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Schrödinger equations and coupled systems with Stein-Weiss convolution parts

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JOSÉ LUANDO DE BRITO SANTOS

Schrödinger equations and coupled systems with Stein-Weiss convolution parts

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

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RESUMO

Neste trabalho, investigamos a existência de soluções positivas para certas classes de equações de Schrödinger e sistemas acoplados com não linearidades do tipo Stein-Weiss. No caso escalar, analisamos classes de equações que envolvem perturbações no termo de Stein-Weiss com potencial que pode se anular no infinito ou ser constante igual a 1. Consideramos tanto o caso de uma não linearidade geral, com crescimento subcrítico que satisfaz certas condições apropriadas, quanto o caso homogêneo crítico no sentido da desigualdade de Stein-Weiss. Além disso, exploramos duas classes de sistemas acoplados. A primeira classe envolve um sistema linear, com potenciais que podem se anular no infinito e não linearidades gerais com crescimento subcrítico, também atendendo a condições específicas. A segunda classe trata-se de um sistema não linear acoplado, cujas não linearidades gerais apresentam crescimento exponencial crítico no sentido da desigualdade de Trudinger-Moser. Estudamos a existência de soluções positivas e a regularidade das soluções para este sistema. Para alcançar os resultados, empregamos métodos variacionais, utilizando técnicas de minimização sobre a variedade de Nehari, truncamentos combinados com a técnica de penalização de Del Pino e Felmer, e o método de iteração de Moser para obter estimativas L^{∞}. Além disso, ao lidar com o sistema não linear acoplado, apresentamos uma alternativa aos argumentos padrão, baseada em uma variante do princípio de criticalidade simétrica de Palais, em vez dos argumentos tradicionais de vanishing-nonvanishing e shifted sequences de Lions, que não são aplicáveis, devido o duplo peso presente na convolução do tipo Stein-Weiss.

Palavras-chaves: Não linearidade do tipo Stein-Weiss. Interação não local com peso duplo. Expoente supercrítico. Iteração de Moser. Crescimento exponencial crítico. Desigualdade de Trudinger-Moser.

ABSTRACT

In this work, we investigate the existence of positive solutions for certain classes of Schrödinger equations and coupled systems with Stein-Weiss type nonlinearities. In the scalar case, we analyze classes of equations that involve perturbations in the Stein-Weiss term with a potential that may vanish at infinity or remain constant at 1. We consider both the case of a general nonlinearity with subcritical growth that satisfies certain appropriate conditions, and the critical homogeneous case in the sense of the Stein-Weiss inequality. Additionally, we explore two classes of coupled systems. The first class involves a linear system with potentials that may vanish at infinity and general nonlinearities with subcritical growth, also meeting specific conditions. The second class deals with a coupled nonlinear system, where the general nonlinearities exhibit critical exponential growth in the sense of the Trudinger-Moser inequality. We study the existence of positive solutions and the regularity of solutions for this system. To achieve these results, we employ variational methods, utilizing techniques such as minimization over the Nehari manifold, truncations combined with the penalization technique of Del Pino and Felmer, and Moser's iteration method to obtain $L^{\infty}-$estimates. Furthermore, when dealing with the coupled nonlinear system, we present an alternative to the standard arguments, based on a variant of Palais symmetric criticality principle, instead of the traditional vanishing-nonvanishing and shifted sequences arguments of Lions, which are not applicable, due to the double weight present in the Stein-Weiss type convolution.

Keywords: Stein-Weiss type nonlinearity. Double weighted nonlocal interaction. Supercritical exponent. Moser iteration. Critical exponential growth. Trudinger-Moser inequality

CONTENTS

1 INTRODUCTION

In 1958, Stein and Weiss proved the so-called weighted Hardy-Littlewood-Sobolev inequality, which generalizes the classical Hardy-Littlewood-Sobolev inequality in (SOBOLEV, 1938) by inserting two weights $|x|^{-\beta}$ and $|y|^{-\alpha}$. Being more precise, it was proved in (STEIN; WEISS, 1958) that there is a sharp constant $C(r,s,N,\alpha,\beta,\mu)$ such that

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(y)h(x)}{|y|^{\alpha}|x-y|^{\mu}|x|^{\beta}}\,\mathrm{d}y\mathrm{d}x\right|\leqslant C(r,s,N,\alpha,\beta,\mu)\|g\|_r\|h\|_s, \tag{1.1}$$

where $g\in L^r(\mathbb{R}^N)$, $h\in L^s(\mathbb{R}^N)$ and r,s,α,β, μ satisfy the following conditions:

$$1<r,s<+\infty,\quad 0<\mu<N,\quad \alpha+\beta\geq 0,\quad 0<\alpha+\beta+\mu\leq N,$$
$$\frac{1}{r}+\frac{1}{s}+\frac{\alpha+\beta+\mu}{N}=2\quad\text{and}\quad 1-\frac{1}{r}-\frac{\mu}{N}<\frac{\alpha}{N}<1-\frac{1}{r}.$$

In particular, if we consider $g=h=|u|^q$, $\alpha=\beta$ and $r=s$, in inequality (1.1), then $2\alpha+\mu\leqslant N$ and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^q|u(x)|^q}{|y|^{\alpha}|x-y|^{\mu}|x|^{\alpha}}\,\mathrm{d}y\mathrm{d}x<\infty,$$

provided

$$2_{*\alpha,\mu}:=\frac{2N-2\alpha-\mu}{N}\leqslant q\leqslant\frac{2N-2\alpha-\mu}{N-2}=:2^*_{\alpha,\mu}, \tag{1.2}$$

where $2^*_{\alpha,\mu}$ is called the upper critical Sobolev exponent $(N\geqslant 3)$ and $2_{*\alpha,\mu}$ the lower critical Sobolev exponent by (DU; GAO; YANG, 2022), in the sense of (1.1). If $N=2$ then $2^*_{\alpha,\mu}=\infty$. For details, see Remark 2.1.16 in Chapter 2 of this thesis.

Given its importance in applications to harmonic analysis and partial differential equations, the Stein-Weiss inequality has recently attracted considerable attention. In particular, there has been a surge in research on elliptic problems motivated by this inequality. A notable example is the work by (DU; GAO; YANG, 2022), where the authors studied the following equation involving the upper critical Sobolev exponent

$$-\Delta u=\frac{1}{|x|^{\alpha}}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_{\alpha,\mu}}}{|y|^{\alpha}|x-y|^{\mu}}\mathrm{d}y\right)|u|^{2^*_{\alpha,\mu}-2}u,\quad\text{in }\mathbb{R}^N, \tag{1.3}$$

where the convolutionary nonlinearity with double weight is called the Stein–Weiss type. They established the existence of nontrivial solutions and investigated qualitative properties of the solutions of (1.3), such as, regularity and symmetry. Moreover, the authors have studied the existence of nontrivial solutions for the following class of equations

$$-\Delta u+u=\frac{1}{|x|^{\alpha}}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p}}{|y|^{\alpha}|x-y|^{\mu}}\mathrm{d}y\right)|u|^{p-2}u,\quad\text{in }\mathbb{R}^N, \tag{1.4}$$

where $N \geqslant 3$, $0 < \mu < N$, $\alpha \geqslant 0$, $0 < 2\alpha + \mu \leqslant N$ and for $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$. It is obtained regularity, existence, symmetry of solutions and nonexistence for $p \geqslant 2^*_{\alpha,\mu}$ or $2_{*\alpha,\mu} \geqslant p$. They still established a nonlocal version of the concentration-compactness principle. In (DE ALBUQUERQUE; SANTOS, 2023), the authors have considered the Schrödinger problem involving Stein-Weiss type nonlinearity and a potential which may vanishes at infinity. In (BISWAS; GOYAL; SREENADH, 2023b) it was considered the case when the nonlinearity has critical exponential growth in Trudinger-Moser inequality sense. For quasilinear critical Kirchhoff-Schrödinger Stein-Weiss problem, we refer (BISWAS; GOYAL; SREENADH, 2023a) for existence of infinitely many nontrivial solutions via concentration-compactness argument. A fractional Kirchhoff Hardy problem combining weighted Choquard and singular nonlinearity is discussed in (GOYAL; SHARMA, 2022). We refer to (ZHANG X. TANG, 2021) for an Anisotropic Choquard problem. In (YANG; ZHOU, 2021) the existence, nonexistence, regularity, symmetry and asymptotic behavior of solutions for a coupled Schrödinger system with Stein-Weiss type convolution part have been explored.

Equation (1.4) when $\alpha = \beta = 0$, reduces to the following classic Choquard equation

$$-\Delta u + u = \left(\frac{1}{|x|^{\mu}} * |u|^p\right)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $*$ denotes the convolution operator. The term $\frac{1}{|x|^{\mu}}$ can be reinterpreted as the classical Riesz potential. Consequently, equation (1.5) is closely related to the Choquard equation, which arises from the study of Bose-Einstein condensation. This relationship can be utilized to describe finite-range many-body interactions among particles. Equation (1.5) has a strong physical meaning and appears in several physical contexts, for example, in the relevant case in which $p = 2$, $N = 3$ and $\mu = 1$, equation (1.5) boils down to the special case

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right)u, \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

Equation (1.6) is called Choquard-Pekar equation. This equation appeared first in 1954, (PEKAR, 1954), where Pekar studied the quantum theory of a polaron at rest. Successively, in 1976, Choquard adopted this equation to characterize an electron trapped in its own hole, in a certain approximation to the Hartree-Fock theory for the one-component plasma, see (LIEB, 1976/77). In 1996, Penrose (PENROSE, 1996), proposed, in a particular case, a model of self-gravitating matter in a programme in which quantum state reduction is regarded as a gravitational phenomenon and, in that context, it is referred as Schrödinger-Newton equation. Finally we observe that if u solves equation (1.6), then the function Ψ defined by

$\Psi(t,x) = e^{itu}(x)$ is a solitary wave of the focusing time-dependent Hartree equation

$$i\Psi_t + \Delta\Psi = -\left(\frac{1}{|x|} * |\Psi|^2\right)\Psi, \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3.$$

Thus, (1.6) is also known as the stationary nonlinear Hartree equation. In the pioneering work (LIEB, 1976/77), proved the existence and uniqueness of positive solutions to (1.6). Later, multiplicity results for (1.6) were obtained by (LIONS, 1980; LIONS, 1982) by variational methods.

Motivated by the physical relevance, many authors studied the equation (1.5) in dimension $N \geq 3$ and and nonlinearity with subcritical or critical polynomial growth in the sense of Hardy-Littlewood-Sobolev inequality. In (ACKERMANN, 2004) has considered Schrödinger equation with nonlocal superlinear part and shown the existence of infinitely many solutions. In (MOROZ; SCHAFTINGEN, 2013), the authors have studied the existence of positive ground state solution as well as they have discussed the regularity and decaying behavior of such solutions. Moreover, considering a suitable control on the potential term in (MOROZ; SCHAFTINGEN, 2015b), they have also studied existence and nonexistence results where the convolution term involves critical exponent with respect to the Hardy-Littlewood-Sobolev inequality. In (MOROZ; SCHAFTINGEN, 2015a), it is proved the existence of a ground state solution under assumptions of Berestycki-Lions type. We also refer the readers to (BUFFONI; JEANJEAN; STUART, 1993; ALVES; YANG, 2014; ALVES et al., 2016a; ALVES; YANG, 2016; ALVES; FIGUEIREDO; YANG, 2016; SCHAFTINGEN; XIA, 2017; GAO; YANG, 2018; DU; YANG, 2019; GAO et al., 2020; CINGOLANI; GALLO; TANAKA, 2022) and references therein, specially (MOROZ; SCHAFTINGEN, 2017), for a meaningful review of Choquard equations.

Naturally, the results were extended to coupled systems. For instance, in (XU; MA; XING, 2020) the authors studied the existence and asymptotic behavior of vector solutions for the following class of linearly coupled Choquard-type systems

$$\begin{cases} -\Delta u + \lambda_1 u = \left(\displaystyle\int_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^\mu}\,\mathrm{d}y\right) f(u(x)) + \lambda v, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \left(\displaystyle\int_{\mathbb{R}^N} \frac{G(v(y))}{|x-y|^\mu}\,\mathrm{d}y\right) g(v(x)) + \lambda u, & \text{in } \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $\lambda_1, \lambda_2, \lambda > 0$ and nonlinearities f, g with polynomial growth. For other works concerned with coupled systems involving Choquard type equations, we refer the readers to (CHEN; LIU, 2018; DE ALBUQUERQUE et al., 2019; XU; MA; XING, 2020; SUN, 2021) and references therein.

Inspired by the preceding discussion, we decided to study in this thesis certain classes of Schrödinger equations and coupled systems with Stein-Weiss type nonlinearities. In the following, we outline the main results that will be explored in the subsequent chapters of this thesis.

In Chapter 2, inspired by (GAO; YANG, 2017; AO, 2019; PAN; LIU; TANG, 2022), we are going to study the existence of ground states for the following class of perturbations of the Stein-Weiss convolution

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^s}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{s-2}u}{|x|^\alpha} + f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

where $N \geqslant 3$, $0<\mu<N$, $\alpha \geqslant 0$, $0 < 2\alpha+\mu \leqslant N$, $\frac{2N-2\alpha-\mu}{N} =: 2_{*\alpha,\mu} < s \leqslant 2^*_{\alpha,\mu} := \frac{2N-2\alpha-\mu}{N-2}$ and under suitable assumptions on different types of nonlinearities f. In (PAN; LIU; TANG, 2022), the authors examined a variant of Problem (1.7) without the double weight, i.e., when $\alpha = 0$. They further assumed that $s = 2^*_{0,\mu}$, $f(x,u) = \lambda |u|^q$ with $2 < q < 2^*$, and proved the existence of a positive radial solution for sufficiently large λ. In the case where $\lambda = 1$, the author in (AO, 2019) established the existence of a nontrivial solution. Our objective is to investigate Problem (1.7) involving double weighted nonlocal terms, specifically in the scenario where $\alpha \neq 0$, considering different types of f, namely:

(f_1) $f(x,u) = \lambda \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}$, $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$;

(f_2) $f(x,u) = \lambda \frac{u}{|x|^2}$;

(f_3) $f(x,u) = \lambda |u|^{p-2}u$, $2 < p < 2^*$;

(f_4) $f(x,u) = \lambda |u|^{q-2}u$, $q \geqslant 2^*$,

where $\lambda > 0$ is a parameter. When $\lambda = 0$, Problem (1.7) becomes (1.4) without the term f with $s = 2^*_{\alpha,\mu}$ and as proved in (DU; GAO; YANG, 2022), there does not exist any nontrivial solution. Furthermore, we also study the version of Problem (1.7) with f satisfying (f_1) above, however for the critical case in the sense of the Sobolev inequality, i.e., we consider the following class of Schrödinger equations

$$-\Delta u + u = |u|^{2^*-2}u + \lambda \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}, \quad \text{in } \mathbb{R}^N, \tag{1.8}$$

where $2^* = \frac{2N}{N-2}$. We emphasize that the key to addressing the critical case of the double weighted nonlocal interaction, together with the hypotheses about f, was the utilization of the minimizing function $U(x)$ of $S_{\alpha,\mu}$ (see (2.12) in Chapter 2) as established in (DU; GAO; YANG, 2022, Theorem 1.3), this function satisfies (1.3), with

$$\int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x = S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}.$$

In addition to this challenge, we draw attention to the case (f_4), where dealing with the supercritical exponent $q \geqslant 2^*$ requires the application of a truncation argument to define the associated energy functional properly. For further details, refer to Subsection 2.5.1 of Chapter 2. Our approach to solving Problems (1.7) and (1.8) is variational, relying on a minimization technique over the Nehari manifold. Additionally, to address the case (f_4) in Problem (1.7), we combine the truncation argument with $L^\infty-$estimates. The results presented in Chapter 2 are novel and extend the existing solutions found in (AO, 2019; PAN; LIU; TANG, 2022; DU; GAO; YANG, 2022) in the following ways:

1. If $f \neq 0$, $\alpha = 0$, then our results complete the picture of (AO, 2019; PAN; LIU; TANG, 2022), for $2_{*\alpha,\mu} < s \leqslant 2^*_{\alpha,\mu}$;
2. If $f \neq 0$, $\alpha \neq 0$, then our results complete the picture of (AO, 2019; PAN; LIU; TANG, 2022; DU; GAO; YANG, 2022).

In Chapter 3, drawing inspiration from (ALVES; SOUTO, 2012; ALVES; FIGUEIREDO; YANG, 2016; CARDOSO; DOS PRAZERES; SEVERO, 2020; DE ALBUQUERQUE; SANTOS, 2023), we explore the existence of positive solutions for the following class of Schrödinger equations with Stein-Weiss type nonlinearity.

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{f(u)}{|x|^\alpha} + \psi(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.9}$$

where the potential $V : \mathbb{R}^N \to \mathbb{R}$ decays to zero at infinity and F is the primitive of function f. Later, we will introduce the assumptions on $V(x)$, f and ψ. Furthermore, we also study a special version of Problem (1.9) with the same ψ, however the nonlinearity f assumes the homogeneous critical case in the sense of the Stein-Weiss inequality (1.1). Precisely, we consider the following class of Schrödinger equations

$$-\Delta u + V(x)u = \frac{1}{2^*_{\alpha,\mu}} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u(x)|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} + \psi(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.10}$$

where $2^*_{\alpha,\mu} = \frac{2N-2\alpha-\mu}{N-2}$ and the potential $V(x)$ is a radial function, i.e., $V(|x|) = V(x)$, for all $x \in \mathbb{R}^N$. In order to find solutions of Problems (1.9) and (1.10), our intention is to study it considering the following types of hypotheses about ψ:

(ψ_1) $\psi(x,u) = \lambda(x)|u|^{q-2}u$, and $\lambda(x)$ a nonnegative function such that $\lambda(x) \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$;

(ψ_2) $\psi(x,u) = \dfrac{\lambda}{q}\left(\displaystyle\int_{\mathbb{R}^N} \frac{|u|^q}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\dfrac{|u|^{q-2}u}{|x|^\alpha}$, $\lambda \geqslant 0$ is a parameter

with $q \geqslant 2^*_{\alpha,\mu}$ and $f(u)$ is a general nonlinearity with subcritical growth satisfying some appropriate conditions and with $V(x)$ that may vanish at infinity, i.e., $V(x) \to 0$ as $|x| \to \infty$, in the sense of (ALVES; SOUTO, 2012), where the authors studied existence of solutions for the following equation

$$-\Delta u + V(x)u = g(u), \quad \text{in } \mathbb{R}^N, \tag{1.11}$$

where $V(x)$ has the following decay behavior

$$\frac{1}{R^4}\inf_{|x|\geqslant R}|x|^4V(x) \geqslant \Lambda > 0. \tag{1.12}$$

It is important to mention that (ALVES; SOUTO, 2012) was extended in several directions, for instance: (DO Ó; GLOSS; SANTANA, 2015) for quasilinear problems, (ALVES; FIGUEIREDO; YANG, 2016) for Choquard-type equation, (DO Ó; SOUTO; UBILLA, 2020) for Kirchhoff-type equation, (DE ALBUQUERQUE; SILVA; SOUSA, 2022) fractional linearly coupled Choquard-type system, etc. In these works, the decay (1.12) was adapted to the respective class of problems. Inspired by (ALVES; SOUTO, 2012), the existence of solutions is obtained by applying variational methods jointly with the penalization method in the spirit of (DEL PINO; FELMER, 1996). Our focus is to study the Problem (1.9), assuming that the potential $V(x)$ satisfies the following hypotheses:

(V_1) $V(x)$ is positive and there exists $R_0 > 1$ such that

$$\frac{1}{R_0^{(q-2)(N-2)}}\inf_{|x|\geqslant R_0}|x|^{(q-2)(N-2)}V(x) =: \Lambda > 0;$$

We will assume that $f : \mathbb{R} \to \mathbb{R}_+$ is a nonzero continuous function and satisfies the following general hypotheses:

(f_1) $\lim_{t\to 0^+}\dfrac{tf(t)}{t^q} < \infty$ and $\lim_{t\to\infty}\dfrac{tf(t)}{t^p} = 0$ for some $p \in \left(1, \dfrac{2(N-\alpha-\mu)}{N-2}\right)$;

(f_2) there exists $\theta \in (2, \min\left\{2^*_{\alpha,\mu}, 4\right\})$, such that $0 < \theta F(t) \leqslant 2f(t)t$, for all $t \geqslant 0$, where $F(t) = \int_0^t f(\tau)\mathrm{d}\tau$.

To finish the Chapter 3, we will study the Problem (1.10), with the following hypotheses about the potential $V(x)$:

(V_2) $V(x)$ radial (i.e., $V(|x|) = V(x)$), and there exists $\hat{R}_0 > 1$ such that

$$\inf_{|x| \geqslant \hat{R}_0} |x|^{(2^*_{\alpha,\mu}-2)(\frac{N-2}{2})} V(x) =: \hat{\Lambda} > 0.$$

In order to deal with the nonlocal term and supercritical term in Problems (1.9) and (1.10), we use the following hypotheses for α, μ:

$$N \geqslant 3, \quad 0 < \mu < N, \quad \alpha \geqslant 0, \quad 0 < 2\alpha + \mu < \min\left\{\frac{N+2}{2}, 4\right\}.$$

We encountered some obstacles while investigating Problems (1.9) and (1.10), primarily due to the presence of the Stein-Weiss term combined with a supercritical term, which introduced additional difficulties. Firstly, because of the supercritical term, we needed to apply a truncation argument to properly define the associated energy functional. Unlike in (CARDOSO; DOS PRAZERES; SEVERO, 2020), however, we were unable to simultaneously apply truncation and penalization to both terms. Our approach involved applying truncation to the supercritical power and penalization to the Stein-Weiss term separately. This strategy, detailed in Subsection 3.2.1, introduced significant difficulties throughout the chapter.

To overcome these difficulties, we introduced two auxiliary problems to restore some compactness, while carefully controlling the terms λ and Λ to connect the solution of the auxiliary problem with the original Problem (1.9). We applied similar arguments to Problem (1.10). In addressing these problems, our approach combined truncation arguments, an adapted version of the penalization method, and $L^\infty-$estimates.

Our main contribution in this chapter lies in our ability to handle three challenging scenarios simultaneously: the case where $q \geqslant 2^*_{\alpha,\mu}$, potentials $V(x)$ that may vanish at infinity, and the double weighted nonlocal terms, where we still consider the critical case. In this context, we emphasize the following:

1. If $\psi \equiv 0$, $\alpha = 0$, $0 < \mu < \min\{\frac{N+2}{2}, 4\}$, $N \geqslant 3$, $q \geqslant 2^*_{0,\mu} := 2^*_\mu = \dfrac{2N-\mu}{N-2}$, then our results complete the picture of (ALVES; FIGUEIREDO; YANG, 2016);

2. If (ψ_1) holds with $\lambda(x) \equiv \lambda$, $\alpha = 0$, $0 < \mu < \frac{N+2}{2}$, $q \geqslant 2^*$, then our results extend and complement the previous item;

3. If $\psi \equiv 0$, $\alpha \neq 0$, $q \geqslant 2^*_{\alpha,\mu}$, then our results complement (DE ALBUQUERQUE; SANTOS, 2023);

4. The case $\psi \not\equiv 0$ extends and complements the previous items.

In Chapter 4, motivated by the work in (DE ALBUQUERQUE; SILVA; SOUSA, 2022; DE ALBUQUERQUE; SANTOS, 2023), we investigate the existence of solution for the following class of linear coupled systems involving Stein-Weiss type nonlinearities

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = \left(\int_{\mathbb{R}^N} \frac{F_1(u_1)}{|y|^\alpha |x-y|^\mu} \mathrm{d}y\right) \frac{f_1(u_1)}{|x|^\alpha} + \lambda(x)u_2, & \text{in } \mathbb{R}^N \\ -\Delta u_2 + V_2(x)u_2 = \left(\int_{\mathbb{R}^N} \frac{F_2(u_2)}{|y|^\alpha |x-y|^\mu} \mathrm{d}y\right) \frac{f_2(u_2)}{|x|^\alpha} + \lambda(x)u_1, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.13}$$

where $N \geqslant 3, 0 < \mu < N, \alpha \geqslant 0, 0 < 2\alpha + \mu < \min\left\{\frac{N+2}{2}, 4\right\}$ and F_i is the primitive of function f_i. We consider continuous functions $V_1(x), V_2(x)$ that may decay to zero at infinity and are related with the coupling function by

$$0 < \lambda(x) \leqslant \delta \min\{V_1(x), V_2(x)\}, \quad \delta \in \left(0, \frac{1}{2}\right), \quad \forall x \in \mathbb{R}^N,$$

where V_i and f_i satisfy hypotheses similar to (V_1), $(f_1) - (f_2)$ of Problem (1.9), i.e., for $i = 1, 2$, we have:

$(V_{i,1})$ $V_i(x)$ is positive and there exists $R_0 > 1$ such that

$$\frac{1}{R_0^{(q_i-2)(N-2)}} \inf_{|x| \geq R_0} |x|^{(q_i-2)(N-2)} V_i(x) =: \Lambda_i > 0;$$

$(f_{i,1})$ $\lim_{t\to 0^+} \dfrac{tf_i(t)}{t^{q_i}} < \infty$ and $\lim_{t\to\infty} \dfrac{tf_i(t)}{t^{p_i}} = 0$ for some $p_i \in \left(1, \dfrac{2(N-\alpha-\mu)}{N-2}\right)$;

$(f_{i,2})$ there exists $\theta_i \in (2, 4)$, such that $0 < \theta_i F_i(t) \leqslant 2f_i(t)t$, for all $t \geqslant 0$, where $F_i(t) = \int_0^t f_i(\tau)\mathrm{d}\tau$.

This class of systems imposes some difficulties. The first one is the presence of the Stein-Weiss terms which are nonlocal. Moreover, this class of systems is also characterized by its lack of compactness inherent to problems defined on unbounded domains. In order to overcome such difficulties, we use the Del Pino and Felmer penalization method, in which we introduce an auxiliary problem where we are able to recover some compactness and obtain a solution. After that, we prove an $L^\infty-$estimate which jointly with regularity theory, we obtain a positive vector solution for System (1.13). In this way, in solving the system we will take an approach based on variational method combined with with penalization technique and $L^\infty-$estimates. Our main contribution in this chapter is to complete the study done by the authors in (DE ALBUQUERQUE; SANTOS, 2023; ALVES; FIGUEIREDO; YANG, 2016), in the following aspects:

1. If $\lambda = 0$, $f_1 = f_2$ and $u_1 = u_2$, then System (1.13) boils down to the class of scalar equations in (1.9) when $\psi \equiv 0$;

2. If $\alpha = 0$, $0 < \mu < \min\{\frac{N+2}{2}, 4\}$, $N \geqslant 3$, $q \geqslant 2^*_{0,\mu} := 2^*_\mu = \dfrac{2N-\mu}{N-2}$, then our results complete the picture of (ALVES; FIGUEIREDO; YANG, 2016);

3. If $\alpha \neq 0$, $q \geqslant 2^*_{\alpha,\mu}$, then our results complement (DE ALBUQUERQUE; SANTOS, 2023);

4. To the best of our knowledge, this is the first work to consider coupled Schrödinger systems with Stein-Weiss type nonlinearities involving potentials that decay to zero at infinity.

Finally, in Chapter 5, inspired by the work of (ALVES; SHEN, 2023; BISWAS; GOYAL; SREENADH, 2023b), our focus is to study the following class of coupled system involving doubly weighted nonlocal interaction.

$$\begin{cases} -\Delta_N u + |u|^{N-2}u = \left(\displaystyle\int_{\mathbb{R}^N} \frac{F(u(y))}{|y|^\beta |x-y|^\mu}\,\mathrm{d}y\right) \dfrac{f(u(x))}{|x|^\beta} + \lambda p |u|^{p-2}u|v|^q, & \text{in } \mathbb{R}^N, \\ -\Delta_N v + |v|^{N-2}v = \left(\displaystyle\int_{\mathbb{R}^N} \frac{G(v(y))}{|y|^\beta |x-y|^\mu}\,\mathrm{d}y\right) \dfrac{g(v(x))}{|x|^\beta} + \lambda q |u|^p|v|^{q-2}v, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.14}$$

where $N \geq 2$, $0 < \mu < N$, $\lambda > 0$, $\beta \geq 0$, $0 < 2\beta + \mu < N$, $p > \frac{N}{2}, q > \frac{N}{2}$, $p + q > N$, $\Delta_N u = \text{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator, $f(s)$, $g(s)$ have critical growth of Trudinger-Moser type, $F(s), G(s)$ are the primitives of $f(s)$, $g(s)$ respectively.

From a mathematical point of view, the cases involving $N-$Laplacian (for $N \geq 3$) or Laplacian (for $N = 2$) are particularly very interesting as the corresponding Sobolev embedding yields $W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for all $q \geq N$, but $W^{1,N}(\mathbb{R}^N) \not\subset L^\infty(\mathbb{R}^N)$. In these cases, the Pohozaev-Trudinger-Moser inequality (CAO, 1992) (see (MOSER, 1971; POHOZAEV, 1965) for the bounded domain case) serves as an alternative to the Sobolev inequality, allowing us to establish the sharp maximal growth for functions in $W^{1,N}(\mathbb{R}^N)$ as follows.

Proposition 1.0.1. ***(Pohozaev-Trudinger-Moser inequality, (CAO, 1992))*** *If $\alpha > 0$, $N \geq 2$ and $u \in W^{1,N}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} (\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u))\,\mathrm{d}x < \infty,$$

where

$$S_{N-2}(\alpha, u) = \sum_{m=0}^{N-2} \frac{\alpha^m |u|^{\frac{mN}{N-1}}}{m!}.$$

Moreover, if $\|\nabla u\|_N^N \leq 1$, $\|u\|_N \leq M < \infty$ *and* $\alpha < \alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$*, where* ω_{N-1} *is the surface area of* $(N-1)-$*dimensional unit sphere, then there exists a constant* $C = C(\alpha, M, N) > 0$ *such that*

$$\int_{\mathbb{R}^N} (\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u))\,\mathrm{d}x \leq C.$$

In the sense of the Pohozaev-Trudinger-Moser inequality, we say that a function $h : \mathbb{R} \to \mathbb{R}$ has α_0- critical exponential growth at $+\infty$, if there exists $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{h(s)}{\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)} = \begin{cases} 0, & \text{if} \quad \alpha > \alpha_0, \\ +\infty, & \text{if} \quad \alpha < \alpha_0. \end{cases} \tag{1.15}$$

This definition of criticality was introduced by Adimurthi and Yadava (ADIMURTHI; YADAVA, 1990), see also (FIGUEIREDO; MIYAGAKI OLIMPIO; RUF, 1995). There are a few works considering Stein-Weiss term and a nonlinearity with critical exponential growth, see for example (ALVES; SHEN, 2023) and (YUAN et al., 2023). For works considering Choquard type equations and nonlinearities with critical exponential growth, we refer the readers to (ALVES et al., 2016b; YANG, 2018; ALBUQUERQUE; FERREIRA; SEVERO, 2021; QIN; TANG, 2021; SHEN; RADULESCU; YANG, 2022) and references therein.

With this in mind, in the same spirit as (1.15), we suppose that the nonlinearities f and g have α_0- critical exponential growth at $+\infty$ and the following hypotheses:

(a) f and g are continuous, $f(s) = g(s) = 0$ if $s \leq 0$ and $f(s) > 0, g(s) > 0$ if $s > 0$. Also

$$\lim_{s\to 0^+} \frac{f(s)}{s^{\frac{2N-2\beta-\mu}{2}-1}} = \lim_{s\to 0^+} \frac{g(s)}{s^{\frac{2N-2\beta-\mu}{2}-1}} = 0;$$

(b) $\displaystyle\liminf_{|s|\to\infty} \frac{F(s)}{e^{\alpha_0 s^{\frac{N}{N-1}}}} = \liminf_{|s|\to\infty} \frac{G(s)}{e^{\alpha_0 s^{\frac{N}{N-1}}}} = \beta_0 > 0;$

(c) there exist $s_0, M_0 > 0$ and $m_0 \in (0, 1]$ such that

$$0 < s^{m_0} F(s) \leq M_0 f(s), \quad \forall\ s \geq s_0;$$

(d) the functions $s \mapsto f(s)/s^{N-1}$ and $s \mapsto g(s)/s^{N-1}$ are increasing for $s > 0$;

(e) there exists $\theta \in \Big(N, p+q\Big]$ such that $0 < \theta F(s) \leq f(s)s$ and $0 < \theta G(s) \leq g(s)s$, for all $s > 0$.

In order to deal with the coupling terms in System (1.14), we use the following hypotheses for p, q and N:

$$N > 2, \quad p, q > \frac{N}{2} \quad \text{and} \quad p + q > N.$$

Beyond the challenges posed by the nonlocal and critical growth behavior of the nonlinearity, we encountered several technical difficulties while studying Systems (1.14). These include:

(i) The nonlocal term is not periodic for $\beta \neq 0$. For this reason, the standard approach based on Lions' vanishing-nonvanishing argument is not applicable anymore;

(ii) Showing the solution of System (1.14) to be vectorial is not obvious and requires a careful treatment;

(iii) For $N > 2$, due to the lack of Hilbert space structure, a variant of Palais principle of symmetric criticality is needed, see Appendix A for details;

(iv) For the critical exponential growth in the general case $N > 2$, even some obvious results require some careful analysis throughout the chapter. Needless to mention Lemma 5.2.11 as an example.

The main contributions of this chapter are as follows:

1. The results presented here complete the framework established in (DE ALBUQUERQUE et al., 2019) in any dimension $N > 2$. We complement and extend some works which consider Choquard type problems with critical exponential growth, such as (CHEN; TANG, 2022);

2. In case of dimension $N = 2$, the existence result obtained in (ALVES; SHEN, 2023) can be achieved from the study of asymptotic behavior of solutions of (S_λ) as $\lambda \searrow 0$, described in Section 4.2 of (DE ALBUQUERQUE et al., 2024) in details;

3. Even for scalar case (when $\lambda = 0$) the results of this chapter are new and complement (ALVES; SHEN, 2023) for dimensions $N > 2$;

4. This chapter offers an alternative to the standard arguments based on Lions' vanishing-nonvanishing and shifted sequences, which are not applicable for $\beta \neq 0$ Instead, we utilize a variant of Palais principle of symmetric criticality.

Given its motivational significance and frequent application in this thesis, we will precisely state the weighted Hardy-Littlewood-Sobolev inequality, commonly referred to as the Stein-Weiss inequality.

Proposition 1.0.2. *(**Weighted Hardy-Littlewood-Sobolev inequality, (STEIN; WEISS, 1958)**) Let $1 < r, s < +\infty$, $0 < \mu < N$, $\alpha + \beta \geq 0$, $0 < \alpha + \beta + \mu \leq N$, $g \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$. Then, there exists a sharp constant $C(r, s, N, \alpha, \beta, \mu)$ such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(y)h(x)}{|y|^\alpha |x-y|^\mu |x|^\beta} \mathrm{d}y \mathrm{d}x \right| \leq C(r, s, N, \alpha, \beta, \mu) \|g\|_r \|h\|_s, \tag{1.16}$$

where

$$\frac{1}{r} + \frac{1}{s} + \frac{\alpha + \beta + \mu}{N} = 2$$

and

$$1 - \frac{1}{r} - \frac{\mu}{N} < \frac{\alpha}{N} < 1 - \frac{1}{r}.$$

In addition, for all $h \in L^s(\mathbb{R}^N)$, we have

$$\left\| \int_{\mathbb{R}^N} \frac{h(y)}{|y|^\alpha |x-y|^\mu |x|^\beta} \mathrm{d}y \right\|_t \leq C(t, s, N, \alpha, \beta, \mu) \|h\|_s,$$

where t verifies

$$1 + \frac{1}{t} = \frac{1}{s} + \frac{\alpha + \beta + \mu}{N} \quad \text{and} \quad \frac{\alpha}{N} < \frac{1}{t} < \frac{\alpha + \mu}{N}.$$

Note that if $\alpha = \beta = 0$, then (1.16) reduces to the classical Hardy-Littlewood-Sobolev inequality, see (SOBOLEV, 1938; LIEB; LOSS, 2001). For other references involving the Stein-Weiss inequality, see (HAN; LU; ZHU, 2012; NGÔ, 2021) and references therein.

To conclude the thesis, in Appendix A, we present, for $N > 2$, due to the lack of Hilbert space structure, a variant of Palais principle of symmetric criticality, which was essential for obtaining the main result of Chapter 5.

2 EXISTENCE OF POSITIVE SOLUTIONS FOR PERTURBATIONS OF THE DOUBLE WEIGHTED NONLOCAL INTERACTION PART WITH CRITICAL OR SUBCRITICAL EXPONENTS

In this chapter, our main goal is to study the following class of Schrödinger equations involving double weighted nonlocal

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^s}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{s-2}u}{|x|^\alpha} + f(x,u), \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where $2_{*\alpha,\mu} < s \leqslant 2^*_{\alpha,\mu}$, $N \geqslant 3$, $0 < \mu < N$, $\alpha \geqslant 0$, $2\alpha + \mu \leqslant N$. The lower bound $2_{*\alpha,\mu}$ is called *lower critical exponent* and the upper bound $2^*_{\alpha,\mu}$ is called *upper critical exponent* in the sense of the weighted Hardy-Littlewood-Sobolev inequality, see (1.2). Here f in (2.1) is a nonlinearity satisfying certain assumptions. Later, we will specify the assumptions on f. Furthermore, we also study the version of the Problem (2.1) for the critical case in the sense of the Sobolev inequality. Precisely, we consider the following class of Schrödinger equations

$$-\Delta u + u = |u|^{2^*-2}u + \lambda\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}, \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $2^* = \frac{2N}{N-2}$ is the critical exponent for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to $L^{2^*}(\mathbb{R}^N)$ and $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$.

2.1 ASSUMPTIONS AND MAIN RESULTS

Inspired by (GAO; YANG, 2017; AO, 2019; PAN; LIU; TANG, 2022), we study the existence of solutions for Problem (2.1), considering different types of f:

(f_1) $f(x,u) = \lambda\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}$, $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$;

(f_2) $f(x,u) = \lambda\frac{u}{|x|^2}$;

(f_3) $f(x,u) = \lambda|u|^{p-2}u$, $2 < p < 2^*$;

(f_4) $f(x,u) = \lambda|u|^{q-2}u$, $q \geqslant 2^*$;

where $\lambda > 0$ is a parameter.

Throughout this chapter, let $H^1(\mathbb{R}^N)$ be the usual Sobolev space endowed with the usual inner product and norm

$$\langle u,v\rangle := \int_{\mathbb{R}^N}(\nabla u\nabla v + uv)\,\mathrm{d}x \quad \text{and} \quad \|u\| = \langle u,u\rangle^{\frac{1}{2}}, \quad \forall u,v \in H^1(\mathbb{R}^N).$$

As the results will be proved by variational methods, the energy functional $\mathcal{F}: H^1(\mathbb{R}^N) \longrightarrow \mathbb{R}$ associated to Problem (2.1) is given by

$$\begin{aligned}\mathcal{F}(u) =& \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|u|^2\right)\mathrm{d}x\\ &-\frac{1}{2s}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^s}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^s}{|x|^\alpha}\,\mathrm{d}x-\int_{\mathbb{R}^N}F(x,u)\,\mathrm{d}x,\end{aligned}$$

where $F(x,u)=\int_0^u f(x,\tau)\,\mathrm{d}\tau$ and f is one of the nonlinearities $(f_1)-(f_4)$. Note that, for any of the functions $(f_1)-(f_3)$, one may deduce that $\mathcal{F}\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ (see Remark 2.1.17 and Lemma 2.1.18 below) with

$$\begin{aligned}\mathcal{F}'(u)v =& \int_{\mathbb{R}^N}\left(\nabla u\nabla v+uv\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^s}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{s-2}u}{|x|^\alpha}v\,\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)v\,\mathrm{d}x,\end{aligned}$$

for each $v\in H^1(\mathbb{R}^N)$. The case (f_4) is studied separately and we will discuss with more details in Section 2.5.

Definition 2.1.1. *We say that a function $u\in H^1(\mathbb{R}^N)$ is a weak solution of Problem* (2.1)*, if there holds*

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\nabla u\nabla v\mathrm{d}x+uv\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^s}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{s-2}u}{|x|^\alpha}v\,\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)v\,\mathrm{d}x=0,\quad \forall\, v\in H^1(\mathbb{R}^N).\end{aligned}$$

Thus the weak solutions of (2.1) are precisely the critical points of $\mathcal{F}$.

At this point, we emphasize that the presence of the Stein-Weiss term jointly with the conditions on f bring additional difficulties. The main difficulties when dealing with this problem lie in the fact that Lions' vanishing argument is not applicable, due to the non-periodic characteristic of Stein-Weiss term and the lack of compactness due to the unboundedness of the domain $\mathbb{R}^N$. To overcome this hurdle, we restrict the energy functional on radial Sobolev space

$$H^1_{\mathrm{rad}}(\mathbb{R}^N)=\left\{u\in H^1(\mathbb{R}^N): u(x)=u(|x|)\right\},$$

endowed with the norm $\|\cdot\|$ induced by $H^1(\mathbb{R}^N)$. Thus, throughout the chapter we will consider Problem (2.1) in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and if u is a point critical of functional $\mathcal{F}$ restricted to $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, then using the principle of symmetric criticality due to Palais (WILLEM, 1996, Theorem 1.28), we conclude that u is a point critical of $\mathcal{F}$ (see Remark 2.1.3 below).

Now, we define the Nehari manifold as

$$\mathcal{N} = \left\{ u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\} : \mathcal{F}'(u)u = 0 \right\}.$$

The radially symmetric critical points of the functional $\mathcal{F}$ must lie on the Nehari manifold $\mathcal{N}$. Therefore, to establish the existence of a solution for Problem (2.1), we consider the following constrained minimization problem:

$$c := \inf_{\mathcal{N}} \mathcal{F}. \tag{2.3}$$

Definition 2.1.2. *We will say that a solution is a nontrivial radial ground state solution (or positive radial ground state) if its energy is minimal among all the nontrivial radial solutions (or all the nontrivial radial positive solutions) of Problem* (2.1)*.*

We shall prove that if the infimum in (2.3) is attained by u, then u is a radial ground state solution of (2.1).

Similarly to Problem (2.1), we define a weak solution for Problem (2.2). See the next sections for more details.

Remark 2.1.3. *We consider Problems* (2.1) *and* (2.2) *in* $H^1_{\mathrm{rad}}(\mathbb{R}^N)$*. If* u *is a point critical of functional* $\mathcal{F}$ *restricted to* $H^1_{\mathrm{rad}}(\mathbb{R}^N)$*, then* u *is a critical of* $\mathcal{F}$*. In fact, we consider the action of group of linear transformations* $G = O(N)$ *on* $H^1(\mathbb{R}^N)$*, according to the definition in (WILLEM, 1996, Definition 1.23), then this action is isometric. Moreover,* $\mathcal{F}$ *is invariant* $(\mathcal{F}(gu) = \mathcal{F}(u))$*, since that*

$$\|gu\|^2 = \|u\|^2, \ \forall g \in G$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|gu(y)|^q |gu(x)|^q}{|y|^\alpha |x-y|^\mu |x|^\alpha}\,\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^q |u(x)|^q}{|y|^\alpha |x-y|^\mu |x|^\alpha}\,\mathrm{d}y\mathrm{d}x, \ \forall g \in G, 2_{*\alpha,\mu} \leqslant q \leqslant 2^*_{\alpha,\mu}$$

where above we use the change of variables theorem, the fact that $\mathbb{R}^N$ *is G-invariant and that* g *is an orthogonal liner transformation. It follows from of principle of symmetric criticality (WILLEM, 1996, Theorem 1.28) that* u *is a point critical of functional* $\mathcal{F}$*. See a more general version in the Appendix A of the thesis.*

The main results of this chapter can be stated as follows.

For Problem (2.1) in $\mathbb{R}^N$, we will first consider the case with subcritical nonlocal term (f_1), then Problem (2.1) becomes

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^s}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y \right) \frac{|u|^{s-2}u}{|x|^\alpha} + \lambda \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu}\mathrm{d}y \right) \frac{|u|^{p-2}u}{|x|^\alpha}, \tag{2.4}$$

and we have the following existence result.

Theorem 2.1.4. *Assume that* $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$ *and* $s = 2^*_{\alpha,\mu}$. *Then, Problem* (2.4) *has a nontrivial radial ground state solution if either*

(i) $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$, $2\alpha + \mu = N$, $N \geqslant 3$, $\lambda > 0$ *and* $\alpha \neq 0$,

(ii) $2_{*\alpha,\mu} < \frac{N+2-2\alpha-\mu}{N-2} < p < 2^*_{\alpha,\mu}$, $N = 3, 4$ *and* $\lambda > 0$,

(iii) $2_{*\alpha,\mu} < \frac{2N-2-2\alpha-\mu}{N-2} < p < 2^*_{\alpha,\mu}$, $N \geqslant 5$ *and* $\lambda > 0$,

or

(iv) $2_{*\alpha,\mu} < p \leqslant \frac{N+2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N = 3, 4$ *and* λ *sufficiently large,*

(v) $2_{*\alpha,\mu} < p \leqslant \frac{2N-2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N \geqslant 5$ *and* λ *sufficiently large.*

Theorem 2.1.5. *Let* $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ *be a nontrivial radial ground state solution of Problem* (2.4) *obtained in Theorem 2.1.4 and assume that* $0 < 2\alpha + \mu < \min\{\frac{N+2}{2}, 4\}$. *Then,* $u \in L^\infty(\mathbb{R}^N)$ *if either*

(i) $2_{*\alpha,\mu} < \max\{\frac{N+2-2\alpha-\mu}{N-2}, \frac{4}{2^*_{\alpha,\mu}}\frac{N+2-2\alpha-\mu}{N-2}\} < p < 2^*_{\alpha,\mu}$, $N = 3, 4$ *and* $\lambda > 0$,

(ii) $2_{*\alpha,\mu} < \max\{\frac{4}{2^*_{\alpha,\mu}}\frac{N+2-2\alpha-\mu}{N-2}, \frac{2N-2-2\alpha-\mu}{N-2}\} < p < 2^*_{\alpha,\mu}$, $N \geqslant 5$ *and* $\lambda > 0$,

or

(iii) $2_{*\alpha,\mu} < \frac{4}{2^*_{\alpha,\mu}}\frac{N+2-2\alpha-\mu}{N-2} \leqslant p \leqslant \frac{N+2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N = 3$, $2\alpha + \mu < 2$ *and* λ *sufficiently large.*

In addition, $u \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$, *for some* $\gamma \in (0, 1)$ *and* u *is positive.*

We are also interested in the Problem (2.1) with Hardy potential term (f_2), i.e.,

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^s}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{s-2}u}{|x|^\alpha} + \lambda\frac{u}{|x|^2}, \quad \text{in } \mathbb{R}^N, \tag{2.5}$$

in which, we establish the following existence result.

Theorem 2.1.6. *Assume that* $\lambda \in (0, 1)$, $s = 2^*_{\alpha,\mu}$ *and* $N > 4$. *Then, Problem* (2.5) *has a nontrivial radial ground state solution.*

We also study Problem (2.1) with subcritical local term (f_3) with $\lambda = 1$, i.e.,

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^s}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{|u|^{s-2}u}{|x|^\alpha} + |u|^{p-2}u, \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

For this case, we establish the following existence result.

Theorem 2.1.7. *Assume that* $2 < p < 2^*$, $s = 2^*_{\alpha,\mu}$ *and* $N \geqslant 3$. *Then, Problem* (2.6) *has a nontrivial radial ground state solution.*

Theorem 2.1.8. *Let* $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ *be a nontrivial radial ground state solution of Problem* (2.6) *obtained in Theorem 2.1.7 and assume that* $0 < 2\alpha + \mu < \min\{\frac{N+2}{2}, 4\}$. *Then, Problem* (2.6) *has a nonnegative radial ground state solution* $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ *for some* $\gamma \in (0,1)$. *Moreover, the radial ground state is positive.*

Furthermore, we will study Problem (2.1) with supercritical local term (f_4), i.e.,

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^s}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{|u|^{s-2}u}{|x|^\alpha} + \lambda |u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{2.7}$$

where $q \geqslant 2^*$ and $\lambda > 0$ is a parameter. We precisely have the following result.

Theorem 2.1.9. *Assume that* $q \geqslant 2^*$ *and* $s = 2^*_{\alpha,\mu}$. *Then, there exists* $\lambda_0 > 0$ *such that if* $\lambda \in (0, \lambda_0]$, *then Problem* (2.7) *has a positive radial ground state solution.*

Lastly we will study Problem (2.2) with subcritical nonlocal term (f_1), i.e.,

$$-\Delta u + u = |u|^{2^*-2}u + \lambda \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \mathrm{d}y\right) \frac{|u|^{p-2}u}{|x|^\alpha}, \quad \text{in } \mathbb{R}^N, \tag{2.8}$$

where we have the following existence result.

Theorem 2.1.10. *Assume that* $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$. *Then, Problem* (2.8) *has a nontrivial radial ground state solution if either*

(i) $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$, $2\alpha + \mu = N$, $N \geqslant 3$, $\lambda > 0$ *and* $\alpha \neq 0$,

(ii) $2_{*\alpha,\mu} < \frac{N+2-2\alpha-\mu}{N-2} < p < 2^*_{\alpha,\mu}$, $N = 3, 4$ *and* $\lambda > 0$,

(iii) $2_{*\alpha,\mu} < \frac{2N-2-2\alpha-\mu}{N-2} < p < 2^*_{\alpha,\mu}$, $N \geqslant 5$ *and* $\lambda > 0$,

or

(iv) $2_{*\alpha,\mu} < p \leqslant \frac{N+2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N = 3, 4$ *and* λ *sufficiently large,*

(v) $2_{*\alpha,\mu} < p \leqslant \frac{2N-2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N \geqslant 5$ *and* λ *sufficiently large.*

Theorem 2.1.11. *Assume that* $2_{*\alpha,\mu} < s < 2^*_{\alpha,\mu}$. *Then, Problems* (2.4), (2.5) (2.6) *and* (2.7) *has a nontrivial radial ground state solution. Furthermore, if we assume that* $0 < 2\alpha + \mu < \min\{\frac{N+2}{2}, 4\}$, *then Problems* (2.4), (2.6) *and* (2.7) *has a radial ground state solution* $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\gamma}_{\rm loc}(\mathbb{R}^N)$ *for some* $\gamma \in (0,1)$, *which is positive.*

Now, we list some remarks.

Remark 2.1.12. *The main contributions of this chapter are the following:*

1. *If* $f \neq 0$, $\alpha = 0$, *then our results complete the picture of (AO, 2019; PAN; LIU; TANG, 2022), for* $2_{*\alpha,\mu} < s \leqslant 2^*_{\alpha,\mu}$;
2. *If* $f \neq 0$, $\alpha \neq 0$, *then our results complete the picture of (AO, 2019; PAN; LIU; TANG, 2022; DU; GAO; YANG, 2022).*

Thus, the results presented of this chapter are new and completes and extends the results of the existence of solutions in the works of (AO, 2019; PAN; LIU; TANG, 2022; DU; GAO; YANG, 2022). The approach is variational and based on minimization technique over the Nehari manifold. Moreover, we combine this approach, to deal with the local term (f_4) *in Problem* (2.7), *truncation arguments with* $L^\infty-$*estimates.*

Remark 2.1.13. *For Problem* (2.1) *with* $s = 2^*_{\alpha,\mu}$ *without the term* f, *as proved in (DU; GAO; YANG, 2022, Theorem 1.10), there does not exist any nontrivial solution. For this, the authors established the following Pahožaev identity,*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = \frac{2N-2\alpha-\mu}{2p}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^q}{|x|^\alpha}\,\mathrm{d}x,$$

where $u \in W^{2,2}_{\rm loc}(\mathbb{R}^N) \cap L^{\frac{2Np}{2N-2\alpha-\mu}}(\mathbb{R}^N)$ *is a positive solution of* (2.1) *for* $f \equiv 0$.

Remark 2.1.14. *In the presence of potential* $V(x)$ *that may decay to zero at infinity, we show in the Chapter 3 of the thesis the existence of a positive solution to the Problem* (2.1) *without the term* f. *We precisely consider the following class of Schrödinger equations with Stein-Weiss type nonlinearity*

$$-\Delta u + V(x)u = \frac{1}{2^*_{\alpha,\mu}}\left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u(x)|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} + \psi(x,u), \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 3$, $0 < \mu < N$, $\alpha \geqslant 0$, $0 < 2\alpha + \mu < \min\{\frac{N+2}{2}, 4\}$, *the potential* V *is a radial function, i.e.,* $V(|x|) = V(x)$, *for all* $x \in \mathbb{R}^N$ *and* ψ *is a nonlinearity satisfying certain assumptions including the case* $\psi \equiv 0$. *The approach was based on variational methods combined with penalization techniques and* $L^\infty-$*estimates, see Chapter 3 for more details.*

Remark 2.1.15. *In the proof of Theorems 2.1.5, 2.1.8 and 2.1.9, we shall use the Moser's iteration method in combination with Proposition 1.0.2. However, due to the presence of the weights* $\frac{1}{|x|^\alpha}$ *and* $\frac{1}{|y|^\alpha}$*, the convolutions*

$$\frac{1}{|x|^\alpha}\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y \tag{2.9}$$

introduces difficulty in establishing $L^\infty-$*estimates involving solution of our problems. For this reason, we require the assumption* $0<2\alpha+\mu<\min\left\{\frac{N+2}{2},4\right\}$ *to carry out Moser's iteration method.*

Remark 2.1.16. *In view of Proposition 1.0.2, if* $g=h=F$, $\alpha=\beta$, $s=r$, *then we obtain*

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\alpha}\,\mathrm{d}x\leqslant C(N,\alpha,\mu)\|F(u)\|_s^2, \tag{2.10}$$

where $s>1$ *is defined by* $\dfrac{2}{s}+\dfrac{2\alpha+\mu}{N}=2$, *i.e.,* $s=\dfrac{2N}{2N-2\alpha-\mu}$. *This means we must require* $F(u)\in L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N)$. *By considering* $N\geqslant 3$ *and the pure power* $F(t)=|t|^q$, *we may use Sobolev embedding when*

$$\frac{2Nq}{2N-2\alpha-\mu}\in[2,2^*],$$

i.e., when the exponent q *satisfies*

$$2_{*\alpha,\mu}:=\frac{2N-2\alpha-\mu}{N}\leqslant q\leqslant\frac{2N-2\alpha-\mu}{N-2}=:2^*_{\alpha,\mu},$$

where $2^*_{\alpha,\mu}$ *is called the upper critical Sobolev exponent and* $2_{*\alpha,\mu}$ *the lower critical Sobolev exponent by (DU; GAO; YANG, 2022), in the sense of inequality* (1.16) *(Proposition 1.0.2), due to the following inequalities*

$$\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac{1}{2^*_{\alpha,\mu}}}\leqslant C^*(N,\alpha,\mu)\int_{\mathbb{R}^N}|u|^{2^*}\,\mathrm{d}x, \tag{2.11}$$

$$\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2_{*\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2_{*\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac{1}{2_{*\alpha,\mu}}}\leqslant C_*(N,\alpha,\mu)\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x.$$

In view of (2.11), *we define*

$$\|u\|_{\alpha,\mu}:=\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac{1}{22^*_{\alpha,\mu}}},$$

which turns out to be a norm on $L^{2^*}(\mathbb{R}^N)$ *and use* $S_{\alpha,\mu}$ *to denote the best constant*

$$S_{\alpha,\mu}:=\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x}{\left[\displaystyle\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac{1}{2^*_{\alpha,\mu}}}}, \tag{2.12}$$

where $S_{\alpha,\mu}$ *is related to the nonlocal Euler-Lagrange equation* (1.3).

From (2.11), *for all* $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, *we know*

$$\|u\|_{\alpha,\mu}^2 \leqslant C(\alpha,\mu,N)^{\frac{1}{2^*_{\alpha,\mu}}}\|u\|_{2^*}^2.$$

Then

$$S_{\alpha,\mu} \geqslant \frac{S}{C(N,\alpha,\mu)^{\frac{1}{2^*_{\alpha,\mu}}}} > 0,$$

where S *is the best Sobolev constant for the embedding of* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *into* $L^{2^*}(\mathbb{R}^N)$,

$$S\left(\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant \int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x. \tag{2.13}$$

Remark 2.1.17. *In view of* (2.10) *and Hölder's inequality, for any* $u,v \in H^1(\mathbb{R}^N)$ *and* $2_{*\alpha,\mu} \leqslant q \leqslant 2^*_{\alpha,\mu}$, *we obtain*

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^q}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{q-2}u}{|x|^\alpha}v\,\mathrm{d}x\\
&\qquad\leqslant C(N,\alpha,\mu)\left(\int_{\mathbb{R}^N}|u|^{\frac{2Nq}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\
&\qquad\quad\times\left(\int_{\mathbb{R}^N}|u|^{\frac{2N(q-1)}{2N-2\alpha-\mu}}|v|^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\
&\qquad\leqslant C(N,\alpha,\mu)\left(\int_{\mathbb{R}^N}|u|^{\frac{2Nq}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\
&\qquad\quad\times\left(\int_{\mathbb{R}^N}|u|^{\frac{2Nq}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}\frac{q-1}{q}}\left(\int_{\mathbb{R}^N}|v|^{\frac{2Nq}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}\frac{1}{q}}\\
&\qquad= C(N,\alpha,\mu)\|u\|^{2q-1}_{\frac{2Nq}{2N-2\alpha-\mu}}\|v\|_{\frac{2Nq}{2N-2\alpha-\mu}},
\end{aligned}$$

i.e., the integral

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^q}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{q-2}u}{|x|^\alpha}v\,\mathrm{d}x$$

is well defined if $2_{*\alpha,\mu} \leqslant q \leqslant 2^*_{\alpha,\mu}$, *by Sobolev embedding.*

In order to be able to deal with the term $f(u) = \dfrac{u}{|x|^2}$ in equation (2.5), we need to recall the Hardy inequality in (AZORERO; ALONSO, 1998, Lemma 2.1).

Lemma 2.1.18. *Assume* $N \geqslant 3$. *If* $u \in H^1(\mathbb{R}^N)$, *then*

(1) $\dfrac{u}{|x|} \in L^2(\mathbb{R}^N)$.

(2) ***(Hardy inequality)***

$$\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}\mathrm{d}x \leqslant C_{N,2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x,$$

where the constant $C_{N,2} = \left(\dfrac{2}{N-2}\right)^2$ *is optimal.*

Remark 2.1.19. *We emphasize that we will study the results previous for the critical case $s = 2^*_{\alpha,\mu}$, having observed that the results involving the case $2_{\alpha,\mu} < s < 2^*_{\alpha,\mu}$ are naturally adaptable.*

The remainder of this chapter is organized as follows: In Section 2.2, we introduce the Nehari manifold associated with Problem (2.4), outline its key properties, and derive several estimates and convergence lemmas, including a nonlocal version of the Brézis-Lieb Lemma. Using a minimization method on the Nehari manifold, we obtain a nontrivial radial ground state solution, thereby proving Theorem 2.1.4. To establish Theorem 2.1.5, we apply Moser's iteration to ensure the regularity of the solution obtained in Theorem 2.1.4, and subsequently use the Maximum Principle to demonstrate that the solution is positive. Section 2.3 focuses on the proof of Theorem 2.1.6, employing a similar approach to that used in proving Theorem 2.1.4, but under the assumption that the parameter λ is small. Theorems 2.1.7, 2.1.8, and 2.1.10 are proven in Sections 2.4 and 2.6, respectively, using a method analogous to that in Section 2.2. Finally, in Section 2.5, following the ideas developed by (CHABROWSKI; YANG, 1997), we truncate the local term in Problem (2.7) and define an auxiliary problem with subcritical growth based on this truncation. We then apply the approach from Section 2.2 to find a solution to the auxiliary problem. Using Moser's iteration method, we introduce a suitable $L^\infty-$estimate for the solution to the auxiliary problem, ultimately showing that it is indeed a solution to the original problem, thereby proving Theorem 2.1.9.

2.2 THE NONLOCAL PERTURBATION

The main goal in the present section is to prove Theorems 2.1.4 and 2.1.5. Thus, we will study the existence of solution for the following equation involving Stein-Weiss type critical nonlinearity in $\mathbb{R}^N$ with nonlocal term (f_1):

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} + \lambda\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}, \quad (2.14)$$

where $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$ and $\lambda > 0$.

We introduce the energy functional associated with Problem (2.14):

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ &- \frac{\lambda}{2p}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}\,\mathrm{d}x, \end{aligned}$$

where, according to Remark 2.1.17, I_λ belongs to $C^1(H^1(\mathbb{R}^N),\mathbb{R})$, and its derivative is given by

$$I'_\lambda(u)v = \int_{\mathbb{R}^N}(\nabla u\nabla v + uv)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{2^*-2}u}{|x|^\alpha}v\,\mathrm{d}x$$
$$- \lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|y-x|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}v\,\mathrm{d}x. \tag{2.15}$$

Note that solutions of (2.14) correspond to the critical points of the energy functional I_λ. We consider I_λ on $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. If u is a point critical of I_λ when restricted to $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, then u is a critical of I_λ, see Remark 2.1.3. A necessary condition for $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ to be a critical point of I_λ is that $I'_\lambda(u)u = 0$. This condition defines the Nehari manifold:

$$\mathcal{N}_\lambda = \left\{u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\} : I'_\lambda(u)u = 0\right\}.$$

Notice that if $u \in \mathcal{N}_\lambda$, then

$$\|u\|^2 = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + \lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}\,\mathrm{d}x. \tag{2.16}$$

Now, we consider the following constrained minimization problem:

$$c_\lambda := \inf_{\mathcal{N}_\lambda} I_\lambda. \tag{2.17}$$

We will prove that if the infimum in (2.17) is attained by u, then u is a radial ground state solution of (2.14). It is important to emphasize that the number c_λ is well defined by proving that I_λ is bounded from below on $\mathcal{N}_\lambda$ and the set $\mathcal{N}_\lambda$ is non-empty. Indeed, for $u \in \mathcal{N}_\lambda$, from (2.16), we obtain

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ & - \frac{1}{2p}\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}\,\mathrm{d}x \\ =& \left(\frac{1}{2} - \frac{1}{2p}\right)\|u\|^2 + \left(\frac{1}{2p} - \frac{1}{22^*_{\alpha,\mu}}\right)\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \geqslant 0, \end{aligned} \tag{2.18}$$

provided $1 < p < 2^*_{\alpha,\mu}$, i.e., I_λ is bounded from below on $\mathcal{N}_\lambda$.

As a consequence of the next lemma, the set $\mathcal{N}_\lambda \neq \emptyset$.

Lemma 2.2.1. *For any $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$, we have that*

(i) *there exists a unique $t_0 > 0$, depending on u, such that*

$$t_0 u \in \mathcal{N}_\lambda \quad \textit{and} \quad \max_{t\geq 0} I_\lambda(tu) = I_\lambda(t_0 u);$$

(ii) there exists a constant $\delta > 0$ such that $\|u\| \geqslant \delta$, for any $u \in \mathcal{N}_\lambda$;

(iii) $c_\lambda = \inf\limits_{\mathcal{N}} I_\lambda > 0$.

Proof. (i) Let $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\}$ be fixed and consider the function $g : [0, +\infty) \to \mathbb{R}$ defined by

$$\begin{aligned} g(t) = I_\lambda(tu) =& \frac{t^2}{2}\|u\|^2 - \frac{t^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\ & - \frac{t^{2p}\lambda}{2p} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \,\mathrm{d}x, \end{aligned}$$

then $g'(t) = I'_\lambda(tu)tu$. Hence, t_0 is a positive critical point of g if and only if $t_0 u \in \mathcal{N}_\lambda$. Since $1 < p < 2^*_{\alpha,\mu}$, we conclude that $g(t) < 0$ for $t > 0$ sufficiently large. On the other hand, combining the Remark 2.1.17 and Sobolev embedding, we have

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \leqslant C\|u\|^{22^*_{\alpha,\mu}}, \tag{2.19}$$

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \,\mathrm{d}x \leqslant \tilde{C}\|u\|^{2p}, \tag{2.20}$$

which imply that

$$g(t) \geqslant t^2\|u\|^2 \left[\frac{1}{2} - \frac{t^{22^*_{\alpha,\mu}-2}}{22^*_{\alpha,\mu}} \tilde{C}\|u\|^{22^*_{\alpha,\mu}-2} - \frac{t^{2p-2}}{2p} \lambda \tilde{C}\|u\|^{2p-2} \right] > 0,$$

provided $t > 0$ is sufficiently small. Thus g has maximum points in $(0, \infty)$. Suppose that there exists $t_1, t_2 > 0$ such that $g'(t_1) = g'(t_2) = 0$. Since every critical point of g satisfies

$$\begin{aligned} \|u\|^2 =& t^{22^*_{\alpha,\mu}-2} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\ & + t^{2p-2}\lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \,\mathrm{d}x, \end{aligned}$$

we deduce,

$$\begin{aligned} 0 = & \left(t_1^{22^*_{\alpha,\mu}-2} - t_2^{22^*_{\alpha,\mu}-2} \right) \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\ & + \left(t_1^{2p-2} - t_2^{2p-2} \right) \lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \,\mathrm{d}x. \end{aligned}$$

Therefore, since both terms in parentheses have the same sign if $t_1 \neq t_2$ and we also have $\lambda > 0$, it follows that $t_1 = t_2$ and the proof of (i) is complete.

(ii) For any $u \in \mathcal{N}_\lambda$, together (2.16) with (2.19) and (2.20), we obtain

$$\|u\|^2 \leqslant C\|u\|^{22^*_{\alpha,\mu}} + \lambda \tilde{C}\|u\|^{2p}.$$

Hence, $1 \leqslant C\|u\|^{22^*_{\alpha,\mu}-2} + \lambda\tilde{C}\|u\|^{2p-2}$, which implies that (ii) holds.

(iii) Combining (2.18) with (ii) we obtain

$$\begin{aligned} I_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{2p}\right)\|u\|^2 + \left(\frac{1}{2p} - \frac{1}{22^*_{\alpha,\mu}}\right)\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2p}\right)\delta^2 > 0, \end{aligned}$$

provided $1 < p < 2^*_{\alpha,\mu}$. Thus, for the arbitrary of $u \in \mathcal{N}_\lambda$, we infer

$$c_\lambda = \inf_{\mathcal{N}_\lambda} I_\lambda(u) \geqslant \inf_{\mathcal{N}_\lambda}\delta^2 > 0,$$

which finishes the proof. □

The next lemma is crucial in our arguments, because it establishes an important estimate involving the level c_λ.

Lemma 2.2.2. *The level c_λ satisfies*

$$0 < c_\lambda < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)}S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}},$$

if either

(i) $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$, $2\alpha+\mu = N$, $N \geqslant 3$,

(ii) $2_{*\alpha,\mu} < \frac{N+2-2\alpha-\mu}{N-2} < p < 2^*_{\alpha,\mu}$, $N = 3,4$,

(iii) $2_{*\alpha,\mu} < \frac{2N-2-2\alpha-\mu}{N-2} < p < 2^*_{\alpha,\mu}$, $N \geqslant 5$,

or

(iv) $2_{*\alpha,\mu} < p \leqslant \frac{N+2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N = 3,4$ *and λ sufficiently large,*

(v) $2_{*\alpha,\mu} < p \leqslant \frac{2N-2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N \geqslant 5$ *and λ sufficiently large.*

Proof. We will divide the proof into two cases.

Case 1. In what follows, the proof only includes items (i), (ii) and (iii).

For $\varepsilon > 0$ define

$$U_\varepsilon(x) = \varepsilon^{\frac{2-N}{2}}U(\frac{x}{\varepsilon}),$$

where $U(x)$ is a minimizant of $S_{\alpha,\mu}$ (see (DU; GAO; YANG, 2022, Theorem 1.3)) and satisfies

$$-\Delta u = \left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha}, \quad \text{in } \mathbb{R}^N$$

with

$$\int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x = S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}. \tag{2.21}$$

Applying the change of variable theorem, we observe that

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x, \\ \int_{\mathbb{R}^N} |U_\varepsilon|^2 \,\mathrm{d}x = \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x, \\ \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\ = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x, \\ \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U_\varepsilon|^p}{|x|^\alpha} \,\mathrm{d}x \\ = \varepsilon^{(2-N)p+2N-2\alpha-\mu} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U|^p}{|x|^\alpha} \,\mathrm{d}x. \end{cases} \tag{2.22}$$

Now, arguing as in the proof of Lemma 2.2.1 (i), implies that there exists $t_\varepsilon > 0$ such that

$$t_\varepsilon U_\varepsilon \in \mathcal{N}_\lambda \quad \text{and} \quad \max_{t \geqslant 0} g(t) = g(t_\varepsilon) = I_\lambda(t_\varepsilon U_\varepsilon).$$

Furthermore, t_ε is unique. In view of $t_\varepsilon U_\varepsilon \in \mathcal{N}_\lambda$ and since $\lambda > 0$, we deduce

$$\begin{aligned} 2\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |U_\varepsilon|^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |U_\varepsilon|^2 \,\mathrm{d}x \\ = t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\ + t_\varepsilon^{2(p-1)} \lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U_\varepsilon|^p}{|x|^\alpha} \,\mathrm{d}x \\ \geqslant t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \end{aligned} \tag{2.23}$$

and combining with (2.21)-(2.22), we obtain

$$2S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x \geqslant t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}, \tag{2.24}$$

whence we have $0 < t_\varepsilon \leqslant 2^{\frac{1}{2(2^*_{\alpha,\mu}-1)}}$, for ε small enough.

Claim. $t_\varepsilon \to 1$, as $\varepsilon \to 0$.

In fact, by (2.21)-(2.22), it follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x - t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\ &= -\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x + t_\varepsilon^{2(p-1)} \varepsilon^{(2-N)p+2N-2\alpha-\mu} \lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U|^p}{|x|^\alpha} \,\mathrm{d}x \\ &\quad \to 0, \end{aligned} \tag{2.25}$$

as $\varepsilon \to 0$, implying that $t_\varepsilon \to 1$ as $\varepsilon \to 0$, which proves the claim.

Finally, combining once more (2.21)-(2.22), we see that

$$\begin{aligned}
c_\lambda &\leqslant \max_{t\geqslant 0} I_\lambda(tU_\varepsilon) = I_\lambda(t_\varepsilon U_\varepsilon) \\
&= \left(\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}}\right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x \\
&\quad - \frac{t^{2p}}{2p}\varepsilon^{(2-N)p+2N-2\alpha-\mu}\lambda \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U|^p}{|x|^\alpha}\,\mathrm{d}x \\
&\leqslant \max_{t>0}\left(\frac{t^2}{2} - \frac{t^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}}\right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x \\
&\quad - \frac{t^{2p}}{2p}\varepsilon^{(2-N)p+2N-2\alpha-\mu}\lambda \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U|^p}{|x|^\alpha}\,\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x \\
&\quad - \frac{t^{2p}}{2p}\varepsilon^{(2-N)p+2N-2\alpha-\mu}\lambda \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U|^p}{|x|^\alpha}\,\mathrm{d}x,
\end{aligned}$$

i.e.,

$$\begin{aligned}
c_\lambda &\leqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x \\
&\quad - \frac{t^{2p}}{2p}\varepsilon^{(2-N)p+2N-2\alpha-\mu}\lambda \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U|^p}{|x|^\alpha}\,\mathrm{d}x. \qquad (2.26)
\end{aligned}$$

But, since $\lambda > 0$ and $(2-N)p + 2N - 2\alpha - \mu < 2$ for every choice of p in (i), (ii) and (iii), we have for ε small enough that

$$\frac{t_\varepsilon^2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x - \frac{t^{2p}}{2p}\varepsilon^{(2-N)p+2N-2\alpha-\mu}\lambda \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U|^p}{|x|^\alpha}\,\mathrm{d}x < 0.$$

This together with (2.26), one has

$$c_\lambda < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}.$$

Therefore, the proof of Case 1 is complete.

