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ESTEVAN LUIZ DA SILVA

The existence of solutions for some classes of nonlinear elliptic equations and systems with sub-natural growth terms

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The existence of solutions for some classes of nonlinear elliptic equations and systems with sub-natural growth terms

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BANCA EXAMINADORA

____

Prof. Dr. João Marcos Bezerra do Ó (Orientador)
Universidade Federal da Paraíba

____

Prof. Dr. José Carlos de Albuquerque Melo Júnior (Examinador Interno)
Universidade Federal de Pernambuco

____

Prof. Dr. Everaldo Souto de Medeiros (Examinador Externo)
Universidade Federal da Paraíba

____

Prof. Dr. Olimpio Hiroshi Miyagaki (Examinador Externo)
Universidade Federal de São Carlos

____

Prof. Dr. Jefferson Abrantes dos Santos (Examinador Externo)
Universidade Federal de Campina Grande

To my grandmothers (in memoriam) Rute Ferreira and Virtuosa Maria da Silva.

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ABSTRACT

Global pointwise estimates are obtained for quasilinear Lane-Emden-type systems involving measures in the "sublinear growth" rate. Necessary and sufficient conditions are presented for the existence of positive solutions to a class of systems of quasilinear elliptic equations involving measures in the "sublinear growth" rate expressed in terms of Wolff's potential. Our approach is based on recent advances due to T. Kilpeläinen and J. Malý in the potential theory. Also, we are interested in a class of k-Hessian Lane-Emden type systems with measure data in the "sublinear growth" rate. We give global pointwise estimates of the so-called Brezis–Kamin type in terms of Wolff potentials, which allows us to obtain necessary and sufficient conditions for the existence of positive solutions. This method enables us to treat several kinds of problems, such as equations involving general quasilinear operators and fractional Laplacian, or fully nonlinear k-Hessian operators.

Further, we present a sufficient condition in terms of Wolff potentials for the existence of a finite energy solution to measure data (p, q)-Laplacian equation in the "sublinear growth" rate. We prove that such a solution is minimal. Besides, we show a necessary condition in terms of a suitably generalized potential of the Wolff-type for the eventual solutions, not necessarily of finite energy. Our main tools are integral inequalities closely associated with (p, q)-Laplacian equations with measure data, and pointwise potential estimates which allow us to obtain bounds of solutions. This method enables us to treat other nonlinear elliptic problems involving general quasilinear operators.

Keywords: Nonlinear elliptic equations; Wolff potentials; p-Laplacian; Fractional Laplacian; k-Hessian equations; Lane-Emden system; Integral equations; Orlicz-Sobolev spaces; Measure data problems.

RESUMO

Estimativas pontuais globais são obtidas para sistemas quasi-lineares do tipo Lane-Emden envolvendo medidas na taxa de "crescimento sublinear". Condições necessárias e suficientes são apresentadas para a existência de soluções positivas para uma classe de sistemas de equações elípticas quase-linear envolvendo medidas numa taxa de "crescimento sublinear" expressadas em termos do potencial de Wolff. Nossa abordagem é baseada em avanços recentes devido a T. Kilpeläinen e J. Malý na teoria do potencial. Além disso, estamos interessados em um classe de sistemas do tipo Lane-Emden regidos pelo operador k-Hessiano também com medidas num "crescimento sublinear". Fornecemos estimativas pontuais globais do chamado tipo Brezis-Kamin em termos de potenciais Wolff, o que nos permite obter condições necessárias e suficientes para a existência de soluções positivas a esse sistema. Este método nos permite tratar vários tipos de problemas, como equações envolvendo operadores quase-lineares mais gerais e o Laplaciano fracionário, ou operadores totalmente não-lineares do tipo k-Hessiano.

Em um viés parecido, apresentamos uma condição suficiente em termos de potenciais de Wolff para a existência de uma solução de energia finita para uma equação do tipo (p, q)-Laplaciano com medidas no "crescimento sublinear". Provamos que tal solução é mínima. Além disso, exibimos uma condição necessária em termos de um potencial adequadamente generalizado do tipo Wolff para eventuais soluções desse problema, não necessariamente de energia finita. Nossas principais ferramentas são as desigualdades de integrais intimamente associadas a equações do regidas pelo operador (p, q)-Laplaciano, com medidas, e estimativas pontuais de potenciais que permitem obter limitações para estas soluções. Este método nos permite tratar outras formas de problemas elípticas não-lineares envolvendo operadores quase-lineares mais gerais.

Palavras-chaves: Equações elípitcas não-lineares; Potenciais de Wolff; p-Laplaciano; Laplaciano Fracionário; Equações integrais; Espaços de Orlicz-Sobolev; Problemas com medidas dadas.

LIST OF SYMBOLS

$\Omega \subseteq \mathbb{R}^n$ stands a domain.

As usual, we use the letters c, $\tilde{c}$, C, and $\tilde{C}$, with or without subscripts, to denote different constants.

$A \approx B$ means that there exists a constant $c > 0$ such that $c^{-1} B \leq A \leq c B$.

$\chi_E :=$ the characteristic function of a set E.

$M^+(\Omega) :=$ the set of all nonnegative Radon measures σ defined on Ω.

Often, we use the Greek letters μ and ω to denote Radon measures.

We denote by $\mathrm{d}\sigma$-a.e in $\mathbb{R}^n$, when a property holds almost everywhere in $\mathbb{R}^n$ in the sense of the measure σ.

$C(\Omega) :=$ the set of all continuous functions in Ω.

$C_c^\infty(\Omega) :=$ the set of all infinitely differentiable functions with compact support in Ω.

$L^s_{\mathrm{loc}}(\Omega, \mathrm{d}\mu) :=$ the local L^s space with respect to $\mu \in M^+(\Omega)$, $s > 0$. If μ is the Lebesgue measure, we write $L^s_{\mathrm{loc}}(\Omega)$.

$W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) :=$ the space of all functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ which admit weak derivatives $\partial_i u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for $i = 1, \ldots, n$.

$W^{-1,p'}_{\mathrm{loc}}(\Omega) :=$ the dual of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, where $p' = p/(p-1)$.

We denote by $\sigma(E) = \int_E \mathrm{d}\sigma$ the measure of any σ-measurable subset E of Ω. When σ is the Lebesgue measure, we write $|E| = \sigma(E)$.

$B(x,t):=$ the open ball of radius $t > 0$ centered at $x \in \mathbb{R}^n$.

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1 INTRODUCTION

This thesis aims to use tools of nonlinear potential theory to prove the existence of solutions to systems of quasilinear elliptic equations, systems of fully nonlinear elliptic equations, and quasilinear elliptic equations with Orlicz growth. All these problems will be of the Lane-Emden type and considered in the "sublinear growth" rate.

Fix $n \geq 3$ and let $\sigma \in M^+(\mathbb{R}^n)$. To make the explanation more precise, we first give necessary and sufficient conditions for the existence of *solutions* for:

($\mathbf{I}$) systems of the Lane-Emden type, including the following model problem for quasilinear elliptic equations

$$\begin{cases} -\Delta_p u = \sigma\, v^{q_1}, \quad u > 0, \quad \text{in} \quad \mathbb{R}^n, \\ -\Delta_p v = \sigma\, u^{q_2}, \quad v > 0, \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \qquad \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{S_1}$$

where $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f)$ is the p-Laplacian, $1 < p < \infty$;

($\mathbf{II}$) systems of fully nonlinear elliptic equations of the Lane-Emden type

$$\begin{cases} F_k[-u] = \sigma\, v^{q_1}, \quad u > 0, \quad \text{in} \quad \mathbb{R}^n, \\ F_k[-v] = \sigma\, u^{q_2}, \quad v > 0, \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \qquad \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{S_2}$$

where $F_k[u]$ is the k-Hessian operator for the range $1 \leq k < n/2$ defined as the sum of $k \times k$ minors of the Hessian matrix D^2u. We analyze Systs. (S_1) and (S_2) in the sub-natural growth, that is, the cases $0 < q_i < p - 1$ for $i = 1, 2$ in (S_1), and $0 < q_i < k$ for $i = 1, 2$ in (S_2). We can see that when $u = v$ and $q_1 = q_2 = q$, Syst. (S_2) can be reduced to the following equation:

$$\begin{cases} F_k[-u] = \sigma\, u^q \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{P_1}$$

A *solution* to Systs. (S_1) or (S_2) is understood in the potential-theoretic sense. Namely, a solution to Syst. (S_1) will be a pair of nonnegative p-superhamonic functions (u, v) satisfying (S_1) in the measure sense; in fact, under Wolff's inequality (see (2.1.2) below), we will show that $(u, v) \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \times W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ and it satisfies Syst. (S_1) in the distributional sense. While a solution to Syst. (S_2) will be a pair of k-subharmonic (k-convex) (u, v), that is a pair (u, v)

of k-subharmonic functions, satisfying (S_2) in the measure sense. For the precise definitions, see Sections 2.1 and 2.2, respectively.

Next, we provide a sufficient condition for the existence of a *finite energy solution* for:

$(\mathbf{III})$ the quasilinear elliptic equations of the following type (p,q)-Laplace operator:

$$-\Big(\Delta_p u + \Delta_q u\Big) = \sigma\left(u^{\gamma(p-1)} + u^{\gamma(q-1)}\right) \quad \text{in} \quad \mathbb{R}^n,$$

where $2 \le p < q < \infty$, and γ is a constant satisfying:

$$0 < \gamma < \min\left\{\frac{p-1}{q-1}, \frac{1}{q-p}\right\}. \tag{A_1}$$

In particular, $\gamma < 1$, which describes exactly the situation of "sublinear growth". Here, it is suitable to treat the previous equation in the Orlicz setting, seeing this equation as

$$-\mathrm{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \sigma\, g(u^\gamma) \quad \text{in} \quad \mathbb{R}^n, \tag{P_2}$$

where $g : [0,\infty) \to [0,\infty)$ is the function given by

$$g(t) = t^{p-1} + t^{q-1}. \tag{A_2}$$

If $p = q$, we have $g(t) = t^{p-1}$ and Eq. (P_2) becomes in

$$-\Delta_p u = \sigma\, u^{\gamma(p-1)} \quad \text{in} \quad \mathbb{R}^n, \tag{1.0.1}$$

here we set $0 < \gamma < 1$. We highlight that for additional examples of quasilinear operators, we recommend referring to, for instance, [Ó 1997, Montenegro 1999]. Because of Eq. (P_2), our approach employs tools of the theory of Orlicz Spaces. For the precise definitions of the Orlicz spaces considered below, see Section 2.3. By a *finite energy solution* to (P_2), we mean a nonnegative function $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ satisfying weakly (P_2), where $\mathcal{D}^{1,G}(\mathbb{R}^n)$ is the homogeneous Sobolev-Orlicz space defined as the space of all functions $u \in L^G_{\mathrm{loc}}(\mathbb{R}^n)$, which admit weak derivatives $\partial_k u \in L^G(\mathbb{R}^n)$ for $k = 1, \ldots, n$. To be more precise, setting $f(t) = g(t^\gamma)$ and $F(t) = \int_0^t f(s)\,\mathrm{d}s$ for $t \ge 0$, a nonnegative function u is called a finite energy solution to (P_2) if $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$, and it satisfies (P_2) weakly, that is

$$\int_{\mathbb{R}^n} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\varphi\,\mathrm{d}x = \int_{\mathbb{R}^n} f(u)\varphi\,\mathrm{d}\sigma \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n).$$

Note that finite energy solutions to (P_2) are critical points of the functional

$$J(v) = \int_{\mathbb{R}^n} G(|\nabla v|)\,\mathrm{d}x - \int_{\mathbb{R}^n} F(|v|)\,\mathrm{d}\sigma, \quad v \in \mathcal{D}^{1,G}(\mathbb{R}^n). \tag{1.0.2}$$

We are interested in nontrivial solutions of finite energy.

Our results are new even for systems of nonlinear elliptic equations involving the classical Laplace operator (like (S_1)), corresponding to the case $p = 2$; or for nonnegative $\sigma \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, here $\mathrm{d}\sigma = \sigma\,\mathrm{d}x$.

The main idea is to apply *potential estimates* in the problems above since they will be seen in the sense of measures. Potential estimates are well-established and precise tools in the analysis of measure data elliptic partial differential equations. Roughly speaking, potential estimates allow us to deal with integral equations instead of elliptic partial differential equations.

To study Systs. (S_1) and (S_2), we employ some elements of nonlinear potential theory, accurately on the systematic use of the Wolff potential $\mathbf{W}_{\alpha,p}\sigma$ defined by

$$\mathbf{W}_{\alpha,p}\sigma(x) = \int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}, \quad x \in \mathbb{R}^n, \tag{1.0.3}$$

where $1 < p < \infty,\ 0 < \alpha < n/p$. Observe that $\mathbf{W}_{\alpha,p}\sigma$ is always positive for $\sigma \not\equiv 0$ which may be ∞. In fact, by [Cao and Verbitsky 2017, Corollary 3.2], $\mathbf{W}_{\alpha,p}\sigma(x) \not\equiv \infty$ for all $x \in \mathbb{R}^n$ if and only if

$$\int_1^\infty \left(\frac{\sigma(B(0,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} < \infty. \tag{1.0.4}$$

This means that $\mathbf{W}_{\alpha,p}\sigma(x) < \infty$ for all $x \in \mathbb{R}^n$ if and only if $\mathbf{W}_{\alpha,p}\sigma(x_0) < \infty$ for some $x_0 \in \mathbb{R}^n$ (we may take $x_0 = 0$).

The same reasoning applies to solving Eq. (P_2). To be precise, we consider the potential of Wolff-type $\mathbf{W}_G\sigma$ defined by

$$\mathbf{W}_G\sigma(x) = \int_0^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right) \mathrm{d}t. \tag{1.0.5}$$

Observe that $\mathbf{W}_G\sigma$ may be infinity. In Section 4.1 below, we characterize the finiteness of $\mathbf{W}_G\sigma$ similarly to (1.0.4).

The next sections will play a motivational role in our work. If the reader is interested in seeing the main results of this thesis soon, they can be found in Section 1.3.

1.1 BRIEFLY HISTORY ON POTENTIAL ESTIMATES

Potential estimates arise naturally as an extension of representation formulas via fundamental solutions. It is known that representation formulas via fundamental solutions are an assertive tool for establishing the qualitative properties of solutions to linear elliptic equations. Eventually, these lead to consider linear Riesz potentials and singular integrals. Let us deal with the classical Poisson equation, which is the simplest example, given by

$$-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^n.$$

Here μ, which is eventually taken to be a measure, is for simplicity assumed to be a smooth and compactly supported function, while u is the unique solution which decays to zero at infinity. The point we are interested in now is that u can be recovered via convolution with the so-called fundamental solution. This means that the following representation formula holds:

$$u(x) = \frac{1}{c(n)} \int_{\mathbb{R}^n} G(x,y)\,\mathrm{d}\mu(y), \tag{1.1.1}$$

where $G(x,y) = |x-y|^{-(n-2)},\ c(n) = n(n-2)|B(0,1)|$ is a normalization constant. From now on, this normalization constant will be dropped for the sake of convenience. The uniqueness of u follows, for instance, from [Evans 1998, Theorem 9, Chapter 2, Section 3], since $G(x,y) \to 0$ as $|x| \to \infty$ uniformly in $y \in \mathbb{R}^n \setminus \{x\}$.

Remark 1.1.1. If $n = 2$, $c(n) = 2\pi$ and $G(x,y) = \ln|x-y|$, which is unbounded as $|x| \to \infty$, and so may be $\int_{\mathbb{R}^2} G(x,y)\,\mathrm{d}\mu(y)$. For this reason, unless otherwise stated we assume that $n \geq 3$.

The formula (1.1.1) allows to redirect the study of solutions to the analysis of a related integral equation. Clearly, the laplacian operator Δ is a particular case of p-laplacian operator Δ_p with $p = 2$. However, no corresponding integral representation (1.1.1) is available for Δ_p when $p \neq 2$, despite the existence of fundamental solutions given by

$$G_p(x,y) = \begin{cases} \dfrac{1}{|x-y|^{\frac{n-p}{p-1}}} & \text{if} \quad 1 < p < n, \\ \log|x-y| & \text{if} \quad p = n. \end{cases}$$

When $\mu \in M^+(\mathbb{R}^n)$, formula (1.1.1) still holds as a consequence of Riesz representation theorem (see [Armitage and Gardiner 2001, Corollary 4.3.3] for the details). Note that formula (1.1.1) can be seen in terms of the Newtonian potential of μ, that is (1.1.1) says $u(x) = \mathbf{I}_2\mu(x)$ for all $x \in \mathbb{R}^n$, where the Newtonian potential $\mathbf{I}_2$ potential is defined by

$$\mathbf{I}_2\mu(x) = \int_{\mathbb{R}^n} \frac{\mathrm{d}\mu(y)}{|x-y|^{n-2}}.$$

To recover formula (1.1.1) for other types of elliptic operators, it was necessary to "expand" the space of solutions (i.e. weaken the notion of solution), and deal with the measure data problem. For that, Wolff potentials (1.0.3) plays the role of extending the formula (1.1.1), through which we will relate elliptic problems with suitable integral equations.

In the context of quasilinear problems, Wolff potentials with $\alpha = 1$ appeared in the notable works of T. Kilpeläinen and J. Malý [Kilpeläinen and Malý 1992, Kilpeläinen and Malý 1994]. Indeed, from [Kilpeläinen and Malý 1994, Corollary 4.13], there exists a constant $K = K(n,p) \geq 1$ such that it holds

$$K^{-1}\mathbf{W}_{1,p}\mu(x) \leq u(x) \leq K\,\mathbf{W}_{1,p}\mu(x) \quad \forall x \in \mathbb{R}^n, \tag{1.1.2}$$

provided u is a nonnegative solution in the potential-theoretic sense of

$$\begin{cases} -\Delta_p u = \mu, \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.1.3}$$

See Section 2.1 for more details. Moreover, u exists if and only if $\mathbf{W}_{1,p}\mu < \infty$, or equivalently,

$$\int_1^\infty \left(\frac{\mu(B(0,t))}{t^{n-p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} < \infty. \tag{1.1.4}$$

Wolff potentials with $\alpha = 2k/(k+1)$ and $p = k+1$, for $k < n/2$, in the context of fully nonlinear problems, was considered in [Labutin 2002], where D. A. Labutin proved that if $\mu \in M^+(\mathbb{R}^n)$ and w is a nonnegative k-subharmonic solution of the equation

$$\begin{cases} F_k[-w] = \mu \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} w(x) = 0, \end{cases}$$

then there exists a constant $K = K(n,k) \geq 1$ such that

$$K^{-1}\mathbf{W}_{\frac{2k}{k+1},k+1}\mu(x) \leq w(x) \leq K\,\mathbf{W}_{\frac{2k}{k+1},k+1}\mu(x) \quad \forall x \in \mathbb{R}^n. \tag{1.1.5}$$

In particular, such w exists if and only if $\mathbf{W}_{2k/(k-1),k+1}\mu < \infty$, or equivalently

$$\int_1^\infty \left(\frac{\mu(B(0,t))}{t^{n-2k}}\right)^{\frac{1}{k}} \frac{\mathrm{d}t}{t} < \infty. \tag{1.1.6}$$

In the Orlicz setting, a type of Orlicz counterpart of this result was established with the potential (1.0.5) in [Malý 2003, Chlebicka, Giannetti and Zatorska-Goldstein 2023] to problems like:

$$\begin{cases} -\mathrm{div}\left(\dfrac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu \quad \text{in} \quad \mathbb{R}^n, \\ \inf\limits_{\mathbb{R}^n} u = 0. \end{cases}$$

For our purposes, using ideas of [Chlebicka, Giannetti and Zatorska-Goldstein 2023], we establish a suitable Wolff potential estimate in terms of (1.0.5) similar to (1.1.3) (see Theorem 2.3.19 below).

For an overview of Wolff potentials and their applications in Analysis and PDE, see [Adams and Hedberg 1996, Hedberg and Wolff 1983, Kuusi and Mingione 2014, Maz'ya 2011, Heinonen, Kilpeläinen and Martio 2006] and references therein.

1.2 STATE OF THE ART

Sublinear elliptic equations with measurable coefficients have been investigated by many authors. In [Brezis and Kamin 1992], H. Brezis and S. Kamin established a necessary and sufficient condition for the existence of a *bounded* solution to the sublinear problem

$$-\Delta u = \sigma(x) u^q \quad \text{in} \quad \mathbb{R}^n \tag{1.2.1}$$

with $0 < q < 1,\ 0 \not\equiv \sigma \geq 0$ and $\sigma \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. For bounded solutions of (1.2.1), by using the Green function of the Laplacian (the Newtonian potential) in balls of $\mathbb{R}^n$, they proved global pointwise estimates of the form

$$c^{-1} \left(\mathbf{I}_2 \sigma\right)^{\frac{1}{1-q}} \leq u \leq c\, \mathbf{I}_2 \sigma, \tag{1.2.2}$$

where $c > 0$ is a constant independent of u.

Notice that when $q_1 = q_2$ and $u = v$, Syst. (S_1) reduces to the following problem dealing with the single equation

$$\begin{cases} -\Delta_p u = \sigma\, u^q \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x| \to \infty} u(x) = 0. \end{cases} \tag{1.2.3}$$

Using estimates (1.1.3), D. Cao and I. Verbistky in [Cao and Verbitsky 2016] obtained pointwise estimates of *Brezis-Kamin* type for solutions of quasilinear elliptic equations to eq. (1.2.3). Indeed, by a capacity condition, they gave a sufficient condition in terms of $\mathbf{W}_{1,p}\sigma$ to the existence of a nonnegative solution to eq. (1.2.3) satisfying bilateral estimates in terms of Wolff potential $\mathbf{W}_{1,p}\sigma$, where $\sigma \in M^+(\mathbb{R}^n)$. Also, a necessary and sufficient condition for the existence of such a solution was presented in terms only of Wolff's potential. Estimates of Brezis-Kamin type are potential estimates similar to (1.2.2), see potential estimates (1.3.4) below. See also [Cao and Verbitsky 2017] for some related results.

Furthermore, note that the previous equation can be seen as eq. (1.0.1) with $q = \gamma(p-1)$. In [Dat and Verbitsky 2015], C. Dat and I. Verbistky were able to exhibit a necessary and sufficient condition in terms of $\mathbf{W}_{1,p}\sigma$ to construct a solution to (1.0.1), which is finite energy in view of functional J given in (1.0.2), that is

$$J(v) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla v|^p \,\mathrm{d}x - \frac{1}{1+q}\int_{\mathbb{R}^n} v^{1+q}\,\mathrm{d}\sigma.$$

1.3 MAIN RESULTS

The assertions below are the compilation of our results, which are motivated by some earlier ideas developed in [Dat and Verbitsky 2015, Cao and Verbitsky 2017, Cao and Verbitsky 2016, Heinonen, Kilpeläinen and Martio 2006, Kilpeläinen and Malý 1994, Kilpeläinen and Malý 1992, Trudinger and Wang 1997, Trudinger and Wang 1999, Trudinger and Wang 2002, Labutin 2002, Phuc and Verbitsky 2009, Phuc and Verbitsky 2008].

1.3.1 Systems of Wolff potentials

To examine systems of quasilinear elliptic equations like Syst. (S_1), and fully nonlinear elliptic systems of the form (S_2), first we study in a general framework the existence of a pair of nonnegative functions $(u, v) \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma) \times L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ satisfying pointwise the following system of integral equations involving the Wolff potentials:

$$\begin{cases} u = \mathbf{W}_{\alpha,p}\left(v^{q_1}\mathrm{d}\sigma\right) & \text{in} \quad \mathbb{R}^n, \\ v = \mathbf{W}_{\alpha,p}\left(u^{q_2}\mathrm{d}\sigma\right) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{SI}$$

where $0 < q_i < p-1$ for $i = 1,\, 2$, that is

$$\begin{cases} u(x) = \displaystyle\int_0^\infty \left(\frac{\int_{B(x,t)} v^{q_1}(y)\,\mathrm{d}\sigma(y)}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}, \\ v(x) = \displaystyle\int_0^\infty \left(\frac{\int_{B(x,t)} u^{q_2}(y)\,\mathrm{d}\sigma(y)}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}, \qquad \forall\, x \in \mathbb{R}^n. \end{cases}$$

We prove the existence of solutions to Syst. (SI) using the sub and super solutions method. Because of this result, by using the method of successive approximations, we show the existence of *solutions* to Systs. (S_1) and (S_2), jointly with sharp global pointwise bounds of solutions. Indeed, in view of potential estimates known, solutions of (SI) work like an upper barrier, which will allow us to control the successive approximations.

For the next two theorems, we assume that (1.0.4) holds. To state our first result in a precise way, let us recall the notion of Riesz capacity of a compact set $E \subset \mathbb{R}^n$,

$$\mathrm{cap}_{\mathbf{I}_\alpha,p}(E) = \inf\{\|f\|_{L^p}^p : f \in L^p(\mathbb{R}^n),\ f \geq 0,\ \mathbf{I}_\alpha f \geq 1 \text{ on } E\}, \tag{1.3.1}$$

where $\mathbf{I}_\alpha \sigma$ is the Riesz potential of order α defined for $0 < \alpha < n$ by

$$\mathbf{I}_\alpha \sigma(x) = \int_{\mathbb{R}^n} \frac{\mathrm{d}\sigma(y)}{|x-y|^{n-\alpha}}, \quad x \in \mathbb{R}^n. \tag{1.3.2}$$

See [Adams and Hedberg 1996, Chapter 2] for an overview of several types of capacity. We will show in Lemma 3.1.2 below that if there exists a nontrivial solution to Syst. (SI), then σ must be absolutely continuous with respect to $\mathrm{cap}_{\mathbf{I}_\alpha,p}$. Indeed, for our main results, we will impose that σ satisfies the following strongest condition

$$\sigma(E) \leq C_\sigma \,\mathrm{cap}_{\alpha,p}(E) \quad \text{for all compact sets } E \subset \mathbb{R}^n. \tag{1.3.3}$$

Theorem 1.3.1. *Let* $1 < p < \infty$, $0 < q_i < p-1$, $i = 1, 2$, $0 < \alpha < n/p$ *and* $\sigma \in M^+(\mathbb{R}^n)$ *satisfying* (1.0.4) *and* (1.3.3). *Then there exists a solution* (u, v) *to Syst.* (SI) *such that*

$$\begin{aligned} c^{-1}\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}} &\leq u \leq c\left(\mathbf{W}_{\alpha,p}\sigma + \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}}\right), \\ c^{-1}\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}} &\leq v \leq c\left(\mathbf{W}_{\alpha,p}\sigma + \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}}\right), \end{aligned} \tag{1.3.4}$$

where $c = c(n, p, q_1, q_2, \alpha, C_\sigma) > 0$. *Furthermore,* $u, v \in L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, *for every* $s > 0$.

Remark 1.3.2. Based on the assumption (1.0.4), we show that all nontrivial solutions to Syst. (SI) satisfy the lower bounds in (1.3.4). For the upper bounds in (1.3.4), we also use hypothesis (1.3.3), which will be decisive in building our argument.

The next theorem brings a necessary and sufficient on σ to the existence of solutions (u, v) to (SI) which enjoy the bilateral estimates (1.3.4). Suppose $\sigma \in M^+(\mathbb{R}^n)$ satisfies

$$\begin{aligned} \mathbf{W}_{\alpha,p}\left(\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)(p-1+q_1)q_2}{(p-1)^2-q_1q_2}} \mathrm{d}\sigma\right) &\leq \lambda\left(\mathbf{W}_{\alpha,p}\sigma + \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}}\right), \\ \mathbf{W}_{\alpha,p}\left(\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)(p-1+q_2)q_1}{(p-1)^2-q_1q_2}} \mathrm{d}\sigma\right) &\leq \lambda\left(\mathbf{W}_{\alpha,p}\sigma + \left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}}\right), \end{aligned} \tag{1.3.5}$$

here the right-hand sides are finite almost everywhere in $\mathbb{R}^n$, and λ is a positive constant. In general, condition (1.3.5) is weaker than (1.3.3) (see Remark 1.3.5 below). However, we can still construct solutions for Syst. (SI) satisfying (1.3.4) and show that (1.3.5) is necessary for the existence of such solutions.

Theorem 1.3.3. *Let $1 < p < \infty$, $0 < q_i < p-1$, $i = 1, 2$, and $0 < \alpha < n/p$. If $\sigma \in M^+(\mathbb{R}^n)$ satisfies* (1.0.4) *and* (1.3.5)*, then there exists a solution (u, v) to Syst. (SI) such that* (1.3.4) *holds with a positive constant $c = c(n, p, \alpha, q_1, q_2, \lambda)$. Conversely, suppose that there exists a nontrivial solution (u, v) to Syst. (SI) satisfying* (1.3.4)*. Then* (1.3.5) *holds with λ depending only on p, q_1, q_2 and c.*

Remark 1.3.4. For the case $q_1 = q_2$, one can see that the solution (u, v) obtained in Theorem 1.3.1 is such that $u = v$, and satisfies

$$c^{-1}\Big(\mathbf{W}_{\alpha,p}\sigma\Big)^{\frac{p-1}{p-1-q}} \leq u \leq c\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right), \tag{1.3.6}$$

where $q := q_1$. Thus, [Cao and Verbitsky 2016, Theorem 3.2] is a corollary of Theorem 1.3.1 for $q_1 = q_2$. Also, [Cao and Verbitsky 2016, Theorem 3.3] is a corollary of Theorem 1.3.3, since in the case $q_1 = q_2$ condition (1.3.5) is written as follows

$$\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{\frac{q(p-1)}{p-1-q}}\,\mathrm{d}\sigma\right) \leq \lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}\right) < \infty. \tag{1.3.7}$$

Remark 1.3.5. As commented above, condition (1.3.5) is weaker than (1.3.3). Indeed, let $p = 2$, $0 < q < 1$ and $0 < 2\alpha < n$. Setting $q_1 = q_2 = q$, condition (1.3.5) becomes (1.3.7). Let $\mathrm{d}\sigma = \sigma\,\mathrm{d}x$ for $\sigma(x) = |x|^{-s}\chi_{B\setminus\{0\}}(x)$, with $2\alpha < s < n - (n-2\alpha)q$; here $B = B(0,1)$. For p, q and α given above, being σ radial, condition (1.3.7) is characterized by

$$\varlimsup_{|x|\to 0} \frac{|x|^{-(n-2\alpha)(1-q)}\int_{|y|<|x|}|y|^{-(n-2\alpha)q}\,\mathrm{d}\sigma(y)}{\int_{|y|\geq|x|}|y|^{-(n-2\alpha)}\,\mathrm{d}\sigma(y)} < \infty. \tag{1.3.8}$$

For details on (1.3.8), see [Cao and Verbitsky 2016, Proposition 5.2]. For $|x| < 1$, one has

$$\begin{aligned}\int_{|y|<|x|}|y|^{-(n-2\alpha)q}\,\mathrm{d}\sigma(y) &= \int_{|y|<|x|}|y|^{-[(n-2\alpha)q+s]}\,\mathrm{d}y\\ &= c_1(n,\alpha,s,q)\,|x|^{-(n-2\alpha)q-s+n},\end{aligned}$$

where in the last equality was used polar coordinates, since $s + (n-2\alpha)q < n$ (see [Folland 1999, Corollary 2.52]). Thereby,

$$\begin{aligned}|x|^{-(n-2\alpha)(1-q)}\int_{|y|<|x|}|y|^{-(n-2\alpha)q}\,\mathrm{d}\sigma(y) &= c_1\,|x|^{-(n-2\alpha)q-s+n}|x|^{-(n-2\alpha)(1-q)}\\ &= c_1\,|x|^{-(n-2\alpha)+(n-2\alpha)q-(n-2\alpha)q-s+n}\\ &= c_1\,|x|^{2\alpha-s}.\end{aligned} \tag{1.3.9}$$

Moreover, by polar coordinates,

$$\int_{|y|\ge|x|} |y|^{-(n-2\alpha)}\,\mathrm{d}\sigma(y) = \int_{|x|\le|y|<1} |y|^{-(n-2\alpha)-s}\,\mathrm{d}y$$
$$= c_2\int_{|x|}^{1} r^{-n+2\alpha-s+n-1}\,\mathrm{d}r = c_2(1-|x|^{2\alpha-s}). \tag{1.3.10}$$

Combining (1.3.9) with (1.3.10), we arrive at

$$\overline{\lim_{|x|\to 0}}\, \frac{|x|^{-(n-2\alpha)(1-q)}\int_{|y|<|x|}|y|^{-(n-2\alpha)q}\,\mathrm{d}\sigma(y)}{\int_{|y|\ge|x|}|y|^{-(n-2\alpha)}\,\mathrm{d}\sigma(y)} = c_3(n,\alpha,q,s)\,\overline{\lim_{|x|\to 0}}\,\frac{|x|^{2\alpha-s}}{1-|x|^{2\alpha-s}} < \infty.$$

Thus, σ satisfies (1.3.8), whence it satisfies (1.3.7).

On the other hand, fix $0<R<1$, by polar coordinates

$$\sigma(\overline{B(0,R)}) = \int_{B(0,R)} |y|^{-s}\,\mathrm{d}y = c_4\int_0^R r^{-s+n-1}\,\mathrm{d}r = c_5\,R^{n-s}.$$

From [Adams and Hedberg 1996, equation 2.6.1], $\mathrm{cap}_{\mathbf{I}_\alpha,2}(\overline{B(0,R)}) \approx R^{n-2\alpha}$. Consequently, σ can not satisfies (1.3.3), since $s>2\alpha$ implies $R^{n-s}>R^{n-2\alpha}$, whenever $R<1$.

1.3.2 Applications in systems of quasilinear elliptic equations

We first recall the p-capacity for compact subsets E of $\mathbb{R}^n$,

$$\mathrm{cap}_p(E) = \inf\left\{\|\nabla\varphi\|_{L^p}^p : \varphi\in C_c^\infty(\mathbb{R}^n), \varphi\ge 1 \text{ on } E\right\}.$$

From [Adams and Hedberg 1996, Proposition 2.3.13], one has $\mathrm{cap}_{\mathbf{I}_1,p}(E)\approx \mathrm{cap}_p(E)$ for all compact sets E.

Let $\sigma\in M^+(\mathbb{R}^n)$ satisfying the capacity condition

$$\sigma(E)\le C_\sigma\,\mathrm{cap}_p(E) \quad \text{for all compact sets } E\subset\mathbb{R}^n. \tag{1.3.11}$$

Under assumption (1.3.11), we prove the existence of a *minimal* positive *p-superharmonic solution* to Syst. (S_1), more precisely, a pair $(u,v)\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)\times W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ such that u, v are positive and p-superharmonic functions in $\mathbb{R}^n$, and satisfies Syst. (S_1) in the distributional sense. Here a minimal solution (u,v) to Syst. (S_1) means that for any solution (w_1,w_2) to (S_1), we have $w_1\ge u$ and $w_2\ge v$ almost everywhere in $\mathbb{R}^n$. See Section 2.1 for the definition of solutions to Syst. (S_1) and p-superharmonic functions. Here, and subsequently, $\mathbf{W}_p$ will denote $\mathbf{W}_{1,p}$

Theorem 1.3.6. *Let $1 < p < n$ and $0 < q_i < p-1$ for $i = 1, 2$. Suppose $\sigma \in M^+(\mathbb{R}^n)$ satisfies* (1.1.4) *and* (1.3.11)*. Then there exists a minimal p-superharmonic positive solution $(u, v) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \times W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ to Syst. (S_1) such that almost everywhere we have*

$$\begin{aligned} c^{-1}\left(\mathbf{W}_p\sigma\right)^{\frac{(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}} &\leq u \leq c\left(\mathbf{W}_p\sigma + \left(\mathbf{W}_p\sigma\right)^{\frac{(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}}\right), \\ c^{-1}\left(\mathbf{W}_p\sigma\right)^{\frac{(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}} &\leq v \leq c\left(\mathbf{W}_p\sigma + \left(\mathbf{W}_p\sigma\right)^{\frac{(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}}\right), \end{aligned} \tag{1.3.12}$$

where $c = c(n, p, q_1, q_2, C_\sigma) > 0$. If $p \geq n$, there are no nontrivial solutions to Syst. (S_1) in $\mathbb{R}^n$.

Remark 1.3.7. In view of Theorem D and Lemma 3.1.3, the lower estimates in (1.3.12) hold for any distributional solution (u, v) to (S_1). Nevertheless, the upper estimates in (1.3.12) are obtained only for the minimal solution.

As in Theorem 1.3.3, we may assume a weaker condition on σ than (1.3.11) and still assure the existence of solutions to Syst. (S_1) satisfying (1.3.12). The following theorem deals with this.

Theorem 1.3.8. *Let $\sigma \in M^+(\mathbb{R}^n)$, $1 < p < n$ and $0 < q_i < p-1$ for $i = 1, 2$. Then there exists a p-superharmonic positive solution (u, v) to Syst. (S_1) satisfying* (1.3.12) *if and only if there exists $\lambda > 0$ such that almost everywhere we have*

$$\begin{aligned} \mathbf{W}_p\left(\left(\mathbf{W}_p\sigma\right)^{\frac{q_2(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}}\right) &\leq \lambda\left(\mathbf{W}_p\sigma + \left(\mathbf{W}_p\sigma\right)^{\frac{(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}}\right) < \infty, \\ \mathbf{W}_p\left(\left(\mathbf{W}_p\sigma\right)^{\frac{q_1(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}}\right) &\leq \lambda\left(\mathbf{W}_p\sigma + \left(\mathbf{W}_p\sigma\right)^{\frac{(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}}\right) < \infty. \end{aligned} \tag{1.3.13}$$

Our approach considers also solutions to systems of equations involving the fractional Laplacian of the form

$$\begin{cases} (-\Delta)^\alpha u = \sigma\, v^{q_1}, \quad v > 0 \quad \text{in} \quad \mathbb{R}^n, \\ (-\Delta)^\alpha v = \sigma\, u^{q_2}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n, \\ \varliminf\limits_{|x|\to\infty} u(x) = 0, \qquad \varliminf\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{1.3.14}$$

with $0 < \alpha < n/2$ and $q_1,\ q_2 \in (0,1)$. A solution (u, v) to (1.3.14) is understood in the sense

$$\begin{cases} u(x) = \dfrac{1}{c(n,\alpha)}\mathbf{I}_{2\alpha}(v^{q_1}\mathrm{d}\sigma)(x), \quad x \in \mathbb{R}^n, \\ v(x) = \dfrac{1}{c(n,\alpha)}\mathbf{I}_{2\alpha}(u^{q_2}\mathrm{d}\sigma)(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where $\mathbf{I}_{2\alpha}$ is the Riesz Potential defined in (1.3.2) and $c(n, \alpha)$ is a normalization constant given by

$$c(n,\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}}\Gamma(\alpha)}{\Gamma(\frac{n-2\alpha}{2})}$$

(see for instance [Lischke et al. 2020]). As usual, the normalization constant will be dropped for the sake of convenience. Thus, Syst. (1.3.14) is equivalent (up to a constant) to Syst. (SI), since

$$\mathbf{I}_{2\alpha}\sigma(x) = \int_{\mathbb{R}^n} \frac{\mathrm{d}\sigma(y)}{|x-y|^{n-2\alpha}} = (n-2\alpha)\int_0^\infty \frac{\sigma(B(x,t))}{t^{n-2\alpha}}\frac{\mathrm{d}t}{t} = (n-2\alpha)\mathbf{W}_{\alpha,2}\sigma(x).$$

The second previous equality follows from [Malý and Ziemer 1997, Lemma 1.27]. Therefore, the following theorem is a special case of Theorem 1.3.1 with $p = 2$.

Theorem 1.3.9. *Let* $0 < 2\alpha < n$, $0 < q_i < 1$, $i = 1, 2$. *Suppose* $\sigma \in M^+(\mathbb{R}^n)$ *satisfies both* (1.0.4) *and* (1.3.3) *with* $p = 2$. *Then Syst.* (1.3.14) *admits a solution such that*

$$\begin{aligned} c^{-1}\left(\mathbf{I}_{2\alpha}\sigma\right)^{\frac{1+q_1}{1-q_1q_2}} &\le u \le c\left(\mathbf{I}_{2\alpha}\sigma + \left(\mathbf{I}_{2\alpha}\sigma\right)^{\frac{1+q_1}{1-q_1q_2}}\right), \\ c^{-1}\left(\mathbf{I}_{2\alpha}\sigma\right)^{\frac{1+q_2}{1-q_1q_2}} &\le v \le c\left(\mathbf{I}_{2\alpha}\sigma + \left(\mathbf{I}_{2\alpha}\sigma\right)^{\frac{1+q_2}{1-q_1q_2}}\right), \end{aligned} \tag{1.3.15}$$

where $c = c(n, q_1, q_2, \alpha, C_\sigma) > 0$. *Furthermore,* $u, v \in L^s_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, *for every* $s > 0$.

Remark 1.3.10. Suppose $\mathbf{I}_{2\alpha}\sigma \in L^\infty(\mathbb{R}^n)$ as in [Brezis and Kamin 1992], then assumption (1.3.3) holds for $p = 2$. Indeed, in view of [Verbitsky 1999, Theorem 1.11], there exists a constant $C_0 > 0$ which depends only on n and α, such that, for all compact sets $E \subset \mathbb{R}^n$,

$$\begin{aligned} \sigma(E) = \int_E \mathrm{d}\sigma = \int_E \mathbf{W}_{\alpha,2}\sigma\,\frac{\mathrm{d}\sigma}{\mathbf{W}_{\alpha,2}\sigma} &= \int_E \mathbf{I}_{2\alpha}\sigma\,\frac{\mathrm{d}\sigma}{\mathbf{I}_{2\alpha}\sigma} \\ &\le \|\mathbf{I}_{2\alpha}\sigma\|_{L^\infty}\int_E \frac{\mathrm{d}\sigma}{\mathbf{I}_{2\alpha}\sigma} \\ &\le \|\mathbf{I}_{2\alpha}\sigma\|_{L^\infty}\, C_0\, \mathrm{cap}_{\alpha,2}(E), \end{aligned}$$

and this shows condition (1.3.3) for $p = 2$. Thus, under assumption $\mathbf{I}_{2\alpha}\sigma \in L^\infty(\mathbb{R}^n)$, from Theorem 1.3.9, it follows that Syst. (1.3.14) possess a *bounded* nontrivial solution (u, v) satisfying the so-called Brezis–Kamin estimate (1.3.15). In particular, Theorem 1.3.9, with $\alpha = 1$ and $q_1 = q_2$, recover [Brezis and Kamin 1992, Theorem 1] together with the estimate (1.2.2), provided $\mathbf{I}_2\sigma \in L^\infty(\mathbb{R}^n)$.

