbf{u}}, D^\alpha \mathbf{v}^{\mathbf{u}})_{L^2} - \sum_{|\alpha|\le 2}(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{u}}) - \rho(\mathbf{u}\cdot\nabla)D^\alpha \mathbf{v}^{\mathbf{u}}, D^\alpha \mathbf{v}^{\mathbf{u}})_{L^2}$$
$$:= R_{7,1} + R_{7,2}.$$

Using integration by parts, the incompressible condition $\operatorname{div}\ \mathbf{u} = 0$, Cauchy-Schwarz's inequality, estimates (2.6) and (2.8) and the Proposition 4.1, we have

$$\begin{aligned}|R_{7,1}| = \left|\frac{1}{2}\sum_{|\alpha|\le 2}\int_{\mathbb{R}^2}(\mathbf{u}\cdot\nabla\rho)|D^\alpha \mathbf{v}^{\mathbf{u}}|^2\,dx\right| &\le \frac{1}{2}\sum_{|\alpha|\le 2}\|\mathbf{u}\cdot\nabla\rho\|_2\|D^\alpha \mathbf{v}^{\mathbf{u}}\|_2^2\\ &\le C\|\mathbf{u}\|_\infty\|\nabla\rho\|_2\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2\\ &\le C\|\mathbf{u}\|_{H^3}\|\nabla\rho\|_{H^2}\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2 \le C\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2.\end{aligned}$$

Now using also (2.5) and Hölder's inequality, we find

$$\begin{aligned}|R_{7,2}| &\le \sum_{|\alpha|=1}|(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{u}}) - \rho(\mathbf{u}\cdot\nabla)D^\alpha \mathbf{v}^{\mathbf{u}}, D^\alpha \mathbf{v}^{\mathbf{u}})_{L^2}|\\ &\quad+\sum_{|\alpha|=2}|(D^\alpha(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{u}}) - \rho(\mathbf{u}\cdot\nabla)D^\alpha \mathbf{v}^{\mathbf{u}}, D^\alpha \mathbf{v}^{\mathbf{u}})_{L^2}|\\ &\le C\|\nabla(\rho\mathbf{u})\|_\infty\|\nabla \mathbf{v}^{\mathbf{u}}\|_2^2 + C(\|D^2(\rho\mathbf{u})\|_4\|\nabla\mathbf{v}^{\mathbf{u}}\|_4 + \|\nabla(\rho\mathbf{u})\|_\infty\|\nabla\mathbf{v}^{\mathbf{u}}\|_2)\|D^2\mathbf{v}^{\mathbf{u}}\|_2\\ &\le C(\|\nabla(\rho\mathbf{u})\|_\infty + \|D^2(\rho\mathbf{u})\|_4)\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2\\ &\le C(\|\nabla(\rho\mathbf{u})\|_\infty + \|D^2(\rho\mathbf{u})\|_{H^1})\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2\\ &\le C[\|\mathbf{u}\|_{H^2}\|\nabla\rho\|_{H^2} + \|\mathbf{u}\|_{H^3}(1+\|\nabla\rho\|_{H^2})]\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2\\ &\le C(1+\|\nabla\rho\|_{H^2})\|\mathbf{u}\|_{H^3}\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2 \le C\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2.\end{aligned}$$

Thus, $|R_7| \le C\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2$.

Similarly we find that $|R_9| \le C\|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2$ and $|R_{14}| \le C\|\mathbf{v}^{\mathbf{b}}\|_{H^2}^2$. If we repeat the same process to obtain the bound for R_5, we get $|R_{10}| \le C(\|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2 + \|\mathbf{v}_t^{\mathbf{w}}\|_{H^1}^2)$. Analogously how we obtain the estimate for R_6, one obtains $|R_{12}| \le C\|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2$.

For estimate the term R_{11}, we use Cauchy-Schwarz and Young's inequalities to get

$$\begin{aligned}|R_{11}| \le 2\chi\sum_{|\alpha|\le 2}\|D^\alpha\mathbf{w}^0\|_2\|D^\alpha\mathbf{v}^{\mathbf{w}}\|_{H^2} &\le 2\chi\|\mathbf{w}^0\|_{H^3}\|\mathbf{v}^{\mathbf{w}}\|_{H^2}\\ &\le C(\chi^2\|\mathbf{w}^0\|_{H^3}^2 + \|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2)\\ &\le C(\bar{\mu} + \|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2)\end{aligned}$$

since $\chi^2 \le \chi \le \bar{\mu}$. Using the assumption $\operatorname{div}\ \mathbf{b} = 0$ and

$$\sum_{|\alpha|\le 2}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{v}^{\mathbf{b}}, D^\alpha\mathbf{v}^{\mathbf{u}})_{L^2} + \sum_{|\alpha|\le 2}((\mathbf{b}\cdot\nabla)D^\alpha\mathbf{v}^{\mathbf{u}}, D^\alpha\mathbf{v}^{\mathbf{b}})_{L^2} = 0$$

we can rewrite $R_8 + R_{13}$ as

$$\sum_{|\alpha|\leq 2}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{v}^\mathbf{b})-(\mathbf{b}\cdot\nabla)D^\alpha\mathbf{v}^\mathbf{b},D^\alpha\mathbf{v}^\mathbf{u})_{L^2}+\sum_{|\alpha|\leq 2}(D^\alpha((\mathbf{b}\cdot\nabla)\mathbf{v}^\mathbf{u})-(\mathbf{b}\cdot\nabla)D^\alpha\mathbf{v}^\mathbf{u},D^\alpha\mathbf{v}^\mathbf{b})_{L^2}.$$