Case 2. For this case, we assume λ sufficiently large, and the proof in what follows will include items (iv) and (v). In fact, for $t > 0$, we define

$$\begin{aligned}
g_\lambda(t) :=& I_\lambda(tU_\varepsilon) = \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla U_\varepsilon(x)|^2 \,\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{R}^N} |U_\varepsilon(x)|^2 \,\mathrm{d}x \\
&- \frac{t^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\
&- \lambda\frac{t^{2p}}{2p} \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U_\varepsilon|^p}{|x|^\alpha}\,\mathrm{d}x. \qquad (2.27)
\end{aligned}$$

It is well known that for $t > 0$ large enough $g_\lambda(t) < 0$, for $t > 0$ small enough $g_\lambda(t) > 0$. This implies there exists $t_\lambda > 0$ such that

$$t_\lambda U_\varepsilon \in \mathcal{N} \text{ and } c_\lambda \leqslant \max_{t \geqslant 0} g_\lambda(t) = g_\lambda(t_\lambda), \tag{2.28}$$

$$\begin{aligned} \|U_\varepsilon\|^2 =& t_\lambda^{2(2^*_{\alpha,\mu}-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \mathrm{d}y \right) \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \mathrm{d}x \\ &+ \lambda t_\lambda^{2(p-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha |x-y|^\mu} \mathrm{d}y \right) \frac{|U_\varepsilon|^p}{|x|^\alpha} \mathrm{d}x \end{aligned} \tag{2.29}$$

and, we deduce

$$\|U_\varepsilon\|^2 \geqslant \lambda t_\lambda^{2(p-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha |x-y|^\mu} \mathrm{d}y \right) \frac{|U_\varepsilon|^p}{|x|^\alpha} \mathrm{d}x.$$

This implies that

$$t_\lambda \leqslant \left(\frac{\|U_\varepsilon\|^2}{\lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha |x-y|^\mu} \mathrm{d}y \right) \frac{|U_\varepsilon|^p}{|x|^\alpha} \mathrm{d}x} \right)^{\frac{1}{2(p-1)}}.$$

Follows that $\lim_{\lambda \to \infty} t_\lambda = 0$. Finally, from (2.27)-(2.29), we observe that

$$\begin{aligned} \max_{t \geqslant 0} g_\lambda(t) =& \frac{t_\lambda^2}{2} \|U_\varepsilon\|^2 - \frac{t_\lambda^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^{2^*_{\alpha,\mu}} |U_\varepsilon(y)|^{2^*_{\alpha,\mu}}}{|x|^\alpha |y-x|^\mu |y|^\alpha} \mathrm{d}x\mathrm{d}y \\ &- \lambda \frac{t_\lambda^{2p}}{2p} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha |x-y|^\mu} \mathrm{d}y \right) \frac{|U_\varepsilon|^p}{|x|^\alpha} \mathrm{d}x \to 0, \end{aligned}$$

as $\lambda \to \infty$. This means there exists $\lambda_0 > 0$ such that when $\lambda \geqslant \lambda_0$, one can always get

$$c_\lambda < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}$$

and this proves Case 2. Thus, in both cases, estimate (2.17) holds true . Finishing the proof of lemma. □

In the following, we will explore some convergence lemmas that are crucial for proving that every $(PS)_c-$sequence has a convergent subsequence. Specifically, the following definition will be employed throughout the thesis:

Definition 2.2.3. *A sequence $(u_n)_n \subset X$ (X is a Banach space) is called Palais-Smale sequence for $I \in C^1(X, \mathbb{R})$ at level $c \in \mathbb{R}$ ($(PS)_c-$sequence for short), if there holds $I(u_n) \to c$ and $\|I'(u_n)\| \to 0$, as $n \to \infty$. We say that I satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ ($(PS)_c-$condition for short) if any $(PS)_c-$sequence has a convergent subsequence in X.*

Lemma 2.2.4. *((WILLEM, 2022, Proposition 5.4.7)) Let $N \geqslant 3$, $s \in (1, \infty)$ and $(u_n)_n \in L^s(\mathbb{R}^N)$. If $(u_n)_n$ is a bounded sequence in $L^s(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, then $u_n \rightharpoonup u$ in $L^s(\mathbb{R}^N)$.*

The next Lemma is a nonlocal version of the Brézis-Lieb convergence lemma (see (BRÉZIS; LIEB, 1983)).

Lemma 2.2.5. *((DU; GAO; YANG, 2022, Lemma 2.2)) Let $N \geqslant 3$, $\alpha \geqslant 0$, $0 < \mu < N$, $2\alpha + \mu \leqslant N$ and $2_{*_{\alpha,\mu}} \leqslant p \leqslant 2^*_{\alpha,\mu}$. If $(u_n)_n$ is a bounded sequence in $L^{\frac{2Np}{2N-2\alpha-\mu}}(\mathbb{R}^N)$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, then*

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u_n-u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n-u|^p}{|x|^\alpha}\,\mathrm{d}x + o_n(1).$$

Lemma 2.2.6. *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y \rightharpoonup \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y, \quad \text{in} \quad L^{\frac{2N}{\mu+2\alpha}}(\mathbb{R}^N),$$

for all $2_{*_{\alpha,\mu}} \leqslant p \leqslant 2^*_{\alpha,\mu}$. In addition, the sequence $\left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\right)_n$ is bounded in $L^{\frac{2N}{\mu+2\alpha}}(\mathbb{R}^N)$.

Proof. Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, we have that $u_n \to u$ in $L^t(\mathbb{R}^N)$, for $t \in (2, 2^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. Hence, $|u_n|^p \to |u|^p$ a.e. in $\mathbb{R}^N$. By continuous Sobolev embedding $(|u_n|^p)_n$ is bounded in $L^{\frac{2N}{2N-\mu-\alpha}}(\mathbb{R}^N)$ and by Lemma 2.2.4, we obtain

$$|u_n|^p \rightharpoonup |u|^p \quad \text{in} \quad L^{\frac{2N}{2N-\mu-\alpha}}(\mathbb{R}^N).$$

In light of Proposition 1.0.2, we see that the operator

$$L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N) \ni h \longmapsto \int_{\mathbb{R}^N} \frac{h}{|x|^\alpha|x-y|^\mu|y|^\alpha}\,\mathrm{d}y \in L^{\frac{2N}{\mu+2\alpha}}(\mathbb{R}^N)$$

is a linear bounded operator from $L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\mu+2\alpha}}(\mathbb{R}^N)$, which assures us

$$\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y \rightharpoonup \int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y, \quad \text{in} \quad L^{\frac{2N}{\mu+2\alpha}}(\mathbb{R}^N).$$

$\square$

Lemma 2.2.7. *Let $N \geqslant 3$, $\alpha \geqslant 0$, $0 < \mu < N$, $2\alpha + \mu \leqslant N$. If $(u_n)_n$ is a sequence in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, then for any $v \in H^1(\mathbb{R}^N)$ and $2_{*\alpha,\mu} \leqslant p \leqslant 2^*_{\alpha,\mu}$,*

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{p-2}u_n}{|x|^\alpha}v\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}v\,\mathrm{d}x + o_n(1).$$

Proof. In what follows, we will adapt some ideas from the case $\alpha = 0$ in (AO, 2019, Theorem 1.1). Since $u_n \rightharpoonup u$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, we have that $u_n \to u$ in $L^t(\mathbb{R}^N)$, for $t \in (2, 2^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. By continuous Sobolev embedding $(|u_n|^{p-2}u_n)_n$ is bounded in $L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N)$ and by Lemma 2.2.4, obtain

$$|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u \quad \text{in} \quad L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N).$$

Now, by Egorov's Theorem, $|u_n|^{p-2}u_n \to |u|^{p-2}u$ a.e. uniformly in K, where $K \subset\subset \mathbb{R}^N$. Hence, $|u_n|^{p-2}u_n \to |u|^{p-2}u$ measure on K. Since $|u_n|^{p-2}u_n \to |u|^{p-2}u$ in measure on K, for any $\varepsilon > 0$, $\delta > 0$, there exists some $n_0 > 0$ such that any $n > n_0$, we have

$$\operatorname{meas}\left\{\ x \in K : \left||u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x)\right| > \delta\right\}\ < \varepsilon. \tag{2.30}$$

Setting $\psi \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp}(\psi) \subset K$ and

$$K_1 := \{x \in K : \left||u_n(x)|^{p-2}u(x) - |u(x)|^{p-2}u(x)\right| > \delta\}.$$

Claim. It is true that

$$o_n(1) = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{p-2}u_n}{|x|^\alpha}\psi\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}\psi\,\mathrm{d}x =: \mathcal{Q}_n.$$

Indeed, we observe that

$$\begin{aligned}
\mathcal{Q}_n \leqslant& \left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p-|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}\psi\,\mathrm{d}x\right| \\
&+ \left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{p-2}u_n-|u|^{p-2}u}{|x|^\alpha}\psi\,\mathrm{d}x\right| \\
\leqslant& \left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p-|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}\psi\,\mathrm{d}x\right| \\
&+ \int_{K_1}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\left|\frac{|u_n|^{p-2}u_n-|u|^{p-2}u}{|x|^\alpha}\psi\right|\,\mathrm{d}x \\
&+ \int_{K\setminus K_1}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\left|\frac{|u_n|^{p-2}|u_n-|u|^{p-2}u}{|x|^\alpha}\psi\right|\,\mathrm{d}x \\
=:& \mathcal{Q}_n^1 + \mathcal{Q}_n^2 + \mathcal{Q}_n^3.
\end{aligned} \tag{2.31}$$

We will estimate $\mathcal{Q}_n^1$, $\mathcal{Q}_n^2$ and $\mathcal{Q}_n^3$. In fact, since $|u|^{p-2}u\psi \in L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N)$, it follows from Lemma 2.2.6 that $\mathcal{Q}_n^1 \to 0$. Moreover, by Lemma 2.2.6, there exists $C > 0$ such that

$$\left\| \int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha |x-y|^\mu |x|^\alpha} \,\mathrm{d}y \right\|_{\frac{2N}{\mu+2\alpha}} \leqslant C. \tag{2.32}$$

This and from Hölder's inequality, it follows that

$$\begin{aligned}
\mathcal{Q}_n^2 \leqslant & \left[\int_{K_1} \left| \left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha |x-y|^\mu |x|^\alpha} \,\mathrm{d}y \right) \right|^{\frac{2N}{\mu+2\alpha}} \right]^{\frac{\mu+2\alpha}{2N}} \\
& \times \left(\int_{K_1} \left| |u_n|^{p-2}u_n - |u|^{p-2}u\psi \right|^{\frac{2N}{2N-2\alpha-\mu}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \\
\leqslant & C \left(\int_{K_1} \left| |u_n|^{p-2}u_n - |u|^{p-2}u\psi(x) \right|^{\frac{2N}{2N-2\alpha-\mu}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \\
\leqslant & C \left(\int_{K_1} \left| |u_n|^{p-2}u_n - |u|^{p-2}u \right|^{\frac{2N}{2N-2\alpha-\mu}\frac{p}{p-1}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}\frac{p-1}{p}} \\
& \times \left(\int_{K_1} \left| \psi \right|^{\frac{2Np}{2N-2\alpha-\mu}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}\frac{1}{p}}.
\end{aligned} \tag{2.33}$$

Now, to get an estimate for $\mathcal{Q}_n^3$, we observe that

$$|u_n(x)|^{p-2}u_n(x) - |u|^{p-2}u(x) \leqslant \delta, \quad \text{in} \quad K \backslash K_1,$$

which jointly with (2.32) and Hölder inequality implies that

$$\mathcal{Q}_n^3 \leqslant \delta C \left(\int_{K\backslash K_1} \left| \psi(x) \right|^{\frac{2N}{2N-2\alpha-\mu}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \leqslant \delta C \left(\int_{\mathbb{R}^N} \left| \psi(x) \right|^{\frac{2N}{2N-2\alpha-\mu}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}}$$

Combining with (2.31) and (2.33), we obtain

$$\begin{aligned}
\mathcal{Q}_n \leqslant & C \left(\int_{K_1} \left| |u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x) \right|^{\frac{2N}{2N-2\alpha-\mu}\frac{p}{p-1}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}\frac{p-1}{p}} \\
& \times \left(\int_{K_1} \left| \psi(x) \right|^{\frac{2Np}{2N-2\alpha-\mu}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}\frac{1}{p}} + \delta C \left(\int_{\mathbb{R}^N} \left| \psi(x) \right|^{\frac{2N}{2N-2\alpha-\mu}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}}.
\end{aligned}$$

By the arbitrariness of ε and δ, we have $\mathcal{Q}_n \to 0$. Which proves the claim.

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u_n|^{p-2}u_n}{|x|^\alpha} v \,\mathrm{d}x = & \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{p-2}u}{|x|^\alpha} v \,\mathrm{d}x \\
& + o_n(1),
\end{aligned}$$

for any $v \in H^1(\mathbb{R}^N)$ and $2_{*\alpha,\mu} \leqslant p \leqslant 2^*_{\alpha,\mu}$. □

Lemma 2.2.8. *Let $N \geqslant 3$, $\alpha \geqslant 0$, $0 < \mu < N$, $2\alpha + \mu \leqslant N$ and $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$. If $(u_n)_n$ is a sequence in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u_n|^p}{|x|^\alpha} \, \mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \, \mathrm{d}x + o_n(1).$$

Proof. Combining Proposition 1.0.2 with Hölder's inequality, we infer that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u_n|^{p-2} u_n (u_n - u)}{|x|^\alpha} \, \mathrm{d}x \\
\leqslant & C(N,\alpha,\mu) \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{2N-2\alpha-\mu}} \, \mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \left(\int_{\mathbb{R}^N} \left| |u_n|^{p-1} (u_n - u) \right|^{\frac{2N}{2N-2\alpha-\mu}} \, \mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \\
\leqslant & C(N,\alpha,\mu) \|u_n\|^p_{\frac{2Np}{2N-2\alpha-\mu}} \|u_n\|^{p-1}_{\frac{2Np}{2N-2\alpha-\mu}} \|u_n - u\|_{\frac{2Np}{2N-2\alpha-\mu}}.
\end{aligned} \tag{2.34}$$

Similarly

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u|^{p-2} u (u_n - u)}{|x|^\alpha} \, \mathrm{d}x \leqslant C \|u_n - u\|_{\frac{2Np}{2N-2\alpha-\mu}}, \tag{2.35}$$

where $C := C(N,\alpha,\mu) \|u\|^{2p-1}_{\frac{2Np}{2N-2\alpha-\mu}}$. Since $u_n \rightharpoonup u$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, we have that $u_n \to u$ in $L^t(\mathbb{R}^N)$, for $t \in (2, 2^*)$. Hence, one has that

$$\|u_n - u\|_{\frac{2Np}{2N-2\alpha-\mu}} = o_n(1), \quad \forall p \in (2_{*\alpha,\mu}, 2^*_{\alpha,\mu}),$$

which together with (2.34)-(2.35), we infer

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u_n|^p}{|x|^\alpha} \, \mathrm{d}x - \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \, \mathrm{d}x \right| \\
= & \left| \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^p}{|x|^\alpha |y-x|^\mu |y|^\alpha} \, \mathrm{d}y \right) |u_n|^{p-2} u_n u \, \mathrm{d}x - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y)|^p |u(x)|^p}{|x|^\alpha |y-x|^\mu |y|^\alpha} \, \mathrm{d}y \mathrm{d}x \right| + o_n(1) \\
=: & \mathcal{Q}_n + o_n(1),
\end{aligned}$$

where taking $v = u$ in Lemma 2.2.7, we obtain $\mathcal{Q}_n = o_n(1)$. Finishing the proof of the lemma. □

Lemma 2.2.9. *If $(u_n)_n$ is a $(PS)_c-$sequence of the constrained functional $I_\lambda|_{\mathcal{N}}$ is also a $(PS)_{c_\lambda}-$sequence of I_λ, namely, if $(u_n)_n$ in $\mathcal{N}_\lambda$ satisfies*

$$I_\lambda(u_n) = c_\lambda + o_n(1) \quad \textit{and} \quad I'_\lambda|_{\mathcal{N}_\lambda}(u_n) = o_n(1),$$

then $I'_\lambda(u_n) = o_n(1)$.

Proof. Initially, we will check that $(u_n)_n$ is bounded. In fact, since $(u_n)_n \subset \mathcal{N}_\lambda$ and $I_\lambda(u_n) = c_\lambda + o_n(1)$, we have

$$\begin{aligned}\|u_n\|^2 = &\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\\ &+\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}c_\lambda + o_n(1) =&\frac{1}{2}\|u_n\|^2 - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\\ &-\frac{1}{2p}\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x.\end{aligned}$$

Whence it follows from $p < 2^*_{\alpha,\mu}$ that

$$\begin{aligned}c_\lambda + o_n >&\frac{1}{2}\|u_n\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\\ &-\frac{1}{2p}\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x\\ =&\left(\frac{1}{2}-\frac{1}{2p}\right)\|u_n\|^2,\end{aligned} \tag{2.37}$$

thus $(u_n)_n$ is bounded. Now, we prove that $\mathcal{N}_\lambda$ is a C^1-manifold. Let $J_\lambda : H^1(\mathbb{R}^N) \longrightarrow \mathbb{R}$ be the C^1-functional defined by $J_\lambda(u) = I'_\lambda(u)u$. Then $\mathcal{N}_\lambda = J_\lambda^{-1}(0)$. If $u_n \in \mathcal{N}_\lambda$, then it follows from (2.36) that

$$\begin{aligned}J'_\lambda(u_n)u_n =&2\|u_n\|^2 - 22^*_{\alpha,\mu}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\\ &-2p\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x\\ \leqslant&2\|u\|^2 - 2p\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\\ &-2p\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x\\ =&(2-2p)\|u\|^2,\end{aligned}$$

which together with Lemma 2.2.1 (ii) and since $2 < p$ implies that

$$J'_\lambda(u)u \leqslant (2-2p)\delta^2 < 0. \tag{2.38}$$

Thus, 0 is a regular value of J_λ and therefore $\mathcal{N}_\lambda$ is a C^1-manifold. By the Lagrange multiplier theorem, there exists a sequence $(t_n)_n \subset \mathbb{R}$ such that

$$o_n(1) = I'_\lambda|_{\mathcal{N}_\lambda}(u_n) = I'_\lambda(u_n) - t_n J'_\lambda(u_n), \tag{2.39}$$

implying $t_n J_\lambda'(u_n)u_n = o_n(1)$. Using once more (2.36) and since that $\lambda(22^*_{\alpha,\mu} - 2p) > 0$, we see

$$\begin{aligned} J_\lambda'(u_n)u_n =& 2\|u_n\|^2 - 22^*_{\alpha,\mu}\left[\|u_n\|^2 - \lambda \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x\right] \\ &- 2p\lambda \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x \\ \geqslant& (2 - 22^*_{\alpha.\mu})\|u_n\|^2. \end{aligned}$$

This and by (2.38), it follows that $(J_\lambda'(u_n)u_n)_n$ is bounded thanks to the boundedness of $(u_n)_n$. Hence, $t_n \to 0$, and from (2.39), we achieve that $I_\lambda'(u_n) \to 0$. Finishing the proof of the lemma. □

In the following lemma, we prove that the functional I_λ satisfies the $(PS)_{c_\lambda}-$condition when c_λ meets the estimate provided in Lemma 2.2.2. This condition is crucial for obtaining a nontrivial solution to Problem (2.14). Specifically, we present the following result.

Lemma 2.2.10. *If*

$$c_\lambda < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}},$$

then I_λ satisfies the $(PS)_{c_\lambda}-$condition.

Proof. Let $(u_n)_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be a $(PS)_{c_\lambda}-$sequence for I_λ. Thus, the sequence $(u_n)_n$ is bounded in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and, up to a subsequence, $u_n \rightharpoonup u$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, which implies that

$$\int_{\mathbb{R}^N} \nabla u_n \nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} u_n v\,\mathrm{d}x = \int_{\mathbb{R}^N} \nabla u \nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} uv\,\mathrm{d}x + o_n(1). \tag{2.40}$$

Consequently, in light of Lemma 2.2.7, we obtain $I_\lambda'(u_n)v = I_\lambda'(u)v + o_n(1)$ and since that $I_\lambda'(u_n)v = o_n(1)$, we deduce for all $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$,

$$\begin{aligned} \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\,\mathrm{d}x =& \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_\alpha}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_\alpha-2}u}{|x|^\alpha}v\,\mathrm{d}x \\ &+ \lambda \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}v\,\mathrm{d}x, \end{aligned} \tag{2.41}$$

i.e., u is a solution of the Problem (2.14). Taking $u = v$ in (2.41), implies that

$$\begin{aligned} \mathcal{I}_\lambda(u) =& \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right)\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ &+ \lambda\left(\frac{1}{2} - \frac{1}{2p}\right)\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}\,\mathrm{d}x \geqslant 0, \end{aligned} \tag{2.42}$$

since that $\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} > 0$ and $\lambda(\frac{1}{2} - \frac{1}{2p}) > 0$.

Let $v_n := u_n - u$. Thus,

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1) \tag{2.43}$$

and combining Proposition 1.0.2 with Sobolev embedding, we see

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^p}{|x|^\alpha}\,\mathrm{d}x \leqslant C(N,\alpha,\mu)\|v_n\|^{2p}_{\frac{2Np}{2N-2\alpha-\mu}} = o_n(1). \tag{2.44}$$

Now, by Lemma 2.2.5, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\\ &+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1)\end{aligned} \tag{2.45}$$

and

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^p}{|x|^\alpha}\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}\,\mathrm{d}x + o_n(1), \tag{2.46}$$

which together with the fact that $o_n(1) = I'_\lambda(u_n)u_n$, we deduce

$$\begin{aligned}o_n(1) =& I'_\lambda(u_n)u_n = \|v_n\|^2 + \|u\|^2\\ &-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\\ &-\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}\,\mathrm{d}x + o_n(1)\\ =& I'_\lambda(u)u + \|v_n\|^2 - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1)\\ =& \|v_n\|^2 - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1).\end{aligned} \tag{2.47}$$

Suppose that $\|v_n\|^2 \to b$, then

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \to b.$$

By the definition of the best constant $S_{\alpha,\mu}$ in (2.12)

$$S_{\alpha,\mu}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac{1}{2^*_{\alpha,\mu}}} \leqslant \int_{\mathbb{R}^N}|\nabla v_n|^2\,\mathrm{d}x,$$

which yields $b^{\frac{1}{2^*_{\alpha,\mu}}} S_{\alpha,\mu} \leqslant b$. This implies that either $b=0$ or $b \geqslant S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} > 0$. Now, using (2.42)-(2.46), we observe

$$\begin{aligned}
c_\lambda + o_n(1) =& I_\lambda(u_n)\\
=& \frac{1}{2}\|u_n\|^2 - \frac{1}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x\\
& - \frac{1}{2p}\lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u_n|^p}{|x|^\alpha} \,\mathrm{d}x\\
=& \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x\\
& - \frac{1}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x\\
& - \frac{1}{2p}\lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \,\mathrm{d}x + o_n(1)\\
=& I_\lambda(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x + o_n(1)\\
=& I_\lambda(u) + \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) b + o_n(1)\\
\geqslant& \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) b + o_n(1),
\end{aligned} \tag{2.48}$$

i.e., $c_\lambda \geqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) b$. Therefore, if $b \geqslant S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}$, then

$$\frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} = \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} \leqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) b \leqslant c_\lambda,$$

which contradicts with the fact that $c_\lambda < \dfrac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}$. Thus, $b=0$ and

$$\|u_n - u\| \to 0,$$

as $n \to \infty$, this completes the proof. □

Proof of Theorem 2.1.4. Since I_λ bounded from below on $\mathcal{N}_\lambda$, we have by Ekeland's variational principle, (WILLEM, 1996, Theorem 2.4), there exists a sequence $(u_n)_n$ in $\mathcal{N}_\lambda$ satisfying

$$I_\lambda(u_n) = c_\lambda + o_n(1) \quad \text{and} \quad I'_\lambda|_{\mathcal{N}_\lambda}(u_n) = o_n(1).$$

By Lemma 2.2.9, $I'_\lambda(u_n) = o_n(1)$. In light of Lemma 2.2.10, we infer that $u_n \to u$ strongly in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Thus, $I'_\lambda(u) = 0$ e $I_\lambda(u) = c_\lambda > 0$. We conclude that, $u \neq 0$ is a radial ground state solution of (2.4). This ends the proof of theorem. □

Proof of Theorem 2.1.5. In the following we will show that each u obtained in Theorem 2.1.4 belongs to $L^\infty(\mathbb{R}^N)$. For this, we shall use the Moser's iteration method and according to the Theorem 2.1.5, let us recall the following hypotheses for α, μ and p: $0<\mu<N$, $\alpha \geqslant 0$, $N \geqslant 3$,

$$0<2\alpha+\mu<\min\left\{\frac{N+2}{2},4\right\} \quad \text{and} \quad 2_{*\alpha,\mu}<\frac{4}{2^*_{\alpha,\mu}}\frac{N+2-2\alpha-\mu}{N-2}\leqslant p<2^*_{\alpha,\mu}. \tag{2.49}$$

It is important to mention that, in order to handle the double weight in the Stein-Weiss term and apply the iteration process, we use (2.49) in combination with Proposition 1.0.2.

Lemma 2.2.11. *Let u be the solution of* (2.4) *obtained in Theorem 2.1.4. Assume that* (2.49) *is satisfied. Then, $u \in L^\infty(\mathbb{R}^N)\cap C^{1,\gamma}_{\text{loc}}(\mathbb{R}^N)$, for some $\gamma\in(0,1)$.*

Proof. For $L>0$, we define $\phi_L = uu_L^{2(\beta-1)}$ and $w_L = uu_L^{(\beta-1)}$, where $u_L=\min\{u,L\}$. By taking $\phi_L = uu_L^{2(\beta-1)}$ as test function in (2.15), where $\beta>1$ will be chosen later, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\nabla u\nabla(uu_L^{2(\beta-1)})\,\mathrm{d}x+\int_{\mathbb{R}^N}|u|^2u_L^{2(\beta-1)}\,\mathrm{d}x&\\ -\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha}uu_L^{2(\beta-1)}\,\mathrm{d}x&\\ -\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{p}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}uu_L^{2(\beta-1)}\,\mathrm{d}x&=0,\end{aligned}$$

which implies

$$\begin{aligned}\int_{\mathbb{R}^N}u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x=&-2(\beta-1)\int_{\mathbb{R}^N}u_L^{2(\beta-1)-1}u\nabla u\nabla u_L\,\mathrm{d}x\\ &+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha}uu_L^{2(\beta-1)}\,\mathrm{d}x\\ &+\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{p}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}uu_L^{2(\beta-1)}\,\mathrm{d}x\\ &-\int_{\mathbb{R}^N}|u|^2u_L^{2(\beta-1)}\,\mathrm{d}x.\end{aligned}\tag{2.50}$$

Since

$$2(\beta-1)\int_{\mathbb{R}^N}u_L^{2(\beta-1)-1}u\nabla u\nabla u_L\,\mathrm{d}x=2(\beta-1)\int_{\{u\leqslant L\}}u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x\geqslant 0,\tag{2.51}$$

it follows from (2.50) that

$$\begin{aligned}\int_{\mathbb{R}^N}u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x\leqslant&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}u_L^{2(\beta-1)}\,\mathrm{d}x\\ &+\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{p}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{p}}{|x|^\alpha}u_L^{2(\beta-1)}\,\mathrm{d}x.\end{aligned}\tag{2.52}$$

By (2.13), note that

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant S^{-1}\int_{\mathbb{R}^N} |\nabla w_L|^2\,\mathrm{d}x = S^{-1}\int_{\mathbb{R}^N} \nabla(uu_L^{(\beta-1)})\,\mathrm{d}x,$$

Thus,

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant & S^{-1}\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x + S^{-1}(\beta-1)^2\int_{\mathbb{R}^N} |u|^2u_L^{2(\beta-2)}|\nabla u_L|^2\,\mathrm{d}x \\ \leqslant & S^{-1}\beta^2\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x + S^{-1}\beta^2\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x \\ = & 2S^{-1}\beta^2\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x, \end{aligned} \tag{2.53}$$

where we have used $\nabla u_L = 0$ in $\{u > L\}$, $u = u_L$ in $\{u \leqslant L\}$ and $\beta > 1$. According to (2.52) and (2.53), we obtain

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2S^{-1}\beta^2\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}u_L^{2(\beta-1)}\,\mathrm{d}x \tag{2.54}$$

$$+\,2S^{-1}\beta^2\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}u_L^{2(\beta-1)}\,\mathrm{d}x. \tag{2.55}$$

Next, we estimate the right-hand side of (2.54)-(2.55). By (2.37), we get $\|u\|^2 \leqslant \frac{2p}{p-1}c_\lambda =: M$ and from (2.13), $\|u\|^2_{2^*} \leqslant S^{-1}M$. Combining this with Proposition 1.0.2, we deduce

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}u_L^{2(\beta-1)}\,\mathrm{d}x \leqslant \tilde{C}_2\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}},$$

where $\tilde{C}_2 := C(N,\alpha,\mu)(S^{-1}M)^{\frac{2^*_{\alpha,\mu}}{2}}$. Similarly, we deduce

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}u_L^{2(\beta-1)}\,\mathrm{d}x \leqslant \tilde{C}_3\left(\int_{\mathbb{R}^N}(|u|^pu_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}},$$

where $\tilde{C}_3 := C(N,\alpha,\mu)\hat{C}^pM^{\frac{p}{2}}$ and we are denoting by $\hat{C}$ the constant of the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$, for all $s \in [2, 2^*]$. Using these last two estimates in (2.54)-(2.55) and denoting by $C = 2S^{-1}\max\{\tilde{C}_2, \tilde{C}_3\}$, we estimate that

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant & C\beta^2\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \\ & + C\beta^2\lambda\left(\int_{\mathbb{R}^N}(|u|^pu_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \\ =: & C\beta^2(I_1 + \lambda I_2). \end{aligned} \tag{2.56}$$

In the sequence, we estimate I_1 and I_2. Recalling that

$$(a+b)^{p_1} < a^{p_1} + b^{p_1}, \quad \forall a, b > 0,\ p_1 \in (0,1),$$

for any $K>0$, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} &< \left(\int_{\{|u|\leqslant K\}}(|u|^{2^*_{\alpha,\mu}-2}|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &\quad+\left(\int_{\{|u|>K\}}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &=:I_1^1+I_1^2. \end{aligned} \tag{2.57}$$

Note that

$$I_1^1\leqslant K^{2^*_{\alpha,\mu}-2}\left(\int_{\{|u|\leqslant K\}}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \tag{2.58}$$

and using Hölder's inequality with exponents $\frac{2^*_{\alpha,\mu}}{2^*_{\alpha,\mu}-2}$ and $\frac{2^*_{\alpha,\mu}}{2}$ one deduce

$$I_1^2\leqslant\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}. \tag{2.59}$$

This and estimates (2.57)-(2.58), imply that

$$\begin{aligned}I_1\leqslant& C_1\left(\int_{\{|u|\leqslant K\}}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.\end{aligned} \tag{2.60}$$

Now we will estimate I_2. Analogously, from Hölder's inequality and (2.49), we obtain

$$\begin{aligned}I_2\leqslant& C_2\left(\int_{\{|u|\leqslant K\}}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+\left(\int_{\{|u|>K\}}|u|^{(p-2)\frac{2^*_{\alpha,\mu}}{2^*_{\alpha,\mu}-2}}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.\end{aligned} \tag{2.61}$$

Combining (2.56), (2.60) and (2.61), we derive

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\leqslant& C\beta^2\left(\int_{\mathbb{R}^N}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+C\beta^2\left[\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}+\lambda\left(\int_{\{|u|>K\}}|u|^{(p-2)\frac{2^*_{\alpha,\mu}}{2^*_{\alpha,\mu}-2}}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\right]\\ &\times\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.\end{aligned} \tag{2.62}$$

Taking into account (2.49) and since $u \in L^s(\mathbb{R}^N)$, for all $s \in [2, 2^*]$, we may fix $K > 0$ such that

$$\begin{cases} \left(\int_{\{|u|>K\}} |u|^{2^*} \mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \leqslant \dfrac{1}{4C\beta^2}, \\ \left(\int_{\{|u|>K\}} |u|^{(p-2)\frac{2^*_{\alpha,\mu}}{2^*_{\alpha,\mu}-2}} \mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \leqslant \dfrac{1}{4C\beta^2\lambda}. \end{cases} \tag{2.63}$$

Combining (2.63) with (2.62) implies that

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2C\beta^2 \left(\int_{\mathbb{R}^N} (|u|^2 u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}} \mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}. \tag{2.64}$$

Claim. $u \in L^{2^*\beta}(\mathbb{R}^N)$, for $\beta = \frac{2^*_{\alpha,\mu}}{2}$.

In fact, since $u_L \leqslant |u|$ and recalling $w_L = u u_L^{(\beta-1)}$, it follows from (2.64) that

$$\left(\int_{\mathbb{R}^N} (|u| u_L^{\frac{2^*_{\alpha,\mu}-2}{2}})^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2C\beta^2 \left(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} < \infty. \tag{2.65}$$

By taking the limit as $L \to \infty$ we conclude that

$$\int_{\mathbb{R}^N} |u|^{2^* \frac{2^*_{\alpha,\mu}}{2}} \mathrm{d}x < \infty,$$

which proves the claim.

Now, using that $u_L \leqslant |u|$ and passing to the limit as $L \to \infty$ in (2.64), we obtain

$$\|u\|_{2^*\beta}^{2\beta} \leqslant C\beta^2 \left(\int_{\mathbb{R}^N} |u|^{2\beta \frac{2^*}{2^*_{\alpha,\mu}}} \mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}}{2^*} \frac{2\beta}{2\beta}} = C\beta^2 \|u\|_{q^*_{\alpha,\mu}\beta}^{2\beta}, \tag{2.66}$$

or equivalently,

$$\|u\|_{2^*\beta} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\beta}} \beta^{\frac{1}{\beta}} \|u\|_{q^*_{\alpha,\mu}\beta}, \tag{2.67}$$

where $q^*_{\alpha,\mu} := \frac{22^*}{2^*_{\alpha,\mu}}$.

The next step is using inequality (2.67) to obtain $u \in L^\infty(\mathbb{R}^N)$, through an iterative process. For this purpose, we follow three steps.

First step. If $\beta = \gamma_1 := \frac{2^*}{q^*_{\alpha,\mu}}$, then (2.67) becomes

$$\|u\|_{2^*\gamma_1} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\gamma_1}} \gamma_1^{\frac{1}{\gamma_1}} \|u\|_{2^*}, \quad 2^*\gamma_1 = \gamma_1^2 q^*_{\alpha,\mu}, \tag{2.68}$$

that is, $u^{\gamma_1^2} \in L^{q^*_{\alpha,\mu}}(\mathbb{R}^N)$.

Second step. If $\beta = \gamma_2 := \gamma_1^2$, then $q^*_{\alpha,\mu}\gamma_2 = q^*_{\alpha,\mu}\gamma_1^2 = 2^*\gamma_1$ and (2.67) becomes

$$\|u\|_{2^*\gamma_2} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\gamma_2}} \gamma_2^{\frac{1}{\gamma_2}} \|u\|_{q^*_{\alpha,\mu}\gamma_2},$$

i.e.,

$$\|u\|_{2^*\gamma_2} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\gamma_2}} (\gamma_2)^{\frac{1}{\gamma_2}} \|u\|_{2^*\gamma_1},$$

which jointly with (2.68) yields that

$$\|u\|_{2^*\gamma_2} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}} (\gamma_1)^{\frac{1}{\gamma_1}} (\gamma_2)^{\frac{1}{\gamma_2}} \|u\|_{2^*}. \tag{2.69}$$

Since $2^*\gamma_2 = 2^*\gamma_1^2 = 2^*\gamma_1\gamma_1 = q^*_{\alpha,\mu}\gamma_1^3$, it follows that $u^{\gamma_1^3} \in L^{q^*_{\alpha,\mu}}(\mathbb{R}^N)$.

Third step. If $\beta = \gamma_3 := \gamma_1^3$, then $q^*_{\alpha,\mu}\gamma_3 = q^*_{\alpha,\mu}\gamma_1^3 = 2^*\gamma_2$ and (2.67) becomes

$$\|u\|_{2^*\gamma_3} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\gamma_3}} \gamma_3^{\frac{1}{\gamma_3}} \|u\|_{q^*_{\alpha,\mu}\gamma_3},$$

i.e.,

$$\|u\|_{2^*\gamma_3} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\gamma_3}} (\gamma_3)^{\frac{1}{\gamma_3}} \|u\|_{2^*\gamma_2},$$

which jointly with (2.69) yields that

$$\|u\|_{2^*\gamma_3} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\frac{1}{\gamma_3}} (\gamma_1)^{\frac{1}{\gamma_1}} (\gamma_2)^{\frac{1}{\gamma_2}} (\gamma_3)^{\frac{1}{\gamma_3}} \|u\|_{2^*}.$$

Since $2^*\gamma_3 = 2^*\gamma_1^3 = (2^*\gamma_1\gamma_1)\gamma_1 = q^*_{\alpha,\mu}\gamma_1^3$, it follows that $u^{\gamma_1^4} \in L^{q^*_{\alpha,\mu}}(\mathbb{R}^N)$.

Inductively, if we consider $\beta = \gamma_m := \gamma_1^m$, then $q^*_{\alpha,\mu}\gamma_{m+1} = 2^*\gamma_m$ and (2.67) becomes

$$\|u\|_{2^*\gamma_m} \leqslant [C^{\frac{1}{2}}]^{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\cdots+\frac{1}{\gamma_m}} \gamma_1^{\frac{1}{\gamma_1}} \gamma_2^{\frac{1}{\gamma_2}} \cdots \gamma_m^{\frac{1}{\gamma_m}} \|u\|_{2^*}, \tag{2.70}$$

and we deduce that $u^{\gamma_1^{(m+1)}} \in L^{q^*_{\alpha,\mu}}(\mathbb{R}^N)$ for all $m \in \mathbb{N}$. Recalling that $\gamma_1 = \frac{2^*}{q^*_{\alpha,\mu}} > 1$, then

$$\begin{cases} \lim\limits_{m\to\infty} 2^*\gamma_m = 2^* \lim\limits_{m\to\infty} \gamma_1^m = 2^* \lim\limits_{m\to\infty} \beta^m = \infty, \\ \displaystyle\sum_{j=1}^{\infty} \frac{1}{\gamma_j} = \sum_{j=1}^{\infty} \left(\frac{1}{\gamma_1}\right)^j = \frac{1}{\gamma_1 - 1}, \\ \displaystyle\sum_{j=1}^{\infty} \frac{j}{\gamma_j} = \sum_{j=1}^{\infty} \frac{j}{\gamma_1^j} = \frac{\gamma_1}{(\gamma_1-1)^2}, \\ \gamma_1^{\frac{1}{\gamma_1}} \gamma_2^{\frac{1}{\gamma_2}} \cdots \gamma_m^{\frac{1}{\gamma_m}} = \gamma_1^{\frac{1}{\gamma_1}} \gamma_1^{\frac{2}{\gamma_1^2}} \cdots \gamma_1^{\frac{m}{\gamma_1^m}} \leqslant \gamma_1^{\sum_{j=1}^{\infty} \frac{j}{\gamma_1^j} = \frac{\gamma_1}{(\gamma_1-1)^2}}. \end{cases}$$

By taking the limit as $m \to \infty$ in (2.70), leads to

$$\|u\|_\infty \leqslant M_1 \|u\|_{2^*}, \tag{2.71}$$

where $M_1 := C^{\frac{1}{2(\gamma_1-1)}} \gamma_1^{\frac{\gamma_1}{(\gamma_1-1)^2}}$. Thus, using regularity theory (see for instance (TOLKSDORF, 1984, Theorem 1)), we obtain $u \in C^{1,\gamma}_{\text{loc}}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$. $\square$

Finally, since $|u| \in \mathcal{N}_\lambda$ and $I_\lambda(|u|) = I_\lambda(u)$, we have that $|u|$ is a nonnegative solution of Problem (2.4). We denote $u_1 = |u|$. Therefore, in light of Strong Maximum Principle, we conclude that u_1 is positive. This finishes the proof of the Theorem 2.1.5. $\square$

2.3 THE HARDY POTENTIAL PERTURBATION

In this section, we aim to prove Theorem 2.1.6. Thus, we will study the existence of solutions for the following equation involving Stein-Weiss convolutions and Hardy potential term (f_2):

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} + \lambda \frac{u}{|x|^2}, \quad \text{in } \mathbb{R}^N. \tag{2.72}$$

The energy functional associated with (2.72) is given by

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\ & - \frac{1}{2}\lambda \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \,\mathrm{d}x \end{aligned}$$

and, in view of Lemma 2.1.18 and Remark 2.1.17, I_λ is well defined on $H^1(\mathbb{R}^N)$, belongs to $C^1(H^1(\mathbb{R}^N), \mathbb{R})$, and its derivative given by

$$\begin{aligned} I'_\lambda(u)v =& \int_{\mathbb{R}^N} (\nabla u \nabla v + uv) \,\mathrm{d}x - \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} v \,\mathrm{d}x \\ & - \lambda \int_{\mathbb{R}^N} \frac{uv}{|x|^2} \,\mathrm{d}x. \end{aligned}$$

Thus, weak solutions of (2.72) are precisely the critical points of I_λ.

As a consequence of Lemma 2.1.18, we have the following result.

Lemma 2.3.1. *Assume that (f_2) holds. Then, for all $N \geqslant 3$, there exists $C_{N,2,\lambda} > 0$ such that*

$$\|u\|^2 - \int_{\mathbb{R}^{\mathbb{N}}} f(x,u) \,\mathrm{d}x \geqslant C_{N,2,\lambda} \|u\|^2, \quad \forall u \in H^1(\mathbb{R}^N), \tag{2.73}$$

where $C_{N,2,\lambda} := 1 - \lambda C_{N,2} > 0$, if $\in \lambda \in (0,1)$ for all $N > 4$ or if $\lambda = 1$, $C_{N,2,1} > 0$, for $N = 3, 4$.

Proof. According to Lemma 2.1.18, we have

$$\|u\|^2 - \int_{\mathbb{R}^{\mathbb{N}}} f(x,u) \,\mathrm{d}x \geqslant (1 - \lambda C_{N,2}) \|u\|^2. \tag{2.74}$$

For $N > 4$ and $\lambda \in (0,1)$, let us note that $(1 - C_{N,2}) > 0$. Now, for $N = 3,4$, we suppose $\lambda = 1$ and we see that $(1 - \lambda C_{N,2}) > 0$. Therefore, in both cases, it follows that (2.73) holds true. $\square$

Now, we consider the Nehari set associated our Problem (2.72) as follows

$$\mathcal{N}_\lambda = \left\{ u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\} : I'_\lambda(u)u = 0 \right\}.$$

For any $u \in \mathcal{N}_\lambda$, implies that

$$\|u_n\|^2 - \lambda \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,\mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y \right) \frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x,$$

which jointly with Lemma 2.73, we infer from $C_{N,2,\lambda} > 0$ and $\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} > 0$ that

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\left(\|u\|^2 - \lambda \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \right) - \frac{1}{22^*_{\alpha,\mu}} \left(\|u_n\|^2 - \lambda \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \right) \\ \geqslant& \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) (1 - C_{N,2,\lambda})\|u\|^2 \geqslant 0. \end{aligned} \tag{2.75}$$

Hence, we may we consider the following constrained minimizing problem:

$$c_\lambda := \inf_{\mathcal{N}_\lambda} I_\lambda. \tag{2.76}$$

We shall prove that if the infimum in (2.76) is attained by u, then u is a radial ground state solution of Problem (2.72).

Similar to Lemma 2.2.1, we infer the following result.

Lemma 2.3.2. *For each $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$, we have that*

(i) *there exists a unique $t_0 > 0$, depending on u, such that*

$$t_0 u \in \mathcal{N}_\lambda \quad \textit{and} \quad \max_{t \geq 0} I_\lambda(tu) = I_\lambda(t_0 u);$$

(ii) *there exists a constant $\delta > 0$ such that $\|u\| \geqslant \delta$, for any $u \in \mathcal{N}_\lambda$;*

(iii) $c_\lambda = \inf_{\mathcal{N}_\lambda} I_\lambda > 0$.

Proof. Using the same arguments explored in proof of Lemma 2.2.1, if $g_\lambda(t) := I_\lambda(tu)$, then we may see that $g_\lambda(t) < 0$ for $t > 0$ sufficiently large and, combining (2.19) with Lemma 2.3.1, implies that

$$g_\lambda(t) \geqslant t^2\|u\|^2 \left[\frac{1}{2} C_{N,2,\lambda} - \frac{t^{22^*_{\alpha,\mu}-2}}{22^*_{\alpha,\mu}} C \|u\|^{22^*_{\alpha,\mu}-2} \right] > 0,$$

provided $t > 0$ is sufficiently small. Thus g_λ has a unique t_0 maximum point in $(0, \infty)$ if and only if $t_0 u \in \mathcal{N}_\lambda$. The proof of (i) is complete.

(ii) For any $u \in \mathcal{N}$, together (2.19) with Lemma 2.73, we obtain

$$C_{N,2,\lambda}\|u\|^2 \leqslant C\|u\|^{22^*_{\alpha,\mu}}.$$

Hence, $C_{N,2,\lambda} \leqslant C\|u\|^{22^*_{\alpha,\mu}-2}$, which implies that (ii) holds.

(iii) In view of (2.75) and (ii), it follows that

$$I_\lambda(u) \geqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right)(1 - \lambda C_{N,2})\|u\|^2 \geqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right)(1 - \lambda C_{N,2})\delta^2 > 0.$$

Therefore, for the arbitrary of $u \in \mathcal{N}_\lambda$, we infer $c_\lambda = \inf_{\mathcal{N}_\lambda} I_\lambda(u) \geqslant \inf_{\mathcal{N}_\lambda} \delta^2 > 0$, which finishes the proof. □

The next lemma, as well as Lemma 2.2.2, establishes a key estimate involving the level c_λ, which is essential for confirming that the solution to the Problem (2.72) is nontrivial. The result will be derived for sufficiently small λ.

Lemma 2.3.3. *Let $N > 4$ and $\lambda \in (0, 1)$. Then, the level c_λ satisfies*

$$0 < c_\lambda < \frac{N + 2 - 2\alpha - \mu}{2(2N - 2\alpha - \mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}, \tag{2.77}$$

for $\lambda > 0$ small enough.

Proof. Arguing as in the proof Lemma 2.2.1, it follows that replacing

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U_\varepsilon|^p}{|x|^\alpha}\,\mathrm{d}x = \varepsilon^{(2-N)p+2N-2\alpha-\mu}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U|^p}{|x|^\alpha}\,\mathrm{d}x$$

by

$$\int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^2}{|x|^2}\,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{|U(x)|^2}{|x|^2}\,\mathrm{d}x$$

in (2.22) and (2.23), we obtain similar estimates,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla U|^2\,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} |U_\varepsilon|^2\,\mathrm{d}x = \varepsilon^2\int_{\mathbb{R}^N} |U|^2\,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^2}{|x|^2}\,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^2}\,\mathrm{d}x \end{cases} \tag{2.78}$$

and

$$2S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \, \mathrm{d}x \geqslant t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}.$$

Thus, we achieved $0 < t_\varepsilon \leqslant 2^{\frac{1}{2(2^*_{\alpha,\mu})-1}}$, for ε small enough.

We will consider that $\varepsilon < \lambda$.

Claim. $t_\varepsilon \to 1$, as $\varepsilon \to 0$, once that $\lambda \to 0$.

In fact, from Lemma 2.1.18 and (2.21),

$$\lambda \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^2} \mathrm{d}x \leqslant \lambda C_{N,2} \int_{\mathbb{R}^N} |\nabla U|^2 \mathrm{d}x \leqslant \lambda C_{N,2} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} =: \lambda C_{N,\alpha,\mu}, \tag{2.79}$$

where $C_{N,\alpha,\mu}$ does not depend on λ. Similar to (2.25), by (2.21), (2.78) and (2.79), it follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla U|^2 \, \mathrm{d}x - t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \, \mathrm{d}x \\ &= -\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \, \mathrm{d}x + \lambda \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^2} \, \mathrm{d}x \to 0, \end{aligned}$$

as $\varepsilon \to 0$, once that $\lambda \to 0$ implying that $t_\varepsilon \to 1$. Which proves the claim.

Finally, combining once more (2.21) and (2.78), we see that

$$\begin{aligned} c_\lambda &\leqslant \max_{t \geqslant 0} I_\lambda(tU_\varepsilon) = I_\lambda(t_\varepsilon U_\varepsilon) \\ &\leqslant \max_{t>0} \left(\frac{t^2}{2} - \frac{t^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}} \right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2} \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \, \mathrm{d}x - \frac{t_\varepsilon^2}{2} \lambda \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^2} \, \mathrm{d}x, \end{aligned}$$

i.e.,

$$c_\lambda \leqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2} \varepsilon^2 \int_{\mathbb{R}^N} |U(x)|^2 \, \mathrm{d}x - \frac{t_\varepsilon^2}{2} \lambda \int_{\mathbb{R}^N} \frac{|U(x)|^2}{|x|^2} \, \mathrm{d}x. \tag{2.80}$$

But, since we are assuming $\varepsilon < \lambda$, we have for λ small enough that

$$\frac{t_\varepsilon^2}{2} \varepsilon^2 \int_{\mathbb{R}^N} |U(x)|^2 \, \mathrm{d}x - \frac{t_\varepsilon^2}{2} \lambda \int_{\mathbb{R}^N} \frac{|U(x)|^2}{|x|^2} \, \mathrm{d}x < 0.$$

This together with (2.80) implies that

$$c_\lambda < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}.$$

Therefore, the proof is complete. □

Similar to the proof of Lemma 2.2.9 and Lemma 2.2.10, we have the following conclusions.

Lemma 2.3.4. *If $(u_n)_n$ is a $(PS)_{c_\lambda}-$sequence of the constrained functional $I_\lambda|_{\mathcal{N}_\lambda}$ is also a $(PS)_{c_\lambda}-$sequence of I_λ, namely, if $(u_n)_n$ in $\mathcal{N}_\lambda$ satisfies*

$$I_\lambda(u_n) = c + o_n(1) \quad \text{and} \quad I'_\lambda|_{\mathcal{N}_\lambda}(u_n) = o_n(1),$$

then $I'_\lambda(u_n) = o_n(1)$.

Lemma 2.3.5. *If*

$$c_\lambda < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}},$$

then, I_λ satisfies the $(PS)_{c_\lambda}-$condition.

Proof. Let $(u_n)_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be a $(PS)_{c_\lambda}-$sequence for I_λ. Then, according to the proof of Lemma 2.2.9, the sequence $(u_n)_n$ is bounded in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Hence, we have, from Lemma 2.1.18

$$\frac{u_n}{|x|} \rightharpoonup \frac{u}{|x|} \quad \text{in} \quad L^2(\mathbb{R}^N),$$

i.e.,

$$\int_{\mathbb{R}^N} \frac{u_n}{|x|}\frac{v}{|x|}\,\mathrm{d}x \to \int_{\mathbb{R}^N} \frac{u}{|x|}\frac{v}{|x|}\,\mathrm{d}x, \quad \forall \frac{v}{|x|} \in L^2(\mathbb{R}^N),$$

which leads us to infer

$$\int_{\mathbb{R}^N} \frac{u_n v}{|x|^2} v\,\mathrm{d}x \to \int_{\mathbb{R}^N} \frac{uv}{|x|^2}\,\mathrm{d}x, \quad \forall v \in H^1_{\mathrm{rad}}(\mathbb{R}^N).$$

Now, arguing as in the proof of Lemma 2.2.7, we obtain $I'_\lambda(u)v = 0$ and $I_\lambda(u) \geqslant 0$. Let $v_n := u_n - u$ and $\|w\|_H^2 := \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2}\,\mathrm{d}x$. Then, by Brézis-Lieb Lemma (BRÉZIS; LIEB, 1983) and Lemma 2.2.5

$$\begin{cases} \|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1), \\ \|u_n\|_H^2 = \|v_n\|_H^2 + \|u\|_H^2 + o_n(1), \\ \displaystyle\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ \displaystyle= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ \displaystyle\quad + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1). \end{cases}$$

Furthermore, similarly to the estimate (2.47), we have

$$\begin{aligned} o_n(1) = I'_\lambda(u_n)u_n =& \|v_n\|^2 - \lambda\|v_n\|_H^2 \\ &- \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1). \end{aligned}$$

Suppose that $\|v_n\|^2 - \lambda\|v_n\|_H^2 \to b$, then

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \to b.$$

Similarly to the estimate (2.48), we obtain

$$\begin{aligned}c_\lambda + o_n(1) =& I_\lambda(u) + \frac{1}{2}\left(\|v_n\|^2 - \lambda\|v_n\|_H^2\right)\\&-\frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1)\\=& I_\lambda(u) + \left(\frac{1}{2}-\frac{1}{22^*_{\alpha,\mu}}\right)b + o_n(1) \geqslant \left(\frac{1}{2}-\frac{1}{22^*_{\alpha,\mu}}\right)b + o_n(1).\end{aligned}$$

Thus, we finish the proof of the lemma by arguing as in the final part of the proof of Lemma 2.2.10. □

Proof of Theorem 2.1.6. Using Lemmas 2.3.4 and 2.3.5, the proof of Theorem 2.1.6 is similar to that of Theorem 2.1.4. Thus, we finish the proof of the theorem. □

2.4 THE LOCAL PERTURBATION

In this section we will study the existence of solutions for the following equation involving Stein-Weiss type critical nonlinearity with local perturbation (f_3):

$$-\Delta u + u = \left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} + |u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{2.81}$$

where $2 < p < 2^*$.

To avoid repetition, we focus only on the results that differ from the case involving the nonlocal term. We begin by noting that the energy functional $I : H^1(\mathbb{R}^N) \longrightarrow \mathbb{R}$, defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x,$$

is differentiable, and its critical points correspond to the solutions of Problem (2.81). In other words, the solutions of (2.81) satisfy

$$\begin{aligned}I'(u)v =& \int_{\mathbb{R}^N}(\nabla u\nabla v + uv)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha}v\,\mathrm{d}x\\&-\int_{\mathbb{R}^N}|u|^{p-2}uv\,\mathrm{d}x = 0.\end{aligned} \tag{2.82}$$

We introduce the Nehari manifold associated our Problem (2.81), which is defined as follows

$$\mathcal{N} = \left\{u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\} : I'(u)u = 0\right\}$$

and the level

$$c := \inf_{\mathcal{N}} I. \tag{2.83}$$

We shall prove that if the infimum in (2.83) is attained by u, then u is a radial ground state solution of (2.81).

For the manifold $\mathcal{N}$ defined above, we have the following result.

Lemma 2.4.1. *For each $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$, we have that*

(i) *there exists a unique $t_0 > 0$, depending on u, such that*

$$t_0 u \in \mathcal{N} \quad \textit{and} \quad \max_{t \geq 0} I_\lambda(tu) = I(t_0 u);$$

(ii) *there exists a constant $\delta > 0$ such that $\|u\| \geqslant \delta$, for all $u \in \mathcal{N}_\lambda$;*

(iii) $c = \inf_{\mathcal{N}} I > 0.$

Proof. Exploring similar arguments in the proof of $(i) - (ii)$ in Lemma 2.2.1, we will omit the proof of $(i) - (ii)$ and we will only prove the item (iii). In fact, for each $u \in \mathcal{N}$, we see

$$\begin{aligned} I(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x \\ =& \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|^2 + \left(\frac{1}{p} - \frac{1}{22^*_{\alpha,\mu}} \right) \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \qquad (2.84) \\ =& \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) \|u\|^2 + \left(\frac{1}{22^*_{\alpha,\mu}} - \frac{1}{p} \right) \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x. \qquad (2.85) \end{aligned}$$

Combining with (ii), if $\frac{1}{p} - \frac{1}{22^*_{\alpha,\mu}} > 0$, then we have that (2.84) implies

$$I(u) \geqslant \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|^2 \geqslant \left(\frac{1}{2} - \frac{1}{p} \right) \delta_1^2 > 0$$

and, on the other hand, if $\frac{1}{22^*_{\alpha,\mu}} - \frac{1}{p} > 0$, then (2.85) implies

$$I(u) \geqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) \|u\|^2 \geqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) \delta_1^2 > 0.$$

Take $\delta_0 = \min\left\{ \left(\frac{1}{2} - \frac{1}{p} \right), \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) \right\}$, then for the arbitrary of $u \in \mathcal{N}$, we infer

$$c = \inf_{\mathcal{N}} I(u) \geqslant \inf_{\mathcal{N}} \delta_0 \delta_1^2 > 0,$$

which finishes the proof. □

We also need the following result similar to Lemma 2.2.10.

Lemma 2.4.2. *The level c satisfies*

$$c < \frac{N+2-2\alpha-\mu}{2(2N-,2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}.$$

Proof. The proof follows the lines of Case 1 in Lemma 2.2.10, replacing

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|U_\varepsilon|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U_\varepsilon|^p}{|x|^\alpha}\,\mathrm{d}x = \varepsilon^{(2-N)p+2N-2\alpha-\mu}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|U|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U|^p}{|x|^\alpha}\,\mathrm{d}x$$

by

$$\int_{\mathbb{R}^N}|U_\varepsilon|^p\,\mathrm{d}x = \varepsilon^{\frac{2N-p(N-2)}{2}}\int_{\mathbb{R}^N}|U|^p\,\mathrm{d}x$$

in (2.22). □

With natural adaptations to the proof of Lemmas 2.2.9 and 2.2.10, the following results hold.

Lemma 2.4.3. *If $(u_n)_n$ is a $(PS)_c-$sequence of the constrained functional $I|_{\mathcal{N}}$ is also a $(PS)_c-$sequence of I, namely, if $(u_n)_n$ in $\mathcal{N}$ satisfies*

$$I(u_n) = c + o_n(1) \quad \textit{and} \quad I'_{|\mathcal{N}}(u_n) = o_n(1),$$

then $I'(u_n) = o_n(1)$.

Lemma 2.4.4. *If*

$$c < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}},$$

then, I satisfies the $(PS)_c-$condition.

Proof of Theorem 2.1.7. Using Lemmas 2.4.3 and 2.4.4, the proof of Theorem 2.1.7 is similar to that of Theorem 2.1.4. Thus, we finish the proof of the theorem. □

Proof of Theorem 2.1.8. Thus, as in the previous section, the next lemma is crucial to complete the proof of this theorem.

Lemma 2.4.5. *Let u be the solution of* (2.6) *obtained in Theorem 2.1.4. Then, $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$.*

Proof. According to the arguments presented in the proof of Lemma 2.2.11, and using $\phi_L = uu_L^{2(\beta-1)}$, where $u_L = \min\{u, L\}$, as a test function in (2.82), we derive the following estimate

$$\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2S^{-1}\beta^2\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right)|u|^{2^*_{\alpha,\mu}-2}|u|^2u_L^{2(\beta-1)}\,\mathrm{d}x \quad (2.86)$$

$$+ 2S^{-1}\beta^2\int_{\mathbb{R}^N}|u|^{p-2}|u|^2u_L^{2(\beta-1)}\,\mathrm{d}x. \quad (2.87)$$

Next, we estimate the right-hand side of (2.86)-(2.87). Since u is a solution of Problem (2.6), we have that (2.82) holds. Hence, (2.84) and (2.85) hold, whence it follows

$$\|u\|^2 \leqslant \delta_0^{-1}c =: M \quad \text{where} \quad \delta_0 = \min\left\{\left(\frac{1}{2}-\frac{1}{p}\right), \left(\frac{1}{2}-\frac{1}{22^*_{\alpha,\mu}}\right)\right\}. \tag{2.88}$$

Combining this with (2.13), we have $\|u\|_{2^*} \leqslant S^{-1}M$. Further, by applying Proposition 1.0.2, we deduce

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right)|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)}\mathrm{d}x \leqslant \tilde{C}_2\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}},$$

where $\tilde{C}_2 := C(N,\alpha,\mu)(S^{-1}M)^{\frac{2^*_{\alpha,\mu}}{2}}$, which jointly with (2.86) yields that

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant&\tilde{C}_2 2S^{-1}\beta^2\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+\tilde{C}_2 2S^{-1}\beta^2\int_{\mathbb{R}^N}|u|^{p-2}|u|^2u_L^{2(\beta-1)}\,\mathrm{d}x\\ =&:2\tilde{C}_2S^{-1}\beta^2(I_1+I_2).\end{aligned} \tag{2.89}$$

Next, we estimate I_1 and I_2. Following a similar approach to the estimates in (2.57)-(2.59), we obtain that

$$\begin{aligned}I_1 \leqslant& C\left(\int_{\{|u|\leqslant K\}}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.\end{aligned} \tag{2.90}$$

Now we will estimate I_2. By (2.88) and Hölder's inequality, it follows that

$$\begin{aligned}I_2 \leqslant&\|u\|^{p-2}_{\frac{2N(p-2)}{2\alpha+\mu}}\left(\int_{\mathbb{R}^N}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ \leqslant& M^{\frac{p-2}{2}}\hat{C}^{p-2}\left(\int_{\mathbb{R}^N}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}},\end{aligned} \tag{2.91}$$

where $\hat{C}$ denotes the constant of the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$, for all $s \in [2,2^*]$. Combining (2.89), (2.90) and (2.91), we derive

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant& C\beta^2(1+M^{\frac{p-2}{2}}\hat{C}^{p-2})\left(\int_{\mathbb{R}^N}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+C\beta^2\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.\end{aligned}$$

Note that we may obtain estimates similar to (2.63)-(2.65) and thus we reach

$$\|u\|_{2^*\beta} \leqslant [C^{\frac{1}{2}}(1+M^{\frac{p-2}{2}}\hat{C}^{p-2})^{\frac{1}{2}}]^{\frac{1}{\beta}}\beta^{\frac{1}{\beta}}\|u\|_{q^*_{\alpha,\mu}\beta}, \quad \text{where} \quad q^*_{\alpha,\mu} := \frac{22^*}{2^*_{\alpha,\mu}}. \tag{2.92}$$

In the next step, by applying inequality (2.92) and following a similar approach as in steps 1, 2, and 3 of Lemma 2.2.11 in (2.68)-(2.70), we obtain the following estimate

$$\|u\|_\infty \leqslant (1 + M^{\frac{p-2}{2}} \hat{C}^{p-2})^{\frac{1}{2(\gamma_1 - 1)}} M_1 \|u\|_{2^*},$$

where $M_1 = C^{\frac{1}{2(\gamma_1 - 1})} \gamma_1^{\frac{\gamma_1}{(\gamma_1 - 1)^2}}$. Hence, by regularity theory (see for instance (TOLKSDORF, 1984, Theorem 1)), we have that $u \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$, for some $\gamma \in (0, 1)$. □

Finally, since $|u| \in \mathcal{N}_\lambda$ and $I_\lambda(|u|) = I_\lambda(u)$, we have that $|u|$ is a nonnegative solution of Problem (2.6). We denote $u_1 = |u|$ Therefore, in light of Strong Maximum Principle, we conclude that u_1 is positive. This finishes the proof of the Theorem 2.1.8. □

2.5 THE SUPERCRITICAL PERTURBATION

In this subsection, we focus on studying the existence of positive solution for Problem (2.1) with supercritical term. Precisely, we consider

$$-\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}-2} u}{|x|^\alpha} + \lambda |u|^{q-2} u, \quad \text{in } \mathbb{R}^N, \tag{2.93}$$

where $q \geqslant 2^*$ and λ is positive parameter.

One of the main challenges in studying the problem above is the potential loss of compactness, given that we are working in the whole space $\mathbb{R}^N$. Additionally, variational methods cannot be directly applied to the problem under consideration, as the energy functional associated with Problem (2.93) is given by

$$\begin{aligned} \mathcal{F}_\lambda(u) =& \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2 \right) \mathrm{d}x - \frac{1}{2 2^*_{\alpha,\mu}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \, \mathrm{d}x \\ & - \frac{\lambda}{q} \int_{\mathbb{R}^N} |u|^q \, \mathrm{d}x, \end{aligned}$$

is not well defined for $q > 2^*$. It is not hard to check that $\mathcal{F}_\lambda$ is well defined in $H^1(\mathbb{R}^N)$, if and only, $q = 2^*$.

Associated with Problem (2.93), let us consider the following constrained minimizing problem:

$$c_\lambda := \inf_{\mathcal{N}_\lambda} \mathcal{F}_\lambda \quad \text{where} \quad \mathcal{N}_\lambda = \left\{ u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \backslash \{0\} : \mathcal{F}'_\lambda(u) u = 0 \right\}. \tag{2.94}$$

For the reasons discussed earlier, (2.94) is well defined only for $q = 2^*$.

As pointed out, we are not able to work variationally directly on the energy functional associated to (2.93) when $2^* < q$. For this reason, we will make an appropriate truncation, similar to the one used in the papers (RABINOWITZ, 1973/74; CHABROWSKI; YANG, 1997) and, thus introduce an auxiliary problem where we have a well defined variational structure and we recover some compactness, see next sections for more details. Throughout the text we will consider Problem (2.93) in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and, if u is a weak solution of the Problem (2.93) restricted to $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, then u is a weak solution of the Problem 2.93, see Remark 2.1.3.

2.5.1 The auxiliary Problem

As already noted above, in order to apply minimax methods to obtain a solutions for (2.93), we consider an auxiliary problem. We start by introducing the truncation in the supercritical local term. Given by $\kappa \in \mathbb{N}$, we define the function $g_\kappa : \mathbb{R} \to \mathbb{R}$ by

$$g_\kappa(t) = \begin{cases} 0, & \text{if } \ t \leqslant 0, \\ t^{q-1}, & \text{if } \ 0 \leqslant t \leqslant \kappa, \\ \kappa^{q-p} t^{p-1}, & \text{if } \ t \geqslant \kappa, \end{cases} \tag{2.95}$$

where

$$\frac{2\alpha+\mu}{N} + 2 \leqslant p \leqslant \frac{2\alpha+\mu}{N-2} + 2. \tag{2.96}$$

It is not hard to check that g_κ admits the following inequalities:

$$|g_\kappa(t)| \leqslant \kappa^{q-p} t^{p-1}, \, \forall\, t \geqslant 0. \tag{g_1}$$

Moreover, denoting $G_\kappa(t) = \int_0^t g_\kappa(\tau)\,\mathrm{d}\tau$, there holds

$$G_\kappa(t) = \int_0^t g_\kappa(\tau)\,\mathrm{d}\tau = \begin{cases} 0, & \text{if } \ t \leqslant 0, \\ \dfrac{t^q}{q}, & \text{if } \ 0 \leqslant t \leqslant \kappa, \\ \dfrac{1}{p}\kappa^{q-p} t^p + \left(\dfrac{1}{q} - \dfrac{1}{p}\right)\kappa^q, & \text{if } \ t \geqslant \kappa. \end{cases} \tag{2.97}$$

By simple computations, G_κ admits the following inequalities:

$$|G_\kappa(t)| \leqslant \frac{1}{p} k^{q-p} t^p, \quad \forall\, t \geqslant 0. \tag{G_1}$$

Now, related to g_κ, we shall consider the auxiliary problem

$$\Delta u + u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right) \frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} + \lambda g_\kappa(u), \quad \text{in } \mathbb{R}^N. \tag{$A_{\lambda,\kappa}$}$$

Thus, we say that a function $u \in H^1(\mathbb{R}^N)$ is a weak solution of auxiliary Problem $(A_{\lambda,\kappa})$ if and only if

$$\int_{\mathbb{R}^N} (\nabla u \nabla \phi + u\phi)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha}\phi\,\mathrm{d}x \\ - \lambda \int_{\mathbb{R}^N} g_\kappa(u)\phi\,\mathrm{d}x = 0, \quad \forall \phi \in H^1(\mathbb{R}^N). \tag{2.98}$$

Remark 2.5.1. *According to* (2.95)*, it is important to note that if u is a weak solution of the Problem $(A_{\lambda,\kappa})$ and satisfies $|u(x)| \leqslant \kappa$ for all $x \in \mathbb{R}^N$, then u is a weak solution of Problem* (2.93)*.*

The energy functional $\mathcal{I}_{\lambda,\kappa} : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated with Problem $(A_{\lambda,\kappa})$ is given by

$$\mathcal{I}_{\lambda,\kappa}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\,\mathrm{d}x - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ - \lambda\int_{\mathbb{R}^N} G_\kappa(u)\,\mathrm{d}x$$

and, in view of the assumptions on G_κ above and Proposition (1.0.2), $\mathcal{I}_{\lambda,\kappa}$ is well defined.

We consider $(A_{\lambda,\kappa})$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. If u is a point critical of functional $\mathcal{I}_{\lambda,\kappa}$ restricted to $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, then u is a point critical of $\mathcal{I}_{\lambda,\kappa}$, see Remark 2.1.3, for more details.

Next, we shall discuss some properties on the Nehari manifold associated auxiliary Problem $(A_{\lambda,\kappa})$, which is defined as follows

$$\mathcal{N}_{\lambda,\kappa} = \left\{ u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\} : \mathcal{I}'_{\lambda,\kappa}(u)u = 0 \right\}.$$

Notice that if $u \in \mathcal{N}_{\lambda,\kappa}$, then

$$\|u\|^2 = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + \lambda\int_{\mathbb{R}^N} g_\kappa(u)u\,\mathrm{d}x. \tag{2.99}$$

Now we define the following constrained minimizing problem:

$$c_{\lambda,\kappa} := \inf_{\mathcal{N}_{\lambda,\kappa}} \mathcal{I}_{\lambda,\kappa}. \tag{2.100}$$

The number $c_{\lambda,\kappa}$ is well defined by proving that $\mathcal{I}_{\lambda,\kappa}$ is bounded from below on $\mathcal{N}_{\lambda,\kappa}$ and the set $\mathcal{N}_{\lambda,\kappa}$ is non-empty. Refer to Lemmas 2.5.3 and 2.5.4 below. We shall prove that if the infimum in (2.100) is attained by u, then u is a radial ground state solution of $(A_{\lambda,\kappa})$.