1.3.3 Applications in systems of fully nonlinear elliptic equations

Here, $\mathbf{W}_k$ will denote $\mathbf{W}_{2k/(k+1),k+1}$. As in Theorem 1.3.6, the next theorem assures the existence of a *minimal* solution to Syst. (S_2). By a *minimal* solution to Syst. (S_2) we mean a pair of nonnegative functions (u, v) which solves Syst. (S_2), and if (w_1, w_2) is any other solution to Syst. (S_2), then $w_1 \ge u$ and $w_2 \ge v$ pointwise in $\mathbb{R}^n$.

Theorem 1.3.11. *Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying* (1.0.4) *and* (1.3.3) *with $\alpha = 2k/(k+1)$ and $p = k+1$. Then there exists a minimal solution (u, v) to Syst. (S_2) satisfying*

$$\begin{aligned} c^{-1}\left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_1)}{k^2-q_1q_2}} &\leq u \leq c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_1)}{k^2-q_1q_2}}\right), \\ c^{-1}\left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_2)}{k^2-q_1q_2}} &\leq v \leq c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_2)}{k^2-q_1q_2}}\right), \end{aligned} \tag{1.3.16}$$

where $c = c(n, k, q_1, q_2, \alpha, C_\sigma) > 0$. In addition, $u, v \in L^r_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, for all $r > 0$.

Theorem 1.3.12. *Let $1 \leq k < n/2$, $0 < q_i < k$, $i = 1, 2$ and $\sigma \in M^+(\mathbb{R}^n)$.*

(**i**) *If there exists a solution (u, v) to Syst. (S_2) satisfying* (1.3.16) *for some $c > 0$, then there exists a constant $\lambda = \lambda(n, k, q_1, q_2, c) > 0$ such that, for almost everywhere, it holds*

$$\begin{aligned} \mathbf{W}_k\left(\left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_1)q_2}{k^2-q_1q_2}}\mathrm{d}\sigma\right) &\leq \lambda\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_1)}{k^2-q_1q_2}}\right) < \infty, \\ \mathbf{W}_k\left(\left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_2)q_1}{k^2-q_1q_2}}\mathrm{d}\sigma\right) &\leq \lambda\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k(k+q_2)}{k^2-q_1q_2}}\right) < \infty. \end{aligned} \tag{1.3.17}$$

(**ii**) *Suppose that σ satisfies* (1.3.17) *for some $\lambda > 0$. Then there exists a solution (u, v) to Syst. (S_2) satisfying* (1.3.16)*, where $c = c(n, k, q_1, q_2, \lambda)$.*

From the result above, we conclude that condition (1.3.17) is necessary and sufficient for the existence of a solution to Syst. (S_2) satisfying (1.3.16). The following result is a simple consequence of the preceding theorems by taking $q_1 = q_2$. We highlight that it was already announced in [Cao and Verbitsky 2017, Cao and Verbitsky 2016].

Corollary 1.3.13. *Let $1 \leq k < n/2$, $0 < q < k$ and $\sigma \in M^+(\mathbb{R}^n)$.*

(**i**) *If $\mathbf{W}_k\sigma < \infty$ and* (1.3.3) *hold with $\alpha = 2k/(k+1)$ and $p = k+1$, then there exists a k-subharmonic nonnegative solution to (P_1) satisfying*

$$c^{-1}\left(\mathbf{W}_k\sigma\right)^{\frac{k}{k-q}} \leq u \leq c\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k}{k-q}}\right), \tag{1.3.18}$$

where $c = c(n, k, q, \sigma) > 0$.

(**ii**) *Let u be a k-subharmonic nonnegative solution to (P_1), satisfying* (1.3.18)*. Then there exists a constant $\lambda = \lambda(n, k, q, c) > 0$ such that, for almost everywhere, it holds*

$$\mathbf{W}_k\left(\left(\mathbf{W}_k\sigma\right)^{\frac{kq}{k-q}}\mathrm{d}\sigma\right) \leq \lambda\left(\mathbf{W}_k\sigma + \left(\mathbf{W}_k\sigma\right)^{\frac{k}{k-q}}\right) < \infty. \tag{1.3.19}$$

(**iii**)*Suppose that σ satisfies* (1.3.19) *for some $\lambda > 0$. Then there exists a k-subharmonic nonnegative solution to (P_1) satisfying* (1.3.18)*, where $c = c(n, k, q, \lambda) > 0$.*

1.3.4 Application in quasilinear elliptic equations with Orlicz growth

The main idea is the same as the two previous subsections. Let us first observe that when $p = q$, note that $\mathbf{W}_G\sigma$, defined in (1.0.5), becomes the so-called Wolff potential

$$\mathbf{W}_G\sigma(x) = \int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)^{\frac{1}{p-1}} \mathrm{d}t = \int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} = \mathbf{W}_p\sigma(x). \quad (1.3.20)$$

To solve (P_2), we first examine in a general framework the following integral equation involving the Wolff potential

$$u = \mathbf{W}_G\Big(f(u)\mathrm{d}\sigma\Big) \quad \text{in} \quad \mathbb{R}^n \quad (S)$$

where $f(t) = g(t^\gamma)$, and u is a nonnegative σ-measurable function, belonging to $L^f_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$. The existence of solutions to (S) is proved by using the sub and super solutions method. Finally, to prove the existence of solutions to (P_2) (finite energy), we use the method of successive approximations, where solutions to (S) perform like an upper barrier which will allow us to control the successive approximations. As we will see, no one additional relation of p and q will be imposed, much less on the ratio q/p, which is usual when studying problems involving the (p, q)-Laplace operator. Our results are new even for nonnegative functions $\sigma \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, here $\mathrm{d}\sigma = \sigma\,\mathrm{d}x$.

We will require that $\sigma \in M^+(\mathbb{R}^n)$ satisfies the following condition

$$\begin{cases} \mathbf{W}_p\sigma^{\frac{1}{1-\gamma}},\, \mathbf{W}_q\sigma^{\frac{1}{1-\gamma}} & \in \ L^F(\mathbb{R}^n, \mathrm{d}\sigma), \\ \mathbf{W}_p\sigma^{\frac{p-1}{p-1-\gamma(q-1)}},\, \mathbf{W}_q\sigma^{\frac{p-1}{p-1-\gamma(q-1)}} & \in \ L^F(\mathbb{R}^n, \mathrm{d}\sigma), \\ \mathbf{W}_p\sigma^{\frac{q-1}{q-1-\gamma(p-1)}},\, \mathbf{W}_q\sigma^{\frac{q-1}{q-1-\gamma(p-1)}} & \in \ L^F(\mathbb{R}^n, \mathrm{d}\sigma). \end{cases} \quad (1.3.21)$$

This condition extends partially the condition stated in [Dat and Verbitsky 2015]; see Remark 4.4.1 below. We can now formulate our main results.

Theorem 1.3.14. *Let g be given by (A_2). Let $\sigma \in M^+(\mathbb{R}^n)$ with $\mathbf{W}_G\sigma < \infty$ in $\mathbb{R}^n$. The equation (S) has a nontrivial solution $u \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ whenever* (1.3.21) *holds.*

It is our interest to know whether condition (1.3.21) can be shortened in a more uncomplicated condition (see (4.1.31) below). We will apply the previous theorem to obtain solutions to Eq. (P_2).

Theorem 1.3.15. *Suppose $2 \le p < q < n$. Let g be given by (A_2). Let $\sigma \in M^+(\mathbb{R}^n)$ with $\mathbf{W}_G\sigma < \infty$ in $\mathbb{R}^n$. Suppose that* (1.3.21) *holds. Then there exists a nontrivial solution*

$u \in \mathcal{D}^{1,G}(\mathbb{R}^n) \cap L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ *to* (P_2)*. Furthermore,* u *is minimal. For* $q \geq n$*,* (P_2) *has only the trivial solution* $u = 0$*.*

1.4 STRATEGIES AND DIFFICULTIES

Variational methods, in general, do not apply well when dealing with problems in sublinear growth with the p-Laplace operator. For example, the classical *Ambrosetti-Rabinowitz's condition* fails in the Brezis-Kamin problem (eq. (1.2.1)). Indeed, in the case $p = 2$ and $0 < q < 1$, any constant $\theta > 2$ must satisfies

$$\frac{\theta}{1+q} t^{1+q} = \theta \int_0^t s^q \,\mathrm{d}s > t^{1+q} = t \cdot t^q \quad \forall t \geq 0,$$

since $\theta/(1+q) > 1$. This brings difficulties to make use of the classical *Mountain Pass Theorem* to obtain solutions to eq. (1.2.1). Furthermore, one can check that there is no functional $I : (u, v) \mapsto I(u, v)$ associated with Syst. (S_1) whose critical points are solutions to Syst. (S_1). Thereby, no one variational methods apply to looking for solutions to Syst. (S_1). There are other techniques to overcome this difficulty. Usual tools are, for instance, topological methods with fixed point arguments (see [Amann 1976]), sub-super solutions, a priori estimates, and approximate methods. Once one has a solution between sub-super solutions a natural question is localizing extremal solutions, i.e., maximal or minimal solutions.

There are some challenges in studying Hessian Lane-Emden-type systems with measures of the form Syst (S_2) because our criteria for the solvability of Hessian equations on the entire space depends on the existence result in subdomains with specific geometric conditions, which goes back to the celebrated work [Caffarelli, Nirenberg and Spruck 1985]. Moreover, one needs to develop accurate estimates that are more involved than the ones we can find in the literature.

Some challenges arise naturally in studying quasilinear equations of the form (P_2). Indeed, since it is not assumed to enjoy homogeneity: $g(|\lambda\,\xi|) = |\lambda|^{p-1} g(|\xi|)$, the computations are more complicated; in particular our class of solutions is not invariant to scalar multiplication. When σ is the null measure, the existence of solutions and its regularity to (P_2), i.e. critical points of J, is known only for the ratio q/p sufficiently close to 1 in dependence on n [Marcellini 1989, Marcellini 1991]; this together with the lack of homogeneity difficulties to show *priori estimates* (local boundness of the function) in general case $2 \leq p < q < \infty$ for solutions to (P_2).

To overcome these difficulties, our approach makes use accurately of the systematic use of Wolff potentials in light of potential estimates well-established. We do not only present necessary and sufficient conditions on σ to prove the existence of solutions to Systs. (S_1), (S_2) and Eq. (P_2), but we show that the solution obtained here is minimal, in the pointwise sense. Further, the solution to Systs. (S_1) and (S_2) satisfies sharp global pointwise estimates of Brezis–Kamin type in terms of the Wolff potential (1.0.3), whereas the solution to Eq. (P_2) satisfies a lower global pointwise in terms of the Wolff-type potential (1.0.5).

1.5 RELATED WORKS

Recently, studies of systems involving the p-Laplace operator with measure-valued right-hand side have been done. In this respect, in [Kuusi and Mingione 2018], T. Kuusi and G. Mingione used the notion SOLA (Solution Obtained as Limits of Approximations) to propose a vectorial version of [Boccardo and Gallouët 1989], where this notion was introduced. They use a different approach from ours to establish local upper pointwise potential estimates for $W^{1,p-1}_{\mathrm{loc}}$-vectorial solutions in terms of Wolff's potential if $p > 2-1/n$, and its gradient in terms of Riesz's potential if $p > 2$, under the standard assumptions $|\sigma|(\mathbb{R}^n) < \infty$ (see also [Kuusi and Mingione 2016]). We emphasize that the present work brings a global and more accurate estimate depending only on Wolff's potential of σ of some distributional solutions to Syst. (S_1) than the one in [Kuusi and Mingione 2018]. G. Dolzmann, N. Hungerbühler and S. Müller in [Dolzmann, Hungerbühler and Müller 1997] proved the existence of distributional solutions for p-harmonic functions and established the Lorentz space estimates for such solutions. I. Chlebicka, Y. Youn, and A. Zatorska-Goldstein [Chlebicka, Youn and Zatorska-Goldstein 2023] applied the approach introduced in [Kuusi and Mingione 2018] to study solutions to measure data elliptic systems involving operators of the divergence form with Orlicz growth. They provided pointwise estimates for the solutions expressed in terms of a nonlinear potential of generalized Wolff-type (1.0.5).

Moreover, studies of systems involving the k-Hessian operator also have been done. See for example [Zhang et al. 2023, Yang and Bai 2022, Bidaut-Véron, Nguyen and Véron 2020] and references therein. In measure setting, M. Véron, Q. Nguyen and L. Véron, in [Bidaut-Véron, Nguyen and Véron 2020], proposed to study nonlinear systems with super-natural growth in $\mathbb{R}^n$, as a counterpart of [Phuc and Verbitsky 2008]. They could give necessary and sufficient conditions for existence expressed in terms of Riesz and Bessel capacities, together with bilateral estimates in terms of Wolff potentials.

Problems involving elliptic operators ruled by Orlicz-Sobolev spaces and general p, q-growth have been systematically investigated in the literature, whose right-hand side is a Radon measure, starting with the papers [Boccardo and Gallouët 1992, Boccardo and Gallouët 1989], where operators modelled upon the p-Laplacian are treated ($p = q$). We refer to some contributions on this topic [Malý 2003, Mingione 2007, Cianchi and Maz'ya 2017, Baroni 2015, Byun and Youn 2018, Chlebicka, Giannetti and Zatorska-Goldstein 2023] and references therein. In this respect, potential estimates are well-established and precise tools in the analysis of these kinds of problems. I. Chlebicka, F. Giannetti, and A. Zatorska-Goldstein in [Chlebicka, Giannetti and Zatorska-Goldstein 2023] were able to establish sharp pointwise bounds expressed in terms of (1.0.5) for a broad class of solutions to problems with Orlicz growth. They also provided powerful corollaries by giving regular consequences for the local behavior of solutions, in particular when measuring data satisfies conditions expressed in the relevant scales of generalized Lorentz, Marcinkiewicz, or Morrey type. Our method relies on these potential estimates in a particular form like the right-side in (P_2). This work was intended as an attempt to motivate the study equations like (P_2), via linking of integral equations (S), with the technique of potentials of Wolff-type $\mathbf{W}_G\sigma$ performing ideas of [Dat and Verbitsky 2015]. We mention that a different approach from ours is the notion of SOLA in [Boccardo and Gallouët 1989], which works well when the Radon measure on the right-side is bounded and is known *local upper bounds* for the eventual solutions.

1.6 OUTLINE

The main body of this thesis is organized as follows.

Chapter 2 presents some preliminary definitions and results on the Nonlinear Potential Theory used in this work. For convenience, we divide it into three sections, first, we develop a framework for standard quasilinear elliptic equations (modeled to the p-Laplace operator). The second one is devoted to the fully nonlinear elliptic equations ruled by the k-Hessian operator. The last one deals with Orlicz's growth, and for this reason, it gives a brief exposition of the Theory of Orlicz spaces and finally introduces the nonlinear potential theory in the Orlicz-Sobolev spaces.

In Chapter 3, we produce a framework to solve Syst. (SI) by proving Theorem 1.3.1 and Theorem 1.3.3. Then by making use of suitable potential estimates, we provide the proofs of

Theorem 1.3.6, Theorem 1.3.8, Theorem 1.3.9 Theorem 1.3.11, Theorem 1.3.12, and Corollary 1.3.13. We also suggest some questions related to Syst. (S_1), with further problems.

In Chapter 4, we will look more closely at Eq. (S), and indicate how the solution obtained from this equation in a suitable Orlicz space may be used to build a finite energy solution to Eq. (P_2). For that, first, we prove Theorem 1.3.14, and we provide detailed auxiliary results to prove Theorem 1.3.15. Our techniques do not appeal to any relation on p and q, much less on ratio q/p. Besides, we present some remarks concerning Theorem 1.3.15.

In Appendix A, we bring precise proof of the potential estimate used to prove Theorem 1.3.15.

2 PRELIMINARIES

This chapter aims to summarize without proof the relevant material on Wolff potentials, and nonlinear potential theory that will be later used in this manuscript. Also, some necessary background definitions are introduced. All results cited here will contain references to make our exposition more precise.

Remind the definition of Wolff potential (1.0.3), we highlight that our approach does not include the Lebesgue measure as an admissible Radon measure. Indeed, if σ is the Lebesgue measure, then

$$\begin{aligned}\int_1^\infty \left(\frac{\sigma(B(0,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} &= \int_1^\infty \left(\frac{|B(0,t)|}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\ &= c\int_1^\infty \left(\frac{t^n}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\ &= c\int_1^\infty t^{\frac{\alpha p}{p-1}-1}\,\mathrm{d}t = c\,\frac{p-1}{\alpha p} t^{\frac{\alpha p}{p-1}}\Big|_1^\infty = \infty,\end{aligned}$$

whenever $\alpha p > 0$. Thus, from (1.0.4), $\mathbf{W}_{\alpha,p}\sigma \equiv \infty$ if $\mathrm{d}\sigma = \mathrm{d}x$.

Let us start with the following result [Cao and Verbitsky 2016, Lemma 2.1].

Lemma A. *Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying (1.3.3). Then, for every $r > 0$,*

$$\int_E (\mathbf{W}_{\alpha,p}\sigma_E)^r\,\mathrm{d}\sigma \le c\,\sigma(E) \quad \textit{for all compact sets } E \subset \mathbb{R}^n, \tag{2.0.1}$$

where c is a positive constant which depends on $n,\ p,\ \alpha,\ r$ and C_σ. Moreover, if (2.0.1) holds for a given $r > 0$, then (1.3.3) holds with $C = C(n,p,\alpha,r,c)$; hence (2.0.1) holds for every $r > 0$.

It is through this equivalence that we will be able to exhibit a nontrivial solution to (SI). In particular, Lemma A implies that for all open balls B,

$$\int_B (\mathbf{W}_{\alpha,p}\sigma_B)^r\,\mathrm{d}\sigma \le c\,\sigma(B). \tag{2.0.2}$$

This follows by Fatou's lemma. Indeed, let B be an open ball and consider an increasing sequence of balls $\{B_i\}$ satisfying

$$B = \bigcup_{i=1}^\infty \overline{B}_i, \qquad \overline{B}_i \subset \overline{B}_{i+1} \subset B \quad \forall\, i.$$

By condition (2.0.1),

$$\int_{\overline{B}_i} (\mathbf{W}_{\alpha,p}\sigma_{\overline{B}_i})^r\,\mathrm{d}\sigma \le c\,\sigma(\overline{B}_i) \le c\,\sigma(B) \quad \forall\, i.$$

Using Fatou's lemma, we conclude that

$$\int_B (\mathbf{W}_{\alpha,p}\sigma_B)^r \,\mathrm{d}\sigma = \int_B \lim_{i\to\infty} \chi_{\overline{B}_i} (\mathbf{W}_{\alpha,p}\sigma_{\overline{B}_i})^r \,\mathrm{d}\sigma$$
$$\leq \varliminf_{i\to\infty} \int_B \chi_{\overline{B}_i} (\mathbf{W}_{\alpha,p}\sigma_{\overline{B}_i})^r \,\mathrm{d}\sigma = \varliminf_{i\to\infty} \int_{\overline{B}_i} (\mathbf{W}_{\alpha,p}\sigma_{\overline{B}_i})^r \,\mathrm{d}\sigma \leq c\,\sigma(B),$$

since $\left\{\chi_{\overline{B}_i}(\mathbf{W}_{\alpha,p}\sigma_{\overline{B}_i})^r\right\}$ is an increasing sequence of nonnegative functions which converges pointwise to $(\mathbf{W}_{\alpha,p}\sigma_B)^r$ in B.

2.1 NONLINEAR POTENTIAL THEORY OF QUASILINEAR ELLIPTIC EQUATIONS

We begin this section by recalling some basic definitions and results for easy reference. For details see [Kilpeläinen and Malý 1992, Kilpeläinen and Malý 1994, Heinonen, Kilpeläinen and Martio 2006].

Definition 2.1.1. Let Ω be a domain in $\mathbb{R}^n$ and $1 < p < \infty$.

$(\mathbf{i})$ We define the p-Laplace operator for $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ in a distributional sense as follows

$$\langle \Delta_p w, \varphi\rangle = \langle \mathrm{div}\left(|\nabla w|^{p-2}\nabla w\right), \varphi\rangle = -\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla\varphi \,\mathrm{d}x, \quad \forall \varphi \in C^\infty_c(\Omega).$$

$(\mathbf{ii})$ We say that $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a distributional solution of the homogeneous p-laplacian equation (p-harmonic) if $\langle -\Delta_p w, \varphi\rangle = 0,\ \forall \varphi \in C^\infty_c(\Omega)$.
$(\mathbf{iii})$ We define a supersolution $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ in Ω if $\langle -\Delta_p w, \varphi\rangle \geq 0$ for all nonnegative $\varphi \in C^\infty_c(\Omega)$.

Next, we extend the notion of the distributional solutions for the equation $-\Delta_p w = \mu$ where w does not necessarily belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$ and μ is in the dual space $W^{-1,p'}_{\mathrm{loc}}(\Omega)$. Indeed, we will understand solutions in the following potential-theoretic sense using p-superharmonic functions.

Definition 2.1.2. A function $w : \Omega \to (-\infty, \infty) \cup \{\infty\}$ is p-superharmonic in Ω if

$(\mathbf{i})$ w is lower semicontinuous,
$(\mathbf{ii})$ w is not identically infinite in any component of Ω,
$(\mathbf{iii})$ for each open subset D compactly contained in Ω and each p-harmonic function h in D such that $h \in C(\overline{D})$ and $h \leq w$ in ∂D implies $h \leq w$ in D.

We denote by $\mathcal{S}_p(\Omega)$ for the class of all p-superharmonic functions in Ω.

For $w \in \mathcal{S}_p(\Omega)$ we define its truncation as follows

$$T_k(w) = \min(k, \max(w, -k)), \quad \forall k > 0.$$

We mention that $T_k(w) \in W^{1,p}_{\mathrm{loc}}(\Omega)$ although w does not necessarily belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$. It is well known that $T_k(w)$ is a supersolution in Ω, for all $k > 0$, in the sense of Definition 2.1.1 item (3).

Following [Heinonen, Kilpeläinen and Martio 2006], we consider the generalized gradient for $w \in \mathcal{S}_p(\Omega)$ (see Definition 2.1.2), as the following pointwise limit

$$Dw(x) = \lim_{k\to\infty} \nabla(T_k(w))(x) \quad \text{almost everywhere in } \Omega.$$

If $p > 2 - 1/n$, one checks that Dw is the distributional gradient of w (see [Heinonen, Kilpeläinen and Martio 2006, page 154]). Using [Kilpeläinen and Malý 1992, Theorem 1.15], we see that for $w \in \mathcal{S}_p(\Omega)$ and $1 \le r < n/(n-1)$, we have $|Dw|^{p-1} \in L^r_{\mathrm{loc}}(\Omega)$. In particular, $|Dw|^{p-2}Dw \in L^r_{\mathrm{loc}}(\Omega)$. Because of this fact, we can define the p-Laplace operator in a distributional sense for $w \in \mathcal{S}_p(\Omega)$ as follows

$$\langle -\Delta_p w, \varphi \rangle = \int_\Omega |Dw|^{p-2} Dw \cdot \nabla\varphi \, \mathrm{d}x, \quad \forall \varphi \in C^\infty_c(\Omega).$$

Therefore, by the Riesz Representation Theorem [Lieb and Loss 2001, Theorem 6.22], there exists a unique measure $\mu = \mu[w] \in M^+(\Omega)$ such that $-\Delta_p w = \mu[w]$. In the literature, $\mu[w]$ is called the Riesz measure of w.

Definition 2.1.3. For $\sigma \in M^+(\Omega)$, we say that w is a solution in the potential-theoretic sense to the equation

$$-\Delta_p w = \sigma \quad \text{in} \quad \Omega$$

if $w \in \mathcal{S}_p(\Omega)$ and $\mu[w] = \sigma$.

In light of Definition 2.1.3, if $\sigma \in M^+(\Omega)$, then a pair (u, v) is a solution (in the potential-theoretic sense) to the system

$$\begin{cases} -\Delta_p u = \sigma\, v^{q_1} & \text{in} \quad \Omega, \\ -\Delta_p v = \sigma\, u^{q_2} & \text{in} \quad \Omega \end{cases} \tag{2.1.1}$$

whenever u and v are nonnegative functions and

$$\begin{cases} u \in \mathcal{S}_p(\Omega) \cap L^{q_2}_{\mathrm{loc}}(\Omega, \mathrm{d}\sigma), \\ v \in \mathcal{S}_p(\Omega) \cap L^{q_1}_{\mathrm{loc}}(\Omega, \mathrm{d}\sigma), \\ \mathrm{d}\mu[u] = v^{q_1}\mathrm{d}\sigma, \\ \mathrm{d}\mu[v] = u^{q_2}\mathrm{d}\sigma. \end{cases}$$

Definition 2.1.4. Let $\sigma \in M^+(\mathbb{R}^n)$. A pair (u, v) is called a supersolution to Syst. (S_1) if u and v are nonnegative functions and

$$\begin{cases} u \in \mathcal{S}_p(\mathbb{R}^n) \cap L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma), \\ v \in \mathcal{S}_p(\mathbb{R}^n) \cap L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma), \\ \displaystyle\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla\varphi \,\mathrm{d}x \geq \int_{\mathbb{R}^n} v^{q_1} \varphi \,\mathrm{d}\sigma, \\ \displaystyle\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla\varphi \,\mathrm{d}x \geq \int_{\mathbb{R}^n} u^{q_2} \varphi \,\mathrm{d}\sigma, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n),\, \varphi \geq 0. \end{cases}$$

The notion of *subsolution* to Syst. (S_1) is defined similarly by replacing "$\geq$" by "$\leq$" in Definition 2.1.4.

Remark 2.1.5. In view of Definitions 2.1.2 and 2.1.4, supersolutions (or solution) to Syst. (S_1) are supersolutions in $\mathbb{R}^n$. Indeed, if (u, v) is a supersolution to (S_1) in the sense of Definition 2.1.4 with u, $v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, then u and v are supersolutions in $\mathbb{R}^n$ in the sense of Definition 2.1.2.

Next, we will employ some fundamental results of the potential theory of quasilinear elliptic equations. The following lemma will be used to prove that a pointwise limit of a sequence of p-superharmonic functions is, indeed, a p-superharmonic (see [Heinonen, Kilpeläinen and Martio 2006, Lemma 7.3]).

Lemma B. *Suppose that $\{w_j\}$ is a sequence of p-superharmonic functions in Ω. If the sequence $\{w_j\}$ either inscreasing or converges uniformly on compact subsets in Ω, then in each component of Ω the pointwise limit function $w = \lim_{j\to\infty} w_j$ is a p-superharmonic function unless $w \equiv \infty$.*

The following weak continuity of the p-Laplacian result is due to N. S. Trudinger and X.-J. Wang [Trudinger and Wang 2002] and will be used to prove the existence of p-superharmonic solutions to quasilinear equations.

Theorem C. *Let $\{w_j\}$ be a sequence of nonnegative p-superharmonic functions in Ω. Suppose that $\{w_j\}$ converge pointwise to w where is finite almost everywhere and p-superharmonic function in Ω. Then $\mu[w_j]$ converges weakly to $\mu[w]$, that is,*

$$\lim_{j\to\infty} \int_\Omega \varphi \,\mathrm{d}\mu[w_j] = \int_\Omega \varphi \,\mathrm{d}\mu[w], \quad \forall \varphi \in C_c^\infty(\Omega).$$

Next, we estate a crucial result on pointwise estimates of nonnegative p-superharmonic functions in terms of Wolff's potential due to T. Kilpeläinen and J. Malý in [Kilpeläinen and Malý 1994, Corollary 4.13].

Theorem D. *Let w be a p-superharmonic function in $\mathbb{R}^n$ with $\underline{\lim}_{|x|\to\infty} w(x) = 0$. If $1 < p < n$ and $\omega = \mu[w]$, that is, $-\Delta_p w = \omega$, then there exists a constant $K \geq 1$ depending only on n and p such that*

$$K^{-1}\mathbf{W}_p\omega(x) \leq w(x) \leq K\,\mathbf{W}_p\omega(x), \quad \forall x \in \mathbb{R}^n.$$

Observe that we can assume, without loss of generality, that any p-superharmonic function w in $\mathbb{R}^n$ can be chosen to be quasicontinuous in $\mathbb{R}^n$, that is, w is a continuous function in $\mathbb{R}^n$ except in a set of p-capacity zero. See more details in [Heinonen, Kilpeläinen and Martio 2006, Chapter 7].

Next, let us express the local version of *Wolff's inequality* in the case $\Omega = \mathbb{R}^n$:

$$\mu \in M^+(\mathbb{R}^n) \cap W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n) \Longleftrightarrow \int_B \mathbf{W}_p\mu_B \,\mathrm{d}\mu < \infty \quad \text{for all balls } B, \tag{2.1.2}$$

where $B = B(x,R)$, $\mu_B = \chi_B\mu$ and $W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n) = W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)^*$ is the dual Sobolev space; see [Adams and Hedberg 1996, Theorem 4.55].

The following result is established in [Cao and Verbitsky 2017, Lemma 3.3], and in combination with (2.1.2), will be crucial to prove that the solutions to Syst. (S_1) belong to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \times W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma E. *Let $1 < p < n$ and $\omega \in M^+(\mathbb{R}^n) \cap W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$. Suppose that w is a nonnegative p-superharmonic solution to*

$$\begin{cases} -\Delta_p w = \omega \quad \textit{in} \quad \mathbb{R}^n, \\ \underline{\lim}_{|x|\to\infty} \; w(x) = 0. \end{cases}$$

Then $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\omega)$.

In view of Theorem D and (2.1.2), one can check a converse type result of Lemma E, that is if $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ solves $-\Delta_p w = \omega$ in the distributional sense with $\omega \in M^+(\mathbb{R}^n)$, then $\omega \in W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$. Indeed, being $w \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ a weak solution to $-\Delta_p w = \omega$, with $\omega \in M^+(\mathbb{R}^n)$, we have that w is a p-supersolution. From [Heinonen, Kilpeläinen and Martio 2006, Theorem 7.25 (iv)], we may consider w as a p-superhamonic function, whence by Theorem D

$$\int_B \mathbf{W}_p\omega_B \,\mathrm{d}\omega \leq \int_B \mathbf{W}_p\omega \,\mathrm{d}\omega \leq K^{-1}\int_B w \,\mathrm{d}\omega < \infty \quad \text{for all balls } B.$$

Employing (2.1.2), our assertion follows.

To finish this section, we recall a basic fact on Wolff's potential (see [Cao and Verbitsky 2017, Corollary 3.2 (iii)]): If $1 < p < \infty$ and $0 < \alpha < n/p$, for any $\omega \in M^+(\mathbb{R}^n)$, it holds

$$\varliminf_{|x|\to\infty} \mathbf{W}_{\alpha,p}\omega(x) = 0. \tag{2.1.3}$$

Therefore, in view of (2.1.3), any nontrivial solution (u, v) to (SI) satisfies

$$\varliminf_{|x|\to\infty} u(x) = \varliminf_{|x|\to\infty} v(x) = 0, \tag{2.1.4}$$

provided that (u, v) enjoy the property in (1.3.4).

2.2 NONLINEAR POTENTIAL THEORY OF FULLY NONLINEAR ELLIPTIC EQUATIONS: k-HESSIAN MEASURES

We begin this section by recalling some basic definitions and results due to N. S. Trudinger and X.-J. Wang [Trudinger and Wang 1997, Trudinger and Wang 1999, Trudinger and Wang 2002] and D. A. Labutin [Labutin 2002]. These results were summarized in [Phuc and Verbitsky 2009, Section 4] (see also [Phuc and Verbitsky 2008, Section 7]), in which we state some of them for the convenience of the reader. Let Ω be an open set in $\mathbb{R}^n$. For $k = 1, \ldots, n$ and $w \in C^2(\Omega)$, $k-$Hessian operator, $F_k[w]$ is defined as follows

$$F_k[w] = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix D^2w on $\mathbb{R}^n$, with $k = 1, 2, \ldots, n$. Alternatively, we may write $F_k[w] = [D^2w]_k$, where $[A]_k$ denotes the sum of the $k\times k$ principal minors of an $n \times n$ matrix A. For example, $F_1[w] = \Delta w$ is the Laplacian operator of u, and $F_n[w] = \det D^2w$ is the Monge–Ampère operator of w.

Definition 2.2.1. A function $w : \Omega \to (-\infty, \infty) \cup \{-\infty\}$ is k-subharmonic in Ω if

$(\mathbf{i})$ w is upper semicontinuous,

$(\mathbf{ii})$ $w \not\equiv -\infty$ in any component of Ω,

$(\mathbf{iii})$ for each open subset D compactly contained in Ω and each $\varphi \in C^2(D) \cap C(\overline{D})$ satisfying $F_k[\varphi] \ge 0$ in D, the following implication holds

$$w \le \varphi \text{ on } \partial D \Longrightarrow w \le \varphi \text{ in } D.$$

We denote by $\Phi^k(\Omega)$ the class of all k-subharmonic functions in Ω.

The following weak convergence result is essential to potential theory related to k-Hessian operators, and it was proved by N. S. Trudinger and X.-J. Wang [Trudinger and Wang 1999, Theorem 1.1].

Theorem F. *For each $w \in \Phi^k(\Omega)$, there exists a Radon measure $\mu_k[w]$ such that*

$(\mathbf{i})$ $\mu_k[w] = F_k[w]$ *for $w \in C^2(\Omega)$, i.e. $\mu_k[w] = F_k[w]\,\mathrm{d}x$, and*

$(\mathbf{ii})$ *if $\{w_j\}$ is a sequence of k-subharmonic functions which converges almost everywhere to w, then the sequence of the corresponding measures $\{\mu_k[w_j]\}$ converges to $\mu_k[w]$ weakly, that is,*

$$\lim_{j\to\infty}\int_\Omega \varphi\,\mathrm{d}\mu_k[w_j] = \int_\Omega \varphi\,\mathrm{d}\mu_k[w] \quad \forall\varphi \in C_c^\infty(\Omega).$$

We mention that the measure $\mu_k[w]$ in Theorem F is said to be the k-Hessian measure associated with $w \in \Phi^k(\Omega)$, whence we may write $F_k[w]$ in place of $\mu_k[w]$ even in the case where w does not belong to $C^2(\Omega)$.

Remark 2.2.2. A Harnack-type convergence theorem follows from $(\mathbf{ii})$ in Theorem F: If $\{w_j\} \subset \Phi^k(\Omega)$ is a nonincreasing sequence, then in each component of Ω the pointwise limit function $w = \lim_{j\to\infty} w_j$ is a k-subharmonic function and $\mu_k[w_j]$ converges to $\mu_k[w]$ weakly. Indeed, in each component of Ω, we have $w \in \Phi^k(\Omega)$ by definition of k-subharmonic functions.

Definition 2.2.3. For $\sigma \in M^+(\Omega)$, we say that w is a solution to the equation

$$F_k[w] = \sigma \quad \text{in} \quad \Omega,$$

in the potential-theoretic sense, whenever

$$w \in \Phi^k(\Omega) \quad \text{and} \quad \mu_k[w] = \sigma.$$

The k-Hessian measure is an important tool in potential theory for $\Phi^k(\Omega)$. The next result is a direct consequence of [Labutin 2002, Theorem 2.1] due to D. A. Labutin, which brings a global pointwise estimate for functions in $\Phi^k(\Omega)$ in terms of Wolff potentials $\mathbf{W}_k$. See also [Phuc and Verbitsky 2009, Theorem 4.3].

Theorem G. *Let $w \geq 0$ be such that $-w \in \Phi^k(\mathbb{R}^n)$, where $1 \leq k < n/2$. Assume that*

$$\mu = F_k[-w] \quad \textit{and} \quad \lim_{|x|\to\infty} w(x) = 0.$$

Then, there exists $K = K(n,k) > 0$ such that

$$K^{-1}\,\mathbf{W}_k\mu(x) \le w(x) \le K\,\mathbf{W}_k\mu(x) \quad \forall x \in \mathbb{R}^n.$$

In view of Definition 2.1.2, we present the precise definition of a solution to Syst. (S_2) in the potential-theoretic sense.

Definition 2.2.4. Let $1 \le k \le n$ and $q_1, q_2 > 0$. For $\sigma \in M^+(\mathbb{R}^n)$, we say that a pair of nonnegative functions (u, v) is a solution to Syst. (S_2) whenever u and v satisfy

$$\begin{cases} -u \in \Phi^k(\mathbb{R}^n), & u \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma), \\ -v \in \Phi^k(\mathbb{R}^n), & v \in L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma), \\ \mathrm{d}\mu_k[u] = v^{q_1}\mathrm{d}\sigma, & \mathrm{d}\mu_k[v] = u^{q_2}\mathrm{d}\sigma, \\ \varliminf\limits_{|x|\to\infty} u(x) = 0, & \varliminf\limits_{|x|\to\infty} v(x) = 0. \end{cases}$$

The remainder of this section will be devoted to establishing relevant existence results for Hessian equations with measure data. Following [Trudinger 1995, Trudinger and Wang 1997, Trudinger and Wang 1999, Trudinger 1994], we say that $\Omega \subset \mathbb{R}^n$ is a *bounded uniformly $(k-1)$-convex domain in $\mathbb{R}^n$* whenever Ω is bounded, $\partial\Omega \in C^2$, and $H_j[\partial\Omega] > 0$ for $j = 0, \dots, k-1$, where $H_j[\partial\Omega]$ denotes the j-mean curvature of the boundary $\partial\Omega$, more precisely

$$H_j[\partial\Omega] = \sum \kappa_1 \cdots \kappa_{i_j},$$

here $\kappa_1, \dots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ and the preceding sum is taken over increasing j-tuples $(i_1, \dots, i_j) \subset \{1, \dots, n-1\}$. The curvature $\kappa_1, \dots, \kappa_{n-1}$ are normalized so that they are positive on spheres, and for $j = 0$ we set $H_0[\partial\Omega] = 1$. In particular, a ball $B(x, R)$ is an example of bounded uniformly $(k-1)$-convex domains in $\mathbb{R}^n$.

Let $\mu \in M^+(\Omega)$ and consider the following fully nonlinear problem:

$$\begin{cases} F_k[-w] = \mu & \text{in} \quad \Omega, \\ \qquad w = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2.1}$$

where w is a nonnegative function such that $-w \in \Phi^k(\Omega)$ and it is continuous near $\partial\Omega$. Under a suitable assumption on μ, the existence of solutions to (2.2.1) is the content of the following lemma [Phuc and Verbitsky 2009, Lemma 4.4].

Lemma H. *Let Ω be a bounded uniformly $(k-1)$-convex domain in $\mathbb{R}^n$. Suppose $\mu \in M^+(\Omega)$ can be decomposed as a sum $\mu = f\mathrm{d}x + \mu_0$, where μ_0 is a nonnegative measure compactly supported in Ω, and $f \ge 0$ belongs to $L^s(\Omega)$, for* (**i**) $s > n/2$ *if* $1 \le k \le n/2$, *or*

(**ii**) $s = 1$ *if* $n/2 < k \le n$.

Then there exists $w \ge 0$ *such that* $-w \in \Phi^k(\Omega) \cap C(\overline{\Omega})$, *and* w *solves* (2.2.1).

In our argument to prove Theorem 1.3.11, we use a comparison principle due to N. S. Trudinger and X.-J. Wang in [Trudinger and Wang 2002, Theorem 4.1]. For its statement, let us to introduce the so-called k-Hessian capacity $\mathrm{cap}_k(\cdot, \Omega)$ defined for a compact set $E \subset \Omega$ by

$$\mathrm{cap}_k(E, \Omega) = \sup\left\{ \int_E \mathrm{d}\mu_k[u] : u \in \Phi^k(\Omega),\ -1 < u < 0 \ \text{ in } \ \Omega \right\}.$$

We say that measure $\omega \in M^+(\Omega)$ is *continuous with respect to capacity* $\mathrm{cap}_k(\cdot, \Omega)$ if $\omega(E) = 0$ whenever $\mathrm{cap}_k(E, \Omega) = 0$ for all compact sets $E \subset \Omega$.

Theorem I. *Let* $w_1,\ w_2 \in \Phi^k(\Omega)$ *such that* $\mu_k[w_1]$ *and* $\mu_k[w_2]$ *are continuous with respect to capacity* $\mathrm{cap}_k(\cdot, \Omega)$. *Suppose* $w_1 \le w_2$ *continuously on* $\partial\Omega$. *If* $\mu_k[w_1] \ge \mu_k[w_2]$, *then* $w_1 \le w_2$ *in* Ω.

Note that one can combine Lemma H with Theorem I to prove an existence and uniqueness result of solutions to (2.2.1), provided Ω is a bounded uniformly $(k-1)$-convex domain in $\mathbb{R}^n$.