Then, similarly how we bound $R_{7,2}$ we find that

$$R_8 + R_{13} \leq C(\|\mathbf{v}^\mathbf{u}\|_{H^2}^2 + \|\mathbf{v}^\mathbf{b}\|_{H^2}^2).$$

Finally, recalling the definitions of $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ (see $(4.43)_1$, $(4.43)_2$ and $(4.43)_3$), using the Cauchy-Schwarz and Young's inequalities and the Proposition 4.1, we estimate R_{15}, R_{16} and R_{17} as

$$\begin{aligned}|R_{15}|+|R_{16}|+|R_{17}| &\leq C(\|\mathbf{v}^\mathbf{u}\|_{H^2}^2+\|\mathbf{v}^\mathbf{w}\|_{H^2}^2+\|\mathbf{v}^\mathbf{b}\|_{H^2}^2)+C(\|\mathbf{F}\|_{H^2}^2+\|\mathbf{G}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^2)\\ &\leq C(\bar{\mu}+\|\mathbf{v}^\mathbf{u}\|_{H^2}^2+\|\mathbf{v}^\mathbf{w}\|_{H^2}^2+\|\mathbf{v}^\mathbf{b}\|_{H^2}^2+\|\sigma\|_{H^2}^2)+C\|\sigma\nabla(p^0)/\rho^0\|_{H^2}^2\end{aligned} \tag{4.54}$$

since $\chi \leq \bar{\mu}$. From the equation $(4.3)_1$ it is easy to see that

$$\begin{aligned}\|\sigma\nabla(p^0)/\rho^0\|_{H^2}^2 &\leq C\|\sigma\|_{H^2}^2(\|\mathbf{u}_t^0\|_{H^2}^2+\|(\mathbf{u^0}\cdot\nabla)\mathbf{u^0}\|_{H^2}^2+\|(\mathbf{b^0}\cdot\nabla)\mathbf{b^0}\|_{H^2}^2+\|\mathbf{f}\|_{H^2}^2)\\ &\leq C\|\sigma\|_{H^2}^2(\|\mathbf{u}_t^0\|_{H^2}^2+\|\mathbf{u^0}\|_{H^2}^2\|\nabla\mathbf{u^0}\|_{H^2}^2+\|\mathbf{b^0}\|_{H^2}^2\|\nabla\mathbf{b^0}\|_{H^2}^2+\|\mathbf{f}\|_{H^3}^2)\\ &\leq C\|\sigma\|_{H^2}^2(1+\|\mathbf{u}_t^0\|_{H^2}^2+\|\mathbf{f}\|_{H^3}^2).\end{aligned} \tag{4.55}$$

Due to estimates (4.54) and (4.55), we obtain

$$|R_{15}|+|R_{16}|+|R_{17}| \leq C(\bar{\mu}+\|\mathbf{v}^\mathbf{u}\|_{H^2}^2+\|\mathbf{v}^\mathbf{w}\|_{H^2}^2+\|\mathbf{v}^\mathbf{b}\|_{H^2}^2)+C\|\sigma\|_{H^2}^2(1+\|\mathbf{u}_t^0\|_{H^2}^2+\|\mathbf{f}\|_{H^3}^2).$$

Thus, replacing the estimates for $R_i, i = 1, \cdots, 17$ into identity (4.53), we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\sum_{|\alpha|\leq 2}\|\sqrt{\rho}D^\alpha\mathbf{v}^\mathbf{u}\|_2^2+\sum_{|\alpha|\leq 2}\|\sqrt{\rho}D^\alpha\mathbf{v}^\mathbf{w}\|_2^2+\|\mathbf{v}^\mathbf{b}\|_{H^2}^2\right) &\leq C(\bar{\mu}+\gamma+\nu+\|\mathbf{v}^\mathbf{u}\|_{H^2}^2+\|\mathbf{v}_t^\mathbf{u}\|_{H^1}^2\\ &+\|\mathbf{v}^\mathbf{w}\|_{H^2}^2+\|\mathbf{v}_t^\mathbf{w}\|_{H^1}^2+\|\mathbf{v}^\mathbf{b}\|_{H^2}^2)+C\|\sigma\|_{H^2}^2(1+\|\mathbf{u}_t^0\|_{H^2}^2+\|\mathbf{f}\|_{H^3}^2).\end{aligned} \tag{4.56}$$

Integrating the inequality (4.56) over $[0,t]$, using the estimates (4.6), (4.49) and the Proposition 4.1, we concludes that

$$\begin{aligned}\|\sigma(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^\mathbf{u}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^\mathbf{w}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^\mathbf{b}(\cdot,t)\|_{H^2}^2 &\leq C(\bar{\mu}+\gamma+\nu)T_0\\ &+C\int_0^t(\|\mathbf{v}_t^\mathbf{u}(\cdot,s)\|_{H^1}^2+\|\mathbf{v}_t^\mathbf{w}(\cdot,s)\|_{H^1}^2)\,ds\\ +C\int_0^t(\|\sigma(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^\mathbf{u}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^\mathbf{w}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^\mathbf{b}(\cdot,s)\|_{H^2}^2)\,ds.\end{aligned} \tag{4.57}$$

since $\mathbf{v}^{\mathbf{u}}(\mathbf{x},0)=\mathbf{v}^{\mathbf{w}}(\mathbf{x},0)=\mathbf{v}^{\mathbf{b}}(\mathbf{x},0)=0$, for all $\mathbf{x}\in\mathbb{R}^3$.