Remark 2.5.2. *We emphasize that if the infimum in* (2.100) *is attained by $u := u_{\lambda,\kappa}$, i.e., $\mathcal{I}_{\lambda,\kappa}(u) = c_{\lambda,\kappa}$, satisfying $|u(x)| \leqslant \kappa$, for all $x \in \mathbb{R}^N$ and for κ, λ which we will choose* *appropriately later, then, according to* (2.97)

$$
\begin{aligned}
c_{\lambda,\kappa} =& \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|u|^2)\,\mathrm{d}x - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\
&-\lambda\int_{\mathbb{R}^N}G_\kappa(u)\,\mathrm{d}x \\
=& \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|u|^2)\,\mathrm{d}x - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\
&-\frac{\lambda}{q}\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x \\
=& \mathcal{F}_\lambda(u).
\end{aligned}
$$

Furthermore, thanks to Remark 2.5.1, with appropriately chosen κ and λ, from (2.94)*, we obtain*

$$
\begin{aligned}
\mathcal{N}_{\lambda,\kappa} =& \left\{u\in H^1_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\} : \mathcal{I}'_{\lambda,\kappa}(u)u=0\right\} = \left\{u\in H^1_{\mathrm{rad}}(\mathbb{R}^N)\backslash\{0\} : \mathcal{F}'_\lambda(u)u=0\right\} \\
=& \mathcal{N}_\lambda
\end{aligned}
$$

and $c_{\lambda,\kappa} := \inf_{\mathcal{N}_{\lambda,\kappa}}\mathcal{I}_{\lambda,\kappa} = \inf_{\mathcal{N}_\lambda}\mathcal{F}_\lambda =: c_\lambda$. This leads us to infer that u is a radial ground state solution of (2.93)*, which motivates us to study the auxiliary Problem $(A_{\lambda,\kappa})$.*

Lemma 2.5.3. *The functional $\mathcal{I}_{\lambda,\kappa}$ is bounded from below on $\mathcal{N}_{\lambda,\kappa}$.*

Proof. In view of (2.95), (2.96) and (2.97), there exists $\theta\in(2,p)$ such that

$$
\begin{aligned}
g_\kappa(t)t-\theta G_\kappa(t) =& \begin{cases} 0, & \text{if } t\leqslant 0, \\ t^q\left(\dfrac{q-\theta}{q}\right), & \text{if } 0\leqslant t\leqslant\kappa, \\ k^{q-p}t^p\left(1-\dfrac{\theta}{p}\right)+\theta\kappa^q\left(\dfrac{1}{p}-\dfrac{1}{q}\right), & \text{if } t\geqslant\kappa, \end{cases} \\
\geqslant& 0. \qquad (2.101)
\end{aligned}
$$

For $u\in\mathcal{N}_{\lambda,\kappa}$, from (2.101), we obtain

$$
\begin{aligned}
\mathcal{I}_{\lambda,\kappa}(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N}G_\kappa(u)\,\mathrm{d}x \\
=& \left(\frac{1}{2}-\frac{1}{22^*_{\alpha,\mu}}\right)\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\
&+\frac{\lambda}{2}\int_{\mathbb{R}^N}\left(g_\kappa(u)u-2G_\kappa(u)\right)\,\mathrm{d}x \geqslant 0, \qquad (2.102)
\end{aligned}
$$

provided $1<2^*_{\alpha,\mu}$, which finishes the proof. □

Throughout this section, we assume that

$$N \geqslant 3, \quad 0 < \mu < N, \quad \alpha \geqslant 0, \quad 0 < 2\alpha + \mu < \min\left\{\frac{N+2}{2}, 4\right\}. \tag{2.103}$$

This and by (2.96), we get

$$p \leqslant \frac{2\alpha+\mu}{N-2} + 2 < 22^*_{\alpha,\mu}. \tag{2.104}$$

As a consequence of the next lemma, the set $\mathcal{N}_{\lambda,\kappa} \neq \emptyset$.

Lemma 2.5.4. *For any $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$, we have that*

(i) *there exists a unique $t_0 > 0$, depending on u, such that*

$$t_0 u \in \mathcal{N}_{\lambda,\kappa} \quad \textit{and} \quad \max_{t\geq 0} \mathcal{I}_{\lambda,\kappa}(tu) = \mathcal{I}_{\lambda,\kappa}(t_0 u);$$

(ii) *there exists a constant $\delta > 0$ such that $\|u\| \geqslant \delta$, for any $u \in \mathcal{N}_{\lambda,\kappa}$. In particular*

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x > \delta^2;$$

(iii) $c_{\lambda,\kappa} = \inf\limits_{\mathcal{N}_{\lambda,\kappa}} \mathcal{I}_{\lambda,\kappa} > 0.$

Proof. (i) First, we note that (2.95) ensures $\dfrac{g_\kappa(t)}{t}$ is increasing for $t > 0$. Hence, for all $t_1, t_2, s \in (0, +\infty)$, if $t_1 < t_2$, then $\dfrac{g_\kappa(t_1 s)s}{t_1} < \dfrac{g_\kappa(t_2 s)s}{t_2}$, implying that the function

$$\frac{1}{t}\int_{\mathbb{R}^N} g_\kappa(tu)u\,\mathrm{d}x \quad \text{is increasing for} \quad t > 0. \tag{2.105}$$

Let $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)\setminus\{0\}$ be fixed and consider the function $\varphi_{\lambda,\kappa} : [0, +\infty) \to \mathbb{R}$ defined by

$$\varphi_{\lambda,\kappa}(t) = \mathcal{I}_{\lambda,\kappa}(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} G_\kappa(tu)\,\mathrm{d}x,$$

then $\varphi'_{\lambda,\kappa}(t) = \mathcal{I}'_{\lambda,\kappa}(tu)tu$. Thus, $\varphi'_{\lambda,\kappa}(t) = 0$ if only if $\mathcal{I}'_{\lambda,\kappa}(tu)tu = 0$, implying that

$$\|u\|^2 = t^{22^*_{\alpha,\mu}-2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + \frac{\lambda}{t}\int_{\mathbb{R}^N} g_\kappa(tu)u\,\mathrm{d}x. \tag{2.106}$$

Hence, t_0 is a positive critical point of $\varphi_{\lambda,\kappa}$ if and only if $t_0 u \in \mathcal{N}_{\lambda,\kappa}$. Moreover, from (2.105), we may infer that the right-hand side of (2.106) is an increasing function on $t > 0$. Recalling the definition of G_κ in (2.97) and since $p < 22^*_{\alpha,\mu}$, we obtain

$$\lim_{t\to\infty} \frac{G_\kappa(t)}{t^{22^*_{\alpha,\mu}}} = 0, \tag{2.107}$$

which leads us to deduce that $\varphi_{\lambda,\kappa}(t) < 0$ for $t > 0$ sufficiently large. On the other hand, combining the Proposition 1.0.2 and Sobolev embedding, we have

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^{\alpha}}\,\mathrm{d}x \leqslant C\|u\|^{22^*_{\alpha,\mu}} \tag{2.108}$$

and by (G_1)

$$\int_{\mathbb{R}^N} G_\kappa(tu)\,\mathrm{d}x \leqslant \frac{1}{p}\kappa^{q-p}t^p\|u\|_p^p \leqslant \frac{1}{p}\kappa^{q-p}\tilde{C}t^p\|u\|^p, \tag{2.109}$$

which implies that

$$\varphi_{\lambda,\kappa}(t) \geqslant t^2\|u\|^2\left[\frac{1}{2} - \frac{t^{22^*_{\alpha,\mu}-2}}{22^*_{\alpha,\mu}}C\|u\|^{22^*_{\alpha,\mu}-2} - \frac{t^{p-2}}{p}\lambda\kappa^{q-p}\tilde{C}\|u\|^{p-2}\right] > 0,$$

provided $t > 0$ is sufficiently small. Thus $\varphi_{\lambda,\kappa}$ has maximum points in $(0,\infty)$. Suppose that there exists $t_1, t_2 > 0$ such that $\varphi'_{\lambda,\kappa}(t_1) = \varphi'_{\lambda,\kappa}(t_2) = 0$. Since every critical point of $\varphi_{\lambda,\kappa}$ satisfies (2.106), we see

$$\begin{aligned} 0 = {} & \left(t_1^{22^*_{\alpha,\mu}-2} - t_2^{22^*_{\alpha,\mu}-2}\right)\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^{\alpha}}\,\mathrm{d}x \\ & + \lambda\int_{\mathbb{R}^N}\left(\frac{g_\kappa(t_1u)u}{t_1} - \frac{g_\kappa(t_2u)u}{t_2}\right)\mathrm{d}x. \end{aligned}$$

Therefore, since both terms in parentheses have the same sign if $t_1 \neq t_2$, it follows that $t_1 = t_2$ and the proof of (i) is complete.

(ii) For any $u \in \mathcal{N}_{\lambda,\kappa}$, together (2.99) with (2.108) and by (g_1),

$$\int_{\mathbb{R}^N} g_\kappa(u)u\,\mathrm{d}x \leqslant \kappa^{q-p}\|u\|_p^p \leqslant \kappa^{q-p}\tilde{C}\|u\|^p,$$

it follows that $\|u\|^2 \leqslant C\|u\|^{22^*_{\alpha,\mu}} + \lambda\kappa^{q-p}\tilde{C}\|u\|^p$. Hence, we have that $0 < 1 \leqslant C\|u\|^{22^*_{\alpha,\mu}-2} + \frac{\lambda}{p}\kappa^{q-p}\tilde{C}\|u\|^{p-2}$, which implies that (ii) holds.

(iii) Combining (2.102) with (ii) we obtain

$$\mathcal{I}_{\lambda,\kappa}(u) = \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right)\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^{\alpha}}\,\mathrm{d}x \geqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right)\delta^2 > 0,$$

provided $2 < 22^*_{\alpha,\mu}$. Thus, for the arbitrary of $u \in \mathcal{N}_{\lambda,\kappa}$, we achieved

$$c_{\lambda,\kappa} = \inf_{\mathcal{N}_{\lambda,\kappa}} \mathcal{I}_{\lambda,\kappa}(u) \geqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right)\delta^2 > 0,$$

finishing the proof of the lemma. □

Similar to the Lemma 2.2.2, we establish an important estimate involving the level $c_{\lambda,\kappa}$.

Lemma 2.5.5. *For any λ and κ, the level $c_{\lambda,\kappa}$ satisfies*

$$c_{\lambda,\kappa} < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}.$$

Proof. Again, for $\varepsilon > 0$ define $U_\varepsilon(x) = \varepsilon^{\frac{2-N}{2}} U(\frac{x}{\varepsilon})$, where $U(x)$ is a minimizant of $S_{\alpha,\mu}$ (see (DU; GAO; YANG, 2022, Theorem 1.3)) and satisfies

$$-\Delta u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{|u|^{2^*_{\alpha,\mu}-2} u}{|x|^\alpha}, \quad \text{in } \mathbb{R}^N$$

with

$$\int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x = S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}. \tag{2.110}$$

Applying the change of variable theorem, we observe that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} |U_\varepsilon|^2 \,\mathrm{d}x = \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} G_\kappa(U_\varepsilon) \,\mathrm{d}x = \varepsilon^N \int_{\mathbb{R}^N} G_\kappa(\varepsilon^{\frac{2-N}{2}} U) \,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} |U_\varepsilon|^p \,\mathrm{d}x = \varepsilon^{\frac{2N-p(N-2)}{2}} \int_{\mathbb{R}^N} |U|^p \,\mathrm{d}x, \quad 2 > \frac{2N-p(N-2)}{2} > 0. \end{cases} \tag{2.111}$$

Now, arguing as in the proof of Lemma 2.5.4 (i), it follows that there exists $t_\varepsilon > 0$ such that

$$t_\varepsilon U_\varepsilon \in \mathcal{N}_{\lambda,\kappa} \quad \text{and} \quad \max_{t \geqslant 0} g(t) = g(t_\varepsilon) = I_\lambda(t_\varepsilon U_\varepsilon).$$

Furthermore, t_ε is unique. Since $t_\varepsilon U_\varepsilon \in \mathcal{N}_{\lambda,\kappa}$ and $\lambda > 0$, we deduce

$$\begin{aligned} 2\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |U_\varepsilon|^2 \,\mathrm{d}x \geqslant{}& \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |U_\varepsilon|^2 \,\mathrm{d}x \\ ={}& t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\ &+ \lambda \int_{\mathbb{R}^N} \frac{g_\kappa(t_\varepsilon U_\varepsilon)}{t_\varepsilon} U_\varepsilon \,\mathrm{d}x \\ \geqslant{}& t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{|U_\varepsilon|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \end{aligned} \tag{2.112}$$

and combining with (2.110)-(2.111), we obtain

$$2S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x \geqslant t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}, \tag{2.113}$$

whence we have $0 < t_\varepsilon \leqslant 2^{\frac{1}{2(2^*_{\alpha,\mu}-1)}}$, for ε small enough.

Claim. $t_\varepsilon \to 1$, as $\varepsilon \to 0$.

In fact, first, from (g_1) and (2.111), we have that

$$\begin{aligned}
&\left| -\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x + \lambda \int_{\mathbb{R}^N} \frac{g_\kappa(t_\varepsilon U_\varepsilon)}{t_\varepsilon} U_\varepsilon \,\mathrm{d}x \right| \\
\leqslant & \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x + \lambda \kappa^{q-p} t_\varepsilon^{p-2} \int_{\mathbb{R}^N} |U_\varepsilon|^p \,\mathrm{d}x \\
\leqslant & \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x + \lambda \kappa^{q-p} t_\varepsilon^{p-2} \varepsilon^{\frac{2N-p(N-2)}{2}} \int_{\mathbb{R}^N} |U|^p \,\mathrm{d}x \to 0, \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

which jointly with (2.110), (2.111) and since $t_\varepsilon U_\varepsilon \in \mathcal{N}_{\lambda,\kappa}$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x - t_\varepsilon^{2(2^*_{\alpha,\mu}-1)} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \,\mathrm{d}x \\
&= -\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x + \lambda \int_{\mathbb{R}^N} \frac{g_\kappa(t_\varepsilon U_\varepsilon)}{t_\varepsilon} U_\varepsilon \,\mathrm{d}x \to 0,
\end{aligned}$$

as $\varepsilon \to 0$, implying that $t_\varepsilon \to 1$ as $\varepsilon \to 0$. Which proves the claim.

Finally, using (2.110) and (2.111), we see that

$$\begin{aligned}
c_{\lambda,\kappa} &\leqslant \max_{t \geqslant 0} I_\lambda(tU_\varepsilon) = I_\lambda(t_\varepsilon U_\varepsilon) \\
&= \left(\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}} \right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2} \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x - \lambda \varepsilon^N \int_{\mathbb{R}^N} G_\kappa(t_\varepsilon \varepsilon^{\frac{2-N}{2}} U) \,\mathrm{d}x \\
&\leqslant \max_{t>0} \left(\frac{t^2}{2} - \frac{t^{22^*_{\alpha,\mu}}}{22^*_{\alpha,\mu}} \right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2} \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x - \lambda \varepsilon^N \int_{\mathbb{R}^N} G_\kappa(t_\varepsilon \varepsilon^{\frac{2-N}{2}} U) \,\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2} \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x - \lambda \varepsilon^N \int_{\mathbb{R}^N} G_\kappa(t_\varepsilon \varepsilon^{\frac{2-N}{2}} U) \,\mathrm{d}x,
\end{aligned}$$

i.e.,

$$c_\lambda \leqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}} \right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} + \frac{t_\varepsilon^2}{2} \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x - \lambda \varepsilon^N \int_{\mathbb{R}^N} G_\kappa(t_\varepsilon \varepsilon^{\frac{2-N}{2}} U) \,\mathrm{d}x. \tag{2.114}$$

On the other hand, using the definition in (2.97), for $\hat{s}$ large enough, we may infer

$$G_k(\hat{s}) = \frac{1}{p} \kappa^{q-p} \hat{s}^p + \left(\frac{1}{q} - \frac{1}{p} \right) \kappa^q.$$

Let $\hat{s} := \frac{s}{\varepsilon^{\frac{N-2}{2}}}$, then

$$\varepsilon^N G_\kappa \left(\frac{s}{\varepsilon^{\frac{N-2}{2}}} \right) = \varepsilon^{\frac{2N-p(N-2)}{2}} \frac{1}{p} \kappa^{q-p} s^p + \varepsilon^N \left(\frac{1}{q} - \frac{1}{p} \right) \kappa^q, \quad \forall s > 0,$$

whence it follows that

$$\lim_{\varepsilon \to 0^+} \varepsilon^N G_\kappa \left(\frac{s}{\varepsilon^{\frac{N-2}{2}}} \right) = 0, \quad \forall s > 0.$$

Hence and since $\lambda > 0$, $2 > \frac{2N-p(N-2)}{2} > 0$, we have for ε small enough that

$$\frac{t_\varepsilon^2}{2}\varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \, \mathrm{d}x - \lambda \int_{\mathbb{R}^N} \left(\varepsilon^{\frac{2N-p(N-2)}{2}} \frac{1}{p} \kappa^{q-p} U^p + \varepsilon^N \left(\frac{1}{q} - \frac{1}{p} \right) \kappa^q \right) \mathrm{d}x < 0.$$

This together with (2.114), one has

$$c_{\lambda,\kappa} < \frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}.$$

Therefore, the proof is complete. □

We also have the following result similar to Lemma 2.2.9.

Lemma 2.5.6. *If $(u_n)_n$ is a $(PS)_{c_{\lambda,\kappa}}-$sequence of the constrained functional $\mathcal{I}_{\lambda,\kappa}|_{\mathcal{N}_{\lambda,\kappa}}$ is also a $(PS)_{c_{\lambda,\kappa}}-$sequence of $\mathcal{I}_{\lambda,\kappa}$, namely, if $(u_n)_n$ in $\mathcal{N}_{\lambda,\kappa}$ satisfies*

$$\mathcal{I}_{\lambda,\kappa}(u_n) = c_{\lambda,\kappa} + o_n(1) \quad \text{and} \quad \mathcal{I}'_{\lambda,\kappa}|_{\mathcal{N}_{\lambda,\kappa}}(u_n) = o_n(1), \tag{2.115}$$

then $\mathcal{I}'_{\lambda,\kappa}(u_n) = o_n(1)$.

Proof. Initially, we will check that $(u_n)_n$ is bounded. In fact, from (2.101), (2.115) and since $\theta \in (2, p)$ where $p < 22^*_{\alpha,\mu}$, it follows that

$$\begin{aligned} c_{\lambda,\kappa} + o_n(1) =& \mathcal{I}_{\lambda,\kappa}(u_n) - \frac{1}{\theta}\mathcal{I}'_{\lambda,\kappa}(u_n)u_n \\ =& \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|^2 + \left(\frac{1}{\theta} - \frac{1}{22^*_{\alpha,\mu}}\right) \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu} \mathrm{d}y \right) \frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha} \mathrm{d}x \\ &+ \lambda \int_{\mathbb{R}^N} \left(\frac{g_\kappa(u_n)u_n}{\theta} - G_\kappa(u_n) \right) \mathrm{d}x \\ \geqslant& \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|^2, \end{aligned} \tag{2.116}$$

thus $(u_n)_n$ is bounded. Now, we prove that $\mathcal{N}_{\lambda,\kappa}$ is a C^1-manifold. Let $J_{\lambda,\kappa} : H^1(\mathbb{R}^N) \longrightarrow \mathbb{R}$ be the C^1-functional defined by $J_{\lambda,\kappa}(u) = \mathcal{I}'_{\lambda,\kappa}(u)u$. Then $\mathcal{N}_{\lambda,\kappa} = J^{-1}_{\lambda,\kappa}(0)$. From (2.95), we deduce

$$\begin{aligned} g_\kappa(t)t - g'_\kappa(t)t^2 =& \begin{cases} 0, & \text{if } \ t \leqslant 0, \\ [1-(q-1)]\, t^q, & \text{if } \ 0 \leqslant t \leqslant \kappa, \\ [1-(p-1)]\, \kappa^{q-p} t^p, & \text{if } \ t \geqslant \kappa, \end{cases} \\ \leqslant& 0. \end{aligned} \tag{2.117}$$

For any fixed $u \in \mathcal{N}_{\lambda,\kappa}$, it follows from (2.117) that

$$\begin{aligned} J'_{\lambda,\kappa}(u)u =& 2\|u\|^2 - 22^*_{\alpha,\mu}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ &- \lambda\int_{\mathbb{R}^N}\left(g'_\kappa(u)u^2 + g_\kappa(u)u\right)\,\mathrm{d}x \\ =& (2-22^*_{\alpha,\mu})\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\ &+ \lambda\int_{\mathbb{R}^N}\left(g_\kappa(u)u - g'_\kappa(u)u^2\right)\,\mathrm{d}x \\ \leqslant& (2-22^*_{\alpha,\mu})\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x, \end{aligned} \tag{2.118}$$

which together with (ii) in Lemma 2.5.4 and since $1 < 2^*_{\alpha,\mu}$ implies that

$$J'_{\lambda,\kappa}(u)u \leqslant (2-22^*_{\alpha,\mu})\delta^2 < 0. \tag{2.119}$$

Thus, 0 is a regular value of $J_{\lambda,\kappa}$ and therefore $\mathcal{N}_{\lambda,\kappa}$ is a C^1-manifold. By the Lagrange multiplier theorem, there exists a sequence $(t_n)_n \subset \mathbb{R}$ such that

$$o_n(1) = \mathcal{I}'_{\lambda,\kappa}|_{\mathcal{N}_{\lambda,\kappa}}(u_n) = \mathcal{I}'_{\lambda,\kappa}(u_n) - t_nJ'_{\lambda,\kappa}(u_n), \tag{2.120}$$

implying

$$t_nJ'_{\lambda,\kappa}(u_n)u_n = o_n(1). \tag{2.121}$$

We claim that there exists $\delta_0 > 0$ such that

$$|J'_{\lambda,\kappa}(u_n)u_n| > \delta_0, \quad \forall n \in \mathbb{N}. \tag{2.122}$$

In fact, by (2.118), it follows that

$$-J'_{\lambda,\kappa}(u_n)u_n \geqslant (22^*_{\alpha,\mu}-2)\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \geqslant 0,$$

thus, assuming by contradiction that $J'_{\lambda,\kappa}(u_n)u_n \to 0$, we get

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \to 0,$$

but this is an absurd, from (ii) in Lemma 2.5.4. By (2.121) and (2.122), $t_n \to 0$, and from (2.120), we achieve that $\mathcal{I}'_{\lambda,\kappa}(u_n) \to 0$. Finishing the proof of the lemma. □

Let $(u_n)_n$ be a $(PS)_{c_{\lambda,\kappa}}-$sequence for $\mathcal{I}_{\lambda,\kappa}$. It follows from Lemma 2.5.6 that $(u_n)_n$ is bounded in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Hence we may assume, passing to a subsequence if necessary, as

$n \to \infty$

$$
\begin{cases}
u_n \rightharpoonup u, & \text{weakly in } H^1_{\mathrm{rad}}(\mathbb{R}^N),\\
u_n \to u, & \text{strongly in } L^s(\mathbb{R}^N),\ 2 < s < 2^*,\\
u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^N,\\
|u_n(x)|, |u(x)| \le h(x), & \text{for some } h \in L^s(\mathbb{R}^N),
\end{cases} \tag{2.123}
$$

and

$$\|u_n\|^2 \leqslant \frac{2\theta}{\theta - 2}(c_{\lambda,\kappa} + 1), \quad \forall\, n \geqslant n_0.$$

Lemma 2.5.7. *If* $u_n \rightharpoonup u$ *in* $H^1_{\mathrm{rad}}(\mathbb{R}^N)$*, then*

(i) $g_\kappa(u_n)v \to g_\kappa(u)v$ *in* $L^1(\mathbb{R}^N)$*, for all* $v \in H^1(\mathbb{R}^N)$*;*

(ii) $g_\kappa(u_n - u)(u_n - u) \to 0$ *in* $L^1(\mathbb{R}^N)$*. In addition*

$$\int_{\mathbb{R}^N} g_\kappa(u_n)u_n \,\mathrm{d}x - \int_{\mathbb{R}^N} g_\kappa(u_n - u)(u_n - u)\,\mathrm{d}x = \int_{\mathbb{R}^N} g_\kappa(u)u\,\mathrm{d}x + o_n(1).$$

Proof. Let us define $f_{n,\kappa}(x) = g_\kappa(u_n)v - g_\kappa(u)v$. In view of (2.95) and (2.123), $f_{n,\kappa}(x) \to 0$ a.e. in $\mathbb{R}^N$, as $n \to \infty$ and from Hölder's inequality, there exists $h_\kappa \in L^1(\mathbb{R}^N)$ such that $|f_{n,\kappa}(x)| \leqslant h_\kappa(x)$. Hence, applying Lebesgue Dominated Convergence Theorem, (i) holds true. Let $v_n := u_n - u$. Then, (ii) is true by arguments similar to the proof of (i). □

In order to obtain a nontrivial solution, the next result, similar to Lemma 2.2.10, is crucial.

Lemma 2.5.8. *If*

$$c_{\lambda,\kappa} < \frac{N + 2 - 2\alpha - \mu}{2(2N - 2\alpha - \mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}},$$

then, $\mathcal{I}_{\lambda,\kappa}$ *satisfies the* $(PS)_{c_{\lambda,\kappa}}$*−condition for all* $\lambda > 0$ *and* $\kappa > 0$*.*

Proof. Let $(u_n)_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be a $(PS)_{c_{\lambda,\kappa}}$−sequence for $\mathcal{I}_{\lambda,\kappa}$. Then, the sequence $(u_n)_n$ is bounded in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and, up to a subsequence, $u_n \rightharpoonup u$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, which implies that

$$\int_{\mathbb{R}^N} \nabla u_n \nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} u_n v\,\mathrm{d}x = \int_{\mathbb{R}^N} \nabla u \nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} uv\,\mathrm{d}x + o_n(1).$$

In light of Lemmas 2.2.7 and 2.5.7, we obtain $\mathcal{I}'_{\lambda,\kappa}(u_n)v = \mathcal{I}'_{\lambda,\kappa}(u)v + o_n(1)$ and since that $\mathcal{I}'_{\lambda,\kappa}(u_n)v = o_n(1)$, we deduce for all $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\,\mathrm{d}x = &\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_\alpha}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right) \frac{|u|^{2^*_\alpha - 2}u}{|x|^\alpha} v\,\mathrm{d}x\\ &+ \lambda \int_{\mathbb{R}^N} g_\kappa(u)v\,\mathrm{d}x,\end{aligned} \tag{2.124}$$

i.e., u is solution of the Problem (2.93). Taking $u = v$ in (2.124), we obtain from (2.102) that $\mathcal{I}_{\lambda,\kappa}(u) \geqslant 0$. Now, let $v_n := u_n - u$. Then,

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1) \tag{2.125}$$

and using Lemmas 2.2.5 and 2.5.7, we see

$$\begin{aligned}
o_n(1) =& \mathcal{I}'_{\lambda,\kappa}(u_n)u_n = \|v_n\|^2 + \|u\|^2 \\
& - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\
& - \lambda\int_{\mathbb{R}^N} g_\kappa(u)u\,\mathrm{d}x + o_n(1) \\
=& \mathcal{I}'_{\lambda,\kappa}(u)u + \|v_n\|^2 - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1) \\
=& \|v_n\|^2 - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1).
\end{aligned}$$

Suppose that $\|v_n\|^2 \to b$, then

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \to b.$$

By the definition of the best constant $S_{\alpha,\mu}$ in (2.12)

$$S_{\alpha,\mu}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac{1}{2^*_{\alpha,\mu}}} \leqslant \int_{\mathbb{R}^N} |\nabla v_n|^2\,\mathrm{d}x,$$

which yields $b^{\frac{1}{2^*_{\alpha,\mu}}} S_{\alpha,\mu} \leqslant b$. This implies that either $b = 0$ or $b \geqslant S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} > 0$. From (2.125), Lemmas 2.2.7, 2.5.7 and since $\mathcal{I}_{\lambda,\kappa}(u) \geqslant 0$, we may infer

$$\begin{aligned}
c_{\lambda,\kappa} + o_n(1) =& \mathcal{I}_{\lambda,\kappa}(u_n) \\
=& \frac{1}{2}\|u_n\|^2 - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} G_\kappa(u_n)\,\mathrm{d}x \\
=& \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x \\
& - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} G_\kappa(u)\,\mathrm{d}x + o_n(1) \\
=& \mathcal{I}_{\lambda,\kappa}(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|v_n|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|v_n|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x + o_n(1) \\
=& \mathcal{I}_{\lambda,\kappa}(u) + \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right)b + o_n(1) \\
\geqslant& \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right)b + o_n(1),
\end{aligned}$$

i.e., $c_{\lambda,\kappa} \geqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right) b$. Therefore, if $b \geqslant S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}$, then

$$\frac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} = \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right) S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}} \leqslant \left(\frac{1}{2} - \frac{1}{22^*_{\alpha,\mu}}\right) b \leqslant c_{\lambda,\kappa},$$

which contradicts with the fact that $c_{\lambda,\kappa} < \dfrac{N+2-2\alpha-\mu}{2(2N-2\alpha-\mu)} S_{\alpha,\mu}^{\frac{2N-2\alpha-\mu}{N+2-2\alpha-\mu}}$. Thus, $b = 0$ and

$$\|u_n - u\| \to 0,$$

as $n \to \infty$, this completes the proof. □

Lemma 2.5.9. *The functional $\mathcal{I}_{\lambda,\kappa}$ has a nonnegative critical point $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\mathcal{I}_{\lambda,\kappa}(u) = c_{\lambda,\kappa}$, i.e., u is a nonnegative radial ground state solution for auxiliary Problem $(A_{\lambda,\kappa})$.*

Proof. In view of (2.116), $\mathcal{I}_{\lambda,\kappa}$ is coercive. Hence, $\mathcal{I}_{\lambda,\kappa}$ bounded from below on $\mathcal{N}_{\lambda,\kappa}$. By Ekeland's variational principle, (WILLEM, 1996, Theorem 2.4), there exists a sequence $(u_n)_n$ in $\mathcal{N}_{\lambda,\kappa}$ satisfying

$$\mathcal{I}_{\lambda,\kappa}(u_n) = c_{\lambda,\kappa} + o_n(1) \quad \text{and} \quad \mathcal{I}'_{\lambda,\kappa}|_{\mathcal{N}_\lambda}(u_n) = o_n(1).$$

By Lemma 2.5.6, $\mathcal{I}'_{\lambda,\kappa}(u_n) = o_n(1)$. In light of Lemma 2.5.8, we infer that $u_n \to u$ strongly in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ as $n \to \infty$. Thus, $\mathcal{I}'_{\lambda,\kappa}(u) = 0$ e $\mathcal{I}_{\lambda,\kappa}(u) = c_{\lambda,\kappa} > 0$. We conclude that, $u \neq 0$ is a radial ground solution of $(A_{\lambda,\kappa})$. By using $u^- := \max\{-u, 0\}$ as test function in (2.98), we deduce from (2.95) that

$$\begin{aligned}
0 = &\int_{\mathbb{R}^N} (\nabla u \nabla u^- + u u^-)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} u^-\,\mathrm{d}x \\
&- \int_{\mathbb{R}^N} g_\kappa(u) u^-\,\mathrm{d}x \\
= &\int_{\mathbb{R}^N} (|\nabla u^-|^2 + |u^-|^2)\,\mathrm{d}x - \int_{\{u>0\}}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} u^-\,\mathrm{d}x \\
&- \int_{\{u<0\}}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} u^-\,\mathrm{d}x - \int_{\{u<0\}} g_\kappa(u) u^-\,\mathrm{d}x \\
&- \int_{\{u>0\}} g_\kappa(u) u^-\,\mathrm{d}x \\
= &\|u^-\| + \int_{\{u<0\}}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u^2}{|x|^\alpha}\,\mathrm{d}x \\
\geqslant &\|u^-\|,
\end{aligned}$$

i.e., the nontrivial weak solution u is nonnegative. □

2.5.1.1 $L^\infty-$estimates

By virtue of Lemma 2.5.9, the auxiliary Problem $(A_{\lambda,\kappa})$ admits a solution $u := u_{\lambda,\kappa}$ for all $\lambda > 0$ and $\kappa > 0$. In what follows, we derive a uniform estimate for the norm of the solution $u_{\lambda,\kappa}$ of Problem $(A_{\lambda,\kappa})$. To achieve this, we introduce the functional $\mathcal{I}_0 : H^1_{\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\mathcal{I}_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) \mathrm{d}x - \frac{1}{22^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}}}{|x|^\alpha}\,\mathrm{d}x.$$

Moreover, we denote by c_0 the level of the Mountain Pass associated with the functional $\mathcal{I}_0$, i.e.,

$$0 < c_0 := \inf_{\gamma\in\Gamma_0}\max_{t\in[0,1]} \mathcal{I}_0(\gamma(t)),$$

where

$$\Gamma_0 := \left\{\gamma \in C([0,1], H^1_{\mathrm{rad}}(\mathbb{R}^N) : \gamma(0) = 0 \ \text{ and } \ \mathcal{I}_0(\gamma(1)) < 0\right\}.$$

Here, it is important to emphasize that c_0 is independent of the choice of λ and κ. In addition, $c_{\lambda,\kappa} \leqslant c_0$.

Lemma 2.5.10. *Let $u_{\lambda,\kappa}$ be the critical point of $\mathcal{I}_{\lambda,\kappa}$ obtained in Lemma 2.5.9. Then, there exists a constant M (which depends only on N,θ,μ,α,p and independent of λ and κ) such that*

$$\|u_{\lambda,\kappa}\|^2 \leqslant \frac{2\theta}{\theta-2}c_0 =: M.$$

In particular, by (2.12) *we have that $\|u_{\lambda,\kappa}\|^2_{2^*} \leqslant S^{-1}M$.*

Proof. In view of estimate (2.116) and recalling that $c_{\lambda,\kappa} \leqslant c_0$, we have that

$$c_0 \geqslant c_{\lambda,\kappa} = \mathcal{I}_{\lambda,\kappa}(u_{\lambda,\kappa}) - \frac{1}{\theta}\mathcal{I}'_{\lambda,\kappa}(u_{\lambda,\kappa})u_{\lambda,\kappa} \geqslant \frac{\theta-2}{2\theta}\|u_{\lambda,\kappa}\|^2$$

and the proof is finished. □

The next lemma plays a crucial role in our arguments, since it establishes an important estimate involving the $L^\infty-$norm of the solution of the auxiliary Problem $(A_{\lambda,\kappa})$. For this purpose, we shall use Moser's iteration method. However, as mentioned in Remark 2.1.15, the Stein-Weiss type convolutions, present in the problems of this thesis, does not admit boundedness in the entire space $\mathbb{R}^N$. In this direction, to deal with double weight in the Stein-Weis term, and apply the iteration process, we use (2.103) combined with the Proposition 1.0.2.

Lemma 2.5.11. *Let $u_{\lambda,k}$ be the solution of $(A_{\lambda,\kappa})$ obtained in Lemma 2.5.9. Then, there exist $C_1 > 0$ and $M_1 > 0$ (which depends only on N, θ, μ, α, p and independent of λ and κ) such that*

$$\|u_{\lambda,\kappa}\|_\infty \leqslant (1+\lambda\kappa^{q-p})^{C_1} M_1 \|u_{\lambda,k}\|_{2^*}.$$

Proof. Building on the arguments presented in the proof of Lemma 2.2.11, we derive the following estimate, using the $\phi_L = uu_L^{2(\beta-1)}$ as test function in (2.98):

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2S^{-1}\beta^2 \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu |x|^\alpha} \mathrm{d}y\right) |u|^{2^*_{\alpha,\mu}-1} u u_L^{2(\beta-1)} \,\mathrm{d}x \quad (2.126)$$

$$+ 2S^{-1}\beta^2 \lambda \int_{\mathbb{R}^N} g_\kappa(u) u u_L^{2(\beta-1)} \,\mathrm{d}x. \quad (2.127)$$

Next, we estimate the right-hand side of (2.126). By combining Proposition 1.0.2 and Hölder's inequality with Lemma 2.5.10, we deduce

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu |x|^\alpha} \mathrm{d}y\right) |u|^{2^*_{\alpha,\mu}} u_L^{2(\beta-1)} \mathrm{d}x \leqslant \tilde{C}_2 \left(\int_{\mathbb{R}^N} (|u|^{2^*_{\alpha,\mu}} u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}} \mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}, \quad (2.128)$$

where $\tilde{C}_2 := C(N,\theta,\alpha,\mu)(S^{-1}M)^{\frac{2^*_{\alpha,\mu}}{2}}$, which jointly with (2.126) yields that

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |w_L|^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} &\leqslant \tilde{C}_2 2S^{-1}\beta^2 \left(\int_{\mathbb{R}^N} (|u|^{2^*_{\alpha,\mu}} u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}} \mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \\ &\quad + \tilde{C}_2 2S^{-1}\beta^2 \lambda \int_{\mathbb{R}^N} g_\kappa(u) u u_L^{2(\beta-1)} \,\mathrm{d}x \\ &=: 2\tilde{C}_2 S^{-1}\beta^2 (I_1 + \lambda I_2). \end{aligned} \quad (2.129)$$

In the sequence, we estimate I_1 and I_2. Similarly to estimates (2.57)-(2.59), we obtain the following estimate for I_1,

$$\begin{aligned} I_1 &\leqslant K^{2^*_{\alpha,\mu}-2} \left(\int_{\{|u|\leqslant K\}} (|u|^2 u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}} \mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \\ &\quad + \left(\int_{\{|u|>K\}} |u|^{2^*} \mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \left(\int_{\mathbb{R}^N} |w_L|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}}. \end{aligned} \quad (2.130)$$

By (g_1), it follows from Hölder's inequality and Lemma 2.5.10 that

$$\begin{aligned} I_2 &\leqslant k^{q-p} \int_{\mathbb{R}^N} |u|^{p-2} u^2 u_L^{2(\beta-1)} \,\mathrm{d}x \\ &\leqslant \kappa^{q-p} \|u\|^{p-2}_{\frac{2N(p-2)}{2\alpha+\mu}} \left(\int_{\mathbb{R}^N} (|u|^2 u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}} \mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \\ &\leqslant \lambda\kappa^{q-p} M^{\frac{p-2}{2}} \hat{C}^{p-2} \left(\int_{\mathbb{R}^N} (|u|^2 u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}} \mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}, \end{aligned} \quad (2.131)$$

where $\hat{C}$ denotes the constant of the embedding $H^1_{\rm rad}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$, for all $s \in [2, 2^*]$.

Denote $C := 2S\tilde{C}_2$ in (2.129). In view of (2.130) and (2.131) we derive

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant C\beta^2(1 + \lambda\kappa^{q-p} M^{\frac{p-2}{2}} \hat{C}^{p-2}) \left(\int_{\mathbb{R}^N} (|u|^2 u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}} \mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}$$
$$+ C\beta^2 \left(\int_{\{|u|>K\}} |u|^{2^*} \mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \left(\int_{\mathbb{R}^N} |w_L|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}}. \tag{2.132}$$

Since $u \in L^{2^*}(\mathbb{R}^N)$, we may fix $K > 0$ such that

$$\left(\int_{\{|u|>K\}} |u|^{2^*} \mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \leqslant \frac{1}{2C\beta^2}. \tag{2.133}$$

Note that we can obtain estimates analogous to (2.63)-(2.67), and thus we arrive at

$$\|u\|_{2^*\beta} \leqslant [C^{\frac{1}{2}}(1 + \lambda\kappa^{q-p})^{\frac{1}{2}}]^{\frac{1}{\beta}} \beta^{\frac{1}{\beta}} \|u\|_{q^*_{\alpha,\mu}\beta}, \quad \text{where} \quad q^*_{\alpha,\mu} := \frac{22^*}{2^*_{\alpha,\mu}} \tag{2.134}$$

The next step is using inequality (2.134) to obtain the desired L^∞–estimate, through an iterative process. For this purpose, following a similar approach as in steps 1, 2, and 3 of Lemma 2.2.11 in (2.68)-(2.70), we obtain the following estimate

$$\|u_{\lambda,k}\|_\infty \leqslant (1 + \lambda\kappa^{q-p})^{C_1} M_1 \|u_{\lambda,k}\|_{2^*}, \tag{2.135}$$

where $C_1 := \frac{1}{2(\gamma_1 - 1)}$ and $M_1 := C^{\frac{1}{2(\gamma_1-1)}} \gamma_1^{\frac{\gamma_1}{(\gamma_1-1)^2}}$. This finishes the proof. □

Remark 2.5.12. *Let $u \in H^1_{\rm rad}(\mathbb{R}^N)$ be the nonnegative solution obtained in Lemma 2.5.9. In view of Lemma 2.5.11 and regularity theory (see for instance (TOLKSDORF, 1984, Theorem 1)), we have that $u \in C^{1,\gamma}_{\rm loc}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$. Therefore, in light of Strong Maximum Principle, we conclude that u is positive.*

Proof of Theorem 2.1.9. At this point, in view of (2.135), we are able to find suitable values of λ and κ such that the following inequality holds true

$$\|u_{\lambda,\kappa}\|_\infty \leqslant (1 + \lambda\kappa^{q-p})^{C_1} M_1 (SM)^{\frac{1}{2}} \leqslant \kappa.$$

In fact, we shall verify that $(1 + \lambda\kappa^{q-p})^{C_1} M_1 (SM)^{\frac{1}{2}} \leqslant \kappa$, or equivalently,

$$\lambda\kappa^{q-p} \leqslant \left(\frac{\kappa}{M_1 (SM)^{\frac{1}{2}}}\right)^{\frac{1}{C_1}} - 1.$$

Consider $\kappa > 0$ such that

$$\left(\frac{\kappa}{M_1 (SM)^{\frac{1}{2}}}\right)^{\frac{1}{C_1}} - 1 > 0$$

and fix $\lambda_0^* > 0$ satisfying

$$\lambda \leqslant \lambda_0^* \leqslant \left[\left(\frac{\kappa}{M_1(SM)^{\frac{1}{2}}}\right)^{\frac{1}{C_1}} - 1\right]\frac{1}{\kappa^{q-p}}.$$

Thus, taking $\kappa_0 > M_1(SM)^{\frac{1}{2}}$, we obtain $\lambda_0^* > 0$, such that

$$\|u_{\lambda,k_0}\|_\infty \leqslant \kappa_0, \tag{2.136}$$

for all $\lambda \in (0, \lambda_0^*]$. Therefore, by (2.136), it follows from definition of g_{κ_0} that

$$g_{\kappa_0}(u_{\lambda,\kappa_0}) = \lambda|u_{\lambda,\kappa_0}|^{q-2}u_{\lambda,\kappa_0} \quad \text{and} \quad G_{\kappa_0}(u_{\lambda,\kappa_0}) = \frac{\lambda}{q}|u_{\lambda,\kappa_0}|^q, \quad \forall\lambda \in (0, \lambda^*]. \tag{2.137}$$

Consequently, by (2.95), (2.137) and since u_{λ,κ_0} is a critical point of $\mathcal{I}_{\lambda,\kappa_0}$, we reach

$$\begin{aligned}
0 = \mathcal{I}'_{\lambda,\kappa_0}(u_{\lambda,\kappa_0})\phi &= \int_{\mathbb{R}^N} \nabla u_{\lambda,\kappa_0}\nabla\phi \,\mathrm{d}x + \int_{\mathbb{R}^N} u_{\lambda,\kappa_0}\phi \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u_{\lambda,\kappa_0}|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u_{\lambda,\kappa_0}|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha}\phi \,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} g_{\kappa_0}(u_{\lambda,\kappa_0})\phi \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \nabla u_{\lambda,\kappa_0}\nabla\phi \,\mathrm{d}x + \int_{\mathbb{R}^N} u_{\lambda,\kappa_0}\phi \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|u_{\lambda,\kappa_0}|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u_{\lambda,\kappa_0}|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha}\phi \,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} |u_{\lambda,\kappa_0}|^{q-1}\phi \,\mathrm{d}x.
\end{aligned}$$

for all $\phi \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Therefore, from Remarks 2.5.1 and 2.5.2, we conclude that u_{λ,κ_0} is a positive radial ground state solution of Problem (2.93), which finishes the proof of Theorem 2.1.9. □

2.6 THE CRITICAL SOBOLEV CASE WITH NONLOCAL PERTURBATION

In this section, we follow ideas from the previous sections and prove Theorem 2.1.10. Thus, our main goal is to study the existence result for problem

$$-\Delta u + u = |u|^{2^*-2}u + \lambda\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}, \quad \text{in } \mathbb{R}^N, \tag{2.138}$$

where $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$, $2^* = \frac{2N}{N-2}$ and $\lambda > 0$.

Associated with equation (2.138), we define the energy functional $I_\lambda : H^1(\mathbb{R}^N) \longrightarrow \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + |u|^2\right)\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^p}{|x|^\alpha}\mathrm{d}x.$$

In other words, the solutions of (2.138) satisfy

$$\begin{aligned}
I'_\lambda(u)v &= \int_{\mathbb{R}^N}(\nabla u\nabla v + uv)\,\mathrm{d}x - \int_{\mathbb{R}^N}|u|^{2^*-2}uv \,\mathrm{d}x \\
&\quad - \lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u|^p}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{p-2}u}{|x|^\alpha}v \,\mathrm{d}x = 0.
\end{aligned}$$

This leads us to define the Nehari manifold associated with Problem (2.138) as follows

$$\mathcal{N}_\lambda = \left\{ u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \backslash \{0\} : I'_\lambda(u)u = 0 \right\}.$$

For any $u \in \mathcal{N}_\lambda$,

$$\|u\|^2 = \int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x + \lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \,\mathrm{d}x,$$

which implies that

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x - \frac{\lambda}{2p} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \,\mathrm{d}x \\ =& \left(\frac{1}{2} - \frac{1}{2p} \right) \|u\|^2 + \lambda \left(\frac{1}{2p} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x \qquad (2.139) \\ =& \left(\frac{1}{2} - \frac{1}{2^*} \right) \|u\|^2 + \lambda \left(\frac{1}{2^*} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \,\mathrm{d}x. \qquad (2.140) \end{aligned}$$

If $\frac{1}{2p} - \frac{1}{2^*} > 0$, then (2.139) implies $I_\lambda(u) > 0$; if $\frac{1}{2p} - \frac{1}{2^*} \leqslant 0$, then (2.140) implies $I_\lambda(u) \geqslant 0$. Consequently, we take $\delta_0 := \min\left\{ \left(\frac{1}{2} - \frac{1}{2^*}\right), \left(\frac{1}{2} - \frac{1}{2p}\right) \right\}$, then for any $u \in \mathcal{N}_\lambda$, we infer $I_\lambda(u) \geqslant \delta_0 \|u\|^2 \geqslant 0$.

We consider the following constrained minimizing problem:

$$c_\lambda := \inf_{\mathcal{N}_\lambda} I_\lambda. \qquad (2.141)$$

We shall prove that if the infimum in (2.141) is attained by u, then u is a radial ground state solution of (2.138). Using the same arguments explored in proof of Lemma 2.2.1, we can show that $\mathcal{N}_\lambda$ satisfies:

Lemma 2.6.1. *For each $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$, we have that*

(i) $\mathcal{N}_\lambda \neq \emptyset$;

(ii) *there exists a constant $\delta > 0$ such that $\|u\| \geqslant \delta$, for all $u \in \mathcal{N}_\lambda$;*

(iii) $c_\lambda = \inf_{\mathcal{N}_\lambda} I_\lambda > 0$.

As Lemma 2.2.2, the next lemma establishes an important estimate involving the level c_λ. In this way we introduce the function, for $\varepsilon > 0$, $U_\varepsilon(x) := \varepsilon^{\frac{2-N}{2}} U(\frac{x}{\varepsilon})$, where $U(x)$ is a minimizant of S and satisfies

$$-\Delta u = u^{2^*-2}u, \quad \text{in } \mathbb{R}^N$$

with

$$\int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |U|^{2^*} \,\mathrm{d}x = S^{\frac{N}{2}}. \qquad (2.142)$$

Notice that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla U|^2 \,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} |U_\varepsilon|^2 \,\mathrm{d}x = \varepsilon^2 \int_{\mathbb{R}^N} |U|^2 \,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} |U_\varepsilon|^{2^*} \,\mathrm{d}x = \int_{\mathbb{R}^N} |U|^{2^*} \,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U_\varepsilon|^p}{|x|^\alpha} \,\mathrm{d}x = \\ \displaystyle= \varepsilon^{(2-N)p+2N-2\alpha-\mu} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|U|^p}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{|U|^p}{|x|^\alpha} \,\mathrm{d}x. \end{cases} \tag{2.143}$$

Therefore, we have the following estimates involving level c_λ, namely:

Lemma 2.6.2. $0 < c_\lambda < \dfrac{1}{N} S^{\frac{N}{2}}$, *if either*

(i) $2_{*\alpha,\mu} < p < 2^*_{\alpha,\mu}$, $2\alpha + \mu = N$, $N \geq 3$;

(ii) $2_{*\alpha,\mu} < \frac{N+2-2\alpha-\mu}{N-2} < p < 2^*_{\alpha,\mu}$, $N = 3, 4$;

(iii) $2_{*\alpha,\mu} < \frac{2N-2-2\alpha-\mu}{N-2} < p < 2^*_{\alpha,\mu}$, $N \geq 5$;

or

(iv) $2_{*\alpha,\mu} < p \leqslant \frac{N+2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N = 3, 4$ *and* λ *sufficiently large;*

(v) $2_{*\alpha,\mu} < p \leqslant \frac{2N-2-2\alpha-\mu}{N-2} < 2^*_{\alpha,\mu}$, $N \geq 5$ *and* λ *sufficiently large.*

Proof. We will use similar arguments as in the proof of Lemma 2.2.2. Accordingly, we will divide the proof into two cases.

Case 1. In what follows, the proof only includes items (i), (ii) and (iii).

Initially, observe that arguing as in the proof of Lemma 2.6.1 (i), implies that there exists $t_\varepsilon > 0$ such that

$$t_\varepsilon U_\varepsilon \in \mathcal{N}_\lambda \quad \text{and} \quad \max_{t \geq 0} g(t) = g(t_\varepsilon) = I_\lambda(t_\varepsilon U_\varepsilon).$$

Since that $t_\varepsilon U_\varepsilon \in \mathcal{N}_\lambda$, we deduce

$$2 \int_{\mathbb{R}^N} |\nabla U_\varepsilon(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |U_\varepsilon(x)|^2 \,\mathrm{d}x \geqslant t_\varepsilon^{2^*-2} \int_{\mathbb{R}^N} |U_\varepsilon(x)|^{2^*} \,\mathrm{d}x$$

and combining with (2.142)-(2.143), we obtain $0 < t_\varepsilon \leqslant 2^{\frac{1}{2^*-2}}$, for ε small enough.

Claim. $t_\varepsilon \to 1$, as $\varepsilon \to 0$.

For (2.142)-(2.143), we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla U|^2\,\mathrm{d}x-t_\varepsilon^{2^*-2}\int_{\mathbb{R}^N}|U_\varepsilon|^{2^*}\,\mathrm{d}x\\ =&-\varepsilon^2\int_{\mathbb{R}^N}|U|^2\,\mathrm{d}x+t_\varepsilon^{2(p-1)}\varepsilon^{(2-N)p+2N-2\alpha-\mu}\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|U_\varepsilon|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U_\varepsilon|^p}{|x|^\alpha}\,\mathrm{d}x\\ &\to 0,\ \text{as } \varepsilon\to 0,\end{aligned}$$

implying that $t_\varepsilon \to 1$ as $\varepsilon \to 0$, which proves the claim.

Finally, combining once more (2.142)-(2.143), we see that

$$\begin{aligned}c_\lambda \leqslant \max_{t\geq 0} I_\lambda(tU_\varepsilon) = I_\lambda(t_\varepsilon U_\varepsilon) \leqslant \left(\frac{1}{2}-\frac{1}{2^*}\right)S^{\frac{N}{2}}+\frac{t_\varepsilon^2}{2}\varepsilon^2\int_{\mathbb{R}^N}|U|^2\,\mathrm{d}x\\ -\frac{t_\varepsilon^{2p}}{2p}\varepsilon^{(2-N)p+2N-2\alpha-\mu}\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|U_\varepsilon|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U_\varepsilon|^p}{|x|^\alpha}\,\mathrm{d}x.\end{aligned}$$

Since $(2-N)p+2N-2\alpha-\mu<2$, we have for ε small enough that

$$\frac{t_\varepsilon^2}{2}\varepsilon^2\int_{\mathbb{R}^N}|U|^2\,\mathrm{d}x-\frac{t_\varepsilon^{2p}}{2p}\varepsilon^{(2-N)p+2N-2\alpha-\mu}\lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|U_\varepsilon|^p}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|U_\varepsilon|^p}{|x|^\alpha}\,\mathrm{d}x<0.$$

Therefore, $c_\lambda < \frac{1}{N}S^{\frac{N}{2}}$, the proof is complete for this case.

Case 2. Now, for λ sufficiently large the proof of items (iv) and (v) follows the same arguments explored in Case 2 of Lemma 2.2.2. With this the proof of the lemma is complete. □

We also have the following results similar to Lemma 2.2.9 and 2.2.10.

Lemma 2.6.3. *If $(u_n)_n$ is a $(PS)_{c_\lambda}-$sequence of the constrained functional $I_\lambda|_{\mathcal{N}_\lambda}$ is also a $(PS)_{c_\lambda}-$sequence of I_λ, namely, if $(u_n)_n$ in $\mathcal{N}_\lambda$ satisfies*

$$I_\lambda(u_n)=c_\lambda+o_n(1)\quad \textit{and}\quad I'_\lambda|_{\mathcal{N}_\lambda}(u_n)=o_n(1)$$

then $I'_\lambda(u_n)=o_n(1)$.

For the sake of completeness of the thesis, we will prove the following lemma.

Lemma 2.6.4. *The functional I_λ satisfies the $(PS)_{c_\lambda}-$condition with $c_\lambda < \frac{1}{N}S^{\frac{N}{2}}$.*

Proof. Let $(u_n)_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be a $(PS)_{c_\lambda}-$sequence for I_λ. Then, the sequence $(u_n)_n$ is bounded in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Hence, we have

$$\int_{\mathbb{R}^N}\nabla u_n\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^N}u_nv\,\mathrm{d}x=\int_{\mathbb{R}^N}\nabla u\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^N}uv\,\mathrm{d}x+o_n(1).$$

Since $u_n \rightharpoonup u$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, we have that $u_n \to u$ a.e. in $\mathbb{R}^N$. Since $(u_n)_n$ is bounded in $L^{2^*}(\mathbb{R}^N)$, $(|u_n|^{2^*-2}u_n)_n$ is bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$ and thus by Lemma 2.2.4, $|u_n|^{2^*-2}u_n \rightharpoonup |u|^{2^*-2}u$ in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$, i.e.,

$$\int_{\mathbb{R}^N} |u_n|^{2^*-2}u_n v \, \mathrm{d}x = \int_{\mathbb{R}^N} |u|^{2^*-2}uv \, \mathrm{d}x + o_n(1), \quad \forall v \in L^{2^*}(\mathbb{R}^N),$$

in particular,

$$\int_{\mathbb{R}^N} |u_n|^{2^*-2}u_n v \, \mathrm{d}x = \int_{\mathbb{R}^N} |u|^{2^*-2}uv \, \mathrm{d}x + o_n(1), \quad \forall v \in H^1_{\mathrm{rad}}(\mathbb{R}^N).$$

In light of Lemma 2.2.7, we obtain $I'_\lambda(u_n)v = I'_\lambda(u)v + o_n(1)$, for all $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and since that $I'_\lambda(u_n)v = o_n(1)$, for all $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$, we deduce for all $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ that

$$\begin{aligned}\int_{\mathbb{R}^N} (\nabla u \nabla v + uv) \, \mathrm{d}x = &\int_{\mathbb{R}^N} |u|^{2^*-2}uv \, \mathrm{d}x \\ &+ \lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u|^{p-2}u}{|x|^\alpha} v \, \mathrm{d}x, \end{aligned} \tag{2.144}$$

i.e., u is solution of the Problem (2.138). Taking $u = v$ in (2.144), we may infer that $I_\lambda(u) \geq 0$. Let $v_n := u_n - u$. Then, by Brézis-Lieb Lemma in (BRÉZIS; LIEB, 1983), we have

$$\begin{aligned} &\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1), \\ &\int_{\mathbb{R}^N} |u_n|^{2^*} \, \mathrm{d}x = \int_{\mathbb{R}^N} |v_n|^{2^*} \, \mathrm{d}x + \int_{\mathbb{R}^N} |u|^{2^*} \, \mathrm{d}x + o_n(1). \end{aligned} \tag{2.145}$$

Combining Proposition 1.0.2 with Sobolev embedding, we see

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|v_n|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|v_n|^p}{|x|^\alpha} \, \mathrm{d}x \leqslant C(N, \alpha, \mu) \|v_n\|^{2p}_{\frac{2Np}{2N-2\alpha-\mu}} = o_n(1). \tag{2.146}$$

By using (2.145)-(2.146) and Lemma 2.2.5, we deduce

$$\begin{aligned} o_n(1) = &I'_\lambda(u_n)u_n = \|v_n\|^2 + \|u\|^2 - \int_{\mathbb{R}^N} |u|^{2^*} \, \mathrm{d}x - \int_{\mathbb{R}^N} |v_n|^{2^*} \, \mathrm{d}x \\ &- \lambda \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{|u|^p}{|x|^\alpha} \, \mathrm{d}x + o_n(1) \\ = &\|v_n\|^2 - \int_{\mathbb{R}^N} |v_n|^{2^*} \, \mathrm{d}x + o_n(1). \end{aligned}$$

Suppose that $\|v_n\|^2 \to b$, then $\int_{\mathbb{R}^N} |v_n|^{2^*} \, \mathrm{d}x \to b$. By (2.13), which yields $b^{\frac{2}{2^*}} S \leqslant b$. This implies that either $b = 0$ or $b \geq S^{\frac{N}{2}} > 0$. Now, using $I_\lambda(u) \geq 0$ and (2.145)-(2.146), the result follows through similar arguments as in the proof of Lemma 2.2.10. □

Proof of Theorem 2.1.10. Using Lemmas 2.6.3 and 2.6.4, the proof of Theorem 2.1.10 is similar to that of Theorem 2.1.4. Thus, we finish the proof of the theorem. □

3 SUPERCRITICAL SCHRÖDINGER EQUATIONS WITH VANISHING POTENTIAL AND DOUBLE WEIGHTED NONLOCAL INTERACTION PART

This chapter focuses on the study of the existence of positive solutions for the following class Schrödinger equations involving double weighted nonlocal interaction

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\alpha} + \psi(x,u), \quad \text{in } \mathbb{R}^N, \tag{P}$$

where $N \geqslant 3$, $0 < \mu < N$, $\alpha \geqslant 0$, $0 < 2\alpha + \mu < \min\{\frac{N+2}{2}, 4\}$, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous and positive potential and F is the primitive of function f. Later, we will introduce the assumptions on $V(x)$, f and ψ. Furthermore, we also study the version of the Problem (P) with the same ψ, however the nonlinearity f assumes the homogeneous critical case in the sense of the weighted Hardy-Littlewood-Sobolev inequality. Precisely, we consider the following class of Schrödinger equations

$$-\Delta u + V(x)u = \frac{1}{2^*_{\alpha,\mu}}\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,\mu}}}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{2^*_{\alpha,\mu}-2}u}{|x|^\alpha} + \psi(x,u), \quad \text{in } \mathbb{R}^N, \tag{Q}$$

where $2^*_{\alpha,\mu} = \frac{2N-2\alpha-\mu}{N-2}$ and the potential $V(x)$ is a radial function, i.e., $V(|x|) = V(x)$, for all $x \in \mathbb{R}^N$.

3.1 ASSUMPTIONS AND MAIN RESULTS

Inspired by the works of (ALVES; SOUTO, 2012; ALVES; FIGUEIREDO; YANG, 2016; CARDOSO; DOS PRAZERES; SEVERO, 2020), we study Problem (P) under the assumption that $V(x)$ a positive continuous function. We adopt the following notation: $m = \max_{|x|\leqslant 1} V(x)$, and we introduce the function $\Lambda : (1,\infty) \to [0,\infty)$ defined by

$$\Lambda(R) = \frac{1}{R^{(q-2)(N-2)}} \inf_{|x|\geqslant R} |x|^{(q-2)(N-2)} V(x). \tag{V_1}$$

Moreover, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{t\to 0^+} \frac{tf(t)}{t^q} < \infty, \tag{f_1}$$

for $q \geqslant 2^* = \frac{2N}{N-2}$ and

$$\lim_{t\to\infty} \frac{tf(t)}{t^p} = 0, \tag{f_2}$$

for $p \in (1, \frac{2(N-\alpha-\mu)}{N-2})$. By considering that $0 < 2\alpha+\mu < \min\{\frac{N+2}{2}, 4\}$, one may conclude that the interval $(1, \frac{2(N-\alpha-\mu)}{N-2})$ is nonempty. In view of (f_1) and the fact that $\frac{2(N-\alpha-\mu)}{N-2} < 2^*_{\alpha,\mu} < 2^*$, there hold

$$\lim_{t\to 0} \frac{tf(t)}{t^{2^*_{\alpha,\mu}}} = 0 \quad \text{and} \quad \lim_{t\to 0} \frac{tf(t)}{t^p} = 0. \tag{3.1}$$

In this chapter, we will also consider the case in which $2^*_{\alpha,\mu} \leqslant q$ in (f_1), i.e.,

$$\lim_{t\to 0^+} \frac{tf(t)}{t^q} < \infty, \quad \text{for } 2^*_{\alpha,\mu} \leqslant q. \tag{f_1'}$$

Using assumptions (f_1'), (f_2) and (3.1), there exists $c_0 > 0$ such that

$$|tf(t)| \leqslant c_0|t|^{2^*_{\alpha,\mu}}, \quad |tf(t)| \leqslant c_0|t|^q, \quad |tf(t)| \leqslant c_0|t|^p, \quad \forall t \in \mathbb{R}. \tag{3.2}$$

We also assume that f satisfies the Ambrosetti-Rabinowitz condition, i.e., there exists $\theta \in (2, \min\left\{2^*_{\alpha,\mu}, 4\right\})$, such that

$$0 < \theta F(t) := \theta \int_0^t f(\tau)\mathrm{d}\tau \leqslant 2f(t)t, \quad \forall t > 0. \tag{f_3}$$

Since we are going to look for positive solutions, we will assume that $f(t) = 0$, for all $t \leqslant 0$.

On the other hand, in order to study Problem (Q), we will assume that the potential $V(x)$ is a radial function, i.e., $V(|x|) = V(x)$, for all $x \in \mathbb{R}^N$ and we introduce the function $W : (1, \infty) \to [0, \infty)$ defined by

$$W(R) = \inf_{|x|\geq R} |x|^{(2^*_{\alpha,\mu}-2)(\frac{N-2}{2})}V(x). \tag{V_2}$$

Throughout the chapter, we are going to study the existence of solutions for Problems (P) and (Q), under the following assumptions regarding ψ:

(ψ_1) $\psi(x, u) = \lambda(x)|u|^{q-2}u$, $q \geqslant 2^*_{\alpha,\mu}$, $\lambda(x)$ is a nonnegative function such that $\lambda(x) \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$;

(ψ_2) $\psi(x, u) = \dfrac{\lambda}{q}\left(\displaystyle\int_{\mathbb{R}^N} \frac{|u|^q}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\dfrac{|u|^{q-2}u}{|x|^\alpha}$, $q \geqslant 2^*_{\alpha,\mu}$ and $\lambda \geqslant 0$ is a parameter.

Furthermore, we assume the following assumptions for α and μ:

$$N \geqslant 3, \quad 0 < \mu < N, \quad \alpha \geqslant 0, \quad 0 < 2\alpha + \mu < \min\left\{\frac{N+2}{2}, 4\right\}. \tag{3.3}$$

To present the main results of this chapter, we now introduce the normed space suitable for solving Problem (P). In fact, due to the presence of $V(x)$ in Problem (P), we defined the subspace of $\mathcal{D}^{1,2}(\mathbb{R}^N)$

$$E := \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x < \infty\right\},$$

which is a Hilbert space when endowed with the inner product and its correspondent norm

$$\langle u, v\rangle := \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x) uv)\,\mathrm{d}x \quad \text{and} \quad \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

Since $V(x)$ is positive, the embedding $E \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous.

As in Chapter 2, we denote by S as the best Sobolev constant for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, i.e.,

$$S\left(\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x. \tag{3.4}$$

With this in mind, we have the following definition.

Definition 3.1.1. *We say that a function $u \in E$ is a weak solution of Problem (P), if there holds*

$$\int_{\mathbb{R}^N} (\nabla u \nabla \phi + V(x) u\phi)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{F(u)}{|x|^{\alpha}}\phi\,\mathrm{d}x - \int_{\mathbb{R}^N} \psi(x,u)\phi\,\mathrm{d}x = 0, \quad \forall\,\phi \in E. \tag{3.5}$$

Now, by noting that due to the presence of $V(|x|)$ in Problem (Q), we replace the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ by $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ and consider

$$E_{\mathrm{rad}} := \left\{u \in \mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2\,\mathrm{d}x < \infty\right\}.$$

Similarly to (3.5), we define a weak solution for Problem (Q), see Section 3.4 for more details.

Regarding Problem (P), we have the following results:

Theorem 3.1.2. *Suppose the case (ψ_1) and that f satisfies $(f_1')-(f_3)$. There are $\varepsilon_0, \Lambda_0 > 0$ such that if $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0$ and $\Lambda \geqslant \Lambda_0$, then Problem (P) has a positive solution.*

Corollary 3.1.3. *Suppose the case (ψ_1) with $\lambda(x) \equiv \lambda$, $\alpha = 0$ and that f satisfies $(f_1)-(f_3)$. There are $\lambda_0^*, \Lambda_0 > 0$ such that if $\lambda \in [0, \lambda_0^*)$ and $\Lambda \geqslant \Lambda_0$, then Problem (P) has a positive solution.*

Theorem 3.1.4. *Suppose the case (ψ_2) and that f satisfies $(f_1')-(f_3)$. There are $\lambda_0, \Lambda_0 > 0$ such that if $\lambda \in [0, \lambda_0)$ and $\Lambda \geqslant \Lambda_0$, then Problem (P) has a positive solution.*

With respect to Problem (Q), we find the following result:

Theorem 3.1.5. *Suppose that (ψ_1) (or (ψ_2)) holds. There are $\hat{\varepsilon}_0, \hat{\Lambda}_0 > 0$ such that if $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \hat{\varepsilon}_0$ (or if $\lambda \in [0, \hat{\lambda}_0)$) and $\hat{\Lambda} \geqslant \hat{\Lambda}_0$, then Problem (Q) has a radial positive solution.*

One of the challenges in studying the problems above is the lack of compactness, as we are working in the entire space $\mathbb{R}^N$. Moreover, as noted in Section 2.5 of Chapter 2, variational methods cannot be directly applied to these problems. For example, the energy functional associated with Problem (P), given by

$$\begin{aligned}\mathcal{F}(u) =& \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x \\ &- \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\alpha}\,\mathrm{d}x - \int_{\mathbb{R}^N}\Psi(x,u)\,\mathrm{d}x,\end{aligned} \tag{3.6}$$

where $\Psi(x,u) = \int_0^u \psi(x,\tau)\,\mathrm{d}\tau$ and ψ is one of the nonlinearities $(\psi_1) - (\psi_2)$, is not well defined for $q > 2^*_{\alpha,\mu}$. For more details, see Remark 3.1.7 below and the following sections of this chapter. A similar difficulty arises in Problem (Q).

Now we list some remarks on this chapter.

Remark 3.1.6. *The main contributions of this chapter are the following:*

1. *If $\psi \equiv 0$, $\alpha = 0$, $0 < \mu < \min\{\frac{N+2}{2}, 4\}$, $N \geqslant 3$, $q \geqslant 2^*_{0,\mu} := 2^*_\mu = \dfrac{2N-\mu}{N-2}$, then our results complete the picture of (ALVES; FIGUEIREDO; YANG, 2016);*

2. *If $\psi \not\equiv 0$, (ψ_1) holds with $\lambda(x) \equiv \lambda$, $\alpha = 0$, $0 < \mu < \frac{N+2}{2}$, $q \geqslant 2^*$, then our results extends and complements the previous item ;*

3. *If $\psi \equiv 0$, $\alpha \neq 0$, $q \geqslant 2^*_{\alpha,\mu}$, then our results complement (DE ALBUQUERQUE; SANTOS, 2023);*

4. *The case $\psi \not\equiv 0$ extends and complements the previous items.*

The approach is based on variational methods combined with penalization techniques and $L^\infty-$estimates.

Remark 3.1.7. *In view of* (3.2) *and* (f_3)*, we have*

$$F(t) \leqslant \frac{2c_0}{\theta}|t|^{2^*_{\alpha,\mu}}, \tag{3.7}$$

and noticing that $\frac{2N}{2N-2\alpha-\mu}2^*_{\alpha,\mu} = 2^*$, we conclude that for each $u \in E$, ensures that

$$\int_{\mathbb{R}^N}|F(u)|^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x \leqslant \left(\frac{2c_0}{\theta}\right)^{\frac{2N}{2N-2\alpha-\mu}}\int_{\mathbb{R}^N}|u|^{2^*}\,\mathrm{d}x < \infty, \tag{3.8}$$

$$\int_{\mathbb{R}^N}|f(u)u|^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x \leqslant c_0^{\frac{2N}{2N-2\alpha-\mu}}\int_{\mathbb{R}^N}|u|^{2^*}\,\mathrm{d}x < \infty, \tag{3.9}$$

which implies that

$$f(u)u, F(u) \in L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N), \quad \forall u \in E.$$

In light of the Proposition 1.0.2, (2.10), (3.7), (3.8), (3.9) *and Sobolev embedding, there hold*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))F(u(x))}{|y|^\alpha |x-y|^\mu |x|^\alpha}\,\mathrm{d}y\mathrm{d}x \leqslant C\|u\|^{22^*_{\alpha,\mu}}, \quad \forall u \in E, \tag{3.10}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))f(u(x))u(x)}{|y|^\alpha |x-y|^\mu |x|^\alpha}\,\mathrm{d}y\mathrm{d}x \leqslant \tilde{C}\|u\|^{22^*_{\alpha,\mu}}, \quad \forall u \in E. \tag{3.11}$$

According to (3.10)*, the energy functional* $\mathcal{F} : E \to \mathbb{R}$ *given in* (3.6) *associated with* (P)*, is well defined in* E*, if and only,* $q = 2^*_{\alpha,\mu}$ *in* (ψ_1) *and* (ψ_2)*. In order to study Problem* (P) *when* $2^*_{\alpha,\mu} < q$*, we cannot directly apply variational methods to the functional* $\mathcal{F}$*. To deal with this technical difficulty, we introduce an appropriate truncation technique, similar to the approach used in (RABINOWITZ, 1973/74; CHABROWSKI; YANG, 1997). This method was also employed in Chapter 2, Section 2.5.*

Remark 3.1.8. *An example of potential* $V(x)$ *that satisfies our assumptions is given by*

$$V(x) = \begin{cases} \varrho_1, & \text{if } \ |x| \leqslant \varrho_1 + \varrho_2, \\ |x| - \varrho_2, & \text{if } \ \varrho_1 + \varrho_2 < |x| \leqslant R, \\ \frac{R^{(q-2)(N-2)}}{|x|^{(q-2)(N-2)}}(R - \varrho_2), & \text{if } \ |x| \geqslant R, \end{cases}$$

where $\varrho_1, \varrho_2 > 0$ *and* $0 < \varrho_1 + \varrho_2 < R$*. A function* f *that satisfies* (f_1')*-*(f_3) *is given by*

$$f(t) = \begin{cases} 0, & \text{if } \ t \leqslant 0, \\ |t|^{q-1}, & \text{if } \ 0 < t < 1, \\ |t|^{p-1}, & \text{if } \ t \geqslant 1. \end{cases}$$

The next sections of the chapter are organized as follows: Section 3.2 is devoted to the proof of Theorem 3.1.2. We establish the variational structure, the penalized problems related to (P), we apply Mountain Pass Theorem to obtain the existence of nonnegative solution for the auxiliary problem, we introduce a suitable $L^\infty-$estimate for the solution of the auxiliary problem to conclude that the solution of the auxiliary problem is positive and, in fact, a solution for (P). We conclude Section 3.2 with the proof of Corollary 3.1.3. The proofs of Theorems 3.1.4 and 3.1.5 are performed in Sections 3.3 and 3.4, respectively.