Remark 2.2.5. If $\sigma \in M^+(\mathbb{R}^n)$ satisfies the Riesz capacity condition (1.3.3) with $\alpha = 2k/(k+1)$ and $p = k+1$, then σ is continuous with respect to capacity $\mathrm{cap}_k(\cdot) := \mathrm{cap}_k(\cdot, \mathbb{R}^n)$. Indeed, for $\alpha > 0$, we denote by $\mathbf{G}_\alpha\sigma$ the Bessel potential of order α, see [Adams and Hedberg 1996, Section 1.2.4] for its definition. Similarly to Riesz capacity (1.3.1), we define the Bessel capacity of a compact set $E \subset \mathbb{R}^n$,

$$\mathrm{cap}_{\mathbf{G}_\alpha, p}(E) = \inf\{\|f\|_{L^p}^p : f \in L^p(\mathbb{R}^n),\ f \ge 0,\ \mathbf{G}_\alpha f \ge 1 \text{ on } E\}.$$

By [Phuc and Verbitsky 2008, Theorem 2.20], $\mathrm{cap}_{\mathbf{G}_\alpha, p}$ is equivalent to the k-Hessian capacity cap_k for $\alpha = 2k/(k+1)$ and $p = k+1$, with $k < n/2$, i.e., there exits a constant $C = C(n,k) \ge 1$ such that, for all compact sets $E \subset \mathbb{R}^n$, it holds

$$C^{-1}\,\mathrm{cap}_k(E) \le \mathrm{cap}_{\mathbf{G}_{\frac{2k}{k+1}}, k+1}(E) \le C\,\mathrm{cap}_k(E).$$

On the other hand, since $\mathbf{G}_\alpha\sigma \le \mathbf{I}_\alpha\sigma$ for all $\alpha > 0$ and $\sigma \in M^+(\mathbb{R}^n)$, one has $\mathrm{cap}_{\mathbf{I}_\alpha, p} \le \mathrm{cap}_{\mathbf{G}_\alpha, p}$. The assertion follows by combining the two previous inequalities with $\alpha = 2k/(k+1)$ and $p = k+1$.

The next result [Phuc and Verbitsky 2009, Lemma 4.7] shows a monotonicity-type result to Hessian equations required in the argument in the proof of Theorem 1.3.11. In what follows, we set $B_R = B(0,R)$ for $R>0$.

Lemma J. *Let $\mu_1,\ \mu_2 \in M^+(\mathbb{R}^n)$ satisfying $\mu_1 \le \mu_2$, and $\mathbf{W}_k\mu_2 < \infty$. Let $w_1 \ge 0$ be a function satisfying $-w_1 \in \Phi^k(\mathbb{R}^n)$ and*

$$\begin{cases} F_k[-w_1] = \mu_1 \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} w_1 = 0. \end{cases}$$

Suppose w_1 is a limit of a subsequence of $\{w_1^i\}$ pointwise almost everywhere in $\mathbb{R}^n$, where w_1^i is continuous near ∂B_{i+1} and it satisfies

$$\begin{cases} -\, w_1^i \in \Phi^k(B_{i+1}), \\ F_k[-w_1^i] = \mu_{1,i} \quad \text{in} \quad B_{i+1}, \\ \qquad w_1^i = 0 \qquad \text{on} \quad \partial B_{i+1}, \end{cases}$$

here $\mu_{1,i} := \chi_{B_i}\mu_1$. Then there exists a function w_2 satisfying

$$\begin{cases} \ w_2 \ge w_1,\ -w_2 \in \Phi^k(\mathbb{R}^n), \\ F_k[-w_2] = \mu_2 \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} w_2 = 0. \end{cases}$$

Moreover, w_2 is also a pointwise almost everywhere limit of a subsequence of $\{w_2^i\}$, where w_2 is continuous near ∂B_{i+1} and it satisfies

$$\begin{cases} -\, w_2^i \in \Phi^k(B_{i+1}), \\ F_k[-w_2^i] = \mu_{2,i} \quad \text{in} \quad B_{i+1}, \\ \qquad w_2^i = 0 \qquad \text{on} \quad \partial B_{i+1}. \end{cases}$$

2.3 NONLINEAR POTENTIAL THEORY FOR ORLICZ SETTING

For an overview of Orlicz space theory, we refer to the books [Adams and Fournier 2003, Krasnosel'skii and Rutickii 1961, Rao and Ren 1991, Harjulehto and Hästö 2019] and references therein. Let us remark that some of their references deal with generalized Orlicz growth.

2.3.1 Young Functions

Definition 2.3.1 (Young function)**.** A function $G : [0,\infty) \to [0,\infty)$ is said to be a Young function if G is convex, strictly increasing and satisfies $G(0) = 0$.

Definition 2.3.2 (N-functions and its Conjugate)**.** A Young function $G : [0,\infty) \to [0,\infty)$ is an N-function if

$$\lim_{t\to 0^+} \frac{G(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{G(t)}{t} = \infty.$$

The function $G^*(s) := \sup_{t>0}\{st - G(t)\}$, for $s \in [0,\infty)$, is called the complementary function of G.

Equivalently, we may define G^* by

$$G^*(s) = \int_0^s g^{-1}(r)\,\mathrm{d}r, \quad s \geq 0,$$

where $g = G'$. Clearly, by definition, it holds a *Young's inequality*

$$ts \leq G(t) + G^*(s) \quad \forall t, s \geq 0. \tag{2.3.1}$$

Equality occurs in (2.3.1) if and only if either $t = g^{-1}(s)$ or $s = g(t)$. In addition, it holds $(G^*)^*(t) = G(t)$ for all $t \geq 0$.

Unless otherwise stated, we assume that all N-functions G in this work belong to $C^2(0,\infty)$ where $g = G'$ satisfies

$$p - 1 \leq \frac{tg'(t)}{g(t)} \leq q - 1 \quad \forall t > 0. \tag{2.3.2}$$

for some $1 < p \leq q < \infty$. In particular, the following bound holds

$$p \leq \frac{tg(t)}{G(t)} \leq q \quad \forall t > 0, \tag{2.3.3}$$

Indeed, by (2.3.2), $t \mapsto g(t)/t^{p-1}$ is a nondecreasing function and $t \mapsto g(t)/t^{q-1}$ is a nonincreasing function. Consequently, for all $t > 0$

$$G(t) = \int_0^t g(s)\,\mathrm{d}s = \int_0^t s^{p-1}\frac{g(s)}{s^{p-1}}\,\mathrm{d}s \leq \frac{tg'(t)}{p},$$
$$G(t) = \int_0^t g(s)\,\mathrm{d}s = \int_0^t s^{q-1}\frac{g(s)}{s^{q-1}}\,\mathrm{d}s \geq \frac{tg'(t)}{q}.$$

In customary terminology, condition (2.3.3) is known as Δ_2 *and* ∇_2 *condition*. A typical example of a function that satisfies (2.3.3) is our model $G(t) = t^p/p + t^q/q$, with $g(t) = t^{p-1} + t^{q-1}$, whenever $p \geq 2$. The inequalities (2.3.3) have some basic applications. In the next lemma, we emphasize some of that, where we omit the easy proof which can be found e.g. in [Malý 2003, Section 2], or [Lee and Lee 2021, Lemma 2.10].

Lemma K. *Suppose* g *satisfies* (2.3.2)*. Then* (2.3.3) *holds. In particular,* $t \mapsto G(t)/t^p$ *is a nondecreasing function, and* $t \mapsto G(t)/t^q$ *is a nonincreasing function, for* $t > 0$*. Moreover,*

(**i**) *for all* $\alpha \geq 0$ *and* $t > 0$*,*

$$\min\{\alpha^{p-1}, \alpha^{q-1}\}g(t) \leq g(\alpha t) \leq \max\{\alpha^{p-1}, \alpha^{q-1}\}g(t), \tag{2.3.4}$$

$$\min\{\alpha^{p}, \alpha^{q}\}G(t) \leq G(\alpha t) \leq \max\{\alpha^{p}, \alpha^{q}\}G(t), \tag{2.3.5}$$

$$\left(\frac{p}{q}\right)^{\frac{1}{p-1}} \min\{\alpha^{\frac{1}{p-1}}, \alpha^{\frac{1}{q-1}}\}g^{-1}(t) \leq g^{-1}(\alpha t) \leq \left(\frac{q}{p}\right)^{\frac{1}{p-1}} \max\{\alpha^{\frac{1}{p-1}}, \alpha^{\frac{1}{q-1}}\}g^{-1}(t), \tag{2.3.6}$$

$$\min\{\alpha^{\frac{p}{p-1}}, \alpha^{\frac{q}{q-1}}\}G^*(t) \leq G^*(\alpha t) \leq \max\{\alpha^{\frac{p}{p-1}}, \alpha^{\frac{q}{q-1}}\}G^*(t); \tag{2.3.7}$$

(**ii**) *for all* $t > 0$*, setting* $c = g^{-1}(1)$*, it holds*

$$g^{-1}(t) \leq c\left(\frac{q}{p}\right)^{\frac{1}{p-1}}\left(t^{\frac{1}{p-1}} + t^{\frac{1}{q-1}}\right), \tag{2.3.8}$$

(**iii**) *for all* $t > 0$*, it holds*

$$(p-1)G(t) \leq G^*(g(t))) \leq q^{\frac{p}{p-1}}G(t). \tag{2.3.9}$$

2.3.2 Orlicz Space

Definition 2.3.3. The Orlicz space $L^G(\Omega)$ is understood as the set

$$L^G(\Omega) := \left\{u \in L^0(\Omega) : \int_\Omega G(\lambda|u|)\,\mathrm{d}x < \infty \text{ for some } \lambda > 0\right\}.$$

$L^G(\Omega)$ is a Banach space with the Luxemburg norm

$$\|u\|_{L^G} := \inf_{\lambda>0}\left\{\lambda > 0 : \int_\Omega G\left(\frac{|u|}{\lambda}\right)\mathrm{d}x \leq 1\right\}.$$

Remark 2.3.4. When $p = q$ in (2.3.3), i.e. $G(t) = t^p/p$, the Lebesgue space $L^p(\Omega)$ coincides with the Orlicz space $L^G(\Omega)$ as Banach spaces, where

$$\|u\|_{L^p} = p^{\frac{1}{p}}\|u\|_{L^G}.$$

See details, for instance, in [Adams and Fournier 2003, Chapter 8].

In view of (2.3.3), by [Harjulehto and Hästö 2019, Lemma 3.1.3], we have

$$L^G(\Omega) = \left\{u \in L^0(\Omega) : \int_\Omega G(|u|)\,\mathrm{d}x < \infty\right\}.$$

From [Harjulehto and Hästö 2019, Corollary 3.2.8], it holds

$$\|u\|_{L^G} \leq \int_\Omega G(|u|)\,\mathrm{d}\sigma + 1 \quad \forall u \in L^G(\Omega). \tag{2.3.10}$$

Furthermore, putting $\rho_G(u) = \int_\Omega G(|u|)\,\mathrm{d}x$, the following lemma establishes the relation between $\rho_G(\cdot)$ and norm $\|\cdot\|_{L^G}$ [Harjulehto and Hästö 2019, Lemma 3.2.9]. The function $\rho_G(\cdot)$ is called *modular*.

Lemma L. *For all $u \in L^G(\Omega)$, it holds*

$$\min\{\rho_G(u)^{\frac{1}{p}}, \rho_G(u)^{\frac{1}{q}}\} \leq \|u\|_{L^G} \leq \max\{\rho_G(u)^{\frac{1}{p}}, \rho_G(u)^{\frac{1}{q}}\},$$
$$\min\{\|u\|^p_{L^G}, \|u\|^q_{L^G}\} \leq \rho_G(u) \leq \max\{\|u\|^p_{L^G}, \|u\|^q_{L^G}\}.$$

The following result [Harjulehto and Hästö 2019, Lemma 3.2.11] is the generalization of the classical Hölder's inequality in Lebesgue spaces to Orlicz spaces.

Lemma M. *For all $u \in L^G(\Omega)$ and $v \in L^{G^*}(\Omega)$, it holds*

$$\left| \int_\Omega uv\,\mathrm{d}x \right| \leq 2\|u\|_{L^G}\|v\|_{L^{G^*}}.$$

Remark 2.3.5. Under condition (2.3.3), $L^G(\Omega)$ is reflexive, separable, and uniformly convex. In light of Hölder's inequality, the dual space $\left(L^G(\Omega)\right)^*$ coincides with $L^{G^*}(\Omega)$ (see [Harjulehto and Hästö 2019, Chapter 2, Section 3] for more details).

Definition 2.3.6. Let $\{u_j\} \in L^G(\Omega)$ be a sequence and let $u \in L^G(\Omega)$. We say that u_j converges weakly to u in $L^G(\Omega)$ if

$$\lim_{j\to\infty} \int_\Omega u_j v\,\mathrm{d}x = \int_\Omega uv\,\mathrm{d}x \quad \forall v \in L^{G^*}(\Omega).$$

As usual, we write $u_j \rightharpoonup u$ in $L^G(\Omega)$. The weak convergence of vector-valued functions in $L^G(\Omega; \mathbb{R}^n)$ has an obvious interpretation regarding the coordinate functions.

The following result will be useful in some of our arguments [Benyaiche and Khlifi 2021, Theorem 2.1]

Theorem N. *Let $\{u_j\}$ be a bounded sequence in $L^G(\Omega)$. If $u = \lim_j u_j$ pointwise in Ω (almost everywhere), then $u_j \rightharpoonup u$ in $L^G(\Omega)$.*

Definition 2.3.7. The Orlicz-Sobolev space $W^{1,G}(\Omega)$ is understood as the set of all functions $u \in L^G(\Omega)$ which admit weak derivatives $\partial_i u \in L^G(\Omega)$ for $i = 1, \ldots, n$; that is

$$W^{1,G}(\Omega) = \{u \in L^G(\Omega) : |\nabla u| \in L^G(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,G}} = \|u\|_{L^G} + \|\nabla u\|_{L^G}.$$

By $W_0^{1,G}(\Omega)$ we denote the closure of $C_c^\infty(\Omega)$ in $W^{1,G}(\Omega)$. As usual, $W_{\text{loc}}^{1,G}(\Omega)$ is the set of all functions u such that $u \in W^{1,G}(U)$ for all open subset U compactly contained in Ω.

Remark 2.3.8. Similar to the preceding remark, assuming (2.3.3), $W^{1,G}(\Omega)$ is a Banach space, separable, uniformly convex, and reflexive (see [Harjulehto and Hästö 2019, Theorem 6.1.4]). This also holds for $W_0^{1,G}(\Omega)$.

Next, we state a *modular* Poincaré inequality [Lieberman 1991, Lemma 2.2], which will be useful in our results.

Lemma O. *Let B_R be a ball with a radius R. There exists a constant $c = c(n, p, q) > 0$ such that*

$$\int_{B_R} G\left(\frac{|u|}{R}\right) \mathrm{d}x \leq c \int_{B_R} G(|\nabla u|) \, \mathrm{d}x \quad \forall u \in W_0^{1,G}(B_R).$$

Definition 2.3.9. The homogeneous Sobolev-Orlicz space $\mathcal{D}^{1,G}(\Omega)$ is understood as the set

$$\mathcal{D}^{1,G}(\Omega) = \{u \in W_{\text{loc}}^{1,G}(\Omega) : |\nabla u| \in L^G(\Omega)\}.$$

In this space, we have the following seminorm

$$\|u\|_{\mathcal{D}^{1,G}} = \|\nabla u\|_{L^G}. \tag{2.3.11}$$

Remark 2.3.10. Consider $\Omega = \mathbb{R}^n$.

(i) If $q \geq n$ in (2.3.3), all constants functions belong to $\mathcal{D}^{1,G}(\mathbb{R}^n)$. Indeed, appealing to [Harjulehto and Hästö 2019, Lemma 3.7.7], from (2.3.3) we have the following inclusions

$$\mathcal{D}^{1,p}(\mathbb{R}^n) \cap \mathcal{D}^{1,q}(\mathbb{R}^n) \subset \mathcal{D}^{1,G}(\mathbb{R}^n) \subset \mathcal{D}^{1,p}(\mathbb{R}^n) + \mathcal{D}^{1,q}(\mathbb{R}^n), \tag{2.3.12}$$

where $\mathcal{D}^{1,p}(\mathbb{R}^n)$, $\mathcal{D}^{1,q}(\mathbb{R}^n)$ are the classical homogenous Sobolev spaces, and $\mathcal{D}^{1,p}(\mathbb{R}^n) + \mathcal{D}^{1,q}(\mathbb{R}^n) := \{u + v : u \in \mathcal{D}^{1,p}(\mathbb{R}^n),\, v \in \mathcal{D}^{1,q}(\mathbb{R}^n)\}$. The space $\mathcal{D}^{1,q}(\mathbb{R}^n)$ retains all constants functions if $q \geq n$. This is elucidated in [Malý and Ziemer 1997, page 48] for $q = n$, while the case $q > n$ follows from the classical Morrey inequality. Consequently, the inclusions in (2.3.12) imply the desired fact.

(ii) When $1 < p \leq q < n$, in light of the classical Sobolev Inequality (see, for instance, [Malý and Ziemer 1997, Corollary 1.77]), the only constant function in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ and $\mathcal{D}^{1,q}(\mathbb{R}^n)$ is the zero function, which also occurs in $\mathcal{D}^{1,G}(\mathbb{R}^n)$ by (2.3.12). Hence the assignment (2.3.11) defines a norm on $\mathcal{D}^{1,G}(\mathbb{R}^n)$. Moreover, by using the standard mollifier functions, analysis similar to that in the proof of [Harjulehto and Hästö 2019, Theorem 6.4.4] allows us to infer that $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}^{1,G}(\mathbb{R}^n)$, whenever (2.3.11) is a norm.

Definition 2.3.11. $(\mathbf{i})$ A continuous function $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ is called an G-harmonic function in Ω if it satisfies $\mathrm{div}\Big(g(|\nabla u|)/|\nabla u|\nabla u\Big) = 0$ weakly in Ω, that is

$$\int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = 0 \quad \forall \varphi \in C_c^\infty(\Omega). \tag{2.3.13}$$

$(\mathbf{ii})$ We say that $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ is a G-supersolution in Ω if $-\mathrm{div}\Big(g(|\nabla u|)/|\nabla u|\nabla u\Big) \geq 0$ weakly, that is u satisfies

$$\int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\varphi \,\mathrm{d}x \geq 0 \quad \forall \varphi \in C_c^\infty(\Omega),\ \varphi \geq 0.$$

Finally, we say that $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ is a G-subsolution in Ω if $-u$ is a G-supersolution in Ω.

Existence and uniqueness of harmonic functions are proven in [Benyaiche and Khlifi 2021, Chlebicka and Karppinen 2021, Fan 2012]. The following lemma gives a version of the comparison principle for G-supersolutions and G-subsolutions [Chlebicka and Zatorska-Goldstein 2022, Lem. 3.5].

Lemma P. *Let $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ be a G-supersolution and $v \in W^{1,G}_{\mathrm{loc}}(\Omega)$ be a G-subsolution in Ω. If $\min\{u - v, 0\} \in W^{1,G}_0(\Omega)$, then $u \geq v$ almost everywhere in Ω.*

Let μ be a Radon measure (not necessarily nonnegative), and consider the following quasilinear elliptic equation with data measure

$$-\mathrm{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu \quad \text{in} \quad \Omega. \tag{2.3.14}$$

Definition 2.3.12. A function $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ is a solution to (2.3.14) if

$$\int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \varphi \,\mathrm{d}\mu \quad \forall \varphi \in C_c^\infty(\Omega).$$

Note that if μ is nonnegative, then a solution to (2.3.14) is a G-supersolution, in sense of Definition 2.3.12. The following result will be needed in Section 4.2, and it deals with the existence of solutions to (2.3.14). It is a consequence of [Benyaiche and Khlifi 2023, Theorem 4.3].

Theorem Q. *Let μ be a Radon measure in $\left(W_0^{1,G}(\Omega)\right)^*$. Then there exists a unique $u \in W_0^{1,G}(\Omega)$ satisfying* (2.3.14).

Now, we introduce the notion of the G-capacity of a compact subset of $\Omega \subseteq \mathbb{R}^n$ following [Lee and Lee 2021]. The G-capacity will be used to ensure that if $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ is a solution to eq. (2.3.14), then $u = 0$ whether $q \geq n$ in (2.3.3).

Definition 2.3.13. Let $E \subset \Omega$ be a compact subset, we define $\operatorname{cap}_G(E, \Omega)$, G-capacity of E with respect to Ω by

$$\operatorname{cap}_G(E, \Omega) = \inf \left\{ \int_\Omega G(|\nabla \varphi|) \, \mathrm{d}x : \varphi \in C_c^\infty(\Omega), \ \varphi \geq 1 \text{ on } E \right\}.$$

We set $\operatorname{cap}_G(E) = \operatorname{cap}_G(E, \mathbb{R}^n)$ when $\Omega = \mathbb{R}^n$.

Obseve that for $G_p(t) = t^p/p$, $\operatorname{cap}_{G_p}(\cdot, \Omega)$ coincides with the usual p-capicty with respect to Ω, see for instance [Heinonen, Kilpeläinen and Martio 2006, Adams and Hedberg 1996, Malý and Ziemer 1997]. In general, we may define equivalently

$$\operatorname{cap}_G(E, \Omega) = \inf \left\{ \int_\Omega G(|\nabla \varphi|) \, \mathrm{d}x : \varphi \in \mathcal{D}^{1,G}(\Omega), \ \varphi \geq 1 \text{ in a neighborhood of } E \right\}.$$

This follows by the same method as in the proof of [Malý and Ziemer 1997, Theorem 2.3 (iii)]. Note that by Remark 2.3.10, $\operatorname{cap}_G(E) = 0$ for all compact set $E \subset \mathbb{R}^n$, since the function 1 belongs to $\mathcal{D}^{1,G}(\mathbb{R}^n)$ if $q \geq n$ in (2.3.3).

Remark 2.3.14. Suppose $q \geq n$ and let $\mu \in M^+(\mathbb{R}^n) \cap \left(\mathcal{D}^{1,G}(\mathbb{R}^n)\right)^*$, then a solution u to (2.3.14) in $\mathcal{D}^{1,G}(\mathbb{R}^n)$ must be constant. To see this, we show first that μ must be *absolutely continuous* with respect to the G-capacity, that is $\mu(E) = 0$ whenever $\operatorname{cap}_G(E) = 0$ for all compact sets $E \subset \mathbb{R}^n$. Fix $E \subset \mathbb{R}^n$ a compact set and let $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on E. Then, by testing (2.3.14) with such φ and combining Cauchy-Schwarz Inequality with Hölder's inequality (Lemma M), it follows

$$\begin{aligned} \mu(E) &= \int_E \varphi \, \mathrm{d}\mu \leq \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu \\ &= \int_{\mathbb{R}^n} \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, \mathrm{d}x \leq 2 \|g(|\nabla u|)\|_{L^{G^*}} \|\nabla \varphi\|_{L^G}. \end{aligned}$$

From (2.3.9) and Lemma L, we have

$$\mu(E) \leq 2\, q^{\frac{p}{p-1}} \left(\rho_G(|\nabla u|) + 1 \right) \max \left\{ \left(\rho_G(|\nabla \varphi|) \right)^{\frac{1}{p}}, \left(\rho_G(|\nabla \varphi|) \right)^{\frac{1}{q}} \right\},$$

where $\rho_G(\cdot)$ is the modular function. Consequently,

$$\mu(E) \le C\, \max\left\{\Big(\mathrm{cap}_G(E)\Big)^{\frac{1}{p}}, \Big(\mathrm{cap}_G(E)\Big)^{\frac{1}{q}}\right\},$$

with $C = C\Big(p, q, \rho_G(|\nabla u|)\Big) > 0$, which shows that μ is absolutely continuous with respect to the G-capacity. Next, since $q \ge n$, $\mathrm{cap}_G(E)$ vanishes for all compact set $E \subset \mathbb{R}^n$, whence $\mu = 0$ by the inner regularity of μ. From a Liouville-type theorem [Araya and Mohammed 2019, Theorem 4.1], u must be constant.

2.3.3 Superharmonic functions

Let $\mu \in \Big(W^{1,G}_{\mathrm{loc}}(\Omega)\Big)^*$. Here we extend the notion of the distributional solutions to (2.3.14), where u does not necessarily belong to $W^{1,G}_{\mathrm{loc}}(\Omega)$. To be more precise, we will understand solutions in the following potential-theoretic sense using G-superharmonic functions.

Definition 2.3.15. A function $u : \Omega \to (-\infty, \infty) \cup \{\infty\}$ is G-superharmonic in Ω if

$(\mathbf{i})$ u is lower semicontinuous,
$(\mathbf{ii})$ u is not identically infinite in any component of Ω,
$(\mathbf{iii})$ for each open subset V compactly contained in Ω and each G-harmonic function h in V such that $h \in C(\overline{V})$, and $h \le u$ in ∂V, implies that $h \le u$ in V.

We denote $\mathcal{S}_G(\Omega)$ the class of all G-superharmonic functions in Ω. Notice that for $G_p(t) = t^p/p$, one has $\mathcal{S}_{G_p}(\Omega) = \mathcal{S}_p(\Omega)$ for all $1 < p < \infty$.

For $u \in \mathcal{S}_G(\Omega)$ we define its truncation as follows

$$T_k(u) = \min(k, \max(u, -k)), \quad \forall k > 0.$$

From [Chlebicka and Zatorska-Goldstein 2022, Lemma 4.6], $\{T_k(u)\}$ is a sequence of G-supersolutions in Ω. Then there exists a unique measurable function $Z_u : \Omega \to \mathbb{R}^n$ satisfying

$$Z_u(x) = \lim_{k\to\infty} \nabla\Big(T_k(u)\Big)(x) \quad \text{almost everywhere in } \Omega.$$

We denote Z_u by Du and call it a *generalized gradient* of u. See details in [Chlebicka and Zatorska-Goldstein 2022, Remark 4.13]. If $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$, then clearly $Du = \nabla u$ since in this case $\nabla\Big(T_k(u)\Big) = \chi_{\{-k<u<k\}}\nabla u$.

Note that by the Riesz Representation Theorem [Lieb and Loss 2001, Theorem 6.22], there exists a unique measure $\mu[T_k(u)] \in M^+(\Omega)$ satisfying

$$\int_\Omega \frac{g(|\nabla T_k(u)|)}{|\nabla T_k(u)|}\nabla T_k(u)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}\mu[T_k(u)], \quad \forall\varphi\in C_c^\infty(\Omega), \varphi\ge 0.$$

On the other hand, by [Chlebicka and Zatorska-Goldstein 2022, Lemma 4.12], the sequence $\{g(\nabla T_k(u))\}$ is bounded in $L^1(B)$, for all open ball $B\subset\Omega$. Consequently, by Fatou's lemma, $g(|Du|)\in L^1_{\mathrm{loc}}(\Omega)$, and since $Du = \lim_k \nabla\Big(T_k(u)\Big)$ (pointwise), it holds

$$\int_\Omega \frac{g(|Du|)}{|Du|}Du\cdot\nabla\varphi\,\mathrm{d}x = \lim_{k\to\infty}\int_\Omega \frac{g(|\nabla T_k(u)|)}{|\nabla T_k(u)|}\nabla T_k(u)\cdot\nabla\varphi\,\mathrm{d}x, \quad \forall\varphi\in C_c^\infty(\Omega).$$

Therefore, by the Riesz Representation Theorem, there exists a unique measure $\mu = \mu[u] \in M^+(\Omega)$ such that

$$\int_\Omega \frac{g(|Du|)}{|Du|}Du\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}\mu, \quad \forall\varphi\in C_c^\infty(\Omega), \varphi\ge 0.$$

This means

$$-\mathrm{div}\bigg(\frac{g(|Du|)}{|Du|}Du\bigg) = \mu \quad \text{in} \quad \Omega.$$

In the literature, $\mu[u]$ is called the *Riesz measure* of u.

Definition 2.3.16. For $\sigma\in M^+(\Omega)$, we say that u is a solution in the potential-theoretic sense to the equation

$$-\mathrm{div}\bigg(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\bigg) = \sigma \quad \text{in} \quad \Omega$$

if $u\in\mathcal{S}_G(\Omega)$ and $\mu[u]=\sigma$.

Let $f(t) = g(t^\gamma)$, $t\ge 0$. In light of Definition 2.3.16, if $\sigma\in M^+(\Omega)$, then a function u is a solution (in the potential-theoretic sense) to the equation

$$-\mathrm{div}\bigg(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\bigg) = \sigma\, f(u) \quad \text{in} \quad \Omega \tag{2.3.15}$$

whenever u is nonnegative and it satisfies

$$\begin{cases} u\in\mathcal{S}_G(\Omega)\cap L^f_{\mathrm{loc}}(\Omega,\mathrm{d}\sigma), \\ \mathrm{d}\mu[u] = f(u)\mathrm{d}\sigma. \end{cases} \tag{2.3.16}$$

Definition 2.3.17. Let $\sigma\in M^+(\mathbb{R}^n)$. A function u is a supersolution to (P_2) if u is nonnegative and satisfies

$$\begin{cases} u\in\mathcal{S}_G(\mathbb{R}^n)\cap L^f_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}\sigma), \\ \displaystyle\int_{\mathbb{R}^n} \frac{g(|Du|)}{|Du|}Du\cdot\nabla\varphi\,\mathrm{d}x \ge \int_{\mathbb{R}^n}\varphi f(u)\,\mathrm{d}\sigma \quad \forall\varphi\in C_c^\infty(\mathbb{R}^n), \varphi\ge 0. \end{cases} \tag{2.3.17}$$

Finally, the notion of solution to (P_2) is defined similarly by replacing "$\ge$" with "$=$" in (2.3.17).

Definition 2.3.18. Let $\sigma \in M^+(\mathbb{R}^n)$. A function u is a solution of finite energy to (P_2) whenever u is a nonnegative solution to (P_2) and $u \in \mathcal{D}^{1,G}(\mathbb{R}^n) \cap L^F(\mathbb{R}^n, \mathrm{d}\sigma)$.

Next, we will state some fundamental results of the potential theory of quasilinear elliptic equations with Orlicz growth, like (2.3.14). We start with the following theorem that will be used to prove that a pointwise limit of a sequence of G-superharmonic functions is a G-superharmonic function [Chlebicka and Zatorska-Goldstein 2022, Theorem 2].

Theorem R (Harnack's Principle)**.** *Let $\{u_j\}$ be a sequence of G-superharmonic functions, with each u_j finite almost everywhere in Ω. If $\{u_j\}$ is nondecreasing, then the pointwise limit function $u = \lim_j u_j$ is G-superharmonic function, unless $u \equiv \infty$. Moreover, if u_j is nonnegative for all $j \geq 1$, then up to a subsequence one has $Du = \lim_j Du_j$ in the set $\{u < \infty\}$.*

The following theorem describes the main technical result which will be decisive in linking supersolution to (P_2) with solutions to (S), provided g is the function given by (A_2). Its proof is postponed to Appendix A.

Theorem 2.3.19. *Let g be the function given by (A_2). Suppose that u is a G-superharmonic function in $B(x_0, 2R)$, and let $\mu = \mu[u] \in M^+(B(x_0, 2R))$. Then there exist constants $C_1 > 0$ and $C_2 > 0$ depending only on n, p and q such that*

$$C_1 \mathbf{W}_G^R \mu(x_0) \leq u(x_0) \leq C_2 \Big(\inf_{B(x_0,R)} u + \mathbf{W}_G^R \mu(x_0) \Big). \tag{2.3.18}$$

Here $\mathbf{W}_G^R \sigma$ is the R-truncated Wolff potential of $\sigma \in M^+(\mathbb{R}^n)$ defined by

$$\mathbf{W}_G^R \sigma(x) = \int_0^R g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}} \right) \mathrm{d}t, \quad x \in \mathbb{R}^n.$$

For our purpose, it will need the following consequence of Theorem 2.3.19.

Corollary 2.3.20. *Let g be the function given by (A_2). Suppose that u is a G-superharmonic function in $\mathbb{R}^n$ with $\inf_{\mathbb{R}^n} u = 0$, and let $\mu = \mu[u] \in M^+(\mathbb{R}^n)$. Then there exists constant $K \geq 1$ depending only on n, p and q such that*

$$K^{-1} \, \mathbf{W}_G \mu(x) \leq u(x) \leq K \, \mathbf{W}_G \mu(x) \quad \forall x \in \mathbb{R}^n.$$

Proof. Fix $x \in \mathbb{R}^n$. Then clearly u is G-superharmonic in $B(x, 2R)$ for all $R > 0$. By Theorem 2.3.19, there exist constants $C_1 = C_1(n,p,q) > 0$ and $C_2 = C_2(n,p,q) > 0$ such that

$$C_1 \mathbf{W}_G^R \mu(x) \leq u(x) \leq C_2 \Big(\inf_{B(x,R)} u + \mathbf{W}_G^R \mu(x) \Big) \quad \forall R > 0.$$

Since $\inf_{\mathbb{R}^n} u = 0$, $\lim_{R\to\infty} \inf_{B(x,R)} u = 0$. Consequently, being C_1 and C_2 independent of R, letting $R \to \infty$ in the previous bounds, we arrive at

$$C_1 \mathbf{W}_G \mu(x) \le u(x) \le C_2 \mathbf{W}_G \mu(x).$$

Setting $K = \max\{C_2, (C_1)^{-1}, 1\}$, Corollary 2.3.20 is proved since x was arbitrary. □

The following lemma brings the relationship between G-superhamonic functions in Ω and G-supersolutions to eq. (2.3.13). To be more precise, we have that a G-supersolution to eq. (2.3.14) is always a G-superhamonic function (almost everywhere).

Lemma 2.3.21. *Suppose that u is a G-supersolution to eq.* (2.3.14) *in Ω satisfying*

$$u(x) = \operatorname{ess} \varliminf_{y\to x} u(y) \quad \textit{for all } x \in \Omega. \tag{2.3.19}$$

Then u is a G-superhamonic function in Ω

Proof. The proof is similar in spirit to [Heinonen, Kilpeläinen and Martio 2006, Theorem 7.16]. Let $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ be a G-supersolution to eq. (2.3.14). By [Harjulehto and Hästö 2019, Lemma 6.1.6], $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, which implies that u is locally essentially bounded from below and from above, i.e. $-\infty < u < \infty$ locally almost everywhere. In particular, u cannot be identically ∞ in any component of Ω. The lower semicontinuity of u follows from(2.3.19).

Now, let V be an open subset compactly contained in Ω and $h \in C(\overline{V})$ be such that $h \le u$ on ∂V. Fix $\varepsilon > 0$ and choose an open set U compactly contained in V such that $u > h - \varepsilon$ in $V \setminus U$. Since the function $w := \min\{u + \varepsilon - h, 0\}$ has compact support in $V \setminus U$, it holds $w \in W^{1,G}_0(V)$ (see Lemma Z below). Consequently, the comparison principle (Lemma P) ensures that $u \ge h - \varepsilon$ almost everywhere in V. Using (2.3.19) again,

$$u(x) = \operatorname{ess} \varliminf_{y\to x} u(y) \ge \operatorname{ess} \varliminf_{y\to x} h(y) - \varepsilon = h(x) - \varepsilon \quad \forall x \in V.$$

Therefore, the lemma is proved since ε was arbitrary. □

Remark 2.3.22. We show in the previous lemma that each G-supersolution u in Ω can be redefined in a set of Lebesgue-measure zero such that the property (2.3.19) holds. Accordingly, the function

$$\tilde{u}(x) := \operatorname{ess} \varliminf_{y\to x} u(x), \quad x \in \Omega,$$

is a G-superharmonic function in Ω satisfying $\tilde{u} = u$ almost everywhere in Ω. From this, it follows that each G-supersolution can be redefined in a set of measure zero such that the previous limit holds. Thereby, a G-supersolution will be treated as a G-superharmonic function. In particular, when $\Omega = \mathbb{R}^n$, Corollary 2.3.20 holds for G-supersolutions in $\mathbb{R}^n$.

The following result is the Orlicz growth version of [Hedberg and Wolff 1983, Theorem 1] (Wolff's inequality). This result is [Chlebicka, Giannetti and Zatorska-Goldstein 2023, Theorem 3], and it gives a condition to nonnegative Radon measure μ belonging to $\left(W_0^{1,G}(\Omega)\right)^*$ in terms of the Wolff potential $\mathbf{W}_G\mu$, provided Ω is bounded and $\operatorname{supp}\mu \subset \Omega$. Here $\left(W_0^{1,G}(\Omega)\right)^*$ means the dual space of $W_0^{1,G}(\Omega)$.

Theorem S. *Suppose that Ω is bounded. Let $\mu \in M^+(\Omega)$ with $\operatorname{supp}\mu \subset \Omega$. Then*

$$\mu \in \left(W_0^{1,G}(\Omega)\right)^* \iff \int_\Omega \mathbf{W}_G^R\mu \,\mathrm{d}\mu < \infty \quad \textit{for some } R > 0.$$

3 SYSTEM WITH WOLFF POTENTIALS AND APPLICATIONS

This chapter is devoted to providing the proof of Theorems 1.3.1, 1.3.3, 1.3.6, 1.3.8, 1.3.9 1.3.11, 1.3.12, and Corollary 1.3.13. Our strategy is based on linking Syst. (SI) with Systs. (S_1) and (S_2) in view of the potential estimate Theorems D and G, respectively. We divide our arguments into three sections in the sequel, where we prove the existence of solutions Syst. (SI), Syst. (S_1) and Syst. (S_2), respectively. We also indicate some questions related to Syst. (S_1).

3.1 SYSTEM WITH WOLFF POTENTIALS

Definition 3.1.1. We say that a pair of nonnegative functions $(v, u) \in L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma) \times L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ is a *supersolution* to (SI) if satisfies (pointwise)

$$\begin{cases} u(x) \geq \mathbf{W}_{\alpha,p}\left(v^{q_1}\mathrm{d}\sigma\right)(x), & \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n, \\ v(x) \geq \mathbf{W}_{\alpha,p}\left(u^{q_2}\mathrm{d}\sigma\right)(x), & \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n. \end{cases} \tag{3.1.1}$$

The notion of *solution* or *subsolution* to (SI) is defined similarly by replacing "$\geq$" by "$=$" or "$\leq$" in (3.1.1), respectively.

We start showing that if there exists a nontrivial supersolution (u, v) to (SI), then σ must be absolutely continuous with respect to the (α, p)-capacity $\mathrm{cap}_{\alpha,p}$, see (1.3.1). In view of Theorem D and next lemma we see that if Syst. (S_1) has a nontrivial p-superharmonic supersolution, then σ is absolutely continuous with respect to the p-capacity cap_p, since $\mathrm{cap}_p(E) \approx \mathrm{cap}_{1,p}(E)$ for all compact sets E [Adams and Hedberg 1996, Proposition 2.3.13].

Lemma 3.1.2. *Let* $1 < p < \infty,\ 0 < q_i < p-1,\ i = 1, 2,\ 0 < \alpha < n/p$, *and* $\sigma \in M^+(\mathbb{R}^n)$. *Suppose there is a nontrivial supersolution* (u, v) *to* (SI). *Then there exists a positive constant* C *depending only on* n, p, q_1, q_2 *and* α *such that for every compact set* $E \subset \mathbb{R}^n$,

$$\sigma(E) \leq C\Big[\mathrm{cap}_{\alpha,p}(E)^{\frac{q_1}{p-1}}\Big(\int_E v^{q_1}\,\mathrm{d}\sigma\Big)^{\frac{p-1}{p-1-q_1}} + \mathrm{cap}_{\alpha,p}(E)^{\frac{q_2}{p-1}}\Big(\int_E u^{q_2}\,\mathrm{d}\sigma\Big)^{\frac{p-1}{p-1-q_2}}\Big]. \tag{3.1.2}$$

Proof. We first recall the following result [Verbitsky 1999, Theorem 1.11]: for any $\mu \in M^+(\mathbb{R}^n)$ it holds

$$\int_E \frac{\mathrm{d}\mu}{(\mathbf{W}_{\alpha,p}\mu)^{p-1}} \leq C_0\,\mathrm{cap}_{\alpha,p}(E),$$

where $C_0 = C_0(n,p,\alpha)$ is a positive constant. Taking $\mathrm{d}\mu = v^{q_1}\mathrm{d}\sigma$, we obtain

$$\int_E v^{q_1} u^{-(p-1)}\,\mathrm{d}\sigma \le \int_E \frac{\mathrm{d}\mu}{(\mathbf{W}_{\alpha,p}\mu)^{p-1}} \le C_0\,\mathrm{cap}_{\alpha,p}(E),$$

since $u \ge \mathbf{W}_{\alpha,p}\mu$. Thus,

$$\begin{aligned}\int_{E\cap\{v\ge u\}} v^{q_1-p+1}\,\mathrm{d}\sigma &\le \int_{E\cap\{v\ge u\}} v^{q_1}u^{-(p-1)}\,\mathrm{d}\sigma \\ &\le \int_E v^{q_1}u^{-(p-1)}\,\mathrm{d}\sigma \le C_0\,\mathrm{cap}_{\alpha,p}(E).\end{aligned} \tag{3.1.3}$$

Setting $\beta = q_1(p-1-q_1)/(p-1)$ and using the Hölder's inequality with exponents $r = (p-1)/q_1$ and $r' = (p-1)/(p-1-q_1)$, we deduce

$$\begin{aligned}\sigma(E\cap\{v\ge u\}) &= \int_{E\cap\{v\ge u\}} v^{-\beta}v^{\beta}\,\mathrm{d}\sigma \\ &\le \left(\int_{E\cap\{v\ge u\}} v^{-\beta r}\,\mathrm{d}\sigma\right)^{\frac{1}{r}} \left(\int_{E\cap\{v\ge u\}} v^{\beta r'}\,\mathrm{d}\sigma\right)^{\frac{1}{r'}} \\ &= \left(\int_{E\cap\{v\ge u\}} v^{q_1-p+1}\,\mathrm{d}\sigma\right)^{\frac{q_1}{p-1}} \left(\int_{E\cap\{v\ge u\}} v^{q_1}\,\mathrm{d}\sigma\right)^{\frac{p-1-q_1}{p-1}},\end{aligned}$$

since $-\beta r = q_1 - p + 1$ and $\beta r' = q_1$. By (3.1.3),

$$\sigma(E\cap\{v\ge u\}) \le C_0^{\frac{q_1}{p-1}} \left(\mathrm{cap}_{\alpha,p}(E)\right)^{\frac{q_1}{p-1}} \left(\int_E v^{q_1}\,\mathrm{d}\sigma\right)^{\frac{p-1-q_1}{p-1}}.$$

Similarly, we also obtain

$$\int_{E\cap\{u\ge v\}} u^{q_2-p+1}\,\mathrm{d}\sigma \le C_0\,\mathrm{cap}_{\alpha,p}(E),$$

and consequently

$$\sigma(E\cap\{u\ge v\}) \le C_0^{\frac{q_2}{p-1}} \left(\mathrm{cap}_{\alpha,p}(E)\right)^{\frac{q_2}{p-1}} \left(\int_E u^{q_2}\,\mathrm{d}\sigma\right)^{\frac{p-1-q_2}{p-1}}.$$

Since $\sigma(E) \le \sigma(E\cap\{v\ge u\}) + \sigma(E\cap\{u\ge v\})$, picking $C = \max\{C_0, 1\}$, (3.1.2) follows. □

Notice that in Syst. (3.1.1), placing the second inequality $v \ge \mathbf{W}_{\alpha,p}(u^q\mathrm{d}\sigma)$ in the first one, we obtain

$$u \ge \mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}(u^{q_2}\mathrm{d}\sigma))^{q_1}\,\mathrm{d}\sigma\right) \quad \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n. \tag{3.1.4}$$

Thus, if (u,v) solves (3.1.1), then u solves (3.1.4). In the next Lemma, we will use this perception to obtain a lower bound for supersolutions of (SI) in terms of Wolff potentials.