Step 3**.**Since $\operatorname{div}\ \mathbf{v}_t^{\mathbf{u}}=(\operatorname{div}\ \mathbf{v}^{\mathbf{u}})_t=0$, then

$$(\nabla q,\mathbf{v}_t^{\mathbf{u}})_{L^2}=-(q,\operatorname{div}\ \mathbf{v}_t^{\mathbf{u}})=0.$$

Therefore, multiplying the equation $(4.43)_1$ by $\mathbf{v}_t^{\mathbf{u}}$ in $\mathbf{L}^2$, we have

$$\begin{aligned}\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}(\cdot,t)\|_2^2+\frac{\bar{\mu}}{2}\frac{d}{dt}\|\nabla\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_2^2&=\chi(\nabla\times\mathbf{w},\mathbf{v}_t^{\mathbf{u}})_{L^2}+\bar{\mu}(\Delta\mathbf{u}^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}-(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{u}},\mathbf{v}_t^{\mathbf{u}})_{L^2}\\&+((\mathbf{b}\cdot\nabla)\mathbf{v}^{\mathbf{b}},\mathbf{v}_t^{\mathbf{u}})_{L^2}-(\rho(\mathbf{v}^{\mathbf{u}}\cdot\nabla)\mathbf{u}^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}\\&+(\sigma\nabla(p^0)/\rho^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}+((\mathbf{v}^{\mathbf{b}}\cdot\nabla)\mathbf{b}^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}\\&+(\sigma/\rho^0(\mathbf{b}^0\cdot\nabla)\mathbf{b}^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}\end{aligned}\tag{4.58}$$

We will estimate each term on the right side of identity (4.58). Similarly to the proof of Lemma 4.6, we obtain

$$\begin{aligned}|\chi(\nabla\times\mathbf{w},\mathbf{v}_t^{\mathbf{u}})_{L^2}|&\le C\chi\|\nabla\mathbf{w}\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\le C\bar{\mu}\|\mathbf{w}\|_{H^3}\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\\|\bar{\mu}(\Delta\mathbf{u}^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}|&\le C\bar{\mu}\|\Delta\mathbf{u}^0\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\le C\bar{\mu}\|\mathbf{u}^0\|_{H^3}\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\\|(\rho(\mathbf{u}\cdot\nabla)\mathbf{v}^{\mathbf{u}},\mathbf{v}_t^{\mathbf{u}})_{L^2}|&\le C\|\mathbf{u}\|_\infty\|\nabla\mathbf{v}^{\mathbf{u}}\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\le C\|\mathbf{u}\|_{H^3}\|\mathbf{v}^{\mathbf{u}}\|_{H^2}\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\\|((\mathbf{b}\cdot\nabla)\mathbf{v}^{\mathbf{b}},\mathbf{v}_t^{\mathbf{u}})_{L^2}|&\le C\|\mathbf{b}\|_\infty\|\nabla\mathbf{v}^{\mathbf{b}}\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\le C\|\mathbf{b}\|_{H^3}\|\mathbf{v}^{\mathbf{b}}\|_{H^2}\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\\|(\rho(\mathbf{v}^{\mathbf{u}}\cdot\nabla)\mathbf{u}^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}|&\le C\|\mathbf{v}^{\mathbf{u}}\|_\infty\|\nabla\mathbf{u}^0\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\le C\|\mathbf{v}^{\mathbf{u}}\|_{H^2}\|\mathbf{u}^0\|_{H^3}\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\\|(\sigma\nabla(p^0)/\rho^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}|&\le C\|\sigma\|_{H^2}(1+\|\mathbf{u}_t^0\|_{H^2}+\|\mathbf{f}\|_{H^3})\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\\|((\mathbf{v}^{\mathbf{b}}\cdot\nabla)\mathbf{b}^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}|&\le C\|\mathbf{v}^{\mathbf{b}}\|_\infty\|\nabla\mathbf{b}^0\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\le C\|\mathbf{v}^{\mathbf{b}}\|_{H^2}\|\mathbf{b}^0\|_{H^3}\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2.\end{aligned}$$

And for Cauchy-Shwarz's inequality and (2.8), we get

$$\begin{aligned}|(\sigma/\rho^0(\mathbf{b}^0\cdot\nabla)\mathbf{b}^0,\mathbf{v}_t^{\mathbf{u}})_{L^2}|\le C\|\sigma/\rho^0(\mathbf{b}^0\cdot\nabla)\mathbf{b}^0\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2&\le C\|\sigma/\rho^0\mathbf{b}^0\|_\infty\|\nabla\mathbf{b}^0\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\\&\le C\|\sigma\|_{H^2}\|\mathbf{b}^0\|_{H^2}\|\nabla\mathbf{b}^0\|_2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2\\&\le C\|\sigma\|_{H^2}\|\mathbf{b}^0\|_{H^3}^2\|\sqrt{\rho}\mathbf{v}_t^{\mathbf{u}}\|_2.\end{aligned}$$