3.2 THE SUPERCRITICAL LOCAL PERTURBATION

In this section, we explore the existence of positive solutions for Problem (P) involving the supercritical term (ψ_1). Precisely, we consider

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\alpha} + \lambda(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{3.12}$$

where $q \geqslant 2^*_{\alpha,\mu}$, $\lambda(x)$ is a nonnegative function such that $\lambda(x) \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$. As pointed out in Remark 3.1.7, we are not able to work variationally directly on the energy functional associated to (3.12). For this reason, we introduce auxiliary problems where we have a well defined variational structure and we recover some compactness.

3.2.1 The auxiliary Problems (A_κ) and (B_κ)

In order to apply minimax methods to obtain a solutions for (P) with supercritical local term (ψ_1), we consider two auxiliary problems. Initially, we will introduce a truncation in the function given in (ψ_1). In fact, given a natural number κ, we define the function $h_\kappa : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$h_\kappa(x,t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ \lambda(x)t^{q-1}, & \text{if } 0 \leqslant t \leqslant \kappa, \\ \lambda(x)\kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}-1}, & \text{if } t \geqslant \kappa. \end{cases} \tag{3.13}$$

Observe that h_κ admits the following inequalities:

$$|h_\kappa(x,t)| \leqslant \lambda(x)\kappa^{q-2^*_{\alpha,\mu}} t^{2^*-1} \quad \text{and} \quad |h_\kappa(x,t)| \leqslant \lambda(x)\kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}-1}, \ \forall\, t \geqslant 0. \tag{h_1}$$

Moreover, denoting $H_\kappa(x,t) = \int_0^t h_\kappa(x,\tau)\,\mathrm{d}\tau$, there holds

$$H_\kappa(x,t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ \dfrac{\lambda(x)}{q} t^q, & \text{if } 0 \leqslant t \leqslant \kappa, \\ \dfrac{\lambda(x)}{2^*_{\alpha,\mu}}\kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}} + \lambda(x)\left(\dfrac{1}{q} - \dfrac{1}{2^*_{\alpha,\mu}}\right)\kappa^q, & \text{if } t \geqslant \kappa. \end{cases} \tag{3.14}$$

Thus, H_κ admits the following inequalities:

$$|H_\kappa(x,t)| \leqslant \frac{\lambda(x)}{2^*}\kappa^{q-2^*_{\alpha,\mu}} t^{2^*} \quad \text{and} \quad |H_\kappa(x,t)| \leqslant \frac{\lambda(x)}{2^*_{\alpha,\mu}}\kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}}, \quad \forall\, t \geqslant 0. \tag{H_1}$$

By using (3.13), (3.14) and the fact that $q \geqslant 2^*_{\alpha,\mu} > \theta$, one may check that

$$h_\kappa(x,t)t - \theta H_\kappa(x,t) = \begin{cases} 0, & \text{if } \; t \leqslant 0, \\ \lambda(x)t^q\left(\frac{q-\theta}{q}\right), & \text{if } \; 0 \leqslant t \leqslant \kappa, \\ \lambda(x)\kappa^{q-2^*_{\alpha,\mu}}t^{2^*_{\alpha,\mu}}\Big(1-\frac{\theta}{2^*_{\alpha,\mu}}\Big) + \theta\lambda(x)\kappa^q\Big(\frac{1}{2^*_{\alpha,\mu}} - \frac{1}{q}\Big), & \text{if } \; t \geqslant \kappa, \end{cases}$$
$$\geqslant 0, \tag{3.15}$$

which ensures the following Ambrosetti-Rabinowitz type condition

$$h_\kappa(x,t)t - \theta H_\kappa(x,t) \geqslant 0, \quad \forall t \in \mathbb{R}, \tag{3.16}$$

leading us to infer in the existence of constants $d_1, d_2 > 0$, such that for all $t > 0$,

$$H_\kappa(x,t) \geqslant d_1 t^\theta - d_2. \tag{3.17}$$

Now, related to κ, we shall consider the auxiliary problem

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\alpha} + h_\kappa(x,u), \quad \text{in } \mathbb{R}^N. \tag{A_κ}$$

Thus, we say that a function $u \in E$ is a weak solution of auxiliary Problem (A_κ), if

$$\int_{\mathbb{R}^N} (\nabla u \nabla \phi + V(x)u\phi)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\alpha}\phi\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^N} H_\kappa(x,u)\phi\,\mathrm{d}x = 0, \quad \forall \phi \in E.$$

It is important to note that if u is a weak solution of the auxiliary Problem (A_κ) and satisfies $|u(x)| \leqslant \kappa$ for all $x \in \mathbb{R}^N$, then u is a weak solution of Problem (P). This motivates us to study the auxiliary Problem (A_κ).

The energy functional $\mathcal{I}_\kappa : E \to \mathbb{R}$ associated with Problem (A_κ) is given by

$$\mathcal{I}_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\alpha}\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^N} H_\kappa(x,u)\,\mathrm{d}x$$

and in view of the assumptions above regarding the function κ and since $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$, $\mathcal{I}_\kappa$ is well defined.

However, there are at least two difficulties in dealing with the auxiliary Problem (A_κ). The first is the lack of compactness, i.e., is to assure that the energy functional $\mathcal{I}_\kappa$ satisfies the Palais-Smale condition. The second one is to prove uniform estimates (independent of

$\|\lambda\|_{\frac{2N}{2\alpha+\mu}}$ and κ) for the solutions of (A_κ). In order to overcome such difficulty, we introduce a new auxiliary problem where we can restore some compactness. For this purpose, we use the penalization method introduced in (DEL PINO; FELMER, 1996) and adapted in (ALVES; SOUTO, 2012; ALVES; FIGUEIREDO; YANG, 2016). For $\ell > 1$ and $R > 1$ to be determined later, we define the function $\hat{f} : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\hat{f}(x,t) = \begin{cases} f(t), & \text{if } \ell f(t) \leqslant V(x)t, \\ \dfrac{V(x)}{\ell}t, & \text{if } \ell f(t) > V(x)t, \end{cases} \tag{3.18}$$

$\hat{f}(x,t) = 0$, for all $x \in \mathbb{R}^N$ if $t \leqslant 0$ and

$$g(x,t) = \begin{cases} f(t), & \text{if } |x| \leqslant R, \\ \hat{f}(x,t), & \text{if } |x| > R. \end{cases} \tag{3.19}$$

Observe that for all $t \in \mathbb{R}$, the following inequalities hold:

$$g(x,t) \leqslant f(t), \qquad \forall\, x \in \mathbb{R}^N, \tag{3.20}$$

$$G(x,t) \leqslant F(t), \qquad \forall\, x \in \mathbb{R}^N, \tag{3.21}$$

$$g(x,t) \leqslant \frac{V(x)}{\ell}t, \quad \text{if } |x| > R, \tag{3.22}$$

$$G(x,t) \leqslant \frac{V(x)}{2\ell}t^2, \quad \text{if } |x| > R, \tag{3.23}$$

$$G(x,t) = F(t), \qquad \text{if } |x| \leqslant R. \tag{3.24}$$

Hence, it follows from (f_3), (3.2), (3.20) and (3.21) that for all $t \in \mathbb{R}$

$$G(x,t) \leqslant \frac{2c_0}{\theta}|t|^p, \quad G(x,t) \leqslant \frac{2c_0}{\theta}|t|^{2^*_{\alpha,\mu}} \quad \text{and} \quad g(x,t)t \leqslant c_0|t|^{2^*_{\alpha,\mu}}, \quad \forall\, x \in \mathbb{R}^N. \tag{3.25}$$

Moreover, combining (f_3) and (3.24) we obtain

$$\frac{1}{\theta}g(x,t)t - \frac{1}{2}G(x,t) \geqslant 0, \quad \forall\, |x| \leqslant R \text{ and } \forall\, t \geqslant 0 \tag{3.26}$$

and joining (3.20) and (3.21) with (3.10) and (3.11), one have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(y,u)G(x,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\mathrm{d}x \leqslant C\|u\|^{22^*_{\alpha,\mu}}, \quad \forall u \in E, \tag{3.27}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(y,u)g(x,u)u}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\mathrm{d}x \leqslant \tilde{C}\|u\|^{22^*_{\alpha,\mu}}, \quad \forall u \in E. \tag{3.28}$$

Using the previous notations, we introduce the following auxiliary problem

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha} + h_\kappa(x,u), \quad \text{in } \mathbb{R}^N, \tag{B_κ}$$

where $G(x,t)=\int_0^t g(x,\tau)\,\mathrm{d}\tau$.

We say that a function $u\in E$ is a weak solution of the auxiliary Problem (B_κ), if satisfies

$$\int_{\mathbb{R}^N}(\nabla u\nabla\phi+V(x)u\phi)\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha}\phi\,\mathrm{d}x$$
$$-\int_{\mathbb{R}^N}h_\kappa(x,u)\phi\,\mathrm{d}x=0,\quad\forall\phi\in E. \tag{3.29}$$

The energy functional associated with Problem (B_κ) is given by

$$\mathcal{J}_\kappa(u)=\frac{1}{2}\|u\|^2-\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(x,u)}{|x|^\alpha}\,\mathrm{d}x-\int_{\mathbb{R}^N}H_\kappa(x,u)\,\mathrm{d}x.$$

In light of our assumptions one may conclude that $\mathcal{J}_\kappa$ is well defined, belongs to $C^1(E,\mathbb{R})$, and its derivative given by

$$\mathcal{J}_\kappa'(u)v=\int_{\mathbb{R}^N}(\nabla u\nabla v+V(x)uv)\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha}v\,\mathrm{d}x$$
$$-\int_{\mathbb{R}^N}h_\kappa(x,u)v\,\mathrm{d}x.$$

Thus, weak solutions of (B_κ) are precisely the critical points of $\mathcal{J}_\kappa$.

It is also worth mentioning that the auxiliary Problem (B_κ) is strongly related to Problem (A_κ). In fact, if u is a solution of (B_κ) which verifies $\ell f(u(x))\leqslant V(x)u(x)$ for all $|x|\geqslant R$, then $g(x,u)=f(u)$ and u is also a solution for Problem (A_κ), which motivates us to study the auxiliary Problem (B_κ). The crucial role here is that working on Problem (B_κ) we are able to restore some compactness.

3.2.2 Existence of solutions for the auxiliary Problem (B_κ)

In this subsection, we examine the existence of nonnegative solutions for the auxiliary Problem (B_κ). Next, we shall prove that $\mathcal{J}_\kappa$ verifies the mountain pass geometry stated in the following lemma:

Lemma 3.2.1. *The functional $\mathcal{J}_\kappa$ satisfies the following conditions:*

(i) *there exist $\delta,\rho>0$ such that $\mathcal{J}_\kappa(v)\geqslant\delta$ if $\|v\|=\rho$;*

(ii) *there exists $e\in E$ such that $\|e\|>\rho$ and $\mathcal{J}_\kappa(e)<0$.*

Proof. In view of (H_1) and the fact that $\lambda\in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$, using Hölder's inequality and (3.4), we reach

$$\int_{\mathbb{R}^N}H_\kappa(x,u)\,\mathrm{d}x\leqslant\frac{\kappa^{q-2^*_{\alpha,\mu}}}{2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}(S^{-1})^{\frac{2^*_{\alpha,\mu}}{2}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)|u|^2)\,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}}{2}}, \tag{3.30}$$

which together with (3.27), leads us to

$$\mathcal{J}_\kappa(u) \geqslant \frac{1}{2}\|u\|^2 - C\|u\|^{22^*_{\alpha,\mu}} - C_1\|u\|^{2^*_{\alpha,\mu}},$$

where $C_1 := \frac{\kappa^{q-2^*_{\alpha,\mu}}}{2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}} S^{\frac{2^*_{\alpha,\mu}}{2}}$. Therefore, from (3.3), we have that $2^*_{\alpha,\mu} > 2$. Hence, we choose ρ small enough such that $\mathcal{J}_\kappa(u) \geqslant \delta > 0$ for all $u \in E$ with $\|u\| = \rho$, i.e., (i) holds.

To prove part (ii), we fix $\phi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, $\phi \geqslant 0$ in $\mathbb{R}^N$, $\operatorname{supp}(\phi) \subset \bar{B}_1(0)$ and we set

$$\mathcal{A}(t) = \Psi\left(\frac{t\phi}{\|\phi\|}\right) > 0, \text{ for } t > 0,$$

where

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(x,u)}{|x|^\alpha}\,\mathrm{d}x.$$

Since $G(\cdot,\phi) = F(\phi)$ in $\operatorname{supp}(\phi) \subset B_R(0)$, it follows from (3.26) that

$$\begin{aligned}\mathcal{A}'(t) =& \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G\left(y,\frac{t\phi}{\|\phi\|}\right)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g\left(x,\frac{t\phi}{\|\phi\|}\right)}{|x|^\alpha}\frac{\phi}{\|\phi\|}\,\mathrm{d}x\\ \geqslant& \frac{\theta}{t}\frac{1}{2}\int_{\operatorname{supp}(\phi)}\left(\int_{\mathbb{R}^N}\frac{G\left(y,\frac{t\phi}{\|\phi\|}\right)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G\left(x,\frac{t\phi}{\|\phi\|}\right)}{|x|^\alpha}\,\mathrm{d}x\\ =& \frac{\theta}{t}\mathcal{A}(t), \quad \forall t > 0. \end{aligned} \tag{3.31}$$

By integrating (3.31) on $[1, s\|\phi\|]$ with $s > \frac{1}{\|\phi\|}$, we obtain

$$\ln\left(\frac{\mathcal{A}(s\|\phi\|)}{\mathcal{A}(1)}\right) \geqslant \ln\left(\|\phi\|^\theta s^\theta\right),$$

which infers that $\mathcal{A}(s\|\phi\|) \geqslant A(1)\|\phi\|^\theta s^\theta$. Thus,

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(x,s\phi)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(y,s\phi)}{|x|^\alpha}\,\mathrm{d}x \geqslant \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G\left(x,\frac{\phi}{\|\phi\|}\right)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G\left(y,\frac{\phi}{\|\phi\|}\right)}{|x|^\alpha}\,\mathrm{d}x\|\phi\|^\theta s^\theta.$$

By (3.17), it follows that

$$-\int_{\mathbb{R}^N} H_\kappa(x,s\phi)\,\mathrm{d}x \leqslant -d_1 s^\theta\int_{\operatorname{supp}(\phi)}\phi^\theta\,\mathrm{d}x + d_2|\operatorname{supp}(\phi)|. \tag{3.32}$$

Consequently,

$$\begin{aligned}\mathcal{J}_\kappa(s\phi) \leqslant& \frac{1}{2}\|\phi\|^2 s^2 - \frac{1}{2}\|\phi\|^\theta s^\theta\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G\left(x,\frac{\phi}{\|\phi\|}\right)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G\left(y,\frac{\phi}{\|\phi\|}\right)}{|x|^\alpha}\,\mathrm{d}x\\ &- d_1 s^\theta\int_{\operatorname{supp}(\phi)}\phi^\theta\,\mathrm{d}x + d_2|\operatorname{supp}(\phi)|\\ =:& C_1 s^2 - s^\theta(C_2 + C_3) + d_2|\operatorname{supp}(\phi)|, \text{ for } s > \frac{1}{\|\phi\|},\end{aligned}$$

which implies that $\mathcal{J}_\kappa(s\phi) \to -\infty$, as $s \to \infty$, since $\theta > 2$. Finally, assertion (ii) follows for $e = s\phi$ with s large enough. □

According to Definition 2.2.3, by applying a version of Mountain Pass Theorem that does not require the (PS) condition, as in (WILLEM, 1996), we obtain a Palais-Smale sequence $(u_n)_n \subset E$ such that

$$\mathcal{J}_\kappa(u_n) \to c_\kappa \quad \text{and} \quad \mathcal{J}_\kappa'(u_n) \to 0, \tag{3.33}$$

where c_κ is the mountain pass level characterized by

$$0 < c_\kappa := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{J}_\kappa(\gamma(t)) \tag{3.34}$$

where

$$\Gamma_\kappa := \Big\{ \gamma \in C([0,1], E) : \gamma(0) = 0 \ \text{ and } \ \mathcal{J}_\kappa(\gamma(1)) < 0 \Big\}.$$

Now, we introduce the functional $\mathcal{I}_0 : H_0^1(B_1(0)) \to \mathbb{R}$ given by

$$\mathcal{I}_0(u) = \frac{1}{2}\int_{B_1(0)} |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{B_1(0)} m|u|^2 \,\mathrm{d}x - \frac{1}{2}\int_{B_1(0)} \left(\int_{B_1(0)} \frac{F(u)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{F(u)}{|x|^\alpha} \,\mathrm{d}x,$$

where $m = \max_{|x|\leqslant 1} V(x)$. Moreover, we denote by d the level of the mountain pass value associated with the functional $\mathcal{I}_0$, i.e.,

$$0 < d := \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} \mathcal{I}_0(\gamma(t)),$$

where

$$\Gamma_0 := \Big\{ \gamma \in C([0,1], H_0^1(B_1(0))) : \gamma(0) = 0 \ \text{ and } \ \mathcal{I}_0(\gamma(1)) < 0 \Big\}.$$

Here, it is important to emphasize that d is independent of the choice of ℓ, R, $\lambda(x)$ and κ. Moreover, $c_\kappa \leqslant d$.

Lemma 3.2.2. *Assume that conditions* (f_3) *and* (3.15) *hold. Then,*

$$\begin{aligned} \frac{1}{\theta}\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{g(x,u)}{|x|^\alpha} u \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{G(x,u)}{|x|^\alpha} \,\mathrm{d}x \\ =: \mathcal{A}_1 - \mathcal{A}_2 \geqslant 0 \end{aligned} \tag{3.35}$$

and by (3.16)

$$\int_{\mathbb{R}^N} \left(\frac{1}{\theta} h_\kappa(x,u)u - \frac{1}{2} H_\kappa(x,u) \right) \mathrm{d}x \geqslant 0. \tag{3.36}$$

Proof. Initially, note that

$$\begin{aligned} \mathcal{A}_1 - \mathcal{A}_2 = & \int_{\{|x|>R\}} \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{\left[\frac{1}{\theta} g(x,u)u - \frac{1}{2} G(x,u) \right]}{|x|^\alpha} \,\mathrm{d}x \\ & + \int_{\{|x|\leqslant R\}} \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{\left[\frac{1}{\theta} f(u)u - \frac{1}{2} F(u) \right]}{|x|^\alpha} \,\mathrm{d}x, \end{aligned}$$

whence follows from (f_3) and $G(\cdot,t)\geqslant 0$ for all $t\in\mathbb{R}$ that

$$\mathcal{A}_1-\mathcal{A}_2\geqslant\int_{\{|x|>R\}}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{\left[\frac{1}{\theta}g(x,u)u-\frac{1}{2}G(x,u)\right]}{|x|^{\alpha}}\,\mathrm{d}x.$$

On the other hand, defining $\Omega_{V(x)}:=\{\ell f(u(x))\leqslant V(x)u(x)\}$, we use (3.26), (3.22)-(3.23) to deduce

$$\begin{aligned}
&\int_{\{|x|>R\}}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{\left[\frac{1}{\theta}g(x,u)u-\frac{1}{2}G(x,u)\right]}{|x|^{\alpha}}\,\mathrm{d}x\\
=&\int_{\{|x|>R\}\cap\Omega^c_{V(x)}}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{\left[\frac{1}{\theta}g(x,u)u-\frac{1}{2}G(x,u)\right]}{|x|^{\alpha}}\,\mathrm{d}x\\
&+\int_{\{|x|>R\}\cap\Omega_{V(x)}}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{\left[\frac{1}{\theta}g(x,u)u-\frac{1}{2}G(x,u)\right]}{|x|^{\alpha}}\,\mathrm{d}x\\
\geqslant&\int_{\{|x|>R\}\cap\Omega^c_{V(x)}}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{\left[\frac{1}{\theta}g(x,u)u-\frac{1}{2}G(x,u)\right]}{|x|^{\alpha}}\,\mathrm{d}x\\
\geqslant&\int_{\{|x|>R\}\cap\Omega^c_{V(x)}}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^{\alpha}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{1}{|x|^{\alpha}}\frac{1}{\ell}\left(\frac{1}{\theta}-\frac{1}{4}\right)V(x)u^2\,\mathrm{d}x\\
\geqslant&0,
\end{aligned}$$

where we are using the fact that $\left(\frac{1}{\theta}-\frac{1}{4}\right)>0$. Thus, we conclude that (3.35) holds true. The estimate in (3.36) follows directly from (3.16), and the proof is done. □

Lemma 3.2.3. *Let $(u_n)_n$ be a $(PS)_{c_\kappa}-$sequence for $\mathcal{J}_\kappa$. Then, $(u_n)_n$ is bounded in E and there exists $n_0\in\mathbb{N}$ such that*

$$\|u_n\|^2\leqslant\frac{2\theta}{\theta-2}(d+1),\quad\forall\,n\geqslant n_0.$$

Proof. By using (3.35) and (3.36), we deduce

$$\mathcal{J}_\kappa(u_n)-\frac{1}{\theta}\mathcal{J}'_\kappa(u_n)u_n\geqslant\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2=\frac{\theta-2}{2\theta}\|u_n\|^2. \tag{3.37}$$

Note that

$$-\frac{1}{\theta}\mathcal{J}'_\kappa(u_n)u_n\leqslant\left|\frac{1}{\theta}\mathcal{J}'_\kappa(u_n)u_n\right|\leqslant\frac{1}{\theta}\|\mathcal{J}'_\kappa(u_n)\|\|u_n\|. \tag{3.38}$$

Since $(u_n)_n$ is a $(PS)_{c_\kappa}-$sequence for $\mathcal{J}_\kappa$, we see

$$c_\kappa+\|u_n\|\geqslant\frac{\theta-2}{2\theta}\|u_n\|^2,$$

for n sufficiently large. Thus, $(u_n)_n$ is bounded in E. For $n\in\mathbb{N}$ sufficiently large, which jointly with (3.37) implies that

$$\|u_n\|^2\leqslant\frac{2\theta}{\theta-2}(c_\kappa+1)\leqslant\frac{2\theta}{\theta-2}(d+1)$$

and the lemma is proved. □

Let $(u_n)_n$ be a $(PS)_{c_\kappa}-$sequence for $\mathcal{J}_\kappa$. It follows from Lemma 3.2.3 that $(u_n)_n$ is bounded in E. Hence we may assume, passing to a subsequence if necessary, as $n \to \infty$

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } E, \\ u_n \to u, & \text{strongly in } L^p_{loc}(\mathbb{R}^N),\ 1 < p < 2^*, \\ u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^N, \\ |u_n(x)|, |u(x)| \le h(x), & \text{for some } h \in L^p_{loc}(\mathbb{R}^N), \end{cases} \tag{3.39}$$

and

$$\|u_n\|^2 \leqslant \frac{2\theta}{\theta - 2}(d+1) =: \bar{C}, \quad \forall\, n \geqslant n_0. \tag{3.40}$$

As a consequence,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \,\mathrm{d}x,\ \int_{\mathbb{R}^N} V(x)|u_n|^2 \,\mathrm{d}x \leqslant \|u_n\|^2 \leqslant \bar{C}, \quad \forall\, n \geqslant n_0. \tag{3.41}$$

Combining this with (3.4), we get the estimate

$$\left(\int_{\mathbb{R}^N} |u_n|^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant S^{-1}\|\nabla u_n\|_2^2 \leqslant S^{-1}\bar{C} =: C_*, \quad \forall\, n \geqslant n_0. \tag{3.42}$$

Moreover, by (3.27) and (3.41), for all $n \geqslant n_0$, we have that

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{G(x,u_n)}{|x|^\alpha} \,\mathrm{d}x \leqslant C\bar{C}^{2^*_{\alpha,\mu}}. \tag{3.43}$$

3.2.2.1 Compactness results

In this subsection, we will establish some results that are very significant in this chapter, in order to verify that the functional $\mathcal{J}_\kappa$ satisfies the $(PS)_{c_\kappa}-$condition. It is important to mention that in the case $\alpha = 0$, see for instance (ALVES; FIGUEIREDO; YANG, 2016), the boundedness of the nonlocal term

$$\int_{\mathbb{R}^N} \frac{G(y,u)}{|x-y|^\mu} \,\mathrm{d}y,$$

over the set

$$\mathcal{B} := \left\{ u \in E : \|u\|^2 \leqslant \frac{2\theta}{\theta - 2}(d+1) \right\},$$

plays a very crucial role in the arguments. However, as pointed out in Remark 2.1.15 of Chapter 2, the Stein-Weiss type convolution does not admit such boundedness. Thus, we adapt the argument by studying the boundedness of the term

$$\mathcal{K}(u)(x) := \int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y. \tag{3.44}$$

Lemma 3.2.4. *For all $u \in \mathcal{B}$, we have that*

$$\mathcal{K}(u)(x) := \int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \in L^\infty(\mathbb{R}^N). \tag{3.45}$$

In addition, there exists $\ell_0 > 0$, which is independent of R, such that

$$\frac{\sup_{u\in\mathcal{B}} \|\mathcal{K}(u)(x)\|_\infty}{\ell_0} < \frac{1}{2}.$$

Proof. To prove (3.45), we follow some ideas from (DU; GAO; YANG, 2022, Theorem 2.7) and (DE ALBUQUERQUE; SANTOS, 2023, Lemma 3.3). For $u \in \mathcal{B}$ and $r > 0$, we write

$$|\mathcal{K}(u)(x)| \leqslant \int_{B_r(0)} \frac{|G(y,u)|}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y + \int_{B_r^c(0)} \frac{|G(y,u)|}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y. \tag{3.46}$$

On the one hand, for $x \in B_{2r}^c(0)$ and (3.25), we have $|x-y| > |y|$ and

$$\int_{B_r(0)} \frac{|G(y,u)|}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \leqslant \frac{2c_0}{\theta} \int_{B_r(0)} \frac{|u|^p}{|y|^{\mu+\alpha}} \, \mathrm{d}y.$$

Choosing $k := \frac{2^*}{p}$, it follows from Hölder's inequality and Sobolev embedding that

$$\begin{aligned} \frac{2c_0}{\theta} \int_{B_r(0)} \frac{|u|^p}{|y|^{\mu+\alpha}} \, \mathrm{d}y \leqslant & \frac{2c_0}{\theta} \left(\int_{B_r(0)} |u|^{2^*} \, \mathrm{d}y \right)^{\frac{1}{k}} \left(\int_{B_r(0)} \left(\frac{1}{|y|^{\mu+\alpha}} \right)^{\frac{k}{k-1}} \mathrm{d}y \right)^{\frac{k-1}{k}} \\ \leqslant & C\|u\|^p \left(\int_0^r |\tilde{r}|^{N-1-(\alpha+\mu)\frac{k}{k-1}} \, \mathrm{d}\tilde{r} \right)^{\frac{k-1}{k}} \\ \leqslant & C_1 \left(\int_0^r |\tilde{r}|^{N-1-(\alpha+\mu)\frac{k}{k-1}} \, \mathrm{d}\tilde{r} \right)^{\frac{k-1}{k}} := C_2 < \infty, \end{aligned}$$

where we have used that $N-1-(\alpha+\mu)\frac{k}{k-1} > -1$. For $x \in B_{2r}(0)$, using the arguments as above, we observe

$$\begin{aligned} \int_{B_r(0)} \frac{|G(y,u)|}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \leqslant & \int_{B_r(0)} \frac{|G(y,u)|}{|y|^{\mu+\alpha}} \, \mathrm{d}y + \int_{B_{3r}(x)} \frac{|G(y,u)|}{|x-y|^{\mu+\alpha}} \, \mathrm{d}y \\ \leqslant & C_2 + C_3 \left(\int_0^{3r} |\tilde{r}|^{N-1-(\alpha+\mu)\frac{k}{k-1}} \, \mathrm{d}\tilde{r} \right)^{\frac{k-1}{k}} < \infty. \end{aligned}$$

Hence, for each $x \in \mathbb{R}^N$, we get

$$\int_{B_r(0)} \frac{|G(y,u)|}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y < \infty. \tag{3.47}$$

On the other hand, we write

$$\int_{B_r^c(0)} \frac{|G(y,u)|}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y = \int_{B_r^c(0)\cap B_r(x)} \frac{|G(y,u)|}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y + \int_{B_r^c(0)\cap B_r^c(x)} \frac{|G(y,u)|}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y =: \mathcal{I}_1 + \mathcal{I}_2.$$

Arguing as in the preceding estimates, we deduce from (3.25) that

$$\mathcal{I}_1 \leqslant \frac{1}{r^\alpha} \int_{B_r(x)} \frac{|G(y,u)|}{|x-y|^\mu} \, \mathrm{d}y \leqslant \frac{2c_0}{\theta} \frac{1}{r^\alpha} \int_{B_r(x)} \frac{|u|^p}{|x-y|^\mu} \, \mathrm{d}y < \infty.$$

Now, choosing $q_1 = \frac{2N}{2N-2\alpha-\mu}, q_2 = \frac{2N}{\alpha}, q_3 = \frac{2N}{\mu}$ satisfying $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \leq 1$, it follows from Hölder's inequality that

$$\mathcal{I}_2 \leqslant \left(\int_{B_r^c(0)} |G(y,u)|^{\frac{2N}{2N-2\alpha-\mu}} \, \mathrm{d}y \right)^{\frac{2N-2\alpha-\mu}{2N}} \left(\int_{B_r^c(0)} \frac{1}{|y|^{2N}} \, \mathrm{d}y \right)^{\frac{\alpha}{2N}} \left(\int_{B_r^c(x)} \frac{1}{|x-y|^{2N}} \, \mathrm{d}y \right)^{\frac{\mu}{2N}}$$
$$\leqslant \left(\int_{B_r^c(0)} |u|^{2^*} \, \mathrm{d}y \right)^{\frac{2N-2\alpha-\mu}{2N}} \left(\int_{B_r^c(0)} \frac{1}{|y|^{2N}} \, \mathrm{d}y \right)^{\frac{\alpha}{2N}} \left(\int_{B_r^c(x)} \frac{1}{|x-y|^{2N}} \, \mathrm{d}y \right)^{\frac{\mu}{2N}} < \infty.$$

Thus, we obtain

$$\int_{B_r^c(0)} \frac{|G(y,u)|}{|y|^{\alpha}|x-y|^{\mu}} \, \mathrm{d}y < \infty. \tag{3.48}$$

Hence, (5.40), , (5.42) imply that $\mathcal{K}(u) \in L^{\infty}(\mathbb{R}^N)$. Hence, there exists $C_0 > 0$ such that

$$\sup_{u \in \mathcal{B}} \|\mathcal{K}(u)(x)\|_{\infty} \leqslant C_0. \tag{3.49}$$

Since (3.49) holds, there exists $\ell_0 > 0$ such that

$$\frac{\sup_{u \in \mathcal{B}} \|\mathcal{K}(u)(x)\|_{\infty}}{\ell_0} \leqslant \frac{C_0}{\ell_0} < \frac{1}{2}$$

and this completes the proof. □

Throughout this chapter, we assume $\ell > \ell_0 > 0$ in the auxiliary Problem (B_κ).

Remark 3.2.5. *It is important to note that the Lemma 3.2.4 remains true for a bounded sequence $(u_n)_n \subset E$.*

We emphasize here that the next result does not require the assumption that $(u_n)_n$ is a $(PS)_c-$sequence.

Lemma 3.2.6. *If $u_n \rightharpoonup u$ in E, as $n \to \infty$, then, passing to a subsequence if necessary, we have*

$$\lim_{n \to \infty} \int_{B_{\bar{R}}(0)} \left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^{\alpha}|x-y|^{\mu}} \, \mathrm{d}y \right) \frac{g(x,u_n)}{|x|^{\alpha}} (u_n - u) \, \mathrm{d}x = 0, \tag{3.50}$$
$$\lim_{n \to \infty} \int_{B_{\bar{R}}(0)} h_\kappa(x,u_n)(u_n - u) \, \mathrm{d}x = 0, \tag{3.51}$$

for any $\bar{R} > 0$.

Proof. Firstly, we claim that exists $C > 0$ such that

$$\left\| \int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^{\alpha}|x-y|^{\mu}|x|^{\alpha}} \, \mathrm{d}y \right\|_{\frac{2N}{\mu+2\alpha}} \leqslant C, \quad \forall n \in \mathbb{N}. \tag{3.52}$$

In fact, we observe that $(G(y,u_n))_n$ is bounded in $L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N)$, $u_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$ and G is continuous, thus we infer that

$$G(y,u_n)\rightharpoonup G(y,u),\quad \text{in } L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N).$$

In light of Proposition 1.0.2, we see that the operator

$$L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N)\ni h\longmapsto \int_{\mathbb{R}^N}\frac{h}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\in L^{\frac{2N}{\mu+2\alpha}}(\mathbb{R}^N)$$

is a linear bounded operator from $L^{\frac{2N}{2N-2\alpha-\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\mu+2\alpha}}(\mathbb{R}^N)$, which assures us

$$\int_{\mathbb{R}^N}\frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\rightharpoonup\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y,\quad \text{in}\quad L^{\frac{2N}{\mu+2\alpha}}(\mathbb{R}^N).$$

Thus, there exists $C>0$ such that (3.52) holds.

For any fixed $\bar{R}>0$, using Hölder's inequality with exponents $\frac{2N}{\mu+2\alpha}$ and $\frac{2N}{2N-2\alpha-\mu}$, it follows from (3.52) that

$$\begin{aligned}
&\int_{B_{\bar{R}}(0)}\left(\int_{\mathbb{R}^N}\frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\right)g(x,u_n)(u_n-u)\,\mathrm{d}x\\
\leqslant&\left\|\int_{\mathbb{R}^N}\frac{G(x,u_n)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\right\|_{\frac{2N}{\mu+2\alpha}}\left(\int_{B_{\bar{R}}(0)}\left|g(x,u)(u_n-u)\right|^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\
\leqslant&C\left(\int_{B_{\bar{R}}(0)}\left|g(x,u)(u_n-u)\right|^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}=:\tilde{L}_n.
\end{aligned}\tag{3.53}$$

According to the growth condition in (3.25), we have $g(x,u_n)=f(u_n)\leqslant c_0|u_n|^{2^*_{\alpha,\mu}-2}|u_n|$ in $B_{\bar{R}}(0)$, which combined with Hölder's inequality, we reach

$$\begin{aligned}
\tilde{L}_n\leqslant&Cc_0\left(\int_{B_{\bar{R}}(0)}\left(|u_n|^{2^*_{\alpha,\mu}-2}\right)^{\frac{2N}{2N-2\alpha-\mu}}\left(|u_n||u_n-u|\right)^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\
\leqslant&Cc_0\left(\int_{B_{\bar{R}}(0)}|u_n|^{2^*}\,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{B_{\bar{R}}(0)}(|u_n||u_n-u|)^{\frac{2^*}{2}}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\\
\leqslant&C_*^{\frac{2^*_{\alpha,\mu}-2}{2}}Cc_0\left(\int_{B_{\bar{R}}(0)}\left(|u_n||u_n-u|\right)^{\frac{2^*}{2}}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\to 0,\quad \text{as } n\to\infty,
\end{aligned}\tag{3.54}$$

where we are using (3.39), (3.42), the fact that $1<\frac{2^*}{2}<2^*$ and applying Lebesgue Dominated Convergence Theorem. It follows from (3.53) and (3.54) that (3.50) holds true.

In order to prove (3.51), we combine the growth condition in (h_1), the fact that

$\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ and Hölder's inequality to obtain the following estimate

$$\begin{aligned}
&\int_{B_r(0)} h_\kappa(x,u_n)(u_n-u)\,\mathrm{d}x \leqslant \kappa^{q-2^*_{\alpha,\mu}} \int_{B_r(0)} \lambda(x)|u_n|^{2^*_{\alpha,\mu}-2}|u_n||u_n-u|\,\mathrm{d}x \\
\leqslant& \kappa^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} \left[\int_{B_r(0)} \left(|u_n|^{2^*_{\alpha,\mu}-2}\right)^{\frac{2N}{2N-2\alpha-\mu}} \left(|u_n||u_n-u|\right)^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right]^{\frac{2N-2\alpha-\mu}{2N}} \\
\leqslant& \kappa^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} \left[\int_{B_r(0)} |u_n|^{2^*}\,\mathrm{d}x\right]^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \left[\int_{B_r(0)} (|u_n||u_n-u|)^{\frac{2^*}{2}}\,\mathrm{d}x\right]^{\frac{2}{2^*}} \\
\leqslant& \kappa^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} C_*^{\frac{2^*_{\alpha,\mu}-2}{2}} \left[\int_{B_r(0)} (|u_n||u_n-u|)^{\frac{2^*}{2}}\,\mathrm{d}x\right]^{\frac{2}{2^*}} \to 0, \quad \text{as } n\to\infty,
\end{aligned}$$

where we have used (3.39), $1 < \frac{2^*}{2} < 2^*$ and applied Lebesgue Dominated Convergence Theorem. Thus, we see that (3.51) holds. The proof of Lemma is done. □

Now, let $\eta_r \in C^\infty(\mathbb{R}^N)$ be defined as

$$\begin{cases}
\eta_r(x) = \begin{cases} 0, & \text{in } B_r(0), \\ 1, & \text{in } B^c_{2r}(0), \end{cases} \\
0 \leqslant \eta_r(x) \leqslant 1, \qquad \text{in } \mathbb{R}^N, \\
|\nabla \eta_r(x)| \leqslant \frac{C}{r}, \qquad \text{in } \mathbb{R}^N,
\end{cases} \tag{3.55}$$

for some $C>0$ independent of r with $r>R$ and we also need the following lemma:

Lemma 3.2.7. *Let $(u_n)_n$ be a $(PS)_{c_\kappa}-$sequence for $\mathcal{J}_\kappa$. Then,*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \eta_r^2(|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x \leqslant \frac{C_1}{2r^\alpha} + \frac{C_2}{r}\left(\int_{\{r\leqslant|x|\leqslant 2r\}} |u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&+ 4\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \kappa^{q-2^*_{\alpha,\mu}} C_*^{\frac{2^*_{\alpha,\mu}-2}{2}} S^{-1} \left(\int_{\mathbb{R}^N} \eta_r^2(|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x + \int_{\mathbb{R}^N} u_n^2|\nabla\eta_r|^2\,\mathrm{d}x\right),
\end{aligned} \tag{3.56}$$

where C_* was defined in (3.42) *and η_r was introduced in* (3.55). *In addition for each $\kappa>0$, consider $\varepsilon_0^* := \varepsilon_0^*(\kappa) > 0$ such that if $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^*$, then*

$$4\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \kappa^{q-2^*_{\alpha,\mu}} C_*^{\frac{2^*_{\alpha,\mu}-2}{2}} S^{-1} \leqslant \frac{1}{2}. \tag{3.57}$$

Consequently, (3.56) *becomes*

$$\begin{aligned}
\frac{1}{2}\limsup_{n\to\infty} \int_{\mathbb{R}^N} \eta_r^2(|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x \leqslant& \frac{C_1}{2r^\alpha} + \frac{C_2}{r}\left(\int_{\{r\leqslant|x|\leqslant 2r\}} |u|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&+ \frac{C_3}{r^2}\int_{\{r\leqslant|x|\leqslant 2r\}} |u|^2\,\mathrm{d}x \\
=:& \frac{C_1}{2r^\alpha} + C_2\left(\frac{\mathcal{R}(r)}{r^2}\right)^{\frac{1}{2}} + C_3\frac{\mathcal{R}(r)}{r^2},
\end{aligned} \tag{3.58}$$

where C_1, C_2 and C_3 are positive constants. Therefore, for each $\varepsilon > 0$, there exists $r = r(\varepsilon) > R$ verifying

$$\limsup_{n\to\infty} \int_{B^c_{2r}(0)} (|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x < \varepsilon, \tag{3.59}$$

$$\limsup_{n\to\infty} \int_{B^c_{2r}(0)} \left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u_n)}{|x|^\alpha}u_n\,\mathrm{d}x \leqslant C_1\varepsilon, \tag{3.60}$$

$$\limsup_{n\to\infty} \int_{B^c_{2r}(0)} \left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u_n)}{|x|^\alpha}u\,\mathrm{d}x \leqslant C_2\varepsilon, \tag{3.61}$$

$$\limsup_{n\to\infty} \int_{B^c_{2r}(0)} h_\kappa(x,u_n)u_n\,\mathrm{d}x \leqslant C_{1,k}\varepsilon, \tag{3.62}$$

$$\limsup_{n\to\infty} \int_{B^c_{2r}(0)} h_\kappa(x,u_n)u\,\mathrm{d}x \leqslant C_{2,k}\varepsilon, \tag{3.63}$$

whenever (3.57) *holds true.*

Proof. In view of (3.39), we may assume, passing to a subsequence if necessary, $u_n \rightharpoonup u$ weakly in E and the sequence $(u_n)_n$ satisfies (3.40), (3.41), (3.42) and (3.43). By (3.55) the sequence $(u_n\eta_r^2)_n$ is bounded in E. Hence, $\mathcal{J}'_{\lambda,\kappa}(u_n)u_n\eta_r^2 = o_n(1)$, whence it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} \eta_r^2(|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x \leqslant &\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u_n)}{|x|^\alpha}u_n\eta_r^2\,\mathrm{d}x\\ &+\left|-2\int_{\mathbb{R}^N} u_n\nabla u_n\nabla\eta_r\,\mathrm{d}x\right| + \int_{\mathbb{R}^N} h_\kappa(x,u_n)u_n\eta_r^2 + o_n(1)\\ :=&\mathcal{Q}_1(n,r) + \mathcal{Q}_2(n,r) + \mathcal{Q}_{3,\kappa}(n,r) + o_n(1). \end{aligned}\tag{3.64}$$

To estimate $\mathcal{Q}_1(n,r)$, we consider the following two cases:

- If $\alpha \neq 0$, then from (3.22), (3.41), (3.55) and using Lemma 3.2.4, it follows that

$$\begin{aligned}\mathcal{Q}_1(n,r) \leqslant&\frac{1}{r^\alpha}\left|\int_{\{|x|\geqslant r\}}\left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{V(x)}{\ell_0}|u_n(x)|^2\eta_r^2\,\mathrm{d}x\right|\\ \leqslant&\frac{1}{r^\alpha}\int_{\{|x|\geqslant r\}} \frac{\sup_{u_n\in\mathcal{B}}\|K(u_n)(x)\|_\infty}{\ell_0}V(x)|u_n|^2\,\mathrm{d}x\\ \leqslant&\frac{1}{2r^\alpha}\int_{\mathbb{R}^N} V(x)|u_n|^2\,\mathrm{d}x\\ \leqslant&\frac{1}{2r^\alpha}\bar{C}.\end{aligned}$$

- If $\alpha = 0$, then from (3.22), (3.41), (3.55) and using Lemma 3.2.4 again, one has

$$\begin{aligned}\mathcal{Q}_1(n,r) \leqslant&C\int_{\{|x|\geqslant r\}}\left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|x-y|^\mu}\,\mathrm{d}y\right)g(x,u_n)u_n\eta_r^2\,\mathrm{d}x\\ \leqslant&\int_{\{|x|\geqslant r\}} \frac{\sup_{u_n\in\mathcal{B}}\|K(u_n)(x)\|_\infty}{\ell_0}V(x)|u_n|^2\eta_r^2\,\mathrm{d}x\\ \leqslant&\frac{1}{2}\int_{\mathbb{R}^N} \eta_r^2(|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x.\end{aligned}$$

Hence, in both cases, (3.64) becomes, respectively

$$\int_{\mathbb{R}^N} \eta_r^2(|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x \leqslant \frac{1}{2r^\alpha}\tilde{C} + \mathcal{Q}_2(n,r) + \mathcal{Q}_{3,\kappa}(n,r) + o_n(1), \tag{3.65}$$

$$\frac{1}{2}\int_{\mathbb{R}^N} \eta_r^2(|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x \leqslant \mathcal{Q}_2(n,r) + \mathcal{Q}_{3,\kappa}(n,r) + o_n(1). \tag{3.66}$$

Next, we will do the proof only for estimate (3.65), because the proof for estimate (3.66) is similar.

Using once more (3.41) and (3.55), thanks to Hölder's inequality, we get the following estimate for $\mathcal{Q}_2(n,r)$,

$$\begin{aligned}
\mathcal{Q}_2(n,r) \leqslant& 2\frac{C}{r}\left(\int_{\{r\leqslant|x|\leqslant 2r\}} |u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\{r\leqslant|x|\leqslant 2r\}} |\nabla u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\
\leqslant& 2\bar{C}^{\frac{1}{2}}\frac{C}{r}\left(\int_{\{r\leqslant|x|\leqslant 2r\}} |u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.67}$$

Thus, since $u_n \to u$ in $L^2(\{r \leqslant |x| \leqslant 2r\})$, we have

$$\limsup_{n\to\infty} \mathcal{Q}_2(n,r) \leqslant 2\bar{C}^{\frac{1}{2}}\frac{C}{r}\left(\int_{\{r\leqslant|x|\leqslant 2r\}} u^2\,\mathrm{d}x\right)^{\frac{1}{2}} =: C_2\left(\frac{\mathcal{R}(r)}{r^2}\right)^{\frac{1}{2}}. \tag{3.68}$$

By assumption (h_1), thanks to Hölder's inequality, from (3.4) and (3.42), we get the following

$$\begin{aligned}
\mathcal{Q}_{3,\kappa}(n,r) \leqslant& \kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}\left(\int_{\mathbb{R}^N}(|u_n|^{2^*_{\alpha,\mu}-2}|u_n|^2\eta_r^2)^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\
\leqslant& \kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2}}\left(\int_{\mathbb{R}^N}|u_n\eta_r|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\\
\leqslant& \kappa^{q-2^*_{\alpha,\mu}}C_*^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}\left(\int_{\mathbb{R}^N}u_n^{2^*}\eta_r^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\\
\leqslant& \kappa^{q-2^*_{\alpha,\mu}}C_*^{\frac{2^*_{\alpha,\mu}-2}{2}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}S^{-1}\int_{\mathbb{R}^N}|\eta_r\nabla u_n + u_n\nabla\eta_r|^2\,\mathrm{d}x\\
\leqslant& \kappa^{q-2^*_{\alpha,\mu}}C_*^{\frac{2^*_{\alpha,\mu}-2}{2}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}S^{-1}2^2\int_{\mathbb{R}^N}\eta_r^2(|\nabla u_n|^2+V(x)|u_n|^2)\,\mathrm{d}x\\
&+\kappa^{q-2^*_{\alpha,\mu}}C_*^{\frac{2^*_{\alpha,\mu}-2}{2}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}S^{-1}2^2\int_{\mathbb{R}^N}u_n^2|\nabla\eta_r|^2\,\mathrm{d}x,
\end{aligned} \tag{3.69}$$

which together with (3.65) and (3.67) implies that (3.56) holds.

In order to prove (3.58), for each $\kappa > 0$, let $\varepsilon_0^* = \varepsilon_0^*(\kappa) > 0$ be such that if $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^*$, then (3.57) holds. According to (3.57) and (3.69) we see that

$$\begin{aligned}
\mathcal{Q}_{3,\kappa}(n,r) \leqslant& \frac{1}{2}\int_{\mathbb{R}^N}\eta_r^2(|\nabla u_n|^2+V(x)|u_n|^2)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}u_n^2|\nabla\eta_r|^2\,\mathrm{d}x\\
=:& \frac{1}{2}\int_{\mathbb{R}^N}\eta_r^2(|\nabla u_n|^2+V(x)|u_n|^2)\,\mathrm{d}x + \mathcal{Q}_4(n,r).
\end{aligned} \tag{3.70}$$

Noticing that $\mathbb{R}^N = B_r(0) \cup \{r \leqslant |x| \leqslant 2r\} \cup B^c_{2r}(0)$ and using that $\nabla \eta_r = 0$ in B^c_{2r}, $|\nabla \eta_r(x)| \leqslant \frac{C}{r}$ and since $u_n \to u$ in $L^2(\{r \leqslant |x| \leqslant 2r\})$, thanks to (3.39), we have that

$$\limsup_{n\to\infty} \mathcal{Q}_4(n,r) \leqslant \frac{C^2}{2r^2} \int_{\{r\leqslant|x|\leqslant 2r\}} |u|^2 \,\mathrm{d}x =: C_3 \frac{\mathcal{R}(r)}{r^2}. \tag{3.71}$$

Putting together (3.65), (3.70) and (3.71), then (3.58) holds true.

From (3.71), applying Sobolev embedding

$$C_3\mathcal{R}(r) \leqslant \frac{C^2}{2r^2} \left(\int_{\{r\leqslant|x|\leqslant 2r\}} |u|^{2^*} \,\mathrm{d}x \right)^{\frac{2}{2}} |\{r \leqslant |x| \leqslant 2r\}|^{\frac{2}{N}} \tag{3.72}$$

and since $|\{r \leqslant |x| \leqslant 2r\}| \leqslant |B_{2r}| = \omega_N 2^N r^N$, where ω_N is the volume of the unitary ball in $\mathbb{R}^N$, it follows that

$$\lim_{r\to\infty} C_3 \frac{\mathcal{R}(r)}{r^2} = 0. \tag{3.73}$$

Recalling (3.68), from (3.73), we reach

$$\lim_{r\to\infty} C_2 \left(\frac{\mathcal{R}(r)}{r^2} \right)^{\frac{1}{2}} = 0. \tag{3.74}$$

By (3.55), we have $\eta_r = 1$ in $B^c_{2r}(0)$, implying that

$$\int_{B^c_{2r}(0)} (|\nabla u_n|^2 + V(x)|u_n|^2) \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} \eta_r^2 (|\nabla u_n|^2 + V(x)|u_n|^2) \,\mathrm{d}x. \tag{3.75}$$

This together with (3.58) and (3.73)-(3.75), it follows that for any fixed $\varepsilon > 0$, we can choose $r(\varepsilon) > R > 0$ such that

$$\limsup_{n\to\infty} \int_{B^c_{2r}(0)} (|\nabla u_n|^2 + V(x)|u_n|^2) \,\mathrm{d}x < \varepsilon,$$

whenever (3.57) holds true. Therefore, (3.59) follows.

Henceforth, we assume that (3.57) holds. From (3.25) and noticing that $g(\cdot\,, t)t \in L^{\frac{2N}{2N-2\alpha-\mu}}$, thanks to (3.52) and applying Hölder's inequality combined with (3.4) there holds

$$\begin{aligned}
\int_{B^c_{2r}(0)} \left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{g(x,u_n)}{|x|^\alpha} u_n \,\mathrm{d}x \leqslant{}& \left\| \int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha |x-y|^\mu |x|^\alpha} \,\mathrm{d}y \right\|_{\frac{2N}{\mu+2\alpha}} \\
&\times \left(\int_{B^c_{2r}(0)} |g(x,u_n)u_n|^{\frac{2N}{2N-2\alpha-\mu}} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \\
\leqslant{}& C \left(\int_{B^c_{2r}(0)} |u_n|^{2^*} \,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \\
\leqslant{}& C S^{\frac{2^*}{2} \frac{2N-2\alpha-\mu}{2N}} \left(\int_{B^c_{2r}(0)} |\nabla u_n|^2 + V(x)|u_n|^2 \,\mathrm{d}x \right)^{\frac{2^*}{2} \frac{2N-2\alpha-\mu}{2N}}.
\end{aligned}$$

Putting this together with (3.59), we infer that for given $\varepsilon > 0$, there exists $r(\varepsilon) > 0$ such that

$$\limsup_{n\to\infty} \int_{B^c_{2r}(0)} \left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right) \frac{g(x,u_n)}{|x|^\alpha} u_n \,\mathrm{d}x \leqslant C_1 \varepsilon.$$

By a similar arguments, we obtain

$$\begin{aligned}\int_{B^c_{2r}(0)} \left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right) \frac{g(x,u_n)}{|x|^\alpha} u \,\mathrm{d}x \leqslant & \left\| \int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha |x-y|^\mu |x|^\alpha}\,\mathrm{d}y \right\|_{\frac{2N}{\mu+2\alpha}} \\ & \times \left(\int_{B^c_{2r}(0)} |g(x,u_n)u|^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \\ \leqslant & C\left(\int_{B^c_{2r}(0)} (|u_n|^{2^*_{\alpha,\mu}-1}u)^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}.\end{aligned}$$

On the other hand, applying Hölder's inequality combined with (3.4), we obtain

$$\left(\int_{B^c_{2r}(0)} (|u_n|^{2^*_{\alpha,\mu}-1}u)^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \leqslant \hat{C}\left(\int_{B^c_{2r}(0)} |\nabla u_n|^2 + V(x)|u_n|^2\,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-1}{2}},$$

where $\hat{C} := \|u\|_{2^*}(S^{-1})^{\frac{2^*_{\alpha,\mu}-1}{2}}$. These last two estimates take us

$$\int_{B^c_{2r}(0)} \left(\int_{\mathbb{R}^N} \frac{G(y,u_n)}{|y|^\alpha |x-y|^\mu |x|^\alpha}\,\mathrm{d}y\right) g(x,u_n) u \,\mathrm{d}x \leqslant \hat{C}\left(\int_{B^c_{2r}(0)} |\nabla u_n|^2 + V(x)|u_n|^2\,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-1}{2}}.$$

Thus, (3.60) and (3.61) hold.

To prove (3.62), we will use (3.59). In fact, from (h_1), Hölder's inequality and the light of (3.4), there holds

$$\begin{aligned}\int_{B^c_{2r}(0)} h_\kappa(x,u_n)u_n\,\mathrm{d}x \leqslant & \kappa^{q-2^*_{\alpha,\mu}} \int_{B^c_{2r}(0)} \lambda(x)|u_n|^{2^*_{\alpha,\mu}}\,\mathrm{d}x \\ \leqslant & \kappa^{q-2^*_{\alpha,\mu}}(S^{-1})^{\frac{2^*_{\alpha,\mu}}{2}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} \left(\int_{B^c_{2r}(0)} |\nabla u_n|^2 + V(x)|u_n|^2\,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}}{2}},\end{aligned} \tag{3.76}$$

which jointly with (3.59) we conclude that for given $\varepsilon > 0$, there exists $r(\varepsilon) > 0$ such that

$$\limsup_{n\to\infty} \int_{B^c_r(0)} h_\kappa(x,u_n)u_n\,\mathrm{d}x \leqslant C_{1,\kappa}\varepsilon,$$

where $C_{1,\kappa} := \|\lambda\|_{\frac{2N}{2\alpha+\mu}} \kappa^{q-2^*_{\alpha,\mu}}(S^{-1})^{\frac{2^*_{\alpha,\mu}}{2}}$.

Similarly, by (h_1) and applying Hölder's inequality together with (3.4), we see that

$$\begin{aligned}\int_{B^c_{2r}(0)} h_\kappa(x,u_n)u\,\mathrm{d}x \leqslant & \kappa^{q-2^*_{\alpha,\mu}} \int_{B^c_{2r}(0)} \lambda(x)|u_n|^{2^*_{\alpha,\mu}-1}u\,\mathrm{d}x \\ \leqslant & C_{2,\kappa}\left(\int_{B^c_{2r}(0)} |\nabla u_n|^2 + V(x)|u_n|^2\,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-1}{2}},\end{aligned} \tag{3.77}$$

where $C_{2,\kappa} := \kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}\|u\|_{2^*}(S^{-1})^{\frac{2^*_{\alpha,\mu}-1}{2}}$. Consequently, it follows that (3.63) holds. This completes the proof of the lemma. $\square$

In what follows, we emphasize an important remark that we will use in Subsection 3.4.1 of this chapter during our study of Problem (Q). To verify this remark, we will make use of a result due to Berestycki and Lions, which provides a characterization for the functions belonging to $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$.

Lemma 3.2.8. *(BERESTYCKI; LIONS, 1983, Radial Lemma A.III) Let $N \geqslant 3$. Every function u in $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ is almost everywhere equal to a function $U(x)$, continuous for $x \neq 0$. Furthermore, there is a constant $C := C_N$ such that for all $u \in \mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$, we obtain*

$$|U(x)| \leqslant C\frac{\|\nabla u\|_2}{|x|^{\frac{N-2}{2}}}, \quad |x| \geqslant 1.$$

Remark 3.2.9. *If we replace the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ by $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$, then in view of the Cauchy-Schwarz inequality, see (LIEB; LOSS, 2001, Theorem 9.8), we can estimate $\mathcal{Q}_1(n,r)$ as follows*

$$\begin{aligned}
\mathcal{Q}_1(n,r) \leqslant & \left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{g(y,u_n)u_n\eta_r^2}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u_n)u_n\eta_r^2}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac12}\\
&\times\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(x,u_n)}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac12}\\
\leqslant & C^{\frac12}\bar{C}^{\frac{2^*_{\alpha,\mu}}{2}}\left[\int_{\{|x|\geqslant r\}}\left(\int_{\{|y|\geqslant r\}}\frac{g(y,u_n)u_n\eta_r^2}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u_n)u_n\eta_r^2}{|x|^\alpha}\,\mathrm{d}x\right]^{\frac12}\\
\leqslant & \frac{1}{r^{\frac{\alpha}{2}}}C^{\frac12}\bar{C}^{\frac{2^*_{\alpha,\mu}}{2}}\left[\int_{\{|x|\geqslant r\}}\left(\int_{\{|y|\geqslant r\}}\frac{|u_n(y)|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{V(x)}{\ell_0}|u_n|^2\,\mathrm{d}x\right]^{\frac12},
\end{aligned} \tag{3.78}$$

where we use (3.22) *and* (3.43). *On the other hand, we observe that*

$$\begin{aligned}
\int_{\{|y|\geqslant r\}}\frac{|u_n(y)|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y =& \int_{\{|y|\geqslant r\}\cap\{|x-y|>1\}}\frac{|u_n(y)|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\\
&+\int_{\{|y|\geqslant r\}\cap\{|x-y|\leqslant 1\}}\frac{|u_n(y)|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y =: \mathcal{P}_1(n,r)+\mathcal{P}_2(n,r).
\end{aligned}$$

It follows from Hölder's inequality and (3.42) *that*

$$\begin{aligned}
\mathcal{P}_1(n,r) \leqslant & \left(\int_{\mathbb{R}^N}|u_n|^{2^*}\,\mathrm{d}y\right)^{\frac{2N-2\alpha-\mu}{2N}}\left(\int_{|y|\geqslant r}\frac{1}{|y|^{2N}}\,\mathrm{d}y\right)^{\frac{\alpha}{2N}}\left(\int_{\{|x-y|>1\}}\frac{1}{|x-y|^{2N}}\,\mathrm{d}y\right)^{\frac{\mu}{2N}}\\
\leqslant & \bar{C}^{\frac{2^*_{\alpha,\mu}}{2}}\left(\frac{\omega_{N-1}}{Nr^N}\right)^{\frac{\alpha}{2N}}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\mu}{2N}}\\
=: & C(\omega_{N-1},\alpha,\mu,N)\frac{1}{r^{\frac{\alpha}{2}}},
\end{aligned}$$

where ω_{N-1} is the surface area of $(N-1)-$dimensional unit sphere. Now, to estimate $\mathcal{P}_2(n,r)$, we will use the Radial Lemma 3.2.8, as follows

$$\begin{aligned}\mathcal{P}_2(n,r) \leqslant& (C\|\nabla u_n\|_2)^{2^*_{\alpha,\mu}} \int_{\{|y|\geqslant r\}\cap\{|x-y|\leqslant 1\}} \frac{1}{|y|^{\frac{N-2}{2}}|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\\ \leqslant& \frac{(C\|\nabla u_n\|_2)^{2^*_{\alpha,\mu}}}{r^{\frac{N-2}{2}+\alpha}} \int_{\{|y|\geqslant r\}\cap\{|x-y|\leqslant 1\}} \frac{1}{|x-y|^\mu}\,\mathrm{d}y\\ \leqslant& \frac{C^{2^*_{\alpha,\mu}}\bar{C}^{\frac{2^*_{\alpha,\mu}}{2}}}{r^{\frac{N-2}{2}+\alpha}} \int_{\{|x-y|\leqslant 1\}} \frac{1}{|x-y|^\mu}\,\mathrm{d}y\\ \leqslant& C(\omega_{N-1},\alpha,\mu,N)\frac{1}{r^{\frac{N-2}{2}+\alpha}},\end{aligned}$$

where we used (3.41) *and the fact that $N>\mu$. By apllying these estimates in* (3.78) *and using* (3.41)*, we see that*

$$\begin{aligned}\mathcal{Q}_1(n,r) \leqslant& \frac{1}{r^{\frac{\alpha}{2}}}C^{\frac{1}{2}}\bar{C}^{\frac{2^*_{\alpha,\mu}}{2}}C(\omega_{N-1},\alpha,\mu,N)\left(\frac{1}{r^{\frac{\alpha}{2}}}+\frac{1}{r^{\frac{N-2}{2}+\alpha}}\right)\frac{1}{\ell_0}\left[\int_{\{|x|\geqslant r\}} V(x)|u_n|^2\,\mathrm{d}x\right]^{\frac{1}{2}}\\ \leqslant& C\left(\frac{1}{r^{\frac{3\alpha}{4}}}+\frac{1}{r^{\frac{N-2}{4}+\alpha}}\right),\end{aligned}$$

where $C := C^{\frac{1}{2}}\bar{C}^{\frac{2^*_{\alpha,\mu}}{2}}C(\omega_{N-1},\alpha,\mu,N)$. In this way, we observe that we obtain the same result as in Lemma 3.2.7, however, without using Lemma 3.2.4. The argument for $\mathcal{Q}_2(n,r)$ and $\mathcal{Q}_{3,\kappa}(n,r)$ remain the same.

Lemma 3.2.10. *Let $(u_n)_n$ be a $(PS)_{c_\kappa}-$sequence for $\mathcal{J}_\kappa$. Then, we have*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u_n)}{|x|^\alpha}(u_n-u)\,\mathrm{d}x=0, \tag{3.79}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_\kappa(x,u_n)(u_n-u)\,\mathrm{d}x=0. \tag{3.80}$$

Proof. In view of Lemma 3.2.7, it is enough to prove that

$$\lim_{n\to\infty}\int_{B_{2r}(0)}\left(\int_{\mathbb{R}^N}\frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u_n)}{|x|^\alpha}(u_n-u)\,\mathrm{d}x=0, \tag{3.81}$$

$$\lim_{n\to\infty}\int_{B_{2r}(0)} h_\kappa(x,u_n)(u_n-u)\,\mathrm{d}x=0. \tag{3.82}$$

The proof of (3.81) and (3.82) follows from Lemma 3.2.6 taking $\bar{R}\geqslant 2r$. □

Henceforth, we assume that $\|\lambda\|_{\frac{2N}{2\alpha+\mu}}$ and κ satisfy (3.57) for the auxiliary problem (B_κ).

Lemma 3.2.11. *The functional $\mathcal{J}_\kappa$ satisfies the $(PS)_{c_\kappa}-$condition.*

Proof. Let $(u_n)_n \subset E$ be a $(PS)_{c_\kappa}-$sequence for $\mathcal{J}_\kappa$. In view of Lemma 3.2.3, the sequence $(u_n)_n$ is bounded in E and, up to a subsequence, $u_n \rightharpoonup u$ weakly in E. Since $\mathcal{J}'_\kappa(u_n)u_n = \mathcal{J}'_\kappa(u_n)u = o_n(1)$, it follows that

$$\|u_n - u\|^2 = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u_n)}{|x|^\alpha}(u_n-u)\,\mathrm{d}x + \int_{\mathbb{R}^N} h_\kappa(x,u_n)(u_n-u)\,\mathrm{d}x + o_n(1),$$

which jointly with Lemma 3.2.10 implies that $\mathcal{J}_\kappa$ satisfies the $(PS)_{c_\kappa}-$condition. □

Lemma 3.2.12. *The functional $\mathcal{J}_\kappa$ has a nonnegative critical point $u \in E$ such that $\mathcal{J}_\kappa(u) = c_\kappa$, i.e., u is a nonnegative mountain pass solution for Problem (B_κ).*

Proof. In view of Lemmas 3.2.1 and 3.2.11, Problem (B_κ) admits a nontrivial solution of mountain pass type. By using $u^- := \max\{-u, 0\}$ as test function in (3.29), we deduce from (3.13) and (3.19) that

$$\begin{aligned}
0 =& \int_{\mathbb{R}^N}(\nabla u\nabla u^- + V(x)uu^-)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\right)g(x,u)u^-\,\mathrm{d}x\\
&- \int_{\mathbb{R}^N} h_\kappa(x,u)u^-\,\mathrm{d}x\\
=& \int_{\mathbb{R}^N}(|\nabla u^-|^2 + V(x)|u^-|^2)\,\mathrm{d}x - \int_{\{u>0\}}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\right)g(x,u)u^-\,\mathrm{d}x\\
&- \int_{\{u<0\}}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\right)g(x,u)u^-\,\mathrm{d}x - \int_{\{u<0\}} h_\kappa(x,u)u^-\,\mathrm{d}x\\
&- \int_{\{u>0\}} h_\kappa(x,u)u^-\,\mathrm{d}x\\
=& \|u^-\|,
\end{aligned}$$

i.e., the nontrivial weak solution u is nonnegative. □

3.2.3 $L^\infty-$estimates

By virtue of (3.57) in Lemma 3.2.7 and thanks to Lemma 3.2.12, the auxiliary Problem (B_κ) admits a solution $u := u_{\lambda,\kappa}$ for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ and $\kappa > 0$ with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^* := \varepsilon_0^*(\kappa)$. In what follows, we deduce a uniform estimate for the norm of the solution $u_{\lambda,\kappa}$ of Problem (B_κ). Precisely, we prove the following result:

Lemma 3.2.13. *Let $u_{\lambda,\kappa}$ be the critical point of $\mathcal{J}_\kappa$ obtained in Lemma 3.2.12. Then, there exists a constant M (which depends only on $N, \theta, \mu, \alpha, p, m$ and independent of ℓ, $\|\lambda\|_{\frac{2N}{2\alpha+\mu}}$,*

κ and R) such that

$$\|u_{\lambda,\kappa}\|^2 \leqslant \frac{2\theta}{\theta-2} d =: M, \quad \forall\, \lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$$

with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^*$. In particular, by (3.4) we have that

$$\|u_{\lambda,\kappa}\|_{2^*}^2 \leqslant S^{-1}M < S^{-1}\bar{C}, \quad \forall\, \lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$$

with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^*$.

Proof. In view of estimate (3.37) and recalling that $c_\kappa \leqslant d$, we have that

$$d \geqslant c_\kappa = \mathcal{J}_\kappa(u_{\lambda,\kappa}) - \frac{1}{\theta}\mathcal{J}_\kappa'(u_{\lambda,\kappa})u_{\lambda,\kappa} \geqslant \frac{\theta-2}{2\theta}\|u_{\lambda,\kappa}\|^2$$

and the proof is finished. □

The next Lemma plays a crucial role in our arguments, since it establishes an important estimate involving the $L^\infty-$norm of the solution of the auxiliary Problem (B_κ). For this purpose, we shall use Moser's iteration method.

Lemma 3.2.14. *Let $u_{\lambda,k}$ be the solution of (B_κ) obtained in Lemma 3.2.12. Then, there exist $C_1 > 0$ and $M_1 > 0$ (which depends only on N, θ, μ, α, p, m and independent of ℓ, $\|\lambda\|_{\frac{2N}{2\alpha+\mu}}$, κ and R) such that if $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^*$, then*

$$\|u_{\lambda,\kappa}\|_\infty \leqslant (1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{C_1} M_1 \|u_{\lambda,k}\|_{2^*},$$

where ε_0^* was established in (3.57).