Lemma 3.1.3. *If (u,v) is a nontrivial supersolution to (SI), then there exist a positive constant c, which depends only on n, p, α, q_1 and q_2, such that the inequalities below holds*

$$u(x) \geq C\left(\mathbf{W}_{\alpha,p}\sigma(x)\right)^{\frac{(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}}, \qquad \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n,$$
$$v(x) \geq C\left(\mathbf{W}_{\alpha,p}\sigma(x)\right)^{\frac{(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}}, \qquad \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n.$$

Before proving Lemma 3.1.3, let us recall the following result [Cao and Verbitsky 2017, Lemma 3.5].

Lemma T. *Let $\omega \in M^+(\mathbb{R}^n)$. For every $r>0$ and for all $x \in \mathbb{R}^n$, it holds*

$$\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\omega)^r \mathrm{d}\omega\right)(x) \geq \kappa^{\frac{r}{p-1}}\left(\mathbf{W}_{\alpha,p}\omega(x)\right)^{\frac{r}{p-1}+1}, \tag{3.1.5}$$

where κ depends only on n, p and α.

Proof of Lemma 3.1.3. We will prove the lower estimate for u, the first coordinate of (u,v). The lower estimate for v is entirely analogous. First, we prove the following claim.

Claim 1. Let $\sigma \in M^+(\mathbb{R}^n)$ and let w be a nontrivial solution to (3.1.4). If there exist $c>0$ and $\delta>0$ such that

$$w(x) \geq c\left(\mathbf{W}_{\alpha,p}\sigma(x)\right)^{\delta}, \quad \forall x \in \mathbb{R}^n,$$

then

$$w(x) \geq c^{\frac{q_1q_2}{(p-1)^2}}\kappa^{\frac{q_1(p-1+2q_2\delta)}{(p-1)^2}}\left(\mathbf{W}_{\alpha,p}\sigma(x)\right)^{\frac{q_1}{p-1}\left(\frac{q_2}{p-1}\delta+1\right)+1}, \quad \forall x \in \mathbb{R}^n, \tag{3.1.6}$$

where $\kappa = \kappa(n,p,\alpha) > 0$ is the constant in Lemma T.

Indeed, if $w \geq c\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\delta}$, we can estimate for all $x \in \mathbb{R}^n$,

$$\begin{aligned} w(x) &= \mathbf{W}_{\alpha,p}\left(\left(\mathbf{W}_{\alpha,p}(u^{q_2}\mathrm{d}\sigma)\right)^{q_1}\mathrm{d}\sigma\right)(x) \\ &\geq c^{\frac{q_1q_2}{(p-1)^2}}\mathbf{W}_{\alpha,p}\left[\left(\mathbf{W}_{\alpha,p}(\mathbf{W}_{\alpha,p}\sigma)^{q_2\delta}\mathrm{d}\sigma\right)\right)^{q_1}\mathrm{d}\sigma\right](x). \end{aligned}$$

Using Lemma T twice with $\omega=\sigma$, $r=q_2\delta$ and $r=(\delta q_2/(p-1)+1)q_1$ respectively in the previous estimate, we deduce that

$$\begin{aligned} w(x) &\geq c^{\frac{q_1q_2}{(p-1)^2}}\mathbf{W}_{\alpha,p}\left[\left(\kappa^{\frac{q_2}{p-1}\delta}\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\frac{q_2}{p-1}\delta+1}\right)^{q_1}\mathrm{d}\sigma\right](x) \\ &= c^{\frac{q_1q_2}{(p-1)^2}}\kappa^{\frac{q_1q_2}{(p-1)^2}\delta}\mathbf{W}_{\alpha,p}\left[\left(\mathbf{W}_{\alpha,p}\sigma\right)^{\left(\frac{q_2}{p-1}\delta+1\right)q_1}\mathrm{d}\sigma\right](x) \\ &\geq c^{\frac{q_1q_2}{(p-1)^2}}\kappa^{\frac{q_1q_2}{(p-1)^2}\delta}\kappa^{\frac{q_1}{p-1}\left(\frac{q_2}{p-1}\delta+1\right)}\left(\mathbf{W}_{\alpha,p}\sigma(x)\right)^{\frac{q_1}{p-1}\left(\frac{q_2}{p-1}\delta+1\right)+1}, \end{aligned}$$

which completes the proof of Claim 1.

Now, fix $x \in \mathbb{R}^n$ and $R > |x|$. Let $B = B(0, R)$, $\sigma_B = \chi_B \sigma$ and $\mu \in M^+(\mathbb{R}^n)$ defined by $\mu = u^{q_2}\sigma_B$. From (3.1.4),

$$\begin{aligned} u(x) &\geq \mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}(u^{q_2}\mathrm{d}\sigma))^{q_1}\,\mathrm{d}\sigma\right)(x) \\ &\geq \mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\mu)^{q_1}\,\mathrm{d}\sigma\right)(x) \\ &\geq \mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\mu)^{q_1}\,\mathrm{d}\sigma_B\right)(x). \end{aligned} \tag{3.1.7}$$

We first obtain a lower bound for $\mathbf{W}_{\alpha,p}\mu(z)$,

$$\begin{aligned} \mathbf{W}_{\alpha,p}\mu(z) &= \int_0^\infty \left(\frac{\mu(B(z,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \geq \int_R^\infty \left(\frac{\mu(B(z,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\ &= c_0 \int_R^\infty \left(\frac{\mu(B(z,2s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}s}{s}, \end{aligned}$$

where $c_0 = 2^{-(n-\alpha p)/(p-1)}$. Note that if $z \in B$ and $s \geq R$, $B(0,s) \subset B(z,2s)$. Thus for $z \in B$

$$\begin{aligned} \mathbf{W}_{\alpha,p}\mu(z) &\geq 2^{-\frac{n-\alpha p}{p-1}} \int_R^\infty \left(\frac{\mu(B(0,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}s}{s} = 2^{-\frac{n-\alpha p}{p-1}} \int_R^\infty \left(\frac{\int_{B(0,s)} u^{q_2}\mathrm{d}\sigma_B}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}s}{s} \\ &= 2^{-\frac{n-\alpha p}{p-1}} \int_R^\infty \left(\frac{\int_{B(0,s)\cap B} u^{q_2}\mathrm{d}\sigma}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}s}{s} = 2^{-\frac{n-\alpha p}{p-1}} \int_R^\infty \left(\frac{\int_B u^{q_2}\mathrm{d}\sigma}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}s}{s} \\ &= 2^{-\frac{n-\alpha p}{p-1}} \left(\int_B u^{q_2}\mathrm{d}\sigma\right)^{\frac{1}{p-1}} \int_R^\infty \left(\frac{1}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}s}{s} =: A(R) \end{aligned} \tag{3.1.8}$$

Let $c_1 = A(R)^{q_1/(p-1)}$. By (3.1.7), it follows from (3.1.8) that

$$u(x) \geq c_1 \mathbf{W}_{\alpha,p}\sigma_B(x). \tag{3.1.9}$$

With the aid of Claim 1, where we consider σ_B in place of σ and $\delta = 1$, we obtain from (3.1.9) that

$$u(x) \geq c_2 \left(\mathbf{W}_{\alpha,p}\sigma_B(x)\right)^{\delta_2},$$

where

$$c_2 = c_1^{\frac{q_1 q_2}{(p-1)^2}} \kappa^{\frac{q_1(p-1+2q_2)}{(p-1)^2}}, \quad \delta_2 = \frac{q_1}{p-1}\left(\frac{q_2}{p-1}+1\right)+1.$$

In fact, setting $\delta_1 = 1$ and c_1 as above, iterating (3.1.7) and (3.1.9) with Claim 1, we concluded that

$$u(x) \geq c_j \left(\mathbf{W}_{\alpha,p}\sigma_B(x)\right)^{\delta_j}, \tag{3.1.10}$$

where δ_j and c_j, for $j = 2, 3, \ldots$, are given by

$$\delta_j = \frac{q_1}{p-1}\left(\frac{q_2}{p-1}\delta_{j-1}+1\right)+1, \tag{3.1.11}$$

$$c_j = c_{j-1}^{\frac{q_1 q_2}{(p-1)^2}} \kappa^{\frac{q_1(p-1+2q_2\delta_{j-1})}{(p-1)^2}}. \tag{3.1.12}$$

If $\gamma_1 = \lim_{j\to\infty} \delta_j$ and $C_1 = \lim_{j\to\infty} c_j$, letting $j \to \infty$ in (3.1.11) and (3.1.12) is a straightforward computation to conclude that

$$\gamma_1 = \frac{(p-1)(p-1+q_1)}{(p-1)^2 - q_1q_2} \quad \text{and} \quad C_1 = \kappa^{\frac{q_1(p-1)[(p-1)^2+2(p-1)q_2+q_1q_2]}{[(p-1)^2-q_1q_2]^2}}. \tag{3.1.13}$$

Hence letting $j \to \infty$ in (3.1.10), we obtain

$$u(x) \geq C_1 \left(\mathbf{W}_{\alpha,p}\sigma_B(x)\right)^{\gamma_1}, \tag{3.1.14}$$

where we recall that $B = B(0,R)$ and $R > |x|$. Finally, letting $R \to \infty$ in (3.1.14) yields

$$u(x) \geq C_1 \left(\mathbf{W}_{\alpha,p}\sigma(x)\right)^{\frac{(p-1)(p-1+q_1)}{(p-1)^2-q_1q_2}}, \quad \forall x \in \mathbb{R}^n.$$

The same arguments, replacing q_1 by q_2 in (3.1.4) and u by v, yield the lower estimate for v, that is,

$$v(x) \geq \tilde{c}_j \left(\mathbf{W}_{\alpha,p}\sigma_B(x)\right)^{\tilde{\delta}_j}, \tag{3.1.15}$$

where

$$\begin{aligned}
&\tilde{\delta}_1 = 1,\\
&\tilde{c}_1 = 2^{-\frac{n-\alpha p}{p-1}} \left(\int_B v^{q_1} \mathrm{d}\sigma\right)^{\frac{1}{p-1}} \int_R^\infty \left(\frac{1}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}s}{s},\\
&\tilde{\delta}_j = \frac{q_2}{p-1}\left(\frac{q_1}{p-1}\tilde{\delta}_{j-1} + 1\right) + 1, &(3.1.16)\\
&\tilde{c}_j = \tilde{c}_{j-1}^{\frac{q_1q_2}{(p-1)^2}} \kappa^{\frac{q_2(p-1+2q_1\tilde{\delta}_{j-1})}{(p-1)^2}},\ j = 2, 3, \ldots &(3.1.17)
\end{aligned}$$

If $\gamma_2 = \lim_{j\to\infty} \tilde{\delta}_j$ and $\tilde{C}_1 = \lim_{j\to\infty} \tilde{c}_j$, letting $j \to \infty$ in (3.1.16) and (3.1.17), a straightforward computation yields

$$\gamma_2 = \frac{(p-1)(p-1+q_2)}{(p-1)^2 - q_1q_2} \quad \text{and} \quad \tilde{C}_1 = \kappa^{\frac{q_2(p-1)[(p-1)^2+2(p-1)q_1+q_1q_2]}{[(p-1)^2-q_1q_2]^2}}. \tag{3.1.18}$$

Thus, taking the limits $j \to \infty$ and $R \to \infty$ respectively in (3.1.15),

$$v(x) \geq \tilde{C}_1 \left(\mathbf{W}_{\alpha,p}\sigma(x)\right)^{\frac{(p-1)(p-1+q_2)}{(p-1)^2-q_1q_2}}, \quad \forall x \in \mathbb{R}^n.$$

Setting $C = \min\{C_1, \tilde{C}_1\}$, we complete the proof of Lemma 3.1.3. □

The following Lemma is a consequence of (2.0.2).

Lemma 3.1.4. *Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying (1.0.4) and (2.0.1). Then, for all $r > 0$, it holds*

$$\mathbf{W}_{\alpha,p}\sigma \in L^r_{\text{loc}}(\mathbb{R}^n, \mathrm{d}\sigma).$$

Proof. It is enough to prove that $\mathbf{W}_{\alpha,p}\sigma \in L^r(B, \mathrm{d}\sigma)$, for all balls $B \subset \mathbb{R}^n$. Fixed $B = B(x_0, R)$, setting $2B = B(x_0, 2R)$ and denoting by $(2B)^c$ the complement of $2B$ in $\mathbb{R}^n$, we can write

$$\begin{aligned}\int_B (\mathbf{W}_{\alpha,p}\sigma)^r \,\mathrm{d}\sigma &= \int_B \left(\int_0^\infty \left(\frac{\big(\sigma_{2B} + \sigma_{(2B)^c}\big)(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \mathrm{d}t \right)^r \mathrm{d}\sigma \\ &= \int_B \left(\int_0^\infty \left(\frac{\sigma_{2B}(B(x,t)) + \sigma_{(2B)^c}(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \mathrm{d}t \right)^r \mathrm{d}\sigma.\end{aligned}$$

Using the elementary inequality: given $r > 0$, it holds

$$|a_1 + a_2|^r \leq 2^r \left(|a_1|^r + |a_2|^r \right), \quad \forall a_1, a_2 \in \mathbb{R}, \tag{3.1.19}$$

we have

$$\begin{aligned}\int_B (\mathbf{W}_{\alpha,p}\sigma)^r \,\mathrm{d}\sigma \leq & 2^{\frac{r}{p-1}} \int_B \left(\int_0^\infty \left(\frac{\sigma_{2B}(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \mathrm{d}t + \int_0^\infty \left(\frac{\sigma_{(2B)^c}(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \mathrm{d}t \right)^r \mathrm{d}\sigma \\ \leq & c_1 (I_1 + I_2),\end{aligned}$$

where

$$I_1 := \int_B (\mathbf{W}_{\alpha,p}\sigma_{2B})^r \,\mathrm{d}\sigma \quad \text{and} \quad I_2 := \int_B \left(\mathbf{W}_{\alpha,p}\sigma_{(2B)^c} \right)^r \mathrm{d}\sigma$$

and $c_1 = c_1(p, r) > 0$. The estimate of I_1 is a consequence of (2.0.2). Indeed,

$$I_1 \leq c_1 \int_{2B} (\mathbf{W}_{\alpha,p}\sigma_{2B})^r \,\mathrm{d}\sigma \leq c_2\, \sigma(2B) < \infty,$$

where $c_2 = c_2(n, p, r, \alpha, c_\sigma)$.

To estimate I_2, we first observe that $(2B)^c \cap B(y,t) = \emptyset$ for $y \in B$ and $0 < t < R$, whence for $y \in B$

$$\begin{aligned}\mathbf{W}_{\alpha,p}\sigma_{(2B)^c}(y) &= \int_0^\infty \left(\frac{\sigma((2B)^c \cap B(y,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \\ &= \int_R^\infty \left(\frac{\sigma((2B)^c \cap B(y,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \leq \int_R^\infty \left(\frac{\sigma(B(y,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}.\end{aligned}$$

Since $B(y,t) \subset B(0, 2t)$ for $y \in B$ and $t \geq R$, it follows

$$\mathbf{W}_{\alpha,p}\sigma_{(2B)^c}(y) \leq \int_R^\infty \left(\frac{\sigma(B(0,2t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \leq 2^{\frac{n-\alpha p}{p-1}} \int_R^\infty \left(\frac{\sigma(B(0,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}.$$

Applying the previous inequality,

$$I_2 = c_1 \int_B \left(\mathbf{W}_{\alpha,p}\sigma_{(2B)^c} \right)^r \mathrm{d}\sigma \leq c_3 \left(\int_R^\infty \left(\frac{\sigma(B(0,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \right)^r \sigma(B) < \infty.$$

This completes the proof of Lemma 3.1.4. □

3.1.1 Proof of Theorem 1.3.1

We will proceed by the method of sub- and super-solutions. We begin recalling $\gamma_1,\ \gamma_2$ given in (3.1.13) and (3.1.18), that is

$$\gamma_1 = \frac{(p-1)(p-1+q_1)}{(p-1)^2 - q_1 q_2} \quad \text{and} \quad \gamma_2 = \frac{(p-1)(p-1+q_2)}{(p-1)^2 - q_1 q_2},$$

which can be rewritten as follows

$$\gamma_1 = \frac{q_1}{p-1}\gamma_2 + 1 \quad \text{and} \quad \gamma_2 = \frac{q_2}{p-1}\gamma_1 + 1.$$

Claim 2. There exists $\lambda_1 > 0$ sufficiently small such that

$$(\underline{u}, \underline{v}) = \Big(\lambda_1 (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}, \lambda_1 (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\Big)$$

is a subsolution to (SI).

Indeed, using Lemma T,

$$\begin{aligned}
\mathbf{W}_{\alpha,p}(\underline{v}^{q_1}\mathrm{d}\sigma) &= \lambda_1^{\frac{q_1}{p-1}} \mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^{q_1\gamma_2}\mathrm{d}\sigma) \\
&\geq \lambda_1^{\frac{q_1}{p-1}} \kappa^{\frac{q_1}{p-1}\gamma_2} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{q_1}{p-1}\gamma_2 + 1} \\
&= \lambda_1^{\frac{q_1}{p-1}} \kappa^{\frac{q_1}{p-1}\gamma_2} (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}, \\
\mathbf{W}_{\alpha,p}(\underline{u}^{q_2}\mathrm{d}\sigma) &= \lambda_1^{\frac{q_2}{p-1}} \mathbf{W}_{\alpha,p}((\mathbf{W}_{\alpha,p}\sigma)^{q_2\gamma_1}\mathrm{d}\sigma) \\
&\geq \lambda_1^{\frac{q_2}{p-1}} \kappa^{\frac{q_2}{p-1}\gamma_1} (\mathbf{W}_{\alpha,p}\sigma)^{\frac{q_2}{p-1}\gamma_1 + 1} \\
&= \lambda_1^{\frac{q_2}{p-1}} \kappa^{\frac{q_2}{p-1}\gamma_1} (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}.
\end{aligned}$$

Now, choosing $\lambda_1 = \min\{\kappa^{(q_1\gamma_2)/(p-1-q_1)}, \kappa^{(q_2\gamma_2)/(p-1-q_2)}\}$, we deduce

$$\begin{aligned}
\underline{u} &\leq \mathbf{W}_{\alpha,p}(\underline{v}^{q_1}\mathrm{d}\sigma), \\
\underline{v} &\leq \mathbf{W}_{\alpha,p}(\underline{u}^{q_2}\mathrm{d}\sigma),
\end{aligned}$$

which completes the proof of Claim 2.

Claim 3. There exists $\lambda_2 > 0$ sufficiently large such that

$$(\overline{u}, \overline{v}) = \Big(\lambda_2\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}\right), \lambda_2\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right)\Big)$$

is a supersolution to (SI), such that $\overline{u} \geq \underline{u}$ and $\overline{v} \geq \underline{v}$.

Indeed, we begin by construction upper bounds to $\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma)$ and $\mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma)$. For the first one, we need to obtain two estimates, which will be done in two steps. First, arguing as in the proof of Lemma 3.1.4, one can check that there exists $c_1 = c_1(p, q_1, \sigma) > 0$ such that, for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
\int_0^\infty \Big(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{q_1}\,\mathrm{d}\sigma}{t^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t} &\leq \int_0^\infty \Big(\frac{c_1\sigma(B(x,2t))+c_1\sigma(B(x,t))}{t^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t}\\
&\leq c_2\int_0^\infty \Big(\frac{\sigma(B(x,2t))}{t^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t} + c_2\,\mathbf{W}_{\alpha,p}\sigma(x)\\
&\leq c_2\,2^{\frac{n-\alpha p}{p-1}}\int_0^\infty \Big(\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t} + c_2\,\mathbf{W}_{\alpha,p}\sigma(x)\\
&\leq c_3\mathbf{W}_{\alpha,p}\sigma(x), \qquad (3.1.20)
\end{aligned}$$

where $c_3 = c_3(n, p, q_1, \sigma) > 0$. The second step consists of the following estimate

$$\begin{aligned}
\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma &= \int_{B(x,t)}\Big[\int_0^\infty \Big(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y)\\
&\leq c_4\int_{B(x,t)}\Big[\int_0^t \Big(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y)\\
&\quad + c_4\int_{B(x,t)}\Big[\int_t^\infty \Big(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y) =: c_4\,(I_1+I_2)\,,
\end{aligned}$$

where $c_4 = c_4(p, q_1, q_2) > 0$. For $y \in B(x,t)$ and $r \leq t$, we have $B(y,r) \subset B(x,2t)$, whence

$$\begin{aligned}
I_1 &= \int_{B(x,t)}\Big[\int_0^t \Big(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y)\\
&\leq \int_{B(x,2t)}\Big[\int_0^t \Big(\frac{\sigma(B(y,r)\cap B(x,2t))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y)\\
&\leq \int_{B(x,2t)}\Big[\mathbf{W}_{\alpha,p}\sigma_{B(x,2t)}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y).
\end{aligned}$$

By (2.0.2), we deduce

$$I_1 \leq \int_{B(x,2t)}\Big[\mathbf{W}_{\alpha,p}\sigma_{B(x,2t)}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y) \leq c_5\,\sigma(B(x,2t)),$$

where $c_5 = c_5(n, p, q_1, q_2, C_\sigma)$. Now, for $r \geq t$, we have $B(y,r) \subset B(x,2r)$, and consequently

$$\begin{aligned}
I_2 &\leq \int_{B(x,t)}\Big[\int_t^\infty \Big(\frac{\sigma(B(x,2r))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y)\\
&= \sigma(B(x,t))\Big[\int_t^\infty \Big(\frac{\sigma(B(x,2r))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\\
&\leq \sigma(B(x,t))\Big[\int_0^\infty \Big(\frac{\sigma(B(x,2r))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1} \leq c_6\,\sigma(B(x,t))\,[\mathbf{W}_{\alpha,p}\sigma(x)]^{\gamma_2 q_1}\,,
\end{aligned}$$

where $c_6 = c_6(n,p,\alpha,q_1,q_2) = 2^{((n-\alpha p)\gamma_2 q_1)/(p-1)}$. From this, we obtain

$$\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma \le c_7\left[\sigma(B(x,2t)) + (\mathbf{W}_{\alpha,p}\sigma(x))^{\gamma_2 q_1}\,\sigma(B(x,t))\right],$$

where $c_7 = c_7(n,p,q_1,q_2,\alpha,C_\sigma)$. Thus, a combination of (3.1.19), (2.0.2), (3.1.20) with the previous inequality yields

$$\begin{aligned}
\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma)(x) &\le \lambda_2^{\frac{q_1}{p-1}} c_8 \int_0^\infty \left(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{q_1} + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t}\\
&\le \lambda_2^{\frac{q_1}{p-1}} c_9\Big[\mathbf{W}_{\alpha,p}\sigma(x) + \int_0^\infty \left(\frac{\sigma(B(x,2t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t}\\
&\qquad + (\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{q_1}{p-1}\gamma_2}\int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t}\Big]\\
&\le \lambda_2^{\frac{q_1}{p-1}} c_{10}\Big[\mathbf{W}_{\alpha,p}\sigma(x) + (\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{q_1}{p-1}\gamma_2+1}\Big],
\end{aligned} \tag{3.1.21}$$

where $c_{10} = c_{10}(n,p,q_1,q_2,\alpha,\sigma) > 0$.

To estimate $\mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma)$, we use a similar argument. By replacing q_1 by q_2 and γ_2 by γ_1, we deduce

$$\begin{aligned}
&\int_0^\infty \left(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{q_2}\,\mathrm{d}\sigma}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t} \le \tilde{c}_1\mathbf{W}_{\alpha,p}\sigma(x),\\
&\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1 q_2}\,\mathrm{d}\sigma \le \tilde{c}_2\,(\tilde{I}_1 + \tilde{I}_2),
\end{aligned} \tag{3.1.22}$$

where

$$\begin{aligned}
\tilde{I}_1 &= \int_{B(x,t)}\Big[\int_0^t \left(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_1 q_2}\mathrm{d}\sigma(y)\\
&= \int_{B(x,t)}\Big[\int_0^t \left(\frac{\sigma(B(y,r)\cap B(x,2t))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_1 q_2}\mathrm{d}\sigma(y)\\
&\le \int_{B(x,2t)}\Big[\mathbf{W}_{\alpha,p}\sigma_{B(x,2t)}\Big]^{\gamma_1 q_2}\mathrm{d}\sigma(y) \le \tilde{c}_3\,\sigma(B(x,2t))
\end{aligned}$$

and

$$\begin{aligned}
\tilde{I}_2 &= \int_{B(x,t)}\Big[\int_t^\infty \left(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_1 q_2}\mathrm{d}\sigma(y)\\
&\le \int_{B(x,t)}\Big[\int_t^\infty \left(\frac{\sigma(B(x,2r))}{r^{n-\alpha p}}\right)^{\frac{1}{p-1}}\frac{\mathrm{d}r}{r}\Big]^{\gamma_1 q_2}\mathrm{d}\sigma(y)\\
&\le \tilde{c}_4\,\sigma(B(x,t))\,[\mathbf{W}_{\alpha,p}\sigma(x)]^{\gamma_1 q_2},
\end{aligned}$$

which $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$ and $\tilde{c}_4$ are constants depending only on n, p, q_1, q_2 and σ. The previous estimates in combination with (3.1.22) yield

$$\int_{B(x,t)} (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1 q_2}\,\mathrm{d}\sigma \le \tilde{c}_5\left[\sigma(B(x,2t)) + (\mathbf{W}_{\alpha,p}\sigma(x))^{\gamma_1 q_2}\,\sigma(B(x,t))\right],$$

where $\tilde{c}_5 = \tilde{c}_5(n, p, q_1, q_2, \alpha, \sigma)$. Hence, using again (3.1.19), (2.0.2) and the previous inequality, we deduce

$$\mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma)(x) \le \lambda_2^{\frac{q_2}{p-1}}\tilde{c}_6\Big[\mathbf{W}_{\alpha,p}\sigma(x) + (\mathbf{W}_{\alpha,p}\sigma(x))^{\frac{q_1}{p-1}\gamma_2+1}\Big], \tag{3.1.23}$$

where $\tilde{c}_6 = \tilde{c}_6(n, p, q_1, q_2, \alpha, \sigma)$. We recall that $(q_1\gamma_2)/(p-1)+1 = \gamma_1$ and $(q_2\gamma_1)/(p-1)+1 = \gamma_2$. Therefore, picking λ_2 such that

$$\lambda_2 = \max\{c_{10}^{\frac{p-1}{p-1-q_1}}, (\tilde{c}_6)^{\frac{p-1}{p-1-q_2}}, \lambda_1\},$$

we conclude from (3.1.21) and (3.1.23) that

$$\overline{u} \ge \mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma), \quad \overline{u} \ge \underline{u},$$
$$\overline{v} \ge \mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma), \quad \overline{v} \ge \underline{v},$$

which completes the proof of Claim 3.

Now, in order to obtain solutions to (SI), we use a standard iteration argument and Monotone Convergence Theorem. For convenience, we repeat the main idea. Let $u_0 = \underline{u} = \lambda_1(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}$ and $v_0 = \underline{v} = \lambda_1(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}$, where λ_1 is the constant obtained in Claim 2. Clearly, $u_0 \le \overline{u}$ and $v_0 \le \overline{v}$. We set $u_1 = \mathbf{W}_{\alpha,p}(v_0^{q_1}\mathrm{d}\sigma)$ and $v_1 = \mathbf{W}_{\alpha,p}(u_0^{q_2}\mathrm{d}\sigma)$. By Claim 2, we have $u_1 \ge u_0$ and $v_1 \ge v_0$. Let us construct the sequence of pair of functions (u_j, v_j) in $\mathbb{R}^n$, with $(u_j, v_j) \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma) \times L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ such that

$$\begin{cases} u_j = \mathbf{W}_{\alpha,p}(v_{j-1}^{q_1}\mathrm{d}\sigma) & \text{in} \quad \mathbb{R}^n, \\ v_j = \mathbf{W}_{\alpha,p}(u_{j-1}^{q_1}\mathrm{d}\sigma) & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{3.1.24}$$

Indeed, by induction, we can show that the sequences $\{u_j\}$ and $\{v_j\}$ are nondecreasing, with $\underline{u} \le u_j \le \overline{u}$ and $\underline{v} \le v_j \le \overline{v}$ (for $j = 0, 1, \ldots$). Due to Lemma 3.1.4, both $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ belong to $L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma) \times L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, whence $(u_j, v_j) \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma) \times L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ for all $j = 1, 2, 3, \ldots$.

Using the Monotone Convergence Theorem and passing to the limit as $j \to \infty$ in (3.1.24), we see that there exist nonnegative functions $u = \lim u_j$ and $v = \lim v_j$ such that $(u, v) \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma) \times L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, which the pair (u, v) satisfies (SI) with $\underline{u} \le u \le \overline{u}$ and $\underline{v} \le v \le \overline{v}$. Thus for an appropriate constant $c = c(n, p, \alpha, q_1, q_2, C_\sigma)$ we obtain that u and v satisfy (1.3.4). This completes the proof of Theorem 1.3.1.

3.1.2 Proof of Theorem 1.3.3

As in the proof of the previous theorem, we will proceed by the method of sub- and super-solutions. Let $\underline{u} = \lambda_1(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}$ and $\underline{v} = \lambda_1(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}$, where γ_1 and γ_2 are given in (3.1.13) and (3.1.18), respectively. By Claim 2, we already know that $(\underline{u}, \underline{v})$ is a subsolution to (SI) if $\lambda_1 > 0$ is picked to be sufficiently small. It remains now exhibits a supersolution under assumption (1.3.5).

Claim 4. There exists $\lambda_2 > 0$ sufficiently large such that

$$(\overline{u}, \overline{v}) = \Big(\lambda_2\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}\right), \lambda_2\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right)\Big)$$

is a supersolution to (SI), which $\overline{u} \geq \underline{u}$ and $\overline{v} \geq \underline{v}$.

Indeed, using (3.1.19) and assumption (1.3.5), we have the following estimate for $\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma)$:

$$\begin{aligned}\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma) &\leq c_1\lambda_2^{\frac{q_1}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{q_1}\,\mathrm{d}\sigma\right) + c_1\lambda_2^{\frac{q_1}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma\right)\\ &\leq c_1\lambda_2^{\frac{q_1}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{q_1}\,\mathrm{d}\sigma\right) + c_1\lambda_2^{\frac{q_1}{p-1}}\lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}\right), \end{aligned} \tag{3.1.25}$$

where $c_1 = c_1(p, q_1) > 0$. To establish a convenient upper bound to $\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma)$, we need to estimate the first term in the previous inequality. By Hölder's inequality and Young's inequality with the exponent γ_2 and its conjugate

$$\gamma_2' = \frac{(p-1)(p-1+q_2)}{q_2(p-1+q_1)},$$

we obtain

$$\begin{aligned}\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{q_1}\,\mathrm{d}\sigma &\leq \Big(\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma\Big)^{\frac{1}{\gamma_2}}\left[\sigma(B(x,t))\right]^{\frac{1}{\gamma_2'}}\\ &\leq c_2\Big(\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma + \sigma(B(x,t))\Big),\end{aligned} \tag{3.1.26}$$

where $c_2 = c_2(p, q_1, q_2) > 0$.

From (1.0.3) we can write for $x \in \mathbb{R}^n$

$$\mathbf{W}_{\alpha,p}\Big(\big(\mathbf{W}_{\alpha,p}\sigma\big)^{q_1}\mathrm{d}\sigma\Big)(x) = \int_0^\infty\Big(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{q_1}\,\mathrm{d}\sigma}{t^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t},$$

which together with (3.1.19) and (3.1.26) implies

$$\begin{aligned}&\mathbf{W}_{\alpha,p}\Big(\big(\mathbf{W}_{\alpha,p}\sigma\big)^{q_1}\mathrm{d}\sigma\Big)(x) \leq\\ &\qquad c_3\Big[\int_0^\infty\Big(\frac{\int_{B(x,t)}(\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma}{t^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t} + \int_0^\infty\Big(\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{\mathrm{d}t}{t}\Big].\end{aligned}$$

Using condition (1.3.5) in the previous inequality, where $c_3 = c_3(p, q_1, q_2) > 0$, we deduce

$$\begin{aligned}\mathbf{W}_{\alpha,p}\Big(\Big(\mathbf{W}_{\alpha,p}\sigma\Big)^{q_1}\mathrm{d}\sigma\Big)(x) &\leq c_3\left(\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma\right)(x) + \mathbf{W}_{\alpha,p}\sigma(x)\right)\\ &\leq c_3\mathbf{W}_{\alpha,p}\sigma(x) + c_3\lambda\left(\mathbf{W}_{\alpha,p}\sigma(x) + (\mathbf{W}_{\alpha,p}\sigma(x))^{\gamma_1}\right).\end{aligned}$$

The last estimate in combination with (3.1.25) yield

$$\begin{aligned}&\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma) \leq\\ &\qquad \lambda_2^{\frac{q_1}{p-1}}c_1\left[c_3\mathbf{W}_{\alpha,p}\sigma + c_3\lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}\right)\right] + c_1\lambda_2^{\frac{q_1}{p-1}}\lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}\right).\end{aligned}$$

Thus, choosing $c_4 = c_1(c_3 + c_3\lambda + \lambda)$, we concluded that

$$\mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma) \leq \lambda_2^{\frac{q_1}{p-1}}c_4\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}\right). \tag{3.1.27}$$

Now, by a similar argument, we establish an upper bound to $\mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma)$. Again, we can replace q_1 by q_2 and γ_2 by γ_1, to obtain from (3.1.19) and (1.3.5) that

$$\begin{aligned}\mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma) &\leq \tilde{c}_1\lambda_2^{\frac{q_2}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{q_2}\,\mathrm{d}\sigma\right) + \tilde{c}_1\lambda_2^{\frac{q_2}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1 q_2}\,\mathrm{d}\sigma\right)\\ &\leq \tilde{c}_1\lambda_2^{\frac{q_2}{p-1}}\mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{q_2}\,\mathrm{d}\sigma\right) + \tilde{c}_1\lambda_2^{\frac{q_2}{p-1}}\lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right),\end{aligned} \tag{3.1.28}$$

where $\tilde{c}_1 = \tilde{c}_1(p, q_2) > 0$. We also have

$$\mathbf{W}_{\alpha,p}\left(\left(\mathbf{W}_{\alpha,p}\sigma\right)^{q_2}\mathrm{d}\sigma\right) \leq \tilde{c}_2\mathbf{W}_{\alpha,p}\sigma + \tilde{c}_2\lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right), \tag{3.1.29}$$

for some constant $\tilde{c}_2 = \tilde{c}_2(p, q_1, q_2) > 0$. From (1.3.5), (3.1.28) and (3.1.29), it follows

$$\begin{aligned}&\mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma) \leq\\ &\qquad \lambda_2^{\frac{q_2}{p-1}}\tilde{c}_1\left[\tilde{c}_2\mathbf{W}_{\alpha,p}\sigma + \tilde{c}_2\lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right)\right] + \tilde{c}_1\lambda_2^{\frac{q_2}{p-1}}\lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right).\end{aligned}$$

Thus, choosing $\tilde{c}_3 = \tilde{c}_1(\tilde{c}_2 + \tilde{c}_2\lambda + \lambda)$, we concluded that

$$\mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma) \leq \lambda_2^{\frac{q_2}{p-1}}\tilde{c}_3\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right). \tag{3.1.30}$$

Therefore, picking λ_2 such that

$$\lambda_2 = \max\{c_4^{\frac{p-1}{p-1-q_1}}, (\tilde{c}_3)^{\frac{p-1}{p-1-q_2}}, \lambda_1\},$$

we finally see from (3.1.27) and (3.1.30) that

$$\begin{aligned}\overline{u} &\geq \mathbf{W}_{\alpha,p}(\overline{v}^{q_1}\mathrm{d}\sigma), \quad \overline{u} \geq \underline{u},\\ \overline{v} &\geq \mathbf{W}_{\alpha,p}(\overline{u}^{q_2}\mathrm{d}\sigma), \quad \overline{v} \geq \underline{v},\end{aligned}$$

which completes the proof of Claim 4.

Using iterations as in (3.1.24), and the Monotone Convergence Theorem, we ensure that there exists a solution (u, v) to (SI) which satisfies (1.3.4), with $c = c(n, p, q_1, q_2, \alpha, \lambda)$.

Conversely, suppose that there exists a nontrivial solution (u, v) to (SI) such that (1.3.4) holds. By the lower bounds in (1.3.4), we have

$$\begin{aligned} u &= \mathbf{W}_{\alpha,p}(v^{q_1}\mathrm{d}\sigma) \geq (c^{-1})^{\frac{q_1}{p-1}}\mathbf{W}_{\alpha,p}\Big((\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\Big), \\ v &= \mathbf{W}_{\alpha,p}(u^{q_2}\mathrm{d}\sigma) \geq (c^{-1})^{\frac{q_2}{p-1}}\mathbf{W}_{\alpha,p}\Big((\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\Big), \end{aligned}$$

where $c > 0$ is a constant. The previous estimates, in combination with the upper bounds in (1.3.4), yield

$$\begin{aligned} \mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\right) &\leq \lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1}\right) < \infty \text{ - a.e.}, \\ \mathbf{W}_{\alpha,p}\left((\mathbf{W}_{\alpha,p}\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\right) &\leq \lambda\left(\mathbf{W}_{\alpha,p}\sigma + (\mathbf{W}_{\alpha,p}\sigma)^{\gamma_2}\right) < \infty \text{ - a.e.}, \end{aligned}$$

where λ can be defined as $\lambda = \max\{c^{(p-1+q_1)/(p-1)}, c^{(p-1+q_2)/(p-1)}\}$. This completes the proof of Theorem 1.3.3.