Replacing the previous estimates into identity (4.58), using Proposition 4.1 and Young's inequality, one obtains

$$\begin{aligned}m\|\mathbf{v}_t^{\mathbf{u}}(\cdot,t)\|_2^2+\bar{\mu}\frac{d}{dt}\|\nabla\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_2^2\le C[\bar{\mu}+\|\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,t)\|_{H^2}^2\\+\|\sigma(\cdot,t)\|_{H^2}^2(1+\|\mathbf{u}_t^0(\cdot,t)\|_{H^2}^2+\|\mathbf{f}(\cdot,t)\|_{H^3}^2)].\end{aligned}\tag{4.59}$$

Integrating the inequality (4.59) over $[0,t]$, using the estimate (4.49) and the Proposition 4.1, we find

$$m\int_0^t \|\mathbf{v}_t^{\mathbf{u}}(\cdot,s)\|_2^2\,ds+\bar{\mu}\|\nabla\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_2^2\le C\bar{\mu}T_0$$
$$+C\int_0^t(\|\sigma(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{u}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,s)\|_{H^2}^2)\,ds. \tag{4.60}$$

Following a similar process to that one for obtain (4.59), using the equations $(4.43)_2$ and $(4.43)_3$ and recalling that $\gamma\ge\max\{\kappa,\chi\}$, we find the inequalities:

$$m\|\mathbf{v}_t^{\mathbf{w}}(\cdot,t)\|_2^2+\gamma\frac{d}{dt}\|\nabla\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_2^2+\kappa\frac{d}{dt}\|\operatorname{div}\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_2^2+2\chi\frac{d}{dt}\|\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_2^2$$
$$\le C(\gamma+\|\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_{H^2}^2), \tag{4.61}$$

$$\|\mathbf{v}_t^{\mathbf{b}}(\cdot,t)\|_2^2+\nu\frac{d}{dt}\|\nabla\mathbf{v}^{\mathbf{b}}(\cdot,t)\|_2^2\le C(\nu+\|\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,t)\|_{H^2}^2). \tag{4.62}$$

Thus, integrating the inequality (4.61) over $[0,t]$, we obtain

$$m\int_0^t \|\mathbf{v}_t^{\mathbf{w}}(\cdot,s)\|_2^2\,ds+\gamma\|\nabla\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_2^2\le C\gamma T_0+C\int_0^t(\|\mathbf{v}^{\mathbf{u}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{w}}(\cdot,s)\|_{H^2}^2)\,ds. \tag{4.63}$$

Taking the operator D^α on both sides of $(4.43)_1$, multiplying the resulting equation by $D^\alpha\mathbf{v}_t^{\mathbf{u}}$ in $\mathbf{L}^2$, summing over $|\alpha|=1$ and recalling that $\bar{\mu}\ge\chi$, we get that

$$m\|\nabla\mathbf{v}_t^{\mathbf{u}}\|_2^2+\bar{\mu}\frac{d}{dt}\|D^2\mathbf{v}^{\mathbf{u}}\|_2^2\le C[\bar{\mu}\|\mathbf{u}^0\|_{H^3}+(1+\|\nabla\rho\|_{H^2})^2\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2\|\mathbf{u}^0\|_{H^3}^2+\|\nabla\rho\|_{H^2}^2\|\mathbf{v}_t^{\mathbf{u}}\|_2^2$$
$$+(1+\|\nabla\rho\|_{H^2})^2\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2\|\mathbf{u}\|_{H^3}^2+\bar{\mu}\|\mathbf{w}\|_{H^3}^2+\|\mathbf{v}^{\mathbf{b}}\|_{H^2}^2(\|\mathbf{b}\|_{H^3}^2+\|\mathbf{b}^0\|_{H^3}^2)$$
$$+\|\sigma\|_{H^2}^2\Big(\|\nabla\rho^0\|_{H^2}^2\|\mathbf{b}^0\|_{H^3}^4+(1+\|\nabla\rho^0\|_{H^2}^2)(\|\mathbf{b}^0\|_{H^3}^4+(1+\|\nabla\rho^0\|_{H^2}^2)[\|\mathbf{u}_t^0\|_{H^2}^2+\|\mathbf{f}\|_{H^3}^2+\|\mathbf{u}^0\|_{H^3}^4])\Big)].$$

From Proposition 4.1 the above inequality can be rewrite as

$$m\|\nabla\mathbf{v}_t^{\mathbf{u}}\|_2^2+\bar{\mu}\frac{d}{dt}\|D^2\mathbf{v}^{\mathbf{u}}\|_2^2\le C[\bar{\mu}+\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2+\|\mathbf{v}_t^{\mathbf{u}}\|_2^2+\|\mathbf{v}^{\mathbf{b}}\|_{H^2}^2$$
$$+\|\sigma\|_{H^2}^2(1+\|\mathbf{u}_t^0\|_{H^2}^2+\|\mathbf{f}\|_{H^3}^2)]. \tag{4.64}$$