Proof. In what follows we will explore the ideas from the proof of Lemma 2.2.11 from Chapter 2. For the sake of simplicity, we denote $u = u_{\lambda,\kappa}$. For $L > 0$, we define $\phi_L = uu_L^{2(\beta-1)}$ and $w_L = uu_L^{(\beta-1)}$, where $u_L = \min\{u, L\}$. By taking $\phi_L = uu_L^{2(\beta-1)}$ as test function in (3.29), where $\beta > 1$ will be chosen later, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla u \nabla(uu_L^{2(\beta-1)})\,\mathrm{d}x &+ \int_{\mathbb{R}^N} V(x)|u|^2 u_L^{2(\beta-1)}\,\mathrm{d}x \\ &- \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right) g(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x \\ &- \int_{\mathbb{R}^N} h_\kappa(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x = 0,\end{aligned}$$

which implies

$$\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x = -2(\beta-1)\int_{\mathbb{R}^N} u_L^{2(\beta-1)-1}u\nabla u\nabla u_L\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right)g(x,u)uu_L^{2(\beta-1)}\mathrm{d}x + \int_{\mathbb{R}^N} h_\kappa(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x - \int_{\mathbb{R}^N} V(x)|u|^2u_L^{2(\beta-1)}\,\mathrm{d}x. \tag{3.83}$$

Since

$$2(\beta-1)\int_{\mathbb{R}^N} u_L^{2(\beta-1)-1}u\nabla u\nabla u_L\,\mathrm{d}x = 2(\beta-1)\int_{\{u\leqslant L\}} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x \geqslant 0, \tag{3.84}$$

it follows from (3.83) that

$$\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right)g(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x + \int_{\mathbb{R}^N} h_\kappa(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x. \tag{3.85}$$

Note that

$$\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant S^{-1}\int_{\mathbb{R}^N}|\nabla w_L|^2\,\mathrm{d}x = S^{-1}\int_{\mathbb{R}^N}\nabla(uu_L^{(\beta-1)})\,\mathrm{d}x.$$

Thus,

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} &\leqslant S^{-1}\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x + S^{-1}(\beta-1)^2\int_{\mathbb{R}^N}|u|^2u_L^{2(\beta-2)}|\nabla u_L|^2\,\mathrm{d}x\\ &\leqslant S^{-1}\beta^2\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x + S^{-1}\beta^2\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x\\ &= 2S^{-1}\beta^2\int_{\mathbb{R}^N} u_L^{2(\beta-1)}|\nabla u|^2\,\mathrm{d}x,\end{aligned} \tag{3.86}$$

where we have used $\nabla u_L = 0$ in $\{u > L\}$, $u = u_L$ in $\{u \leqslant L\}$ and $\beta > 1$. According to (3.85) and (3.86), we obtain

$$\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2S^{-1}\beta^2\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right)g(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x \tag{3.87}$$

$$+ 2S^{-1}\beta^2\int_{\mathbb{R}^N} h_\kappa(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x. \tag{3.88}$$

Next, we estimate the right-hand side of (3.87). By combining (3.25), Proposition 1.0.2 and Hölder's inequality with Lemma 3.2.13, we deduce

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right)g(x,u)uu_L^{2(\beta-1)}\mathrm{d}x \leqslant \tilde{C}_2\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}, \tag{3.89}$$

where $\tilde{C}_2 := C(N,\theta,\alpha,\mu,c_0)(SM)^{\frac{2^*_{\alpha,\mu}}{2}}$, which jointly with (3.87) yields that

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant& \tilde{C}_2 2S^{-1}\beta^2\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+\tilde{C}_2 2S^{-1}\beta^2\int_{\mathbb{R}^N}h_\kappa(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x\\ =:&\tilde{C}_2 2S^{-1}\beta^2\left(I_1+I_2\right).\end{aligned} \tag{3.90}$$

In the sequence, we estimate I_1 and I_2. Recalling that

$$(a+b)^{p_1}<a^{p_1}+b^{p_1},\quad \forall a,b>0,\ p_1\in(0,1), \tag{3.91}$$

for any $K>0$, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} <& \left(\int_{\{|u|\leqslant K\}}(|u|^{2^*_{\alpha,\mu}-2}|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+\left(\int_{\{|u|>K\}}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ =:&I_1^1+I_1^2.\end{aligned} \tag{3.92}$$

Note that

$$I_1^1\leqslant K^{2^*_{\alpha,\mu}-2}\left(\int_{\{|u|\leqslant K\}}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \tag{3.93}$$

and using Hölder's inequality with exponents $\frac{2^*_{\alpha,\mu}}{2^*_{\alpha,\mu}-2}$ and $\frac{2^*_{\alpha,\mu}}{2}$ one deduce

$$I_1^2\leqslant\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}. \tag{3.94}$$

Now, we estimate the right-hand side of (3.88). By (h_1), we get

$$I_2=\int_{\mathbb{R}^N}h_\kappa(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x\leqslant k^{q-2^*_{\alpha,\mu}}\int_{\mathbb{R}^N}\lambda(x)|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)}\,\mathrm{d}x. \tag{3.95}$$

Since $\lambda\in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$, it follows from Hölder's inequality

$$I_2\leqslant\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}\left(\int_{\mathbb{R}^N}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}. \tag{3.96}$$

For any $K>0$, we obtain

$$\begin{aligned}I_2\leqslant&\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}\left(\int_{\{|u|>K\}}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \qquad (3.97)\\ &+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}\left(\int_{\{|u|\leqslant K\}}(|u|^{2^*_{\alpha,\mu}}u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ =:&\,I_2^1+I_2^2.\end{aligned} \tag{3.98}$$

Arguing as in (3.93) and (3.94), we reach

$$I_2^1 \leqslant \kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}} K^{2^*_{\alpha,\mu}-2}\left(\int_{\{|u|\leqslant K\}}(|u|^2 u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \tag{3.99}$$

and

$$I_2^2 \leqslant \kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}}\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}. \tag{3.100}$$

Denote $C := 2S\tilde{C}_2$ in (3.90). In view of (3.90)–(3.100) we derive

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant & C\beta^2(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})\left(\int_{\mathbb{R}^N}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\ &+C\beta^2(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}.\end{aligned} \tag{3.101}$$

Since $u \in L^{2^*}(\mathbb{R}^N)$, we may fix $K > 0$ such that

$$\left(\int_{\{|u|>K\}}|u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \leqslant \frac{1}{2C\beta^2(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})}. \tag{3.102}$$

Taking into account (3.101) and (3.102) we get

$$\left(\int_{\mathbb{R}^N}|w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2C\beta^2(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})\left(\int_{\mathbb{R}^N}(|u|^2u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}. \tag{3.103}$$

Claim. $u \in L^{2^*\beta}(\mathbb{R}^N)$, for $\beta = \frac{2^*_{\alpha,\mu}}{2}$.

In fact, since $u_L \leqslant |u|$ and recalling $w_L = uu_L^{(\beta-1)}$, it follows from (3.103) that

$$\left(\int_{\mathbb{R}^N}(|u|u_L^{\frac{2^*_{\alpha,\mu}-2}{2}})^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2C\beta^2(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})\left(\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} < \infty. \tag{3.104}$$

By taking the limit as $L \to \infty$ we conclude that

$$\int_{\mathbb{R}^N}|u|^{2^*\frac{2^*_{\alpha,\mu}}{2}}\mathrm{d}x < \infty,$$

which proves the claim.

Now, using that $u_L \leqslant |u|$ and passing to the limit as $L \to \infty$ in (3.103), we obtain

$$\begin{aligned}\|u\|^{2\beta}_{2^*\beta} \leqslant & C\beta^2(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})\left(\int_{\mathbb{R}^N}|u|^{2\beta\frac{2^*}{2^*_{\alpha,\mu}}}\,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}}{2^*}\frac{2\beta}{2\beta}}\\ =& C\beta^2(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})\|u\|^{2\beta}_{q^*_{\alpha,\mu}\beta},\end{aligned} \tag{3.105}$$

or equivalently,

$$\|u\|_{2^*\beta} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\beta}}\beta^{\frac{1}{\beta}}\|u\|_{q^*_{\alpha,\mu}\beta}, \tag{3.106}$$

where $q^*_{\alpha,\mu} := \frac{22^*}{2^*_{\alpha,\mu}}$.

The next step is using inequality (3.106) to obtain the desired L^∞–estimate, through an iterative process. For this purpose, following a similar approach as in steps 1, 2, and 3 of Lemma 2.2.11 in (2.68)-(2.70), we follow three steps.

First step. If $\beta = \gamma_1 := \frac{2^*}{q^*_{\alpha,\mu}}$, then (3.106) becomes

$$\|u\|_{2^*\gamma_1} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\gamma_1}}\gamma_1^{\frac{1}{\gamma_1}}\|u\|_{2^*}, \quad 2^*\gamma_1 = \gamma_1^2 q^*_{\alpha,\mu}, \tag{3.107}$$

that is, $u^{\gamma_1^2} \in L^{q^*_{\alpha,\mu}}(\mathbb{R}^N)$.

Second step. If $\beta = \gamma_2 := \gamma_1^2$, then $q^*_{\alpha,\mu}\gamma_2 = q^*_{\alpha,\mu}\gamma_1^2 = 2^*\gamma_1$ and (3.106) becomes

$$\|u\|_{2^*\gamma_2} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\gamma_2}}\gamma_2^{\frac{1}{\gamma_2}}\|u\|_{q^*_{\alpha,\mu}\gamma_2},$$

i.e.,

$$\|u\|_{2^*\gamma_2} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\gamma_2}}(\gamma_2)^{\frac{1}{\gamma_2}}\|u\|_{2^*\gamma_1},$$

which jointly with (3.107) yields that

$$\|u\|_{2^*\gamma_2} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}}(\gamma_1)^{\frac{1}{\gamma_1}}(\gamma_2)^{\frac{1}{\gamma_2}}\|u\|_{2^*}. \tag{3.108}$$

Since $2^*\gamma_2 = 2^*\gamma_1^2 = 2^*\gamma_1\gamma_1 = q^*_{\alpha,\mu}\gamma_1^3$, it follows that $u^{\gamma_1^3} \in L^{q^*_{\alpha,\mu}}(\mathbb{R}^N)$.

Third step. If $\beta = \gamma_3 := \gamma_1^3$, then $q^*_{\alpha,\mu}\gamma_3 = q^*_{\alpha,\mu}\gamma_1^3 = 2^*\gamma_2$ and (3.106) becomes

$$\|u\|_{2^*\gamma_3} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\gamma_3}}\gamma_3^{\frac{1}{\gamma_3}}\|u\|_{q^*_{\alpha,\mu}\gamma_3},$$

i.e.,

$$\|u\|_{2^*\gamma_3} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\gamma_3}}(\gamma_3)^{\frac{1}{\gamma_3}}\|u\|_{2^*\gamma_2},$$

which jointly with (3.108) yields that

$$\|u\|_{2^*\gamma_3} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\frac{1}{\gamma_3}}(\gamma_1)^{\frac{1}{\gamma_1}}(\gamma_2)^{\frac{1}{\gamma_2}}(\gamma_3)^{\frac{1}{\gamma_3}}\|u\|_{2^*}.$$

Since $2^*\gamma_3 = 2^*\gamma_1^3 = (2^*\gamma_1\gamma_1)\gamma_1 = q^*_{\alpha,\mu}\gamma_1^3$, it follows that $u^{\gamma_1^4} \in L^{q^*_{\alpha,\mu}}(\mathbb{R}^N)$.

Inductively, if we consider $\beta = \gamma_m := \gamma_1^m$, then $q^*_{\alpha,\mu}\gamma_{m+1} = 2^*\gamma_m$ and (3.106) becomes

$$\|u\|_{2^*\gamma_m} \leqslant [C^{\frac{1}{2}}(1+\kappa^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{\frac{1}{2}}]^{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\cdots+\frac{1}{\gamma_m}}\gamma_1^{\frac{1}{\gamma_1}}\gamma_2^{\frac{1}{\gamma_2}}\cdots\gamma_m^{\frac{1}{\gamma_m}}\|u\|_{2^*}, \tag{3.109}$$

and we deduce that $u^{\gamma_1^{(m+1)}} \in L^{q^*_{\alpha,\mu}}(\mathbb{R}^N)$ for all $m \in \mathbb{N}$. Recalling that $\gamma_1 = \frac{2^*}{q^*_{\alpha,\mu}} > 1$, then

$$\begin{cases} \lim\limits_{m\to\infty} 2^*\gamma_m = 2^* \lim\limits_{m\to\infty} \gamma_1^m = 2^* \lim\limits_{m\to\infty} \beta^m = \infty, \\ \displaystyle\sum_{j=1}^{\infty} \frac{1}{\gamma_j} = \sum_{j=1}^{\infty} \left(\frac{1}{\gamma_1}\right)^j = \frac{1}{\gamma_1 - 1}, \\ \displaystyle\sum_{j=1}^{\infty} \frac{j}{\gamma_j} = \sum_{j=1}^{\infty} \frac{j}{\gamma_1^j} = \frac{\gamma_1}{(\gamma_1-1)^2}, \\ \gamma_1^{\frac{1}{\gamma_1}} \gamma_2^{\frac{1}{\gamma_2}} \cdots \gamma_m^{\frac{1}{\gamma_m}} = \gamma_1^{\frac{1}{\gamma_1}} \gamma_1^{\frac{2}{\gamma_1^2}} \cdots \gamma_1^{\frac{m}{\gamma_1^m}} \leqslant \gamma_1^{\sum_{j=1}^{\infty} \frac{j}{\gamma_1^j} = \frac{\gamma_1}{(\gamma_1-1)^2}}. \end{cases}$$

By taking the limit as $m \to \infty$ in (3.109), leads to

$$\|u\|_\infty \leqslant (1 + \kappa^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{C_1} M_1 \|u\|_{2^*}, \tag{3.110}$$

for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^*$, where $C_1 := \frac{1}{2(\gamma_1-1)}$, $M_1 := C^{\frac{1}{2(\gamma_1-1)}} \gamma_1^{\frac{\gamma_1}{(\gamma_1-1)^2}}$ and $u_{\lambda,k} := u$. This finishes the proof. □

At this point, in view of (3.110) and Lemma 3.2.13, we are able to find suitable values of $\|\lambda\|_{\frac{2N}{2\alpha+\mu}}$ and κ such that the following inequality holds true

$$\|u_{\lambda,k}\|_\infty \leqslant (1 + \kappa^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{C_1} M_1 (SM)^{\frac{1}{2}} \leqslant \kappa.$$

In fact, we shall verify that

$$(1 + \kappa^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}})^{C_1} M_1 (SM)^{\frac{1}{2}} \leqslant \kappa,$$

or equivalently,

$$\kappa^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \left(\frac{\kappa}{M_1 (SM)^{\frac{1}{2}}}\right)^{\frac{1}{C_1}} - 1.$$

Consider $\kappa > 0$ such that

$$\left(\frac{\kappa}{M_1 (SM)^{\frac{1}{2}}}\right)^{\frac{1}{C_1}} - 1 > 0$$

and fix $\varepsilon_0^{**} > 0$ satisfying

$$\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^{**} \leqslant \min\left\{\left[\left(\frac{\kappa}{M_1 (SM)^{\frac{1}{2}}}\right)^{\frac{1}{C_1}} - 1\right] \frac{1}{\kappa^{q-2^*_{\alpha,\mu}}}, \varepsilon_0^*\right\}.$$

Thus, taking $\kappa_0 > M_1 (SM)^{\frac{1}{2}}$, we obtain $\varepsilon_0^{**} > 0$, such that

$$\|u_{\lambda,k_0}\|_\infty \leqslant \kappa_0, \tag{3.111}$$

for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^{**}$. Therefore, by (3.111), it follows from definition of h_{κ_0} that

$$h_{\kappa_0}(u_{\lambda,\kappa_0}) = \lambda(x) |u_{\lambda,\kappa_0}|^{q-2} u_{\lambda,\kappa_0}. \tag{3.112}$$

Remark 3.2.15. *Let $u \in E$ be the nonnegative solution obtained in Lemma 3.2.12. In view of Lemma 3.2.14 and regularity theory (see for instance (TOLKSDORF, 1984, Theorem 1)), we have that $u \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$. Therefore, in light of Strong Maximum Principle, we conclude that u is positive.*

***Proof of Theorem 3.1.2**.* In light of Lemma 3.2.12, for each $\kappa > 0$, there exists $\varepsilon_0^* := \varepsilon_0^*(\kappa) > 0$ such that for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0^{**}$, the auxiliary Problem (B_κ) admits a solution $u_{\lambda,\kappa}$ in E. Thereby, in order to prove the existence of solution for the original Problem (P), given that (3.111) holds true, it is sufficient to prove that there exist $R > 1$ and $\varepsilon_0 \in (0, \varepsilon_0^{**}]$ such that following inequality holds:

$$f(u_{\lambda,\kappa_0}) \leqslant \frac{V(x)}{\ell_0} u_{\lambda,\kappa_0}, \quad \forall\, |x| \geqslant R \text{ and } \forall\, \lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$$

with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0$.

Lemma 3.2.16. *For each $R > 1$, let u_{λ,κ_0} be a solution for the auxiliary Problem (B_{κ_0}), such that $\mathcal{J}_{\kappa_0}(u_{\lambda,\kappa_0}) = c_{\kappa_0}$. Then, there exists $\varepsilon_0 \in (0, \varepsilon_0^{**}]$ such that*

$$u_{\lambda,\kappa_0} \leqslant \frac{R^{N-2}}{|x|^{N-2}} \big\| u_{\lambda,\kappa_0} \big\|_\infty \leqslant \frac{R^{N-2}}{|x|^{N-2}} \kappa_0, \quad \forall\, |x| \geqslant R \ \text{ and } \ \forall\, \lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$$

with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0$.

Proof. For the sake of simplicity, we denote $u = u_{\lambda,\kappa_0}$. Let v be the $C^\infty(\mathbb{R}^N \backslash \{0\})$ function

$$v(x) = \frac{R^{N-2} \big\| u \big\|_\infty}{|x|^{N-2}}, \quad x \neq 0.$$

Since $1/|x|^{N-2}$ is harmonic, it follows that $\Delta v(x) = 0$ in $\mathbb{R}^N \backslash \{0\}$. Note that

$$u(x) \leqslant \big\| u \big\|_\infty \leqslant \frac{R^{N-2}}{|x|^{N-2}} \big\| u \big\|_\infty, \quad \forall\, |x| \leqslant R.$$

Let us introduce the function $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ defined by

$$w(x) = \begin{cases} (u - v)^+(x), & \text{if } |x| \geqslant R, \\ 0, & \text{if } |x| \leqslant R. \end{cases} \tag{3.113}$$

By using w as test function we obtain

$$\int_{\mathbb{R}^N} (\nabla u \nabla w + V(x) u w)\, \mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu}\, \mathrm{d}y \right) \frac{g(x,u)}{|x|^\alpha} w\, \mathrm{d}x + \int_{\mathbb{R}^N} h_{\tilde{\lambda},\kappa_0}(x,u) w\, \mathrm{d}x. \tag{3.114}$$

Now, by the definition of w and according to (3.22) and Lemma 3.2.4, it follows from $R^\alpha \geqslant 1$ the following estimate

$$\begin{aligned}\int_{B_R^c(0)} \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |y-x|^\mu} \,\mathrm{d}y\right) \frac{g(x,u)}{|x|^\alpha} w \,\mathrm{d}x - \int_{B_R^c(0)} V(x) uw \,\mathrm{d}x \\ \leqslant \frac{1}{R^\alpha} \int_{B_R^c(0)} \frac{\mathcal{K}(u)(x)}{\ell_0} V(x) uw \,\mathrm{d}x - \int_{B_R^c(0)} V(x) uw \,\mathrm{d}x \\ \leqslant \int_{B_R^c(0)} \left(\frac{\mathcal{K}(u)(x)}{\ell_0} - 1\right) V(x) uw \,\mathrm{d}x \leqslant 0. \end{aligned} \tag{3.115}$$

On the other hand, using the definition of w again, we obtain

$$\int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} \nabla u \nabla w \,\mathrm{d}x - \int_{\mathbb{R}^N} \nabla v \nabla w \,\mathrm{d}x.$$

Since $\Delta v = 0$ in $B_R^c(0)$ and $w = 0$ in $\partial B_R(0)$, there holds

$$\int_{\mathbb{R}^N} \nabla v \nabla w \,\mathrm{d}x = 0.$$

Moreover, by (h_1), we have $|h_{\kappa_0}(x,u)| \leqslant \lambda(x) \kappa_0^{q-2^*_{\alpha,\mu}} |u|^{2^*-2} u$. Using Hölder's inequality twice, we get the following estimate

$$\begin{aligned}\int_{\mathbb{R}^N} h_{\kappa_0}(x,u) w \,\mathrm{d}x \leqslant& \kappa_0^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} \left(\int_{\mathbb{R}^N} (|u|^{2^*_{\alpha,\mu}-2} |u| w)^{\frac{2N}{2N-2\alpha-\mu}} \,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}} \\ \leqslant& \kappa_0^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} \left(\int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \left(\int_{\mathbb{R}^N} |uw|^{\frac{2^*}{2}} \,\mathrm{d}x\right)^{\frac{2}{2^*}} \\ =:& I. \end{aligned} \tag{3.116}$$

Thanks to Young's inequality $2a_1b_1 \leqslant a_1^2 + a_2^2$ and invoking Sobolev embedding together with Lemma 3.2.13, it follows from (3.91) that

$$\begin{aligned} I \leqslant& \kappa_0^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} \left(\int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \frac{1}{2} \left(\int_{\mathbb{R}^N} |u|^{2^*} \,\mathrm{d}x + \int_{\mathbb{R}^N} |w|^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} \\ \leqslant& \frac{1}{2} \kappa_0^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} S^{\frac{2^*_{\alpha,\mu}-2}{2}} M^{\frac{2^*_{\alpha,\mu}-2}{2}} \left(S^{\frac{2^*}{2}} M^{\frac{2^*}{2}} + S^{\frac{2^*}{2}} \|\nabla w\|_2^{\frac{2^*}{2}}\right)^{\frac{2}{2^*}} \\ \leqslant& \frac{1}{2} \kappa_0^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} S^{\frac{2^*_{\alpha,\mu}-2}{2}} M^{\frac{2^*_{\alpha,\mu}-2}{2}} \left(SM + S\|\nabla w\|_2^2\right). \end{aligned} \tag{3.117}$$

Combining (3.114)-(3.117), we see that

$$\begin{aligned}\|\nabla w\|_2^2 \leqslant& \int_{B_R^c(0)} \left(\frac{\mathcal{K}(u)(x)}{\ell_0} - 1\right) V(x) uw \,\mathrm{d}x \\ &+ \frac{1}{2} \kappa_0^{q-2^*_{\alpha,\mu}} \|\lambda\|_{\frac{2N}{2\alpha+\mu}} S^{\frac{2^*_{\alpha,\mu}-2}{2}} M^{\frac{2^*_{\alpha,\mu}-2}{2}} \left(SM + S\|\nabla w\|_2^2\right), \end{aligned}$$

whence it follows that

$$\|\nabla w\|_2^2 - \frac{1}{2}\kappa_0^{q-2^*_{\alpha,\mu}}\|\lambda\|_{\frac{2N}{2\alpha+\mu}} S^{\frac{2^*_{\alpha,\mu}-2}{2}} M^{\frac{2^*_{\alpha,\mu}-2}{2}} \left(SM + S\|\nabla w\|_2^2\right) \leqslant \int_{B_R^c(0)} \left(\frac{\mathcal{K}(u)(x)}{\ell_0} - 1\right) V(x) uw \,\mathrm{d}x \leqslant 0. \tag{3.118}$$

Now, let $\varepsilon_0 > 0$ be such that

$$0 < \varepsilon_0 \leqslant \min\left\{\varepsilon_0^{**}, \frac{2}{\kappa_0^{q-2^*_{\alpha,\mu}} S^{\frac{2^*-2}{2}} M^{\frac{2^*-2}{2}}}\right\}. \tag{3.119}$$

We claim that $\|\nabla w\|_2 = 0$. In fact, suppose by contradiction that $\|\nabla w\|_2 \neq 0$, for some $\lambda_0 \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ with $\|\lambda_0\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0$. Thus, we may rewrite (3.118) as follows

$$\|\nabla w\|_2^2 \left[1 - \frac{1}{2}\|\lambda_0\|_{\frac{2N}{2\alpha+\mu}} \kappa_0^{q-2^*_{\alpha,\mu}} S^{\frac{2^*_{\alpha,\mu}-2}{2}} M^{\frac{2^*_{\alpha,\mu}-2}{2}} \left(\frac{SM}{\|\nabla w\|_2^2} + S\right)\right] \leqslant \int_{B_R^c(0)} \left(\frac{\mathcal{K}(u)(x)}{\ell_0} - 1\right) V(x) uw \,\mathrm{d}x < 0. \tag{3.120}$$

In view of (3.119) there holds

$$1 - \frac{1}{2}\|\lambda_0\|_{\frac{2N}{2\alpha+\mu}} \kappa_0^{q-2^*_{\alpha,\mu}} S^{\frac{2^*_{\alpha,\mu}-2}{2}} M^{\frac{2^*_{\alpha,\mu}-2}{2}} \left(\frac{SM}{\|\nabla w\|_2^2} + S\right) > 0,$$

which jointly with (3.120) leads to a contradiction. Showing that $w \equiv 0$. Recalling the definition of the w function in (3.113), we obtain $|u| \leqslant v$ in $|x| \geqslant R$. This joined with (3.111), implies that

$$u_{\lambda,\kappa_0} \leqslant \frac{R^{N-2}}{|x|^{N-2}}\left\|u_{\lambda,\kappa_0}\right\|_\infty \leqslant \frac{R^{N-2}}{|x|^{N-2}}\kappa_0, \quad \forall\, |x| \geqslant R \ \text{ and } \ \forall\, \lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N) \tag{3.121}$$

with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0$. This ends the proof of lemma. □

We note that (3.121) ensures

$$u_{\lambda,\kappa_0}^{q-2} \leqslant \frac{R^{(N-2)(q-2)}}{|x|^{(N-2)(q-2)}}\left\|u_{\lambda,\kappa_0}\right\|_\infty^{q-2} \leqslant \frac{R^{(N-2)(q-2)}}{|x|^{(N-2)(q-2)}}\kappa_0^{q-2}, \quad \forall\, |x| \geqslant R. \tag{3.122}$$

Furthermore, by fixing κ_0 as in (3.111), there holds

$$\|u_{\lambda,\kappa_0}\| \leqslant \kappa_0, \quad \forall\, \lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N) \tag{3.123}$$

with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0$. By means of (3.2), (3.122) and (3.123), we obtain

$$\begin{aligned} f(u_{\lambda,\kappa_0}) &\leqslant c_0 |u_{\lambda,\kappa_0}|^{q-2} u_{\tilde{\lambda},\kappa_0} \\ &\leqslant c_0 \frac{R^{(N-2)(q-2)}}{|x|^{(N-2)(q-2)}}\left\|u_{\lambda,\kappa_0}\right\|_\infty^{q-2} u_{\lambda,\kappa_0} \\ &\leqslant c_0 \kappa_0^{q-2} \frac{R^{(N-2)(q-2)}}{|x|^{(N-2)(q-2)}} u_{\lambda,\kappa_0}, \quad \forall\, |x| \geqslant R. \end{aligned} \tag{3.124}$$

Now, by fixing $R>1$, it follows from (V_1) that

$$\frac{R^{(N-2)(q-2)}}{|x|^{(N-2)(q-2)}} \leqslant \frac{V(x)}{\Lambda(R)}, \quad \forall\, |x| \geqslant R,$$

which jointly with (3.124) implies that

$$f(u_{\lambda,\kappa_0}) \leqslant \ell_0 c_0 \kappa_0^{q-2} \frac{V(x)}{\Lambda(R)\ell_0} u_{\lambda,\kappa_0}, \quad \forall\, |x| \geqslant R. \tag{3.125}$$

Thus, if $\Lambda(R) \geqslant \Lambda_0 := c_0 \ell_0 \kappa_0^{q-2}$, then we conclude that

$$f(u_{\lambda,\kappa_0}) \leqslant \frac{V(x)}{\ell_0} u_{\lambda,\kappa_0}, \, \forall\, |x| \geqslant R \quad \text{and} \quad \forall\, \lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N) \tag{3.126}$$

with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \varepsilon_0$. Consequently, by (3.13), (3.18), (3.19), (3.123), (3.126) and since u_{λ,κ_0} is a critical point of $\mathcal{J}_{\kappa_0}$, we reach

$$\begin{aligned}
0 = \mathcal{J}'_{\kappa_0}(u_{\lambda,\kappa_0})\phi &= \int_{\mathbb{R}^N} \nabla u_{\lambda,\kappa_0} \nabla \phi \, \mathrm{d}x + \int_{\mathbb{R}^N} V(x) u_{\lambda,\kappa_0} \phi \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(y, u_{\lambda,\kappa_0})}{|y|^\alpha |x-y|^\mu} \mathrm{d}y \right) \frac{g(x, u_{\lambda,\kappa_0})}{|x|^\alpha} \phi \, \mathrm{d}x - \int_{\mathbb{R}^N} h_{\kappa_0}(x, u_{\lambda,\kappa_0}) \phi \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \nabla u_{\lambda,\kappa_0} \nabla \phi \, \mathrm{d}x + \int_{\mathbb{R}^N} V(x) u_{\lambda,\kappa_0} \phi \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_{\lambda,\kappa_0})}{|y|^\alpha |x-y|^\mu} \, \mathrm{d}y \right) \frac{f(u_{\lambda,\kappa_0})}{|x|^\alpha} \phi \, \mathrm{d}x - \int_{\mathbb{R}^N} \lambda(x) |u_{\lambda,\kappa_0}|^{q-1} \phi \, \mathrm{d}x,
\end{aligned}$$

for all $\phi \in E$. Therefore, we conclude that u_{λ,κ_0} is a solution of Problem (P), which finishes the proof of Theorem 3.1.2. □

Proof of the Corollary 3.1.3. Now, by applying the same ideas from the previous sections, we reconsider Problems (A_κ)-(B_κ) with $\alpha = 0$, $\lambda(x) \equiv \lambda$, and $q \geqslant 2^*$. Additionally, we assume that μ satisfies $0 < \mu < \frac{N+2}{2}$, and in the truncation (3.13), we replace $2^*_{\alpha,\mu}$ with 2^* and apply the growth condition given in (h_1), $|h_\kappa(x,t)| \leqslant \lambda \kappa^{q-2^*} t^{2^*-1}$. We emphasize that the functional $\mathcal{J}_\kappa$ still verifies, with natural modifications, Lemmas 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.6, 3.2.7, 3.2.10, 3.2.11 and 3.2.12. Thus, arguing as Lemma 3.2.13, we obtain that there exists $C_* > 0$ such that $u_{\lambda,\kappa}$ is solution of auxiliary Problem (B_κ) i.e., $u_{\lambda,\kappa}$ is a solution of

$$-\Delta u + V(x) u = \mathcal{K}(u) g(x,u) + h_\kappa(x,u), \quad \text{in } \mathbb{R}^N,$$

satisfying $\|u_{\lambda,\kappa}\|_{2^*} \leqslant C_*$, where $\mathcal{K}(u) := \displaystyle\int_{\mathbb{R}^N} \frac{G(y,u)}{|x-y|^\mu} \, \mathrm{d}y$. Moreover, by Lemma 3.2.4, there exists C_1 such that $\|\mathcal{K}(u_{\lambda,\kappa})\|_\infty \leqslant C_1$. Now, arguing as Lemma 3.2.14, we will show that $u_{\lambda,\kappa}$ belongs to $L^\infty(\mathbb{R}^N)$ and we will obtain a convenient $L^\infty-$estimate for $u_{\lambda,\kappa}$. In fact, in view of $\|\mathcal{K}(u_{\lambda,\kappa})\|_\infty \leqslant C_1$, $|g(x,u_{\lambda,\kappa})| \leqslant c_0|u_{\lambda,\kappa}|^{2^*-1}$, $|h_\kappa(x,u_{\lambda,\kappa})| \leqslant \lambda\kappa^{q-2^*}u_{\lambda,\kappa}^{2^*-1}$, from of (3.87)-(3.88) in Lemma 3.2.14 with $\alpha = 0$, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant & 2S^{-1}\beta^2\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(y,u)}{|x-y|^\mu}\mathrm{d}y\right)g(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x\\ & + 2S^{-1}\beta^2\int_{\mathbb{R}^N} h_\kappa(x,u)uu_L^{2(\beta-1)}\,\mathrm{d}x\\ \leqslant & 2S^{-1}\beta^2\left(C_1c_0\int_{\mathbb{R}^N}|u|^{2^*}u_L^{2(\beta-1)}\,\mathrm{d}x + \lambda\kappa^{q-2^*}\int_{\mathbb{R}^N}|u|^{2^*}u_L^{2(\beta-1)}\,\mathrm{d}x\right)\\ \leqslant & C\beta^2\left(1+\lambda\kappa^{q-2^*}\right)\int_{\mathbb{R}^N}|u|^{2^*}u_L^{2(\beta-1)}\,\mathrm{d}x.\end{aligned}$$

Adapting the proof in Lemma 3.2.14, we obtain constants $\hat{C}_1$ and $\hat{M}_1$ such that

$$\|u_{\lambda,\kappa}\|_\infty \leqslant \left(1+\lambda\kappa^{q-2^*}\right)^{\hat{C}_1}\hat{M}_1, \tag{3.127}$$

and fixing $\kappa_0 > \hat{M}_1$, we can obtain $\lambda_0 > 0$ such that $\|u_{\lambda,\kappa_0}\|_\infty \leqslant \kappa$, for all $\lambda \in [0,\lambda_0)$. Therefore, it follows from definition of h_{κ_0} that

$$h_{\kappa_0}(u_{\lambda,\kappa_0}) = \lambda|u_{\lambda,\kappa_0}|^{q-2}u_{\lambda,\kappa_0}.$$

Finally, the proof of Theorem 3.1.3 follows through adaptations of the arguments in Subsection 2.3. □

3.3 THE SUPERCRITICAL NONLOCAL PERTURBATION

In this section, we aim to prove Theorem 3.1.4. Thus, we will study the existence of solutions for the following equation involving Stein-Weiss term in $\mathbb{R}^N$ with supercritical nonlocal term

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\alpha} + \frac{\lambda}{q}\left(\int_{\mathbb{R}^N}\frac{|u|^q}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u|^{q-2}u}{|x|^\alpha}, \tag{$\tilde{P}_\lambda$}$$

where $2^*_{\alpha,\mu} \leqslant q$, $\lambda \geqslant 0$ is a parameter and f satisfies $(f_1)-(f_3)$.

To avoid the repetition, we will only mention the results which are potentially different from the case considered in Theorem 3.1.2. We start by noting that for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, it follows by Proposition 1.0.2 that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^q|u(x)|^q}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\mathrm{d}x < \infty, \quad \text{if} \quad q = 2^*_{\alpha,\mu},$$

i.e., we cannot apply variational methods directly because the functional $\tilde{\mathcal{F}}: E \to \mathbb{R}$, associated with problem $(\tilde{P}_\lambda)$

$$\begin{aligned}\tilde{\mathcal{F}}(u) =& \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\alpha}\,\mathrm{d}x\\ &- \frac{\lambda}{2q}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^q}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{|u(x)|^q}{|x|^\alpha}\,\mathrm{d}x,\end{aligned}$$

is not well defined on E unless $q = 2^*_{\alpha,\mu}$.

3.3.1 The auxiliary Problems $(\tilde{A}_{\lambda,\kappa})$ and $(\tilde{B}_{\lambda,\kappa})$

In order to apply variational methods, we will need to consider the following truncation in the nonlocal term, namely, given $\kappa \in \mathbb{N}$, we define the function $h_\kappa : \mathbb{R} \to \mathbb{R}$ by

$$h_\kappa(t) = \begin{cases} 0, & \text{if } t \leqslant 0,\\ t^{q-1}, & \text{if } 0 \leqslant t \leqslant \kappa,\\ \kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}-1}, & \text{if } t \geqslant \kappa. \end{cases} \tag{3.128}$$

It is not hard to check that h_κ admits the following properties:

$$|h_\kappa(t)| \leqslant \kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}-1}, \quad \forall\, t \geqslant 0, \tag{h_1'}$$

Moreover, denoting $H_\kappa(t) = \int_0^t h_\kappa(\tau)\,\mathrm{d}\tau$, there holds

$$H_\kappa(t) = \begin{cases} 0, & \text{if } t \leqslant 0,\\ \dfrac{t^q}{q}, & \text{if } 0 \leqslant t \leqslant \kappa,\\ \frac{1}{2^*_{\alpha,\mu}}\kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}} + \left(\dfrac{1}{q} - \dfrac{1}{2^*_{\alpha,\mu}}\right)\kappa^q, & \text{if } t \geqslant \kappa. \end{cases} \tag{3.129}$$

and

$$|H_\kappa(t)| \leqslant \frac{1}{2^*_{\alpha,\mu}}\kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}},\ \forall\, t \geqslant 0. \tag{H_1'}$$

By using (3.128), (3.129) and the fact that $q \geqslant 2^*_{\alpha,\mu} > \theta$, one may check that

$$\begin{aligned} h_\kappa(t)t - \theta H_\kappa(t) =& \begin{cases} 0, & \text{if } t \leqslant 0,\\ t^q\left(\dfrac{q-\theta}{q}\right), & \text{if } 0 \leqslant t \leqslant \kappa,\\ \kappa^{q-2^*_{\alpha,\mu}} t^{2^*_{\alpha,\mu}}\left(1 - \frac{\theta}{2^*_{\alpha,\mu}}\right) + \theta\kappa^q\left(\dfrac{1}{2^*_{\alpha,\mu}} - \dfrac{1}{q}\right), & \text{if } t \geqslant \kappa, \end{cases}\\ \geqslant& 0, \end{aligned} \tag{3.130}$$

which ensures the following Ambrosetti-Rabinowitz type condition

$$h_\kappa(t)t - \theta H_\kappa(t) \geqslant 0, \quad \forall\, t \in \mathbb{R}. \tag{3.131}$$

Thus, we have the following auxiliary problem in $\mathbb{R}^N$

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\alpha} + \lambda\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u)}{|x|^\alpha}. \tag{$\tilde{A}_{\lambda,\kappa}$}$$

The energy functional $\tilde{\mathcal{I}}_{\lambda,\kappa} : E \to \mathbb{R}$ associated with Problem $(\tilde{A}_{\lambda,\kappa})$ is given by

$$\begin{aligned}\tilde{\mathcal{I}}_{\lambda,\kappa}(u) =& \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\alpha}\,\mathrm{d}x \\ &- \frac{\lambda}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{H_\kappa(u)}{|x|^\alpha}\,\mathrm{d}x\end{aligned}$$

and in view of assumption (H_1'), $\tilde{\mathcal{I}}_{\lambda,\kappa}$ is well defined. It is evident that if u is a solution of Problem $(\tilde{A}_{\lambda,\kappa})$ and satisfies the estimate $|u(x)| \leqslant \kappa$ for all $x \in \mathbb{R}^N$, then u is a solution of the original problem $(\tilde{P}_\lambda)$.

We also need to introduce truncation (3.18), explored in Subsection 3.2.1. Hence, we introduce another auxiliary problem in $\mathbb{R}^N$, namely,

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha} + \lambda\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u)}{|x|^\alpha}. \tag{$\tilde{B}_{\lambda,\kappa}$}$$

Thus, we say that a function $u \in E$ is a weak solution of the auxiliary Problem $(\tilde{B}_{\lambda,\kappa})$, if satisfies

$$\begin{aligned}\int_{\mathbb{R}^N}(\nabla u\nabla\phi + V(x)u\phi)\,\mathrm{d}x &- \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha}\phi\,\mathrm{d}x \\ &- \lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{H_\kappa(u)}{|x|^\alpha}\phi\,\mathrm{d}x = 0,\end{aligned} \tag{3.132}$$

for all $\phi \in E$. The energy functional associated with Problem $(\tilde{B}_{\lambda,\kappa})$ is given by

$$\begin{aligned}\tilde{\mathcal{J}}_{\lambda,\kappa}(u) =& \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{G(x,u)}{|x|^\alpha}\,\mathrm{d}x \\ &- \frac{\lambda}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{H_\kappa(u)}{|x|^\alpha}\,\mathrm{d}x.\end{aligned}$$

In view of our assumptions one may conclude that $\tilde{\mathcal{J}}_{\lambda,\kappa}$ is well defined, belongs to $C^1(E,\mathbb{R})$ and its derivative given by

$$\begin{aligned}\tilde{\mathcal{J}}_{\lambda,\kappa}'(u)\phi =& \int_{\mathbb{R}^N}(\nabla u\nabla\phi + V(x)u\phi)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha}\phi\,\mathrm{d}x \\ &- \lambda\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u)}{|x|^\alpha}\phi\,\mathrm{d}x.\end{aligned}$$

Thus, weak solutions of $(\tilde{B}_{\lambda,\kappa})$ are precisely the critical points of $\tilde{\mathcal{J}}_{\lambda,\kappa}$.

It is also worth mentioning here that the auxiliary Problem $(\tilde{B}_{\lambda,\kappa})$ is strongly related to Problem $(\tilde{A}_{\lambda,\kappa})$. In fact, if u is a solution of $(\tilde{B}_{\lambda,\kappa})$ which verifies $\ell f(u(x)) \leqslant V(x)u(x)$ for all $|x| \geqslant R$, then $g(x,u) = f(u)$ and u is also a solution for Problem $(\tilde{A}_{\lambda,\kappa})$, which motivates us to study the auxiliary Problem $(\tilde{B}_{\lambda,\kappa})$. The crucial role here is that working on Problem $(\tilde{B}_{\lambda,\kappa})$ we are able to restore some compactness.

3.3.2 Existence of solutions for the auxiliary Problem $(\tilde{B}_{\lambda,\kappa})$

Next, using the same arguments explored in proof of Lemma 3.2.1, we can show that $\tilde{\mathcal{J}}_{\lambda,\kappa}$ verifies the mountain pass geometry. Consequently, there is a $(PS)_{\tilde{c}_{\lambda,\kappa}}$−sequence $(u_n)_n \subset E$ such that

$$\tilde{\mathcal{J}}_{\lambda,\kappa}(u_n) \to \tilde{c}_{\lambda,\kappa} \quad \text{and} \quad \tilde{\mathcal{J}}'_{\lambda,\kappa}(u_n) \to 0,$$

where $\tilde{c}_{\lambda,\kappa}$ is the mountain pass level characterized by

$$0 < \tilde{c}_{\lambda,\kappa} := \inf_{\gamma\in\tilde{\Gamma}} \max_{t\in[0,1]} \tilde{\mathcal{J}}_{\lambda,\kappa}(\gamma(t)) \tag{3.133}$$

where

$$\tilde{\Gamma} := \Big\{\gamma \in C([0,1], E) : \gamma(0) = 0 \ \text{ and } \ \tilde{\mathcal{J}}_{\lambda,\kappa}(\gamma(1)) < 0\Big\}.$$

Now, we introduce the functional $\tilde{I}_0 : H_0^1(B_1(0)) \to \mathbb{R}$ given by

$$\tilde{\mathcal{I}}_0(u) = \frac{1}{2}\int_{B_1(0)} |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{B_1(0)} m|u|^2 \,\mathrm{d}x - \frac{1}{2}\int_{B_1(0)}\left(\int_{B_1(0)} \frac{F(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\alpha}\,\mathrm{d}x,$$

where $m = \max_{|x|\leqslant 1} V(x)$. Denote by $\tilde{d}$ the level of the mountain pass value defined by

$$\tilde{d} := \inf_{u\in H_0^1(B_1(0))} \max_{t\geqslant 0} \mathcal{I}_0(tu) \geqslant \tilde{c}_{\lambda,\kappa}.$$

Analogously to Lemma 3.2.2, we have the following result.

Lemma 3.3.1. *Assume that conditions* (f_3) *and* (3.130) *hold. Then,*

$$\frac{1}{\theta}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha}u\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(x,u)}{|x|^\alpha}\,\mathrm{d}x \geqslant 0 \tag{3.134}$$

and

$$\frac{1}{\theta}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u)}{|x|^\alpha}u\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{H_\kappa(u)}{|x|^\alpha}\,\mathrm{d}x \geqslant 0. \tag{3.135}$$

Proof. The inequality in (3.134) has already been proved in Lemma 3.2.2. Now, by (3.130), we have that (3.135) holds true. □

Arguing as in the proof Lemma 3.2.3, it follows that $(u_n)_n$ is bounded in E with

$$\|u_n\|^2 \leqslant \frac{2\theta}{\theta-2}(\tilde{d}+1) =: \tilde{C}, \quad \forall\, n \in \mathbb{N}$$

and by (3.4), we get the estimate

$$\left(\int_{\mathbb{R}^N} |u_n|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant S^{-1}\|\nabla u_n\|_2^2 \leqslant S^{-1}\tilde{C} =: \tilde{C}_*, \quad \forall\, n \in \mathbb{N}. \tag{3.136}$$

Moreover, the functional $\tilde{\mathcal{J}}_{\lambda,\kappa}$ still verifies, with natural modifications, Lemmas 3.2.6, 3.2.7, 3.2.10, 3.2.11, 3.2.14 and 3.2.16. Indeed, in Lemma 3.2.6, we replace (3.51) by

$$\lim_{n\to\infty}\int_{B_{\bar{R}}(0)}\left(\int_{\mathbb{R}^N}\frac{H_\kappa(u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u_n)}{|x|^\alpha}(u_n-u)\,\mathrm{d}x = 0.$$

This limit is proved by arguing as in the proof of (3.50) in Lemma 3.2.6, given that there exists $C>0$ such that

$$\left\|\int_{\mathbb{R}^N}\frac{H_\kappa(u_n(y))}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\right\|_{\frac{2N}{\mu+2\alpha}} \leqslant C, \quad \forall n \in \mathbb{N}. \tag{3.137}$$

Now, (3.80) of Lemma 3.2.10 we replace by

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{H_\kappa(u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u_n)}{|x|^\alpha}(u_n-u)\,\mathrm{d}x = 0$$

and (3.64) in Lemma 3.2.7 is true with

$$\mathcal{Q}_{3,\lambda,\kappa}(n,r) := \lambda\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{H_\kappa(u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u_n)}{|x|^\alpha}u_n\eta_r^2\,\mathrm{d}x\right|,$$

where we define η_r in (3.55).

To derive estimates analogous to (3.69), (3.76) and (3.77), we use the assumption (H_1') and (h_1') instead of (h_1), as follows, respectively. Utilizing (3.52) and applying Hölder's inequality, in combination with (3.4) and (3.136), we obtain the following

$$\begin{aligned}\mathcal{Q}_{3,\lambda,\kappa}(n,r) \leqslant & \lambda\left\| \int_{\mathbb{R}^N} \frac{H_\kappa(u_n)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\right\|_{\frac{2N}{\mu+2\alpha}} \left(\int_{\mathbb{R}^N} |h_\kappa(u_n)u_n\eta_r^2|^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\
\leqslant & C\kappa^{q-2^*_{\alpha,\mu}}\lambda\left(\int_{\mathbb{R}^N}(|u_n|^{2^*_{\alpha,\mu}-2}|u_n|^2\eta_r^2)^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}\\
\leqslant & C\kappa^{q-2^*_{\alpha,\mu}}\lambda\tilde{C}_*^{\frac{2^*_{\alpha,\mu}-2}{2}}\left(\int_{\mathbb{R}^N} u_n^{2^*}\eta_r^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\\
\leqslant & C\kappa^{q-2^*_{\alpha,\mu}}\lambda\tilde{C}_*^{\frac{2^*_{\alpha,\mu}-2}{2}}S^{-1}\int_{\mathbb{R}^N}|\eta_r\nabla u_n + u_n\nabla\eta_r|^2\,\mathrm{d}x\\
\leqslant & C\kappa^{q-2^*_{\alpha,\mu}}\lambda\tilde{C}_*^{\frac{2^*_{\alpha,\mu}-2}{2}}S^{-1}2^2\int_{\mathbb{R}^N}\eta_r^2(|\nabla u_n|^2+V(x)|u_n|^2)\,\mathrm{d}x\\
& + C\kappa^{q-2^*_{\alpha,\mu}}\lambda\tilde{C}_*^{\frac{2^*_{\alpha,\mu}-2}{2}}S^{-1}2^2\int_{\mathbb{R}^N}u_n^2|\nabla\eta_r|^2\,\mathrm{d}x\\
=: & \mathcal{Q}^1_{3,\lambda,\kappa}(n,r) + \mathcal{Q}^2_{3,\lambda,\kappa}(n,r).\end{aligned}$$

To estimate $\mathcal{Q}^1_{3,\lambda,\kappa}(n,r)$ and $\mathcal{Q}^2_{3,\lambda,\kappa}(n,r)$, we argue as in the proof of the Lemma 3.2.7. Then, similarly, for each $\kappa>0$, we may obtain $\lambda_0^* = \lambda_0^*(\kappa) > 0$ such that

$$4\lambda C\kappa^{2(q-2^*_{\alpha,\mu})}C_*^{\frac{2^*_{\alpha,\mu}-2}{2}}S^{-1} \leqslant \frac{1}{2}, \quad \forall\lambda\in[0,\lambda_0^*]. \tag{3.138}$$

The limits on (3.62) and (3.63) are replaced by the following limits

$$\begin{aligned}\limsup_{n\to\infty}\lambda\int_{B^c_{2r}(0)}\left(\int_{\mathbb{R}^N}\frac{H_\kappa(u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u_n)}{|x|^\alpha}u_n\,\mathrm{d}x \leqslant C_{\lambda,\kappa}\varepsilon,\\
\limsup_{n\to\infty}\lambda\int_{B^c_{2r}(0)}\left(\int_{\mathbb{R}^N}\frac{H_\kappa(u_n)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{h_\kappa(u_n)}{|x|^\alpha}u\,\mathrm{d}x \leqslant C_{\lambda,\kappa}\varepsilon\end{aligned}$$

and which are verified in a similar way to the limits in (3.60) and (3.61).

By adapting the arguments explored in the proof of Lemmas 3.2.11 and 3.2.12, we see that the functional $\tilde{\mathcal{J}}_{\lambda,\kappa}$ satisfies the $(PS)_{\tilde{c}_{\lambda,\kappa}}$–condition and consequently, the functional has a nonnegative critical point $u_{\lambda,\kappa}\in E$ such that $\tilde{\mathcal{J}}_{\lambda,\kappa}(u_{\lambda,\kappa}) = \tilde{c}_{\lambda,\kappa}$, i.e., $u_{\lambda,\kappa}$ is a nonnegative mountain pass solution for Problem $(\tilde{B}_{\lambda,\kappa})$.

Arguing as in the proof of Lemma 3.2.13, we can obtain similar estimates involving the norm of the solution of the auxiliary Problem $(\tilde{B}_{\lambda,\kappa})$. Then, there exist constants $\tilde{M}$ (which depends only on N,θ,μ,α,p,m and independent of ℓ, λ, k and R) and $\tilde{\lambda}_0^* := \tilde{\lambda}_0^*(\kappa) > 0$ such that

$$\|u_{\lambda,\kappa}\|^2 \leqslant \frac{2\theta}{\theta-2}\tilde{d} =: \tilde{M} \quad \text{and} \quad \|u_{\lambda,\kappa}\|^2_{2^*} \leqslant S^{-1}\tilde{M}, \quad \forall\lambda\in[0,\tilde{\lambda}_0^*]. \tag{3.139}$$

In order to obtain the L^∞–estimate for the solution $u_{\lambda,\kappa}$ of auxiliary Problem $(\tilde{B}_{\lambda,\kappa})$, we will make the following adaptations to the proof of Lemma 3.2.14. We start by replacing estimate

(3.88) by

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant 2S^{-1}\beta^2 \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha} u u_L^{2(\beta-1)}\,\mathrm{d}x \tag{3.140}$$

$$+\, 2S^{-1}\beta^2 \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{h_\kappa(u)}{|x|^\alpha} u u_L^{2(\beta-1)}\mathrm{d}x. \tag{3.141}$$

To estimate the right-hand side of (3.140), we combine (3.25), Proposition 1.0.2, the Hölder's inequality and (3.139), as follows, for all $\lambda \in [0, \tilde{\lambda}_0^*]$,

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right) g(x,u) u u_L^{2(\beta-1)}\mathrm{d}x \leqslant C_2\left(\int_{\mathbb{R}^N} (|u|^{2^*_{\alpha,\mu}} u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}$$

and similarly, we estimate (3.141) using (H_1')-(h_1'), as seen

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha|x-y|^\mu|x|^\alpha}\mathrm{d}y\right) h_\kappa(u) u u_L^{2(\beta-1)}\mathrm{d}x \leqslant$$

$$\leqslant \tilde{C}_2 \kappa^{2(q-2^*_{\alpha,\mu})}\lambda\left(\int_{\mathbb{R}^N} (|u|^{2^*_{\alpha,\mu}} u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}},$$

where $C_2 := \frac{2c_0^2}{\theta} C(N,\alpha,\mu)(S^{-1}\tilde{M})^{\frac{2^*_{\alpha,\mu}}{2}}$ and $\tilde{C}_2 := \frac{q}{2^*_{\alpha,\mu}} C(N,\alpha,\mu)(S^{-1}\tilde{M})^{\frac{2^*_{\alpha,\mu}}{2}}$, which jointly with (3.140)-(3.141) and setting $C := 2S\max\left\{C_2, \tilde{C}_2\right\}$, yields that

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant C\beta^2(1+\kappa^{2(q-2^*_{\alpha,\mu})}\lambda)\left(\int_{\mathbb{R}^N} (|u|^{2^*_{\alpha,\mu}} u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}.$$

By combining with (3.92)-(3.94), we derive

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant (1+\kappa^{2(q-2^*_{\alpha,\mu})}\lambda) C\beta^2\left(\int_{\mathbb{R}^N} (|u|^2 u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}$$

$$+\,(1+\kappa^{2(q-2^*_{\alpha,\mu})}\lambda) C\beta^2\left(\int_{\{|u|>K\}} |u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}}\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}. \tag{3.142}$$

Let $K > 0$ be sufficiently large such that

$$(1+\kappa^{2(q-2^*_{\alpha,\mu})}\lambda)\left(\int_{\{|u|>K\}} |u|^{2^*}\mathrm{d}x\right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \leqslant \frac{1}{2C\beta^2}. \tag{3.143}$$

Next, combining (3.143) with (3.142), one has

$$\left(\int_{\mathbb{R}^N} |w_L|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant (1+\kappa^{2(q-2^*_{\alpha,\mu})}\lambda) C\beta^2\left(\int_{\mathbb{R}^N} (|u|^2 u_L^{2(\beta-1)})^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}. \tag{3.144}$$

Claim. $u \in L^{2^*\beta}(\mathbb{R}^N)$ for $\beta = \frac{2^*_{\alpha,\mu}}{2}$, i.e.,

$$\int_{\mathbb{R}^N} |u|^{2^*\frac{2^*_{\alpha,\mu}}{2}}\mathrm{d}x < \infty.$$

In fact, since $u_L \leqslant |u|$ and recalling $w_L = uu_L^{(\beta-1)}$, it follows from (3.144) that

$$\left(\int_{\mathbb{R}^N} (|u|u_L^{\frac{2^*_{\alpha,\mu}-2}{2}})^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \leqslant (1+\kappa^{2(q-2^*_{\alpha,\mu})}\lambda)C\beta^2\left(\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x\right)^{\frac{2N-2\alpha-\mu}{2N}}$$
$$\leqslant C(N,\beta,S,M,\alpha,\mu,\kappa,\lambda) < \infty.$$

By Fatou's lemma, taking the limit as $L \to \infty$, we conclude that

$$\int_{\mathbb{R}^N} |u|^{2^*\frac{2^*_{\alpha,\mu}}{2}}\mathrm{d}x < \infty,$$

which proves the claim. Now, using $u_L \leqslant |u|$ again and by passing to the limit as $L \to \infty$ in (3.144) and arguing as in (3.105), we obtain

$$\|u\|_{2^*\beta} \leqslant [C^{\frac{1}{2}}(1+\kappa^{2(q-2^*_{\alpha,\mu})}\lambda)^{\frac{1}{2}}]^{\frac{1}{\beta}}\beta^{\frac{1}{\beta}}\|u\|_{q^*_{\alpha,\mu}\beta}, \quad \forall\,\lambda \in [0,\tilde{\lambda}_0^*], \tag{3.145}$$

where $q^*_{\alpha,\mu} := \frac{22^*}{2^*_{\alpha,\mu}}$.

By following steps 1, 2 and 3 in (3.107)-(3.111) of Lemma 3.2.14 and applying inequality (3.145), we obtain the desired $L^\infty-$estimate. The resulting estimate is as follows:

$$\|u_{\lambda,k}\|_\infty \leqslant (1+\kappa^{2(q-2^*_{\alpha,\mu})}\lambda)^{\frac{1}{2(\gamma_1-1)}}M_1\|u_{\lambda,k}\|_{2^*}, \quad \forall\,\lambda \in [0,\tilde{\lambda}_0^*],$$

where $M_1 = C^{\frac{1}{2(\gamma_1-1)}})\gamma_1^{\frac{\gamma_1}{(\gamma_1-1)^2}}$.

Finally, arguing as in (3.111) and by (3.139), fixing $\tilde{\kappa}_0 > M_1(S\tilde{M})^{\frac{1}{2}}$, we may obtain $\tilde{\lambda}_0^{**} \in (0,\tilde{\lambda}_0^*]$, such that

$$\|u_{\lambda,\tilde{\kappa}_0}\|_\infty \leqslant (1+\tilde{\kappa}_0^{q-2^*_{\alpha,\mu}}\lambda)^{\frac{1}{\gamma_1-1}}M_1(S\tilde{M})^{\frac{1}{2}} \leqslant \tilde{\kappa}_0, \quad \forall\,\lambda \in [0,\tilde{\lambda}_0^{**}). \tag{3.146}$$

By applying the reasoning from Remark 3.2.15, we conclude that for any $\lambda \in [0,\tilde{\lambda}_0^{**}]$, $u_{\lambda,\tilde{\kappa}_0}$ is positive solution for the problem $(B_{\lambda,\tilde{\kappa}_0})$. Furthermore, by the definition of $h_{\lambda,\tilde{\kappa}_0}$, we have

$$\left(\int_{\mathbb{R}^N}\frac{H_\kappa(u)}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{h_\kappa(u)}{|x|^\alpha} = \frac{\lambda}{q}\left(\int_{\mathbb{R}^N}\frac{|u|^q}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{|u|^{q-2}u}{|x|^\alpha}, \quad x \in \mathbb{R}^N. \tag{3.147}$$

Proof of Theorem 3.1.4. Next, using the same arguments explored in Section 3.2, we can show the existence of solution for the original Problem $(\tilde{P}_\lambda)$. Thus, given that (3.147) holds true, it is sufficient to prove that there exist $R > 1$ and $\lambda_0 \in (0,\tilde{\lambda}_0^{**}]$ such that following inequality holds:

$$f(u_{\lambda,\tilde{\kappa}_0}) \leqslant \frac{V(x)}{\ell_0}u_{\lambda,\tilde{\kappa}_0}, \quad \forall\,|x| \geqslant R \text{ and } \forall\,\lambda \in [0,\lambda_0),$$

where from now on we will fix $\tilde{\kappa}_0$ as in (3.146). We start with the following result similar to Lemma 3.2.16.

Lemma 3.3.2. *For each $R > 1$ and $\lambda > 0$, let $u_{\lambda,\tilde{\kappa}_0}$ be a solution for the auxiliary Problem $(\tilde{B}_{\lambda,\tilde{\kappa}_0})$, such that $\tilde{\mathcal{J}}_{\tilde{\lambda},\kappa_0}(u_{\lambda,\tilde{\kappa}_0}) = c_{\lambda,\tilde{\kappa}_0}$. Then, there exists $\lambda_0 \in (0, \tilde{\lambda}_0^{**}]$ such that*

$$u_{\lambda,\tilde{\kappa}_0} \leqslant \frac{R^{N-2}}{|x|^{N-2}} \big\| u_{\lambda,\tilde{\kappa}_0} \big\|_\infty \leqslant \frac{R^{N-2}}{|x|^{N-2}} \tilde{\kappa}_0, \quad \forall\, |x| \geqslant R \ \text{ and } \ \forall \lambda \in [0, \lambda_0).$$

Proof. For the sake of simplicity, we denote $u = u_{\lambda,\tilde{\kappa}_0}$. By using w, the same function defined in (3.113), as test function in (3.132), we obtain

$$\int_{\mathbb{R}^N} (\nabla u \nabla w + V(x) u w)\,\mathrm{d}x = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |y-x|^\mu}\,\mathrm{d}y \right) \frac{g(x,u)}{|x|^\alpha} w\,\mathrm{d}x + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y \right) \frac{h_\kappa(u)}{|x|^\alpha} w\,\mathrm{d}x. \tag{3.148}$$

By (h_1') and using Hölder's inequality twice, from (3.137) we obtain the following estimate

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu |x|^\alpha}\,\mathrm{d}y \right) h_\kappa(u) w\,\mathrm{d}x \leqslant & \left\| \int_{\mathbb{R}^N} \frac{H_\kappa(u)}{|y|^\alpha |x-y|^\mu |x|^\alpha}\,\mathrm{d}y \right\|_{\frac{2N}{\mu+2\alpha}} \\
& \times \left(\int_{\mathbb{R}^N} |h_\kappa(u) w|^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \\
\leqslant & C\kappa^{q-2^*_{\alpha,\mu}} \lambda \left(\int_{\mathbb{R}^N} (|u|^{2^*_{\alpha,\mu}-2} u w)^{\frac{2N}{2N-2\alpha-\mu}}\,\mathrm{d}x \right)^{\frac{2N-2\alpha-\mu}{2N}} \\
\leqslant & C\kappa^{q-2^*_{\alpha,\mu}} \lambda \left(\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \right)^{\frac{2^*_{\alpha,\mu}-2}{2^*}} \left(\int_{\mathbb{R}^N} |uw|^{\frac{2^*}{2}}\,\mathrm{d}x \right)^{\frac{2}{2^*}} \\
=: & I.
\end{aligned}$$

According to Young's inequality $2a_1 b_1 \leqslant a_1^2 + a_2^2$, and applying the Sobolev embedding theorem along with (3.139), it follows from (3.91) that

$$\begin{aligned}
I \leqslant & C\lambda\kappa_0^{q-2^*_{\alpha,\mu}} \left(\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x \right)^{\frac{2^*-2}{2^*}} \frac{1}{2} \left(\int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x + \int_{\mathbb{R}^N} |w|^{2^*}\,\mathrm{d}x \right)^{\frac{2}{2^*}} \\
\leqslant & \frac{1}{2} C\lambda\kappa_0^{q-2^*_{\alpha,\mu}} (S^{-1})^{\frac{2^*-2}{2}} \tilde{M}^{\frac{2^*-2}{2}} \left((S^{-1})^{\frac{2^*}{2}} \tilde{M}^{\frac{2^*}{2}} + (S^{-1})^{\frac{2^*}{2}} \|\nabla w\|_2^{\frac{2^*}{2}} \right)^{\frac{2}{2^*}} \\
\leqslant & \frac{1}{2} C\lambda\kappa_0^{q-2^*_{\alpha,\mu}} (S^{-1})^{\frac{2^*-2}{2}} \tilde{M}^{\frac{2^*-2}{2}} \left(S^{-1}\tilde{M} + S^{-1} \|\nabla w\|_2^2 \right).
\end{aligned}$$

Now, arguing with the proof of Lemma 3.2.16, we reach the estimate

$$\begin{aligned}
\|\nabla w\|_2^2 - & \frac{1}{2} C\lambda\kappa_0^{q-2^*_{\alpha,\mu}} (S^{-1})^{\frac{2^*-2}{2}} \tilde{M}^{\frac{2^*-2}{2}} \left(S^{-1}\tilde{M} + S^{-1} \|\nabla w\|_2^2 \right) \\
& \leqslant \int_{B_R^c(0)} \left(\frac{\mathcal{K}(u)(x)}{\ell_0} - 1 \right) V(x) u w\,\mathrm{d}x \leqslant 0,
\end{aligned}$$

which is similar to estimate (3.115). Follow the result arguing as in (3.119)-(3.120). □

Finally, in order to conclude the proof of Theorem 3.1.4, it is enough to argue as in (3.122)-(3.126). □

3.4 THE UPPER CRITICAL STEIN-WEISS CASE WITH SUPERCRITICAL LOCAL AND NONLOCAL PERTURBATIONS

In the proof of Theorem 3.1.5, we will only prove item (ψ_1), because the proof of item (ψ_2) is verified in an analogous way. Thus, we study the existence of solutions for the following equation involving Stein-Weiss type critical nonlinearity in $\mathbb{R}^N$ with supercritical term (ψ_1), i.e.,

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\alpha} + \lambda(x)|u|^{q-2}u, \tag{Q}$$

where $F(u) = \frac{|u|^{2^*_{\alpha,\mu}}}{2^*_{\alpha,\mu}}$, $f(u) = |u|^{2^*_{\alpha,\mu}-2}u$, $q \geqslant 2^*_{\alpha,\mu}$, $\lambda(x) \geqslant 0$ and the potential $V(x)$ is a radial function, i.e., $V(|x|) = V(x)$, for all $x \in \mathbb{R}^N$ satisfying (V_2).

We focus on the results which are potentially different from the case involving the function f with subcritical growth. We start by noting that due to the presence of $V(|x|)$, we replace the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ by $\mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)$ and consider

$$E_{\mathrm{rad}} := \left\{u \in \mathcal{D}^{1,2}_{\mathrm{rad}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2\,\mathrm{d}x < \infty\right\}.$$

Associated with (Q), we consider the energy functional $\mathcal{I} : E_{\mathrm{rad}} \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{I}(u) =& \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x - \frac{1}{2(2^*_{\alpha,\mu})^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{\alpha,\mu}}|u(x)|^{2^*_{\alpha,\mu}}}{|y|^\alpha|x-y|^\mu|x|^\alpha}\,\mathrm{d}y\mathrm{d}x \\ &- \frac{1}{q}\int_{\mathbb{R}^N} \lambda(x)|u|^q\,\mathrm{d}x.\end{aligned}$$

3.4.1 The auxiliary Problems $(\hat{A}_\kappa)$ and $(\hat{B}_\kappa)$

Next, arguing with Subsection 3.2.1, we begin to consider functions (3.15)-(3.18) and the respective auxiliary problems associated with each function,

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{F(u(y))}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u(x))}{|x|^\alpha} + h_\kappa(x,u), \quad \text{in } \mathbb{R}^N \tag{$\hat{A}_\kappa$}$$

and

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha} + h_\kappa(x,u), \quad \text{in } \mathbb{R}^N \tag{$\hat{B}_\kappa$}$$

where $F(u) = \frac{|u|^{2^*_{\alpha,\mu}}}{2^*_{\alpha,\mu}}$, $f(u) = |u|^{2^*_{\alpha,\mu}-2}u$, $q \geqslant 2^*_{\alpha,\mu}$. The energy functional associated with problem $(\hat{B}_\kappa)$ is given by

$$\hat{\mathcal{J}}_\kappa(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(x,u)}{|x|^\alpha}\,\mathrm{d}x - \int_{\mathbb{R}^N} h_\kappa(x,u)\,\mathrm{d}x. \tag{3.149}$$

In view of our assumptions one may conclude that $\hat{\mathcal{J}}_\kappa$ is well defined, belongs to $C^1(E,\mathbb{R})$ and its derivative is given by

$$\hat{\mathcal{J}}_\kappa'(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(y,u)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{g(x,u)}{|x|^\alpha} v\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^N} h_\kappa(x,u)v\,\mathrm{d}x.$$

Thus, weak solutions of $(\hat{B}_\kappa)$ are precisely the critical points of $\hat{\mathcal{J}}_\kappa$.

3.4.2 Existence of solutions for the auxiliary Problem $(\hat{B}_\kappa)$

The functional $\mathcal{I}$ still verifies, with natural modifications, Lemmas 3.2.1, 3.2.2, 3.2.3, 3.2.6, 3.2.11 and 3.2.12. For modification to Lemma 3.2.7, see Remark 3.2.9. Thus, $(\hat{B}_\kappa)$ has a nonnegative solution in E_{rad}.

Following a similar reasoning as in the proofs of Lemmas 3.2.13 and 3.2.14, we can obtain analogous estimates involving the norm of $u_{\lambda,\kappa}$ and the $L^\infty-$estimate for $u_{\lambda,\kappa}$. Then, by fixing $\kappa_0 > 0$, we can determine $\hat{\varepsilon}_0 := \hat{\varepsilon}_0(\kappa)$ such that for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ with $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \hat{\varepsilon}_0$, we conclude

$$\|u_{\lambda,\kappa_0}\|^2 \leqslant M \quad \text{and} \quad \|u_{\lambda,\kappa_0}\|_\infty \leqslant \kappa_0, \tag{3.150}$$

for some $M > 0$.

Arguing with Remark 3.2.15, we conclude that for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ satisfying $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \hat{\varepsilon}_0$, u_{λ,κ_0} is a positive solution for the Problem $(\hat{B}_{\kappa_0})$. Moreover, by (3.150) and using the definition of h_{κ_0}, we have

$$h_{\kappa_0}(u_{\lambda,\kappa_0}) = \lambda(x)|u_{\lambda,\kappa_0}|^{q-2}u_{\lambda,\kappa_0}. \tag{3.151}$$

Proof of Theorem 3.1.5. To prove the existence of a solution for the original Problem (Q), in light of (3.151), it is sufficient to show that there exists $R > 1$ such that the following inequality holds:

$$f(u_{\lambda,\kappa_0}) \leqslant \frac{V(x)}{\ell_0} u_{\lambda,\kappa_0}, \quad \forall\, |x| \geqslant R,$$

whenever $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \hat{\varepsilon}_0$ for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$.

Fixing $R > 1$, by hypothesis (V_2), we have

$$\frac{1}{|x|^{(\frac{N-2}{2})(2^*_{\alpha,\mu}-2)}} \leqslant \frac{V(x)}{W(R)}, \quad \forall\, |x| \geqslant R. \tag{3.152}$$

In light of the Lemma 3.2.8 and (3.150), for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ satisfying $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \hat{\varepsilon}_0$, we deduce

$$|u_{\lambda,\kappa_0}(x)| \leqslant C\frac{\|\nabla u_{\lambda,\kappa_0}\|_2}{|x|^{\frac{N-2}{2}}} \leqslant C\frac{M^{\frac{1}{2}}}{|x|^{\frac{N-2}{2}}} =: \frac{C}{|x|^{\frac{N-2}{2}}}, \quad \forall\, |x| \geqslant R.$$

This implies that

$$f(u_{\lambda,\kappa_0}(x)) = |u_{\lambda,\kappa_0}(x)|^{2^*_{\alpha,\mu}-2}u_{\lambda,\kappa_0} \leqslant \frac{C^{2^*_{\alpha,\mu}-2}}{|x|^{\frac{(N-2)}{2}(2^*_{\alpha,\mu}-2)}}u_{\lambda,\kappa_0}(x),$$

which jointly with (3.152), we see that

$$f(u_{\lambda,\kappa_0}(x)) \leqslant \frac{\ell_0 C^{2^*_{\alpha,\mu}-2}V(x)}{\ell_0 W(R)}u_{\lambda,\kappa_0}(x), \quad \forall |x| \geqslant R,$$

for all $\lambda \in L^{\frac{2N}{2\alpha+\mu}}(\mathbb{R}^N)$ satisfying $\|\lambda\|_{\frac{2N}{2\alpha+\mu}} \leqslant \hat{\varepsilon}_0$. Thus, if $W(R) \geqslant W_0 := \ell_0 C^{2^*_{\alpha,\mu}-2}$, then we can infer that

$$f(u_{\lambda,\kappa_0}) \leqslant \frac{V(x)}{\ell_0}u_{\lambda,\kappa_0},\ \forall\, |x| \geqslant R$$

implying that $\mathcal{I}'(u) = 0$ in E_{rad}. Now, using the principle of symmetric criticality due to Palais (WILLEM, 1996, Theorem 1.28), we conclude that $\mathcal{I}'(u) = 0$ in E, thus finishing the proof of Theorem 3.1.5. □

4 ON LINEARLY COUPLED SYSTEMS OF SCHRÖDINGER EQUATIONS WITH DOUBLE WEIGHTED NONLOCAL INTERACTION PART AND POTENTIAL VANISHING AT INFINITY

The main objective of this chapter is to study the following class of coupled systems with the presence of the doubly weighted nonlocal interaction

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = \left(\displaystyle\int_{\mathbb{R}^N} \frac{F_1(u_1)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{f_1(u_1)}{|x|^\alpha} + \lambda(x)u_2, & \text{in } \mathbb{R}^N \\ -\Delta u_2 + V_2(x)u_2 = \left(\displaystyle\int_{\mathbb{R}^N} \frac{F_2(u_2)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{f_2(u_2)}{|x|^\alpha} + \lambda(x)u_1, & \text{in } \mathbb{R}^N, \end{cases} \tag{S}$$

where $N \geq 3$, $0 < \mu < N$, $\alpha \geq 0$, $0 < 2\alpha + \mu < \min\{\frac{N+2}{2}, 4\}$ and F_i is the primitive of function f_i. We consider continuous functions $V_1(x), V_2(x)$ that may decay to zero at infinity and are related with the coupling function by

$$0 < \lambda(x) \leqslant \delta \min\{V_1(x), V_2(x)\}, \quad \delta \in \left(0, \frac{1}{2}\right), \quad \forall x \in \mathbb{R}^N,$$

where V_i and f_i satisfy hypotheses similar to (f_1)-(f_3) of Problem (P) in Chapter 3. In the next section, we will specify the assumptions on $V_i(x)$ and f_i.