3.2 QUASILINEAR LANE-EMDEN TYPE SYSTEMS WITH SUB-NATURAL GROWTH TERMS

3.2.1 Proof of Theorem 1.3.6

First, we assume that $1 < p < n$. The argument is based on the method of successive approximations. Suppose (1.1.4) and (1.3.11). Let K be the constant given in Theorem D. By Theorem 1.3.1, with $\alpha = 1$ and $K^{p-1}\sigma$ in place of σ, there exists a nontrivial solution $(\tilde{u}, \tilde{v})$ to the system

$$\begin{cases} \tilde{u} = K\,\mathbf{W}_p(\tilde{v}^{q_1}\mathrm{d}\sigma) \quad \text{in} \quad \mathbb{R}^n, \\ \tilde{v} = K\,\mathbf{W}_p(\tilde{u}^{q_2}\mathrm{d}\sigma) \quad \text{in} \quad \mathbb{R}^n. \end{cases} \tag{3.2.1}$$

Since $(\tilde{u}, \tilde{v})$ satisfies

$$\begin{aligned} c^{-1}\Big(K\,\mathbf{W}_p\sigma\Big)^{\gamma_1} &\leq \tilde{u} \leq c\Big(K\,\mathbf{W}_p\sigma + \Big(K\,\mathbf{W}_p\sigma\Big)^{\gamma_1}\Big), \\ c^{-1}\Big(K\,\mathbf{W}_p\sigma\Big)^{\gamma_2} &\leq \tilde{v} \leq c\Big(K\,\mathbf{W}_p\sigma + \Big(K\,\mathbf{W}_p\sigma\Big)^{\gamma_2}\Big), \end{aligned} \tag{3.2.2}$$

using (2.1.4), it follows $\underline{\lim}_{|x|\to\infty} \tilde{u}(x) = \underline{\lim}_{|x|\to\infty} \tilde{v}(x) = 0$; here γ_1 and γ_2 are as in (3.1.13) and (3.1.18), respectively. From Lemma 3.1.4, $\tilde{u}$ and $\tilde{v}$ belong to $L^s_{\rm loc}(\mathbb{R}^n, \mathrm{d}\sigma)$ for all $s > 0$, hence by Hölder's inequality

$$\int_B \mathbf{W}_p(\tilde{v}^{q_1}\mathrm{d}\sigma)\tilde{v}^{q_1}\,\mathrm{d}\sigma = K^{-1}\int_B \tilde{u}\,\tilde{v}^{q_1}\,\mathrm{d}\sigma$$
$$\leq K^{-1}\|\tilde{u}\|_{L^s(B,\mathrm{d}\sigma)}\|\tilde{v}^{q_1}\|_{L^{s'}(B,\mathrm{d}\sigma)} < \infty,$$

for all ball $B \subset \mathbb{R}^n$ and, similarly,

$$\int_B \mathbf{W}_p(\tilde{u}^{q_2}\mathrm{d}\sigma)\tilde{u}^{q_2}\,\mathrm{d}\sigma = K^{-1}\|\tilde{u}^{q_2}\|_{L^s(B,\mathrm{d}\sigma)}\|\tilde{v}\|_{L^{s'}(B,\mathrm{d}\sigma)} < \infty.$$

Using (2.1.2), $\tilde{u}^{q_2}\mathrm{d}\sigma$, $\tilde{v}^{q_1}\mathrm{d}\sigma \in W^{-1,p'}_{\rm loc}(\mathbb{R}^n)$. By Lemma 3.1.3, with $K^{p-1}\sigma$ in place of σ again, there exist a constant $c_0 = c_0(n,p,\alpha,q_1,q_2) > 0$ such that

$$\tilde{u} \geq c_0\,K^{\gamma_1}(\mathbf{W}_p\sigma)^{\gamma_1},$$
$$\tilde{v} \geq c_0\,K^{\gamma_2}(\mathbf{W}_p\sigma)^{\gamma_2}.$$

Set $u_0 = \varepsilon\,(\mathbf{W}_p\sigma)^{\gamma_1}$ and $v_0 = \varepsilon\,(\mathbf{W}_p\sigma)^{\gamma_2}$, where $\varepsilon > 0$ is a small constant. Using that

$$\gamma_2\frac{q_1}{p-1} + 1 = \gamma_1 \quad \text{and} \quad \gamma_1\frac{q_2}{p-1} + 1 = \gamma_2.$$

and taking ε sufficiently small, we obtain $u_0 \leq \mathbf{W}_p(v_0^{q_1}\mathrm{d}\sigma)$ and $v_0 \leq \mathbf{W}_p(u_0^{q_2}\mathrm{d}\sigma)$. Moreover, choosing $\varepsilon < \min\{c\,K^{-\gamma_1}, c\,K^{-\gamma_2}\}$, we also have $u_0 \leq \tilde{u}$ and $v_0 \leq \tilde{v}$, respectively. Setting $B_i = B(0,2^i)$, where $i = 1,2,\ldots$, we deduce that $v_0^{q_1}\mathrm{d}\sigma$, $u_0^{q_2}\mathrm{d}\sigma \in W^{-1,p'}(B_i)$, since $v_0 \leq \tilde{v}$, $u_0 \leq \tilde{u}$ and $\tilde{u}^{q_2}\mathrm{d}\sigma, \tilde{v}^{q_1}\mathrm{d}\sigma \in W^{-1,p'}_{\rm loc}(\mathbb{R}^n)$. Thus, applying [Heinonen, Kilpeläinen and Martio 2006, Theorem 21.6], there exist unique p-superharmonic solutions u_1^i, $v_1^i \in W^{1,p}_0(B_i)$ to equations

$$-\Delta_p u_1^i = \sigma\,v_0^{q_1}, \quad -\Delta_p v_1^i = \sigma\,u_0^{q_2} \quad \text{in} \quad B_i.$$

Using a comparison principle, [Cao and Verbitsky 2017, Lemma 5.1], the sequences $\{u_1^i\}_i$ and $\{v_1^i\}_i$ are increasing. We set $u_1 = \lim_{i\to\infty} u_1^i$ and $v_1 = \lim_{i\to\infty} v_1^i$. A combination of Lemma B, of weak continuity of the p-Laplace operator (Theorem C) and of Monotone Convergence Theorem, ensure that u_1 and v_1 are p-superharmonic solutions to the equations

$$-\Delta_p u_1 = \sigma\,v_0^{q_1}, \quad -\Delta_p v_1 = \sigma\,u_0^{q_2} \quad \text{in} \quad \mathbb{R}^n.$$

Applying Theorem D,

$$u_1^i \leq K\,\mathbf{W}_p(v_0^{q_1}\mathrm{d}\sigma) \leq K\,\mathbf{W}_p(\tilde{v}^{q_1}\mathrm{d}\sigma) = \tilde{u},$$
$$v_1^i \leq K\,\mathbf{W}_p(u_0^{q_2}\mathrm{d}\sigma) \leq K\,\mathbf{W}_p(\tilde{u}^{q_2}\mathrm{d}\sigma) = \tilde{v},$$

which implies $u_1 \le \tilde{u}$, $v_1 \le \tilde{v}$, and hence by (2.1.4),

$$\lim_{|x|\to\infty} u_1(x) = \lim_{|x|\to\infty} v_1(x) = 0.$$

Using the lower bound in Theorem D, and Lemma T, we obtain

$$\begin{aligned} u_1 &\ge K^{-1}\mathbf{W}_p(v_0^{q_1}\mathrm{d}\sigma) = K^{-1}\varepsilon^{\frac{q_1}{p-1}}\mathbf{W}_p\left((\mathbf{W}_p\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\right) \\ &\ge K^{-1}\varepsilon^{\frac{q_1}{p-1}}\kappa^{\frac{q_1}{p-1}\gamma_2}(\mathbf{W}_p\sigma)^{\frac{q_1}{p-1}\gamma_2+1} = K^{-1}\varepsilon^{\frac{q_1}{p-1}}\kappa^{\frac{q_1}{p-1}\gamma_2}(\mathbf{W}_p\sigma)^{\gamma_1}, \\ v_1 &\ge K^{-1}\mathbf{W}_p(u_0^{q_2}\mathrm{d}\sigma) = K^{-1}\varepsilon^{\frac{q_2}{p-1}}\mathbf{W}_p\left((\mathbf{W}_p\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\right) \\ &\ge K^{-1}\varepsilon^{\frac{q_2}{p-1}}\kappa^{\frac{q_2}{p-1}\gamma_1}(\mathbf{W}_p\sigma)^{\frac{q_2}{p-1}\gamma_1+1} = K^{-1}\varepsilon^{\frac{q_2}{p-1}}\kappa^{\frac{q_2}{p-1}\gamma_1}(\mathbf{W}_p\sigma)^{\gamma_2}, \end{aligned}$$

where K is the constant given in Theorem D and κ is the constant given in Lemma T. Hence, $c_1(\mathbf{W}_p\sigma)^{\gamma_1} \le u_1 \le \tilde{u}$ and $\tilde{c}_1(\mathbf{W}_p\sigma)^{\gamma_2} \le v_1 \le \tilde{v}$, where

$$c_1 = K^{-1}\varepsilon^{\frac{q_1}{p-1}}\kappa^{\frac{q_1}{p-1}\gamma_2}, \quad \tilde{c}_1 = K^{-1}\varepsilon^{\frac{q_2}{p-1}}\kappa^{\frac{q_2}{p-1}\gamma_1}.$$

We notice that $v_0^{q_1}\mathrm{d}\sigma$, $u_0^{q_2}\mathrm{d}\sigma \in W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$, since $v_0 \le \tilde{v}$, $u_0 \le \tilde{u}$ and $\tilde{v}^{q_1}\mathrm{d}\sigma$, $\tilde{u}^{q_2}\mathrm{d}\sigma \in W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$. Thus, from Lemma E, u_1, $v_1 \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

By induction argument, as above, we construct a sequence $\{(u_j, v_j)\}$ of p-superharmonic functions in $\mathbb{R}^n$ with $u_j \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ and $v_j \in L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ for $j = 2, 3, \dots$, satisfying

$$\begin{cases} -\Delta_p u_j = \sigma\, v_{j-1}^{q_1} \quad \text{in} \quad \mathbb{R}^n, \\ -\Delta_p v_j = \sigma\, u_{j-1}^{q_2} \quad \text{in} \quad \mathbb{R}^n, \\ c_j(\mathbf{W}_p\sigma)^{\gamma_1} \le u_j \le \tilde{u}, \quad \tilde{c}_j(\mathbf{W}_p\sigma)^{\gamma_2} \le v_j \le \tilde{v} \quad \text{in} \quad \mathbb{R}^n, \\ 0 \le u_{j-1} \le u_j, \quad 0 \le v_{j-1} \le v_j, \quad u_j, v_j \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n), \\ \lim\limits_{|x|\to\infty} u_j(x) = \lim\limits_{|x|\to\infty} v_j(x) = 0, \end{cases} \tag{3.2.3}$$

where

$$\begin{aligned} c_1 &= K^{-1}\varepsilon^{\frac{q_1}{p-1}}\kappa^{\frac{q_1}{p-1}\gamma_2}, \quad \tilde{c}_1 = K^{-1}\varepsilon^{\frac{q_2}{p-1}}\kappa^{\frac{q_2}{p-1}\gamma_1}, \\ c_j &= K^{-1}(\tilde{c}_{j-1}\kappa^{\gamma_2})^{\frac{q_1}{p-1}}, \quad \tilde{c}_j = K^{-1}(c_{j-1}\kappa^{\gamma_1})^{\frac{q_2}{p-1}}, \quad \text{for } j = 2, 3, \dots. \end{aligned} \tag{3.2.4}$$

Indeed, suppose that $(u_1, v_1), \dots, (u_{j-1}, v_{j-1})$ have been constructed. Since $u_{j-1} \le \tilde{u}$, $v_{j-1} \le \tilde{v}$ and $\tilde{v}^{q_1}\mathrm{d}\sigma$, $\tilde{u}^{q_2}\mathrm{d}\sigma \in M^+(\mathbb{R}^n) \cap W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$, it follows $v_{j-1}^{q_1}\mathrm{d}\sigma$, $u_{j-1}^{q_2}\mathrm{d}\sigma \in M^+(\mathbb{R}^n) \cap W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$. Clearly, $v_{j-1}^{q_1}\mathrm{d}\sigma$, $u_{j-1}^{q_2}\mathrm{d}\sigma \in W^{-1,p'}(B_i)$, and there exists unique p-superharmonic solutions u_j^l and v_j^l to the equations

$$\begin{cases} -\Delta_p u_j^i = \sigma\, v_{j-1}^{q_1} \quad \text{in} \quad B_i, \qquad u_j^i \in W^{1,p}_0(B_i), \\ -\Delta_p v_j^i = \sigma\, u_{j-1}^{q_2} \quad \text{in} \quad B_i, \qquad v_j^i \in W^{1,p}_0(B_i). \end{cases}$$

Arguing by induction, let u^i_{j-1} and v^i_{j-1} be the unique solutions of the equations

$$\begin{cases} -\Delta_p u^i_{j-1} = \sigma\, v^{q_1}_{j-2} \quad \text{in} \quad B_i, \qquad u^i_{j-1} \in W^{1,p}_0(B_i), \\ -\Delta_p v^i_{j-1} = \sigma\, u^{q_2}_{j-2} \quad \text{in} \quad B_i, \qquad v^i_{j-1} \in W^{1,p}_0(B_i). \end{cases}$$

Since $v_{j-2} \le v_{j-1}$ and $u_{j-2} \le u_{j-1}$, it follows from the comparison principle, [Cao and Verbitsky 2017, Lemma 5.1], that $u^i_{j-1} \le u^i_j$ and $v^i_{j-1} \le v^i_j$ for all $i \ge 1$. By Theorem D, we have

$$0 \le u^i_j \le K\,\mathbf{W}_p(v^{q_1}_{j-1}\mathrm{d}\sigma) \le K\,\mathbf{W}_p(\tilde{v}^{q_1}\mathrm{d}\sigma) = \tilde{u} \quad \text{and}$$
$$0 \le v^i_j \le K\,\mathbf{W}_p(u^{q_1}_{j-1}\mathrm{d}\sigma) \le K\,\mathbf{W}_p(\tilde{u}^{q_1}\mathrm{d}\sigma) = \tilde{v},$$

since $v_{j-1} \le \tilde{v}$ and $u_{j-1} \le \tilde{u}$. Using again the comparison principle, we deduce that the sequences $\{u^i_j\}_i$ and $\{v^i_j\}_i$ are increasing. Thus, letting $u_j = \lim_{i\to\infty} u^i_j$ and $v_j = \lim_{i\to\infty} v^i_j$, we obtain that u_j and v_j are p-superharmonic solutions to the equations

$$\begin{cases} -\Delta_p u_j = \sigma\, v^{q_1}_{j-1} \quad \text{in} \quad \mathbb{R}^n, \\ -\Delta_p v_j = \sigma\, u^{q_2}_{j-1} \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

since, as before, we apply Theorem C and the Monotone Convergence Theorem. We also have $u_{j-1} \le u_j$ and $v_{j-1} \le v_j$, since $u^i_{j-1} \le u^i_j$ and $v^i_{j-1} \le v^i_j$ for all $i \ge 1$. Furthermore, $u_j \le \tilde{u}$ and $v_j \le \tilde{v}$, since $u^i_j \le \tilde{u}$ and $v^i_j \le \tilde{v}$ for all $i \ge 1$. By (2.1.4), we obtain

$$\lim_{|x|\to\infty} u_j(x) = \lim_{|x|\to\infty} v_j(x) = 0. \tag{3.2.5}$$

Since $v^{q_1}_{j-1}\mathrm{d}\sigma$ and $u^{q_2}_{j-1}\mathrm{d}\sigma \in W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n)$, it follows from Lemma E and (3.2.5) that u_j, $v_j \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$. Also, applying Theorem D and Lemma T, and arguing by induction, we concluded that

$$\begin{aligned} u_j &\ge K^{-1}\mathbf{W}_p(v^{q_1}_{j-1}\mathrm{d}\sigma) = K^{-1}\mathbf{W}_p\left(\tilde{c}^{q_1}_{j-1}(\mathbf{W}_p\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\right) \\ &\ge K^{-1}\tilde{c}_{j-1}^{\frac{q_1}{p-1}}\kappa^{\frac{q_1}{p-1}\gamma_2}(\mathbf{W}_p\sigma)^{\frac{q_1}{p-1}\gamma_2+1} = c_j(\mathbf{W}_p\sigma)^{\gamma_1} \end{aligned}$$

and, similarly,

$$\begin{aligned} v_j &\ge K^{-1}\mathbf{W}_p(u^{q_2}_{j-1}\mathrm{d}\sigma) = K^{-1}\mathbf{W}_p\left(c^{q_2}_{j-1}(\mathbf{W}_p\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\right) \\ &\ge K^{-1}c_{j-1}^{\frac{q_2}{p-1}}\kappa^{\frac{q_2}{p-1}\gamma_1}(\mathbf{W}_p\sigma)^{\frac{q_2}{p-1}\gamma_1+1} = \tilde{c}_j(\mathbf{W}_p\sigma)^{\gamma_2}. \end{aligned}$$

Now, we set $u = \lim_{j\to\infty} u_j$ and $v = \lim_{j\to\infty} v_j$. By Lemma B, u and v are p-superharmonic functions in $\mathbb{R}^n$. From Theorem C and Monotone Convergence Theorem, we deduce that (u, v)

is a solution to the system

$$\begin{cases} -\Delta_p u = \sigma\, v^{q_1} \quad \text{in} \quad \mathbb{R}^n, \\ -\Delta_p v = \sigma\, u^{q_2} \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

in the sense of (2.1.1). Moreover, $u \le \tilde{u}$ and $v \le \tilde{v}$, and hence

$$\liminf_{|x|\to\infty} u(x) = \liminf_{|x|\to\infty} v(x) = 0.$$

Using Lemma E again, we deduce that $u,\, v \in W^{1,p}_{\rm loc}(\mathbb{R}^n)$ since $v^{q_1}{\rm d}\sigma,\ u^{q_2}{\rm d}\sigma \in W^{-1,p'}_{\rm loc}(\mathbb{R}^n)$. By (3.2.2), there exists a constant $c = c(n, p, q_1, q_2, \alpha, C_\sigma) > 0$ such that

$$u \le c\left(\mathbf{W}_p\sigma + (\mathbf{W}_p\sigma)^{\gamma_1}\right),\ v \le c\left(\mathbf{W}_p\sigma + (\mathbf{W}_p\sigma)^{\gamma_2}\right),$$

and this shows an upper bound in (1.3.12). From (3.2.3), (u, v) satisfies for all $j = 1, 2, \ldots$ the lower bound

$$\begin{aligned} u &\ge c_j\left(\mathbf{W}_p\sigma\right)^{\gamma_1}, \\ v &\ge \tilde{c}_j\left(\mathbf{W}_p\sigma\right)^{\gamma_2}, \end{aligned} \tag{3.2.6}$$

Passing to limit $j \to \infty$ in (3.2.6), with c_j and $\tilde{c}_j$ given in (3.2.4), we obtain

$$\begin{aligned} u &\ge C\left(\mathbf{W}_p\sigma\right)^{\gamma_1}, \\ v &\ge \tilde{C}\left(\mathbf{W}_p\sigma\right)^{\gamma_2}, \end{aligned}$$

where, by direct computation,

$$\begin{aligned} C &= \lim_{j\to\infty} c_j = K^{-\gamma_1}\kappa^{\frac{\gamma_1 q_1(\gamma_1 q_2 + \gamma_2(p-1))}{(p-1)^2}}, \\ \tilde{C} &= \lim_{j\to\infty} \tilde{c}_j = K^{-\gamma_2}\kappa^{\frac{\gamma_2 q_2(\gamma_2 q_1 + \gamma_1(p-1))}{(p-1)^2}}. \end{aligned}$$

This shows a lower bound in (1.3.4).

Now we prove that (u, v) is a minimal solution; that is, if (w_1, w_2) is any nontrivial solution to Syst. (S_1), then $w_1 \ge u$ and $w_2 \ge v$ almost everywhere in $\mathbb{R}^n$. Let (w_1, w_2) be a solution to Syst. (S_1). From Theorem D, it follows

$$\begin{cases} w_1 \ge K^{-1}\mathbf{W}_p(w_2^{q_1}{\rm d}\sigma) \\ w_2 \ge K^{-1}\mathbf{W}_p(w_1^{q_2}{\rm d}\sigma) \end{cases}$$

Using Lemma 3.1.3 with $K^{-(p-1)}\sigma$ in place of σ, we obtain

$$\begin{aligned} w_1 &\ge c\,K^{-\gamma_1}(\mathbf{W}_p\sigma)^{\gamma_1}, \\ w_2 &\ge c\,K^{-\gamma_2}(\mathbf{W}_p\sigma)^{\gamma_2}. \end{aligned}$$

Let $\mathrm{d}\omega_1 = {w_2}^{q_1}\mathrm{d}\sigma$ and $\mathrm{d}\omega_2 = {w_1}^{q_2}\mathrm{d}\sigma$. With the previous constant c, choosing ε such that

$$\varepsilon \leq \min\{c\,K^{-\gamma_1},\, c\,K^{-\gamma_2}\},$$

we have $\mathrm{d}\omega_1 \geq v_0^{q_1}\mathrm{d}\sigma$ and $\mathrm{d}\omega_2 \geq u_0^{q_2}\mathrm{d}\sigma$. Applying [Cao and Verbitsky 2017, Lemma 5.2], we deduce that the functions u_1^i and v_1^i defined as before satisfy in B_i the inequality $u_1^i \leq w_1$ and $v_1^i \leq w_2$ for all $i \geq 1$, and consequently $u_1 = \lim_{i\to\infty} u_1^i \leq w_1$ and $v_1 = \lim_{i\to\infty} v_1^i \leq w_2$. Repeating this argument by induction, we conclude $u_j \leq w_1$ and $v_j \leq w_2$ for every $j \geq 1$. Therefore,

$$u = \lim_{j\to\infty} u_j \leq w_1, \quad v = \lim_{j\to\infty} v_j \leq w_2 \quad \text{a.e. in } \mathbb{R}^n.$$

It remains to prove that if $p \geq n$, there are no nontrivial solutions to Syst. (S_1) in $\mathbb{R}^n$. Let $w \in \mathcal{S}_p(\mathbb{R}^n)$ be a p-superharmonic function in $\mathbb{R}^n$ with $\varliminf_{|x|\to\infty} w(x) = 0$. By [Heinonen, Kilpeläinen and Martio 2006, Theorem 7.48], there exists $c = c(n,p) > 0$ such that

$$\int_{B_r} \frac{|\nabla w|^p}{w^p}\,\mathrm{d}x \leq c\,\mathrm{cap}_p(B_r), \tag{3.2.7}$$

for all balls $B_r := B(0,r)$. With aid of [Heinonen, Kilpeläinen and Martio 2006, Theorem 2.2 (ii)], we infer from [Heinonen, Kilpeläinen and Martio 2006, Theorem 2.19] that $\mathrm{cap}_p(B_r) = 0$ for all balls B_r, provided $p \geq n$. Thus, letting $r \to \infty$ in (3.2.7), we obtain for $p \geq n$ that $\nabla w \equiv 0$ almost everywhere in $\mathbb{R}^n$. Since $\varliminf_{|x|\to\infty} w(x) = 0$, it follows $w \equiv 0$. Thus, for $p \geq n$, all p-superharmonic functions with vanishes at infinity must be the trivial function. This completes the proof of Theorem 1.3.6.

3.2.2 Proof of Theorem 1.3.8

Suppose that there exist p-superharmonic functions u and v satisfying Syst. (S_1) and (1.3.12). By Theorem D,

$$\begin{cases} u \geq K^{-1}\mathbf{W}_p(v^{q_1}\mathrm{d}\sigma), \\ v \geq K^{-1}\mathbf{W}_p(u^{q_2}\mathrm{d}\sigma), \end{cases}$$

which together with Lemma 3.1.3 implies

$$\begin{aligned} u &\geq c_0\,(\mathbf{W}_p\sigma)^{\gamma_1}\,, \\ v &\geq c_0\,(\mathbf{W}_p\sigma)^{\gamma_2}\,, \end{aligned} \tag{3.2.8}$$

where γ_1 and γ_2 are given by (3.1.13) and (3.1.18) respectively. Consequently,

$$\begin{aligned} u &\geq c_1\,\mathbf{W}_p\left((\mathbf{W}_p\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\right), \\ v &\geq c_2\,\mathbf{W}_p\left((\mathbf{W}_p\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\right), \end{aligned}$$

where $c_1 = K^{-1}{c_0}^{q_1}$ and $c_2 = K^{-1}{c_0}^{q_2}$. Thus, because of (1.3.12), it holds

$$u \leq c_3 \left(\mathbf{W}_p\sigma + (\mathbf{W}_p\sigma)^{\gamma_1}\right),$$
$$v \leq c_3 \left(\mathbf{W}_p\sigma + (\mathbf{W}_p\sigma)^{\gamma_2}\right),$$

for some constant $c_3 > 0$. The previous estimates in combination with (3.2.8) and choosing $\lambda = c_3 \max\{{c_1}^{-1}, {c_2}^{-1}\}$ yield

$$\mathbf{W}_p\left((\mathbf{W}_p\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\right) \leq \lambda\left(\mathbf{W}_p\sigma + (\mathbf{W}_p\sigma)^{\gamma_1}\right) < \infty \text{ - a.e.},$$
$$\mathbf{W}_p\left((\mathbf{W}_p\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\right) \leq \lambda\left(\mathbf{W}_p\sigma + (\mathbf{W}_p\sigma)^{\gamma_2}\right) < \infty \text{ - a.e.},$$

which proves (1.3.13).

Conversely, suppose that (1.3.13) holds. Let K be the constant given in Theorem D. By Theorem 1.3.3 with $\alpha = 1$ and $K^{p-1}\sigma$ in place of σ, there exists a nontrivial solution $(\tilde{u}, \tilde{v})$ to Syst. (3.2.1), satisfying (3.2.2). Using the same argument in the proof of Theorem 1.3.6, one can complete that reciprocal of Theorem 1.3.8 holds.

3.2.3 Proof of Theorem 1.3.9

The proof is direct. Indeed, since $\mathbf{I}_{2\alpha}\mu = (n - 2\alpha)\mathbf{W}_{\alpha,2}\mu$ for any $\mu \in M^+(\mathbb{R}^n)$, it follows that Theorem 1.3.9 is a special case of Theorem 1.3.1 with $p = 2$.

3.2.4 Some examples

1. Fix $2 < \delta < n$ and let σ be the function given by

$$\sigma(x) = \frac{1}{1 + |x|^\delta}, \quad x \in \mathbb{R}^n.$$

We claim that $\mathbf{I}_2\sigma \in L^\infty(\mathbb{R}^n)$. Indeed, setting $\mathrm{d}\sigma = \sigma\,\mathrm{d}x$, for $x \in \mathbb{R}^n$,

$$\begin{aligned}
\sigma(B(x,t)) &= \int_{B(x,t)} \mathrm{d}\sigma(y) = \int_{B(x,t)} \frac{1}{1 + |y|^\delta}\,\mathrm{d}y \\
&= c_0 \int_0^t \frac{r^{n-1}}{1 + r^\delta}\,\mathrm{d}r = c_0 \int_0^t \frac{r^{n-\delta} r^{\delta-1}}{1 + r^\delta}\,\mathrm{d}r \\
&\leq c_0\, t^{n-\delta} \int_0^t \frac{r^{\delta-1}}{1 + r^\delta}\,\mathrm{d}r = c_1\, t^{n-\delta} \ln(1 + t^\delta), \quad \forall t > 0,
\end{aligned}$$

where $c_0 = c_0(n) > 0$ and $c_1 = c_0/\delta$. Using definition (1.3.2) and integration by parts, one has

$$\begin{aligned}
\mathbf{I}_2\sigma(x) &= (n-2)\int_0^\infty \frac{\sigma(B(x,t))}{t^{n-2}}\frac{\mathrm{d}t}{t} \\
&\le c_2 \int_0^\infty \frac{t^{n-\delta}\ln(1+t^\delta)}{t^{n-1}}\,\mathrm{d}t \\
&= c_2 \int_0^\infty \frac{\ln(1+t^\delta)}{t^{\delta-1}}\,\mathrm{d}t \\
&= -c_2 \frac{\ln(1+t^\delta)}{t^{\delta-2}}\bigg|_0^\infty + c_2\int_0^\infty \frac{t^{\delta-1}}{(1+t^\delta)t^{\delta-2}}\,\mathrm{d}t \\
&=: c_2\Big(I_1 + I_2\Big),
\end{aligned}$$

where $c_2 = c_2(n,\delta) > 0$. Applying L'Hôpital's rule, I_1 vanishes since $\delta > 2$.

Notice that I_2 can be split into two integrals:

$$I_2 = \int_0^\infty \frac{t}{1+t^\delta}\,\mathrm{d}t = \int_0^1 \frac{t}{1+t^\delta}\,\mathrm{d}t + \int_1^\infty \frac{t}{1+t^\delta}\,\mathrm{d}t.$$

In the first one, since $1+t^\delta > 1$, it follows

$$\int_0^1 \frac{t}{1+t^\delta}\,\mathrm{d}t \le \int_0^1 t\,\mathrm{d}t = \frac{1}{2}.$$

For the second one, since $1+t^\delta > t^\delta$, we have

$$\int_1^\infty \frac{t}{1+t^\delta}\,\mathrm{d}t \le \int_1^\infty t^{1-\delta}\,\mathrm{d}t = (2-\delta)t^{2-\delta}\Big|_1^\infty = \delta - 2.$$

This shows that $I_2 < \infty$. Consequently, $\mathbf{I}_2\sigma(x) < \infty$ for all $x \in \mathbb{R}^n$, and our claim is established.

By Remark 1.3.10, σ satisfies condition (1.3.11). By Theorem 1.3.6, there exists a nontrivial *bounded* solution to Syst. (S_1) with $p = 2$, that is there exists (u,v) satisfying

$$\begin{cases}
-\Delta u(x) = \dfrac{v^{q_1}(x)}{1+|x|^\delta}, \quad u(x) > 0, \qquad \forall\, x \in \mathbb{R}^n, \\
-\Delta v(x) = \dfrac{u^{q_2}(x)}{1+|x|^\delta}, \quad v(x) > 0, \qquad \forall\, x \in \mathbb{R}^n, \\
\lim\limits_{|x|\to\infty} u(x) = \lim\limits_{|x|\to\infty} v(x) = 0,
\end{cases}$$

jointly with the bilateral estimates

$$\begin{aligned}
c^{-1}\left(\mathbf{I}_2\sigma\right)^{\frac{1+q_1}{1-q_1q_2}} &\le u \le c\left(\mathbf{I}_2\sigma + \left(\mathbf{I}_2\sigma\right)^{\frac{1+q_1}{1-q_1q_2}}\right), \\
c^{-1}\left(\mathbf{I}_2\sigma\right)^{\frac{1+q_2}{1-q_1q_2}} &\le v \le c\left(\mathbf{I}_2\sigma + \left(\mathbf{I}_2\sigma\right)^{\frac{1+q_2}{1-q_1q_2}}\right),
\end{aligned}$$

where $0 < q_1,\, q_2 < 1$.

2. Fix $0 < q < 1$ and $0 < 2\alpha < n$. Let $\mathrm{d}\sigma = \sigma\,\mathrm{d}x$ for $\sigma(x) = |x|^{-s}\chi_{B\setminus\{0\}}(x)$, with $2\alpha < s < n - (n-2\alpha)q$; here $B = B(0,1)$. As commented in Remark 1.3.5, σ does not satisfy condition (1.3.3), but condition (1.3.7) still holds. Setting $q_1 = q_2 = q$, combining Remark 1.3.4 with Theorem 1.3.14, it follows that there exists a nonnegative function u satisfying the following fractional Laplacian equation

$$\begin{cases} (-\Delta)^{\alpha} u(x) = \sigma(x)\, u(x)^q, \quad u(x) > 0 \qquad \forall\, x \in \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

Moreover, by (1.3.15), u satisfies

$$c^{-1}\left(\mathbf{I}_{2\alpha}\sigma(x)\right)^{\frac{1}{1-q}} \leq u \leq c\left(\mathbf{I}_{2\alpha}\sigma(x) + \left(\mathbf{I}_{2\alpha}\sigma(x)\right)^{\frac{1}{1-q}}\right) \quad \forall x \in \mathbb{R}^n;$$

here $\mathbf{I}_{2\alpha}\sigma(x)$ is given by

$$\mathbf{I}_{2\alpha}\sigma(x) = \int_{\mathbb{R}^n} \frac{\sigma(y)}{|x-y|^{n-2\alpha}}\,\mathrm{d}y = \int_B \frac{|y|^{-s}}{|x-y|^{n-2\alpha}}\,\mathrm{d}y.$$

3.2.5 Final comments

Remark 3.2.1. For the case $q_1 = q_2$, setting $q = q_1$, we obtain

$$\gamma_1 = \gamma_2 = \frac{(p-1)(p-1+q)}{(p-1)^2 - q^2} = \frac{p-1}{p-1-q}. \tag{3.2.9}$$

The argument of the proof of Theorem 1.3.6 in combination with (3.2.9) implies $u_0 = v_0$. Thus, by induction,

$$u_j = v_j \quad \forall j = 1, 2, \ldots,$$

where (u_j, v_j) were given in (3.2.3) for $j = 1, 2, \ldots$. It follows that $u = v$ and Syst. (S_1) becomes the equation

$$\begin{cases} -\Delta_p u = \sigma\, u^q \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

Therefore, [Cao and Verbitsky 2016, Theorem 1.2] is a corollary of Theorem 1.3.6 for the case $q = q_1 = q_2$.

Remark 3.2.2. We highlight that our main results are entirely based on the Wolff potential estimates. Thus, direct analogous theorems hold for the more general quasilinear $\mathcal{A}$-operator $\mathrm{div}\mathcal{A}(x, \nabla\cdot)$ in place of $\Delta_p\cdot$ in (1.1.3). Precisely, let $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping satisfying the standard structural assumptions, which ensures the growth condition $\mathcal{A}(x,\xi)\cdot\xi \approx |\xi|^p$. This assumption guarantee that the Wolff potential estimates like (1.1.2) hold for the system

$$\begin{cases} -\operatorname{div}\mathcal{A}(x,\nabla u) = \sigma\, v^{q_1}, \quad v>0 \quad \text{in} \quad \mathbb{R}^n, \\ -\operatorname{div}\mathcal{A}(x,\nabla v) = \sigma\, u^{q_2}, \quad u>0 \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = \lim\limits_{|x|\to\infty} v(x) = 0. \end{cases}$$

In such wise, we can conclude similar theorems (see [Kilpeläinen and Malý 1994, Dat and Verbitsky 2015, Trudinger and Wang 2002] for more details).

Here we indicate some questions related to this class of quasilinear problems:

1. Let $\mu_1,\, \mu_2 \in M^+(\mathbb{R}^n)$. Consider the counterpart *inhomogeneous* to (S_1) as follows:

$$\begin{cases} -\Delta_p u = \sigma\, v^{q_1} + \mu_1, \quad u>0 \quad \text{in} \quad \mathbb{R}^n, \\ -\Delta_p v = \sigma\, u^{q_2} + \mu_2, \quad v>0 \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \qquad \lim\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{3.2.10}$$

Does (3.2.10) have a minimal positive solution for any nontrivial measures μ_1, μ_2? Perhaps, new phenomena involving possible interactions between μ_1, μ_2, and σ will occur.

2. Suppose $\sigma \in M^+(\mathbb{R}^n)$ radially symmetric. Thus, the solution to (1.3.14) obtained in Theorem 1.3.9 must be radially symmetric. In addition, since $\mathbf{I}_{2\alpha}\sigma = (n-2\alpha)\mathbf{W}_{\alpha,2}\sigma$, condition (1.3.5) can be rewritten in terms of the Riesz potential. Is it possible to characterize condition (1.3.5), in terms of Riesz potential, for σ radially symmetric, as was done to the single equation in [Cao and Verbitsky 2016, Proposition 5.2]?

3. Let us consider the quasilinear elliptic system

$$\begin{cases} -\Delta_{p_1} u = \sigma\, v^{q_1}, \quad u>0 \quad \text{in} \quad \mathbb{R}^n, \\ -\Delta_{p_2} v = \sigma\, u^{q_2}, \quad v>0 \quad \text{in} \quad \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \qquad \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{3.2.11}$$

where $p_1,\, p_2 \in (1,n]$ and $q_1,\, q_2 < \max\{p_1-1, p_2-1\}$. Does (3.2.11) have a minimal positive solution? In fact, the question is related to the existence of a nontrivial solution to the following system integral

$$\begin{cases} u = \mathbf{W}_{\alpha_1,p_1}(v^{q_1}\mathrm{d}\sigma), \quad \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n, \\ v = \mathbf{W}_{\alpha_2,p_2}(u^{q_2}\mathrm{d}\sigma), \quad \mathrm{d}\sigma\text{-a.e in } \mathbb{R}^n, \end{cases}$$

It is to be expected that the answers should be more complicated because it will be necessary to assume a possible interaction between $\mathbf{W}_{\alpha_1,p_1}\sigma$ and $\mathbf{W}_{\alpha_2,p_2}\sigma$.

3.3 k-HESSIAN LANE-EMDEN TYPE SYSTEMS WITH SUB-NATURAL GROWTH TERMS

3.3.1 Proof of Theorem 1.3.11

Fix $1 \le k < n/2$, and let K be the constant given in Theorem G. Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying (1.3.3) with $\alpha = 2k/(k+1)$ and $p = k+1$. Applying Theorem 1.3.1 with $\alpha = 2k/(k+1)$, $p = k+1$ and $K^k\sigma$ in place of σ, there exists a nontrivial solution $(\tilde{u}, \tilde{v})$ to the system

$$\begin{cases} \tilde{u} = K\,\mathbf{W}_k(\tilde{v}^{q_1}\mathrm{d}\sigma) & \text{in} \quad \mathbb{R}^n, \\ \tilde{v} = K\,\mathbf{W}_k(\tilde{u}^{q_2}\mathrm{d}\sigma) & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{3.3.1}$$

Moreover, $(\tilde{u}, \tilde{v})$ satisfies

$$\begin{aligned} c^{-1}\Big(K\,\mathbf{W}_k\sigma\Big)^{\gamma_1} &\le \tilde{u} \le c\Big(K\,\mathbf{W}_k\sigma + \Big(K\,\mathbf{W}_k\sigma\Big)^{\gamma_1}\Big), \\ c^{-1}\Big(K\,\mathbf{W}_k\sigma\Big)^{\gamma_2} &\le \tilde{v} \le c\Big(K\,\mathbf{W}_k\sigma + \Big(K\,\mathbf{W}_k\sigma\Big)^{\gamma_2}\Big), \end{aligned} \tag{3.3.2}$$

where γ_1 and γ_2 are given by (3.1.13) and (3.1.18), respectively, with $p-1=k$, i.e.,

$$\gamma_1 = \frac{k(k+q_1)}{k^2 - q_1q_2} \quad \text{and} \quad \gamma_2 = \frac{k(k+q_2)}{k^2 - q_1q_2},$$

which can be rewritten as

$$\gamma_1 = \frac{q_1}{k}\gamma_2 + 1 \quad \text{and} \quad \gamma_2 = \frac{q_2}{k}\gamma_1 + 1.$$

Recall from Lemma 3.1.4 that $\mathbf{W}_k\sigma \in L^r_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ for all $r > 0$, whence $\tilde{u} \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ and $\tilde{v} \in L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$.

We set $u_0 = \varepsilon\,(\mathbf{W}_k\sigma)^{\gamma_1}$ and $v_0 = \varepsilon\,(\mathbf{W}_k\sigma)^{\gamma_2}$, where $\varepsilon > 0$ is given by

$$\varepsilon \le \min\{(K^{-1}\kappa^{\frac{q_1\gamma_2}{k}})^{\frac{k}{k-q_1}}, (K^{-1}\kappa^{\frac{q_2\gamma_1}{k}})^{\frac{k}{k-q_2}},\, c^{-1}K^{\gamma_1},\, c^{-1}K^{\gamma_2},\, C\,K^{-\gamma_1},\, C\,K^{-\gamma_2}\}, \tag{3.3.3}$$

where κ is the constant given in Lemma T, c is the constant in (3.3.2), and C is the constant given in Lemma 3.1.3. Similarly to Claim 1, one can show $u_0 \le K^{-1}\mathbf{W}_k(v_0^{q_1}\mathrm{d}\sigma)$ and $v_0 \le K^{-1}\mathbf{W}_k(u_0^{q_2}\mathrm{d}\sigma)$. In addition, $u_0 \le \tilde{u}$ and $v_0 \le \tilde{v}$ by choice of ε in (3.3.3).

Claim 5. There exists a pair of nonnegative functions $(u_1, v_1) \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma) \times L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ satisfying

$$\begin{cases} -u_1,\ -v_1 \in \Phi^k(\mathbb{R}^n), \\ u_1 \geq u_0, \quad v_1 \geq v_0, \\ F_k[-u_1] = \sigma\, v_0^{q_1} \quad \text{in} \quad \mathbb{R}^n, \quad \varliminf_{|x|\to\infty} u_1(x) = 0, \\ F_k[-v_1] = \sigma\, u_0^{q_2} \quad \text{in} \quad \mathbb{R}^n, \quad \varliminf_{|x|\to\infty} v_1(x) = 0, \\ c_1(\mathbf{W}_k\sigma)^{\gamma_1} \leq u_1 \leq \tilde{u}, \quad \tilde{c}_1(\mathbf{W}_k\sigma)^{\gamma_2} \leq v_1 \leq \tilde{v} \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

where

$$c_1 = K^{-1}(\varepsilon\,\kappa^{\gamma_2})^{\frac{q_1}{k}}, \quad \tilde{c}_1 = K^{-1}(\varepsilon\,\kappa^{\gamma_1})^{\frac{q_2}{k}}.$$

Indeed, letting $\sigma_{B_i} = \chi_{B_i}\sigma$, by Lemma H, there exist $u_1^i \geq 0$ and $v_1^i \geq 0$ satisfying

$$\begin{cases} -u_1^i \in \Phi^k(B_{i+1}) \cap C(\overline{B}_{i+1}), & -v_1^i \in \Phi^k(B_{i+1}) \cap C(\overline{B}_{i+1}), \\ F_k[-u_1^i] = \sigma_{B_i} v_0^{q_1} \quad \text{in} \quad B_{i+1}, & F_k[-v_1^i] = \sigma_{B_i} u_0^{q_1} \quad \text{in} \quad B_{i+1}, \\ u_1^i = 0 \quad \text{on} \quad \partial B_{i+1}, & v_1^i = 0 \quad \text{on} \quad \partial B_{i+1}. \end{cases}$$

Using [Phuc and Verbitsky 2009, Theorem 4.2], we obtain

$$\begin{aligned} u_1^i &\leq K\,\mathbf{W}_k(v_0^{q_1}\mathrm{d}\sigma) \leq K\,\mathbf{W}_k(\tilde{v}^{q_1}\mathrm{d}\sigma) = \tilde{u}, \\ v_1^i &\leq K\,\mathbf{W}_k(u_0^{q_2}\mathrm{d}\sigma) \leq K\,\mathbf{W}_k(\tilde{u}^{q_2}\mathrm{d}\sigma) = \tilde{v} \quad \forall i = 1, 2, \dots. \end{aligned} \tag{3.3.4}$$

Extending by zero away from B_{i+1}, we may assume that u_1^i and v_1^i are functions in whole $\mathbb{R}^n$. It follows from Theorem I and Remark 2.2.5 that the sequences $\{u_1^i\}$ and $\{v_1^i\}$ are increasing pointwise in $\mathbb{R}^n$. Consequently, by Remark 2.2.2, the pointwise limits $u_1 := \lim_{i\to\infty} u_1^i$ and $v_1 := \lim_{i\to\infty} v_1^i$ belong to $\Phi^k(\mathbb{R}^n)$, with $\{F_k[-u_1^i]\}_i$ and $\{F_k[-v_1^i]\}_i$ converging weakly to $F_k[-u_1]$ and $F_k[-v_1]$, respectively. On the other hand, by Monotone Convergence Theorem, $\{\sigma_{B_i} v_0^{q_1}\}_i$ and $\{\sigma_{B_i} u_0^{q_2}\}_i$ converge weakly to $\sigma\, v_0^{q_1}$ and $\sigma\, u_0^{q_2}$, respectively. Hence, as Radon measures in $\mathbb{R}^n$,

$$F_k[-u_1] = \sigma\, v_0^{q_1} \quad \text{and} \quad F_k[-v_1] = \sigma\, u_0^{q_2}.$$

Next, from (3.3.4), one has $u_1 \leq \tilde{u}$ and $v_1 \leq \tilde{v}$, whence $u_1 \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n,\, \mathrm{d}\sigma)$, $v_1 \in L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n,\, \mathrm{d}\sigma)$, and by (2.1.4) it holds

$$\varliminf_{|x|\to\infty} u_1(x) = \varliminf_{|x|\to\infty} v_1(x) = 0.$$

Combining Theorem G with Lemma T, we deduce

$$\begin{aligned}
u_1 &\geq K^{-1}\mathbf{W}_k(v_0^{q_1}\mathrm{d}\sigma) = K^{-1}\varepsilon^{\frac{q_1}{k}}\mathbf{W}_k\left((\mathbf{W}_k\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\right)\\
&\geq K^{-1}\varepsilon^{\frac{q_1}{k}}\kappa^{\frac{q_1}{k}\gamma_2}(\mathbf{W}_k\sigma)^{\frac{q_1}{k}\gamma_2+1} = K^{-1}\varepsilon^{\frac{q_1}{k}}\kappa^{\frac{q_1}{k}\gamma_2}(\mathbf{W}_k\sigma)^{\gamma_1},\\
v_1 &\geq K^{-1}\mathbf{W}_k(u_0^{q_2}\mathrm{d}\sigma) = K^{-1}\varepsilon^{\frac{q_2}{k}}\mathbf{W}_k\left((\mathbf{W}_k\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\right)\\
&\geq K^{-1}\varepsilon^{\frac{q_2}{k}}\kappa^{\frac{q_2}{k}\gamma_1}(\mathbf{W}_k\sigma)^{\frac{q_2}{k}\gamma_1+1} = K^{-1}\varepsilon^{\frac{q_2}{k}}\kappa^{\frac{q_2}{k}\gamma_1}(\mathbf{W}_k\sigma)^{\gamma_2},
\end{aligned}$$

By choice of $\varepsilon > 0$, the previous inequalities give $u_1 \geq u_0$ and $v_1 \geq v_0$. This completes the proof Claim 5.