Summing the inequalities (4.59) and (4.64), we find

$$m\|\mathbf{v}_t^{\mathbf{u}}\|_{H^1}^2+\bar{\mu}\frac{d}{dt}\|\nabla\mathbf{v}^{\mathbf{u}}\|_{H^1}^2\le C[\bar{\mu}+\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2+\|\mathbf{v}_t^{\mathbf{u}}\|_2^2+\|\mathbf{v}^{\mathbf{b}}\|_{H^2}^2+\|\sigma\|_{H^2}^2(1+\|\mathbf{u}_t^0\|_{H^2}^2+\|\mathbf{f}\|_{H^3}^2)].$$

Integrating over $[0,t]$, using the estimates (4.49) and (4.60) and the Proposition 4.1, we have

$$\bar{\mu}\|\nabla\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_{H^1}^2+m\int_0^t\|\mathbf{v}_t^{\mathbf{u}}(\cdot,s)\|_{H^1}^2\,ds\le C\bar{\mu}T_0+C\int_0^t(\|\sigma(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{u}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,s)\|_{H^2}^2)\,ds. \tag{4.65}$$

In the same way that we obtained the estimate (4.64), we find

$$m\|\nabla \mathbf{v}_t^{\mathbf{w}}\|_2^2+\gamma\frac{d}{dt}\|D^2\mathbf{v}^{\mathbf{w}}\|_2^2+\kappa\frac{d}{dt}\|\nabla(\operatorname{div}\ \mathbf{v}^{\mathbf{w}})\|_2^2+2\chi\frac{d}{dt}\|\nabla\mathbf{v}^{\mathbf{w}}\|_2^2\leq C[\gamma+\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2 + \|\mathbf{v}_t^{\mathbf{w}}\|_2^2+\|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2]. \tag{4.66}$$

and

$$\|\nabla \mathbf{v}_t^{\mathbf{b}}\|_2^2+\nu\frac{d}{dt}\|D^2\mathbf{v}^{\mathbf{b}}\|_2^2\leq C[\nu+\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}\|_{H^2}^2]. \tag{4.67}$$

Summing the inequalites (4.61) and (4.66), we have

$$m\|\mathbf{v}_t^{\mathbf{w}}\|_{H^1}^2+\gamma\frac{d}{dt}\|\nabla\mathbf{v}^{\mathbf{w}}\|_{H^1}^2+\kappa\frac{d}{dt}\|\operatorname{div}\ \mathbf{v}^{\mathbf{w}}\|_{H^1}^2+2\chi\frac{d}{dt}\|\mathbf{v}^{\mathbf{w}}\|_{H^1}^2\leq C[\gamma+\|\mathbf{v}^{\mathbf{u}}\|_{H^2}^2 + \|\mathbf{v}_t^{\mathbf{w}}\|_2^2+\|\mathbf{v}^{\mathbf{w}}\|_{H^2}^2]$$

Integrating over $[0,t]$ and using the estimate (4.63), we obtain

$$\gamma\|\nabla\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_{H^1}^2+m\int_0^t\|\mathbf{v}_t^{\mathbf{w}}(\cdot,s)\|_{H^1}^2\,ds\leq C\gamma T_0+C\int_0^t(\|\mathbf{v}^{\mathbf{u}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{w}}(\cdot,s)\|_{H^2}^2)\,ds. \tag{4.68}$$

Now, summing the inequalities (4.62), (4.67) and integrating the result over $[0,t]$, we get

$$\nu\|\nabla\mathbf{v}^{\mathbf{b}}(\cdot,t)\|_{H^1}^2+\int_0^t\|\mathbf{v}_t^{\mathbf{b}}(\cdot,s)\|_{H^1}^2\,ds\leq C\nu T_0+C\int_0^t(\|\mathbf{v}^{\mathbf{u}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,s)\|_{H^2}^2)\,ds. \tag{4.69}$$

Combining (4.65), (4.68) and (4.69), we deduce that

$$\bar{\mu}\|\nabla\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_{H^1}^2+\gamma\|\nabla\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_{H^1}^2+\nu\|\nabla\mathbf{v}^{\mathbf{b}}(\cdot,t)\|_{H^1}^2$$

$$+m\textstyle\int_0^t(\|\mathbf{v}_t^{\mathbf{u}}(\cdot,s)\|_{H^1}^2+\|\mathbf{v}_t^{\mathbf{w}}(\cdot,s)\|_{H^1}^2)\,ds+\int_0^t\|\mathbf{v}_t^{\mathbf{b}}(\cdot,s)\|_{H^1}^2\,ds\leq C(\bar{\mu}+\gamma+\nu)T_0$$

$$+C\textstyle\int_0^t(\|\sigma(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{u}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{w}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,s)\|_{H^2}^2)\,ds.$$

It follows, from the above estimate and the inequality (4.57), that

$$\begin{aligned}\|\sigma(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,t)\|_{H^2}^2\leq C(\bar{\mu}+\gamma+\nu)T_0\\ +C\int_0^t(\|\sigma(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{u}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{w}}(\cdot,s)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,s)\|_{H^2}^2)\,ds.\end{aligned} \tag{4.70}$$

Therefore, applying a corollary of Gronwall's lemma [see (AMANN, 2011), page 90, corollary 6.2], we concludes that

$$\|\sigma(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{u}}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{w}}(\cdot,t)\|_{H^2}^2+\|\mathbf{v}^{\mathbf{b}}(\cdot,t)\|_{H^2}^2\leq C(\bar{\mu}+\gamma+\nu)T_0\exp(CT_0).$$

This completes the proof of Theorem 4.1.