The System (S) was motivated by work (DE ALBUQUERQUE; SANTOS, 2023) which corresponds to Problem (P) with $\psi \equiv 0$.

4.1 ASSUMPTIONS AND MAIN RESULTS

Inspired by (ALVES; FIGUEIREDO; YANG, 2016; DE ALBUQUERQUE; SILVA; SOUSA, 2022; DE ALBUQUERQUE; SANTOS, 2023), we will study System (S), considering the following hypotheses about f_i and $V_i(x)$. We assume that $f_i : \mathbb{R} \to \mathbb{R}$ a nonnegative continuous function and satisfies the the hypothesis for $i = 1, 2$:

$$\lim_{t \to 0^+} \frac{t f_i(t)}{t^{q_i}} < +\infty, \tag{$f_{i,1}$}$$

for $q_i \geq 2^*_{\alpha,\mu} = \frac{2N-2\alpha-\mu}{N-2}$ and

$$\lim_{t \to +\infty} \frac{t f_i(t)}{t^{p_i}} = 0, \tag{$f_{i,2}$}$$

for $p_i \in (1, \frac{2(N-\alpha-\mu)}{N-2})$. We also assume that f_i verifies the Ambrosetti-Rabinowitz type condition, i.e., there exists $\theta_i > 2$, such that

$$0 < \theta_i F_i(t) := \theta_i \int_0^t f_i(\tau)\mathrm{d}\tau \leq 2 f_i(t)t, \quad \forall t > 0. \tag{$f_{i,3}$}$$

Henceforth, we will assume that $\theta_i < 4$. Since we are looking for positive solutions, we suppose that $f_i(t) = 0$, for all $t \leq 0$.

By considering that $0 < 2\alpha + \mu < \min\{\frac{N+2}{2}, 4\}$, one may conclude that the interval $(1, \frac{2(N-\alpha-\mu)}{N-2})$ is nonempty. In view of $(f_{i,1})$ and the fact that $\frac{2(N-\alpha-\mu)}{N-2} < 2^*_{\alpha,\mu} < 2^*$ there holds

$$\lim_{t\to 0} \frac{tf_i(t)}{t^{2^*_{\alpha,\mu}}} = 0 \quad \text{and} \quad \lim_{t\to 0} \frac{tf_i(t)}{t^{p_i}} = 0. \tag{4.1}$$

Thus, it follows from $(f_{i,1})$, $(f_{i,2})$ and (4.1) that there exists $c_0 > 0$ such that

$$|tf_i(t)| \leq c_0|t|^{2^*_{\alpha,\mu}}, \quad |tf_i(t)| \leq c_0|t|^{q_i} \quad \text{and} \quad |tf_i(t)| \leq c_0|t|^{p_i}, \quad \forall t \in \mathbb{R}. \tag{4.2}$$

Regarding to the potential $V_i(x)$, we assume that it is a positive continuous function and its related with the coupling function $\lambda(x)$ through the following assumption:

$$0 < \lambda(x) \leqslant \delta \min\{V_1(x), V_2(x)\}, \quad \delta \in \left(0, \frac{1}{2}\right), \quad \forall x \in \mathbb{R}^N, \tag{$V_{i,1}$}$$

where $V_i(x)$ satisfies

$$\Lambda_i(R) = \frac{1}{R^{(q_i-2)(N-2)}} \inf_{|x|\geq R} |x|^{(q_i-2)(N-2)} V_i(x). \tag{$V_{i,2}$}$$

We introduce the notation

$$m_i = \max_{\{|x|\leqslant 1\}} V_i(x).$$

Due to the presence of $V_i(x)$ in System (S), we introduce the subspace of $\mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\mathcal{D}^{1,2}_{V_i}(\mathbb{R}^N) := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_i(x)|u|^2 \,\mathrm{d}x < \infty \right\},$$

which is a Hilbert space when endowed with the inner product and norm

$$\langle u, v\rangle_{V_i} := \int_{\mathbb{R}^N} (\nabla u \nabla v + V_i(x)uv)\,\mathrm{d}x \quad \text{and} \quad \|u\|_{\mathcal{D}^{1,2}_{V_i}} = \langle u, u\rangle^{\frac{1}{2}}_{V_i}.$$

We set the product space $\mathcal{D} := \mathcal{D}^{1,2}_{V_1}(\mathbb{R}^N) \times \mathcal{D}^{1,2}_{V_2}(\mathbb{R}^N)$ is a Hilbert space when endowed with the inner product and norm

$$\langle (u_1, u_2), (v_1, v_2)\rangle = \langle u_1, v_1\rangle_{V_1} + \langle u_2, v_2\rangle_{V_2} \quad \text{and} \quad \|(u_1, u_2)\|^2 = \sum_{i=1}^{2} \|u_i\|^2_{\mathcal{D}^{1,2}_{V_i}}.$$

Definition 4.1.1. *A pair (u_1, u_2) in $\mathcal{D}$ is said to be a weak solution for System (S) if it satisfies*

$$\begin{aligned}\langle (u_1, u_2), (v_1, v_2)\rangle &- \int_{\mathbb{R}^N} \lambda(x)(u_1v_2 + u_2v_1)\,\mathrm{d}x \\ &- \sum_{i=1}^{2} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F_i(u_i)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y \right) \frac{f_i(u_i)}{|x|^\alpha} v_i \,\mathrm{d}x = 0,\end{aligned}$$

for all $(v_1, v_2) \in \mathcal{D}$. A weak solution (u_1, u_2) is called vector solution if $u_1 \neq 0$ and $u_2 \neq 0$.

The energy functional $\mathcal{I} : \mathcal{D} \to \mathbb{R}$ associated to System (S) is given by

$$\begin{aligned}\mathcal{I}(u_1, u_2) = &\frac{1}{2}\|(u_1, u_2)\|^2 - \int_{\mathbb{R}^N} \lambda(x) u_1 u_2 \,\mathrm{d}x \\ &- \frac{1}{2}\sum_{i=1}^{2} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F_i(u_i)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{F_i(u_i)}{|x|^\alpha} \,\mathrm{d}x.\end{aligned} \tag{4.3}$$

Given the assumptions on f_i and by reasoning similar to Remark 3.1.7 of Chapter 3, we may infer from Remark 4.1.6 below that $\mathcal{I}$ is well defined and belongs to $C^1(\mathcal{D}, \mathbb{R})$. Moreover, critical points of $\mathcal{I}$ are weak solutions to system (S) and vice-versa.

Regarding to System (S), we have the following theorem.

Theorem 4.1.2. *Suppose that $N \geq 3$, $0 < \mu < N$, $\alpha \geq 0$, $0 < 2\alpha + \mu < \min\{\frac{N+2}{2}, 4\}$ and that f satisfies $(f_1) - (f_3)$. Then, there exists a constant $\Lambda_0^i = \Lambda_0^i(m_i, \theta_i, \alpha, \mu, p_i, c_0)$ such that if $\Lambda_i(R) > \Lambda_0^i$ for some $R > 1$, System (S) has a positive solution.*

Now we list some remarks on this chapter.

Remark 4.1.3. *Note that if $(u_1, u_2) \in \mathcal{D}\backslash\{(0, 0)\}$ is a solution for System (S), then (u_1, u_2) is vectorial (see Definition (4.1.1)). In fact, if $u_1 = 0$, then it follows from the first equation of (S) that $u_2 = 0$ when $\lambda(x) > 0$.*

Remark 4.1.4. *In order to obtain positive vector solutions, we consider a strictly positive coupling function $\lambda(x)$ in $\mathbb{R}^N$.*

Remark 4.1.5. *Our main contribution in this chapter is to complete the study done by the authors in (DE ALBUQUERQUE; SANTOS, 2023; ALVES; FIGUEIREDO; YANG, 2016), in the following aspects:*

(1) *If $\lambda = 0$, $f_1 = f_2$ and $u_1 = u_2$, then System (S) boils down to the class of scalar equations of Chapter 3, Problem (P) when $\psi \equiv 0$;*

(2) *If $\alpha = 0$, $0 < \mu < \min\{\frac{N+2}{2}, 4\}$, $N \geqslant 3$, $q \geqslant 2^*_{0,\mu} := 2^*_\mu = \dfrac{2N - \mu}{N - 2}$, then our results complete the picture of (ALVES; FIGUEIREDO; YANG, 2016);*

(3) *If $\alpha \neq 0$, $q \geqslant 2^*_{\alpha,\mu}$, then our results complement (DE ALBUQUERQUE; SANTOS, 2023);*

(4) *As far as we know, this is the first work considering coupled Schrödinger systems with Stein-Weiss type nonlinearities involving potential with decay to zero at infinity.*

Remark 4.1.6. *According to* $(V_{i,1})$ *and Young's inequality, we deduce*

$$\begin{aligned}\int_{\mathbb{R}^N} \lambda(x)u_1u_2\,\mathrm{d}x &\leqslant \frac{\delta}{2}\int_{\mathbb{R}^N} \min\{V_1(x),V_2(x)\}u_1^2\,\mathrm{d}x + \frac{\delta}{2}\int_{\mathbb{R}^N} \min\{V_1(x),V_2(x)\}u_2^2\,\mathrm{d}x\\ &\leqslant \frac{\delta}{2}\int_{\mathbb{R}^N} V_1(x)u_1^2\,\mathrm{d}x + \frac{\delta}{2}\int_{\mathbb{R}^N} V_2(x)u_2^2\,\mathrm{d}x\\ &\leqslant \frac{\delta}{2}\|(u_1,u_2)\|^2.\end{aligned}$$

With this, the coupling term is related with the norm by

$$(1+\delta)\|(u_1,u_2)\|^2 \geq \|(u_1,u_2)\|^2 - 2\int_{\mathbb{R}^N}\lambda(x)u_1u_2\,\mathrm{d}x \geq (1-\delta)\,\|(u_1,u_2)\|^2, \tag{4.4}$$

for $\delta \in \left(0,\frac{1}{2}\right)$.

Our approach to showing the existence of a vector solution for (S) is through variational methods combined with penalization technique and $L^\infty-$estimates.

This chapter is organized as follows: In the forthcoming section in order to overcome the lack of compactness, we introduce a penalized system and we obtain solution for this auxiliary problem. Moreover, in Subsection 4.2.2 we study $L^\infty-$estimates and positivity of such solution. Finally, we show that the solution of the auxiliary System (AS) is in fact a solution for System (S).

4.2 THE AUXILIARY SYSTEM (AS)

Associated with System (S), we define the energy functional $\mathcal{I}$ in (4.3). However, similar to the challenge faced in Chapter 3, we encounter a lack of compactness, specifically in ensuring that the energy functional satisfies the Palais-Smale condition. To address this, we also employ the penalization method. To avoid repetition, we will emphasize only the results that differ from those of Problem (A_κ). We start by adapting the functions (3.18)-(3.19) from Section 3.2 to this specific context, resulting in the following auxiliary system

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = \left(\displaystyle\int_{\mathbb{R}^N}\frac{G_1(y,u_1)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g_1(x,u_1)}{|x|^\alpha} + \lambda(x)u_2, & \text{in } \mathbb{R}^N,\\ -\Delta u_2 + V_2(x)u_2 = \left(\displaystyle\int_{\mathbb{R}^N}\frac{G_2(y,u_2)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g_2(x,u_2)}{|x|^\alpha} + \lambda(x)u_1, & \text{in } \mathbb{R}^N,\end{cases} \tag{AS}$$

where G_i and g_i satisfy (3.20)–(3.28).

We say that a function $(u_1,u_2)\in\mathcal{D}$ is a weak solution of the auxiliary System (AS), if satisfies

$$\begin{aligned}\langle (u_1,u_2),(v_1,v_2)\rangle &- \int_{\mathbb{R}^N}\lambda(x)(u_1v_2+u_2v_1)\,\mathrm{d}x\\ &-\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G_i(y,u_i)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g_i(x,u_i)}{|x|^\alpha}v_i\,\mathrm{d}x=0, \end{aligned}\tag{4.5}$$

for all $(v_1,v_2)\in\mathcal{D}$. The energy functional associated with System (AS) is given by

$$\begin{aligned}\mathcal{J}(u_1,u_2)=\frac{1}{2}\|(u_1,u_2)\|^2 &- \int_{\mathbb{R}^N}\lambda(x)u_1u_2\,\mathrm{d}x\\ &-\frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G_i(y,u_i)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G_i(x,u_i)}{|x|^\alpha}\,\mathrm{d}x. \end{aligned}\tag{4.6}$$

In view of our assumptions one may conclude that $\mathcal{J}$ is well defined, belongs to $C^1(\mathcal{D},\mathbb{R})$. Thus, weak solutions of (AS) are precisely the critical points of $\mathcal{J}$.

It is also worth mentioning here that the auxiliary System (AS) is strongly related to System (S). In fact, if (u_1,u_2) is a solution of (AS) verifying $f_i(u_i(x))\leqslant V_i(x)u_i(x)$ for all $|x|\geqslant R$, then $g_i(x,u_i)=f_i(u_i)$ and (u_1,u_2) is also a solution for System (S). This motivates us to study the System (AS).

4.2.1 Existence of solutions for the auxiliary System (AS)

In what follows we will explore some ideas from Chapter 3. We begin by observing that using the ideas from the proof of Lemma 3.2.1, we may prove that $\mathcal{J}$ verifies the mountain pass geometry stated in the following lemma.

Lemma 4.2.1. *The functional $\mathcal{J}$ satisfies the following conditions:*

(i) *there exist $\tau,\eta>0$ such that $\mathcal{J}(u_1,u_2)\geq\eta>0$, for all $(u_1,u_2)\in\mathcal{D}$ satisfying $\|(u_1,u_2)\|=\tau$;*

(ii) *there exists $(\tilde{u}_1,\tilde{u}_2)\in\mathcal{D}$ with $\|(\tilde{u},\tilde{v})\|>\tau$ such that $\mathcal{J}(\tilde{u}_1,\tilde{u}_2)<0$.*

Proof. By using Remark 4.1.6 and similar ideas to Lemma 3.2.1 one may conclude that $\mathcal{J}$ verifies $(i)-(ii)$. □

We will represent the mountain pass level by

$$0<c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}(\gamma(t))\quad\text{where}\quad \Gamma:=\Big\{\gamma\in C([0,1],E):\gamma(0)=0\ \text{ and }\ \mathcal{J}(\gamma(1))<0\Big\}.$$

Now, we introduce the functional $\mathcal{I}_0 : H_0^1(B_1(0)) \times H_0^1(B_1(0)) \to \mathbb{R}$ given by

$$\mathcal{I}_0(u_1, u_2) = \sum_{i=1}^{2} \frac{1}{2}\Bigg[\int_{B_1(0)} |\nabla u_i|^2 \,\mathrm{d}x + \frac{1}{2}\int_{B_1(0)} m_i |u_i|^2 \,\mathrm{d}x$$
$$\int_{B_1(0)}\left(\int_{B_1(0)} \frac{F_i(u_i)}{|y|^\alpha |x-y|^\mu}\,\mathrm{d}y\right)\frac{F_i(u_i)}{|x|^\alpha}\,\mathrm{d}x\Bigg] - \int_{\mathbb{R}^N} \lambda(x) u_1 u_2 \,\mathrm{d}x,$$

where $m_i = \max_{|x|\leqslant 1} V(x)$. Moreover, we denote by d the level of the mountain pass associated with the functional $\mathcal{I}_0$, i.e.,

$$0 < d := \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} \mathcal{I}_0(\gamma(t)),$$

where

$$\Gamma_0 := \Big\{\gamma \in C([0,1], H_0^1(B_1(0))) : \gamma(0) = 0 \ \text{ and } \ \mathcal{I}_0(\gamma(1)) < 0\Big\}.$$

Here, it is important to emphasize that d is independent of the choice of ℓ and R. Moreover, $c \leqslant d$. In view of Lemma 3.2.2 in Chapter 3,

$$\frac{1}{\theta_i}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G_i(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g_i(x,u)}{|x|^\alpha} u\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G_i(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{G_i(x,u)}{|x|^\alpha}\,\mathrm{d}x \geqslant 0.$$

Thus, by Lemma 4.1.6 and following Lemma 3.2.3, $(u_{1,n}, u_{2,n})_n$ is bounded in $\mathcal{D}$ and there exists $n_0 \in \mathbb{N}$ such

$$\|(u_{1,n}, u_{2,n})\|^2 \leqslant \frac{4\theta}{\theta-2}(d+1), \quad \forall\, n \geqslant n_0,$$

where $2 < \theta =: \min\{\theta_1, \theta_2\}$. In what follows,

$$\mathcal{B}_i := \left\{u \in \mathcal{D}^{1,2}_{V_i}(\mathbb{R}^N) : \|u\|^2_{\mathcal{D}^{1,2}_{V_i}} \leqslant \frac{2\theta}{\theta-2}(d+1)\right\},$$

and

$$\mathcal{K}_i(u)(x) := \int_{\mathbb{R}^N} \frac{G_i(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y. \tag{4.7}$$

Analogous to Lemma 3.2.4, for all $u \in \mathcal{B}_i$, we have that

$$\mathcal{K}_i(u)(x) := \int_{\mathbb{R}^N} \frac{G_i(y,u)}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y \in L^\infty(\mathbb{R}^N). \tag{4.8}$$

In addition, there exists $\ell_0 > 0$, which is independent of R, such that

$$\frac{\sup_{u\in\mathcal{B}_i} \|\mathcal{K}_i(u)(x)\|_\infty}{\ell_0} < C(\delta), \tag{4.9}$$

where $0 < C(\delta) < 1-\delta$.

From now on, we assume $\ell > \ell_0 > 0$ in the auxiliary System (AS). Similarly, to the result of Lemma 3.2.6, we may obtain

$$\lim_{n\to\infty}\sum_{i=1}^{2}\int_{B_{\bar{R}(0)}}\left(\int_{\mathbb{R}^N} \frac{G_i(y,u_{i,n})}{|y|^\alpha|x-y|^\mu}\,\mathrm{d}y\right)\frac{g_i(x,u_{i,n})}{|x|^\alpha}(u_{i,n} - u_i)\,\mathrm{d}x = 0, \tag{4.10}$$

where $\bar{R} > 0$. Moreover, it follows from Sobolev compact embedding that

$$\int_{B_{\bar{R}}(0)} \lambda(x) \left[u_{2,n}(u_{1,n} - u_1) + u_{1,n}(u_{2,n} - u_2)\right] \mathrm{d}x = o_n(1). \tag{4.11}$$

Lemma 4.2.2. *The functional $\mathcal{J}$ satisfies the $(PS)_c-$condition.*

Proof. In what follows, we will explore some ideas from (DE ALBUQUERQUE; SANTOS, 2023). From (DE ALBUQUERQUE; SANTOS, 2023, Lemma 3.4 and Lemma 3.5), for each $\varepsilon > 0$, there exists $r = r(\varepsilon) > R$ verifying

$$\limsup_{n\to\infty} \sum_{i=1}^{2} \int_{B_{2r}^c(0)} (|\nabla u_{i,n}|^2 + V_i(x)|u_{i,n}|^2) \,\mathrm{d}x < \varepsilon, \tag{4.12}$$

$$\limsup_{n\to\infty} \sum_{i=1}^{2} \int_{B_{2r}^c(0)} \left(\int_{\mathbb{R}^N} \frac{G_i(y, u_{i,n})}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{g_i(x, u_{i,n})}{|x|^\alpha} u_{i,n} \,\mathrm{d}x \leqslant C_1 \varepsilon, \tag{4.13}$$

$$\limsup_{n\to\infty} \sum_{i=1}^{2} \int_{B_{2r}^c(0)} \left(\int_{\mathbb{R}^N} \frac{G_i(y, u_{i,n})}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y \right) \frac{g_i(x, u_{i,n})}{|x|^\alpha} u_i \,\mathrm{d}x \leqslant C_2 \varepsilon. \tag{4.14}$$

By $(V_{i,1})$ and Höder's inequality, we see

$$\left| \int_{B_{\bar{R}}^c(0)} \lambda(x) u_{2,n}(u_{1,n} - u_1) \,\mathrm{d}x \right| \leqslant \delta \sum_{i=1}^{2} \left[\int_{B_{\bar{R}}^c(0)} (|\nabla u_{i,n}|^2 + V_i(x)|u_{i,n}|^2) \,\mathrm{d}x \right]^{\frac{1}{2}} \|u_{1,n} - u_1\|_{\mathcal{D}^{1,2}_{V_1}}$$

and

$$\int_{B_{\bar{R}}^c(0)} \lambda(x) u_{1,n} \left(u_{2,n} - u_2\right) \,\mathrm{d}x \leqslant \delta \sum_{i=1}^{2} \left[\int_{B_{\bar{R}}^c(0)} (|\nabla u_{i,n}|^2 + V_i(x)|u_{i,n}|^2) \,\mathrm{d}x \right]^{\frac{1}{2}} \|u_{2,n} - u_2\|_{\mathcal{D}^{1,2}_{V_2}}.$$

By using (4.12) and the fact that $(u_{1,n}, u_{2,n})_n$ is a bounded in $\mathcal{D}$ we obtain

$$\int_{B_{\bar{R}}^c(0)} \lambda(x) \left[u_{2,n}(u_{1,n} - u_1) + u_{1,n}(u_{2,n} - u_2)\right] \mathrm{d}x = o_n(1),$$

which together with (4.11), it follows that

$$\int_{\mathbb{R}^N} \lambda(x) \left[u_{2,n}(u_{1,n} - u_1) + u_{1,n}(u_{2,n} - u_2)\right] \mathrm{d}x = o_n(1).$$

Therefore, in view of (4.10), (4.11), (4.13), (4.14) and arguing as in (DE ALBUQUERQUE; SANTOS, 2023, Lemma 3.5), we deduce that $\mathcal{J}$ satisfies the $(PS)_c-$condition. □

Lemma 4.2.3. *The functional $\mathcal{J}$ has a nonnegative critical point $(u_1, u_2) \in \mathcal{D}$ such that $\mathcal{J}(u_1, u_2) = c$, i.e., (u_1, u_2) is a nonnegative mountain pass solution for System (AS).*

Proof. In view of Lemmas 4.2.1 and 4.2.2, System (AS) admits a nontrivial solution of mountain pass type. Let us write $u_i := u_i^+ + u_i^-$. We deduce from $(V_{i,1})$ that

$$\int_{\mathbb{R}^N} \lambda(x)(u_1 u_2^- + u_2 u_1^-) \,\mathrm{d}x \leqslant 2 \int_{\mathbb{R}^N} \lambda(x) u_1^- u_2^- \,\mathrm{d}x \leqslant \delta \|(u_1^-, u_2^-)\|^2.$$

By using (u_1^-, u_2^-) as test function in (4.5), it follows from the fact $f(t) = 0$ for all $t \leqslant 0$ that $\|(u_1^-, u_2^-)\| \leqslant 0$, i.e., the nontrivial weak solution (u_1, u_2) is nonnegative. □

4.2.2 $L^{\infty}-$estimates

In what follows, we deduce a uniform estimate for the norm of the solution of System (AS), obtained in Lemma 4.2.3. Similar to Lemmas 3.2.13 and 3.2.14, we have the following results.

Lemma 4.2.4. *Let (u_1, u_2) be the critical point of $\mathcal{J}$. Then, there exists a constant M (which depends only on $N, \theta_i, \mu, \alpha, p_i, m_i$ and independent of ℓ and R) such that*

$$\|(u_1, u_2)\|^2 \leqslant \frac{4\theta}{\theta - 2} d =: M.$$

Lemma 4.2.5. *Let pair (u_1, u_2) be the solution of (AS). Then, there exists a constant M_0, (which depends only on N, θ_i, μ, α, p_i, m_i independent of ℓ and R) such that*

$$\|(u_1, u_2)\|_{\infty} \leqslant M_1 \|(u_1, u_2)\|_{2^*}.$$

Proof. In what follows, we will explore arguments for the proof of Lemma 3.2.14. For $L > 0$, we define $\phi_{i,L} = u_i u_{i,L}^{2(\beta-1)}$ and $w_{i,L} = u_i u_{i,L}^{(\beta-1)}$, where $u_{i,L} = \min\{u_i, L\}$. By taking $\phi_{i,L} = u_i u_{i,L}^{2(\beta-1)}$ as test function in definition, where $\beta > 1$ will be chosen later, we have

$$\begin{aligned}\sum_{i=1}^{2} \int_{\mathbb{R}^N} \nabla u_i \nabla (u_i u_{i,L}^{2(\beta-1)}) \,\mathrm{d}x &+ \sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i(x) u_i^2 u_{i,L}^{2(\beta-1)} \,\mathrm{d}x \\ &- \int_{\mathbb{R}^N} \lambda(x)(u_1 u_2 u_{2,L}^{2(\beta-1)} + u_2 u_1 u_{1,L}^{2(\beta-1)}) \,\mathrm{d}x \\ &- \sum_{i=1}^{2} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G_i(y, u_i)}{|y|^{\alpha} |y - x|^{\mu}} \,\mathrm{d}y \right) \frac{g_i(x, u_i)}{|x|^{\alpha}} u_i u_{i,L}^{2(\beta-1)} \,\mathrm{d}x = 0,\end{aligned}$$

which implies

$$\begin{aligned}\sum_{i=1}^{2} \int_{\mathbb{R}^N} u_{i,L}^{2(\beta-1)} |\nabla u_i|^2 \,\mathrm{d}x = &- 2(\beta - 1) \sum_{i=1}^{2} \int_{\mathbb{R}^N} u_{i,L}^{2(\beta-1)-1} u_i \nabla u_i \nabla u_{i,L} \,\mathrm{d}x \\ &+ \sum_{i=1}^{2} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G_i(y, u_i)}{|y|^{\alpha} |y - x|^{\mu}} \mathrm{d}y \right) \frac{g_i(x, u_i)}{|x|^{\alpha}} u_i u_{i,L}^{2(\beta-1)} \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \lambda(x)(u_1 u_2 u_{2,L}^{2(\beta-1)} + u_2 u_1 u_{1,L}^{2(\beta-1)}) \,\mathrm{d}x \\ &- \sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i(x) u_i^2 u_{i,L}^{2(\beta-1)} \,\mathrm{d}x. \qquad (4.15)\end{aligned}$$

Since

$$2(\beta - 1) \int_{\mathbb{R}^N} u_{i,L}^{2(\beta-1)-1} u_i \nabla u_i \nabla u_{i,L} \,\mathrm{d}x = 2(\beta - 1) \int_{\{u_i \leq L\}} u_{i,L}^{2(\beta-1)} |\nabla u_i|^2 \,\mathrm{d}x \geq 0,$$

it follows from (4.15) that

$$\sum_{i=1}^{2}\int_{\mathbb{R}^N} u_{i,L}^{2(\beta-1)}|\nabla u_i|^2\,\mathrm{d}x = \sum_{i=1}^{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G_i(y,u_i)}{|y|^\alpha|y-x|^\mu}\mathrm{d}y\right)\frac{g_i(x,u_i)}{|x|^\alpha}u_iu_{i,L}^{2(\beta-1)}\,\mathrm{d}x + \int_{\mathbb{R}^N}\lambda(x)(u_1u_2u_{2,L}^{2(\beta-1)}+u_2u_1u_{1,L}^{2(\beta-1)})\,\mathrm{d}x - \sum_{i=1}^{2}\int_{\mathbb{R}^N}V_i(x)u_i^2u_{i,L}^{2(\beta-1)}\,\mathrm{d}x. \tag{4.16}$$

Note that

$$\|w_{i,L}\|_{2^*}^2 \le C\int_{\mathbb{R}^N}|\nabla w_{i,L}|^2\,\mathrm{d}x,$$

where $C := S^{-1}$ and S denotes the constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Thus,

$$\begin{aligned}\|w_{i,L}\|_{2^*}^2 \leqslant & C\int_{\mathbb{R}^N}u_{i,L}^{2(\beta-1)}|\nabla u_i|^2\,\mathrm{d}x + C(\beta-1)^2\int_{\mathbb{R}^N}|u_i|^2u_{i,L}^{2(\beta-2)}|\nabla u_{i,L}|^2\,\mathrm{d}x\\ \leqslant & C\beta^2\int_{\mathbb{R}^N}u_{i,L}^{2(\beta-1)}|\nabla u_i|^2\,\mathrm{d}x + C\beta^2\int_{\mathbb{R}^N}u_{i,L}^{2(\beta-1)}|\nabla u_i|^2\,\mathrm{d}x\\ \leqslant & C\beta^2\int_{\mathbb{R}^N}u_{i,L}^{2(\beta-1)}|\nabla u_i|^2\,\mathrm{d}x,\end{aligned} \tag{4.17}$$

where we have used that $\nabla u_{i,L} = 0$ in $\{u_i > L\}$, $u_i = u_{i,L}$ in $\{u_i \le L\}$ and $\beta > 1$. By using assumption $(V_{i,1})$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\lambda(x)u_2u_1u_{1,L}^{2(\beta-1)}\,\mathrm{d}x \leqslant & \frac{\delta}{2}\int_{\mathbb{R}^N}\min\{V_1(x),V_2(x)\}u_1^2u_{2,L}^{2(\beta-1)}\,\mathrm{d}x\\ & + \frac{\delta}{2}\int_{\mathbb{R}^N}\min\{V_1(x),V_2(x)\}u_2^2u_{1,L}^{2(\beta-1)}\,\mathrm{d}x\\ \leqslant & \frac{\delta}{2}\int_{\mathbb{R}^N}V_1(x)u_1^2u_{2,L}^{2(\beta-1)}\,\mathrm{d}x\\ & + \frac{\delta}{2}\int_{\mathbb{R}^N}\min\{V_1(x),V_2(x)\}u_2^2u_{1,L}^{2(\beta-1)}\,\mathrm{d}x.\end{aligned}$$

Now, note that

$$\begin{aligned}\int_{\mathbb{R}^N}\min\{V_1(x),V_2(x)\}u_2^2u_{1,L}^{2(\beta-1)}\,\mathrm{d}x \leqslant & \int_{\{u_1\leqslant u_2\}}\min\{V_1(x),V_2(x)\}u_2^2u_{2,L}^{2(\beta-1)}\,\mathrm{d}x\\ & + \int_{\{u_1>u_2\}}\min\{V_1(x),V_2(x)\}u_2^2u_{1,L}^{2(\beta-1)}\,\mathrm{d}x\\ \leqslant & \int_{\mathbb{R}^N}V_2(x)u_2^2u_{2,L}^{2(\beta-1)}\,\mathrm{d}x\\ & + \int_{\mathbb{R}^N}V_1(x)u_1^2u_{1,L}^{2(\beta-1)}\,\mathrm{d}x.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^N}\lambda(x)u_2u_1u_{1,L}^{2(\beta-1)}\,\mathrm{d}x \leqslant \delta\int_{\mathbb{R}^N}V_1(x)u_1^2u_{2,L}^{2(\beta-1)}\,\mathrm{d}x + \frac{\delta}{2}\int_{\mathbb{R}^N}V_2(x)u_2^2u_{1,L}^{2(\beta-1)}\,\mathrm{d}x. \tag{4.18}$$

Analogously, we deduce

$$\int_{\mathbb{R}^N} \lambda(x) u_1 u_2 u_{2,L}^{2(\beta-1)}\,\mathrm{d}x \leqslant \delta \int_{\mathbb{R}^N} V_2(x) u_2^2 u_{2,L}^{2(\beta-1)}\,\mathrm{d}x + \frac{\delta}{2}\int_{\mathbb{R}^N} V_1(x) u_1^2 u_{1,L}^{2(\beta-1)}\,\mathrm{d}x. \tag{4.19}$$

Thus, (4.18) and (4.19) imply that

$$2\int_{\mathbb{R}^N} \lambda(x)(u_1 u_2 u_{2,L}^{2(\beta-1)} + u_2 u_1 u_{1,L}^{2(\beta-1)})\,\mathrm{d}x \leqslant 3\delta \sum_{i=1}^{2}\int_{\mathbb{R}^N} V_i(x) u_i^2 u_{i,L}^{2(\beta-1)}\,\mathrm{d}x,$$

which jointly with (4.16)-(4.17) implies

$$\sum_{i=1}^{2} \|w_{i,L}\|_{2^*}^2 \leq C\beta^2 \sum_{i=1}^{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G_i(y,u_i)}{|y|^\alpha |y-x|^\mu}\mathrm{d}y\right)\frac{g_i(x,u_i)}{|x|^\alpha} u_i u_{i,L}^{2(\beta-1)}\,\mathrm{d}x. \tag{4.20}$$

Finally, arguing as in estimates (3.87)-(3.110) of Lemma 3.2.14 (see also (DE ALBUQUERQUE; SANTOS, 2023, Lemma 4.2)), we see that exists $M_1 > 0$ such that

$$\|(u_1,u_2)\|_\infty \leqslant M_1 \|(u_1,u_2)\|_{2^*}.$$

Therefore, the result follows. □

Remark 4.2.6. *Let $(u_1,u_2) \in \mathcal{D}$ be the nonnegative solution obtained in Lemma 4.2.3. In view of Lemma 4.2.5 and regularity theory (see for instance (TOLKSDORF, 1984, Theorem 1)), we have that $(u_1,u_2) \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N) \times C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$. Therefore, in light of Strong Maximum Principle, we conclude that (u_1,u_2) is positive.*

Proof of Theorem 4.1.2. In light of Lemma 4.2.3, the auxiliary System (AS) admits a solution (u_1,u_2) in $\mathcal{D}$. Thereby, in order to prove the existence of solution for the original System (S), it is sufficient to prove that there exists $R > 1$ such that following inequality holds:

$$f_i(u_i) \leqslant \frac{V_i(x)}{\ell_0} u_i, \quad \forall\, |x| \geqslant R.$$

Lemma 4.2.7. *For each $R > 1$, let (u_1,u_2) be a solution for the auxiliary System (AS), such that $\mathcal{J}(u_1,u_2) = c$. Then,*

$$u_i \leqslant \frac{R^{N-2}}{|x|^{N-2}}\Big\|(u_1,u_2)\Big\|_\infty \leqslant \frac{R^{N-2}}{|x|^{N-2}} M_1, \quad \forall\, |x| \geqslant R.$$

Proof. Let v be the $C^\infty(\mathbb{R}^N \backslash \{0\})$ function

$$v(x) = \frac{R^{N-2}\Big\|(u_1,u_2)\Big\|_\infty}{|x|^{N-2}}, \quad x \neq 0. \tag{4.21}$$

Since $1/|x|^{N-2}$ is harmonic, it follows that $\Delta v(x) = 0$ in $\mathbb{R}^N \backslash \{0\}$. Note that

$$u_i(x) \leqslant \left\|(u_1,u_2)\right\|_\infty \leqslant \frac{R^{N-2}}{|x|^{N-2}}\left\|(u_1,u_2)\right\|_\infty, \quad \forall\, |x| \leqslant R.$$

Let us introduce the function $w_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ defined by

$$w_i(x) = \begin{cases} (u_i - v)^+(x), & \text{if } \ |x| \geqslant R, \\ \quad 0, & \text{if } \ |x| \leqslant R. \end{cases} \tag{4.22}$$

Now, note that

$$\int_{\mathbb{R}^N} V_i(x) w_i^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} V_i(x) u_i w_i \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} V_i(x) u_i^2 \,\mathrm{d}x < \infty. \tag{4.23}$$

By using (w_1, w_2) as test function we obtain

$$\begin{aligned}\sum_{i=1}^{2} \left[\int_{\mathbb{R}^N} (\nabla u_i \nabla w_i + V_i(x) u_i w_i) \,\mathrm{d}x\right] =& \sum_{i=1}^{2} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G_i(y,u_i)}{|y|^\alpha |x-y|^\mu} \,\mathrm{d}y\right) \frac{g_i(x,u_i)}{|x|^\alpha} w_i \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \lambda(x)(u_2 w_1 + u_1 w_2) \,\mathrm{d}x. \end{aligned} \tag{4.24}$$

We recall as mentioned in Section 4.2.1, G_i and g_i satisfy (3.20)–(3.28) of Chapter 3. Thus, from (3.22), $g_i(x,t) \leqslant \frac{V_i(x)}{\ell_0} t$. By the definition of w_i and (4.7), it follows from $R^\alpha > 1$ the following estimate

$$\begin{aligned} &\sum_{i=1}^{2} \left[\int_{B_R^c(0)} \left(\int_{\mathbb{R}^N} \frac{G_i(y,u_i)}{|y|^\alpha |y-x|^\mu} \,\mathrm{d}y\right) \frac{g_i(x,u_i)}{|x|^\alpha} w_i \,\mathrm{d}x - \int_{B_R^c(0)} V_i(x) u_i w_i \,\mathrm{d}x\right] \\ \leqslant &\sum_{i=1}^{2} \left[\frac{1}{R^\alpha} \int_{B_R^c(0)} \frac{\mathcal{K}_i(u_i)(x)}{\ell_0} V_i(x) u_i w_i \,\mathrm{d}x - \int_{B_R^c(0)} V_i(x) u_i w_i \,\mathrm{d}x\right] \\ \leqslant &\sum_{i=1}^{2} \int_{B_R^c(0)} \left(\frac{\mathcal{K}_i(u_i)(x)}{\ell_0} - 1\right) V_i(x) u_i w_i \,\mathrm{d}x. \end{aligned} \tag{4.25}$$

On the other hand, from $(V_{i,1})$, we achieved the following estimate

$$\begin{aligned} \int_{\mathbb{R}^N} \lambda(x) u_2 w_1 \,\mathrm{d}x \leqslant &\delta \int_{\mathbb{R}^N} \min\{V_1(x), V_2(x)\} u_2 w_1 \,\mathrm{d}x \\ =&\delta \int_{\{|x| \geqslant R\} \cap \{u_1 \leqslant u_2\}} \min\{V_1(x), V_2(x)\} u_2 w_1 \,\mathrm{d}x \\ &+ \delta \int_{\{|x| \geqslant R\} \cap \{u_1 \geqslant u_2\}} \min\{V_1(x), V_2(x)\} u_2 w_1 \,\mathrm{d}x \\ \leqslant &\delta \int_{\mathbb{R}^N} V_1(x) u_1 w_1 \,\mathrm{d}x \\ &+ \delta \int_{\mathbb{R}^N} V_2(x) u_2 w_2 \,\mathrm{d}x. \end{aligned} \tag{4.26}$$

Similarly, we also get the following

$$\int_{\mathbb{R}^N} \lambda(x) u_1 w_2 \,\mathrm{d}x \leqslant \delta \int_{\mathbb{R}^N} V_1(x) u_1 w_1 \,\mathrm{d}x + \delta \int_{\mathbb{R}^N} V_2(x) u_2 w_2 \,\mathrm{d}x,$$

which together with (4.26), implies that

$$\int_{\mathbb{R}^N} \lambda(x)(u_2 w_1 + u_1 w_2)\,\mathrm{d}x \leqslant 2\delta \sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i(x) u_i w_i\,\mathrm{d}x. \tag{4.27}$$

On the other hand, using the definition of w_i again, we obtain

$$\int_{\mathbb{R}^N} |\nabla w_i|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} \nabla u_i \nabla w_i\,\mathrm{d}x - \int_{\mathbb{R}^N} \nabla v \nabla w_i\,\mathrm{d}x. \tag{4.28}$$

Since $\Delta v = 0$ in $B_R^c(0)$ and $w_i = 0$ in $\partial B_R(0)$, there holds

$$\int_{\mathbb{R}^N} \nabla v \nabla w_i\,\mathrm{d}x = 0. \tag{4.29}$$

Combining (4.24)-(4.29) with (4.9), we see that

$$\begin{aligned}
\sum_{i=1}^{2} \|\nabla w_i\|_2^2 \leqslant & \sum_{i=1}^{2} \int_{B_R^c(0)} \left(\frac{\mathcal{K}_i(u_i)(x)}{\ell_0} - 1 \right) V_i(x) u_i w_i\,\mathrm{d}x \\
& + 2\delta \sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i(x) u_i w_i\,\mathrm{d}x \\
= & \sum_{i=1}^{2} \int_{B_R^c(0)} \left(\frac{\mathcal{K}_i(u_i)(x)}{\ell_0} - 1 + 2\delta \right) V_i(x) u_i w_i\,\mathrm{d}x < 0.
\end{aligned}$$

Showing that $w_i \equiv 0$. Recalling the definition of the w_i function in (4.22), we obtain $|u_i| \leqslant v$ in $|x| \geqslant R$, which finishes the proof of lemma. □

In view of (4.2) we obtain

$$f_i(u) \leq c_0 |u|^{q_i - 2} u \leq c_0 M_1^{q_i - 2} \frac{R^{(N-2)(q_i-2)}}{|x|^{(N-2)(q_i-2)}} u, \quad \forall\, |x| \geq R.$$

Now fix $R > 1$ such that $\Lambda_i(R) > 0$. Thus, we have that

$$f_i(u) \leq c_0 |u|^{q_i - 2} u \leq \ell_0 c_0 M_1^{q_i - 2} \frac{V_i(x)}{\mathcal{V}_i(R)\ell_0} u, \quad \forall\, |x| \geq R.$$

Set the number $\Lambda_0^i = c_0 \ell_0 M_1^{q_i - 2}$. Let $R > 1$ be such that $\Lambda_i(R) > \Lambda_0^i$. Thus, we conclude that

$$f_i(u) \leq \frac{V_i(x)}{\ell_0} u, \, \forall\, |x| \geq R.$$

Therefore, the proof of the Theorem 4.1.2 is finished. □

5 N-LAPLACIAN COUPLED SYSTEMS INVOLVING DOUBLE WEIGHTED NONLOCAL INTERACTION PART IN $\mathbb{R}^N$: EXISTENCE OF SOLUTIONS AND REGULARITY

The aim of this chapter is to study the following class of coupled system involving Stein-Weiss type nonlinearities

$$\begin{cases} -\Delta_N u + |u|^{N-2}u = \left(\displaystyle\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta |x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta} + \lambda p|u|^{p-2}u|v|^q, & \text{in } \mathbb{R}^N, \\ -\Delta_N v + |v|^{N-2}v = \left(\displaystyle\int_{\mathbb{R}^N} \frac{G(v)}{|y|^\beta |x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta} + \lambda q|u|^p|v|^{q-2}v, & \text{in } \mathbb{R}^N, \end{cases} \tag{S_λ}$$

where $N \geqslant 2$, $0 < \mu < N$, $\lambda > 0$, $\beta \geqslant 0$, $0 < 2\beta + \mu < N$, $p > \frac{N}{2}, q > \frac{N}{2}$, $p + q > N$, $\Delta_N u = \text{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator, $f(s)$, $g(s)$ have critical growth of Trudinger-Moser type, $F(s), G(s)$ are the primitives of $f(s)$, $g(s)$ respectively.

From a mathematical point of view, the cases involving $N-$Laplacian (for $N \geqslant 3$) is particularly very interesting as the corresponding Sobolev embedding yields $W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for all $q \geqslant N$, but $W^{1,N}(\mathbb{R}^N) \not\subset L^\infty(\mathbb{R}^N)$. In these cases, the Pohozaev-Trudinger-Moser inequality (CAO, 1992) (see (MOSER, 1971; POHOZAEV, 1965) for bounded domain case) can be treated as a substitute of Sobolev inequality which helps us to establish the sharp maximal growth on functions in $W^{1,N}(\mathbb{R}^N)$ as follows.

Proposition 5.0.1. ***(Pohozaev-Trudinger-Moser inequality, (CAO, 1992))*** *If $\alpha > 0$, $N \geqslant 2$ and $u \in W^{1,N}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} (\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u))\,\mathrm{d}x < \infty,$$

where

$$S_{N-2}(\alpha, u) = \sum_{m=0}^{N-2} \frac{\alpha^m |u|^{\frac{mN}{N-1}}}{m!}.$$

Moreover, if $\|\nabla u\|_N^N \leqslant 1$, $\|u\|_N \leqslant M < \infty$ and $\alpha < \alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the surface area of $(N-1)$-dimensional unit sphere, then there exists a constant $C = C(\alpha, M, N) > 0$ such that

$$\int_{\mathbb{R}^N} (\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u))\,\mathrm{d}x \leqslant C.$$

In the sense of the Pohozaev-Trudinger-Moser inequality, we say that a function $h : \mathbb{R} \to \mathbb{R}$ has α_0- critical exponential growth at $+\infty$, if there exists $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{h(s)}{\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0. \end{cases}$$

This definition of criticality was introduced by Adimurthi and Yadava (ADIMURTHI; YADAVA, 1990), see also (FIGUEIREDO; MIYAGAKI OLIMPIO; RUF, 1995). There are a few works considering Stein-Weiss term and a nonlinearity with critical exponential growth, see for example (ALVES; SHEN, 2023) and (YUAN et al., 2023). For works considering Choquard type equations and nonlinearities with critical exponential growth, we refer the readers to (ALVES et al., 2016b; YANG, 2018; ALBUQUERQUE; FERREIRA; SEVERO, 2021; QIN; TANG, 2021; SHEN; RADULESCU; YANG, 2022) and references therein.

5.1 ASSUMPTIONS AND MAIN RESULTS

Throughout the chapter, in order to deal with the coupling terms in System (S_λ), we use the following hypotheses for p, q and N:

$$N > 2, \quad p, q > \frac{N}{2} \quad \text{and} \quad p + q > N. \tag{5.1}$$

We consider the following assumptions on the functions $f, g : \mathbb{R} \to \mathbb{R}$:

(a) f and g are continuous, $f(s) = g(s) = 0$ if $s \leqslant 0$ and $f(s) > 0, g(s) > 0$ if $s > 0$. Also

$$\lim_{s \to 0^+} \frac{f(s)}{s^{\frac{2N-2\beta-\mu}{2}-1}} = \lim_{s \to 0^+} \frac{g(s)}{s^{\frac{2N-2\beta-\mu}{2}-1}} = 0;$$

(b) f and g have α_0- critical exponential growth at $+\infty$;

(c) $\liminf_{|s| \to \infty} \dfrac{F(s)}{e^{\alpha_0 s^{\frac{N}{N-1}}}} = \liminf_{|s| \to \infty} \dfrac{G(s)}{e^{\alpha_0 s^{\frac{N}{N-1}}}} = \beta_0 > 0;$

(d) there exist $s_0, M_0 > 0$ and $m_0 \in (0, 1]$ such that

$$0 < s^{m_0} F(s) \leqslant M_0 f(s), \quad \forall\ s \geqslant s_0;$$

(e) the functions $s \mapsto f(s)/s^{N-1}$ and $s \mapsto g(s)/s^{N-1}$ are increasing for $s > 0$;

(f) there exists $\theta \in \Big(N, p + q\Big]$ such that $0 < \theta F(s) \leqslant f(s)s$ and $0 < \theta G(s) \leqslant g(s)s$, for all $s > 0$.

In addition to the challenges posed by the nonlocal and critical growth behavior of the nonlinearity, we outline several technical difficulties encountered in studying Systems (S_λ).

(i) The nonlocal term is not periodic for $\beta \neq 0$. For this reason, the standard approach based on Lions' vanishing-nonvanishing argument is not applicable anymore;

(*ii*) Since the system is not linear, showing the solution of System (S_λ) to be vectorial is not obvious and requires a careful treatment;

(*iii*) For $N > 2$, due to the lack of Hilbert space structure, a variant of Palais principle of symmetric criticality is needed, see Appendix for details;

(*iv*) For the critical exponential growth in the general case $N \geqslant 2$, even some obvious results require some careful analysis throughout the chapter. Needless to mention Lemma 5.2.11 as an example.

In order to present the main results of this chapter, we now introduce the normed space suitable to study System (S_λ). In fact, we denote by $W^{1,N}(\mathbb{R}^N)$ the usual Sobolev space, endowed with the standard scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ as follows

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{N-2} \nabla u \nabla v + |u|^{N-2} uv) \, \mathrm{d}x, \quad \|u\|^N = \int_{\mathbb{R}^N} \left(|\nabla u|^N + |u|^N \right) \mathrm{d}x.$$

We introduce the product space $\mathbb{W}^N := W^{1,N}(\mathbb{R}^N) \times W^{1,N}(\mathbb{R}^N)$ for $N \geqslant 2$ endowed with the standard inner product and norm

$$\langle (u, v), (\phi, \psi) \rangle = \langle u, \phi \rangle + \langle v, \psi \rangle, \quad \|(u, v)\|^N = \|u\|^N + \|v\|^N.$$

One of the major difficulties is that Lions' vanishing argument is not applicable, due to the non-periodic characteristic of Stein-Weiss term. To overcome this hurdle, we restrict ourselves on radial Sobolev space

$$W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N) = \left\{ u \in W^{1,N}(\mathbb{R}^N) : u(x) = u(|x|) \right\}$$

and the radial product space $\mathbb{W}^N_{\mathrm{rad}} := W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N) \times W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$, endowed with the norm induced by $W^{1,N}(\mathbb{R}^N)$ and $\mathbb{W}^N$ respectively. Using a variant as discussed in (KOBAYASHI; OTANI, 2014) of symmetric criticality principle of Palais (see (WILLEM, 1996)), the critical points of $\mathcal{J}_\lambda$ restricted to $\mathbb{W}^N_{\mathrm{rad}}$ turn out to be the critical points of $\mathcal{J}_\lambda$ in $\mathbb{W}^N$. The details and applicability of symmetric criticality principle of Palais can be seen in the Appendix A of this thesis.

Definition 5.1.1. *A pair $(u, v) \in \mathbb{W}^N$ is said to be a weak solution for System (S_λ) if it* *satisfies*

$$\begin{aligned}&\langle (u,v),(\phi,\psi)\rangle - \lambda p\int_{\mathbb{R}^N}|u|^{p-2}u|v|^q\phi\,\mathrm{d}x - \lambda q\int_{\mathbb{R}^N}|u|^p|v|^{q-2}v\psi\,\mathrm{d}x\\ &-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}\phi\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta}\psi\,\mathrm{d}x\\ &=0,\end{aligned}$$

for all $(\phi,\psi)\in\mathbb{W}^N$. *A weak solution* (u,v) *is called vector solution if* $u\neq 0$ *and* $v\neq 0$.

The energy functional $\mathcal{J}_\lambda:\mathbb{W}^N\to\mathbb{R}$ associated to System (S_λ) is given by

$$\begin{aligned}\mathcal{J}_\lambda(u,v)&=\frac{1}{N}\|(u,v)\|^N-\lambda\int_{\mathbb{R}^N}|u|^p|v|^q\,\mathrm{d}x\\ &-\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\beta}\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v)}{|x|^\beta}\,\mathrm{d}x.\end{aligned}\tag{5.2}$$

In view of the α_0-critical growth assumption on f and g, Proposition 1.0.2 and Proposition 5.0.1, $\mathcal{J}_\lambda$ is well defined and belongs to $C^1(\mathbb{W}^N,\mathbb{R})$. Moreover, critical points of $\mathcal{J}_\lambda$ are weak solutions to System (S_λ) and vice-versa, see Section 5.2 for more details.

The main results of this chapter can be stated as follows.

Theorem 5.1.2. *Suppose that* f *and* g *satisfy assumptions* $(a)-(f)$. *Then System* (S_λ) *has a nonnegative solution* (u_λ,v_λ), *which is vectorial if* $\lambda>\lambda_0$, *for some* $\lambda_0>0$. *In addition,* $(u_\lambda,v_\lambda)\in[L^\infty(\mathbb{R}^N)\cap C^{1,\gamma}(\mathbb{R}^N)]^2$, *for some* $\gamma\in(0,1)$ *and* (u_λ,v_λ) *is positive.*

Remark 5.1.3. *The main contributions of this chapter are the following:*

(i) *The results of this chapter complete the picture of (DE ALBUQUERQUE et al., 2019) in any dimension* $N>2$. *We complement and extend some works which consider Choquard type problems with critical exponential growth, for example (CHEN; TANG, 2022);*

(ii) *Even for scalar case (when* $\lambda=0$*) the results of this chapter are new and complement (ALVES; SHEN, 2023) for dimensions* $N>2$*;*

(iii) *This chapter presents an alternative to the standard arguments based on Lions' vanishing-nonvanishing and shifted sequences argument (not applicable if* $\beta\neq 0$*) by utilizing a variant of Palais principle of symmetric criticality.*

Remark 5.1.4. *A solution* $(u,v)\in\mathbb{W}^N_{\mathrm{rad}}$ *of* (S_λ) *is said to be a radial ground state solution if satisfies* $\mathcal{J}_\lambda(u,v)\leqslant\mathcal{J}_\lambda(w,z)$, *for any other nontrivial solution* $(w,z)\in\mathbb{W}^N_{\mathrm{rad}}$. *By considering this notion, we point out that all solution obtained in Theorem 5.1.2 is radial ground state solution.*

Remark 5.1.5. *Note that if $N=2$, $0<\lambda<1$ and $(u,v)\in\mathbb{W}^2\backslash\{(0,0)\}$ is a solution for System (S_λ), then (u,v) is vectorial, i.e., $u\neq 0$ and $v\neq 0$. In fact, if $u=0$, then it follows from the first equation of (S_λ) that $v=0$. This conclusion is no longer trivial to System (S_λ) with $N>2$. In this case we will consider the coupling parameter $\lambda>0$ sufficiently large to prove that the nontrivial solution obtained is vectorial, see Subsection 5.2.5.1.*

The remainder of this chapter is organized as follows: In the forthcoming section we introduce the variational framework and the energy levels associated to System (S_λ). Next, we devoted to the proof of Theorem 5.1.2 and it is divided into several subsections, which are guided in the following order: Mountains pass level, Nehari manifold, level comparison, estimate of the minimax level, compactness results, regularity and finally the existence of a vectorial radial ground state solution for System (S_λ).

5.2 VARIATIONAL FRAMEWORK AND ENERGY LEVELS

We begin this section by proving that the energy functional (5.2) is well defined. In view of assumptions (a) and (b), for any $\epsilon>0,\ r\geqslant 1$ and $\alpha>\alpha_0$, there exists a constant $C(\epsilon,r,\alpha)>0$ such that

$$\begin{aligned}|f(s)|&\leqslant\epsilon|s|^{\frac{2N-2\beta-\mu}{2}-1}+C(\epsilon,r,\alpha)|s|^{r-1}\left[\exp(\alpha s^{\frac{N}{N-1}})-S_{N-2}(\alpha,s)\right],\\|g(s)|&\leqslant\epsilon|s|^{\frac{2N-2\beta-\mu}{2}-1}+C(\epsilon,r,\alpha)|s|^{r-1}\left[\exp(\alpha s^{\frac{N}{N-1}})-S_{N-2}(\alpha,s)\right]\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}|F(s)|&\leqslant\epsilon|s|^{\frac{2N-2\beta-\mu}{2}}+C(\epsilon,r,\alpha)|s|^{r}\left[\exp(\alpha s^{\frac{N}{N-1}})-S_{N-2}(\alpha,s)\right],\\|G(s)|&\leqslant\epsilon|s|^{\frac{2N-2\beta-\mu}{2}}+C(\epsilon,r,\alpha)|s|^{r}\left[\exp(\alpha s^{\frac{N}{N-1}})-S_{N-2}(\alpha,s)\right].\end{aligned}\tag{5.4}$$

Thus, for any u in $W^{1,N}(\mathbb{R}^N)$, it follows from (5.4) that

$$\begin{aligned}\|F(u)\|_{\frac{2N}{2N-2\beta-\mu}}\leqslant&\epsilon C\|u\|_N^{\frac{2N-2\beta-\mu}{2}}\\&+C(\epsilon,r,\alpha)\||u|^r(\exp(\alpha u^{\frac{N}{N-1}})-S_{N-2}(\alpha,u))\|_{\frac{2N}{2N-2\beta-\mu}}.\end{aligned}\tag{5.5}$$

For any $\bar{r}\geqslant 1$ and $\alpha>0$, we have that

$$\left[\exp(\alpha s^{\frac{N}{N-1}})-S_{N-2}(\alpha,u)\right]^{\bar{r}}\leqslant\exp(\bar{r}\alpha s^{\frac{N}{N-1}})-S_{N-2}(\bar{r}\alpha,s),\quad\forall s\in\mathbb{R}.\tag{5.6}$$

Applying Hölder's inequality with $\frac{1}{n}+\frac{1}{m}=1$, Sobolev embedding and using (5.6) we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N}|u|^{\frac{2Nr}{2N-2\beta-\mu}}\left[\exp\left(\alpha u^{\frac{N}{N-1}}\right)-S_{N-2}(\alpha,u)\right]^{\frac{2N}{2N-2\beta-\mu}}\,\mathrm{d}x\\
\leqslant&\left(\int_{\mathbb{R}^N}|u|^{\frac{2Nrm}{2N-2\beta-\mu}}\,\mathrm{d}x\right)^{\frac{1}{m}}\left\{\int_{\mathbb{R}^N}\left[\exp\left(\alpha u^{\frac{N}{N-1}}\right)-S_{N-2}(\alpha,u)\right]^{\frac{2Nn}{2N-2\beta-\mu}}\,\mathrm{d}x\right\}^{\frac{1}{n}}\\
\leqslant&C_1\|u\|^{\frac{2Nr}{2N-2\beta-\mu}}\left\{\int_{\mathbb{R}^N}\left[\exp\left(\frac{2N\alpha n\|u\|^{\frac{N}{N-1}}}{2N-2\beta-\mu}\left(\frac{|u|}{\|u\|}\right)^{\frac{N}{N-1}}\right)\right.\right.\\
&\left.\left.-S_{N-2}\left(\frac{2N\alpha n\|u\|^{\frac{N}{N-1}}}{2N-2\beta-\mu},\frac{|u|}{\|u\|}\right)\right]\mathrm{d}x\right\}^{\frac{1}{n}}.
\end{aligned}\tag{5.7}$$

Now, choosing $\|(u,v)\|<\left(\frac{2N-2\beta-\mu}{2Nn}\frac{\alpha_N}{\alpha}\right)^{\frac{N-1}{N}}$, Proposition 5.0.1 together with (5.7), we obtain

$$\left\{\int_{\mathbb{R}^N}|u|^{\frac{2Nr}{2N-2\beta-\mu}}\left[\exp\left(\alpha u^{\frac{N}{N-1}}\right)-S_{N-2}(\alpha,u)\right]^{\frac{2Nn}{2N-2\beta-\mu}}\,\mathrm{d}x\right\}^{\frac{2N-2\beta-\mu}{2N}}\leqslant C_2\|u\|^r,$$

which jointly with (5.5) imply that

$$\|F(u)\|^2_{\frac{2N}{2N-2\beta-\mu}}\leqslant 4\epsilon^2C^2\|u\|^{2N-2\beta-\mu}+4C_2^2C^2(\epsilon,r,\alpha)\|u\|^{2r}.$$

Gathering the last estimate with Proposition 1.0.2 and defining

$$\hat{C}:=C(N,\beta,\mu)\max\left\{4\epsilon^2C^2,4C_2^2C^2(\epsilon,r,\alpha)\right\},$$

we may infer

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))F(u(x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x\leqslant&C(N,\beta,N)\|F(u)\|^2_{\frac{2N}{2N-2\beta-\mu}}\\
\leqslant&\hat{C}\|(u,v)\|^{2N-2\beta-\mu}+\hat{C}\|(u,v)\|^{2r}<+\infty.
\end{aligned}\tag{5.8}$$

Following the same lines, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(v(y))G(v(x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x\leqslant&C(N,\beta,N)\|F(u)\|^2_{\frac{2N}{2N-2\beta-\mu}}\\
\leqslant&\hat{C}\|(u,v)\|^{2N-2\beta-\mu}+\hat{C}\|(u,v)\|^{2r}<+\infty.
\end{aligned}\tag{5.9}$$

From (5.4), arguing in a similar way to the previous lines, we deduce

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))f(u(x))u(x)}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x<+\infty,\quad\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(v(y))g(v(x))v(x)}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x<+\infty.$$

Therefore, one may conclude that the energy functional $\mathcal{J}_\lambda$ introduced in (5.2) is well defined on $\mathbb{W}^N$, belongs to $C^1(\mathbb{W}^N,\mathbb{R})$ and its derivative is given by

$$\begin{aligned}
\mathcal{J}_\lambda'(u,v)(\phi,\psi)=&\langle(u,v),(\phi,\psi)\rangle-\lambda p\int_{\mathbb{R}^N}|u|^{p-2}u|v|^q\phi\,\mathrm{d}x-\lambda q\int_{\mathbb{R}^N}|u|^p|v|^{q-2}v\psi\,\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}\phi\,\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta}\psi\,\mathrm{d}x,
\end{aligned}$$

for all $(\phi, \psi) \in \mathbb{W}^N$. Therefore, critical points of $\mathcal{J}_\lambda$ are weak solutions to System (S_λ) and vice-versa.

In the next section, in addition to obtaining the existence of a solution, we will compare the following energy and minimax levels:

$$\begin{aligned} c_\lambda &:= \inf_{(u,v)\in\mathcal{S}_\lambda} \mathcal{J}_\lambda(u,v); \\ m_\lambda &:= \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} \mathcal{J}_\lambda(\gamma(t)); \\ c_{\mathcal{N}_\lambda} &:= \inf_{(u,v)\in\mathcal{N}_\lambda} \mathcal{J}_\lambda(u,v); \\ d_\lambda &:= \inf_{(u,v)\in\mathbb{W}^N_{\mathrm{rad}}\backslash\{(0,0)\}} \max_{t\geqslant 0} \mathcal{J}_\lambda(t(u,v)), \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} \mathcal{S}_\lambda &:= \left\{(u,v)\in\mathbb{W}^N_{\mathrm{rad}}\backslash\{(0,0)\} : (u,v) \text{ is a solution of } (S_\lambda)\right\}; \\ \Gamma_\lambda &:= \left\{\gamma\in C([0,1];\mathbb{W}^N_{\mathrm{rad}}) : \gamma(0)=(0,0),\ \mathcal{J}_\lambda(\gamma(1))<0\right\}; \\ \mathcal{N}_\lambda &:= \left\{(u,v)\in\mathbb{W}^N_{\mathrm{rad}}\setminus\{(0,0)\} : \mathcal{J}_\lambda'(u,v)(u,v)=0\right\}. \end{aligned}$$

5.2.1 Mountain pass level

Next, we will show that the energy functional $\mathcal{J}_\lambda$ satisfies the geometric requirement of the Mountain Pass Theorem stated in the following lemma.

Lemma 5.2.1. *The functional $\mathcal{J}_\lambda$ satisfies the following conditions:*

(i) there exist $\xi, \delta > 0$ such that $\mathcal{J}_\lambda(u,v) \geqslant \delta > 0$, for all $(u,v)\in\mathbb{W}^N$ satisfying $\|(u,v)\| = \xi$;

(ii) there exists $(\tilde{u},\tilde{v})\in\mathbb{W}^N$ with $\|(\tilde{u},\tilde{v})\| > \xi$ such that $\mathcal{J}_\lambda(\tilde{u},\tilde{v}) < 0$.

Proof. In view of (5.1), Hölder's inequality and Sobolev embedding, there holds

$$\int_{\mathbb{R}^N} |u|^p|v|^q \,\mathrm{d}x \leqslant \left(\int_{\mathbb{R}^N} |u|^{2p}\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |v|^{2q}\,\mathrm{d}x\right)^{\frac{1}{2}} \leqslant C\|u\|^p\|v\|^q. \tag{5.11}$$

Thus,

$$\int_{\mathbb{R}^N} |u|^p|v|^q \,\mathrm{d}x \leqslant C\|(u,v)\|^{p+q}. \tag{5.12}$$

By combining (5.8), (5.9), (5.12) with (5.2) and since $\|(u,v)\| < \left(\frac{2N-2\beta-\mu}{2Nn}\frac{\alpha_N}{\alpha}\right)^{\frac{N-1}{N}}$, it follows that

$$\mathcal{J}_\lambda(u,v) \geqslant \frac{1}{N}\|(u,v)\|^N - \lambda C\|(u,v)\|^{p+q} - \hat{C}\|(u,v)\|^{2N-2\beta-\mu} - \hat{C}\|(u,v)\|^{2r}.$$

Denoting $C_{\lambda,p,q} = \max\left\{\frac{\lambda C}{p+q}, \hat{C}\right\}$ and considering $\|(u,v)\| = \xi$, we obtain

$$\mathcal{J}_\lambda(u,v) \geqslant \xi^N\left[\frac{1}{N} - \left(1 + \xi^{2N-(p+q)-2\beta-\mu} + \xi^{2r-(p+q)}\right) C_{\lambda,p,q}\xi^{p+q-N}\right].$$

Thus, since that $p+q > N$, $N - 2\beta - \mu > 0$ and $r > N$, we may choose $0 < \xi < \left(\frac{2N-2\beta-\mu}{2Nn}\frac{\alpha_N}{\alpha}\right)^{\frac{N-1}{N}}$ sufficiently small such that

$$1/N - \left(1 + \xi^{2N-(p+q)-2\beta-\mu} + \xi^{2r-(p+q)}\right) C_{\lambda,p,q}\xi^{p+q-N} > 0.$$

Therefore, if $\|(u,v)\| = \xi$, then $\mathcal{J}_\lambda(u,v) \geqslant \delta$, where

$$\delta := \xi^N\left[1/N - \left(1 + \xi^{2N-(p+q)-2\beta-\mu} + \xi^{2r-(p+q)}\right) C_{\lambda,p,q}\xi^{p+q-N}\right] > 0,$$

which finishes the proof of item (i).

Regarding to (ii), let us take a fixed positive function $(u_0, v_0) \in \mathbb{W}^N$. By (d), we have

$$0 \leqslant \lim_{s\to+\infty}\frac{F(s)}{f(s)s} \leqslant \lim_{s\to+\infty}\frac{M_0}{s^{m_0+1}} = 0 \quad \text{and} \quad 0 \leqslant \lim_{s\to+\infty}\frac{G(s)}{g(s)s} \leqslant \lim_{s\to+\infty}\frac{M_0}{s^{m_0+1}} = 0.$$

Hence, for every $\epsilon > 0$, there exists $\bar{s}_\epsilon > 0$ such that

$$F(s) \leqslant \epsilon s f(s) \quad \text{and} \quad G(s) \leqslant \epsilon s g(s), \quad \forall s \geqslant \bar{s}_\epsilon. \tag{5.13}$$

Choosing $\epsilon = 1/r > 0$ in (5.13) such that $r > N$, leads to

$$rF(s) \leqslant sf(s) \quad \text{and} \quad rG(s) \leqslant sg(s), \quad \forall s \geqslant \bar{s}_\epsilon =: \bar{s}. \tag{5.14}$$

Consequently, by (5.14) there are positive constants a_1 and b_1 such that

$$F(s) \geqslant a_1 s^r \quad \text{and} \quad G(s) \geqslant b_1 s^r, \quad \forall s \geqslant \bar{s}. \tag{5.15}$$

Now, for t sufficiently large, using (5.15) with $s = ts_i$, $i = 1, 2$, we may infer for all $s_i > 0$ that the following estimates hold

$$F(ts_1)F(ts_2) \geqslant a_1^2 t^{2r} s_1^r s_2^r \quad \text{and} \quad G(ts_1)G(ts_2) \geqslant b_1^2 t^{2r} s_1^r s_2^r. \tag{5.16}$$

By Proposition 1.0.2, since that $r > N$, the estimates hold true

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_0^r(y)u_0^r(x)}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x \leqslant C(N,\beta,\mu)\|u_0\|^{2r}_{\frac{2Nr}{2N-2\beta-\mu}} < +\infty$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_0^r(y)v_0^r(x)}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x \leqslant C(N,\beta,\mu)\|v_0\|^{2r}_{\frac{2Nr}{2N-2\beta-\mu}} < +\infty.$$

From this and (5.16), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(tu_0(y))F(tu_0(x))}{|y|^{\beta}|x-y|^{\mu}|x|^{\beta}}\,\mathrm{d}y\mathrm{d}x \geqslant a_1^2t^{2r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_0^r(y)u_0^r(x)}{|y|^{\beta}|x-y|^{\mu}|x|^{\beta}}\,\mathrm{d}y\mathrm{d}x$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(tv_0(y))G(tv_0(x))}{|y|^{\beta}|x-y|^{\mu}|x|^{\beta}}\,\mathrm{d}y\mathrm{d}x \geqslant b_1^2t^{2r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_0^r(y)v_0^r(x)}{|y|^{\beta}|x-y|^{\mu}|x|^{\beta}}\,\mathrm{d}y\mathrm{d}x.$$

Hence,

$$\begin{aligned}&\mathcal{J}_\lambda(tu_0,tv_0)\\ \leqslant&\frac{t^N}{N}\|(u_0,v_0)\|^N-\frac{a_1^2t^{2r}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_0^r(y)u_0^r(x)}{|y|^{\beta}|x-y|^{\mu}|x|^{\beta}}\,\mathrm{d}y\mathrm{d}x\\ &-\frac{b_1^2t^{2r}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_0^r(y)v_0^r(x)}{|y|^{\beta}|x-y|^{\mu}|x|^{\beta}}\,\mathrm{d}y\mathrm{d}x =: k_1t^N-k_2t^{2r}-k_3t^{2r},\quad \text{with}\quad k_1,\,k_2,\,k_3>0.\end{aligned}$$

Thus, $\mathcal{J}_\lambda(tu_0,tv_0)\to-\infty$, as $t\to+\infty$, since that $r>N$. Therefore, there exists $(\tilde{u},\tilde{v}):=t(u_0,v_0)\in\mathbb{W}^N$ with $\|(\tilde{u},\tilde{v})\|>\xi$ and $\mathcal{J}_\lambda(\tilde{u},\tilde{v})<0$, we conclude the proof of item (ii). □

Henceforth, for the sake of simplicity, we denote $\mathcal{J}_\lambda=\mathcal{J}_\lambda|_{\mathbb{W}^N_{\mathrm{rad}}}:\mathbb{W}^N_{\mathrm{rad}}\to\mathbb{R}$. In view of Definition 2.2.3, Lemma 5.2.1 and Mountain Pass Theorem (WILLEM, 1996), there exists a Palais-Smale sequence $(u_n,v_n)_n\subset\mathbb{W}^N_{\mathrm{rad}}$ such that

$$\mathcal{J}_\lambda(u_n,v_n)\to m_\lambda\quad\text{and}\quad\mathcal{J}'_\lambda(u_n,v_n)\to0, \tag{5.17}$$

i.e., $(u_n,v_n)_n$ is a $(PS)_{m_\lambda}$ sequence for $\mathcal{J}_\lambda$. Moreover, Lemma 5.2.1 ensures that $m_\lambda>0$.