With the aid of Lemma J, analysis similar to that in the proof of Claim 5 allows us to construct, inductively, a sequence of pairs of nonnegative functions $\{(u_j, v_j)\} \subset L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)\times L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ such that

$$\begin{cases}
-u_j, -v_j \in \Phi^k(\mathbb{R}^n),\\
F_k[-u_j] = \sigma\, v_{j-1}^{q_1} \quad \text{in} \quad \mathbb{R}^n, \quad \underline{\lim}_{|x|\to\infty} u_j(x) = 0,\\
F_k[-v_j] = \sigma\, u_{j-1}^{q_2} \quad \text{in} \quad \mathbb{R}^n, \quad \underline{\lim}_{|x|\to\infty} v_j(x) = 0,\\
0 \leq u_j \leq u_{j+1}, \quad 0 \leq v_j \leq v_{j+1} \quad \text{in} \quad \mathbb{R}^n\\
c_j(\mathbf{W}_k\sigma)^{\gamma_1} \leq u_j \leq \tilde{u}, \quad \tilde{c}_j(\mathbf{W}_k\sigma)^{\gamma_2} \leq v_j \leq \tilde{v} \quad \text{in} \quad \mathbb{R}^n.
\end{cases} \tag{3.3.5}$$

where c_1 and $\tilde{c}_1$ are as above, and for $j = 2, 3, \ldots$

$$c_j = K^{-1}(\tilde{c}_{j-1}\kappa^{\gamma_2})^{\frac{q_1}{k}}, \quad \tilde{c}_j = K^{-1}(c_{j-1}\kappa^{\gamma_1})^{\frac{q_2}{k}}.$$

Indeed, let u_1 and v_1 as in Claim 5. Since $\sigma\, v_1^{q_1} \geq \sigma\, v_0^{q_1}$ and $\sigma\, u_1^{q_2} \geq \sigma\, u_0^{q_2}$, it follows from Lemma J that there exists a pair of nonnegative functions (u_2, v_2) satisfying

$$\begin{cases}
-u_2, -v_2 \in \Phi^k(\mathbb{R}^n)\\
u_2 \geq u_1, \quad v_2 \geq v_1,\\
F_k[-u_2] = \sigma\, v_1^{q_1} \quad \text{in} \quad \mathbb{R}^n, \quad \underline{\lim}_{|x|\to\infty} u_2(x) = 0,\\
F_k[-v_2] = \sigma\, u_1^{q_2} \quad \text{in} \quad \mathbb{R}^n, \quad \underline{\lim}_{|x|\to\infty} v_2(x) = 0.
\end{cases}$$

Applying Theorem G, one has

$$\begin{aligned}
u_2 &\leq K\,\mathbf{W}_k(v_1^{q_1}\mathrm{d}\sigma) \leq K\,\mathbf{W}_k(\tilde{v}^{q_1}\mathrm{d}\sigma) = \tilde{u},\\
v_2 &\leq K\,\mathbf{W}_k(u_1^{q_2}\mathrm{d}\sigma) \leq K\,\mathbf{W}_k(\tilde{u}^{q_2}\mathrm{d}\sigma) = \tilde{v},
\end{aligned}$$

and, by Lemma T,

$$\begin{aligned} u_2 &\geq K^{-1}\mathbf{W}_k(v_1^{q_1}\mathrm{d}\sigma) = K^{-1}\tilde{c}_1^{\frac{q_1}{k}}\mathbf{W}_k\left((\mathbf{W}_k\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\right) \\ &\geq K^{-1}\tilde{c}_1^{\frac{q_1}{k}}\kappa^{\frac{q_1}{k}\gamma_2}(\mathbf{W}_k\sigma)^{\frac{q_1}{k}\gamma_2+1} = c_2\,(\mathbf{W}_k\sigma)^{\gamma_1}, \\ v_2 &\geq K^{-1}\mathbf{W}_k(u_1^{q_2}\mathrm{d}\sigma) = K^{-1}c_1^{\frac{q_2}{k}}\mathbf{W}_k\left((\mathbf{W}_k\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\right) \\ &\geq K^{-1}c_1^{\frac{q_2}{k}}\kappa^{\frac{q_2}{k}\gamma_1}(\mathbf{W}_k\sigma)^{\frac{q_2}{k}\gamma_1+1} = \tilde{c}_2\,(\mathbf{W}_k\sigma)^{\gamma_2}. \end{aligned}$$

Now, suppose (u_j, v_j) has been obtained for some $j > 1$. In the same manner, one shows that there exist a pair of nonnegative functions (u_{j+1}, v_{j+1}) satisfying

$$\begin{cases} -u_{j+1}, -v_{j+1} \in \Phi^k(\mathbb{R}^n) \\ u_{j+1} \geq u_j, \quad v_{j+1} \geq v_j, \\ F_k[-u_{j+1}] = \sigma\, v_j^{q_1} \quad \text{in} \quad \mathbb{R}^n, \quad \underline{\lim}_{|x|\to\infty} u_2(x) = 0, \\ F_k[-v_{j+1}] = \sigma\, u_j^{q_2} \quad \text{in} \quad \mathbb{R}^n, \quad \underline{\lim}_{|x|\to\infty} v_2(x) = 0. \\ c_{j+1}(\mathbf{W}_k\sigma)^{\gamma_1} \leq u_{j+1} \leq \tilde{u}, \qquad \tilde{c}_{j+1}(\mathbf{W}_k\sigma)^{\gamma_2} \leq v_{j+1} \leq \tilde{v}. \end{cases}$$

From the last line in (3.3.5), we deduce that $(u_j, v_j) \in L^{q_2}_{\mathrm{loc}}(\Omega, \mathrm{d}\sigma) \times L^{q_1}_{\mathrm{loc}}(\Omega, \mathrm{d}\sigma)$ for all $j \geq 1$.

We set the pointwise limits $u = \lim_{j\to\infty} u_j$ and $v = \lim_{j\to\infty} v_j$. Using Remark 2.2.2 again, both $-u$ and $-v$ belong to $\Phi^k(\mathbb{R}^n)$. By Monotone Convergence Theorem, $u \in L^{q_2}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ and $v \in L^{q_1}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$. Thus (u, v) satisfies

$$\begin{cases} F_k[-u] = \sigma\, v^{q_1} \quad \text{in} \quad \mathbb{R}^n, \\ F_k[-v] = \sigma\, u^{q_2} \quad \text{in} \quad \mathbb{R}^n. \end{cases}$$

In view of the last line in (3.3.5), letting $j \to \infty$, one can see that $u \leq \tilde{u}$ and $v \leq \tilde{v}$, whence $\underline{\lim}_{|x|\to\infty} u(x) = \underline{\lim}_{|x|\to\infty} v(x) = 0$, which shows that (u, v) solves Syst. (S_2) in sense of Definition 2.2.4. Furthermore, there exists a constant $C_1 = C_1(n, k, q_1, q_2, C_\sigma) > 0$ such that

$$u \leq C_1\left(\mathbf{W}_k\sigma + (\mathbf{W}_k\sigma)^{\gamma_1}\right) \quad \text{and} \quad v \leq C_1\left(\mathbf{W}_k\sigma + (\mathbf{W}_k\sigma)^{\gamma_2}\right),$$

which assures the upper bound in (1.3.16). Since $u \geq c_j(\mathbf{W}_k\sigma)^{\gamma_1}$ and $v \geq \tilde{c}_j(\mathbf{W}_k\sigma)^{\gamma_2}$ for all $j \geq 1$, a passage to the limit $j \to \infty$ implies that

$$u \geq C_2\,(\mathbf{W}_k\sigma)^{\gamma_1} \quad \text{and} \quad v \geq \tilde{C}_2\,(\mathbf{W}_k\sigma)^{\gamma_2},$$

where is straightforward to compute

$$\begin{aligned} C_2 &= \lim_{j\to\infty} c_j = K^{-\gamma_1}\kappa^{\frac{\gamma_1 q_1(\gamma_1 q_2 + \gamma_2 k)}{k^2}}, \\ \tilde{C}_2 &= \lim_{j\to\infty} \tilde{c}_j = K^{-\gamma_2}\kappa^{\frac{\gamma_2 q_2(\gamma_2 q_1 + \gamma_1 k)}{k^2}}, \end{aligned}$$

which shows a lower bound in (1.3.16).

It remains to prove that (u, v) is a minimal solution; that is, if (w_1, w_2) is any nontrivial solution to Syst. (S_2), then $w_1 \geq u$ and $w_2 \geq v$ almost everywhere in $\mathbb{R}^n$. Let (w_1, w_2) be a solution to Syst. (S_2). From Theorem G, it follows

$$\begin{cases} w_1 \geq K^{-1}\mathbf{W}_k(w_2^{q_1}\mathrm{d}\sigma) = \mathbf{W}_k(K^{-k}w_2^{q_1}\mathrm{d}\sigma) \\ w_2 \geq K^{-1}\mathbf{W}_k(w_1^{q_2}\mathrm{d}\sigma) = \mathbf{W}_k(K^{-k}w_1^{q_2}\mathrm{d}\sigma) \end{cases}$$

Using Lemma 3.1.3 with $K^{-k}\sigma$ in place of σ, we obtain

$$\begin{aligned} w_1 &\geq C\left(\mathbf{W}_k(K^{-k}\sigma)\right)^{\gamma_1} = C\left(K^{-1}\mathbf{W}_k\sigma\right)^{\gamma_1} = C\,K^{-\gamma_1}(\mathbf{W}_k\sigma)^{\gamma_1}, \\ w_2 &\geq C\left(\mathbf{W}_k(K^{-k}\sigma)\right)^{\gamma_2} = C\left(K^{-1}\mathbf{W}_k\sigma\right)^{\gamma_2} = C\,K^{-\gamma_2}(\mathbf{W}_k\sigma)^{\gamma_2}. \end{aligned}$$

Let $\mathrm{d}\omega_1 = w_2^{q_1}\mathrm{d}\sigma$ and $\mathrm{d}\omega_2 = w_1^{q_2}\mathrm{d}\sigma$. Notice that by choice of ε in (3.3.3), we have $\mathrm{d}\omega_1 \geq v_0^{q_1}\mathrm{d}\sigma$ and $\mathrm{d}\omega_2 \geq u_0^{q_2}\mathrm{d}\sigma$. By construction of (u_1, v_1), using Lemma J, we deduce that the functions u_1 and v_1 satisfy $u_1 \leq w_1$ and $v_1 \leq w_2$ in $\mathbb{R}^n$. Repeating this argument by induction, we conclude $u_j \leq w_1$ and $v_j \leq w_2$ for every $j \geq 1$. Therefore,

$$u = \lim_{j\to\infty} u_j \leq w_1, \quad v = \lim_{j\to\infty} v_j \leq w_2 \quad \text{in} \quad \mathbb{R}^n.$$

This completes the proof of Theorem 1.3.11.

3.3.2 Proof of Theorem 1.3.12

Fix $1 \leq k < n/2$. Let γ_1 and γ_2 given by (3.1.13) and (3.1.18), respectively, with $p-1 = k$. We begin by proving the statement $(\mathbf{i})$. Let (u, v) be a nontrivial solution to (S_2), in the sense of Definition 2.2.4, satisfying (1.3.16). By Theorem G, (u, v) satisfies the system

$$\begin{cases} u \geq K^{-1}\mathbf{W}_k(v^{q_1}\mathrm{d}\sigma), \\ v \geq K^{-1}\mathbf{W}_k(u^{q_2}\mathrm{d}\sigma). \end{cases}$$

Applying (1.3.16) in the previous inequalities,

$$\begin{aligned} K^{-1}c^{-\frac{q_1}{k}}\mathbf{W}_k\left((\mathbf{W}_k\sigma)^{\gamma_2 q_1}\mathrm{d}\sigma\right) &\leq K^{-1}\mathbf{W}_k(v^{q_1}\mathrm{d}\sigma) \leq u \leq c\left(\mathbf{W}_k\sigma + (\mathbf{W}_k\sigma)^{\gamma_1}\right), \\ K^{-1}c^{-\frac{q_2}{k}}\mathbf{W}_k\left((\mathbf{W}_k\sigma)^{\gamma_1 q_2}\mathrm{d}\sigma\right) &\leq K^{-1}\mathbf{W}_k(u^{q_2}\mathrm{d}\sigma) \leq v \leq c\left(\mathbf{W}_k\sigma + (\mathbf{W}_k\sigma)^{\gamma_2}\right). \end{aligned}$$

Letting $\lambda = K\,\max\{c^{\frac{k+q_1}{k}}, c^{\frac{k+q_2}{k}}\}$, the statement $(\mathbf{i})$ follows.

To establish the statement $(\mathbf{ii})$, let $\lambda > 0$ such that (1.3.17) holds. By Theorem 1.3.3 with $\alpha = 2k/(k+1)$, $p = k+1$ and $K^k\sigma$ in place of σ, there exists a nontrivial solution $(\tilde{u}, \tilde{v})$ to syst. (3.3.1), satisfying (3.3.2) with $c = c_0(n, k, q_1, q_2, \lambda)$. As in the proof of Theorem 1.3.11, the existence of such $(\tilde{u}, \tilde{v})$ yields that there exists a solution (u, v) to (S_2) for which (1.3.16) holds with $c = c_1(n, k, q_1, q_2, \lambda) > 0$. This proves the statement $(\mathbf{ii})$ and completes the proof of Theorem 1.3.12.

3.3.3 Proof of Corollary 1.3.13

Fix $1 \le k < /2$, and let $\sigma \in M^+(\mathbb{R}^n)$. Suppose $q_1 = q_2 < k$. Setting $q = q_1$, from (3.1.13) and (3.1.18), γ_1 and γ_2 should satisfy

$$\gamma_1 = \gamma_2 = \frac{k(k+q)}{k^2 - q^2} = \frac{k}{k-q}.$$

The statements $(\mathbf{i})$, $(\mathbf{ii})$ and $(\mathbf{iii})$ follow by the same methods as in the proofs of Theorem 1.3.11 and Theorem 1.3.12, respectively.

Indeed, suppose (1.3.3). Following the notation used in the proof of Theorem 1.3.11, (3.2.9) yields $u_0 = v_0$, whence $u_1 = v_1$ by Claim 5. Consequently, by induction

$$u_j = v_j \quad \forall j = 1, 2, \ldots,$$

where (u_j, v_j) are given in (3.3.5) for $j = 1, 2, \ldots$. A passage to the limit as before implies that $u = v \in L^q_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ and $-u \in \Phi^k(\mathbb{R}^n)$. Hence Syst. (S_2) reduces to Eq. (P_1), i.e.

$$\begin{cases} F_k[-u] = \sigma\, u^q \quad \text{in} \quad \mathbb{R}^n, \\ \varliminf_{|x|\to\infty} u(x) = 0. \end{cases}$$

Furthermore, u satisfies bilateral estimates (1.3.18), with $c = c(n, k, q, \sigma) > 0$. This shows part $(\mathbf{i})$. The statement $(\mathbf{ii})$ follows from combining the lower bound given in Theorem G with (1.3.18),

$$K^{-1} c^{-\frac{q}{k}} \mathbf{W}_k \left((\mathbf{W}_k \sigma)^{\frac{kq}{k-q}} \mathrm{d}\sigma \right) \le K^{-1} \mathbf{W}_k (u^q \mathrm{d}\sigma) \le u \le c \left(\mathbf{W}_k \sigma + (\mathbf{W}_k \sigma)^{\frac{k}{k-q}} \right),$$

whence we may take $\lambda = K\, c^{\frac{k+q}{k}}$. This establishes the assertion. To prove the statement $(\mathbf{iii})$, we first notice that a combination of Remark 1.3.4 with condition (1.3.19) implies that there exists a solution $\tilde{u} \in L^q_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ to equation

$$\tilde{u} = K\, \mathbf{W}_k (\tilde{u}^q \mathrm{d}\sigma) \quad \text{in} \quad \mathbb{R}^n.$$

Thus, as in the proof of Theorem 1.3.12, there exists a solution u to (P_1) satisfying (1.3.18), with $c = c(n, k, q, \lambda) > 0$. This verify $(\mathbf{iii})$, and finally completes the proof of Corollary 1.3.13.

4 QUASILINEAR ELLIPTIC EQUATIONS WITH ORLICZ GROWTH

In this chapter, we provide the proof of Theorem 1.3.14 and Theorem 1.3.15. We divide our arguments into four sections in the sequel, where we first produce a framework for the proof of Theorem 1.3.14. Next, we establish some preliminary results to deal with Eq. (P_2), and finally, we provide the proof of Theorem 1.3.15. Finally, we present some comments concerning Theorem 1.3.15.

Here and subsequently, g is the function given by (A_2), with $2 \leq p < q < \infty$, and G its primitive, that is

$$g(t) = t^{p-1} + t^{q-1}, \quad G(t) = \frac{t^p}{p} + \frac{t^q}{q}, \quad \forall t \geq 0.$$

Being $p \geq 2$, g is a convex function, and g^{-1} is concave. Notice that g satisfies the "sub-multiplicity" condition

$$g(ab) \leq g(a)g(b) \quad \forall a, b \geq 0, \tag{4.0.1}$$

which implies a "sup-multiplicity" condition to g^{-1}:

$$g^{-1}(ab) \geq g^{-1}(a)g^{-1}(b) \quad \forall a, b \geq 0. \tag{4.0.2}$$

In the general context of N-functions, these conditions are known as Δ'-*contidion* (see for instance [Rao and Ren 1991, page 28]).

4.1 POTENTIAL OF WOLFF-TYPE

We start with some useful estimates for Wolff potentials.

Lemma 4.1.1. *Let* $\sigma \in M^+(\mathbb{R}^n)$*. Fix* $x \in \mathbb{R}^n$ *and* $R > 0$*. Then there exists a constant* $c = c(n,p,q) > 0$ *such that*

$$c^{-1} \int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right) \mathrm{d}t \leq \inf_{B(x,R)} \mathbf{W}_G\sigma \leq c \int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right) \mathrm{d}t. \tag{4.1.1}$$

Proof. We first show the lower bound. Setting $B = B(x,R)$, note that for $y \in B$ we have

$$\begin{aligned}\mathbf{W}_G\sigma(y) &= \int_0^\infty g^{-1}\left(\frac{\sigma(B(y,t))}{t^{n-1}}\right) \mathrm{d}t \geq \int_{2R}^\infty g^{-1}\left(\frac{\sigma(B(y,t))}{t^{n-1}}\right) \mathrm{d}t \\ &= \int_R^\infty g^{-1}\left(\frac{\sigma(B(y,2t))}{2^{n-1}t^{n-1}}\right) \frac{\mathrm{d}t}{2}.\end{aligned}$$

Using (2.3.4) in the previous estimate, we obtain

$$\mathbf{W}_G\sigma(y) \geq c_1 \int_R^\infty g^{-1}\left(\frac{\sigma(B(y,2t))}{t^{n-1}}\right) \mathrm{d}t, \tag{4.1.2}$$

where $c_1 = c_1(n,p,q)$. For $y \in B$, we have $B(y,2t) \supset B(x,t)$ for all $t \geq R$, consequently the lower estimate in (4.1.1) follows from (4.1.2), that is

$$\mathbf{W}_G\sigma(y) \geq c_1 \int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right) \mathrm{d}t \quad \forall y \in B.$$

Now, to show the upper bound in (4.1.1), it is sufficient to check that

$$\frac{1}{|B|}\int_B \mathbf{W}_G\sigma(y)\,\mathrm{d}y \leq c \int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right) \mathrm{d}t. \tag{4.1.3}$$

We write

$$\begin{aligned}\frac{1}{|B|}\int_B \mathbf{W}_G\sigma(y)\,\mathrm{d}y &= \frac{1}{|B|}\int_B \left(\int_0^R g^{-1}\left(\frac{\sigma(B(y,t))}{t^{n-1}}\right) \mathrm{d}t\right) \mathrm{d}y \\ &\quad + \frac{1}{|B|}\int_B \left(\int_R^\infty g^{-1}\left(\frac{\sigma(B(y,t))}{t^{n-1}}\right) \mathrm{d}t\right) \mathrm{d}y \\ &=: I_1 + I_2. \end{aligned} \tag{4.1.4}$$

To estimate I_2, notice that since $B(y,t) \subset B(x,2t)$ for $y \in B$ and $t \geq R$, it follows

$$\begin{aligned} I_2 &\leq \frac{1}{|B|}\int_B \left(\int_R^\infty g^{-1}\left(\frac{\sigma(B(x,2t))}{t^{n-1}}\right) \mathrm{d}t\right) \mathrm{d}y \\ &= \frac{1}{|B|}\int_B \mathrm{d}y \int_R^\infty g^{-1}\left(\frac{\sigma(B(x,2t))}{t^{n-1}}\right) \mathrm{d}t \\ &= \int_{2R}^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{2^{n-1}t^{n-1}}\right) \mathrm{d}t \leq \int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{2^{n-1}t^{n-1}}\right) \mathrm{d}t. \end{aligned}$$

Using (2.3.6) in the previous estimate, yields

$$I_2 \leq c_2 \int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right) \mathrm{d}t, \tag{4.1.5}$$

where $c_2 = c_2(n,p,q) > 0$. Next, to estimate I_1, using Fubuni's Theorem and Jensen's inequality, since g^{-1} is concave, we obtain

$$\begin{aligned} I_1 &= \int_0^R \left(\frac{1}{|B|}\int_B g^{-1}\left(\frac{\sigma(B(y,t))}{t^{n-1}}\right) \mathrm{d}y\right) \mathrm{d}t \\ &\leq \int_0^R g^{-1}\left(\frac{1}{|B|t^{n-1}}\int_B \sigma(B(y,t))\,\mathrm{d}y\right) \mathrm{d}t. \end{aligned} \tag{4.1.6}$$

Notice that $B(y,t) \subset B(x,2R)$ if $y \in B$ and $t \le R$, whence we have from Fubini's Theorem

$$\begin{aligned}\int_B \sigma(B(y,t))\,\mathrm{d}y &= \int_{B(x,R)} \left(\int_{B(y,t)} \mathrm{d}\sigma\right)\mathrm{d}y = \int_{B(x,R)} \left(\int_{B(x,2R)} \chi_{B(y,t)}\,\mathrm{d}y\right)\mathrm{d}\sigma \\ &\le \int_{B(x,2R)} \left(\int_{B(x,2R)} \chi_{B(y,t)}\,\mathrm{d}y\right)\mathrm{d}\sigma = \int_{B(x,2R)} |B(y,t)|\,\mathrm{d}\sigma \\ &= |B(0,1)| t^n\, \sigma(B(x,2R)).\end{aligned}$$

By the preceding estimate and since g^{-1} is an increasing function, it follows from (4.1.6)

$$\begin{aligned}I_1 &\le \int_0^R g^{-1}\left(\frac{\sigma(B(x,2R))}{R^n} t\right) dt \\ &\le \int_0^R g^{-1}\left(\frac{\sigma(B(x,2R))}{R^{n-1}}\right)\mathrm{d}t = g^{-1}\left(\frac{\sigma(B(x,2R))}{R^{n-1}}\right) R. \end{aligned} \tag{4.1.7}$$

Note that if $R \le t \le 2R$, we have $\sigma(B(x,2t)) \supset \sigma(B(x,2R))$, and

$$g^{-1}\left(\frac{\sigma(B(x,2R))}{R^{n-1}}\right) \le g^{-1}\left(\frac{\sigma(B(x,2t))}{(2^{-1}t)^{n-1}}\right).$$

The preceding inequality implies

$$\begin{aligned}g^{-1}\left(\frac{\sigma(B(x,2R))}{R^{n-1}}\right) R &\le \int_R^{2R} g^{-1}\left(\frac{2^{n-1}\sigma(B(x,2t))}{t^{n-1}}\right)\mathrm{d}t \\ &\le \int_R^{\infty} g^{-1}\left(\frac{2^{n-1}\sigma(B(x,2t))}{t^{n-1}}\right)\mathrm{d}t \\ &= \int_{2R}^{\infty} g^{-1}\left(\frac{4^{n-1}\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t \\ &\le c_3 \int_R^{\infty} g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t, \end{aligned} \tag{4.1.8}$$

where in the last inequality was used (2.3.6), with $c_3 = c_3(n,p,q)$. Using (4.1.8) in (4.1.7), we deduce

$$I_1 \le c_3 \int_R^{\infty} g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t. \tag{4.1.9}$$

Thus, because of (4.1.4), (4.1.5) and (4.1.9) provide (4.1.3), which shows the upper bound in (4.1.1). This completes the proof of Lemma 4.1.1. □

As a consequence of Lemma 4.1.1, we will show that $\mathbf{W}_G\sigma < \infty$ if and only if

$$\int_1^{\infty} g^{-1}\left(\frac{\sigma(B(0,t))}{t^{n-1}}\right)\mathrm{d}t < \infty. \tag{4.1.10}$$

Corollary 4.1.2. *Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying* (4.1.10). *Then, for all $x \in \mathbb{R}^n$, $R > 0$, we have*

$$\int_R^{\infty} g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t < \infty. \tag{4.1.11}$$

In particular, $\mathbf{W}_G\sigma < \infty$ *if and only if* (4.1.10) *holds. Furthermore,* $\mathbf{W}_G\sigma \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\lim_{|x|\to\infty} \mathbf{W}_G\sigma(x) = 0. \tag{4.1.12}$$

Proof. First, notice that (4.1.10) implies (4.1.11) for $x = 0$. Indeed, for $R \geq 1$, (4.1.11) is obvious. Suppose that $0 < R < 1$. Then, we have

$$\begin{aligned}\int_R^\infty g^{-1}\left(\frac{\sigma(B(0,t))}{t^{n-1}}\right)\mathrm{d}t &= \int_R^1 g^{-1}\left(\frac{\sigma(B(0,t))}{t^{n-1}}\right)\mathrm{d}t + \int_1^\infty g^{-1}\left(\frac{\sigma(B(0,t))}{t^{n-1}}\right)\mathrm{d}t\\ &\leq (1-R)g^{-1}\left(\frac{\sigma(B(0,1))}{R^{n-1}}\right) + \int_1^\infty g^{-1}\left(\frac{\sigma(B(0,t))}{t^{n-1}}\right)\mathrm{d}t,\end{aligned}$$

where the right-hand side in the previous inequality is finite, which shows (4.1.11) for $0 < R < 1$ and $x = 0$.

Next, fix $x \neq 0$ and let $R \geq |x|$. If $t \geq |x|$, $B(x,t) \subset B(0,2t)$, and consequently

$$\begin{aligned}I_{|x|} := \int_{|x|}^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t &\leq \int_{|x|}^\infty g^{-1}\left(\frac{\sigma(B(0,2t))}{t^{n-1}}\right)\mathrm{d}t\\ &\leq c_1\int_{|x|}^\infty g^{-1}\left(\frac{\sigma(B(0,2t))}{t^{n-1}}\right)\mathrm{d}t < \infty,\end{aligned}$$

where $c_1 = c_1(n,p,q)$. From the preceding estimate, (4.1.11) holds for $R \geq |x|$. The case $R < |x|$ follows from the splitting

$$\begin{aligned}\int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t &= \int_R^{|x|} g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t + I_{|x|}\\ &\leq (|x|-R)g^{-1}\left(\frac{\sigma(B(x,|x|))}{R^{n-1}}\right) + I_{|x|} < \infty.\end{aligned}$$

Thus, (4.1.11) holds for all $x \in \mathbb{R}^n$ and $R > 0$.

Now, by (4.1.3), there exists a constant $c_2 = c_2(n,p,q) > 0$ such that

$$\int_{B(x,R)} \mathbf{W}_G\sigma\,\mathrm{d}y \leq c_2\,|B(x,R)|\int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t, \quad \forall x \in \mathbb{R}^n, R > 0,$$

and this ensure $\mathbf{W}_G\sigma \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, since we already known that (4.1.10) implies (4.1.11) for all $x \in \mathbb{R}^n$ and $R > 0$. We also have by (4.1.1) that $\mathbf{W}_G\sigma < \infty$ if and only if (4.1.11) holds for all $x \in \mathbb{R}^n$ and $R > 0$.

It remains to show (4.1.12). Setting $A_R = \{x \in \mathbb{R}^n : R/2 < |x| < R\}$ for $R > 0$ and arguing as in the proof of the upper estimate in (4.1.1) (with $x = 0$), we deduce that there exists $c_3 = c_3(n,p,q) > 0$ such that

$$\begin{aligned}\inf_{|x|>R/2} \mathbf{W}_G\sigma \leq \inf_{A_R} \mathbf{W}_G\sigma &\leq \frac{1}{|A_R|}\int_{A_R} \mathbf{W}_G\sigma\,\mathrm{d}y\\ &\leq c_3\int_R^\infty g^{-1}\left(\frac{\sigma(B(x,t))}{t^{n-1}}\right)\mathrm{d}t.\end{aligned}$$

Letting $R \to \infty$ in the previous inequality, we concluded (4.1.12). This completes the proof of Corollary 4.1.2. □

The remainder of this section will be devoted to establishing preliminary results to the proof of Theorem 1.3.14. In what follows, we fix

$$\begin{cases} f(t) = g(t^\gamma), \quad F(t) = \displaystyle\int_0^t f(s)\,\mathrm{d}s \quad \forall t \geq 0, \quad \text{that is} \\ f(t) = t^{(p-1)\gamma} + t^{(q-1)\gamma}, \quad F(t) = \dfrac{t^{(p-1)\gamma+1}}{(p-1)\gamma+1} + \dfrac{t^{(q-1)\gamma+1}}{(q-1)\gamma+1} \quad \forall t \geq 0, \end{cases} \tag{4.1.13}$$

where γ satisfies (A_1). Using (2.3.2), one can verify that

$$(p-1)\gamma \leq \frac{tf'(t)}{f(t)} \leq (q-1)\gamma, \quad (p-1)\gamma + 1 \leq \frac{tf(t)}{F(t)} \leq (q-1)\gamma + 1 \quad \forall t > 0.$$

From this, $L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ is a Banach space. Moreover, using [Harjulehto and Hästö 2019, Lemma 3.7.7], we can identify $L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ with $L^{(p-1)\gamma+1}(\mathbb{R}^n, \mathrm{d}\sigma) \cap L^{(q-1)\gamma+1}(\mathbb{R}^n, \mathrm{d}\sigma)$ as Banach spaces, where the intersection is equipped with the norm

$$\|u\|_{L^{(p-1)\gamma+1} \cap L^{(q-1)\gamma+1}} = \max\left\{\|u\|_{L^{(p-1)\gamma+1}}, \|u\|_{L^{(q-1)\gamma+1}}\right\}.$$

We will consider supersolutions to the integral equation (S).

Definition 4.1.3. A *supersolution* to (S) is a nonnegative function $u \in L^f_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ which satisfies (pointwise)

$$u \geq \mathbf{W}_G\Big(f(u)\mathrm{d}\sigma\Big) \quad \text{in} \quad \mathbb{R}^n. \tag{4.1.14}$$

The notion of *solution* or *subsolution* to (S) is defined similarly by replacing "$\geq$" by "$=$" or "$\leq$" in (4.1.14), respectively.

In the following theorem, we obtain a lower bound for supersolutions to (S) in terms of Wolff potentials, whenever (4.0.1) holds.

Theorem 4.1.4. *Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying* (4.1.10) *and let u be a nontrivial supersolution to* (4.1.14)*. Then there exists a constant $0 < C < 1$ depending only n, p, q and γ such that*

$$u(x) \geq C\left(\mathbf{W}_G\sigma(x)\right)^{\frac{1}{1-\gamma}} \quad \forall x \in \mathbb{R}^n. \tag{4.1.15}$$

Let us show the following result before proving Theorem 4.1.4.

Lemma 4.1.5. *Fix $\alpha > 0$ and let $\varphi(t) = g(t^\alpha)$, $t \geq 0$. Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying* (4.1.10)*. Then there exists a constant $0 < \lambda < 1$, which depends only on n, p, q and α, such that*

$$\mathbf{W}_G\Big(\varphi(\mathbf{W}_G\sigma)\mathrm{d}\sigma\Big)(x) \geq \lambda\left(\mathbf{W}_G\sigma(x)\right)^{1+\alpha}, \quad \forall x \in \mathbb{R}^n.$$

Proof of Lemma 4.1.5. By definition (1.0.5), for any $t > 0$

$$\mathbf{W}_G\sigma(y) \geq \int_t^\infty g^{-1}\Big(\frac{\sigma(B(y,s))}{s^{n-1}}\Big)\mathrm{d}s, \quad \forall y \in \mathbb{R}^n.$$

Notice that $B(y, 2s) \supset B(x, s)$ for $y \in B(x, t)$ and $s \geq t$, whence by (2.3.6) and the previous estimate it holds

$$\begin{aligned}
\mathbf{W}_G\sigma(y) &\geq \int_t^\infty g^{-1}\Big(\frac{\sigma(B(y,s))}{s^{n-1}}\Big)\,\mathrm{d}s = \int_{t/2}^\infty g^{-1}\Big(\frac{\sigma(B(y,2s))}{(2s)^{n-1}}\Big)\,\mathrm{d}s \\
&\geq c_1\int_{t/2}^\infty g^{-1}\Big(\frac{\sigma(B(y,2s))}{s^{n-1}}\Big)\,\mathrm{d}s \geq c_1\int_t^\infty g^{-1}\Big(\frac{\sigma(B(y,2s))}{s^{n-1}}\Big)\,\mathrm{d}s \\
&\geq c_1\int_t^\infty g^{-1}\Big(\frac{\sigma(B(x,s))}{s^{n-1}}\Big)\,\mathrm{d}s =: c_1\rho(t), \qquad (4.1.16)
\end{aligned}$$

where $c_1 = c_1(n, p, q) > 0$. Since $\varphi(t) = g(t^\alpha)$ is an increasing function, it follows from (4.1.16) that

$$\begin{aligned}
\mathbf{W}_G\big(\varphi(\mathbf{W}_G\sigma)\mathrm{d}\sigma\big)(x) &\geq \int_0^\infty g^{-1}\Big[\frac{1}{t^{n-1}}\int_{B(x,t)}\varphi\big(\mathbf{W}_G\sigma(y)\big)\mathrm{d}\sigma(y)\Big]\,\mathrm{d}t \\
&\geq \int_0^\infty g^{-1}\Big[\frac{1}{t^{n-1}}\int_{B(x,t)}\varphi(c_1\rho(t))\mathrm{d}\sigma(y)\Big]\,\mathrm{d}t \\
&= \int_0^\infty g^{-1}\Big[\varphi(c_1\rho(t))\frac{\sigma(B(x,t))}{t^{n-1}}\Big]\,\mathrm{d}t \quad \forall x \in \mathbb{R}^n. \qquad (4.1.17)
\end{aligned}$$

By (2.3.4) and (2.3.6), we have respectively $\varphi(c_1\rho(t)) \geq c_2\varphi(\rho(t))$ and $g^{-1}(c_2\varphi(\rho(t))) \geq c_3g^{-1}(\varphi(\rho(t)))$, where $c_2 = c_2(n, p, q, \alpha) > 0$ and $c_3 = c_3(n, p, q, \alpha) > 0$. Using these estimates in (4.1.17), with the aid of (4.0.2), we obtain

$$\begin{aligned}
\mathbf{W}_G\big(\varphi(\mathbf{W}_G\sigma)\mathrm{d}\sigma\big)(x) &\geq c_3\int_0^\infty g^{-1}\Big[\varphi(\rho(t))\frac{\sigma(B(x,t))}{t^{n-1}}\Big]\,\mathrm{d}t \\
&\geq c_3\int_0^\infty g^{-1}(\varphi(\rho(t)))g^{-1}\Big(\frac{\sigma(B(x,t))}{t^{n-1}}\Big)\,\mathrm{d}t. \qquad (4.1.18)
\end{aligned}$$

Note that $g^{-1}(\varphi(\rho(t))) = \rho(t)^\alpha$ and, by Fundamental Theorem of Calculus,

$$\rho'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_t^\infty g^{-1}\Big(\frac{\sigma(B(x,s))}{s^{n-1}}\Big)\,\mathrm{d}s\Big) = -g^{-1}\Big(\frac{\sigma(B(x,t))}{t^{n-1}}\Big),$$

Hence, we may rewrite (4.1.18) as follows:

$$\mathbf{W}_G\big(\varphi(\mathbf{W}_G\sigma)\mathrm{d}\sigma\big)(x) \geq c_3\int_0^\infty \rho(t)^\alpha\,(-\rho'(t))\,\mathrm{d}t.$$

Integrating by parts, we concluded from the previous inequality that

$$\mathbf{W}_G\big(\varphi(\mathbf{W}_G\sigma)\mathrm{d}\sigma\big)(x) \geq \frac{c_3}{1+\alpha}\rho(0)^{1+\alpha} = \frac{c_3}{1+\alpha}\left(\mathbf{W}_G\sigma(x)\right)^{1+\alpha},$$

which completes the proof of Lemma 4.1.5 by taking

$$\lambda = \frac{c_3}{1+\alpha} = \frac{1}{1+\alpha}\Big(\frac{p}{q}\Big)^{\frac{2}{p-1}}\Big(\frac{p}{2^{n-1}q}\Big)^{\frac{\alpha(q-1)}{(p-1)^2}}. \tag{4.1.19}$$

□

Proof of Theorem 4.1.4. The main idea of the proof is to iterate the inequality (4.1.14) with Lemma 4.1.5. First, we prove the following claim.

Claim 1. Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying (4.1.10). Suppose that u is a nontrivial supersolution to (S) such that it holds

$$u(x) \geq c\left(\mathbf{W}_G\sigma(x)\right)^{\delta}, \quad x \in \mathbb{R}^n,$$

where $0 < c < 1$ and $\delta > 0$. Then

$$u(x) \geq \Big(\frac{p}{q}\Big)^{\frac{2}{p-1}} c^{\frac{\gamma(q-1)}{p-1}} \lambda\left(\mathbf{W}_G\sigma(x)\right)^{1+\delta\gamma} \quad x \in \mathbb{R}^n,$$

where λ is the constant given in Lemma 4.1.5.

Indeed, a combination of (2.3.4) and (2.3.6) with Lemma 4.1.5 gives

$$\begin{aligned}
u(x) &\geq \mathbf{W}_G\Big(f(u)\mathrm{d}\sigma\Big) \geq \mathbf{W}_G\Big(f\Big(c\Big(\mathbf{W}_G\sigma(x)\Big)^{\delta}\Big)\mathrm{d}\sigma\Big)\\
&\geq \mathbf{W}_G\left(\frac{p}{q}c^{\gamma(q-1)}g\Big(\Big(\mathbf{W}_G\sigma(x)\Big)^{\delta\gamma}\Big)\mathrm{d}\sigma\right)\\
&\geq \left(\frac{p}{q}\right)^{\frac{2}{p-1}} c^{\frac{\gamma(q-1)}{p-1}}\mathbf{W}_G\Big(g\Big((\mathbf{W}_G\sigma(x))^{\delta\gamma}\Big)\mathrm{d}\sigma\Big) \geq \left(\frac{p}{q}\right)^{\frac{2}{p-1}} c^{\frac{\gamma(q-1)}{p-1}}\lambda\left(\mathbf{W}_G\sigma(x)\right)^{1+\delta\gamma},
\end{aligned}$$

which is our Claim 1.