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APÊNDICE A – BOUNDS FOR THE NONLINEAR TERM $Q(t)$

The integral form for the solution of (3.1) is given by

$$\mathbf{z}(t) = e^{\mathbb{A}t}\mathbf{z}_0 + \int_0^t e^{\mathbb{A}(t-s)}Q(s)\,ds \tag{A.1}$$

where $\mathbb{A}$ and $Q(t)$ are given by (3.96) and (3.100). Denote by $\hat{Q}(\boldsymbol{\xi},t)$ the Fourier transform of $Q(x,t)$, i.e,

$$\hat{Q}(\boldsymbol{\xi},t) = \begin{bmatrix} -\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t) + \mathcal{F}((\mathbf{b}\cdot\nabla)\mathbf{b})(\boldsymbol{\xi},t) \\ -\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{w})(\boldsymbol{\xi},t) \\ -\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{b})(\boldsymbol{\xi},t) + \mathcal{F}((\mathbf{b}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t) \end{bmatrix}. \tag{A.2}$$

First, we will estimate the term $\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t)$. For each $k=1,2$, note that

$$[\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t)]_k = \int_{\mathbb{R}^2} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\left(\sum_{j=1}^{2} u_j D_j u_k\right) d\mathbf{x}.$$

Since $\operatorname{div}\ \mathbf{u}=0$, it follows that

$$\begin{aligned}
[\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t)]_k &= -\int_{\mathbb{R}^2} u_k \sum_{j=1}^{2} D_j(e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}u_j)\,d\mathbf{x} \\
&= \int_{\mathbb{R}^2} u_k \sum_{j=1}^{2} ie^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\boldsymbol{\xi}_j u_j\,d\mathbf{x} - \int_{\mathbb{R}^2} u_k \sum_{j=1}^{2} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} D_j u_j\,d\mathbf{x} \\
&= \int_{\mathbb{R}^2} u_k \sum_{j=1}^{2} ie^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\boldsymbol{\xi}_j u_j\,d\mathbf{x}.
\end{aligned}$$

Then

$$\begin{aligned}
|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t)| \le \sum_{k,j=1}^{2}\int_{\mathbb{R}^2}|\boldsymbol{\xi}_j||u_k u_j|\,d\mathbf{x} \le \sum_{k,j=1}^{2}|\boldsymbol{\xi}_j|\|u_k\|_2\|u_j\|_2 &\le |\boldsymbol{\xi}|\|\mathbf{u}(t)\|_2^2 \\
&\le |\boldsymbol{\xi}|\|\mathbf{z}(t)\|_2^2.
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
|\mathcal{F}((\mathbf{b}\cdot\nabla)\mathbf{b})(\boldsymbol{\xi},t)| &\le |\boldsymbol{\xi}|\|\mathbf{b}(t)\|_2^2 \le |\boldsymbol{\xi}|\|\mathbf{z}(t)\|_2^2, \\
|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{w})(\boldsymbol{\xi},t)| &\le |\boldsymbol{\xi}|\|\mathbf{u}(t)\|_2\|\mathbf{w}(t)\|_2 \le |\boldsymbol{\xi}|\|\mathbf{z}(t)\|_2^2, \\
|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{b})(\boldsymbol{\xi},t)| &\le |\boldsymbol{\xi}|\|\mathbf{u}(t)\|_2\|\mathbf{b}(t)\|_2 \le |\boldsymbol{\xi}|\|\mathbf{z}(t)\|_2^2, \\
|\mathcal{F}((\mathbf{b}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t)| &\le |\boldsymbol{\xi}|\|\mathbf{b}(t)\|_2\|\mathbf{u}(t)\|_2 \le |\boldsymbol{\xi}|\|\mathbf{z}(t)\|_2^2.
\end{aligned}$$

Therefore, from (A.2) we get the following estimate for $\hat{Q}(\boldsymbol{\xi},t)$:

$$|\hat{Q}(\boldsymbol{\xi},t)| \le |\boldsymbol{\xi}|\|\mathbf{z}(t)\|_2^2. \tag{A.3}$$

We can also find another type of bound for $\hat{Q}(\boldsymbol{\xi},t)$ $\hat{Q}(\boldsymbol{\xi},t)$. Indeed, note that

$$\begin{aligned}
|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t)| \le \sum_{k,j=1}^{2}\int_{\mathbb{R}^2}|e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}u_jD_ju_k|\,d\mathbf{x} &= \sum_{k,j=1}^{2}\int_{\mathbb{R}^2}|u_jD_ju_k|\,d\mathbf{x}\\
&\le \sum_{k,j=1}^{k}\|u_j\|_2\|D_ju_k\|_2\\
&\le \|\mathbf{u}(t)\|_2\|\nabla\mathbf{u}(t)\|_2\\
&\le \|\mathbf{z}(t)\|_2\|\nabla\mathbf{z}(t)\|_2
\end{aligned}$$