5.2.2 Nehari manifold

Next, we shall discuss some properties on the Nehari manifold associated our main System (S_λ), which is defined as follows

$$\mathcal{N}_\lambda=\left\{(u,v)\in\mathbb{W}^N_{\mathrm{rad}}\setminus\{(0,0)\}:\mathcal{J}'_\lambda(u,v)(u,v)=0\right\}.$$

Notice that if $(u,v)\in\mathcal{N}_\lambda$, then

$$\begin{aligned}\|(u,v)\|^N=&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u)}{|x|^{\beta}}u\,\mathrm{d}x\\ &+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v)}{|x|^{\beta}}v\,\mathrm{d}x+\lambda(p+q)\int_{\mathbb{R}^N}|u|^p|v|^q\,\mathrm{d}x.\end{aligned} \tag{5.18}$$

We first show that the number $c_{\mathcal{N}_\lambda}=\inf\limits_{(u,v)\in\mathcal{N}_\lambda}\mathcal{J}_\lambda(u,v)$ is well defined by proving that $\mathcal{J}_\lambda$ is bounded from below on $\mathcal{N}_\lambda$ and the set $\mathcal{N}_\lambda$ is non-empty.

Lemma 5.2.2. *The functional $\mathcal{J}_\lambda$ is bounded from below on $\mathcal{N}_\lambda$.*

Proof. In view of assumption (e), we have that

$$f(s)s - NF(s) > 0 \quad \text{and} \quad g(s)s - NG(s) > 0, \quad \forall s > 0, \tag{5.19}$$

which implies that

$$\frac{1}{N}f(s)s - \frac{1}{2}F(s) > 0 \quad \text{and} \quad \frac{1}{N}g(s)s - \frac{1}{2}G(s) > 0, \quad \forall s > 0. \tag{5.20}$$

Therefore, (5.18) and (5.20) lead to

$$\begin{aligned}
&\mathcal{J}_\lambda(u,v) = \frac{1}{N}\|(u,v)\|^N - \lambda \int_{\mathbb{R}^N} |u|^p|v|^q \,\mathrm{d}x \\
&- \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\beta}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v)}{|x|^\beta}\,\mathrm{d}x \\
=&\lambda\left(\frac{p+q}{N} - 1\right)\int_{\mathbb{R}^N} |u|^p|v|^q\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{\left[\frac{1}{N}f(u)u - \frac{1}{2}F(u)\right]}{|x|^\beta}\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{\left[\frac{1}{N}g(u)u - \frac{1}{2}G(u)\right]}{|x|^\beta}\,\mathrm{d}x \geqslant 0,
\end{aligned}$$

which finishes the proof. □

Lemma 5.2.3. *For any $(u,v) \in \mathbb{W}^N_{\mathrm{rad}} \setminus \{(0,0)\}$, there exists a unique $t_0 > 0$, depending on (u,v), such that*

$$(t_0u, t_0v) \in \mathcal{N}_\lambda \quad \textit{and} \quad \max_{t \geqslant 0} \mathcal{J}_\lambda(t(u,v)) = \mathcal{J}_\lambda(t_0(u,v)).$$

Proof. First, we note that (5.19) ensures

$$\frac{F(s)}{s^N}, \frac{G(s)}{s^N}, \quad \text{are increasing for} \quad s > 0$$

and, consequently, $F(s)$ and $G(s)$ are also increasing for $s > 0$. Hence, for all $s_1, s_2, t_1, t_2 \in (0, +\infty)$, if $t_1 < t_2$, then

$$\frac{F(t_1s_1)}{t_1^N s_1}\frac{f(t_1s_2)}{t_1^{N-1}s_2}s_2^2t_1^N s_1 \leqslant \frac{F(t_2s_1)}{t_2^N s_1}\frac{f(t_2s_2)}{t_2^{N-1}s_2}s_2^2t_2^N s_1$$

and

$$\frac{G(t_1s_1)}{t_1^N s_1}\frac{g(t_1s_2)}{t_1^{N-1}s_2}s_2^2t_1^N s_1 \leqslant \frac{G(t_2s_1)}{t_2^N s_1}\frac{g(t_2s_2)}{t_2^{N-1}s_2}s_2^2t_2^N s_1.$$

Hence, the functions

$$\begin{cases}
\dfrac{1}{t^{N-1}}\displaystyle\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(tu)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(tu)}{|x|^\beta}u\,\mathrm{d}x, \\[2ex]
\dfrac{1}{t^{N-1}}\displaystyle\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(tv)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(tv)}{|x|^\beta}v\mathrm{d}x
\end{cases} \tag{5.21}$$

are increasing for $t > 0$.

Let $(u,v) \in \mathbb{W}^N_{\mathrm{rad}} \setminus \{(0,0)\}$ be fixed and consider the function $\varphi : [0,+\infty) \to \mathbb{R}$ defined by $\varphi(t) = \mathcal{J}_\lambda(t(u,v))$. Notice that $\varphi'(t)t = \mathcal{J}'_\lambda(t(u,v))t(u,v)$. Thus, $\varphi'(t) = 0$ if only if $\mathcal{J}'_\lambda(t(u,v))t(u,v) = 0$, implying that

$$\begin{aligned}\|(u,v)\|^N - \lambda(p+q)t^{p+q-N}\int_{\mathbb{R}^N}|u|^p|v|^q\,\mathrm{d}x\\ =&\frac{1}{t^{N-1}}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(tu)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(tu)}{|x|^\beta}u\,\mathrm{d}x\right.\\ &\left.+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(tv)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(tv)}{|x|^\beta}v\,\mathrm{d}x\right].\end{aligned}\tag{5.22}$$

Therefore, t_0 is a positive critical point of φ if and only if $(t_0u, t_0v) \in \mathcal{N}_\lambda$. Moreover, note that (5.21) ensures that the right-hand side of (5.22) is an increasing function on $t > 0$. From (5.8), (5.9) and (5.11), follows by the definition of φ that

$$\begin{aligned}\varphi(t) \geqslant& t^N\|(u,v)\|^N\left[\frac{1}{N} - \lambda C t^{p+q-N}\|(u,v)\|^{p+q-N} - \hat{C}t^{N-2\beta-\mu}\|(u,v)\|^{N-2\beta-\mu}\right.\\ &\left.-\hat{C}t^{2r-N}\|(u,v)\|^{2r-N}\right] > 0,\end{aligned}$$

provided $t > 0$ is sufficiently small. On the other hand, arguing as in the proof of Lemma 5.2.1 (ii), one may check that $\varphi(t) \leqslant \frac{k_1}{N}t^N - k_2t^{2r} - k_3t^{2r}$, where k_1, k_2, k_3 are positive constants. Hence, since $r > N$, we have $\varphi(t) < 0$ for $t > 0$ sufficiently large. Consequently, φ has maximum points in $(0,+\infty)$. Suppose that there exist $t_1, t_2 > 0$ with $t_1 < t_2$ such that $\varphi'(t_1) = \varphi'(t_2) = 0$. In view of (5.22), we have that

$$\begin{aligned}&\frac{1}{t_1^{N-1}}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(t_1u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(t_1u)}{|x|^\beta}u\,\mathrm{d}x\right.\\ &\qquad\left.+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(t_1v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(t_1v)}{|x|^\beta}v\,\mathrm{d}x\right]\\ &\qquad-\frac{1}{t_2^{N-1}}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(t_2u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(t_2u)}{|x|^\beta}u\,\mathrm{d}x\right.\\ &\qquad\left.+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(t_2v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(t_2v)}{|x|^\beta}v\,\mathrm{d}x\right] = 0.\end{aligned}$$

Therefore, since the right-hand side of (5.22) is an increasing function on $t > 0$, it follows that $u = v = 0$, which is impossible and the proof is complete. □

5.2.3 Comparing levels

As a consequence of Lemma 5.2.3, the set $\mathcal{N}_\lambda \neq \emptyset$. Moreover, combining Lemma 5.2.2 with Lemma 5.2.3 and recalling the definition of d_λ in (5.10), we may conclude that

$$c_{\mathcal{N}_\lambda} = \inf_{(u,v)\in\mathcal{N}_\lambda} \mathcal{J}_\lambda(u,v) \leqslant \inf_{(u,v)\in\mathbb{W}^N_{\mathrm{rad}}\setminus\{(0,0)\}} \max_{t\geqslant 0} \mathcal{J}_\lambda(t(u,v)) = d_\lambda. \tag{5.23}$$

The next lemma is crucial to achieve the objective of this section.

Lemma 5.2.4. *For any $(u,v)\in\mathcal{N}_\lambda$, there holds*

$$\max_{t\geqslant 0} \mathcal{J}_\lambda(t(u,v)) \leqslant \mathcal{J}_\lambda(u,v). \tag{5.24}$$

In addition, we have

$$d_\lambda = c_{\mathcal{N}_\lambda} \quad \text{and} \quad m_\lambda \leqslant c_{\mathcal{N}_\lambda}, \tag{5.25}$$

where m_λ was defined in (5.10).

Proof. Firstly, we will verify that for any $(u,v)\in\mathcal{N}_\lambda$, (5.24) holds true. According to (5.18), we may write

$$\begin{aligned}
&\mathcal{J}_\lambda(u,v) - \mathcal{J}_\lambda(t(u,v))\\
=&\frac{1-t^N}{N}\|(u,v)\|^N + \lambda(t^{p+q}-1)\int_{\mathbb{R}^N}|u|^p|v|^q\,\mathrm{d}x\\
&+\frac{1}{2}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(tu)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(tu)}{|x|^\beta}\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\beta}\,\mathrm{d}x\right]\\
&+\frac{1}{2}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(tv)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(tv)}{|x|^\beta}\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v)}{|x|^\beta}\,\mathrm{d}x\right]\\
=&\frac{1-t^N}{N}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}u\,\mathrm{d}x\right.\\
&\left.+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta}v\,\mathrm{d}x\right]\\
&+\left(\frac{1-t^N}{N}(p+q)+t^{p+q}-1\right)\lambda\int_{\mathbb{R}^N}|u|^p|v|^q\,\mathrm{d}x\\
&+\frac{1}{2}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(tu)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(tu)}{|x|^\beta}\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\beta}\,\mathrm{d}x\right]\\
&+\frac{1}{2}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(tv)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(tv)}{|x|^\beta}\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v)}{|x|^\beta}\,\mathrm{d}x\right].
\end{aligned}$$

On the other hand, note that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}u\,\mathrm{d}x+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta}v\,\mathrm{d}x\\
=&\frac{1}{2}\Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))f(u(x))u(x)+F(u(x))f(u(y))u(y)}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(v(y))g(v(x))v(x)+G(v(x))g(v(y))v(y)}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x\Bigg].
\end{aligned}
$$

These last two estimates take us

$$
\begin{aligned}
&\mathcal{J}_\lambda(u,v)-\mathcal{J}_\lambda(t(u,v))\\
=&\frac{1-t^N}{2N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(y))f(u(x))u(x)+F(u(x))f(u(y))u(y)}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x\\
&+\frac{1}{2}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(tu)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(tu)}{|x|^\beta}\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)\,\mathrm{d}y}{|y|^\beta|x-y|^\mu}\right)\frac{F(u)}{|x|^\beta}\,\mathrm{d}x\right]\\
&+\frac{1-t^N}{2N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(v(y))g(v(x))v(x)+G(v(x))g(v(y))v(y)}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x\\
&+\frac{1}{2}\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(tv)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(tv)}{|x|^\beta}\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v)}{|x|^\beta}\,\mathrm{d}x\right]\\
&+\left(\frac{1-t^N}{N}(p+q)+t^{p+q}-1\right)\lambda\int_{\mathbb{R}^N}|u|^p|v|^q\,\mathrm{d}x. \qquad (5.26)
\end{aligned}
$$

Now, for all $t_1,t_2,s_1,s_2\in(0,+\infty)$ we define

$$
\begin{aligned}
h_1(t,t_1,t_2)&:=\frac{1-t^N}{2N}\left[F(t_2)f(t_1)t_1+F(t_1)f(t_2)t_2\right]+\frac{1}{2}\left[F(tt_2)F(tt_1)-F(t_2)F(t_1)\right],\\
h_2(t,s_1,s_2)&:=\frac{1-t^N}{2N}\left[G(s_2)g(s_1)s_1+G(s_1)g(s_2)s_2\right]+\frac{1}{2}\left[G(ts_2)G(ts_1)-G(s_2)G(s_1)\right],\\
h_3(t,t_1,s_1)&:=\left(\frac{1-t^N}{N}(p+q)+t^{p+q}-1\right)\lambda t_1^p s_1^q.
\end{aligned}
$$

Thus, to prove (5.24) it is sufficient to verify that

$$
h_1(t,t_1,t_2),\ h_2(t,s_1,s_2),\ h_3(t,t_1,s_1)\geqslant 0,\quad \forall t\in(0,+\infty). \qquad (5.27)
$$

Following (ALVES; SHEN, 2023, Lemma 5.1), one may prove that $h_1(t,t_1,t_2),h_2(t,s_1,s_2)\geqslant 0$, for all $t\in(0,+\infty)$. Now, a direct computation shows that

$$
\begin{aligned}
\frac{\partial}{\partial t}h_3(t,t_1,s_1)=&(p+q)\left(t^{p+q-1}-t^{N-1}\right)\lambda t_1^p s_1^q\\
&\begin{cases}\geqslant 0, & \text{if}\quad t\in[1,+\infty),\\ <0, & \text{if}\quad t\in(0,1].\end{cases}
\end{aligned}
$$

Consequently, we reach that $h_3(t, t_1, s_1)$ is decreasing on $t \in (0, 1)$ and increasing on $t \in (1, +\infty)$, implying that

$$h_3(t, t_1, s_1) \geqslant \min_{t>0} h(t, t_1, s_1) = h(1, t_1, s_1) = 0.$$

Thus, combining (5.26) with (5.27) we obtain (5.24). Therefore, (5.23) and (5.24) imply $d_\lambda = c_{\mathcal{N}_\lambda}$.

Finally, observe that for any $(u, v) \in \mathbb{W}^N_{\text{rad}}$, there holds $\mathcal{J}_\lambda(t(u, v)) \to -\infty$, as $t \to +\infty$. Let $T > 0$ be such that $\mathcal{J}_\lambda(T(u, v)) < 0$. Hence, the path $\gamma(t) = tT(u, v)$ belongs to Γ. Now using (5.24) and the definition of mountain pass level, for any $(u, v) \in \mathcal{N}_\lambda$, we have

$$m_\lambda \leqslant \max_{t\in[0,1]} \mathcal{J}_\lambda(tT(u, v)) \leqslant \max_{s\geqslant 0} \mathcal{J}_\lambda(s(u, v)) = \mathcal{J}_\lambda(u, v).$$

Therefore, $m_\lambda \leqslant c_{\mathcal{N}_\lambda}$, which finishes the proof. □

Let (u_0, v_0) be a mountain pass solution for System (S_λ) (see Subsection 5.2.5.1). Thus

$$c_\lambda = \inf_{(u,v)\in\mathcal{S}_\lambda} \mathcal{J}_\lambda(u, v) \leqslant \mathcal{J}_\lambda(u_0, v_0) = m_\lambda.$$

Therefore

$$c_\lambda \leqslant m_\lambda. \tag{5.28}$$

Since $\mathcal{S}_\lambda \subset \mathcal{N}_\lambda$, there holds $c_{\mathcal{N}_\lambda} \leqslant c_\lambda$. Combining (5.23), (5.25) and (5.28), we conclude that

$$d_\lambda = c_{\mathcal{N}_\lambda} \leqslant c_\lambda \leqslant m_\lambda \leqslant c_{\mathcal{N}_\lambda} \leqslant d_\lambda$$

which implies that $c_\lambda = m_\lambda = d_\lambda$.

5.2.4 Estimate of the minimax level

In this section we establish a suitable upper bound to the mountain pass minimax level, which is very crucial to obtain compactness results in the next section.

Lemma 5.2.5. *The energy level $c_{\mathcal{N}_\lambda}$ satisfies*

$$c_{\mathcal{N}_\lambda} < C(\alpha_0, \alpha_N, \beta, \mu) := \frac{1}{N}\left(\frac{2N - 2\beta - \mu}{2N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. The technique to show the estimate on minimax level involves the celebrated idea of analysis of Moser functions given by J. Moser (MOSER, 1971). For some fixed $\tilde{\rho} > 0$, consider the following Moser sequence supported in $B_{\tilde{\rho}}$ defined as

$$\overline{u}_n = \frac{1}{\omega_{N-1}^{\frac{1}{N}}} \begin{cases} \log(n)^{\frac{N-1}{N}}, & 0 \leqslant |x| \leqslant \frac{\tilde{\rho}}{n}, \\ \\ \dfrac{\log(\tilde{\rho}/|x|)}{\log(n)^{\frac{1}{N}}}, & \frac{\tilde{\rho}}{n} \leqslant |x| \leqslant \tilde{\rho}, \\ \\ 0, & |x| \geqslant \tilde{\rho}. \end{cases}$$

From direct computations, we have

$$\|\overline{u}_n\|^N = \int_{\mathbb{R}^N} |\nabla \overline{u}_n|^N \,\mathrm{d}x + \int_{\mathbb{R}^N} |\overline{u}_n|^N \,\mathrm{d}x = 1 + O\left(\frac{1}{\log(n)}\right)$$

and thus $\|\overline{u}_n\| = \left[1 + O\left(\frac{1}{\log(n)}\right)\right]^{\frac{1}{N}}$.

By setting $u_n = \frac{\overline{u}_n}{\|\overline{u}_n\|}$, we have $\|u_n\| \leqslant 1$ and

$$u_n = \frac{(\log(n))^{\frac{N-1}{N}} / \omega_{N-1}^{\frac{1}{N}}}{\left[1 + O\left(\frac{1}{\log(n)}\right)\right]^{\frac{1}{N}}} \equiv \frac{(\log(n))^{\frac{N-1}{N}}}{\omega_{N-1}^{\frac{1}{N}}} + a_n, \quad 0 \leqslant |x| \leqslant \frac{\tilde{\rho}}{n}, \tag{5.29}$$

where $(a_n)_n$ is a bounded sequence of non-negative real numbers. It suffices to show that there exists $(u, v) \in \mathbb{W}_{\mathrm{rad}}^N \setminus \{(0,0)\}$ such that $c_{\mathcal{N}_\lambda} \leqslant \max_{t \geqslant 0} \mathcal{J}_\lambda(tu, tv) < C(\alpha_0, \alpha_N, \beta, \mu)$. We claim that there exists $n \in \mathbb{N}$ such that

$$\max_{t \geqslant 0} \mathcal{J}_\lambda(tu_n, 0) < C(\alpha_0, \alpha_N, \beta, \mu). \tag{5.30}$$

Suppose, by contradiction, that for any $n \in \mathbb{N}$ there exists $t_n > 0$, such that

$$\max_{t \in [0,\infty)} \mathcal{J}_\lambda(tu_n, 0) = \mathcal{J}_\lambda(t_n u_n, 0) \geqslant C(\alpha_0, \alpha_n, \beta, \mu). \tag{5.31}$$

Hence, t_n satisfies $\frac{d}{dt}(\mathcal{J}_\lambda(tu_n, 0))|_{t=t_n} = 0$. Since $\|u_n\|^N \leqslant 1$, there holds

$$t_n^N \geqslant \int_{B_{\tilde{\rho}/n}(0)} \left(\int_{B_{\tilde{\rho}/n}(0)} \frac{F(t_n u_n)}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(t_n u_n)}{|x|^\beta} t_n u_n \,\mathrm{d}x. \tag{5.32}$$

It follows from (5.31) and the fact that $F(s) \geqslant 0$ for all $s \in \mathbb{R}$, that

$$N C(\alpha_0, \alpha_n, \beta, \mu) \leqslant N \mathcal{J}_\lambda(t_n u_n, 0) \leqslant t_n^N \|u_n\|^N,$$

i.e.,

$$t_n^N \geqslant \left(\frac{2N - 2\beta - \mu}{2N} \frac{\alpha_N}{\alpha_0} \right)^{N-1}. \tag{5.33}$$

On the other hand, it follows from assumptions (c) and (d) that for each $\epsilon \in (0, \beta_0)$, there exists $R_\epsilon > 0$ such that

$$F(s)f(s)s \geqslant M_0^{-1}(\beta_0 - \epsilon)s^{m_0+1}e^{2\alpha_0 s^{\frac{N}{N-1}}}, \quad \forall s \geqslant R_\epsilon. \tag{5.34}$$

Gathering the estimates (5.29), (5.32), (5.33) and (5.34), we obtain

$$\begin{aligned}
t_n^N \geqslant & \int_{B_{\tilde{\rho}/n}(0)} \int_{B_{\tilde{\rho}/n}(0)} \frac{F(t_n u_n(y))f(t_n u_n(x))t_n u_n(x)}{|y|^\beta |x-y|^\mu |x|^\beta} \mathrm{d}y\mathrm{d}x \\
\geqslant & \int_{B_{\tilde{\rho}/n}(0)} \int_{B_{\tilde{\rho}/n}(0)} \frac{M_0^{-1}(\beta_0 - \epsilon)(t_n u_n)^{m_0+1} e^{2\alpha_0 (t_n u_n)^{\frac{N}{N-1}}}}{|y|^\beta |x-y|^\mu |x|^\beta} \mathrm{d}y\mathrm{d}x \\
\geqslant & \frac{(\beta_0 - \epsilon)t_n^{m_0+1}}{M_0} \left(\frac{\log(n)}{\omega_{N-1}^{\frac{1}{N-1}}} + a_n \right)^{m_0+1} \exp\left(\frac{2\alpha_0 t_n^{\frac{N}{N-1}} \log(n)}{\omega_{N-1}^{\frac{1}{N-1}}} + 2\alpha_0 t_n^{\frac{N}{N-1}} a_n \right) \\
& \times \int_{B_{\tilde{\rho}/n}(0)} \int_{B_{\tilde{\rho}/n}(0)} \frac{\mathrm{d}y\mathrm{d}x}{|y|^\beta |x-y|^\mu |x|^\beta} \\
\geqslant & \frac{(\beta_0 - \epsilon)t_n^{m_0+1}}{M_0} \left(\frac{\log(n)}{\omega_{N-1}^{\frac{1}{N-1}}} + a_n \right)^{m_0+1} \exp\left(\frac{2\alpha_0 t_n^{\frac{N}{N-1}} \log(n)}{\omega_{N-1}^{\frac{1}{N-1}}} + 2\alpha_0 t_n^{\frac{N}{N-1}} a_n \right) \\
& \times C_{N,\beta,\mu} \left(\frac{\tilde{\rho}}{n} \right)^{2N-2\beta-\mu} \\
= & \frac{C_{N,\beta,\mu}(\beta_0 - \epsilon)\rho^{2N-2\beta-\mu}}{M_0 t_n^{-(m_0+1)}} \left(\frac{\log(n)}{\omega_{N-1}^{\frac{1}{N-1}}} + a_n \right)^{m_0+1} \\
& \times \exp\left[\log(n) \left(\frac{2\alpha_0 t_n^{\frac{N}{N-1}}}{\omega_{N-1}^{\frac{1}{N-1}}} - (2N - 2\beta - \mu) \right) + 2\alpha_0 t_n^{\frac{N}{N-1}} a_n \right],
\end{aligned} \tag{5.35}$$

where we used that $n^{-(2N-2\beta-\mu)} = \exp\left[\log(n^{-(2N-2\beta-\mu)})\right] = \exp\left[-(2N-2\beta-\mu)\log(n)\right]$. Note that

$$\begin{aligned}
\left(\frac{\log(n)}{\omega_{N-1}^{\frac{1}{N-1}}} + a_n \right)^{m_0+1} \geqslant \left(\frac{\log(n)}{\omega_{N-1}^{\frac{1}{N-1}}} \right)^{m_0+1} &= \left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \right)^{m_0+1} \log(n)^{m_0+1} \\
&= \left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \right)^{m_0+1} \exp\left((m_0+1)\log(\log(n)) \right),
\end{aligned}$$

which together with the estimate (5.35), we get

$$\begin{aligned}
t_n^N \geqslant & \frac{C_{N,\beta,\mu}\tilde{\rho}^{2N-2\beta-\mu}(\beta_0 - \epsilon)}{M_0 t_n^{-(m_0+1)}} \left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \right)^{m_0+1} \exp[(m_0+1)\log(\log(n))] \\
& \times \exp\left[\log(n) \left(\frac{2\alpha_0 t_n^{\frac{N}{N-1}}}{\omega_{N-1}^{\frac{1}{N-1}}} - (2N - 2\beta - \mu) \right) + 2\alpha_0 t_n^{\frac{N}{N-1}} a_n \right].
\end{aligned} \tag{5.36}$$

Since $(m_0+1)\log(\log(n)) + 2\alpha_0 t_n^{\frac{N}{N-1}} a_n > 0$ and $N-1-m_0>0$, we infer

$$t_n^{(N-1)-m_0} \geqslant \frac{C_{N,\beta,\mu}\tilde{\rho}^{2N-2\beta-\mu}(\beta_0-\epsilon)}{M_0}\left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\right)^{m_0+1} \times \exp\left[\log(n)\left(\frac{2\alpha_0 t_n^{\frac{N}{N-1}}}{\omega_{N-1}^{\frac{1}{N-1}}} - (2N-2\beta-\mu)\right)\right].$$

Consequently, we deduce

$$((N-1)-m_0)\frac{\log(t_n)}{t_n^{\frac{N}{N-1}}} \geqslant \frac{1}{t_n^{\frac{N}{N-1}}}\log\left[\frac{C_{N,\beta,\mu}\tilde{\rho}^{2N-2\beta-\mu}(\beta_0-\epsilon)}{M_0}\left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\right)^{m_0+1}\right] + \log(n)\left(\frac{2\alpha_0}{\omega_{N-1}^{\frac{N}{N-1}}} - \frac{2N-2\beta-\mu}{t_n^{\frac{N}{N-1}}}\right), \tag{5.37}$$

which implies that $(t_n)_n$ is bounded. Therefore, up to a subsequence, $t_n \to \tilde{t}$, as $n\to\infty$.

Claim. $t_n^{\frac{N}{N-1}} \to \dfrac{2N-2\beta-\mu}{2N}\dfrac{\alpha_N}{\alpha_0}$, as $n\to\infty$.

Indeed, otherwise, in view of (5.33), there would be $\delta>0$, such that $t_n^{\frac{N}{N-1}} - \left(\frac{2N-2\beta-\mu}{2N}\frac{\alpha_N}{\alpha_0}\right) \geqslant \delta$. Applying this in (5.37) and remembering that $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, we obtain

$$\begin{aligned}
&((N-1)-m_0)\log(t_n)\\
&\geqslant \log\left[\frac{C_{N,\beta,\mu}\tilde{\rho}^{2N-2\beta-\mu}(\beta_0-\epsilon)}{M_0}\left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\right)^{m_0+1}\right] + \left[\frac{2\alpha_0 t_n^{\frac{N}{N-1}}}{\omega_{N-1}^{\frac{N}{N-1}}} - (2N-2\beta-\mu)\right]\log(n)\\
&\geqslant \log\left[\frac{C_{N,\beta,\mu}\tilde{\rho}^{2N-2\beta-\mu}(\beta_0-\epsilon)}{M_0}\left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\right)^{m_0+1}\right]\\
&\quad + \left[\frac{2\alpha_0\left(\delta + \frac{2N-2\beta-\mu}{2N}\frac{\alpha_N}{\alpha_0}\right)}{\omega_{N-1}^{\frac{1}{N-1}}} - (2N-2\beta-\mu)\right]\log(n)\\
&= \log\left[\frac{C_{N,\beta,\mu}\tilde{\rho}^{2N-2\beta-\mu}(\beta_0-\epsilon)}{M_0}\left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\right)^{m_0+1}\right] + \frac{2\alpha_0\delta}{\omega_{N-1}^{\frac{1}{N-1}}}\log(n)\\
&\geqslant \frac{2\alpha_0\delta}{\omega_{N-1}^{\frac{1}{N-1}}}\log(n) = \log(n^{\delta_N}), \quad \delta_N := \frac{2\alpha_0\delta}{\omega_{N-1}^{\frac{1}{N-1}}},
\end{aligned}$$

which leads to $t_n^{(N-1)-m_0} \geqslant n^{\delta_N}$, where $(N-1)-m_0>0$. Hence, we obtain a contradiction with the fact that $(t_n)_n$ is a bounded sequence. The claim is proved.

According to (5.36), we may write

$$t_n^N \geqslant C_{N,\beta,\mu}\tilde{\rho}^{2N-2\beta-\mu}M_0^{-1}(\beta_0-\epsilon)t_n^{m_0+1}\left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\right)^{m_0+1}\exp\left(\log(\log(n))\right),$$

which jointly with the last claim, imply that

$$\left(\frac{2N-2\beta-\mu}{2N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \geqslant C\left(\frac{2N-2\beta-\mu}{2N}\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}(m_0+1)}\left(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\right)^{m_0+1} \exp\left(\log(\log(n))\right)$$

$$\to \infty, \quad \text{as } n \to \infty,$$

where $C := C_{N,\beta,\mu}\tilde{\rho}^{2N-2\beta-\mu}M_0^{-1}(\beta-\epsilon)$, which is a contradiction. Therefore, (5.30) is valid and this completes the proof. □

The next lemma establishes the existence of $c > 0$ such that $c < C(\alpha_0, \alpha_N, \beta, \mu) := \frac{1}{N}\left(\frac{2N-2\beta-\mu}{2N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$ and a $(PS)_c-$sequence for $\mathcal{J}_\lambda$. The proof follows standard arguments explored, for example, in (CHEN; LIU, 2018, Lemma 2.10).

Lemma 5.2.6. *There exist a constant $c \in (0, c_{\mathcal{N}_\lambda}]$ and a sequence $(u_n, v_n)_n \subset \mathbb{W}^N_{\mathrm{rad}}$ satisfying*

$$\mathcal{J}_\lambda(u_n, v_n) \to c \quad \textit{and} \quad \mathcal{J}_\lambda'(u_n, v_n) \to 0.$$

Proof. Initially we choose $(u_k, v_k)_k \in \mathcal{N}_\lambda$ such that

$$c_{\mathcal{N}_\lambda} \leqslant \mathcal{J}_\lambda(u_k, v_k) < c_{\mathcal{N}_\lambda} + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{5.38}$$

Once $\mathcal{J}_\lambda(tu_k, tv_k) < 0$ for large enough $t > 0$, it follows from Lemma 5.0.1 and Mountain Pass Theorem that there exist $(u_{k,n}, v_{k,n})_n \subset \mathbb{W}^N_{\mathrm{rad}}$ and $c_k \in [\delta, \sup_{t\geqslant 0}\mathcal{J}_\lambda(tu_k, tv_k)]$ such that

$$\mathcal{J}_\lambda(u_{k,n}, v_{k,n}) \to c_k, \quad \mathcal{J}_\lambda'(u_{k,n}, v_{k,n}) \to 0, \quad k \in \mathbb{N}, \tag{5.39}$$

where $\delta > 0$ is given in Lemma 5.2.1. By (5.24), we have $\mathcal{J}_\lambda(tu_k, tv_k) \leqslant \mathcal{J}_\lambda(u_k, v_k)$ for all $t \geqslant 0$, i.e., $\mathcal{J}_\lambda(u_k, v_k) = \sup_{t\geqslant 0}\mathcal{J}_\lambda(tu_k, tv_k)$. Hence, from (5.38) and (5.39), for all $k \in \mathbb{N}$, we obtain

$$\mathcal{J}_\lambda(u_{k,n}, v_{k,n}) \to c_k \in \left[\delta, c_{\mathcal{N}_\lambda} + \frac{1}{k}\right), \quad \mathcal{J}_\lambda'(u_{k,n}, v_{k,n}) \to 0.$$

Now, let $(n_k)_k \subset \mathbb{N}$ be a sequence such that

$$\mathcal{J}_\lambda(u_{k,n_k}, v_{k,n_k}) \in \left[\delta, c_{\mathcal{N}_\lambda} + \frac{1}{k}\right), \quad \mathcal{J}_\lambda'(u_{k,n_k}, v_{k,n_k}) < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Define $(u_k, v_k) := (v_{k,n_k}, v_{k,n_k})$. Therefore, passing to a subsequence if necessary, we reach

$$\mathcal{J}_\lambda(u_n, v_n) \to c \in [\delta, c_{\mathcal{N}_\lambda}] \quad \text{and} \quad \mathcal{J}_\lambda'(u_n, v_n) \to 0$$

and the lemma is proved. □

5.2.5 Compactness results

In this section we will establish compactness results which play a very important role to verify that the functional satisfies the $(PS)_c-$condition. It is important to emphasize that in the case $\beta = 0$, the boundedness of the nonlocal term

$$\int_{\mathbb{R}^N} \frac{F(u)}{|x-y|^\mu}\,\mathrm{d}y, \quad \forall x \in \mathbb{R}^N$$

is strongly used to prove that the functional satisfies the $(PS)_c-$ condition, see for instance (ALVES; YANG, 2014; ALVES et al., 2016b; ALVES; YANG, 2016). However, as emphasized in Remark 2.1.15 in Chapter 2, due to the presence of the double weight $\frac{1}{|x|^\beta}$ and $\frac{1}{|y|^\beta}$, the convolution

$$\frac{1}{|x|^\beta}\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y$$

does not inherit such property in $\mathbb{R}^N$. But, removing the term $\frac{1}{|x|^\beta}$, it is possible to prove boundedness of

$$\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y.$$

This boundedness is one of the tools to show the following convergences:

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x \to \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_0)}{|x|^\beta}\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v_n)}{|x|^\beta}\,\mathrm{d}x \to \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(v_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v_0)}{|x|^\beta}\,\mathrm{d}x.$$

For more details see Lemma 5.2.9.

Lemma 5.2.7. *For any $u \in W^{1,N}(\mathbb{R}^N)$, we have that*

$$\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y,\ \int_{\mathbb{R}^N} \frac{G(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y \in L^\infty(\mathbb{R}^N).$$

Proof. We follow some ideas of Lemma 3.44 in Chapter 4. For $u \in W^{1,N}(\mathbb{R}^N)$ and $R > 0$, we write

$$\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y \leqslant \int_{B_R(0)} \frac{|F(u)|}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y + \int_{B^c_R(0)} \frac{|F(u)|}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y. \tag{5.40}$$

On the one hand, for $x \in B^c_{2R}(0)$, we have $|x-y| > |y|$ and

$$\int_{B_R(0)} \frac{|F(u)|}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y \leqslant \int_{B_R(0)} \frac{|F(u)|}{|y|^{\mu+\beta}}\,\mathrm{d}y.$$

Now, let $p_1, k > 0$ be such that $\beta + \mu < p_1 < N$ and $k := \frac{N}{N-p_1}$. Since $k > 1$, it follows from Sobolev embedding that $F \in L^k(B_R(0))$. Hence, by Hölder's inequality and since $N - 1 - (\alpha + \mu)\frac{k}{k-1} > -1$, we obtain

$$\begin{aligned}\int_{B_R(0)} \frac{|F(u)|}{|y|^{\mu+\beta}}\,\mathrm{d}y &\leqslant \left(\int_{B_R(0)} |F(u)|^k\,\mathrm{d}y\right)^{\frac{1}{k}} \left[\int_{B_R(0)} \left(\frac{1}{|y|^{\mu+\beta}}\right)^{\frac{k}{k-1}}\,\mathrm{d}y\right]^{\frac{k-1}{k}}\\ &\leqslant \|F(u)\|_{L^k(B_R(0))} \left(\int_0^R |r|^{N-1-(\beta+\mu)\frac{k}{k-1}}\,\mathrm{d}r\right)^{\frac{k-1}{k}} := C_2 < +\infty.\end{aligned}$$

For $x \in B_{2R}(0)$, using similar arguments, we deduce

$$\begin{aligned}\int_{B_R(0)} \frac{|F(u)|}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y \leqslant &\int_{B_R(0)} \frac{|F(u)|}{|y|^{\mu+\beta}}\,\mathrm{d}y + \int_{B_{3R}(x)} \frac{|F(u)|}{|x-y|^{\mu+\beta}}\,\mathrm{d}y\\ \leqslant &C_2 + C_3 \left(\int_0^{3R} |r|^{N-1-(\alpha+\mu)\frac{k}{k-1}}\,\mathrm{d}r\right)^{\frac{k-1}{k}} < +\infty.\end{aligned}$$

Hence, for each $x \in \mathbb{R}^N$, we conclude that

$$\int_{B_R(0)} \frac{|F(u)|}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y < +\infty. \tag{5.41}$$

On the other hand, we write

$$\begin{aligned}\int_{B_R^c(0)} \frac{|F(u)|}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y =&\int_{B_R^c(0)\cap B_R(x)} \frac{|F(u)|}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y + \int_{B_R^c(0)\cap B_R^c(x)} \frac{|F(u)|}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\\ =&: \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

Arguing as in the preceding estimates, we deduce

$$\begin{aligned}\mathcal{I}_1 = \int_{B_R^c(0)\cap B_R(x)} \frac{|F(u)|}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y \leqslant &\frac{1}{R^\beta} \int_{B_R^c(0)\cap B_R(x)} \frac{|F(u)|}{|x-y|^{\mu}}\,\mathrm{d}y\\ \leqslant &\frac{1}{R^\beta} \int_{B_R(x)} \frac{|F(u)|}{|x-y|^{\mu}}\,\mathrm{d}y < +\infty.\end{aligned}$$

Now, choosing $q_1 = \frac{2N}{2N-2\beta-\mu}, q_2 = \frac{2N}{\beta}, q_3 = \frac{2N}{\mu}$ satisfying $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \leqslant 1$, it follows from Hölder's inequality that

$$\begin{aligned}\mathcal{I}_2 \leqslant &\left(\int_{B_R^c(0)} |F(u)|^{\frac{2N}{2N-2\beta-\mu}}\,\mathrm{d}y\right)^{\frac{2N-2\beta-\mu}{2N}}\\ &\times \left(\int_{B_R^c(0)} \frac{1}{|y|^{2N}}\,\mathrm{d}y\right)^{\frac{\beta}{2N}} \left(\int_{B_R^c(x)} \frac{1}{|x-y|^{2N}}\,\mathrm{d}y\right)^{\frac{\mu}{2N}} < +\infty.\end{aligned}$$

Thus, we obtain

$$\int_{B_R^c(0)} \frac{|F(u)|}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y < +\infty. \tag{5.42}$$

Therefore, (5.40) jointly with (5.41) and (5.42) imply that the Lemma is proved. □

Lemma 5.2.8. *The $(PS)_c$ sequence $(u_n, v_n)_n$ of $\mathcal{J}_\lambda$ is bounded in $\mathbb{W}^N_{\mathrm{rad}}$.*

Proof. Let $(u_n, v_n)_n \subset \mathbb{W}^N_{\mathrm{rad}}$ be a $(PS)_c-$sequence for $\mathcal{J}_\lambda$, with $c < C(\alpha_0, \alpha_N, \beta, \mu)$, i.e.

$$\mathcal{J}_\lambda(u_n, v_n) = c + o_n(1) \quad \text{and} \quad \mathcal{J}'_\lambda(u_n, v_n) = o_n(1). \tag{5.43}$$

We deduce from (a) and (f) that

$$\begin{aligned}
&\mathcal{J}_\lambda(u_n, v_n) - \frac{1}{\theta}\mathcal{J}'_\lambda(u_n, v_n)(u_n, v_n)\\
&\quad \geqslant \left(\frac{1}{N} - \frac{1}{\theta}\right)\|(u_n, v_n)\|^N + \left(\frac{p+q}{\theta} - 1\right)\lambda\int_{\mathbb{R}^N}|u|^p|v|^q\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{\left[\frac{1}{\theta}f(u_n)u_n - \frac{1}{2}F(u_n)\right]}{|x|^\beta}\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|y-x|^\mu}\,\mathrm{d}y\right)\frac{\left[\frac{1}{\theta}g(v_n)v_n - \frac{1}{2}G(v_n)\right]}{|x|^\beta}\,\mathrm{d}x\\
&\quad \geqslant \left(\frac{1}{N} - \frac{1}{\theta}\right)\|(u_n, v_n)\|^N.
\end{aligned}$$

From $\mathcal{J}'_\lambda(u_n, v_n) = o_n(1)$, for n sufficiently large, we have $-\frac{1}{\theta}\mathcal{J}'_\lambda(u_n, v_n)(u_n, v_n) \leqslant \|(u_n, v_n)\|$. In this way, from last estimate and using the fact that $\mathcal{J}_\lambda(u_n, v_n) = c + o_n(1)$, we see

$$o_n(1) + c + \|(u_n, v_n)\| \geqslant \left(\frac{\theta - N}{\theta N}\right)\|(u_n, v_n)\|^N,$$

for n sufficiently large. Therefore, $(u_n, v_n)_n$ is bounded in $\mathbb{W}^N_{\mathrm{rad}}$. □

In view of the preceding results, if $(u_n, v_n)_n \subset \mathbb{W}^N_{\mathrm{rad}}$ is a $(PS)_c-$sequence for $\mathcal{J}_\lambda$, then $(u_n, v_n)_n$ is bounded and it follows from (5.43) that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}v_n\,\mathrm{d}x\right|\\
&\quad - \left|\|(u_n, v_n)\|^N - (p+q)\lambda\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x\right| \leqslant |\mathcal{J}'{}_\lambda(u_n, v_n)(u_n, v_n)| = o_n(1)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}v_n\,\mathrm{d}x\right|\\
&\leqslant \left|\|(u_n, v_n)\|^N - (p+q)\lambda\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x\right| + o_n(1).
\end{aligned}$$

Putting this together with the boundedness of $(u_n, v_n)_n$ in $\mathbb{W}^N_{\mathrm{rad}}$, there exists constant $K > 0$ such that

$$\begin{cases}
\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\,\mathrm{d}x \leqslant K,\\
\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}v_n\,\mathrm{d}x \leqslant K,
\end{cases} \tag{5.44}$$

which jointly with (5.19) implies that

$$\begin{cases} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{F(u_n)}{|x|^\beta} \, \mathrm{d}x \leqslant \hat{K}, \\ \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{G(v_n)}{|x|^\beta} \, \mathrm{d}x \leqslant \hat{K}. \end{cases} \tag{5.45}$$

Moreover, we may assume, passing to a subsequence if necessary, that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u_0, v_0), & \text{weakly in } \mathbb{W}^N_{\mathrm{rad}}, \\ (u_n, v_n) \to (u_0, v_0), & \text{strongly in } L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), \ p > N, \\ (u_n, v_n) \to (u_0, v_0), & \text{a.e. } x \in \mathbb{R}^N, \\ |u_n| \leqslant h, \, |u_0| \leqslant h, & \text{a.e. } x \in \mathbb{R}^N, \text{ for some } h \in L^p(\mathbb{R}^N), \\ |v_n| \leqslant \tilde{h}, \, |v_0| \leqslant \tilde{h}, & \text{a.e. } x \in \mathbb{R}^N, \text{ for some } \tilde{h} \in L^p(\mathbb{R}^N). \end{cases} \tag{5.46}$$

Gathering (5.44), (5.46) and Fatou's Lemma, one has

$$\begin{cases} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_0)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{f(u_0)}{|x|^\beta} u_0 \, \mathrm{d}x \leqslant K, \\ \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_0)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{g(v_0)}{|x|^\beta} v_0 \, \mathrm{d}x \leqslant K. \end{cases} \tag{5.47}$$

In view of assumption (d), for given $\epsilon > 0$, there exists $\bar{s} := \bar{s}(\epsilon) > 0$ such that

$$F(s) \leqslant \epsilon s f(s) \quad \text{and} \quad G(s) \leqslant \epsilon s g(s), \quad \forall s \geqslant \bar{s}, \tag{5.48}$$

which jointly with (5.44) and (5.47), imply that

$$\begin{cases} \sup_{n\in\mathbb{N}} \int_{\{|u_n|\geqslant\bar{s}\}} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{F(u_n)}{|x|^\beta} \, \mathrm{d}x \leqslant K\epsilon, \\ \int_{\{|u_0|\geqslant\bar{s}\}} \left(\int_{\mathbb{R}^N} \frac{F(u_0)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{F(u_0)}{|x|^\beta} \, \mathrm{d}x \leqslant K\epsilon, \end{cases} \tag{5.49}$$

and

$$\begin{cases} \sup_{n\in\mathbb{N}} \int_{\{|v_n|\geqslant\bar{s}\}} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{G(v_n)}{|x|^\beta} \, \mathrm{d}x \leqslant K\epsilon, \\ \int_{\{|v_0|\geqslant\bar{s}\}} \left(\int_{\mathbb{R}^N} \frac{G(v_0)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{G(v_0)}{|x|^\beta} \, \mathrm{d}x \leqslant K\epsilon. \end{cases} \tag{5.50}$$

Finally, we are in condition to prove the following compactness result:

Lemma 5.2.9. *Let $(u_n, v_n)_n$ be a $(PS)_c-$sequence for $\mathcal{J}_\lambda$ such that $c < C(\alpha_0, \alpha_N, \beta, \mu)$. Then, up to a subsequence, we have*

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{F(u_n)}{|x|^\beta} \, \mathrm{d}x \to \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_0)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{F(u_0)}{|x|^\beta} \, \mathrm{d}x \tag{5.51}$$

and

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{G(v_n)}{|x|^\beta} \, \mathrm{d}x \to \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_0)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{G(v_0)}{|x|^\beta} \, \mathrm{d}x. \tag{5.52}$$

Proof. We follow the ideas from (ALVES; SHEN, 2023, Lemma 4.6). Let us prove (5.51). For the sake of convenience, we will split the domain of integration in the following way

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_0)}{|x|^\beta}\,\mathrm{d}x\right|\\ \leqslant\int_{\{|u_n|\geqslant\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\\ +\int_{\{|u_0|\geqslant\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_0)}{|x|^\beta}\,\mathrm{d}x\\ +\int_{\{|u_0|=\bar{s}\}\cap\{|u_n|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\\ +\left|\int_{\{|u_0|\neq\bar{s}\}\cap\{|u_n|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\right.\\ \left.-\int_{\{|u_0|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_0)}{|x|^\beta}\,\mathrm{d}x\right|,\end{aligned}$$

which together with (5.50), becomes

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_0)}{|x|^\beta}\,\mathrm{d}x\right|\\ &\leqslant 2K\epsilon+\int_{\{|u_0|=\bar{s}\}\cap\{|u_n|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\\ &\quad+\left|\int_{\{|u_0|\neq\bar{s}\}\cap\{|u_n|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\right.\\ &\quad\left.-\int_{\{|u_0|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_0)}{|x|^\beta}\,\mathrm{d}x\right|.\end{aligned}\tag{5.53}$$

It follows from (5.3) and (5.4) that for given $\epsilon>0$, there exists $C(\bar{s})>0$ such that

$$|f(s)|\leqslant C(\bar{s})|s|^{\frac{2N-2\beta-\mu}{2}-1}\quad\text{and}\quad |F(s)|\leqslant C(\bar{s})|s|^{\frac{2N-2\beta-\mu}{2}},\quad \forall|s|\leqslant\bar{s}.\tag{5.54}$$

In order to prove (5.51) we will prove the following claims:

Claim 1. $\limsup_{n\to\infty}\int_{\{|u_0|=\bar{s}\}\cap\{|u_n|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x<C\epsilon.$

Claim 2. There holds

$$\begin{aligned}&\int_{\{|u_0|\neq\bar{s}\}\cap\{|u_n|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\\ &\qquad\to\int_{\{|u_0|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_0)}{|x|^\beta}\,\mathrm{d}x.\end{aligned}$$

Proof of Claim 1. In view of (5.45) and thanks to the Cauchy-Schwarz type inequality in (LIEB; LOSS, 2001, Theorem 9.8), it follows from Proposition 1.16 that

$$\int_{\{|u_0|=\bar{s}\}\cap\{|u_n|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x$$
$$\leqslant \hat{K}^{\frac{1}{2}}C(N,\beta,\mu)^{\frac{1}{2}}\left(\int_{\{|u_0|=\bar{s}\}}\left|F(u_n)\chi_{\{|u_n|<\bar{s}\}}\right|^{\frac{2N}{2N-2\beta-\mu}}\mathrm{d}y\right)^{\frac{2N-2\beta-\mu}{2N}}. \tag{5.55}$$

On the other hand, by (5.46) we note that $|u_n|^N \to |u_0|^N$ in $L^1(\{|u_0|=\bar{s}\})$. Hence, it follows from (5.54) that

$$|F(u_n)\chi_{|u_n|<\bar{s}\}}|^{\frac{2N}{2N-2\beta-\mu}} \leqslant C^{\frac{2N}{2N-2\beta-\mu}}(\bar{s})|u_n|^N \to C^{\frac{2N}{2N-2\beta-\mu}}(\bar{s})|u_0|^N.$$

Therefore, the Lebesgue's Dominated Convergence Theorem implies that

$$\int_{\{|u_0|=\bar{s}\}}\left|F(u_n)\chi_{\{|u_n|<\bar{s}\}}\right|^{\frac{2N}{2N-2\beta+\mu}}\mathrm{d}y \to \int_{\{|u_0|=\bar{s}\}}\left|F(u_0)\chi_{\{|u_0|<\bar{s}\}}\right|^{\frac{2N}{2N-2\beta+\mu}}\mathrm{d}y,$$

which together with (5.48) and (5.55), finishes the proof of Claim 1.

Proof of Claim 2. For the sake of simplicity we define

$$\Theta_n := \int_{\{|u_0|\neq\bar{s}\}\cap\{|u_n|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x$$
$$=\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)\chi_{\{|u_0|\neq\bar{s}\}\cap\{|u_n|<\bar{s}\}}}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x = \int_{\mathbb{R}^N}\frac{\xi_n F(u_n}{|x|^\beta}\,\mathrm{d}x \tag{5.56}$$

and

$$\Theta_0 := \int_{\{|u_0|<\bar{s}\}}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_0)}{|x|^\beta}\,\mathrm{d}x = \int_{\mathbb{R}^N}\frac{\xi_0 F(u_0}{|x|^\beta}\,\mathrm{d}x, \tag{5.57}$$

where

$$\xi_n := \int_{\mathbb{R}^N}\frac{F(u_n)\chi_{\{|u_0|\neq\bar{s}\}\cap\{|u_n|<\bar{s}\}}}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y \quad\text{and}\quad \xi_0 := \int_{\mathbb{R}^N}\frac{F(u_0)\chi_{\{|u_0|<\bar{s}\}}}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y.$$

To prove $\Theta_n \to \Theta_0$, we need following claim:

Claim 2(i). The sequence $(\xi_n)_n$ is uniformly bounded and $\xi_n \to \xi_0$, a.e. in $\mathbb{R}^N$.

Proof of Claim 2(i). Let $R \geqslant 1$ be an arbitrary constant. Thus, from (5.54) and Hölder's inequality, we derive that

$$\begin{aligned}|\xi_n| \leqslant & C(\bar{s}) \left(\int_{\mathbb{R}^N} \frac{|u_n|^{\frac{N(2N-2\beta-\mu)}{2(N-\beta-\mu)}}}{|y|^{\frac{N\beta}{N-\beta-\mu}}} \, \mathrm{d}y \right)^{\frac{N-\beta-\mu}{N}} \left(\int_{|x-y|<R} \frac{\mathrm{d}y}{|x-y|^{\frac{N\mu}{\beta+\mu}}} \right)^{\frac{\beta+\mu}{N}} \\ &+ C(\bar{s}) \left(\int_{\{|x-y|\geqslant R\} \cap \{|y|<1\}} \frac{|u_n|^{\frac{2N-2\beta-\mu}{2}}}{|y|^{\beta}|x-y|^{\mu}} \, \mathrm{d}y \right) \\ &+ C(\bar{s}) \left(\int_{\{|x-y|\geqslant R\} \cap \{|y|\geqslant 1\}} \frac{|u_n|^{\frac{2N-2\beta-\mu}{2}}}{|y|^{\beta}|x-y|^{\mu}} \, \mathrm{d}y \right) \\ =: & \, C(\bar{s}) \left(\mathcal{A}\mathcal{B} + \mathcal{C} + \mathcal{D} \right). \end{aligned} \tag{5.58}$$

It is not hard to see that $\mathcal{B} \leqslant C(\omega_{N-1}, R, \beta, \mu)$. Now, since $|y| \geqslant R \geqslant 1$, from Hölder's inequality we reach

$$\begin{aligned}\mathcal{A} \leqslant & \int_{|y|\geqslant R} |u_n|^{\frac{N(2N-2\beta-\mu)}{2(N-\beta-\mu)}} \, \mathrm{d}y + \int_{|y|<R} \frac{|u_n|^{\frac{N(2N-2\beta-\mu)}{2(N-\beta-\mu)}}}{|y|^{\frac{N\beta}{N-\beta-\mu}}} \, \mathrm{d}y \\ \leqslant & \int_{\mathbb{R}^N} |u_n|^{\frac{N(2N-2\beta-\mu)}{2(N-\beta-\mu)}} \, \mathrm{d}y \\ &+ \left(\int_{\mathbb{R}^N} |u_n|^{\frac{N(2N-2\beta-\mu)(N-\mu)}{2(N-\beta-\mu)(N-2\beta-\mu)}} \, \mathrm{d}y \right)^{\frac{N-2\beta-\mu}{N-\mu}} \left(\int_{|y|\leqslant R} \frac{1}{|y|^{\frac{N(N-\mu)}{2(N-\beta-\mu)}}} \mathrm{d}y \right)^{\frac{2\beta}{N-\mu}} \\ \leqslant & \int_{\mathbb{R}^N} |u_n|^{\frac{N(2N-2\beta-\mu)}{2(N-\beta-\mu)}} \, \mathrm{d}y \\ &+ C(\omega_{N-1}, \beta, \mu, R) \left(\int_{\mathbb{R}^N} |u_n|^{\frac{(2N-2\beta-\mu)(N-\mu)}{(2-\beta-\mu)(N-2\beta-\mu)}} \, \mathrm{d}y \right)^{\frac{N-2\beta-\mu}{N-\mu}}. \end{aligned} \tag{5.59}$$

One can observe that $\frac{N(2N-2\beta-\mu)}{2(N-\beta-\mu)} > N$, $\frac{N(2N-2\beta-\mu)(N-\mu)}{2(N-\beta-\mu)(N-2\beta-\mu)} > N$ and $\frac{N(N-\mu)}{2(N-\beta-\mu)} < N$. Next, we have

$$\begin{aligned}\mathcal{C} \leqslant & \left(\int_{|y|<1} \frac{|u_n|^{\frac{2N(2N-2\beta-\mu)}{2(2N-\mu)}}}{|y|^{\frac{2N\beta}{(2N-\mu)}}} \, \mathrm{d}y \right)^{\frac{(2N-\mu)}{2N}} \left(\int_{|x-y|\geqslant R} \frac{\mathrm{d}y}{|x-y|^{2N}} \right)^{\frac{\mu}{2N}} \\ \leqslant & C(\omega_{N-1}, R) \left(\int_{|y|<1} \frac{|u_n|^{\frac{2N(2N-2\beta-\mu)}{2(2N-\mu)}}}{|y|^{\frac{2N\beta}{2N-\mu}}} \, \mathrm{d}y \right)^{\frac{2N-\mu}{2N}} \\ \leqslant & C(\omega_{N-1}, R) \left(\int_{|y|<1} \frac{\mathrm{d}y}{|y|^{\frac{N(N+2\beta)}{2N-\mu}}} \right)^{\frac{\beta(2N-\mu)}{N(N+2\beta)}} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{(2N-2\beta-\mu)(N+2\beta)}{(2N-\mu)}} \, \mathrm{d}y \right)^{\frac{(2N-\mu)}{2(N+2\beta)}} \\ \leqslant & C^2_{\omega_{N-1}, R, \beta, \mu} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{(2N-2\beta-\mu)(N+2\beta)}{(2N-\mu)}} \, \mathrm{d}y \right)^{\frac{(2N-\mu)}{2(N+2\beta)}}, \end{aligned} \tag{5.60}$$

where we used that $\frac{N(N+2\beta)}{2N-\mu} < N$ and $\frac{(2N-2\beta-\mu)(N+2\beta)}{(2N-\mu)} > N$.

In order to estimate $\mathcal{D}$, we consider the following cases:

(i) If $\mu \leqslant 2\beta$, then $8\beta - \mu > 0$ and observing that $\frac{2N(2N-2\beta-\mu)}{4N-4\beta-3\mu} > N$, then Hölder's inequality leads to

$$\mathcal{D} \leqslant \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N(2N-2\beta-\mu)}{4N-4\beta-3\mu}}\,\mathrm{d}y\right)^{\frac{4N-4\beta-3\mu}{4N}} \left(\int_{|x-y|\geqslant R} \frac{\mathrm{d}y}{|x-y|^{\frac{8N}{7}}}\right)^{\frac{7\mu}{8N}} \left(\int_{|y|\geqslant 1} \frac{\mathrm{d}y}{|y|^{\frac{8N\beta}{8\beta-\mu}}}\right)^{\frac{8\beta-\mu}{8N}}$$
$$\leqslant C^3_{\omega_{N-1},\beta,\mu,R}\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N(2N-2\beta-\mu)}{4N-4\beta-3\mu}}\,\mathrm{d}y\right)^{\frac{4N-4\beta-3\mu}{4N}}.$$

(ii) If $\mu > 2\beta$, then $\frac{2N(2N-2\beta-\mu)}{4N-2\beta-3\mu} > N$, $\frac{4N\mu}{2\beta+3\mu} > N$ and using Hölder's inequality again, we obtain

$$\mathcal{D} \leqslant \int_{\{|x-y|\geqslant R\}} \frac{|u_n|^{\frac{2N-2\beta-\mu}{2}}}{|x-y|^{\mu}}\,\mathrm{d}y$$
$$\leqslant \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N(2N-2\beta-\mu)}{4N-2\beta-3\mu}}\,\mathrm{d}y\right)^{\frac{4N-2\beta-3\mu}{4N}} \left(\int_{|x-y|\geqslant R} \frac{\mathrm{d}y}{|x-y|^{\frac{4N\mu}{2\beta+3\mu}}}\,\mathrm{d}y\right)^{\frac{2\beta+3\mu}{4N}}$$
$$\leqslant C^4_{\omega_{N-1},\beta,\mu,R}\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N(2N-2\beta-\mu)}{4N-2\beta-3\mu}}\,\mathrm{d}y\right)^{\frac{4N-2\beta-3\mu}{4N}}.$$

Consequently, $\mathcal{D}$ is uniformly bounded as $(\|u_n\|)_n$ is bounded. In a similar way, from (5.59) and (5.60), $\mathcal{A}$ and $\mathcal{C}$ are uniformly bounded. Therefore, from (5.58), we conclude that $(\xi_n)_n$ is uniformly bounded. The proof of the fact $\xi_n \to \xi_0$ a.e. in $\mathbb{R}^N$, can be explored following the ideas given in (ALVES; SHEN, 2023, Lemma 4.6), where it also makes use of Lemma 5.2.7.

Now, by (5.56) and (5.57), we see that

$$|\Theta_n - \Theta_0| = \left|\int_{|u_n|\geqslant\bar{s}} |x|^{-\beta}\xi_n F(u_n)\,\mathrm{d}x + \int_{|u_n|<\bar{s}} |x|^{-\beta}\xi_n F(u_n)\,\mathrm{d}x \right.$$
$$\left. - \int_{|u_0|\geqslant\bar{s}} |x|^{-\beta}\xi_0 F(u_0)\,\mathrm{d}x - \int_{|u_0|<\bar{s}} |x|^{-\beta}\xi_0 F(u_0)\,\mathrm{d}x\right|$$

and using (5.50), we obtain

$$\lim_{n\to\infty} |\Theta_n - \Theta_0|$$
$$\leqslant \lim_{n\to\infty}\left|\int_{|u_n|\geqslant\bar{s}} |x|^{-\beta}\xi_n F(u_n)\,\mathrm{d}x - \int_{|u_0|\geqslant\bar{s}} |x|^{-\beta}\xi_0 F(u_0)\,\mathrm{d}x\right|$$
$$+ \lim_{n\to\infty}\left|\int_{|u_n|<\bar{s}} |x|^{-\beta}\xi_n F(u_n)\,\mathrm{d}x - \int_{|u_0|<\bar{s}} |x|^{-\beta}\xi_0 F(u_0)\,\mathrm{d}x\right|. \tag{5.61}$$

Thus, to finish Claim 2, we shall prove that the limit on the right-hand side of the above equation goes to zero. In fact, we may write

$$\left|\int_{|u_n|<\bar{s}} |x|^{-\beta}\xi_n F(u_n)\,\mathrm{d}x - \int_{|u_0|<\bar{s}} |x|^{-\beta}\xi_n F(u_0)\,\mathrm{d}x\right|$$
$$= \left|\int_{\mathbb{R}^N} |x|^{-\beta}\xi_n F(u_n)\chi_{\{|u_n|<\bar{s}\}}\,\mathrm{d}x - \int_{\mathbb{R}^N} |x|^{-\beta}\xi_n F(u_0)\chi_{\{|u_0|<\bar{s}\}}\,\mathrm{d}x\right|,$$

and by (5.54), we see

$$\begin{aligned}|x|^{-\beta}\xi_n F(u_n)\chi_{\{|u_n|<\bar{s}\}} =& |x|^{-\beta}\int_{\mathbb{R}^N}\frac{F(u_n)\chi_{\{|u_0|\neq\bar{s}\}\cap\{|u_n|<\bar{s}\}}}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y F(u_n)\chi_{\{|u_n|<\bar{s}\}}\\ \leqslant& |x|^{-\beta}\int_{\mathbb{R}^N}\frac{|u_n|^{\frac{2N-2\beta-\mu}{2}}}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y|u_n|^{\frac{2N-2\beta-\mu}{2}} =: v_n.\end{aligned}$$

By using (1.16), we have that $\int_{\mathbb{R}^N} v_n\,\mathrm{d}x < +\infty$ and using Claim $2(i)$, we get $\int_{\mathbb{R}^N} v_n\,\mathrm{d}x \to \int_{\mathbb{R}^N} v_0\,\mathrm{d}x$, as $n\to\infty$, where

$$v_0 := |x|^{-\beta}\int_{\mathbb{R}^N}\frac{|u_0|^{\frac{2N-2\beta-\mu}{2}}}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y|u_0|^{\frac{2N-2\beta-\mu}{2}}.$$

Now, from Claim 2(i) and in light of Lebesgue's Dominated Convergence Theorem, we infer that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-\beta}\xi_n F(u_n)\chi_{\{|u_n|<\bar{s}\}}\,\mathrm{d}x = \int_{\mathbb{R}^N}|x|^{-\beta}\xi_0 F(u_0)\chi_{\{|u_0|<\bar{s}\}}\,\mathrm{d}x. \tag{5.62}$$

Finally, applying (5.62) in (5.61) and taking $\bar{s}$ large enough, we get $\Theta_n \to \Theta_0$ and consequently using (5.50), Claim 1 and Claim 2 in (5.53), we obtain the required result. The proof of (5.52) follows analogously. □

Lemma 5.2.10. *Suppose $(u_n, v_n)_n$ be a $(PS)_c$−sequence for $\mathcal{J}_\lambda$ such that $c < C(\alpha_0, \alpha_N, \beta, \mu)$. Then, up to a subsequence, we have*

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}\phi\,\mathrm{d}x \to \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_0)}{|x|^{\beta}}\phi\,\mathrm{d}x \tag{5.63}$$

and

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^{\beta}}\psi\,\mathrm{d}x \to \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_0)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v_0)}{|x|^{\beta}}\psi\,\mathrm{d}x. \tag{5.64}$$

for all $(\phi,\psi)\in C^{\infty}_{0,\mathrm{rad}}(\mathbb{R}^N)\times C^{\infty}_{0,\mathrm{rad}}(\mathbb{R}^N)$.

Proof. Define $\Omega := \operatorname{supp}\phi\cap\operatorname{supp}\psi$. Thus, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}\phi\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_0)}{|x|^{\beta}}\phi\,\mathrm{d}x\right|\\ =&\left|\int_{\Omega}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}\phi\,\mathrm{d}x - \int_{\Omega}\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_0)}{|x|^{\beta}}\phi\,\mathrm{d}x\right|\\ \leqslant&\int_{|u_n|\geqslant\bar{s}}\left|\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}\phi\right|\mathrm{d}x + \int_{|u_0|\geqslant\bar{s}}\left|\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_0)}{|x|^{\beta}}\phi\right|\mathrm{d}x\\ &+\int_{\Omega\cap|u_n|<\bar{s}}\left|\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}\phi\right|\mathrm{d}x\\ &-\int_{\Omega\cap|u_0|<\bar{s}}\left|\left(\int_{\mathbb{R}^N}\frac{F(u_0)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_0)}{|x|^{\beta}}\phi\right|\mathrm{d}x.\end{aligned}$$

For all $\epsilon > 0$, take $s := s_\epsilon = \epsilon^{-1}(K+1)|\phi|_\infty$, where K is the constant defined in (5.44). Then from (5.44), one can observe that

$$\begin{aligned} &\int_{|u_n|\geqslant \bar{s}} \left| \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \right) \frac{f(u_n)}{|x|^\beta} \phi \right| \mathrm{d}x \\ &\leqslant \frac{\epsilon}{K+1} \int_{|u_n|\geqslant \bar{s}} \left| \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \right) \frac{f(u_n)}{|x|^\beta} u_n \right| \mathrm{d}x < \epsilon. \end{aligned}$$

Therefore, (5.63) follows, arguing as in the final part of the proof in (ALVES; SHEN, 2023, Lemma 4.6). Similarly, we may check the convergence in (5.64). □

In order to prove the next compactness result, we introduce a technical lemma. The idea of the proof is similar as in (DO Ó, 1996, Lemma 4).

Lemma 5.2.11. *Let $(u_n, v_n)_n$ be a $(PS)_c-$sequence for $\mathcal{J}_\lambda$. Then, there exists $(u,v) \in \mathbb{W}^N_{\mathrm{rad}}$, such that $(\nabla u_n, \nabla v_n) \to (\nabla u, \nabla v)$ a.e. in $\mathbb{R}^N$. Moreover,*

$$(|\nabla u_n|^{N-2}\nabla u_n, |\nabla v_n|^{N-2}\nabla v_n) \rightharpoonup (|\nabla u|^{N-2}\nabla u, |\nabla v|^{N-2}\nabla v), \tag{5.65}$$

weakly in $(L^{\frac{N}{N-1}}(\mathbb{R}^N))^N \times (L^{\frac{N}{N-1}}(\mathbb{R}^N))^N$, as $n \to +\infty$.

Proof. It follows from Lemma 5.2.8 that the sequence $(u_n, v_n)_n$ is bounded in $\mathbb{W}^N_{\mathrm{rad}}$. Consequently, the sequence

$$(|\nabla u_n|^{N-2}\nabla u_n, |\nabla v_n|^{N-2}\nabla v_n)_n \quad \text{is bounded in} \quad (L^{\frac{N}{N-1}}(\mathbb{R}^N))^N \times (L^{\frac{N}{N-1}}(\mathbb{R}^N))^N.$$

Therefore, up to a sebsequence,

$$(|\nabla u_n|^{N-2}\nabla u_n, |\nabla v_n|^{N-2}\nabla v_n) \rightharpoonup (U,V) \quad \text{weakly in} \quad (L^{\frac{N}{N-1}}(\mathbb{R}^N))^N \times (L^{\frac{N}{N-1}}(\mathbb{R}^N))^N,$$

as $n \to \infty$, for some $(U,V) \in (L^{\frac{N}{N-1}}(\mathbb{R}^N))^N \times (L^{\frac{N}{N-1}}(\mathbb{R}^N))^N$. Moreover, in view of the boundedness of the sequence $(|\nabla u_n|^N + |u_n|^N, |\nabla v_n|^N + |v_n|^N)_n$ in $L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$, there exist measures (nonnegative) μ_1 and μ_2 satisfying

$$(|\nabla u_n|^N + |u_n|^N, |\nabla v_n|^N + |v_n|^N) \to (\mu_1, \mu_2) \quad \text{in} \quad \mathcal{D}'(K) \times \mathcal{D}'(K),$$

up to a subsequence, as $n \to +\infty$ for any compact subset $K \subset \mathbb{R}^N$. We show that $(U,V) = (|\nabla u|^{N-2}\nabla u, |\nabla v|^{N-2}\nabla v)$. For any fixed $\nu > 0$ we define the energy concentration set

$$X_\nu := \left\{ x \in \mathbb{R}^N : \lim_{l\to 0} \lim_{k\to+\infty} \int_{B_l} \left(|\nabla w_k|^N + |w_k|^N \right) dx \geq \nu \right\}.$$

It is not hard to see that X_ν is a finite set. Suppose $X_\nu = \{x_1, x_2, ..., x_n\}$. In order to show (5.65), we prove two claims.

Claim 1. If $\sigma > 0$ is sufficiently small, then

$$\lim_{n\to\infty}\int_\Omega \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\,\mathrm{d}x = \int_\Omega \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}u\,\mathrm{d}x, \quad (5.66)$$

$$\lim_{n\to\infty}\int_\Omega \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}v_n\,\mathrm{d}x = \int_\Omega \left(\int_{\mathbb{R}^N} \frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta}v\,\mathrm{d}x, \quad (5.67)$$

for any $\Omega \subset\subset \mathbb{R}^N \setminus X_\sigma$.

Proof of Claim 1. We just give the proof of (5.66). Proof of (5.67) is similar. Take $x_0 \in \Omega$ arbitrarily such that $\mu_1(B_{r_0}(x_0)) < \sigma$, where $r_0 > 0$. Also, consider a function $\phi \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \le \phi \le 1$ for $x \in \mathbb{R}^N$, $\phi \equiv 1$ in $B_{r_0/2}(x_0)$ and $\phi \equiv 0$ in $\mathbb{R}^N \setminus (B_{r_0}(x_0))$. Then,

$$\lim_{n\to\infty}\int_{B_{r_0}(x_0)}(|\nabla u_n|^N + |u_n|^N)\phi\,\mathrm{d}x = \int_{B_{r_0}(x_0)}\phi\,\mathrm{d}\mu_1 \le \mu_1(B_{r_0}(x_0)) < \sigma,$$

and consequently for $\epsilon > 0$ sufficiently small, we have

$$\int_{B_{r_0}(x_0)}(|\nabla u_n|^N + |u_n|^N)\phi\,\mathrm{d}x \le \sigma(1-\epsilon), \quad n\to\infty. \quad (5.68)$$

Using assumptions (a), (b) and (5.68), we obtain

$$\begin{aligned}\int_{B_{r_0/2}(x_0)}|f(u_n)|^q\,\mathrm{d}x &\leqslant C\int_{B_{r_0/2}(x_0)}\exp\left(\alpha q|u_n|^{\frac{N}{N-1}}\right)\mathrm{d}x\\ &\le C\int_{B_{r_0/2}(x_0)}\exp\left[\alpha q\sigma^{\frac{1}{N-1}}(1-\epsilon)^{\frac{1}{N-1}}\left(\frac{|u_n|^N}{\int_{B_{r_0/2}}(|\nabla u_n|^N+|u_n|^N)\,\mathrm{d}x}\right)^{\frac{1}{N-1}}\right]\mathrm{d}x.\end{aligned}$$

If we choose $q > 1$ sufficiently close to 1 such that $\alpha q\sigma^{\frac{1}{N-1}}(1-\epsilon)^{\frac{1}{N-1}} < (1-\frac{\beta}{N})\frac{\alpha_N}{\alpha_0}$, then from Proposition 5.0.1, it follows

$$\int_{B_{r_0/2}(x_0)}|f(u_n)|^q\mathrm{d}x \le C. \quad (5.69)$$

Next,

$$
\begin{aligned}
&\int_{B_{r_0/2}(x_0)}\left|\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n-\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}u\right|\,\mathrm{d}x\\
&=\int_{B_{r_0/2}(x_0)}\left|\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n-\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u\right.\\
&\quad\left.+\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u-\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}u\right|\mathrm{d}x\\
&\le\int_{B_{r_0/2}(x_0)}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}\,|u_n-u|\,\mathrm{d}x\\
&\quad+\int_{B_{r_0/2}(x_0)}\left|\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u-\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}u\right|\mathrm{d}x\\
&=:\mathcal{F}_1+\mathcal{F}_2.
\end{aligned}
$$

Using Cauchy-Schwarz type inequality in (LIEB; LOSS, 2001, Theorem 9.8) in $\mathcal{F}_1$, we deduce

$$
\begin{aligned}
&\int_{B_{r_0/2}(x_0)}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}\,|u_n-u|\,\mathrm{d}x\\
&\le\left(\int_{B_{r_0/2}(x_0)}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_{B_{r_0/2}(x_0)}\left(\int_{\mathbb{R}^N}\frac{f(u_n)(u_n-u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}|u_n-u|\,\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}
$$

In view of (5.8) and the boundedness of $(\|u_n\|)_n$, we conclude that the first term in the right-hand side of the above inequality is bounded. Next, using (5.69) together with Hölder's inequality with exponents p', p'' (bigger than one but sufficiently close to 1), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f(u_n)(u_n-u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)(u_n-u)\chi_{B_{r_0/2}(x_0)}}{|x|^\beta}\,\mathrm{d}x\\
&\le C_{\beta,\mu}\left(\int_{\mathbb{R}^N}|f(u_n)(u_n-u)\chi_{B_{r_0/2}(x_0)}|^{\frac{2N}{2N-2\beta-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\beta-\mu}{2}}\\
&\le C_{\beta,\mu}C\left(\int_{B_{r_0/2}(x_0)}|u_n-u|^{\frac{2Np''}{2N-2\beta-\mu}}\,\mathrm{d}x\right)^{\frac{2N-2\beta-\mu}{2p''}}.
\end{aligned}
$$

Using compactness of the Sobolev embedding, it can be deduced that $\mathcal{F}_1\to 0$ as $n\to\infty$.