Now, fix $x \in \mathbb{R}^n$ and $R > |x|$, and let $\sigma_B = \chi_B\sigma$, where $B = B(0,R)$. Setting $\mathrm{d}\mu = f(u)\mathrm{d}\sigma$, we estimate $\mathbf{W}_G\mu(z)$ as follows

$$\begin{aligned}
\mathbf{W}_G\mu(z) &= \int_0^\infty g^{-1}\Big(\frac{\mu(B(z,t))}{t^{n-1}}\Big)\,\mathrm{d}t \geq \int_R^\infty g^{-1}\Big(\frac{\mu(B(z,t))}{t^{n-1}}\Big)\,\mathrm{d}t\\
&= \int_{R/2}^\infty g^{-1}\Big(\frac{\mu(B(z,2t))}{(2t)^{n-1}}\Big)2\,\mathrm{d}t\\
&\geq c_1\int_R^\infty g^{-1}\Big(\frac{\mu(B(z,2t))}{(2t)^{n-1}}\Big)\,\mathrm{d}t,
\end{aligned}$$

where $c_1 = c_1(n,p,q) > 0$ was obtained in a light of (2.3.6). Since $B(z,2t) \supset B(0,t)$ for $t \geq R$ and $z \in B$, it follows from the previous inequality that for all $z \in B$

$$\mathbf{W}_G\mu(z) \geq c_1\int_R^\infty g^{-1}\Big(\frac{\mu(B(0,t))}{(2t)^{n-1}}\Big)\mathrm{d}t =: A(R). \tag{4.1.20}$$

We may assume $A(R) < 1$ for $R > 0$ large enough. Thus iterating (4.1.14) with (4.1.20), we obtain for all $x \in \mathbb{R}^n$

$$\begin{aligned} u(x) &\geq \mathbf{W}_G(f(\mathbf{W}_G\mu)\mathrm{d}\sigma_B)(x) \geq \mathbf{W}_G(f(A(R))\mathrm{d}\sigma_B)(x) \\ &\geq \int_0^\infty g^{-1}\left(\frac{f(A(R))\sigma_B(B(x,t))}{t^{n-1}}\right)\mathrm{d}t \\ &\geq A(R)^\gamma\, \mathbf{W}_G\sigma_B(x), \end{aligned} \tag{4.1.21}$$

where in the last line was used (4.0.2). Setting $c_1 = A(R)^\gamma$ and $\delta_1 = 1$, by Claim 1, with σ_B in place of σ, and (4.1.21), we construct a sequence of lower bounds for u as follows:

$$u(x) \geq c_j \left(\mathbf{W}_G\sigma_B(x)\right)^{\delta_j}, \quad x \in \mathbb{R}^n, \tag{4.1.22}$$

where for $j = 2, 3, \ldots$, δ_j and c_j are given by

$$\begin{aligned} \delta_j &= 1 + \gamma\delta_{j-1}, \\ c_j &= \lambda \left(\frac{p}{q}\right)^{\frac{2}{p-1}} c_{j-1}^{\frac{\gamma(q-1)}{p-1}}. \end{aligned} \tag{4.1.23}$$

Since $0 < \gamma < (p-1)/(q-1) \leq 1$, letting the limit $j \to \infty$ in (4.1.23), it is straightforward to conclude that

$$\begin{aligned} \lim_{j\to\infty} \delta_j &= \frac{1}{1-\gamma}, \\ \lim_{j\to\infty} c_j &= \lambda^{\frac{p-1}{p-1-\gamma(q-1)}} \left(\frac{p}{q}\right)^{\frac{2(p-1)}{(p-1)(q-1)-\gamma(q-1)^2}} =: C. \end{aligned} \tag{4.1.24}$$

Passing to the limit as $j \to \infty$ in (4.1.22), we deduce

$$u(x) \geq C \left(\mathbf{W}_G\sigma_B(x)\right)^{\frac{1}{1-\gamma}} \quad \forall x \in \mathbb{R}^n.$$

Since C does not depend on R, the proof of Theorem 4.1.4 is established after letting $R \to \infty$ in the previous inequality. □

Suppose that (4.1.10) holds. In the view of Theorem 4.1.4, if $u \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ is a solution to (S), then $(\mathbf{W}_G\sigma)^{1/(1-\gamma)} \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$, that is

$$\int_{\mathbb{R}^n} F\Big(\mathbf{W}_G\sigma^{\frac{1}{1-\gamma}}\Big)\,\mathrm{d}\sigma < \infty. \tag{4.1.25}$$

Hence condition (4.1.25) is necessary to the existence of solutions to (S) in $L^F(\mathbb{R}^n, \mathrm{d}\sigma)$. However, this condition is far to be sufficient (at least) to ensure such existence in $L^F(\mathbb{R}^n, \mathrm{d}\sigma)$. We will show that (1.3.21) is a sufficient condition to the existence of a solution to (S) in $L^F(\mathbb{R}^n, \mathrm{d}\sigma)$. Before that, the following result will be needed.

Lemma 4.1.6. *Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying* (4.1.10) *and* (1.3.21)*. Then there exists a constant $c > 0$ such that for all $u \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$, $u \geq 0$, it holds*

$$\int_{\mathbb{R}^n} F\Big(\mathbf{W}_G(f(u)\mathrm{d}\sigma)\Big)\,\mathrm{d}\sigma \\ \leq c\Bigg[\Big(\int_{\mathbb{R}^n} F(u)\,\mathrm{d}\sigma\Big)^{\frac{p-1}{q-1}\gamma} + \Big(\int_{\mathbb{R}^n} F(u)\,\mathrm{d}\sigma\Big)^{\gamma} + \Big(\int_{\mathbb{R}^n} F(u)\,\mathrm{d}\sigma\Big)^{\frac{q-1}{p-1}\gamma}\Bigg].$$

The constant c depends only on n, p, q, and the L^F-norms of the Wolff potentials mentioned in (1.3.21)*.*

Proof. We begin the proof with the following claim.

Claim 2. Fix $1 < s < \infty$, $0 < r < s-1$ and $\alpha > r-1$. Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying $\Big(\mathbf{W}_s\sigma\Big)^{(s-1)/(s-1-r)} \in L^{1+\alpha}(\mathbb{R}^n, \mathrm{d}\sigma)$. Then there exists a constant

$$c_0 = c_0\bigg(n, r, s, \alpha, \|\Big(\mathbf{W}_s\sigma\Big)^{\frac{s-1}{s-1-r}}\|_{L^{1+\alpha}}\bigg) > 0$$

such that for all $u \in L^{1+\alpha}(\mathbb{R}^n, \mathrm{d}\sigma)$, $u \geq 0$, it holds

$$\int_{\mathbb{R}^n} \Big(\mathbf{W}_s(u^r\mathrm{d}\sigma)\Big)^{1+\alpha}\,\mathrm{d}\sigma \leq c_0\bigg(\int_{\mathbb{R}^n} u^{1+\alpha}\,\mathrm{d}\sigma\bigg)^{\frac{r}{s-1}}.$$

Indeed, fix $0 \leq u \in L^{1+\alpha}(\mathbb{R}^n, \mathrm{d}\sigma)$. By definition (1.3.20), we have

$$\begin{aligned} \mathbf{W}_s(u^r\mathrm{d}\sigma)(x) &= \int_0^\infty \bigg(\frac{\int_{B(x,t)} u^r\,\mathrm{d}\sigma}{t^{n-1}}\bigg)^{\frac{1}{s-1}}\,\mathrm{d}t \\ &\leq \int_0^\infty \bigg(M_\sigma u^r(x)\frac{\sigma(B(x,t))}{t^{n-1}}\bigg)^{\frac{1}{s-1}}\,\mathrm{d}t \\ &= \Big(M_\sigma u^r(x)\Big)^{\frac{1}{s-1}}\mathbf{W}_s\sigma(x) \quad \forall x \in \mathbb{R}^n, \end{aligned}$$

where $M_\sigma\cdot$ is the centered maximal operator defined by

$$M_\sigma v(x) = \sup_{t>0}\frac{1}{\sigma(B(x,t))}\int_{B(x,t)} v\,\mathrm{d}\sigma, \quad v \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma).$$

Then using the classical Hölder's inequality with expoents $\beta = (s-1)/r$ and $\beta' = (s-1)/(s-1-r)$ in the preceding inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(\mathbf{W}_s(u^r\mathrm{d}\sigma)\Big)^{1+\alpha}\,\mathrm{d}\sigma &\leq \int_{\mathbb{R}^n} \Big(M_\sigma u^r\Big)^{\frac{1+\alpha}{s-1}}\Big(\mathbf{W}_s\sigma\Big)^{1+\alpha}\,\mathrm{d}\sigma \\ &\leq \bigg(\int_{\mathbb{R}^n} \Big(\mathbf{W}_s\sigma\Big)^{\frac{(s-1)(1+\alpha)}{s-1-r}}\,\mathrm{d}\sigma\bigg)^{\frac{s-1-r}{s-1}}\bigg(\int_{\mathbb{R}^n} \Big(M_\sigma u^r\Big)^{\frac{1+\alpha}{r}}\,\mathrm{d}\sigma\bigg)^{\frac{r}{s-1}}. \end{aligned} \tag{4.1.26}$$

Being $(1+\alpha)/r > 1$, $M_\sigma : L^{(1+\alpha)/r}(\mathbb{R}^n, \mathrm{d}\sigma) \to L^{(1+\alpha)/r}(\mathbb{R}^n, \mathrm{d}\sigma)$ is a bounded operator (see for instance [Malý and Ziemer 1997, Theorem 1.22]), that is there exists a constant $\tilde{c} = \tilde{c}(n, r, \alpha) > 0$ such that

$$\left(\int_{\mathbb{R}^n} \Big(M_\sigma u^r \Big)^{\frac{1+\alpha}{r}} \mathrm{d}\sigma \right)^{\frac{r}{s-1}} \leq \tilde{c}^{\frac{r}{s-1}} \left(\int_{\mathbb{R}^n} \Big(u^r \Big)^{\frac{1+\alpha}{r}} \mathrm{d}\sigma \right)^{\frac{r}{s-1}}.$$

Using this in (4.1.26), we arrive at

$$\int_{\mathbb{R}^n} \Big(\mathbf{W}_s(u^r \mathrm{d}\sigma) \Big)^{1+\alpha} \mathrm{d}\sigma \leq c_0 \left(\int_{\mathbb{R}^n} u^{1+\alpha} \, \mathrm{d}\sigma \right)^{\frac{r}{s-1}},$$

with $c_0 = c_0\Big(n, r, s, \alpha, \| \Big(\mathbf{W}_s \sigma \Big)^{(s-1)/(s-1-r)} \|_{L^{1+\alpha}} \Big) > 0$, which proves Claim 2.

Next, fix $0 \leq u \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$. Note that from (2.3.8) and (4.1.13), there exists a constant $c_1 = c_1(p, q, \gamma) > 0$ such that

$$\begin{aligned} \mathbf{W}_G \sigma(x) &\leq c_1 \Big(\mathbf{W}_p \sigma(x) + \mathbf{W}_q \sigma(x) \Big) \quad \forall x \in \mathbb{R}^n, \\ F(t) &\leq c_1 (t^{(p-1)\gamma+1} + t^{(q-1)\gamma+1}) \quad \forall t \geq 0. \end{aligned}$$

Then, combining the previous inequalities, we can show that

$$\begin{aligned} &\int_{\mathbb{R}^n} F\Big(\mathbf{W}_G(f(u)\mathrm{d}\sigma) \Big) \, \mathrm{d}\sigma \\ &\qquad \leq \int_{\mathbb{R}^n} \Big(\mathbf{W}_G(f(u)\mathrm{d}\sigma) \Big)^{(p-1)\gamma+1} \mathrm{d}\sigma + \int_{\mathbb{R}^n} \Big(\mathbf{W}_G(f(u)\mathrm{d}\sigma) \Big)^{(q-1)\gamma+1} \mathrm{d}\sigma \\ &\qquad \leq c_2 \bigg[\int_{\mathbb{R}^n} \Big(\mathbf{W}_p(f(u)\mathrm{d}\sigma) \Big)^{(p-1)\gamma+1} \mathrm{d}\sigma + \int_{\mathbb{R}^n} \Big(\mathbf{W}_q(f(u)\mathrm{d}\sigma) \Big)^{(p-1)\gamma+1} \mathrm{d}\sigma \\ &\qquad\qquad + \int_{\mathbb{R}^n} \Big(\mathbf{W}_p(f(u)\mathrm{d}\sigma) \Big)^{(q-1)\gamma+1} \mathrm{d}\sigma + \int_{\mathbb{R}^n} \Big(\mathbf{W}_q(f(u)\mathrm{d}\sigma) \Big)^{(q-1)\gamma+1} \bigg], \quad (4.1.27) \end{aligned}$$

where $c_2 = c_2(p, q) > 0$. We shall make use of the following elementary inequality [Malý and Ziemer 1997, Lemma 1.1]: given $\delta > 0$, for all $a, b \in \mathbb{R}$ it holds

$$|a+b|^\delta \leq 2^{\delta-1}(|a|^\delta + |b|^\delta).$$

Because of this inequality, reminding of the definition of $f(t)$ in (4.1.13), we will split each integral in (4.1.27) into another two ones:

$$\begin{aligned}
&\int_{\mathbb{R}^n} F\Big(\mathbf{W}_G(f(u)\mathrm{d}\sigma)\Big)\,\mathrm{d}\sigma \\
&\qquad \le c_2\bigg[\int_{\mathbb{R}^n}\Big(\mathbf{W}_p(f(u)\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma+\int_{\mathbb{R}^n}\Big(\mathbf{W}_q(f(u)\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma \\
&\qquad\quad +\int_{\mathbb{R}^n}\Big(\mathbf{W}_p(f(u)\mathrm{d}\sigma)\Big)^{(q-1)\gamma+1}\mathrm{d}\sigma+\int_{\mathbb{R}^n}\Big(\mathbf{W}_q(f(u)\mathrm{d}\sigma)\Big)^{(q-1)\gamma+1}\bigg] \\
&\qquad \le c_3\bigg[\int_{\mathbb{R}^n}\Big(\mathbf{W}_p(u^{(p-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma+\int_{\mathbb{R}^n}\Big(\mathbf{W}_p(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma \\
&\qquad\quad +\int_{\mathbb{R}^n}\Big(\mathbf{W}_q(u^{(p-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma+\int_{\mathbb{R}^n}\Big(\mathbf{W}_q(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma \\
&\qquad\quad +\int_{\mathbb{R}^n}\Big(\mathbf{W}_p(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma+\int_{\mathbb{R}^n}\Big(\mathbf{W}_p(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(q-1)\gamma+1}\mathrm{d}\sigma \\
&\qquad\quad +\int_{\mathbb{R}^n}\Big(\mathbf{W}_q(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma+\int_{\mathbb{R}^n}\Big(\mathbf{W}_q(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(q-1)\gamma+1}\mathrm{d}\sigma\bigg] \\
&\qquad =: c_3\sum_{j=1}^{8} I_j, \qquad (4.1.28)
\end{aligned}$$

where $c_3 = c_3(\gamma, p, q) > 0$. By assumption on γ, one has $(p-1)\gamma/(q-1) < \gamma < (q-1)\gamma/(p-1) < 1$ and $(p-1)\gamma+1 > (q-1)\gamma$. Hence we estimate I_1, I_2, , I_3 and I_4 by applying Claim 2 with $\alpha = (p-1)\gamma$, $r_1 = (p-1)\gamma$, $s_1 = p$, $r_2 = (q-1)\gamma$, $s_2 = p$, $r_3 = (p-1)\gamma$, $s_3 = q$ and $r_4 = (q-1)\gamma$, $s_4 = q$, respectively to deduce

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n}\Big(\mathbf{W}_p(u^{(p-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma \le c_{0,1}\bigg(\int_{\mathbb{R}^n} u^{(p-1)\gamma+1}\,\mathrm{d}\sigma\bigg)^{\gamma}, \\
I_2 &= \int_{\mathbb{R}^n}\Big(\mathbf{W}_p(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma \le c_{0,2}\bigg(\int_{\mathbb{R}^n} u^{(p-1)\gamma+1}\,\mathrm{d}\sigma\bigg)^{\frac{q-1}{p-1}\gamma}, \\
I_3 &= \int_{\mathbb{R}^n}\Big(\mathbf{W}_q(u^{(p-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma \le c_{0,3}\bigg(\int_{\mathbb{R}^n} u^{(p-1)\gamma+1}\,\mathrm{d}\sigma\bigg)^{\frac{p-1}{q-1}\gamma}, \\
I_4 &= \int_{\mathbb{R}^n}\Big(\mathbf{W}_q(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(p-1)\gamma+1}\mathrm{d}\sigma \le c_{0,4}\bigg(\int_{\mathbb{R}^n} u^{(p-1)\gamma+1}\,\mathrm{d}\sigma\bigg)^{\gamma}.
\end{aligned} \qquad (4.1.29)$$

Similarly, to estimate I_5, I_6, I_7 and I_8, we apply Claim 2 with $\alpha = (q-1)\gamma$, $r_5 = (p-1)\gamma$, $s_5 = p$, $r_6 = (q-1)\gamma$, $s_6 = p$, $r_7 = (p-1)\gamma$, $s_7 = q$ and $r_8 = (q-1)\gamma$, $s_8 = q$, respectively to deduce

$$
\begin{aligned}
I_5 &= \int_{\mathbb{R}^n} \Big(\mathbf{W}_p(u^{(p-1)\gamma}\mathrm{d}\sigma)\Big)^{(q-1)\gamma+1}\mathrm{d}\sigma \le c_{0,5}\bigg(\int_{\mathbb{R}^n} u^{(q-1)\gamma+1}\,\mathrm{d}\sigma\bigg)^{\gamma},\\
I_6 &= \int_{\mathbb{R}^n} \Big(\mathbf{W}_p(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(q-1)\gamma+1}\mathrm{d}\sigma \le c_{0,6}\bigg(\int_{\mathbb{R}^n} u^{(q-1)\gamma+1}\,\mathrm{d}\sigma\bigg)^{\frac{q-1}{p-1}\gamma},\\
I_7 &= \int_{\mathbb{R}^n} \Big(\mathbf{W}_q(u^{(p-1)\gamma}\mathrm{d}\sigma)\Big)^{(q-1)\gamma+1}\mathrm{d}\sigma \le c_{0,7}\bigg(\int_{\mathbb{R}^n} u^{(q-1)\gamma+1}\,\mathrm{d}\sigma\bigg)^{\frac{p-1}{q-1}\gamma},\\
I_8 &= \int_{\mathbb{R}^n} \Big(\mathbf{W}_q(u^{(q-1)\gamma}\mathrm{d}\sigma)\Big)^{(q-1)\gamma+1}\mathrm{d}\sigma \le c_{0,8}\bigg(\int_{\mathbb{R}^n} u^{(q-1)\gamma+1}\,\mathrm{d}\sigma\bigg)^{\gamma}.
\end{aligned}
\tag{4.1.30}
$$

Here the constants $c_{0,j} > 0$, $j = 1,\dots,8$, are given by Claim 2, whose depend only on n, p, q, $,\gamma$ and the L^F-norms of the Wolff potentials presented in (1.3.21). We have

$$\max\{t^{(p-1)\gamma+1},\, t^{(q-1)\gamma+1}\} \le c_4 F(t) \quad \forall t > 0,$$

where $c_4 = c_4(p, q, \gamma) > 0$. A combination of (4.1.28) with (4.1.29) and (4.1.30), completes the proof of Lemma 4.1.6. □

Remark 4.1.7. Because of (2.3.8), $\mathbf{W}_G\sigma$ is controlled from above by the sum of $\mathbf{W}_p\sigma$ with $\mathbf{W}_q\sigma$. Consequently, (1.3.21) implies that

$$\mathbf{W}_G\sigma^{\frac{1}{1-\gamma}},\ \mathbf{W}_G\sigma^{\frac{p-1}{p-1-\gamma(q-1)}},\ \mathbf{W}_G\sigma^{\frac{q-1}{q-1-\gamma(p-1)}} \in L^F(\mathbb{R}^n, \mathrm{d}\sigma). \tag{4.1.31}$$

It would be desirable to show that (4.1.31) is a sufficient condition to ensure the existence of solutions to (S) in $L^F(\mathbb{R}^n, \mathrm{d}\sigma)$, but we have not been able to do this.

4.1.1 Proof of Theorem 1.3.14

Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying (4.1.10), that is $\mathbf{W}_G\sigma < \infty$ in $\mathbb{R}^n$. Our proof starts with the assertion that there exists a constant $\varepsilon > 0$ sufficiently small such that

$$u_0(x) = \varepsilon\,(\mathbf{W}_G\sigma(x))^{\frac{1}{1-\gamma}}, \quad x \in \mathbb{R}^n, \tag{4.1.32}$$

is a subsolution to (S). Indeed, combining Lemma 4.1.5 with (2.3.6), we obtain

$$\mathbf{W}_G(f(u_0)\,\mathrm{d}\sigma) \ge \left(\frac{p}{q}\right)^{\frac{2}{p-1}} \lambda\, \varepsilon^{\frac{(q-1)\gamma}{p-1}} (\mathbf{W}_G\sigma)^{\frac{1}{1-\gamma}},$$

where $\lambda > 0$ is the constant given in Lemma 4.1.5. Consequently, picking $\varepsilon > 0$ such that

$$\varepsilon \le \left(\left(\frac{p}{q}\right)^{\frac{2}{p-1}} \lambda\right)^{\frac{p-1}{p-1-\gamma(q-1)}},$$

we concluded that $\mathbf{W}_G(f(u_0)\,\mathrm{d}\sigma) \geq u_0$, which is our assertion.

Now, let us construct a sequence of iterations of functions u_j as follows:

$$u_j = \mathbf{W}_G(f(u_{j-1})\,\mathrm{d}\sigma), \quad j = 1, 2, \ldots \tag{4.1.33}$$

From the previous assertion, $u_1 \geq u_0$ in $\mathbb{R}^n$. Arguing by induction, one has $u_{j-1} \leq u_j$ in $\mathbb{R}^n$ for all $j \geq 1$. By assumption (1.3.21), $u_0 \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$. Consequently, we can verify by induction that $u_j \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ for all $j = 1, 2, \ldots$. Indeed, suppose that $u_{j-1} \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ for some $j \geq 1$. Using Lemma 4.1.6, we have

$$\begin{aligned}\int_{\mathbb{R}^n} F(u_j)\,\mathrm{d}\sigma &= \int_{\mathbb{R}^n} F\Big(\mathbf{W}_G(f(u_{j-1})\mathrm{d}\sigma)\Big)\,\mathrm{d}\sigma \\ &\leq c\bigg[\Big(\int_{\mathbb{R}^n} F(u_{j-1})\,\mathrm{d}\sigma\Big)^{\frac{p-1}{q-1}\gamma} + \Big(\int_{\mathbb{R}^n} F(u_{j-1})\,\mathrm{d}\sigma\Big)^{\gamma} + \Big(\int_{\mathbb{R}^n} F(u_{j-1})\,\mathrm{d}\sigma\Big)^{\frac{q-1}{p-1}\gamma}\bigg] < \infty.\end{aligned}$$

This shows $u_j \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$. Furthermore, being $\{u_j\}$ an increasing sequence (pointwise), it follows from the preceding inequality that

$$\begin{aligned}&\int_{\mathbb{R}^n} F(u_j)\,\mathrm{d}\sigma \\ &\quad\leq c\bigg[\Big(\int_{\mathbb{R}^n} F(u_j)\,\mathrm{d}\sigma\Big)^{\frac{p-1}{q-1}\gamma} + \Big(\int_{\mathbb{R}^n} F(u_j)\,\mathrm{d}\sigma\Big)^{\gamma} + \Big(\int_{\mathbb{R}^n} F(u_j)\,\mathrm{d}\sigma\Big)^{\frac{q-1}{p-1}\gamma}\bigg] \quad \forall j \geq 1.\end{aligned} \tag{4.1.34}$$

We claim that $\{u_j\}$ is uniformly bounded in $L^F(\mathbb{R}^n, \mathrm{d}\sigma)$. Indeed, consider the continuous real function on $[0, \infty)$

$$h(t) = t - c\Big(t^{\frac{p-1}{q-1}\gamma} + t^{\gamma} + t^{\frac{q-1}{p-1}\gamma}\Big).$$

Notice that (4.1.34) is equivalently to $h\Big(\int_{\mathbb{R}^n} F(u_j)\,\mathrm{d}\sigma\Big) \leq 0$ for all $j \geq 1$. Since $(p-1)\gamma/(q-1) < \gamma < (q-1)\gamma/(p-1) < 1$, one has

$$\lim_{t\to\infty} h(t) = \infty.$$

Hence the subset $\{t \geq 0 : h(t) \leq 0\}$ is bounded in $[0, \infty)$, that is there exists a constant $C = C(p, q, \gamma, c) > 0$ such that $h(t) \leq 0$ if and only if $t \leq C$. Thus

$$\int_{\mathbb{R}^n} F(u_j)\,\mathrm{d}\sigma \leq C \quad \forall j \geq 1.$$

Therefore, letting $j \to \infty$ in the previous inequality, by the Monotone Convergence Theorem there exists $u = \lim_{j\to\infty} u_j$ (pointwise) such that $u \in L^F(\mathbb{R}^n,\, \mathrm{d}\sigma)$. Combining Hölder's inequality (Lemma M) with (2.3.9), we arrive $L^F(\mathbb{R}^n, \mathrm{d}\sigma) \subset L^f_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$, whence u satisfies (S). This completes the proof of Theorem 1.3.14.

4.2 SOME AUXILIARY RESULTS

In this section, we establish some preliminary results concerned Eq. (P_2). It is worth pointing out that these results hold for any N-functions G which enjoy the property (2.3.3), with $1 < p \le q < n$. First, we will use the following lemma to ensure the existence of solutions to (P_2).

Lemma 4.2.1. *Let $v \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ be a supersolution to (S). Then $f(v)\,\mathrm{d}\sigma \in \left(\mathcal{D}^{1,G}(\mathbb{R}^n)\right)^*$.*

Proof. First, recal that $\|\nabla \cdot \|_{L^G}$ defines a norm on $\mathcal{D}^{1,G}(\mathbb{R}^n)$ provided $1 < p \le q < n$ in (2.3.3). Let $\omega = \sigma\, f(v) \in M^+(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}^{1,G}(\mathbb{R}^n)$, it is sufficient to show there exists a constant $c > 0$ such that

$$\left| \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\omega \right| \le c\, \|\nabla \varphi\|_{L^G} \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n). \tag{4.2.1}$$

Since $v \ge \mathbf{W}_G(f(v)\mathrm{d}\sigma)$ in $\mathbb{R}^n$ with $v \in L^F(\mathbb{R}^n,\, \mathrm{d}\sigma)$, it follows from (4.1.13) that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{W}_G \omega \,\mathrm{d}\omega = \int_{\mathbb{R}^n} \mathbf{W}_G(f(v)\mathrm{d}\sigma)\,\mathrm{d}\sigma &\le \int_{\mathbb{R}^n} v f(v)\,\mathrm{d}\sigma \\ &\le \tilde{c}_1 \int_{\mathbb{R}^n} F(v)\,\mathrm{d}\sigma < \infty,\end{aligned} \tag{4.2.2}$$

where $\tilde{c}_1 = \tilde{c}_1(p, q, \gamma) > 0$. Let $B_j = B(0, j)$ and let $\sigma_j = \chi_{B_j}\sigma$, for $j > 1$. Setting $\omega_j = f(v)\mathrm{d}\sigma_j$, one has $\operatorname{supp}\omega_j \subset B_j$. By (4.2.2),

$$\int_{B_{j+1}} \mathbf{W}_G \omega_j \,\mathrm{d}\omega_j \le \int_{\mathbb{R}^n} \mathbf{W}_G \omega \,\mathrm{d}\omega < \infty \quad \forall j > 1.$$

Using Theorem S, we infer from previous inequality that $\omega_j \in \left(W_0^{1,G}(B_{j+1})\right)^*$ for all $j > 1$.

Next, let $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset B_{j+1}$, for some $j > 1$. Then there exists a constant $c_j > 0$ such that

$$\left| \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\omega_j \right| \le c_j\, \|\nabla \varphi\|_{L^G}. \tag{4.2.3}$$

Applying Theorem Q, there exists $u_j \in W_0^{1,G}(B_{j+1})$ satisfying eq. (2.3.14) with $\mu = \omega_j$, that is

$$-\mathrm{div}\left(\frac{g(|\nabla u_j|)}{|\nabla u_j|} \nabla u_j \right) = \omega_j \quad \text{in} \quad B_{j+1}. \tag{4.2.4}$$

Claim 3. The constant c_j given in (4.2.3) are uniformly bounded for all $j > 1$.

Indeed, first note that we may assume

$$c_j \le \tilde{c}_2 \left(\int_{B_{j+1}} G(|\nabla u_j|)\,\mathrm{d}x \right) + 1 \quad \forall j > 1, \tag{4.2.5}$$

where $\tilde{c}_2 = \tilde{c}_2(p,q) > 0$. This is seen by testing eq. (4.2.4) with φ and by using a combination of Cauchy–Schwarz inequality and Hölder's inequality (Lemma M) with (2.3.10) and (2.3.9):

$$\begin{aligned}\left|\int_{B_{j+1}} \varphi \,\mathrm{d}\omega_j\right| &= \left|\int_{B_{j+1}} \frac{g(|\nabla u_j|)}{|\nabla u_j|}\nabla u_j \cdot \nabla\varphi \,\mathrm{d}x\right| \\ &\le \int_{B_{j+1}} \left|\frac{g(|\nabla u_j|)}{|\nabla u_j|}\nabla u_j\right| |\nabla\varphi| \,\mathrm{d}x \le \int_{B_{j+1}} g(|\nabla u_j|)|\nabla\varphi| \,\mathrm{d}x \\ &\le 2\|g(\nabla u_j)\|_{L^{G^*}}\|\nabla\varphi\|_{L^G} \le \left[2\left(\int_{B_{j+1}} G^*\big(g(|\nabla u_j|)\big)\,\mathrm{d}x\right)+1\right]\|\nabla\varphi\|_{L^G} \\ &\le \left[\tilde{c}_2\left(\int_{B_{j+1}} G(|\nabla u_j|)\,\mathrm{d}x\right)+1\right]\|\nabla\varphi\|_{L^G}.\end{aligned}$$

Notice that u_j is harmonic in $B_{j+1}\setminus \overline{B}_j$ since $\operatorname{supp}\omega_j \subset B_j$, whence u_j takes continuously zero boundary values on ∂B_{j+1}. Extending u_j by zero away from ∂B_{j+1}, with the aid of Remark 2.3.22, we infer from Lemma P that u_j is a nonnegative (almost everywhere) G-superharmonic in whole $\mathbb{R}^n$. By Corallary 2.3.20, for almost everywhere in $\mathbb{R}^n$ it holds

$$0 \le u_j \le K\,\mathbf{W}_G\omega_j \le K\,\mathbf{W}_G\omega \quad \forall j>1. \tag{4.2.6}$$

Testing eq. (4.2.4) with u_j, a combination of (2.3.3) and (4.2.2) with (4.2.6) yields

$$\begin{aligned}\int_{\mathbb{R}^n} G(|\nabla u_j|)\,\mathrm{d}x &\le \frac{1}{p}\int_{\mathbb{R}^n} g(|\nabla u_j|)|\nabla u_j|\,\mathrm{d}x = \frac{1}{p}\int_{\mathbb{R}^n} u_j\,\mathrm{d}\omega_j \\ &\le \frac{1}{p}\int_{\mathbb{R}^n} u_j\,\mathrm{d}\omega \le \frac{K}{p}\int_{\mathbb{R}^n} \mathbf{W}_G\omega\,\mathrm{d}\omega < \infty \quad \forall j>1.\end{aligned}$$

Consequently, by (4.2.5), are uniformly bounded, with

$$\sup_{j>1} c_j \le \tilde{c}_3\left(\int_{\mathbb{R}^n} \mathbf{W}_G\omega\,\mathrm{d}\omega\right)+1,$$

where $\tilde{c}_3 = \tilde{c}_3(p,q,K) > 0$. This shows Claim 3.

On the other hand, by the Monotone Theorem Convergence, we have

$$\int_{\mathbb{R}^n} \varphi\,\mathrm{d}\omega = \lim_{j\to\infty}\int_{\mathbb{R}^n} \varphi\,\mathrm{d}\omega_j.$$

Thus, letting $j\to\infty$ in (4.2.3), we obtain

$$\left|\int_{\mathbb{R}^n} \varphi\,\mathrm{d}\omega\right| = \lim_{j\to\infty}\left|\int_{\mathbb{R}^n} \varphi\,\mathrm{d}\omega_j\right| \le c\,\|\nabla\varphi\|_{L^G},$$

where $c = \sup_j c_j$. This shows (4.2.1), and proves Lemma 4.2.1. □

The following lemma gives a version of the comparison principle in $\mathcal{D}^{1,G}(\mathbb{R}^n)$.

Lemma 4.2.2. *Let* $\mu, \nu \in M^+(\mathbb{R}^n) \cap \left(\mathcal{D}^{1,G}(\mathbb{R}^n)\right)^*$ *such that* $\mu \le \nu$. *Suppose that* $u, v \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ *are solutions (respectively) to*

$$-\mathrm{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu, \quad -\mathrm{div}\left(\frac{g(|\nabla v|)}{|\nabla v|}\nabla v\right) = \nu \quad \textit{in} \quad \mathbb{R}^n.$$

Then $u \le v$ *almost everywhere in* $\mathbb{R}^n$.

Proof. The proof is standard and relies on the use of the test function $\varphi = (u-v)^+ = \max\{u-v, 0\} \in \mathcal{D}^{1,G}(\mathbb{R}^n)$. Indeed, testing both equations with such φ, one has

$$\int_{\mathbb{R}^n} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu,$$
$$\int_{\mathbb{R}^n} \frac{g(|\nabla v|)}{|\nabla v|}\nabla v \cdot \nabla\varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\nu.$$

Hence

$$\begin{aligned} 0 \le \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\nu - \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu &= \int_{\mathbb{R}^n} \frac{g(|\nabla v|)}{|\nabla v|}\nabla v \cdot \nabla\varphi \,\mathrm{d}x - \int_{\mathbb{R}^n} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\varphi \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \left(\frac{g(|\nabla v|)}{|\nabla v|}\nabla v - \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) \cdot \nabla\varphi \,\mathrm{d}x. \end{aligned} \tag{4.2.7}$$

By [Borowski and Chlebicka 2022, Theorem 1], the following monotonicity for g holds

$$\left(\frac{g(|\xi|)}{|\xi|}\xi - \frac{g(|\eta|)}{|\eta|}\eta\right) \cdot (\xi - \eta) > 0 \quad \forall \xi, \eta \in \mathbb{R}^n, \, \xi \ne \eta. \tag{4.2.8}$$

Since $\nabla\varphi = \nabla(u-v) = \nabla u - \nabla v$ in $\mathrm{supp}\,\varphi \subset \{u > v\}$, it follows from (4.2.7) and (4.2.8) that

$$0 \le \int_{\mathbb{R}^n} \left(\frac{g(|\nabla v|)}{|\nabla v|}\nabla v - \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) \cdot (\nabla u - \nabla v) \,\mathrm{d}x \le 0,$$

which implies that $\nabla\varphi = 0$ almost everywhere in $\mathbb{R}^n$. Thus $\varphi = 0$ almost everywhere in $\mathbb{R}^n$, and proves Lemma 4.2.2. □

The following lemma concerning weak compactness in $\mathcal{D}^{1,G}(\mathbb{R}^n)$, and it will be useful in the proof of Theorem 1.3.15. Before the statement, we recall *the Mazur lemma* (see for instance [Brezis 2011, Corollary 3.8])

Lemma U (The Mazur lemma)**.** *Let* $\mathcal{X}$ *be a normed space and let* $\{x_j\}$ *be a sequence, which converges weakly in* $\mathcal{X}$ *to* x. *Then there exists a sequence* $\{\tilde{x}_j\}$ *made up of convex combinations of the* $\{x_j\}$,

$$\tilde{x}_j := \sum_{k=1}^{j} \lambda_{k,j} x_k, \quad \lambda_{k,j} \ge 0, \quad \sum_{k=1}^{j} \lambda_{k,j} = 1,$$

such that $\{\tilde{x}_j\}$ *converges strongly in* $\mathcal{X}$ *to* x.

Lemma 4.2.3. *Suppose that* $\{u_j\} \subset \mathcal{D}^{1,G}(\mathbb{R}^n)$ *is a sequence converging pointwise to* u *almost everywhere in* $\mathbb{R}^n$*. If* $\{\nabla u_j\}$ *is bounded in* $L^G(\mathbb{R}^n)$*, then* $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ *and* $\nabla u_j \rightharpoonup \nabla u$ *in* $L^G(\mathbb{R}^n)$*.*

Proof. The proof is similar in spirit to the proof of [Heinonen, Kilpeläinen and Martio 2006, Lemma 1.33]. Fix $B \subset \mathbb{R}^n$ a ball. First, we will need the following claim:

Claim 4. Let $\{v_j\}$ be a sequence in $W^{1,G}(B)$, and let $v \in L^G(B)$ and $X : B \to \mathbb{R}^n$ with $|X| \in L^G(B)$ satisfying

$$\begin{aligned} v_j \rightharpoonup v \quad &\text{in} \quad L^G, \\ \nabla v_j \rightharpoonup X \quad &\text{in} \quad L^G. \end{aligned}$$

Then $v \in W^{1,G}(B)$ and $X = \nabla v$.

Indeed, consider the product space $L^G(B) \times L^G(B;\mathbb{R}^n)$ with the norm

$$\|(v, X)\| = \|v\|_{L^G} + \||X|\|_{L^G}.$$

Observe that $(v_j, \nabla v_j)$ converges weakly in $L^G(B) \times L^G(B;\mathbb{R}^n)$ to (v, X). The Mazur lemma (Lemma U) gives that there is a sequence of convex combinations

$$w_j := \sum_{k=1}^{j} \lambda_{k,j}\,(v_k, \nabla v_k), \quad \lambda_{k,j} \geq 0, \quad \sum_{k=1}^{j} \lambda_{k,j} = 1,$$

which converges strongly in $L^G(B) \times L^G(B;\mathbb{R}^n)$ to (v, X). In particular, the sequence $\{\tilde{v}_j\}$, given by

$$\tilde{v}_j = \sum_{k=1}^{j} \lambda_{k,j}\, v_k, \quad \text{is a Cauchy sequence in } W^{1,G}(B).$$

Accordingly, there exist $\tilde{v} \in W^{1,G}(B)$ such that $\tilde{v} = \lim_j \tilde{v}_j$ in $W^{1,G}(B)$. Hence $v = \tilde{v}$ in $L^G(B)$, and $X = \nabla\tilde{v} = \nabla v$, which shows Claim 4.

Next, by Lemma O, the sequence $\{u_j\}$ is bounded in $W^{1,G}(B)$. Since u_j converges pointwise (almost everywhere) to u in B, from Theorem N, $u_j \rightharpoonup u$ in $L^G(B)$. Moreover, a subsequence ∇u_{j_k} converges weakly to ∇u in $L^G(B)$ by Claim 4. Since the weak limit is independent of the choice of the subsequence, it follows that $\nabla u_j \rightharpoonup \nabla u \in L^G(B)$.

On the other hand, being $\{\nabla u_j\}$ bounded in $L^G(\mathbb{R}^n)$, it has a wakly converging subsequence in $L^G(\mathbb{R}^n)$. This subsequence also converge weakly in $L^G(B)$, whence it converges weakly to ∇u, which gives $\nabla u \in L^G(\mathbb{R}^n)$, that is $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$. Therefore, $\nabla u_j \rightharpoonup \nabla u$ in $L^G(\mathbb{R}^n)$ since the weak limit is independent of the subsequence. This proves Lemma 4.2.3. □

The following lemma uses the Monotone Operator Theory to prove the existence of solutions to eq. (2.3.14) in $\mathcal{D}^{1,G}(\mathbb{R}^n)$.

Lemma 4.2.4. *For all* $\mu \in M^+(\mathbb{R}^n) \cap \left(\mathcal{D}^{1,G}(\mathbb{R}^n)\right)^*$*, there exists a unique solution* $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ *to eq.* (2.3.14) *which is a* G*-superharmonic and nonnegative function (almost everywhere).*

Proof. Let us consider the operator $T : \mathcal{D}^{1,G}(\mathbb{R}^n) \to \left(\mathcal{D}^{1,G}(\mathbb{R}^n)\right)^*$ defined by

$$\langle Tu, \varphi \rangle = \int_{\mathbb{R}^n} \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, \mathrm{d}x \quad \forall \varphi \in \mathcal{D}^{1,G}(\mathbb{R}^n),$$

where $\langle \cdot, \cdot \rangle$ denote the usual pairing between $\left(\mathcal{D}^{1,G}(\mathbb{R}^n)\right)^*$ and $\mathcal{D}^{1,G}(\mathbb{R}^n)$. T is well defined. Indeed, by using a combination of Cauchy–Schwarz inequality and Hölder's inequality (Lemma M) with (2.3.10), we obtain

$$\begin{aligned} |\langle Tu, \varphi \rangle| &\leq 2 \| g(|\nabla u|) \|_{L^{G^*}} \| \nabla \varphi \|_{L^G} \\ &\leq c \bigg(\int_{\mathbb{R}^n} G(|\nabla u|) \, \mathrm{d}x + 1 \bigg) \| \varphi \|_{\mathcal{D}^{1,G}}, \end{aligned}$$

which shows Tu is bounded. A classical result [Lions 1969, Théorème 2.1] assures that T is subjective provided T is coercive, and weakly continuous on $\mathcal{D}^{1,G}(\mathbb{R}^n)$, that is if $u = \lim_j u_j$ in $\mathcal{D}^{1,G}(\mathbb{R}^n)$, then $Tu_j \rightharpoonup Tu$ in $\left(\mathcal{D}^{1,G}(\mathbb{R}^n)\right)^*$.

We first verify the weak continuity of T. Fix $\{u_j\} \subset \mathcal{D}^{1,G}(\mathbb{R}^n)$ a sequence that converges (in norm) to an element $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$. Using [Benyaiche and Khlifi 2021, Lemma 2.1], there exists a subsequence of $\{u_j\}$, also denoted $\{u_j\}$ (by abuse of notation), which converges almost everywhere to u in $\mathbb{R}^n$. From this, by continuity of g, we have

$$\frac{g(|\nabla u|)}{|\nabla u|} \nabla u = \lim_{j \to \infty} \frac{g(|\nabla u_j|)}{|\nabla u_j|} \nabla u_j \quad \text{in} \quad \mathbb{R}^n \quad \text{(almost everywhere)}.$$

Since $\{u_j\}$ converges to u in $\mathcal{D}^{1,G}(\mathbb{R}^n)$, it follows that $\{\int_{\mathbb{R}^n} G(|\nabla u_j|) \, \mathrm{d}x\}$ is bounded, and

$$\left\{ \int_{\mathbb{R}^n} G^* \bigg(\frac{g(|\nabla u_j|)}{|\nabla u_j|} \nabla u_j \bigg) \, \mathrm{d}x \right\} \quad \text{is bounded.}$$

By Lemma L, $\{g(|\nabla u_j|)/|\nabla u_j| \, \nabla u_j\}$ is bounded in L^{G^*}. Applying Theorem N,

$$\frac{g(|\nabla u_j|)}{|\nabla u_j|} \nabla u_j \rightharpoonup \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \quad \text{in} \quad L^{G^*}(\mathbb{R}^n),$$

whence our assertion follows since the weak limit is independent of the choice of a subsequence.