Analogously, we obtain that

$$\begin{aligned}
|\mathcal{F}((\mathbf{b}\cdot\nabla)\mathbf{b})(\boldsymbol{\xi},t)| &\le \|\mathbf{b}(t)\|_2\|\nabla\mathbf{b}(t)\|_2 \le \|\mathbf{z}(t)\|_2\|\nabla\mathbf{z}(t)\|_2,\\
|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{w})(\boldsymbol{\xi},t)| &\le \|\mathbf{u}(t)\|_2\|\nabla\mathbf{w}(t)\|_2 \le \|\mathbf{z}(t)\|_2\|\nabla\mathbf{z}(t)\|_2,\\
|\mathcal{F}((\mathbf{u}\cdot\nabla)\mathbf{b})(\boldsymbol{\xi},t)| &\le \|\mathbf{u}(t)\|_2\|\nabla\mathbf{b}(t)\|_2 \le \|\mathbf{z}(t)\|_2\|\nabla\mathbf{z}(t)\|_2,\\
|\mathcal{F}((\mathbf{b}\cdot\nabla)\mathbf{u})(\boldsymbol{\xi},t)| &\le \|\mathbf{b}(t)\|_2\|\nabla\mathbf{u}(t)\|_2 \le \|\mathbf{z}(t)\|_2\|\nabla\mathbf{z}(t)\|_2.
\end{aligned}$$

Then, using these others estimates in (A.2) we find that

$$|\hat{Q}(\boldsymbol{\xi},t)| \le \|\mathbf{z}(t)\|_2\|\nabla\mathbf{z}(t)\|_2. \tag{A.4}$$

Lemma A.1 *Let $\hat{\mathbf{u}}_0 \in L^\infty(\mathbb{R}^2)$ and $|\boldsymbol{\xi}|^{-1}\hat{\mathbf{u}}_0 \in L^\infty(\mathbb{R}^2)$. Then, we have that*

$$\|e^{\Delta t}\mathbf{u}_0\|_2 \le Ct^{-\frac{1}{2}}\|\hat{\mathbf{u}}_0\|_\infty \tag{A.5}$$

and

$$\|e^{\Delta t}\mathbf{u}_0\|_2 \le Ct^{-1}\|\hat{\mathbf{f}}_0\|_\infty \quad \textit{with} \quad \hat{\mathbf{f}}_0(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-1}\hat{\mathbf{u}}_0(\boldsymbol{\xi}). \tag{A.6}$$

Proof: From Parseval's identity, we have that

$$\begin{aligned}
\|e^{\Delta t}\mathbf{u}_0\|_2^2 = (2\pi)^{-1}\int_{\mathbb{R}^2}e^{-2|\boldsymbol{\xi}|^2t}|\hat{\mathbf{u}}_0(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \le (2\pi)^{-1}\|\hat{\mathbf{u}}_0\|_\infty^2\int_{\mathbb{R}^2}e^{-2|\boldsymbol{\xi}|^2t}\,d\boldsymbol{\xi} &\le \|\hat{\mathbf{u}}_0\|_\infty^2\int_0^\infty re^{-2r^2t}\,dr\\
&= \|\hat{\mathbf{u}}_0\|_\infty^2\frac{1}{4t}\int_0^\infty e^{-\tau}\,d\tau\\
&\le Ct^{-1}\|\hat{\mathbf{u}}_0\|_\infty^2
\end{aligned}$$

which implies that

$$\|e^{\Delta t}\mathbf{u}_0\|_2 \le Ct^{-\frac{1}{2}}\|\hat{\mathbf{u}}_0\|_\infty.$$

Set $\hat{\mathbf{f}}_0(\boldsymbol{\xi}) := |\boldsymbol{\xi}|^{-1}\hat{\mathbf{u}}_0(\boldsymbol{\xi})$. From Parseval's identity, we get that

$$\begin{aligned}\|e^{\Delta t}\mathbf{u}_0\|_2^2 = (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-2|\boldsymbol{\xi}|^2 t}|\hat{\mathbf{u}}_0(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} &= (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-2|\boldsymbol{\xi}|^2 t}|\boldsymbol{\xi}|^2|\hat{\mathbf{f}}_0(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\\ &\leq (2\pi)^{-1}\|\hat{\mathbf{f}}_0\|_\infty^2\int_{\mathbb{R}^2}|\boldsymbol{\xi}|^2 e^{-2|\boldsymbol{\xi}|^2 t}\,d\boldsymbol{\xi}\\ &\leq \|\hat{\mathbf{f}}_0\|_\infty^2\int_0^\infty r^3 e^{-2r^2 t}\,dr\\ &= \|\hat{\mathbf{f}}_0\|_\infty^2\frac{1}{8t^2}\int_0^\infty \tau e^{-\tau}\,d\tau\\ &\leq Ct^{-2}\|\hat{\mathbf{f}}_0\|_\infty^2.\end{aligned}$$

Therefore, we conclude that

$$\|e^{\Delta t}\mathbf{u}_0\|_2 \leq Ct^{-1}|\boldsymbol{\xi}|\|\hat{\mathbf{f}}_0\|_\infty.$$

Hence, from Lemma A.1 and estimates (A.3) and (A.4), it follows that

$$\|e^{C\Delta(t-s)}Q(s)\|_2 \leq C(t-s)^{-\frac{1}{2}}\|\mathbf{z}(s)\|_2\|\nabla\mathbf{z}(s)\|_2 \tag{A.7}$$

and

$$\|e^{C\Delta(t-s)}Q(s)\|_2 \leq C(t-s)^{-1}\|\mathbf{z}(s)\|_2^2. \tag{A.8}$$

Lemma A.2 *Let* $|\boldsymbol{\xi}|^{-1}\hat{\mathbf{u}}_0 \in L^\infty(\mathbb{R}^2)$. *Then*

$$\|\nabla e^{\Delta t}\mathbf{u}_0\|_2 \leq Ct^{-\frac{3}{2}}\|\hat{\mathbf{f}}_0\|_\infty \quad \textit{with} \quad \hat{\mathbf{f}}_0(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-1}\hat{\mathbf{u}}_0(\boldsymbol{\xi}). \tag{A.9}$$