The estimate $\mathcal{F}_2\to 0$ as $n\to\infty$ follows from Lemma 5.2.10 by taking $\phi=\chi_{B_{r_0/2}(x_0)}u$ and using density argument.

Claim 2. Let $\Omega_{\epsilon_0}=\{x\in\mathbb{R}^N:\|x-x_j\|\ge\epsilon_0, j=1,2,...,n\}$ and $B(x_i,\epsilon_0)\cap B(x_j,\epsilon_0)=\emptyset$

if $i \neq j$ where $\epsilon_0 > 0$ is fixed small enough. Then,

$$\int_{\Omega_{\epsilon_0}} \left(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\right)(\nabla u_n - \nabla u)\,\mathrm{d}x$$
$$+ \int_{\Omega_{\epsilon_0}} \left(|\nabla v_n|^{N-2}\nabla v_n - |\nabla v|^{N-2}\nabla v\right)(\nabla v_n - \nabla v)\,\mathrm{d}x \to 0, \quad \text{as } n \to \infty.$$

Although the proof of this assertion can be adopted from (DO Ó, 1996), due to presence of Stein-Weiss term, we have added the proof here to make it precise. Consider the functions $\psi_\epsilon^1, \psi_\epsilon^2$ as in (DO Ó, 1996, Lemma 4). Using $(\phi, \psi) = (\psi_\epsilon^1 u_n, \psi_\epsilon^2 v_n)$ in $\mathcal{J}_\lambda'(u_n, v_n)(\phi, \psi) \to 0$, we have

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^{N-2}\nabla u_n \nabla(\psi_\epsilon^1 u_n) + |u_n|^{N-2}u_n(\psi_\epsilon^1 u_n)\right)\mathrm{d}x + \int_{\mathbb{R}^N} \left(|\nabla v_n|^{N-2}\nabla v_n \nabla(\psi_\epsilon^2 v_n)\right.$$
$$\left. + |v_n|^{N-2}v_n(\psi_\epsilon^2 v_n)\right)\mathrm{d}x - \lambda p \int_{\mathbb{R}^N} |u_n|^{p-2}u_n|v_n|^q(\psi_\epsilon^1 u_n)\,\mathrm{d}x - \lambda q \int_{\mathbb{R}^N} |u_n|^p|v_n|^{q-2}v_n(\psi_\epsilon^2 v_n)\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}\phi\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}\psi\,\mathrm{d}x \to 0,$$

as $n \to \infty$, whence it follows that

$$\int_{\mathbb{R}^N} |\nabla u_n|^N \psi_\epsilon^1\,\mathrm{d}x + \int_{\mathbb{R}^N}\left(u_n|\nabla u_n|^{N-2}\nabla u_n \nabla\psi_\epsilon^1 + |u_n|^N\psi_\epsilon^1\right)\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v_n|^N\psi_\epsilon^2\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^N}\left(v_n|\nabla v_n|^{N-2}\nabla v_n\nabla\psi_\epsilon^2 + |v_n|^N\psi_\epsilon^2\,\mathrm{d}x\right)$$
$$- \lambda p \int_{\mathbb{R}^N} |u_n|^p|v_n|^q\psi_\epsilon^1\,\mathrm{d}x - \lambda q\int_{\mathbb{R}^N} |u_n|^p|v_n|^q\psi_\epsilon^2\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\psi_\epsilon^1\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}v_n\psi_\epsilon^2\,\mathrm{d}x$$
$$\leq \epsilon_n\|\psi_\epsilon^1 u_n\|_{W^{1,N}(\mathbb{R}^N)} + \epsilon_n\|\psi_\epsilon^2 v_n\|_{W^{1,N}(\mathbb{R}^N)}, \tag{5.70}$$

where $\epsilon_n \to 0$ as $n \to \infty$. Again using $(\phi, \psi) = (\psi_\epsilon^1 u, \psi_\epsilon^2 v)$ in $\mathcal{J}_\lambda'(u_n, v_n)(\phi, \psi) \to 0$, we see that

$$\int_{\mathbb{R}^N}\left(|\nabla u_n|^{N-2}\nabla u_n\nabla(\psi_\epsilon^1 u) + |u_n|^{N-2}u_n(\psi_\epsilon^1 u)\right)\mathrm{d}x$$
$$+ \int_{\mathbb{R}^N}\left(|\nabla v_n|^{N-2}\nabla v_n\nabla(\psi_\epsilon^2 v) + |v_n|^{N-2}v_n(\psi_\epsilon^2 v)\right)\mathrm{d}x$$
$$- \lambda p\int_{\mathbb{R}^N}|u_n|^{p-2}u_n|v_n|^q(\psi_\epsilon^1 u)\,\mathrm{d}x - \lambda q\int_{\mathbb{R}^N}|u_n|^p|v_n|^{q-2}v_n(\psi_\epsilon^2 v)\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}(u\psi_\epsilon^1)\,\mathrm{d}x - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}(v\psi_\epsilon^2)\,\mathrm{d}x$$
$$\to 0.$$

Therefore,

$$
\begin{aligned}
&-\int_{\mathbb{R}^N}\left(|\nabla u|^{N-2}\psi_\epsilon^1\nabla u_n\nabla u+|\nabla u_n|^{N-2}u\nabla u_n\nabla\psi_\epsilon^1\right)\,\mathrm{d}x-\int_{\mathbb{R}^2}|u_n|^{N-2}u_n(\psi_\epsilon^1 u)\,\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\left(|\nabla v|^{N-2}\psi_\epsilon^2\nabla v_n\nabla v+|\nabla v_n|^{N-2}v\nabla v_n\nabla\psi_\epsilon^2\right)\,\mathrm{d}x-\int_{\mathbb{R}^2}|v_n|^{N-2}v_n(\psi_\epsilon^2 v)\,\mathrm{d}x\\
&+\lambda p\int_{\mathbb{R}^N}|u_n|^{p-2}u_n|v_n|^q(\psi_\epsilon^1 u)\,\mathrm{d}x+\lambda q\int_{\mathbb{R}^N}|u_n|^p|v_n|^{q-2}v_n(\psi_\epsilon^2 v)\,\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}(u\psi_\epsilon^1)\,\mathrm{d}x+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}(v\psi_\epsilon^2)\,\mathrm{d}x\\
&\leq\epsilon_n\|\psi_\epsilon^1 u\|_{W^{1,N}(\mathbb{R}^N)}+\epsilon_n\|\psi_\epsilon^2 v\|_{W^{1,N}(\mathbb{R}^N)}.
\end{aligned}
\tag{5.71}
$$

Now using the strictly convex behavior of the function $C:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ given by $C(u,v)=|u|^N+|v|^N$, we get

$$
0\leq\left(|\nabla u_n|^{N-2}\nabla u_n-|\nabla u|^{N-2}\nabla u\right)(\nabla u_n-\nabla u)+\left(|\nabla v_n|^{N-2}\nabla v_n-|\nabla v|^{N-2}\nabla v\right)(\nabla v_n-\nabla v)
$$

and as a consequence, we deduce

$$
\begin{aligned}
0&\leq\int_{\Omega_{\epsilon_0}}\left(|\nabla u_n|^{N-2}\nabla u_n-|\nabla u|^{N-2}\nabla u\right)(\nabla u_n-\nabla u)\psi_\epsilon^1\,\mathrm{d}x\\
&\quad+\int_{\Omega_{\epsilon_0}}\left(|\nabla v_n|^{N-2}\nabla v_n-|\nabla v|^{N-2}\nabla v\right)(\nabla v_n-\nabla v)\psi_\epsilon^2\,\mathrm{d}x\\
&\leq\int_{\mathbb{R}^N}\left(|\nabla u_n|^{N-2}\nabla u_n-|\nabla u|^{N-2}\nabla u\right)(\nabla u_n-\nabla u)\psi_\epsilon^1\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}\left(|\nabla v_n|^{N-2}\nabla v_n-|\nabla v|^{N-2}\nabla v\right)(\nabla v_n-\nabla v)\psi_\epsilon^2\,\mathrm{d}x,
\end{aligned}
$$

which leads us to

$$
\begin{aligned}
0\leq&\int_{\mathbb{R}^N}\left(|\nabla u_n|^N\psi_\epsilon^1-|\nabla u_n|^{N-2}\nabla u_n\nabla u\psi_\epsilon^1-|\nabla u|^{N-2}\nabla u_n\nabla u\psi_\epsilon^1+|\nabla u|^N\psi_\epsilon^1\right)\,\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\left(|\nabla v_n|^N\psi_\epsilon^2-|\nabla v_n|^{N-2}\nabla v_n\nabla v\psi_\epsilon^2-|\nabla v|^{N-2}\nabla v_n\nabla v\psi_\epsilon^2+|\nabla v|^N\psi_\epsilon^2\right)\,\mathrm{d}x.
\end{aligned}
\tag{5.72}
$$

Next, combining (5.70), (5.71) and (5.72), we obtain

$$
\begin{aligned}
0 \leq & \int_{\mathbb{R}^N} \left(u_n|\nabla u_n|^{N-2}\nabla u_n \nabla \psi_\epsilon^1 + |u_n|^N \psi_\epsilon^1\right) \mathrm{d}x - \int_{\mathbb{R}^N} \left(v_n|\nabla v_n|^{N-2}\nabla v_n \nabla \psi_\epsilon^2 + |v_n|^N \psi_\epsilon^2\right) \mathrm{d}x \\
& + \lambda p \int_{\mathbb{R}^N} |u_n|^p |v_n|^q \psi_\epsilon^1 \, \mathrm{d}x + \lambda q \int_{\mathbb{R}^N} |u_n|^p |v_n|^q \psi_\epsilon^2 \, \mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{f(u_n)}{|x|^\beta} (u_n \psi_\epsilon^1) \, \mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{g(v_n)}{|x|^\beta} (v_n \psi_\epsilon^2) \, \mathrm{d}x \\
& + \epsilon_n \|\psi_\epsilon^1 u_n\|_{W^{1,N}(\mathbb{R}^N)} + \epsilon_n \|\psi_\epsilon^2 v_n\|_{W^{1,N}(\mathbb{R}^N)} \\
& + \int_{\mathbb{R}^N} |\nabla u_n|^{N-2} u \nabla u_n \nabla \psi_\epsilon^1 \, \mathrm{d}x + \int_{R^N} |u_n|^{N-2} u_n (\psi_\epsilon^1 u) \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v_n|^{N-2} v \nabla v_n \nabla \psi_\epsilon^2 \, \mathrm{d}x \\
& + \int_{R^N} |v_n|^{N-2} v_n (\psi_\epsilon^2 v) \, \mathrm{d}x - \lambda p \int_{\mathbb{R}^N} |u_n|^{p-2} u_n |v_n|^q (\psi_\epsilon^1 u) \, \mathrm{d}x \\
& - \lambda q \int_{\mathbb{R}^N} |u_n|^p |v_n|^{q-2} v_n (\psi_\epsilon^2 v) \, \mathrm{d}x \\
& - \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(u_n)}{|x|^\beta} (u \psi_\epsilon^1) \, \mathrm{d}x - \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{g(v_n)}{|x|^\beta} (v \psi_\epsilon^2) \, \mathrm{d}x \\
& + \epsilon_n \|\psi_\epsilon^1 u\|_{W^{1,N}(\mathbb{R}^N)} + \epsilon_n \|\psi_\epsilon^2 v\|_{W^{1,N}(\mathbb{R}^N)} \\
& + \int_{\mathbb{R}^N} \left(|\nabla u|^N \psi_\epsilon^1 + |\nabla v|^N \psi_\epsilon^2 - |\nabla u|^{N-2} \nabla u_n \nabla u \psi_\epsilon^1 - |\nabla v|^{N-2} \nabla v_n \nabla v \psi_\epsilon^1 \right) \mathrm{d}x,
\end{aligned}
$$

from which, equivalently, we may write

$$
\begin{aligned}
0 \leq & \int_{\mathbb{R}^N} |\nabla u_n|^{N-2} \nabla u_n \nabla \psi_\epsilon^1 (u - u_n) \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v_n|^{N-2} \nabla v_n \nabla \psi_\epsilon^2 (v - v_n) \, \mathrm{d}x \\
& - \int_{\mathbb{R}^N} |u_n|^N \psi_\epsilon^1 \, \mathrm{d}x \\
& - \int_{\mathbb{R}^N} |v_n|^N \psi_\epsilon^2 \, \mathrm{d}x + \lambda p \int_{\mathbb{R}^N} |u_n|^p |v_n|^q \psi_\epsilon^1 \, \mathrm{d}x + \lambda q \int_{\mathbb{R}^N} |u_n|^p |v_n|^q \psi_\epsilon^2 \, \mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{f(u_n)}{|x|^\beta} (u_n - u) \psi_\epsilon^1 \, \mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \, \mathrm{d}y \right) \frac{g(v_n)}{|x|^\beta} (v_n - v) \psi_\epsilon^2 \, \mathrm{d}x \\
& + \epsilon_n \|\psi_\epsilon^1 u_n\|_{W^{1,N}(\mathbb{R}^N)} + \epsilon_n \|\psi_\epsilon^2 v_n\|_{W^{1,N}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |u_n|^{N-2} u_n (\psi_\epsilon^1 u) + \int_{\mathbb{R}^N} |v_n|^{N-2} v_n (\psi_\epsilon^2 v) \\
& - \lambda p \int_{\mathbb{R}^N} |u_n|^{p-2} u_n |v_n|^q (\psi_\epsilon^1 u) \, \mathrm{d}x - \lambda q \int_{\mathbb{R}^N} |u_n|^p |v_n|^{q-2} v_n (\psi_\epsilon^2 v) \, \mathrm{d}x \\
& + \epsilon_n \|\psi_\epsilon^1 u\|_{W^{1,N}(\mathbb{R}^N)} + \epsilon_n \|\psi_\epsilon^2 v\|_{W^{1,N}(\mathbb{R}^N)} \\
& + \int_{\mathbb{R}^N} \left(|\nabla u|^{N-2} \psi_\epsilon^1 \nabla u (\nabla u - \nabla u_n) + |\nabla v|^{N-2} \psi_\epsilon^2 \nabla v (\nabla v - \nabla v_n) \right) \mathrm{d}x. \qquad (5.73)
\end{aligned}
$$

Now, we estimate each term on right-hand side of the inequality (5.73) separately. Following the estimates (24) and (25) from (DO Ó, 1996, Lemma 4), we can conclude that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{N-2}\nabla u_n\nabla\psi^1_\epsilon(u-u_n)\,\mathrm{d}x\le 0,\\
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^{N-2}\nabla v_n\nabla\psi^2_\epsilon(v-v_n)\,\mathrm{d}x\le 0,\\
&\int_{\mathbb{R}^N}\psi^1_\epsilon|\nabla u|^{N-2}|\nabla u|(\nabla u-\nabla u_n)\,\mathrm{d}x\to 0,\\
&\int_{\mathbb{R}^N}\psi^2_\epsilon|\nabla v|^{N-2}|\nabla v|(\nabla v-\nabla v_n)\,\mathrm{d}x\to 0,\quad n\to\infty.
\end{aligned}\tag{5.74}$$

Moreover, from Lemma 5.2.10, we have

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}(u_n-u)\psi^1_\epsilon\,\mathrm{d}x\to 0,\tag{5.75}$$

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}(v_n-v)\psi^2_\epsilon\,\mathrm{d}x\to 0.\tag{5.76}$$

Finally, using all the estimates (5.74), (5.75) and (5.76) in (5.73), we conclude that Claim 2 is true. Thus, $(\nabla u_n,\nabla v_n)\to(\nabla u,\nabla v)$, a.e. in $\mathbb{R}^N$. Using this fact and boundedness of $\{|\nabla u_n|^{N-2}\nabla u_n,|\nabla v_n|^{N-2}\nabla v_n\}$ in $(L^{\frac{N}{N-1}}(\mathbb{R}^N))^N\times(L^{\frac{N}{N-1}}(\mathbb{R}^N))^N$, up to a subsequence, we get (5.65) and this completes the proof of Lemma 5.2.11.

□

Lemma 5.2.12. *Under the same conditions of Lemma 5.2.10, we have*

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u\,\mathrm{d}x\to\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}u\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}v\,\mathrm{d}x\to\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta}v\,\mathrm{d}x,$$

for all $(u,v)\in\mathbb{W}^N_{\mathrm{rad}}$.

Proof. For any given $\varepsilon>0$, noting that if $(u,v)\in\mathbb{W}^N_{\mathrm{rad}}$, then we may choose $(\phi_\varepsilon,\psi_\varepsilon)\in[C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)\times C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)]\cap\mathbb{W}^N_{\mathrm{rad}}$ such that

$$\|(\phi_\varepsilon,\psi_\varepsilon)-(u,v)\|<\varepsilon.\tag{5.77}$$

Thus, since $\mathcal{J}'_\lambda(u_n,v_n)(\phi_\varepsilon,\psi_\varepsilon)=o_n(1)$ and $\mathcal{J}'_\lambda(u_n,v_n)(u,v)=o_n(1)$, we have

$$\begin{aligned}
o_n(1)=&\mathcal{J}'_\lambda(u_n,v_n)\left((\phi_\varepsilon,\psi_\varepsilon)-(u,v)\right)\\
=&\int_{\mathbb{R}^N}|\nabla u_n|^{N-2}\nabla u_n\nabla(\phi_\varepsilon-u)\,\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla v_n|^{N-2}\nabla v_n\nabla(\psi_\varepsilon-v)\,\mathrm{d}x\\
&-\lambda p\int_{\mathbb{R}^N}|u_n|^{p-2}u_n|v_n|^q(\phi_\varepsilon-u)\,\mathrm{d}x-\lambda q\int_{\mathbb{R}^N}|u_n|^p|v_n|^{q-2}v_n(\psi_\varepsilon-v)\,\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}(\phi_\varepsilon-u)\,\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}(\psi_\varepsilon-v)\,\mathrm{d}x.
\end{aligned}\tag{5.78}$$

Combining the Cauchy-Schwarz inequality with the Hölder's inequality, we derive

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_n|^{N-2}\nabla u_n \nabla(\phi_\varepsilon - u)\,\mathrm{d}x &\leqslant \left(\int_{\mathbb{R}^N} |\nabla u_n|^N\,\mathrm{d}x\right)^{\frac{N-1}{N}} \left(\int_{\mathbb{R}^N} |\nabla(\phi_\varepsilon - u)|^N\,\mathrm{d}x\right)^{\frac{1}{N}}\\
&\leqslant \|u_n\|^{N-1}\|\phi_\varepsilon - u\|,\\
\int_{\mathbb{R}^N} |\nabla v_n|^{N-2}\nabla v_n \nabla(\psi_\varepsilon - v)\,\mathrm{d}x &\leqslant \left(\int_{\mathbb{R}^N} |\nabla v_n|^N\,\mathrm{d}x\right)^{\frac{N-1}{N}} \left(\int_{\mathbb{R}^N} |\nabla(\psi_\varepsilon - v)|^N\,\mathrm{d}x\right)^{\frac{1}{N}}\\
&\leqslant \|v_n\|^{N-1}\|\psi_\varepsilon - v\|.
\end{aligned} \tag{5.79}$$

Using Hölder's inequality once more, we see that

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_n|^{p-2}u_n|v_n|^q(\phi_\varepsilon - u)\,\mathrm{d}x &\leqslant \left(\int_{\mathbb{R}^N} |u_n|^{p+q}\,\mathrm{d}x\right)^{\frac{p-1}{p+q}} \left(\int_{\mathbb{R}^N} |v_n|^{p+q}\,\mathrm{d}x\right)^{\frac{q}{p+q}}\\
&\quad\times \left(\int_{\mathbb{R}^N} (\phi_\varepsilon - u)^{p+q}\,\mathrm{d}x\right)^{\frac{1}{p+q}},\\
\int_{\mathbb{R}^N} |u_n|^{p}|v_n|^{q-2}v_n(\psi_\varepsilon - v)\,\mathrm{d}x &\leqslant \left(\int_{\mathbb{R}^N} |u_n|^{p+q}\,\mathrm{d}x\right)^{\frac{p}{p+q}} \left(\int_{\mathbb{R}^N} |v_n|^{p+q}\,\mathrm{d}x\right)^{\frac{q-1}{p+q}}\\
&\quad\times \left(\int_{\mathbb{R}^N} (\psi_\varepsilon - v)^{p+q}\,\mathrm{d}x\right)^{\frac{1}{p+q}}.
\end{aligned} \tag{5.80}$$

Once that $(\|(u_n, v_n)\|)_n$ is bounded, there exists $C > 0$, such that $\|(u_n, v_n)\| \leqslant C$ for all $n \in \mathbb{N}$. In particular, we have $\|u_n\| \leqslant C$, $\|v_n\| \leqslant C$, $\|u_n\|_{p+q} \leqslant C$, $\|v_n\|_{p+q} \leqslant C$, for all $n \in \mathbb{N}$. Since the Sobolev embedding $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is continuous, (5.79) and (5.80) become

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_n|^{N-2}\nabla u_n \nabla(\phi_\varepsilon - u)\,\mathrm{d}x &\leqslant C^{N-1}\|\phi_\varepsilon - u\|,\\
\int_{\mathbb{R}^N} |\nabla v_n|^{N-2}\nabla v_n \nabla(\psi_\varepsilon - v)\,\mathrm{d}x &\leqslant C^{N-1}\|\psi_\varepsilon - v\|,\\
\int_{\mathbb{R}^N} |u_n|^{p-2}u_n|v_n|^q(\phi_\varepsilon - u)\,\mathrm{d}x &\leqslant C^{p+q-1}\|\phi_\varepsilon - u\|,\\
\int_{\mathbb{R}^N} |u_n|^{p}|v_n|^{q-2}v_n(\psi_\varepsilon - v)\,\mathrm{d}x &\leqslant C^{p+q-1}\|\psi_\varepsilon - v\|,
\end{aligned}$$

which together with (5.77) and (5.78), imply that

$$\begin{aligned}
o_n(1) \leqslant{}& \varepsilon(2C^{N-1} + 2\lambda C^{p+q-1}) - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}(\phi_\varepsilon - u)\,\mathrm{d}x\\
&- \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}(\psi_\varepsilon - v)\,\mathrm{d}x,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}(\phi_\varepsilon - u)\,\mathrm{d}x\\
&+ \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}(\psi_\varepsilon - v)\,\mathrm{d}x \leqslant \varepsilon C(N,\lambda,p,q) + o_n(1),
\end{aligned} \tag{5.81}$$

where $C(N,\lambda,p,q)=2(C^{N-1}+\lambda C^{p+q-1})$.

Claim. For any given $\varepsilon>0$, there exists $C(N,\lambda,p,q)>0$, such that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}(\phi_\varepsilon-u)\,\mathrm{d}x\\&+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta}(\psi_\varepsilon-v)\,\mathrm{d}x\leqslant C(N,\lambda,p,q)\varepsilon.\end{aligned}\tag{5.82}$$

Indeed, for any $\phi,\psi\in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$ and by (5.46), it follows from of Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned}&\int_{\mathbb{R}^N}|u_n|^{p-2}u_n|v_n|^q\phi\,\mathrm{d}x\to\int_{\mathbb{R}^N}|u|^{p-2}u|v|^q\phi\,\mathrm{d}x,\\&\int_{\mathbb{R}^N}|u_n|^p|v_n|^{q-2}v_n\psi\,\mathrm{d}x\to\int_{\mathbb{R}^N}|u|^p|v|^{q-2}v\psi\,\mathrm{d}x,\\&\int_{\mathbb{R}^N}|u_n|^{N-2}u_n\phi\,\mathrm{d}x\to\int_{\mathbb{R}^N}|u|^{N-2}u\phi\,\mathrm{d}x,\\&\int_{\mathbb{R}^N}|v_n|^{N-2}v_n\psi\,\mathrm{d}x\to\int_{\mathbb{R}^N}|v|^{N-2}v\psi\,\mathrm{d}x,\end{aligned}\tag{5.83}$$

and by Lemma 5.2.11, we see that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla u_n|^{N-2}\nabla u_n\nabla\phi\,\mathrm{d}x\to\int_{\mathbb{R}^N}|\nabla u|^{N-2}\nabla u\nabla\phi\,\mathrm{d}x,\\&\int_{\mathbb{R}^N}|\nabla v_n|^{N-2}\nabla v_n\nabla\psi\,\mathrm{d}x\to\int_{\mathbb{R}^N}|\nabla v|^{N-2}\nabla v\nabla\psi\,\mathrm{d}x,\end{aligned}\tag{5.84}$$

which together with (5.83) and Lemma 5.2.10, imply that

$$\mathcal{J}'_\lambda(u,v)(\varphi,\psi)=0,\quad\forall(\varphi,\psi)\in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)\times C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N).$$

Arguing as in the proof of (5.81), we conclude that Claim is true.

By Lemma 5.2.10, (5.81) and (5.82), we reach

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}u\,\mathrm{d}x\right|\\&\leqslant\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}\varphi_\varepsilon\,\mathrm{d}x\right.\\&\quad\left.-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}\varphi_\varepsilon\,\mathrm{d}x\right|\\&\quad+\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}(\varphi_\varepsilon-u)\,\mathrm{d}x\right|\\&\quad+\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta}(\varphi_\varepsilon-u)\,\mathrm{d}x\right|<2C(N,\lambda,p,q)\varepsilon+o_n(1).\end{aligned}$$

Similarly, we have

$$\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^{\beta}}v\,\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v)}{|x|^{\beta}}v\,\mathrm{d}x\right|$$
$$< 2C(N,\lambda,p,q)\varepsilon+o_n(1),$$

which finishes the proof. □

In particular, arguing as in the proof of Lemma 5.2.12, we conclude that if $u\neq 0$ and $v=0$, then

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}u\,\mathrm{d}x\to\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u)}{|x|^{\beta}}u\,\mathrm{d}x,$$

or if $v\neq 0$ and $u=0$, then

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^{\beta}}v\,\mathrm{d}x\to\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v)}{|x|^{\beta}}v\,\mathrm{d}x.$$

In view of these last two limits, it follows from (BRÉZIS, 2011, Theorem 4.9), up to subsequence that, if $u\neq 0$ and $v=0$, then

$$\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}u_n\to\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u)}{|x|^{\beta}}u, \tag{5.85}$$

a.e. in $\mathbb{R}^N$, or if $v\neq 0$ and $u=0$, then

$$\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^{\beta}}v_n\to\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v)}{|x|^{\beta}}v, \tag{5.86}$$

a.e. in $\mathbb{R}^N$. To prove (5.85), we observe that

$$\left|\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}u_n-\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u)}{|x|^{\beta}}u\right|$$
$$\leqslant\left|\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}\frac{u(u_n-u)}{u}\right|$$
$$+\left|\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}u-\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u)}{|x|^{\beta}}u\right|$$
$$\to 0,\quad \text{a.e. in } \mathbb{R}^N,$$

where we are using (5.46) and (5.85). Similarly, (5.86) is verified.

Now if $u\neq 0$ and $v\neq 0$, then we have that

$$\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^{\beta}}u_n\to\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{f(u)}{|x|^{\beta}}u, \tag{5.87}$$

a.e. in $\mathbb{R}^N$ and

$$\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^{\beta}}v_n\to\left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^{\beta}|x-y|^{\mu}}\,\mathrm{d}y\right)\frac{g(v)}{|x|^{\beta}}v, \tag{5.88}$$

a.e. in $\mathbb{R}^N$.

It follows from Lemma 5.2.1 that the energy functional $\mathcal{J}_\lambda$ restricted to $\mathbb{W}^N_{\rm rad}$ satisfies the mountain pass geometry. Therefore, there exists a $(PS)_{m_\lambda}-$sequence $(u_n, v_n)_n$ in $\mathbb{W}^N_{\rm rad}$ satisfying (5.17), which is bounded as proved in Lemma 5.2.8. Hence, there exists $(u_0, v_0) \in \mathbb{W}^N_{\rm rad}$ such that, up to a subsequence, $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in $\mathbb{W}^N_{\rm rad}$, $(u_n, v_n) \to (u_0, v_0)$ in $L^q(\mathbb{R}^N)$ for all $q > N$. Moreover, combining the boundedness of $(u_n, v_n)_n$ with the fact that $\mathcal{J}'_\lambda(u_n, v_n) = o_n(1)$, we deduce $\mathcal{J}'_\lambda(u_n, v_n)(u_n, v_n) = o_n(1)$. The next Lemma proves that the weak limit is a critical point of $\mathcal{J}_\lambda$ constrained to $\mathbb{W}^N_{\rm rad}$.

Lemma 5.2.13. *Let $(u_n, v_n)_n$ be a $(PS)_c-$sequence for $\mathcal{J}_\lambda$. Then, there exists $(u, v) \in \mathbb{W}^N_{\rm rad} \setminus \{(0,0)\}$, such that $\mathcal{J}'_\lambda(u, v) = 0$.*

Proof. In view of $(u_n, v_n)_n$ is a sequence $(PS)_c$, then $(u_n, v_n)_n$ is bounded in $\mathbb{W}^N_{\rm rad}$. Thus, we may assume, passing to a subsequence if necessary, there exists $(u, v) \in \mathbb{W}^N_{\rm rad}$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in $\mathbb{W}^N_{\rm rad}$, $(u_n, v_n) \to (u, v)$ in $L^{\hat{p}}(\mathbb{R}^N) \times L^{\hat{p}}(\mathbb{R}^N)$ with $\hat{p} > N$ and $(u_n, v_n) \to (u, v)$, a.e. in $\mathbb{R}^N$, see (5.46). Hence, for any $(\phi, \psi) \in C^\infty_{0,\rm rad}(\mathbb{R}^N) \times C^\infty_{0,\rm rad}(\mathbb{R}^N)$, it follows from of Lebesgue's Dominated Convergence Theorem that (5.83) and (5.84) hold, implying that

$$\mathcal{J}'_\lambda(u, v)(\varphi, \psi) = 0, \quad \forall (\varphi, \psi) \in C^\infty_{0,\rm rad}(\mathbb{R}^N) \times C^\infty_{0,\rm rad}(\mathbb{R}^N).$$

By density arguments one may conclude that $\mathcal{J}'_\lambda(u, v)(\varphi, \psi) = 0$, for all $(\varphi, \psi) \in \mathbb{W}^N_{\rm rad}$.

Now, we claim that $(u, v) \neq (0, 0)$. Let us assume by contradiction that $(u, v) = (0, 0)$. From (5.1) and (5.46), we note that

$$|u_n|^p |v_n|^q \to 0 \quad \text{and} \quad |u_n|^p |v_n|^q \leqslant \frac{1}{2}\left(|h|^{2p} + |\tilde{h}|^{2q}\right) =: h_1, \quad \text{a.e. in} \quad \mathbb{R}^N,$$

where $h_1 \in L^1(\mathbb{R}^N)$. Hence, it follows from of Lebesgue's Dominated Convergence Theorem that

$$\int_{\mathbb{R}^N} |u_n|^p |v_n|^q \,\mathrm{d}x = o_n(1). \tag{5.89}$$

It follows from Lemma 5.2.9 that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{F(u_n)}{|x|^\beta} u_n \,\mathrm{d}x = o_n(1), \\ \displaystyle\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{G(v_n)}{|x|^\beta} v_n \,\mathrm{d}x = o_n(1). \end{cases} \tag{5.90}$$

Now, we claim that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\,\mathrm{d}x=o_n(1),\\ \displaystyle\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{g(v_n)}{|x|^\beta}v_n\,\mathrm{d}x=o_n(1).\end{cases} \tag{5.91}$$

Indeed, in view of Cauchy-Schwarz type inequality in (LIEB; LOSS, 2001, Theorem 9.8), we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\,\mathrm{d}x\right|&\\ \leqslant\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f(u_n)u_n}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)u_n}{|x|^\beta}\,\mathrm{d}x\right]^{\frac{1}{2}}&\\ \times\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\right]^{\frac{1}{2}}&.\end{aligned}$$

Thus, in order to conclude (5.91), it is sufficient to prove that there exists a constant $\tilde{C}>0$ such that

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f(u_n)}{|y|^\beta|x-y|^\mu}u_n\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\,\mathrm{d}x<\tilde{C},\quad\forall n\in\mathbb{N}. \tag{5.92}$$

In fact, since $c=\mathcal{J}_\lambda(u_n,v_n)+o_n(1)$, it follows from (5.89) and (5.90) that

$$\begin{aligned}&\limsup_{n\to\infty}\left(\|\nabla v_n\|_N^N+\|\nabla u_n\|_N^N\right)\\ \leqslant&\limsup_{n\to\infty}\left(\|(u_n,v_n)\|^N-N\lambda\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x+N\lambda\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x\right)\\ \leqslant&N\limsup_{n\to\infty}\mathcal{J}_\lambda(u_n,v_n)+N\lambda\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x\\ &+\frac{N}{2}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta}\,\mathrm{d}x\\ &+\frac{N}{2}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v_n)}{|x|^\beta}\,\mathrm{d}x\\ =&Nc.\end{aligned} \tag{5.93}$$

By virtue of Lemma 5.2.5, one has

$$\limsup_{n\to\infty}\left(\|\nabla v_n\|_N^N+\|\nabla u_n\|_N^N\right)\leqslant Nm_\lambda\leqslant\left(\frac{2N-2\beta-\mu}{2N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Hence, there exist $\delta\in(0,1)$ and $n_0\in\mathbb{N}$ such that

$$\left(\|\nabla v_n\|_N^N+\|\nabla u_n\|_N^N\right)^{\frac{1}{N-1}}\leqslant\left(\frac{2N-2\beta-\mu}{2N}\frac{\alpha_N}{\alpha_0}\right)(1-\delta),\quad\forall n\geqslant n_0. \tag{5.94}$$

Using Proposition 1.0.2, we see

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f(u_n)u_n}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)u_n}{|x|^\beta}\,\mathrm{d}x\leqslant C(N,\beta,\mu)\|f(u_n)u_n\|^2_{\frac{2N}{2N-2\beta-\mu}}. \tag{5.95}$$

On the other hand, by (5.3), for any $\epsilon > 0$, $r > 1$ and $\alpha > \alpha_0$, applying Hölder's inequality and using (5.6) to derive

$$
\begin{aligned}
&\|f(u_n)u_n\|^2_{\frac{2N}{2N-2\beta-\mu}} \leqslant \epsilon C\|u_n\|_N^{2N-2\beta-\mu} \\
&+ \bar{C}C(\epsilon,r,\alpha)\left\{\int_{\mathbb{R}^N}|u_n|^{\frac{2Nr}{2N-2\beta-\mu}}\left[\exp\left(\alpha u^{\frac{N}{N-1}}\right)-S_{N-2}(\alpha,u)\right]^{\frac{2N}{2N-2\beta-\mu}}\,\mathrm{d}x\right\}^{\frac{2N-2\beta-\mu}{N}} \\
\leqslant&\epsilon C\|u_n\|_N^{2N-2\beta-\mu} + \bar{C}C(\epsilon,r,\alpha)\left\{\int_{\mathbb{R}^N}\left[\exp\left(\alpha u_n^{\frac{N}{N-1}}\right)-S_{N-2}(\alpha,u_n)\right]^{\frac{2Nt'}{2N-2\beta-\mu}}\,\mathrm{d}x\right\}^{\frac{2N-2\beta-\mu}{Nt'}} \\
&\times\left(\int_{\mathbb{R}^N}|u_n|^{\frac{2Nrt}{2N-2\beta-\mu}}\right)^{\frac{2N-2\beta-\mu}{Nt}} \\
\leqslant&\epsilon C\|u_n\|_N^{2N-2\beta-\mu} + \left\{\int_{\mathbb{R}^N}\left[\exp\left(\frac{2N\alpha t'}{2N-2\beta-\mu}\|\nabla u_n\|_N^{\frac{N}{N-1}}\left(\frac{|u_n|}{\|\nabla u_n\|_N}\right)^{\frac{N}{N-1}}\right)\right.\right. \\
&\left.\left.-S_{N-2}\left(\frac{2N\alpha t'}{2N-2\beta-\mu}\|\nabla u_n\|_N^{\frac{N}{N-1}},\frac{|u_n|}{\|\nabla u_n\|_N}\right)\right]\mathrm{d}x\right\}^{\frac{2N-2\beta-\mu}{Nt'}} \\
&\times C_1\left(\int_{\mathbb{R}^N}|u_n|^{\frac{2Nrt}{2N-2\beta-\mu}}\right)^{\frac{2N-2\beta-\mu}{Nt}}, \qquad (5.96)
\end{aligned}
$$

where $t > 1$ and $\frac{1}{t}+\frac{1}{t'}=1$. From (5.94) and choosing t sufficiently close to 1 and choosing α sufficiently close to α_0, we may infer

$$
\frac{2N\alpha t'}{2N-2\beta-\mu}\left(\|\nabla v_n\|_N^N+\|\nabla u_n\|_N^N\right)^{\frac{1}{N-1}} \leqslant \alpha t'\left(\frac{\alpha_N}{\alpha_0}\right)(1-\delta) < \alpha_N, \quad \forall n \geqslant n_0.
$$

Next, let us recall the Gagliardo-Nirenberg inequality

$$
\|u\|_q^q \leqslant C\|u\|_N^N\|\nabla u\|_N^{\frac{q-N}{q}}, \quad \forall q > N \text{ and } u \in W^{1,N}(\mathbb{R}^N). \qquad (5.97)
$$

In view of Proposition 5.0.1, (5.97) and choosing $r \geqslant \frac{2N-2\beta-\mu}{2}$, we have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^N}|u_n|^{\frac{2Nrt}{2N-2\beta-\mu}}\right)^{\frac{2N-2\beta-\mu}{Nt}}\left\{\int_{\mathbb{R}^N}\left[\exp\left(\frac{2N\alpha t'}{2N-2\beta-\mu}\|\nabla u_n\|_N^{\frac{N}{N-1}}\left(\frac{|u_n|}{\|\nabla u_n\|_N}\right)^{\frac{N}{N-1}}\right)\right.\right. \\
&\qquad\left.\left.-S_{N-2}\left(\frac{2N\alpha t'}{2N-2\beta-\mu}\|\nabla u_n\|_N^{\frac{N}{N-1}},\frac{|u_n|}{\|\nabla u_n\|_N}\right)\right]\mathrm{d}x\right\}^{\frac{2N-2\beta-\mu}{Nt'}} \\
&\leqslant C\|u_n\|_N^{\frac{2N-2\beta-\mu}{t}}\|\nabla u_n\|_N^{2r-\frac{2N-2\beta-\mu}{t}}\left(\frac{\|u_n\|_N}{\|\nabla u_n\|_N}\right)^{\frac{2N-2\beta-\mu}{t'}} \\
&= C\|u_n\|_N^{2N-2\beta-\mu}\|\nabla u_n\|_N^{2r-(2N-2\beta-\mu)},
\end{aligned}
$$

which jointly with (5.96), implies that

$$
\|f(u_n)u_n\|^2_{\frac{2N}{2N-2\beta-\mu}} \leqslant \epsilon C\|u_n\|_N^{2N-2\beta-\mu} + C_1C\|u_n\|_N^{2N-2\beta-\mu}\|\nabla u_n\|_N^{2r-(2N-2\beta-\mu)}.
$$

Gathering the last estimate with (5.95) and noticing that $\|u_n\|_N, \|\nabla u_n\|_N \leqslant \|(u_n,v_n)\|$, we may infer

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{f(u_n)u_n}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u_n)}{|x|^\beta}u_n\,\mathrm{d}x \leqslant C(N,\beta,\mu)\|f(u_n)u_n\|^2_{\frac{2N}{2N-2\beta-\mu}} < \tilde{C}.$$

Therefore, (5.92) is true and consequently we obtain (5.91).

Next, applying (5.89) and (5.91) in $\mathcal{J}_\lambda'(u_n,v_n)(u_n,v_n) = o_n(1)$, yields that

$$\limsup_{n\to\infty}\|(u_n,v_n)\|^N = 0. \tag{5.98}$$

From $c = \mathcal{J}_\lambda(u_n,v_n) + o_n(1)$ and by using (5.89) and (5.90), it follows that

$$0 < Nc = N\limsup_{n\to\infty}\mathcal{J}_\lambda(u_n,v_n) = \limsup_{n\to\infty}\|(u_n,v_n)\|^N,$$

which jointly with (5.98) leads to a contradiction. Therefore, there exists $(u,v) \in \mathbb{W}^N_{\mathrm{rad}}\backslash\{(0,0)\}$ such that $\mathcal{J}_\lambda'(u,v) = 0$. □

Lemma 5.2.14. *Let $(u_n,v_n)_n \subset \mathbb{W}^N_{\mathrm{rad}}$ be a $(PS)_c-$sequence for $\mathcal{J}_\lambda$. Then, there exists $(u,v) \in \mathbb{W}^N_{\mathrm{rad}}\backslash\{(0,0)\}$ such that ${\mathcal{J}_\lambda}'(u,v) = 0$ and $\mathcal{J}_\lambda(u,v) = c_{\mathcal{N}_\lambda}$. In addition, $(u_n,v_n) \to (u,v)$ strongly in $\mathbb{W}^N_{\mathrm{rad}}$.*

Proof. For the same reasons as in Lemma 5.2.13, we may assume, passing to a subsequence if necessary, there exists $(u,v) \in \mathbb{W}^N_{\mathrm{rad}}$ such that $(u_n,v_n) \rightharpoonup (u,v)$ in $\mathbb{W}^N_{\mathrm{rad}}$, $(u_n,v_n) \to (u,v)$ in $L^{\hat{p}}(\mathbb{R}^N)\times L^{\hat{p}}(\mathbb{R}^N)$ with $\hat{p} > N$ and $(u_n,v_n) \to (u,v)$, a.e. in $\mathbb{R}^N$. Similarly to the proof of Lemma 5.2.13, one has $\mathcal{J}_\lambda'(u,v) = 0$. Since $c_{\mathcal{N}_\lambda} > 0$, we split the proof into two cases.

Case 1. $(u,v) = (0,0)$. Similar to the proof of Lemma 5.2.13.

Case 2. $(u,v) \neq (0,0)$. Since $(u,v) \neq (0,0)$, it follows from Lemma 5.2.3 that there exists a unique $t_{(u,v)} > 0$, depending on (u,v), such that

$$(t_{(u,v)}u, t_{(u,v)}v) \in \mathcal{N}_\lambda \quad \text{and} \quad \max_{t\geqslant 0}\mathcal{J}_\lambda(t(u,v)) = \mathcal{J}_\lambda(t_{(u,v)}(u,v)). \tag{5.99}$$

On the other hand, in view of (5.46), Lemma 5.2.9 and (BRÉZIS, 2011, Theorem 4.9), one may deduce, up to a subsequence, that

$$\left(\int_{\mathbb{R}^N}\frac{F(u_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u_n)}{|x|^\beta} \to \left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\beta}, \quad \text{a.e. in } \mathbb{R}^N, \tag{5.100}$$

and

$$\left(\int_{\mathbb{R}^N}\frac{G(v_n)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v_n)}{|x|^\beta} \to \left(\int_{\mathbb{R}^N}\frac{G(v)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v)}{|x|^\beta}, \quad \text{a.e. in } \mathbb{R}^N. \tag{5.101}$$

Since $\mathcal{J}'(u,v)(u,v)=0$ and according to (5.20), (5.85), (5.86), (5.87), (5.88), (5.99), (5.100) and (5.101), it follows from Fatou's Lemma that

$$
\begin{aligned}
m_\lambda \leqslant c_{\mathcal{N}_\lambda} \leqslant & \mathcal{J}_\lambda(u,v) - \frac{1}{N}\mathcal{J}_\lambda'(u,v)(u,v) = \frac{p+q}{N}\lambda \int_{\mathbb{R}^N} |u|^p|v|^q \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} |u|^p|v|^q \,\mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{\left[\frac{1}{N} f(u)u - \frac{1}{2}F(u)\right]}{|x|^\beta} \,\mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{\left[\frac{1}{N} g(v)v - \frac{1}{2}G(v)\right]}{|x|^\beta} \,\mathrm{d}x \\
\leqslant & \liminf_{n\to\infty} \left(\frac{p+q}{N}\lambda \int_{\mathbb{R}^N} |u_n|^p|v_n|^q \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} |u_n|^p|v_n|^q \,\mathrm{d}x \right) \\
& + \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{\left[\frac{1}{N} f(u_n)u_n - \frac{1}{2}F(u_n)\right]}{|x|^\beta} \,\mathrm{d}x \\
& + \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{\left[\frac{1}{N} g(v_n)v_n - \frac{1}{2}G(v_n)\right]}{|x|^\beta} \,\mathrm{d}x \\
= & \liminf_{n\to\infty} \left[\mathcal{J}_\lambda(u_n,v_n) - \frac{1}{N}\mathcal{J}_\lambda'(u_n,v_n)(u_n,v_n) \right] = c \leqslant m_\lambda \leqslant c_{\mathcal{N}_\lambda},
\end{aligned}
$$

which implies that $\mathcal{J}_\lambda(u,v)=c_{\mathcal{N}_\lambda}$. Combining this with Fatou's Lemma and Lemma 5.13, we see that

$$
\begin{aligned}
& \frac{1}{N}\|(u,v)\|^N \leqslant \liminf_{n\to+\infty} \frac{1}{N}\|(u_n,v_n)\|^N \leqslant \limsup_{n\to+\infty} \frac{1}{N}\|(u_n,v_n)\|^N \\
\leqslant & \limsup_{n\to\infty} \Bigg[\mathcal{J}_\lambda(u_n,v_n) + \lambda \int_{\mathbb{R}^N} |u_n|^p|v_n|^q \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u_n)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{F(u_n)}{|x|^\beta} \,\mathrm{d}x \\
& + \frac{1}{2}\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v_n)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{G(v_n)}{|x|^\beta} \,\mathrm{d}x \Bigg] \\
= & c_{\mathcal{N}_\lambda} \\
& + \frac{1}{2}\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{F(u)}{|x|^\beta} \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{G(v)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right) \frac{G(v)}{|x|^\beta} \,\mathrm{d}x \\
= & \frac{1}{N}\|(u,v)\|^N,
\end{aligned}
$$

i.e., $\lim_{n\to\infty}\|(u_n,v_n)\|^N = \|(u,v)\|^N$. Putting this together with the fact that $(u_n,v_n) \rightharpoonup (u,v)$ and $\mathbb{W}^N_{\mathrm{rad}}$ is uniformly convex, it follows that $(u_n,v_n)\to(u,v)$ in $\mathbb{W}^N_{\mathrm{rad}}$. It finishes the proof of the Lemma. □

In view of Proposition A.0.1, one has that if (u,v) is a critical point of $\mathcal{J}_\lambda$ restricted to $\mathbb{W}^N_{\mathrm{rad}}$, then (u,v) is a critical point of $\mathcal{J}_\lambda$ on $\mathbb{W}^N$. Therefore, Lemma 5.2.14 ensures the existence of a nontrivial solution for System (S_λ). It follows from assumption (a) and (5.19) that $F(|s|) \geqslant F(s)$ for all $s \in \mathbb{R}$. Hence, $\mathcal{J}_\lambda(|u|,|v|) \leqslant \mathcal{J}_\lambda(u,v)$, for $(u,v) \in \mathbb{W}^N_{\mathrm{rad}}$. Let $(u,v) \in \mathcal{N}_\lambda$ be the ground state obtained in Lemma 5.2.14. By using Lemma 5.2.3, there exists t_0 such that $(t_0|u|, t_0|v|) \in \mathcal{N}_\lambda$. Thus, from (5.24) we deduce

$$c_{\mathcal{N}_\lambda} \leqslant \mathcal{J}_\lambda(t_0|u|, t_0|v|) \leqslant \mathcal{J}_\lambda(t_0 u, t_0 v) \leqslant \max_{t>0} \mathcal{J}_\lambda(tu, tv) = \mathcal{J}_\lambda(u,v) = c_{\mathcal{N}_\lambda},$$

which implies that $\mathcal{J}_\lambda(t_0|u|, t_0|v|) = c_{\mathcal{N}_\lambda}$. Therefore, we may suppose that (u,v) is a nonnegative solution (or radial ground state solution) for System (S_λ).

Next, we will check that if (u,v) is a solution of System (S_λ), then (u,v) belongs to $[L^\infty(\mathbb{R}^N) \cap C^{1,\gamma}(\mathbb{R}^N)]^2$, for some $\gamma \in (0,1)$.

Lemma 5.2.15. *Let (u,v) be a solution of System (S_λ). Then, $(u,v) \in [L^\infty(\mathbb{R}^N) \cap C^{1,\gamma}(\mathbb{R}^N)]^2$, for some $\gamma \in (0,1)$.*

Proof. We have the following two claims:

Claim 1. If u and v satisfy

$$-\Delta_N u + |u|^{N-2}u = \left(\int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y\right) \frac{f(u)}{|x|^\beta} + \lambda p |u|^{p-2} u |v|^q, \quad \text{in } \mathbb{R}^N, \tag{5.102}$$

then, $u \in L^\infty(\mathbb{R}^N)$.

Claim 2. If u and v satisfy

$$-\Delta_N v + |v|^{N-2}v = \left(\int_{\mathbb{R}^N} \frac{G(v)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y\right) \frac{g(v)}{|x|^\beta} + \lambda q |u|^p |v|^{q-2} v, \quad \text{in } \mathbb{R}^N,$$

then, $v \in L^\infty(\mathbb{R}^N)$.

Proof of the Claim 1. We follow (ALVES; SHEN, 2023, Theorem 1.6). By setting $s = r = \hat{r} = \frac{2N}{2N-2\beta-\mu}$ and $t = \hat{s} = \frac{2N}{2\beta+\mu}$, it is not hard to see that

$$1 + \frac{1}{\hat{s}} = \frac{1}{\hat{r}} + \frac{2\beta+\mu}{N} \quad \text{and} \quad \frac{\beta}{N} < \frac{1}{\hat{s}} < \frac{\beta+\mu}{N}.$$

Since $F(u) \in L^{\hat{r}}(\mathbb{R}^N)$, it follows from Proposition 1.0.2 that

$$\frac{1}{|x|^\beta} \int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \in L^{\hat{s}}(\mathbb{R}^N). \tag{5.103}$$

Now, note that if $\hat{p} \geqslant \frac{2N}{N-2\beta-\mu}$, then $\frac{[(N-2)+N-2\beta-\mu]\hat{p}}{2} \geqslant N$. Hence, it follows from Proposition 5.0.1, (5.4) and Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^N} |f(u)|^{\hat{p}} \,\mathrm{d}x \leqslant & C\varepsilon^{\hat{p}} \int_{\mathbb{R}^N} |u|^{\frac{((N-2)+N-2\beta-\mu)\hat{p}}{2}} \,\mathrm{d}x \\ & + CC(\alpha, r, \epsilon)^{\hat{p}} \int_{\mathbb{R}^N} |u|^{(r-1)\hat{p}} \left[\exp\left(\alpha u^{\frac{N}{N-1}}\right) - S_{N-2}(\alpha, u)\right]^{\hat{p}} \mathrm{d}x < +\infty,\end{aligned}$$

i.e.,

$$f(u) \in L^{\hat{p}}(\mathbb{R}^N), \quad \forall \hat{p} \geqslant \frac{2N}{N-2\beta-\mu}. \tag{5.104}$$

We claim that there exists $\epsilon_0 > 0$ such that

$$h(x) := \frac{1}{|x|^\beta} \int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y f(u) + \lambda p |u|^{p-2} u |v|^q \in L^{\hat{t}}(\mathbb{R}^N), \quad \forall \hat{t} \in ((1-\epsilon_0)\hat{s}, \hat{s}). \tag{5.105}$$

In fact, let $\epsilon_0 > 0$ be such that

$$\frac{(1-\epsilon)\hat{s}}{\epsilon} > \max\left\{\frac{2N}{N-2\beta-\mu}, p+q\right\}, \quad \forall \epsilon \in (0, \epsilon_0). \tag{5.106}$$

Note that $(1-\epsilon)\hat{s} \in ((1-\epsilon_0)\hat{s}, \hat{s})$. One may deduce

$$\begin{aligned} \int_{\mathbb{R}^N} |h|^{(1-\epsilon)\hat{s}} \,\mathrm{d}x \leqslant & C(\epsilon, \hat{s}) \int_{\mathbb{R}^N} \left| \frac{1}{|x|^\beta} \int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right|^{(1-\epsilon)\hat{s}} |f(u)|^{(1-\epsilon)\hat{s}} \,\mathrm{d}x \\ & + C(\epsilon, \hat{s}) \lambda^{(1-\epsilon)\hat{s}} p^{(1-\epsilon)\hat{s}} \int_{\mathbb{R}^N} (|u|^{(p-1)(1-\epsilon)\hat{s}} |v|^{(q-1)(1-\epsilon)\hat{s}} v^{(1-\epsilon)\hat{s}}) \,\mathrm{d}x \\ =: & \, C(\epsilon, \hat{s}) \mathcal{A}_1 + C(\epsilon, \hat{s}) \lambda^{(1-\epsilon)\hat{s}} p^{(1-\epsilon)\hat{s}} \mathcal{A}_2. \end{aligned} \tag{5.107}$$

In view of (5.103), (5.104), (5.106) and Hölder's inequality, we obtain

$$\mathcal{A}_1 \leqslant \left(\int_{\mathbb{R}^N} \left| \frac{1}{|x|^\beta} \int_{\mathbb{R}^N} \frac{F(u)}{|y|^\beta |x-y|^\mu} \,\mathrm{d}y \right|^{\hat{s}} \mathrm{d}x \right)^{1-\epsilon} \left(\int_{\mathbb{R}^N} |f(u)|^{\frac{(1-\epsilon)\hat{s}}{\epsilon}} \,\mathrm{d}x \right)^{\epsilon} < +\infty. \tag{5.108}$$

Moreover, using (5.1), (5.106) and Hölder's inequality, we have

$$\begin{aligned} \mathcal{A}_2 \leqslant & \left(\int_{\mathbb{R}^N} |u|^{2p\frac{(1-\epsilon)\hat{s}}{\epsilon}} \,\mathrm{d}x \right)^{\frac{\epsilon(p-1)}{2p}} \left(\int_{\mathbb{R}^N} |v|^{2(q-1)\frac{(1-\epsilon)\hat{s}}{\epsilon}} \,\mathrm{d}x \right)^{\frac{\epsilon}{2}} \\ & \times \left(\int_{\mathbb{R}^N} |v|^{2p\frac{(1-\epsilon)\hat{s}}{\epsilon}} \,\mathrm{d}x \right)^{\frac{\epsilon}{2p}} < +\infty. \end{aligned} \tag{5.109}$$

Therefore, (5.107), (5.108), (5.109) imply that (5.105) holds.

Finally, in light of standard regularity theory, we conclude that $u \in W^{2,\hat{t}}(\mathbb{R}^N)$, $\hat{t} \in ((1-\epsilon_0)\hat{s}, \hat{s})$. Since that $\frac{1}{\hat{t}} - \frac{2}{N} < 0$ for ϵ_0 sufficiently small, it follows from (BRÉZIS, 2011, Corollary 9.13) that $W^{2,\hat{t}}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$, which finishes the proof of Claim 1. Analogously one may check the proof of Claim 2.

Since the solution $(u, v) \in L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ and $\hat{t} > N$, we can apply (BRÉZIS, 2011, Corollary 9.13) with $m = 2$, $k = 1$, $\gamma = \theta$ and $p = \hat{t}$, to conclude that $(u, v) \in C^{1,\gamma}(\mathbb{R}^N) \times C^{1,\gamma}(\mathbb{R}^N)$. □

5.2.5.1 Vectorial ground state

It is important to emphasize that unlike to the linearly coupled case (see Remark 5.1.5), the solution of System (S_λ) could be semitrivial, i.e., of type $(u,0)$ or $(0,v)$. Now we will prove that for λ sufficiently large the nontrivial solution obtained in Lemma 5.2.14 can not be semitrivial.

Lemma 5.2.16. *There exists $\lambda_0>0$ such that if $\lambda>\lambda_0$, then System (S_λ) admits a solution (u,v) which is vectorial, i.e., $u\neq 0$ and $v\neq 0$.*

Proof. We begin by studying the following equation

$$-\Delta_N u+|u|^{N-2}u=\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{f(u)}{|x|^\beta},\quad \text{in } \mathbb{R}^N. \qquad (S_0^1)$$

Associated to equation (S_0^1), we have the energy functional $\mathcal{J}_0^1: W^{1,N}(\mathbb{R}^N)\to\mathbb{R}$

$$\mathcal{J}_0^1(u)=\frac{1}{N}\|u\|^N-\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(u)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u)}{|x|^\beta}\,\mathrm{d}x.$$

Recalling the proof of Lemma 5.2.1, it is not hard to check that the functional $\mathcal{J}_0^1$ has the mountain pass geometry. We introduce the mountain pass level as

$$m_0^1:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}_0(\gamma(t)),\ \text{ where } \Gamma:=\left\{\gamma\in C([0,1];W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)):\gamma(0)=0,\ \mathcal{J}_0^1(\gamma(1))<0\right\}$$

and the Nehari energy level for equation (S_0^1) given by

$$c^1_{\mathcal{N}_0^1}:=\inf_{u\in\mathcal{N}_0^1}\mathcal{J}_0^1(u)\quad\text{and}\quad \mathcal{N}_0^1:=\left\{u\in W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)\setminus\{0\};(\mathcal{J}_0^1)'(u)u=0\right\}.$$

Using similar arguments from the previous sections one may deduce that $m_0^1=c^1_{\mathcal{N}_0^1}$. Note that the same arguments used in this work holds true for (S_0^1). Thus, let $u_0\in\mathcal{N}_0^1$ be a nonnegative solution for (S_0^1) at Nehari level, i.e., $(\mathcal{J}_0^1)'(u_0)=0$ and $\mathcal{J}_0^1(u_0)=c^1_{\mathcal{N}_0^1}$. Furthermore, when $\lambda=0$, the equation in (5.102) becomes equation (S_0^1). Hence, for $\lambda=0$, we may argue in a similar way to the proof of Lemma 5.2.15 to conclude that $u_0\in L^\infty(\mathbb{R}^N)\cap C^{1,\gamma}(\mathbb{R}^N)$. In light of Strong Maximum Principle, we conclude that u_0 is a positive ground state solution for (S_0^1). By similar arguments used in the proof of Lemma 5.2.3, for any $u_0\in\mathcal{N}_0^1$, we deduce that

$$\begin{aligned}&\mathcal{J}_0^1(tu_0)\ \text{ is increasing for }\ t\in(0,1),\\ &\mathcal{J}_0^1(tu_0)\ \text{ is decreasing for }\ t\in(1,+\infty),\\ &\mathcal{J}_0^1(tu_0)\to-\infty,\ \text{ as }\ t\to+\infty.\end{aligned}$$

Consequently, we have that $\mathcal{J}_0^1(u_0) = \max_{t\geqslant 0}\mathcal{J}_0^1(tu_0)$. Analogously, we can introduce $\mathcal{J}_0^2$, m_0^2, $\mathcal{N}_0^2$, c_0^2 and conclude that there exists a positive solution $v_0 \in \mathcal{N}_0^2$ at Nehari level for the equation

$$-\Delta_N v + |v|^{N-2}v = \left(\int_{\mathbb{R}^N} \frac{G(v)}{|y|^\beta |y-x|^\mu}\,\mathrm{d}y\right)\frac{g(v)}{|x|^\beta} \quad \text{in } \mathbb{R}^N.$$

In view of Lemma 5.2.3, there exists $t_0 > 0$, depending on (u_0, v_0), such that $(t_0u_0, t_0v_0) \in \mathcal{N}_\lambda$. Moreover, since $p+q > N$, we obtain the following estimate

$$\begin{aligned}
c_{\mathcal{N}_\lambda} &\leqslant \max_{t\geqslant 0}\mathcal{J}_\lambda(tu_0, tv_0)\\
&\leqslant \max_{t\geqslant 0}\left\{\frac{1}{N}t^N\|(u_0,v_0)\|^N - \lambda t^{p+q}\int_{\mathbb{R}^N}|u_0|^p|v_0|^q\,\mathrm{d}x\right.\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(tu_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(tu_0)}{|x|^\beta}\,\mathrm{d}x\\
&\quad \left. -\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{G(tv_0)}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(tv_0)}{|x|^\beta}\,\mathrm{d}x\right\}\\
&\leqslant \max_{t\geqslant 0}\left\{\frac{1}{N}t^N\|(u_0,v_0)\|^N - \lambda t^{p+q}\int_{\mathbb{R}^N}|u_0|^p|v_0|^q\,\mathrm{d}x\right\}\\
&= \frac{1}{(p+q)^{\frac{p+q}{p+q-N}}}\left(\frac{p+q}{N}-1\right)\left[\frac{\|(u_0,v_0)\|}{\lambda^{\frac{1}{p+q}}\left(\int_{\mathbb{R}^N}|u_0|^p|v_0|^q\,\mathrm{d}x\right)^{\frac{1}{p+q}}}\right]^{\frac{N(p+q)}{(p+q-N)}} \to 0, \text{ as } \lambda\to+\infty.
\end{aligned}$$

Therefore, there exists $\lambda_0 > 0$, such that

$$c_{\mathcal{N}_\lambda} < \min\left\{c^1_{\mathcal{N}_0^1}, c^2_{\mathcal{N}_0^2}\right\}, \quad \forall\lambda \geqslant \lambda_0. \tag{5.110}$$

Now, let (u,v) be a solution (radial ground state solution) for System (S_λ) at Nehari level, with $\lambda \geqslant \lambda_0$ (see Lemma 5.2.14). Suppose, by contradiction, that this solution is semitrivial, for instance, $(u,0)$. Since $(\mathcal{J}_0^1)'(u)u = \mathcal{J}_\lambda'(u,0)(u,0) = 0$, it follows that $u \in \mathcal{N}_0^1$. Hence,

$$\min\left\{c^1_{\mathcal{N}_0^1}, c^2_{\mathcal{N}_0^2}\right\} \leqslant c^1_{\mathcal{N}_0^1} \leqslant \mathcal{J}_0^1(u) = \mathcal{J}_\lambda(u,0) = c_{\mathcal{N}_\lambda},$$

which contradicts (5.110). Therefore, the solution is vectorial. □

Proof of Theorem 5.1.2. Since (u,v) is a nonnegative solution for System (S_λ) and in view of Lemmas 5.2.15 and 5.2.16, it follows from Strong Maximum Principle that (u,v) is positive, which finishes the proof of Theorem 5.1.2. □

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APPENDIX A – PRINCIPLE OF SYMMETRIC CRITICALITY

In this appendix, we present, for $N > 2$, due to the lack of Hilbert space structure, a variant of Palais principle of symmetric criticality. This results is related to Chapter 5.

Let $O(N)$ denotes the group of orthogonal linear transformations of $\mathbb{R}^N$. Let G be a subgroup of $O(N)$.

The action of G over a real Banach space $(X, \|\cdot\|_X)$ is said to be an isometry on X if $\|gu\|_X = \|u\|_X$, for all $g \in G$, for all $u \in X$.

Let us denote the class of all $G-$invariant C^1 functional on X by

$$C^1_G(X) := \left\{ J \in C^1 : J(gu) = J(u), \quad \text{for all } g \in G \text{ and } u \in X \right\}.$$

The linear subspace of $G-$symmetric (or invariant) points of X and X^* are defined as common fixed points of G

$$Fix(G) := \{u \in X : gu = u, \forall g \in G\},$$

$$Fix(G)_* := \{v \in X^* : gv^* = v^*, \forall g \in G\}.$$

Consider the following principle:

(P) : *Let X be a Banach space and G an isometric action on X. If $\left(J\big|_{Fix(G)} \right)' u = 0$ for all $J \in C^1_G(X)$, then $J'(u) = 0$ and $u \in Fix(G)$.*

Proposition A.0.1. *For a uniformly convex Banach space X the principle (P) holds true.*

Proof. In light of (PALAIS, 1979, Proposition 4.2), it is enough to show that $(Fix(G))^\perp \cap Fix(G)_* = \{0\}$. Let F be a duality map from X to X^*. It is not hard to see that F is surjective as X is reflexive and from uniform convexity of X, one can conclude that F is strictly monotone and hence F is injective (see (CIORANESCU, 1990)). Thus, F^{-1} is a single valued map in X^*, i.e., $F^{-1}(x^*) \in Fix(G)$ for all $x^* \in Fix(G)_*$. Now, if $x^* \in (Fix(G))^\perp \cap Fix(G)_*$, then the facts that $x^* \in (Fix(G))^\perp$ and $F^{-1}(x^*) \in Fix(G)$ imply that $x^* = 0$. This completes the proof. □

Let us recall that the energy functional $\mathcal{J}_\lambda : \mathbb{W}^N \to \mathbb{R}$ associated to System (S_λ) is

$C^1(\mathbb{W}^N, \mathbb{R})$ and it is defined as

$$\begin{aligned}\mathcal{J}_\lambda(u,v) =& \frac{1}{N}\|(u,v)\|^N - \lambda \int_{\mathbb{R}^N} |u|^p|v|^q \,\mathrm{d}x \\ &- \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(u(y))}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{F(u(x))}{|x|^\beta}\,\mathrm{d}x \\ &- \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{G(v(y))}{|y|^\beta|x-y|^\mu}\,\mathrm{d}y\right)\frac{G(v(x))}{|x|^\beta}\,\mathrm{d}x.\end{aligned}$$

It is well known that the space $\mathbb{W}^N$ is reflexive and strictly convex. In what follows, we define the action of G on $\mathbb{W}^N$ as

$$g(u,v)(x) := (u(g^{-1}x), v(g^{-1}x)). \tag{1.1}$$

Note that this action is a continuous map

$$\begin{aligned} G\times X &\to X \\ [g,u] &\mapsto gu \end{aligned}$$

satisfying

(i) $1\cdot u = u$,

(ii) $(gh)u = g(hu)$,

(iii) $u \mapsto gu$ is linear.

Moreover, the action of G is isometric and $Fix(G) = \mathbb{W}^N_{\mathrm{rad}}$. Next, we verify that $\mathcal{J}_\lambda \in C^1_G(\mathbb{W}^N)$, i.e., $\mathcal{J}_\lambda(g(u,v)) = \mathcal{J}_\lambda(u,v)$ for every $g \in G$. Initially, we will prove that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(gu(y))F(gu(x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(y))F(u(x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x, \quad \forall g\in G \tag{1.2}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(gu(y))G(gu(x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))G(u(x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x, \quad \forall g\in G. \tag{1.3}$$

In fact, from the definition (1.1) we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(gu(y))F(gu(x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(g^{-1}y))F(u(g^{-1}x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}x. \tag{1.4}$$

Let us consider the change of variables, $z_1 = g^{-1}x$ and $z_2 = g^{-1}y$, or yet, $gz_1 = x$ and $gz_2 = y$. Since that $g^{-1} \in O(N)$ (i.e., $|\det g^{-1}| = 1$), we have $dz_1 = \mathrm{d}x$ and $dz_2 = \mathrm{d}y$, which jointly with (1.4) imply that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(gu(y))F(gu(x))}{|y|^\beta|x-y|^\mu|x|^\beta}\,\mathrm{d}y\mathrm{d}y &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(z_2))F(u(z_1))}{|gz_2|^\beta|gz_1-gz_2|^\mu|gz_1|^\beta}\,\mathrm{d}z_2\mathrm{d}z_1\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(z_2))F(u(z_1))}{|z_2|^\beta|g(z_1-z_2)|^\mu|z_1|^\beta}\,\mathrm{d}z_2\mathrm{d}z_1\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u(z_2))F(u(z_1))}{|z_2|^\beta|z_1-z_2|^\mu|z_1|^\beta}\,\mathrm{d}z_2\mathrm{d}z_1,\end{aligned}$$

where we used the fact that g is an isometry in $\mathbb{R}^N$, i.e., $|gx| = |x|$, for all $x \in \mathbb{R}^N$. This proves the equality in (1.2). Analogously, it turns out that (1.3) is also true. In a similar way, it is verified

$$\begin{aligned}\|(gu,gv)\|^N &= \int_{\mathbb{R}^N}(|\nabla(gu(x))|^N + |gu(x)|^N)\,\mathrm{d}x + \int_{\mathbb{R}^N}(|\nabla(gv(x))|^N + |gv(x)|^N)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}(|\nabla u(x)|^N + |v(x)|^N)\,\mathrm{d}x + \int_{\mathbb{R}^N}(|\nabla v(x)|^N + |v(x)|^N)\,\mathrm{d}x\\ &= \|(u,v)\|^N\end{aligned}$$

and

$$\int_{\mathbb{R}^N} |gu(x)|^p|gv(x)|^q\,\mathrm{d}x = \int_{\mathbb{R}^N} |u(x)|^p|v(x)|^q\,\mathrm{d}x.$$

Thus, $\mathcal{J}_\lambda$ is $G-$invariant functional. Therefore, it follows from Proposition A.0.1 that the critical points of $\mathcal{J}_\lambda$ restricted to $\mathbb{W}^N_{\mathrm{rad}}$ are critical points of $\mathcal{J}_\lambda$ in $\mathbb{W}^N$.