Proof: From Parserval's identity, we have

$$\begin{aligned}\|\nabla e^{\Delta t}\mathbf{u}_0\|_2^2 = (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-2|\boldsymbol{\xi}|^2 t}|\boldsymbol{\xi}|^2|\hat{\mathbf{u}}_0(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} &= (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-2|\boldsymbol{\xi}|^2 t}|\boldsymbol{\xi}|^4|\hat{\mathbf{f}}_0(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\\ &\leq (2\pi)^{-1}\|\hat{\mathbf{f}}_0\|_\infty^2\int_{\mathbb{R}^2}|\boldsymbol{\xi}|^4 e^{-2|\boldsymbol{\xi}|^2 t}\,d\boldsymbol{\xi}\\ &\leq \|\hat{\mathbf{f}}_0\|_\infty^2\int_0^\infty r^5 e^{-2r^2 t}\,dr\\ &= \|\hat{\mathbf{f}}_0\|_\infty^2\frac{1}{16t^3}\int_0^\infty \tau^2 e^{-\tau}\,d\tau\\ &\leq Ct^{-3}\|\hat{\mathbf{f}}_0\|_\infty^2\end{aligned}$$

which concludes the proof.

Then, from the previous Lemma and (A.3), we find that

$$\|\nabla e^{\Delta(t-s)}Q(s)\|_2 \leq C(t-s)^{-\frac{3}{2}}\|\mathbf{z}(t)\|_2^2. \tag{A.10}$$

APÊNDICE B – MATRIX $\mathbb{A}$ $\times$ HEAT SEMIGROUP

Let $\mathbb{A}$ be the matrix given in (3.96). Note that $\bar{\mathbf{z}}(x,t) = e^{\mathbb{A}t}\mathbf{z}_0(x)$ is solution of the initial value problem:

$$\begin{cases} \bar{\mathbf{z}}_t(\mathbf{x},t) = \mathbb{A}\bar{\mathbf{z}}(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^2,\ t>0, \\ \bar{\mathbf{z}}(\mathbf{x},0) = \mathbf{z}_0(\mathbf{x}), \quad \mathbf{z}_0\in L^2(\mathbb{R}^2). \end{cases}$$

Furthermore, $\mathbf{y}(\mathbf{x},t) = e^{C\Delta t}\mathbf{z}_0(\mathbf{x})$ is solution of the following initial value problem:

$$\begin{cases} \mathbf{y}_t(\mathbf{x},t) = C\Delta\mathbf{y}(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^2,\ t>0, \\ \mathbf{y}(\mathbf{x},0) = \mathbf{z}_0(\mathbf{x}), \quad \mathbf{z}_0\in L^2(\mathbb{R}^2). \end{cases} \tag{B.1}$$

Applying the Fourier transform in (B.1), we find

$$\begin{cases} \hat{\mathbf{y}}_t(\boldsymbol{\xi},t) = -C|\boldsymbol{\xi}|^2\hat{\mathbf{y}}(\boldsymbol{\xi},t), \\ \hat{\mathbf{y}}(\boldsymbol{\xi},0) = \hat{\mathbf{z}}_0(\boldsymbol{\xi}) \end{cases}$$

where $\hat{\mathbf{y}}(\boldsymbol{\xi},t) = e^{-C|\boldsymbol{\xi}|^2t}\hat{\mathbf{z}}_0(\boldsymbol{\xi})$ is solution. Then from Lemma 3.2 and Parseval's identity, we have

$$\begin{aligned} \|e^{\mathbb{A}t}\mathbf{z}_0\|_2^2 = (2\pi)^{-2}\|e^{tM(\boldsymbol{\xi})}\hat{\mathbf{z}}_0\|_2^2 = (2\pi)^{-2}\int_{\mathbb{R}^2}|e^{tM(\boldsymbol{\xi})}\hat{\mathbf{z}}_0(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} &\le (2\pi)^{-2}\int_{\mathbb{R}^2}|e^{\lambda_{\max}(M(\boldsymbol{\xi}))t}\hat{\mathbf{z}}_0(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \\ &\le (2\pi)^{-2}\int_{\mathbb{R}^2}|e^{-C|\boldsymbol{\xi}|^2t}\hat{\mathbf{z}}_0(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \\ &= (2\pi)^{-2}\int_{\mathbb{R}^2}|\hat{\mathbf{y}}(\boldsymbol{\xi},t)|^2\,d\boldsymbol{\xi} \\ &= \|\mathbf{y}(\cdot,t)\|_2^2 = \|e^{C\Delta t}\mathbf{z}_0\|_2^2. \end{aligned}$$

Therefore,

$$\|e^{\mathbb{A}t}\mathbf{z}_0\|_2 \le \|e^{C\Delta t}\mathbf{z}_0\|_2, \quad \forall\ \mathbf{z}_0\in L^2(\mathbb{R}^2). \tag{B.2}$$