Next, we prove that T is coercive. Let $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ with $\|u\|_{\mathcal{D}^{1,G}(\mathbb{R}^n)} \geq 1$. Combining (2.3.3) with Lemma L, we obtain

$$\begin{aligned} \langle Tu, u \rangle = \int_{\mathbb{R}^n} g(|\nabla u|) |\nabla u| \, \mathrm{d}x &\geq p \int_{\mathbb{R}^n} G(|\nabla u|) \, \mathrm{d}x \\ &\geq p \|\nabla u\|_{L^G}^p = p \|u\|_{\mathcal{D}^{1,G}}^p. \end{aligned}$$

Since $p > 1$,

$$\lim_{\|u\|_{\mathcal{D}^{1,G}}\to\infty} \frac{\langle Tu, u\rangle}{\|u\|_{\mathcal{D}^{1,G}}} = \infty,$$

which shows that T is coercive.

Hence, for $\mu \in M^+(\mathbb{R}^n) \cap \left(\mathcal{D}^{1,G}(\mathbb{R}^n)\right)^*$, there exists a $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$, such that $Tu = \mu$, that is

$$\int_{\mathbb{R}^n} \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla\varphi \, \mathrm{d}x = \int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mu \quad \forall \varphi \in \mathcal{D}^{1,G}(\mathbb{R}^n).$$

Thus u is a solution to eq. (2.3.14), and by monotonicity (4.2.8), it should be unique. Moreover, since $\mu \in M^+(\mathbb{R}^n)$, u is a G-supersolution in $\mathbb{R}^n$ in sense of Definition 2.3.12, whence $u \geq 0$ almost everywhere by Lemma P. Because of Remark 2.3.22, we may suppose that u is G-superharmonic in $\mathbb{R}^n$. This completes the proof of Lemma 4.2.4. □

4.3 PROOF OF THEOREM 1.3.15

We need only consider the case $2 \leq p < q < n$, otherwise by Remark 2.3.10 if $q \geq n$, Eq. (P_2) has only the trivial solution $u = c \geq 0$. Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying (1.3.21) and let $K \geq 1$ be the constant given in Corollary 2.3.20. Using Theorem 1.3.14 with

$$\left((q/p)^{1/(p-1)}K\right)^{q-1}\sigma \quad \text{in place of } \sigma,$$

we deduce that there exists a solution $0 \leq v \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ in $\mathbb{R}^n$ to

$$v = \mathbf{W}_G\left(f(v)\left(\frac{q}{p}\right)^{\frac{q-1}{p-1}} K^{q-1}\mathrm{d}\sigma\right) \tag{4.3.1}$$

By (2.3.6), we have

$$\begin{aligned} v &= \mathbf{W}_G\left(f(v)\left(\frac{q}{p}\right)^{\frac{q-1}{p-1}} K^{q-1}\mathrm{d}\sigma\right) \\ &\geq \left(\frac{p}{q}\right)^{\frac{1}{p-1}} \left[\left(\frac{q}{p}\right)^{\frac{q-1}{p-1}} K^{q-1}\right]^{\frac{1}{q-1}} \mathbf{W}_G(f(v)\mathrm{d}\sigma) \\ &= K\,\mathbf{W}_G(f(v)\mathrm{d}\sigma) \quad \text{in} \quad \mathbb{R}^n. \end{aligned} \tag{4.3.2}$$

Consequently, v is a supersolution to (S), since $K \geq 1$. Employing Lemma 4.2.1, $f(v)\mathrm{d}\sigma \in \Big(\mathcal{D}^{1,G}(\mathbb{R}^n)\Big)^*$. Moreover, from Theorem 4.1.4, one has

$$\begin{aligned} v &\geq C\left[\mathbf{W}_G\bigg(\Big(\frac{q}{p}\Big)^{\frac{q-1}{p-1}} K^{q-1}\sigma\bigg)\right]^{\frac{1}{1-\gamma}} \\ &\geq C\left(\frac{p}{q}\right)^{\frac{1}{p-1}}\left[\Big(\frac{q}{p}\Big)^{\frac{q-1}{p-1}} K^{q-1}\right]^{\frac{1}{(q-1)}\frac{1}{1-\gamma}}\Big(\mathbf{W}_G\sigma\Big)^{\frac{1}{1-\gamma}} \\ &= C\left(\frac{q}{p}\right)^{\frac{\gamma}{(p-1)(1-\gamma)}} K^{\frac{1}{1-\gamma}}\Big(\mathbf{W}_G\sigma\Big)^{\frac{1}{1-\gamma}}, \end{aligned}$$

where C is the constant given in Theorem 4.1.4.

Let ε be a positive constant satisfying

$$\varepsilon \leq \min\left\{\left(\frac{q}{p}\right)^{\frac{\gamma}{(p-1)(1-\gamma)}} K^{\frac{1}{1-\gamma}}\, C,\ \Big(K^{-1}\lambda\Big)^{\frac{p-1}{p-1-\gamma(q-1)}},\ K^{-\frac{p-1}{(q-1)(1-\gamma)}}\, C\right\}, \tag{4.3.3}$$

where λ is the constant given in Lemma 4.1.5. Setting, $u_0 = \varepsilon\Big(\mathbf{W}_G\sigma\Big)^{\frac{1}{1-\gamma}}$, it follows $u_0 \leq v$ in $\mathbb{R}^n$, and

$$u_0 \in L^F(\mathbb{R}^n, \mathrm{d}\sigma) \quad \text{and} \quad f(u_0)\mathrm{d}\sigma \in M^+(\mathbb{R}^n) \cap \Big(\mathcal{D}^{1,G}(\mathbb{R}^n)\Big)^*.$$

From Lemma 4.2.4, there exists a unique G-superharmonic function $u_1 \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ satisfying

$$-\mathrm{div}\bigg(\frac{g(|\nabla u_1|)}{|\nabla u_1|}\nabla u_1\bigg) = \sigma\, f(u_0) \quad \text{in} \quad \mathbb{R}^n.$$

Being $u_1 \geq 0$, a combination of Corollary 2.3.20 with (4.3.2), yields

$$u_1 \leq K\,\mathbf{W}_G(f(u_0)\mathrm{d}\sigma) \leq K\,\mathbf{W}_G(f(v)\mathrm{d}\sigma) \leq v.$$

From this, $u_1 \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ and $f(u_1)\mathrm{d}\sigma \in M^+(\mathbb{R}^n) \cap \Big(\mathcal{D}^{1,G}(\mathbb{R}^n)\Big)^*$. In addition, using Lemma 4.1.5 and (2.3.6), we obtain that

$$\begin{aligned} u_1 &\geq K^{-1}\,\mathbf{W}_G(f(u_0)\mathrm{d}\sigma) = K^{-1}\,\mathbf{W}_G\Big(f\Big(\varepsilon\Big(\mathbf{W}_G\sigma\Big)^{\frac{1}{1-\gamma}}\Big)\mathrm{d}\sigma\Big) \\ &\geq K^{-1}\varepsilon^{\frac{q-1}{p-1}\gamma}\,\mathbf{W}_G\Big(f\Big(\mathbf{W}_G\sigma\Big)^{\frac{1}{1-\gamma}}\mathrm{d}\sigma\Big) \\ &\geq K^{-1}\varepsilon^{\frac{q-1}{p-1}\gamma}\lambda\Big(\mathbf{W}_G\sigma\Big)^{\frac{1}{1-\gamma}}. \end{aligned}$$

By choice of ε, we deduce that $u_1 \geq u_0$. Summarizing, we have

$$\begin{cases} u_1 \in \mathcal{D}^{1,G}(\mathbb{R}^n) \cap L^F(\mathbb{R}^n, \mathrm{d}\sigma), \\ f(u_1)\mathrm{d}\sigma \in M^+(\mathbb{R}^n) \cap \Big(\mathcal{D}^{1,G}(\mathbb{R}^n)\Big)^*, \\ -\,\mathrm{div}\bigg(\dfrac{g(|\nabla u_1|)}{|\nabla u_1|}\nabla u_1\bigg) = \sigma\, f(u_0) \qquad \text{in} \quad \mathbb{R}^n, \\ \quad u_0 \leq u_1 \leq v \quad \text{almost everywhere} \quad \text{in} \quad \mathbb{R}^n. \end{cases}$$

Now, we construct by induction a sequence of nonnegative functions u_j, for $j = 1, 2, \ldots$, satisfying

$$\begin{cases} u_j \in \mathcal{D}^{1,G}(\mathbb{R}^n) \cap L^F(\mathbb{R}^n, \mathrm{d}\sigma), & \forall j > 1, \\ f(u_j)\mathrm{d}\sigma \in M^+(\mathbb{R}^n) \cap \Big(\mathcal{D}^{1,G}(\mathbb{R}^n)\Big)^*, & \forall j > 1, \\ -\operatorname{div}\Big(\dfrac{g(|\nabla u_j|)}{|\nabla u_j|}\nabla u_j\Big) = \sigma\, f(u_{j-1}) \quad \text{in} \quad \mathbb{R}^n, & \forall j > 1, \\ \quad u_{j-1} \le u_j \le v \quad \text{almost everywhere} \quad \text{in} \quad \mathbb{R}^n, & \forall j > 1, \\ \sup\limits_{j>1} \displaystyle\int_{\mathbb{R}^n} G(|\nabla u_j|)\,\mathrm{d}x \le c \int_{\mathbb{R}^n} F(v)\,\mathrm{d}\sigma, \end{cases} \tag{4.3.4}$$

where $c = c(p, q, \gamma) > 0$.

Indeed, suppose that u_j has been obtained for some $j > 1$. In the same manner as the case $j = 1$, since $u_j \le v$ in $\mathbb{R}^n$, it follows from Lemma 4.2.1 that

$$f(u_j)\mathrm{d}\sigma \in M^+(\mathbb{R}^n) \cap \Big(\mathcal{D}^{1,G}(\mathbb{R}^n)\Big)^*.$$

Using Lemma 4.2.4, there exists a unique nonnegative G-superharmonic function $u_{j+1} \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ satisfying

$$-\operatorname{div}\Big(\frac{g(|\nabla u_{j+1}|)}{|\nabla u_{j+1}|}\nabla u_{j+1}\Big) = \sigma\, f(u_j) \quad \text{in} \quad \mathbb{R}^n. \tag{4.3.5}$$

Since $f(u_{j-1})\mathrm{d}\sigma \le f(u_j)\mathrm{d}\sigma$, by Lemma 4.2.2, $u_j \le u_{j+1}$ in $\mathbb{R}^n$. Applying Corollary 2.3.20,

$$u_{j+1} \le K\,\mathbf{W}_G(f(u_j)\mathrm{d}\sigma) \le K\,\mathbf{W}_G(f(v)\mathrm{d}\sigma) \le v.$$

Testing (4.3.5) with u_{j+1}, using (2.3.3) and (4.1.13), we have

$$\begin{aligned} \int_{\mathbb{R}^n} G(|\nabla u_{j+1}|)\,\mathrm{d}x &\le \frac{1}{p}\int_{\mathbb{R}^n} g(|\nabla u_{j+1}|)|\nabla u_{j+1}|\,\mathrm{d}x \\ &= \frac{1}{p}\int_{\mathbb{R}^n} \frac{g(|\nabla u_{j+1}|)}{|\nabla u_{j+1}|}\nabla u_{j+1}\cdot\nabla u_{j+1}\,\mathrm{d}x \\ &= \frac{1}{p}\int_{\mathbb{R}^n} u_{j+1} f(u_j)\,\mathrm{d}\sigma \le \frac{1}{p}\int_{\mathbb{R}^n} v f(v)\,\mathrm{d}\sigma \\ &\le \frac{(q-1)\gamma+1}{p}\int_{\mathbb{R}^n} F(v)\,\mathrm{d}\sigma. \end{aligned}$$

Thus, the sequence (4.3.4) has been constructed.

We set $u = \lim_j u_j$ in $\mathbb{R}^n$ (pointwise). Hence $u \le v$ almost everywhere and, by Lemma 4.2.3, $u \in \mathcal{D}^{1,G}(\mathbb{R}^n)$ with $\nabla u_j \rightharpoonup \nabla u$ in $L^G(\mathbb{R}^n)$. As in the proof of Lemma 4.2.4, the last line in (4.3.4) gives

$$\frac{g(|\nabla u_j|)}{|\nabla u_j|}\nabla u_j \quad \text{is uniformly bounded in} \quad L^{G^*}(\mathbb{R}^n).$$

From Harnack's Principle (Theorem R), up to a subsequence, $\nabla u = \lim_j \nabla u_j$ pointwise in $\mathbb{R}^n$, since $u < \infty$. Consequently, the continuity of the function g and Theorem N assure

$$\frac{g(|\nabla u_j|)}{|\nabla u_j|}\nabla u_j \rightharpoonup \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \quad \text{in} \quad L^{G^*}(\mathbb{R}^n). \tag{4.3.6}$$

On the other hand, by the Monotone Convergence Theorem, one has

$$\lim_{j\to\infty}\int_{\mathbb{R}^n}\varphi f(u_j)\,\mathrm{d}\sigma = \int_{\mathbb{R}^n}\varphi f(u)\,\mathrm{d}\sigma \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n). \tag{4.3.7}$$

Therefore, letting $j \to \infty$ in the first line of (4.3.4), from (4.3.6) and (4.3.7), $u \in \mathcal{D}^{1,G}(\mathbb{R}^n) \cap L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ is a solution to (P_2).

It remains to prove the minimality of such u. Suppose that $0 \le w \in \mathcal{D}^{1,G}(\mathbb{R}^n) \cap L^f_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\sigma)$ is any nontrivial solution to Eq. (P_2). First, we will show $w \in L^F(\mathbb{R}^n, \mathrm{d}\sigma)$, that is $\int_{\mathbb{R}^n} F(w)\,\mathrm{d}\sigma < \infty$. Note that $f(w)\mathrm{d}\sigma \in M^+(\mathbb{R}^n)\cap\Big(\mathcal{D}^{1,G}(\mathbb{R}^n)\Big)^*$. By density of $C_c^\infty(\mathbb{R}^n)$ in $\mathcal{D}^{1,G}(\mathbb{R}^n)$, there exists a sequence $\{\varphi_j\} \subset C_c^\infty(\mathbb{R}^n)$ with $w = \lim_j \varphi_j$ in $\mathcal{D}^{1,G}(\mathbb{R}^n)$. It follows from (4.1.13) that

$$\begin{aligned}\int_{\mathbb{R}^n} F(w)\,\mathrm{d}\sigma &\le \frac{1}{(p-1)\gamma+1}\int_{\mathbb{R}^n} wf(w)\,\mathrm{d}\sigma = \frac{\langle f(w)\mathrm{d}\sigma, w\rangle}{(p-1)\gamma+1}\\ &= \lim_{j\to\infty}\frac{\langle f(w)\mathrm{d}\sigma, \varphi_j\rangle}{(p-1)\gamma+1} = \frac{1}{(p-1)\gamma+1}\lim_{j\to\infty}\int_{\mathbb{R}^n}\frac{g(|\nabla w|)}{|\nabla w|}\nabla w\cdot\nabla\varphi_j\,\mathrm{d}x,\end{aligned}$$

where the last equality follows by testing the equation of w with each φ_j. Since $\nabla\varphi \rightharpoonup \nabla w$ in $L^G(\mathbb{R}^n)$ and $g(|\nabla w|)/|\nabla w|\,\nabla w \in L^{G^*}(\mathbb{R}^n)$, we deduce

$$\begin{aligned}\int_{\mathbb{R}^n} F(w)\,\mathrm{d}\sigma &\le \frac{1}{(p-1)\gamma+1}\int_{\mathbb{R}^n}\frac{g(|\nabla w|)}{|\nabla w|}\nabla w\cdot\nabla w\,\mathrm{d}x\\ &= \frac{1}{(p-1)\gamma+1}\int_{\mathbb{R}^n} g(|\nabla w|)|\nabla w|\,\mathrm{d}x\\ &\le \frac{q}{(p-1)\gamma+1}\int_{\mathbb{R}^n} G(|\nabla w|)\,\mathrm{d}x < \infty.\end{aligned}$$

Next, by Corollary 2.3.20, we have

$$w \ge K^{-1}\,\mathbf{W}_G(f(w)\mathrm{d}\sigma) \ge \mathbf{W}_G(f(w)K^{-(p-1)}\mathrm{d}\sigma),$$

where in the last inequality was used (2.3.6), since $K \ge 1$. Using Theorem 4.1.4 with $K^{-(p-1)}\sigma$ in place of σ, we obtain that

$$w \ge C\Big(\mathbf{W}_G(K^{-(p-1)}\sigma)\Big)^{\frac{1}{1-\gamma}} \ge C\,K^{-\frac{p-1}{(q-1)(1-\gamma)}}\Big(\mathbf{W}_G\sigma\Big)^{\frac{1}{1-\gamma}}.$$

By choice of ε in (4.3.3), one has $u_0 \le w$ in $\mathbb{R}^n$, whence $f(u_0)\mathrm{d}\sigma \le f(w)\mathrm{d}\sigma$ in $M^+(\mathbb{R}^n)\cap\Big(\mathcal{D}^{1,G}(\mathbb{R}^n)\Big)^*$. From Lemma 4.2.2, $u_1 \le w$ almost everywhere in $\mathbb{R}^n$. Proceeding by induction, we deduce that $u_j \leq w$ almost everywhere in $\mathbb{R}^n$ for all $j \geq 1$. Thus

$$u = \lim_{j\to\infty} u_j \leq w \quad \text{almost everywhere in} \quad \mathbb{R}^n.$$

This shows the minimality of u and completes the proof of Theorem 1.3.15.

4.4 FINAL COMMENTS

Remark 4.4.1. For the case $p = q$ in assumption (A_2), γ only satisfies $0 < \gamma < 1$. Setting $r = \gamma(p-1)$, obliviously

$$\frac{1}{1-\gamma} = \frac{p-1}{p-1-\gamma(q-1)} = \frac{q-1}{q-1-\gamma(p-1)} = \frac{p-1}{p-1-r}. \tag{4.4.1}$$

Hence condition (1.3.21) reduces to

$$\Big(\mathbf{W}_p\sigma\Big)^{\frac{p-1}{p-1-r}} \in L^{1+r}(\mathbb{R}^n, \mathrm{d}\sigma). \tag{4.4.2}$$

Moreover, for $p = q$, Eq. P_2 becomes in (1.0.1), that is

$$-\Delta_p u = \sigma\, u^r \quad \text{in} \quad \mathbb{R}^n. \tag{4.4.3}$$

Thus Theorem 1.3.15 is a partial extension of [Dat and Verbitsky 2015, Theorem 3.8], when for the case $p = q$. Notice the similarity between eq. (4.4.3) and equation in (1.2.3).

Observe that, by (4.4.1), condition (4.1.31) coincides with condition (4.4.2) if $p = q$. As it was mentioned in Remark 4.1.7, would be interesting to show that condition (4.1.31) is sufficient to guarantee the existence of a nontrivial solution to Eq. (P_2) in $\mathcal{D}^{1,G}(\mathbb{R}^n) \cap L^F(\mathbb{R}^n, \mathrm{d}\sigma)$, whether $2 \leq p < q < n$. The motivation for this is that in [Dat and Verbitsky 2015] was proved that condition (4.4.2) is not only necessary but also sufficient to ensure the existence of a nontrivial solution to (4.4.3) in $\mathcal{D}^{1,p}(\mathbb{R}^n) \cap L^{1+r}(\mathbb{R}^n, \mathrm{d}\sigma)$, when $p < n$. Certainly, an answer to this question relies on the refinement of Lemma 4.1.6.

Remark 4.4.2. The same conclusion of Theorem 1.3.15 can be drawn for the $\mathcal{A}$-equation

$$-\mathrm{div}\Big(\mathcal{A}(x, \nabla u)\Big) = \sigma\, g(u^\gamma) \quad \text{in} \quad \mathbb{R}^n, \tag{4.4.4}$$

where $\mathcal{A}$ is a Carathéodory regular vector field satisfying the Orlicz growth $\mathcal{A}(x,\xi)\cdot\xi \approx G(|\xi|)$, where G is the primitive of g, given by (A_2).

To proceed formally, suppose (2.3.2) and let $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping satisfying the following conditions:

$$\begin{aligned} &x \longmapsto \mathcal{A}(x,\xi) \quad \text{is measurable for all } \xi \in \mathbb{R}^n, \\ &\xi \longmapsto \mathcal{A}(x,\xi) \quad \text{is continuous for almost everywhere } x \in \mathbb{R}^n, \end{aligned} \tag{4.4.5}$$

and there exist structural constants $\alpha > 0$ and $\beta > 0$ such that for almost everywhere $x \in \mathbb{R}^n$, for all ξ, $\eta \in \mathbb{R}^n$, $\xi \neq \eta$, it holds

$$\begin{aligned} &\mathcal{A}(x,\xi)\cdot\xi \geq \alpha\, G(|\xi|), \quad |\mathcal{A}(x,\xi)| \leq \beta\, g(|\xi|), \\ &\Big(\mathcal{A}(x,\xi) - \mathcal{A}(x,\eta)\Big)\cdot(\xi-\eta) > 0. \end{aligned} \tag{4.4.6}$$

In particular, $\mathcal{A}(x,0) = 0$ for almost everywhere $x \in \mathbb{R}^n$. A typical example is what was treated in this chapter:

$$\mathcal{A}_0 : (x,\xi) \longmapsto \mathcal{A}_0(x,\xi) = \frac{g(|\xi|)}{|\xi|}\xi.$$

Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Similarly to Definition 2.3.11, we say that a continuous function $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ is $\mathcal{A}$-*harmonic* in Ω if it satisfies $-\mathrm{div}\Big(\mathcal{A}(x,\nabla u)\Big) = 0$ weakly in Ω, that is

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = 0 \quad \forall\varphi \in C_c^\infty(\Omega).$$

A function $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ is called $\mathcal{A}$-*supersolution* in Ω if it satisfies $-\mathrm{div}\Big(\mathcal{A}(x,\nabla u)\Big) \geq 0$ weakly in Ω, and by $\mathcal{A}$-*subsolution* in Ω we mean a function u such that $-u$ is $\mathcal{A}$-*supersolution* in Ω. The classes of $\mathcal{A}$-*superharmonic* and $\mathcal{A}$-*subharmonic* functions are defined likewise Definition 2.3.15. We denote $\mathcal{S}_{\mathcal{A}}(\Omega)$ the set of all $\mathcal{A}$-superharmonic functions in Ω.

According to the above definitions, we mention that all basic facts stated in Section 2.3, and the lemmas of Section 4.2, remain true for quasilinear elliptic $\mathcal{A}$-equations with measure data, that is equations of the type

$$-\mathrm{div}\Big(\mathcal{A}(x,\nabla u)\Big) = \mu \quad \text{in} \quad \Omega. \tag{4.4.7}$$

For more details, see [Benyaiche and Khlifi 2021, Chlebicka and Karppinen 2021, Fan 2012, Chlebicka and Zatorska-Goldstein 2022, Benyaiche and Khlifi 2023]. In particular, for g given by (A_2), Corollary 2.3.20 is still true for $\mathcal{A}$-equations with measure data. In Appendix A we deal with this approach. Summarizing, we have the following theorem

Theorem 4.4.3. *Let $\mathcal{A}$ be a mapping satisfying* (4.4.5) *and* (4.4.6) *with g given by* (A_2)*. Under the hypotheses of Theorem 1.3.15, there exists a nonnegative solution $u \in \mathcal{D}^{1,G}(\mathbb{R}^n) \cap L^F(\mathbb{R}^n, \mathrm{d}\sigma)$ to* (4.4.4)*, provided* (1.3.21) *holds. Moreover, u is minimal.*

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APPENDIX A – PROOF OF THEOREM 2.3.19

The argument follows from a series of versions of Harnack inequalities. Our proof is inspired by the ideas in [Chlebicka, Giannetti and Zatorska-Goldstein 2023] and [Korte and Kuusi 2010], where the case $G_p(t) = t^p/p$ is treated.

In fact, we will prove an extending version of Theorem 2.3.19. Namely, *let* $\mathcal{A}$ *be a mapping satisfying* (4.4.5) *and* (4.4.6) *with* g *given by* (A_2), *suppose* u *is a nonnegative* $\mathcal{A}$*-superharmonic in* $B(x_0, 2R)$, *then estimate* (2.3.18) *holds for*

$$\mu = \mu[u] = -\mathrm{div}\Big(\mathcal{A}(x, \nabla u)\Big). \tag{A.0.1}$$

To simplify notation, we set $B_R = B(x_0, R)$ for $R > 0$, and $\kappa B_R = B(x_0, \kappa R)$ for any $\kappa > 0$.

Remark A.0.1. We emphasize that (2.3.18) is an estimate only at x_0, the center of B_{2R}. Hence we may reduce the proof of the theorem significantly to a more restricted case. Namely, we only consider the class of continuous $\mathcal{A}$-supersolutions functions.

Indeed, since u is a nonnegative $\mathcal{A}$-superharmonic in B_{2R}, from [Chlebicka and Zatorska-Goldstein 2022, Proposition 4.5], there exists a nondecreasing sequence of nonnegative functions $\{u_j\} \subset C(\overline{B}_R) \cap \mathcal{S}_{\mathcal{A}}(B_R)$ satisfying $u_j = 0$ on ∂B_R (for $j = 1, 2, \ldots$) and

$$u = \lim_{j\to\infty} u_j \quad \text{in} \quad B_R \text{ (pointwise)}. \tag{A.0.2}$$

By [Chlebicka and Zatorska-Goldstein 2022, Lemma 4.6], u_j is an $\mathcal{A}$-supersolution in B_R, for all $j \geq 1$. This implies that $u_j \in W^{1,G}_{\mathrm{loc}}(B_R)$, and $Du_j = \nabla u_j$ for all $j \geq 1$. By Theorem R, we have $Du = \lim_j \nabla u_j$ pointwise in B_R, possibly passing to a subsequence. On the other hand, notice that $\{u_j\}$ is bounded in B_R (pointwise), since $u_j \leq u$ in B_R and $u_j = 0$ on ∂B_R for all $j \geq 1$. Extending u_j by zero away from ∂B_R, we may consider u_j as an $\mathcal{A}$-supersolution in B_{3R} for all $j \geq 1$. From this, an appeal to [Chlebicka and Karppinen 2021, Lemma 5.2] assure that there exists $c_0 = c_0(p, q, \alpha, \beta) > 0$ such that

$$\begin{aligned}\int_{B_R} G(|\nabla u_j|)\,\mathrm{d}x &\leq c_0 \int_{B_{2R}} G\left(\frac{\mathrm{osc}_{B_{2R}} u_j}{R}\right) \mathrm{d}x \leq c_0 \int_{B_{2R}} G\left(\frac{\sup_{B_{2R}} u_j}{R}\right) \mathrm{d}x \\ &\leq c_1\, R^n G\left(\frac{\sup_{B_{2R}} u_j}{R}\right) \quad \forall j \geq 1,\end{aligned}$$

where $c_1 = c_1(n, p, q, \alpha, \beta) > 0$. Consequently, $\{\nabla u_j\}$ is bounded in $L^G(B_R)$, and by Theorem N, $\nabla u_j \rightharpoonup Du$ in $L^G(B_R)$. In particular, $Du \in L^G(B_R)$. From (4.4.5) and (4.4.6), we deduce that

$$\mathcal{A}(x, \nabla u_j) \rightharpoonup \mathcal{A}(x, Du) \quad \text{in} \quad L^{G^*}(B_R).$$

This follows by the same method as in (4.3.6). Thus we also have the weak convergence of corresponding measures $\mu_j := \mu[u_j]$ to $\mu := \mu[u]$ in B_R:

$$\int_{B_R} \varphi \,\mathrm{d}\mu = \int_{B_R} \mathcal{A}(x, Du) \cdot \nabla\varphi \,\mathrm{d}x = \lim_{j\to\infty} \int_{B_R} \mathcal{A}(x, \nabla u_j) \cdot \nabla\varphi \,\mathrm{d}x$$
$$= \lim_{j\to\infty} \int_{B_R} \varphi \,\mathrm{d}\mu_j \quad \forall \varphi \in C_c^\infty(B_R).$$

The previous equality implies that μ_j converges weakly* to μ, in measure sense. From this, using [Bogachev 2018, Theorem 2.2.5], it follows that

$$\varliminf_{j\to\infty} \mu_j(B(x_0, s)) \geq \mu(B(x_0, s)) \quad \forall s \leq R, \tag{A.0.3}$$

$$\varlimsup_{j\to\infty} \mu_j(\overline{B(x_0, s)}) \leq \mu(\overline{B(x_0, s)}) \quad \forall s \leq R. \tag{A.0.4}$$

Now, suppose the bounds in (2.3.18) holds for u_j, $j = 1, 2, \ldots$. Using (A.0.3), we obtain from the lower bound in (2.3.18) and from (A.0.2) that

$$u(x_0) = \lim_{j\to\infty} u_j(x_0) \geq C_1 \lim_{j\to\infty} \mathbf{W}_G^R \mu_j(x_0) = C_1 \lim_{j\to\infty} \int_0^R g^{-1}\left(\frac{\mu_j(B(x_0, s))}{s^{n-1}}\right) \mathrm{d}s$$
$$= C_1 \int_0^R g^{-1}\left(\frac{\varliminf_{j\to\infty} \mu_j(B(x_0, s))}{s^{n-1}}\right) \mathrm{d}s \geq C_1 \int_0^R g^{-1}\left(\frac{\mu(B(x_0, s))}{s^{n-1}}\right) \mathrm{d}s,$$

which shows the lower bound in (2.3.18) for $u(x_0)$. To verify the upper bound in (2.3.18) for $u(x_0)$, first notice that $\inf_{B_R} u_j \leq \inf_{B_R} u$ for all $j \geq 1$. Combining this with (A.0.4) and (A.0.2), one has

$$u(x_0) = \lim_{j\to\infty} u_j(x_0) \leq C_2\left(\inf_{B_R} u + \varlimsup_{j\to\infty} \mathbf{W}_G^R \mu_j(x_0)\right)$$
$$\leq C_2\left(\inf_{B_R} u + \varlimsup_{j\to\infty} \int_0^R g^{-1}\left(\frac{\mu_j(\overline{B(x_0, s)})}{s^{n-1}}\right) \mathrm{d}s\right)$$
$$= C_2\left(\inf_{B_R} u + \int_0^R g^{-1}\left(\frac{\varlimsup_{j\to\infty} \mu_j(\overline{B(x_0, s)})}{s^{n-1}}\right) \mathrm{d}s\right)$$
$$\leq C_2\left(\inf_{B_R} u + \int_0^R g^{-1}\left(\frac{\mu(\overline{B(x_0, s)})}{s^{n-1}}\right) \mathrm{d}s\right).$$

The upper bound in (2.3.18) for $u(x_0)$ follows from the fact

$$\int_0^R g^{-1}\left(\frac{\mu(\overline{B(x_0, s)})}{s^{n-1}}\right) \mathrm{d}s = \int_0^R g^{-1}\left(\frac{\mu(B(x_0, s))}{s^{n-1}}\right) \mathrm{d}s. \tag{A.0.5}$$

To prove (A.0.5), first note that the function $t \mapsto \mu(B(x_0, t))$ is monotone in $t \geq 0$, whence the set

$$A = \{t_0 > 0 : t \mapsto \mu(B(x_0, t)) \text{ is discontinuous in } t_0\}$$

is enumerable (see for instance [Royden 1988, Chapter 6, Theorem 1]). On the other hand, we infer from [Royden 1988, Chapter 17, Proposition 2] that for all $s > 0$

$$\lim_{r\to s^-} \mu(B(x_0, r)) = \mu(B(x_0, s)) \quad \text{and}$$
$$\lim_{r\to s^+} \mu(B(x_0, r)) = \mu(\overline{B(x_0, s)}).$$

Consequently, $A = \{t_0 > 0 : \mu(\partial B(x_0, t_0)) \neq 0\}$, since $\mu(\overline{B(x_0, s)}) = \mu(B(x_0, s)) + \mu(\partial B(x_0, s))$, and $t \mapsto \mu(B(x_0, t))$ is continuous on $[0, R] \setminus A$. Since A is a set of measure zero, (A.0.5) follows. This completes the assertion of Remark A.0.1.

Let us list some preliminary results. Except for the Harnack inequalities given in (A.0.6)-(A.0.9) below, these results hold for any N-functions G which satisfies (2.3.3). The following type-Caccioppoli estimate [Chlebicka, Giannetti and Zatorska-Goldstein 2023, Proposition 3.24] will be useful to show the lower bound in (2.3.18). As usual, $\Omega \subseteq \mathbb{R}^n$ means a domain.

Lemma V. *If $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$ is a nonnegative $\mathcal{A}$-subsolution, then there exists a constant $C = C(p, q, \alpha, \beta) > 0$, such that*

$$\int_\Omega G(|\nabla u|)\eta^q \,\mathrm{d}x \le C \int_\Omega G(u|\nabla \eta|) \,\mathrm{d}x \quad \forall \eta \in C_c^\infty(\Omega).$$

The following Minimum and Maximum Principles [Chlebicka and Zatorska-Goldstein 2022, Corollaries 4.15 and 4.16] will be a helpful ingredient to work together with Harnack's inequalities, and consequently to prove the bounds in (2.3.18).

Lemma W. *Suppose $\Omega \subset \mathbb{R}^n$ bounded and let $D \subset \Omega$ be a connect open subset compactly contained in Ω.*

$(\mathbf{a})$ *Suppose u is $\mathcal{A}$-superharmonic function and finite (almost everywhere) in Ω. Then*

$$\inf_D u = \inf_{\partial D} u.$$

$(\mathbf{b})$ *Suppose u is $\mathcal{A}$-subharmonic function and finite (almost everywhere) in Ω. Then*

$$\sup_D u = \sup_{\partial D} .$$

In what follows, we use the abbreviation

$$\fint_\Omega u \,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x, \quad \Omega \subset \mathbb{R}^n \text{ bounded.}$$

Next, we state versions of Harnack's inequalities, which will play a crucial role in the proof of bounds in (2.3.18). Recall $W^{1,G}(B_{2R}) \subset W^{1,p}(B_{2R})$ [Harjulehto and Hästö 2019, Lemma 6.1.6]. Suppose g is given by (A_2). With the aid of [Chlebicka and Zatorska-Goldstein 2022, Lemma 3.7 and Corollary 3.8], we infer from [Baroni, Colombo and Mingione 2015, Theorem 2.5] the weak Harnack inequality for $\mathcal{A}$-supersolutions in B_{2R} and for $\mathcal{A}$-harmonic functions in B_{2R}, respectively. This is the content of the following theorem.

Theorem X. *Let g be given by (A_2). Let u be a nonnegative $\mathcal{A}$-supersolution in B_{2R}. Then there exist constants $c_0 = c_0(n, p, q, \alpha, \beta) > 0$ and $s_0 = s_0(n, p, q, \alpha, \beta) \in (0, 1)$ such that*

$$\left(\fint_{B_{2R}} u^{s_0} \, \mathrm{d}x \right)^{\frac{1}{s_0}} \leq c_0 \inf_{B_R} u. \tag{A.0.6}$$

Furthermore, if u is $\mathcal{A}$-harmonic in B_{2R}, there exists $c = c(n, p, q, \alpha, \beta) > 0$ such that

$$\sup_{B_R} u \leq c \inf_{B_R} u. \tag{A.0.7}$$

Combining Lemma W with (A.0.7), we establish the following result, which will be useful in the proof of the upper bound (2.3.18).

Corollary A.0.2. *Let g be given by (A_2). Suppose u is $\mathcal{A}$-harmonic in $B_{3R/2} \setminus B_R$. Then there exists a constant $c = c(n, p, q, \alpha, \beta) > 0$ such that*

$$\sup_{\partial B_{\frac{4}{3}R}} u \leq c \inf_{\partial B_{\frac{4}{3}R}} u. \tag{A.0.8}$$

Proof. Let $\varepsilon > 0$ be a constant sufficiently small for which

$$\overline{B}_R \subset B_{\frac{4}{3}R-\varepsilon} \subset \overline{B}_{\frac{4}{3}R+\varepsilon} \subset B_{\frac{3}{2}R}.$$

Then u is $\mathcal{A}$-harmonic in the annulus $A_\varepsilon := B_{4R/3+\varepsilon} \setminus B_{4R/3-\varepsilon}$. Recall that $B_{3R/2} \setminus B_R = \{z : R < |z - x_0| < 3/2R\}$. We claim that there exists a constant $\delta > 0$ sufficiently small such that for all $x \in A_\varepsilon$, and for all $y \in B(x, \delta)$, it holds $y \in A_\varepsilon$. Indeed, on the contrary, we would find sequences $x_i \in A_\varepsilon$ and $y_i \in B(x_i, 1/i)$, satisfying either $|y_i - x_0| \leq R$ or $|y_i - x_0| \geq 3/2R$. By choice of ε, A_ε is compactly contained in $B_{3R/2} \setminus B_R$, whence we may assume that x_i converges to a $\overline{x}$ in $\overline{A}_\varepsilon$. Thus, y_i converges to $\overline{x}$, which implies that either $|\overline{x} - x_0| \leq R$ or $|\overline{x} - x_0| \geq 3/2R$. This contradicts the fact that $\overline{x} \in \overline{A}_\varepsilon$, and verifies the claim.

We may cover A_ε with finite number of balls of the form $\{B(x_i, \delta)\}$, $x_i \in A_\varepsilon$, $i = 1, \ldots, N$. From (A.0.7), it follows

$$\sup_{B(x_i,\delta)} u \leq c \inf_{B(x_i,\delta)} u \quad \forall i = 1, \ldots, N.$$

Using Lemma W, we deduce that

$$\begin{aligned}\sup_{\partial B_{\frac{4}{3}R+\varepsilon}} u \le \sup_{\partial A_\varepsilon} u = \sup_{A_\varepsilon} u \le \sup_{\bigcup_{i=1}^N B(x_i,\delta)} u \\ \le c \inf_{\bigcup_{i=1}^N B(x_i,\delta)} u \le c \inf_{A_\varepsilon} u \le c \inf_{\partial A_\varepsilon} u \le c \inf_{\partial B_{\frac{4}{3}R+\varepsilon}} u.\end{aligned}$$

Letting $\varepsilon \to 0$, the corollary follows since u is continuous in $B_{3R/2} \setminus B_R$. □

The same reasoning applies to the following case: *if u is an $\mathcal{A}$-harmonic in $B_{2R} \setminus B_{5R/4}$, then there exists $c = c(n,p,q,\alpha,\beta) > 0$ such that*

$$\sup_{\partial B_{\frac{4}{3}R}} u \le c \inf_{\partial B_{\frac{4}{3}R}} u \tag{A.0.9}$$

The following result involves the *Poisson modification* of superharmonic functions, which together with (A.0.6) and (A.0.7) will be decisive in the proof of the upper bound (2.3.18). For its definition, we begin by recalling the following existence and uniqueness of $\mathcal{A}$-harmonic functions [Chlebicka and Zatorska-Goldstein 2022, Proposition 3.1]:

Let Ω be a bounded domain in $\mathbb{R}^n$. If $w \in W^{1,G}(\Omega) \cap C(\Omega)$, then there exists a unique solution $u \in W^{1,G}(\Omega) \cap C(\Omega)$ to problem

$$\begin{cases} -\operatorname{div}\Big(\mathcal{A}(x,\nabla u)\Big) = 0 & \text{in} \quad \Omega, \\ u = w & \text{on} \quad \partial\Omega. \end{cases} \tag{A.0.10}$$

In the second line of (A.0.10), we interpret $u = w$ on $\partial\Omega$ in a weak sense, that is $u - w \in W_0^{1,G}(\Omega)$. We state the notion of a bounded *regular* set Ω. A boundary point $y_0 \in \partial\Omega$ is called *regular* if it satisfies the following property: for any solution u to (A.0.10), with $w \in W^{1,G}(\Omega) \cap C(\overline{\Omega})$, it holds

$$\lim_{\substack{x\to y_0\\ x\in\Omega}} u(x) = w(y_0).$$

This means that a boundary point is regular if at this point the boundary value of any $\mathcal{A}$-harmonic function is attained not only in the distributional sense but also pointwise. A bounded set is called *regular* if all of its boundary points are regular.

There is a known criterion, so-called *the Wiener Criterion*, elucidates that the boundary regularity of a solution to (A.0.10) at $y_0 \in \partial\Omega$ can be characterized by a geometric quantity on $\partial\Omega$ at y_0. We refer the reader to [Lee and Lee 2021] for more details. For instance, by [Lee and Lee 2021, Remark 3.8], bounded Lipshitz domains are regular sets. In particular, balls and annuli are regular sets.

Now let Ω' be a bounded domain and $\Omega \subset \Omega'$ an open subset compactly contained in Ω' with $\overline{\Omega}$ regular. Let u be an $\mathcal{A}$-superharmonic and finite (almost everywhere) in Ω', i.e. $u \in \mathcal{S}_{\mathcal{A}}(\Omega')$. For $x \in \Omega$, we define

$$u_\Omega(x) = \inf \left\{ v(x) : v \in \mathcal{S}_{\mathcal{A}}(\Omega), \quad \varliminf_{\substack{z \to y_0 \\ z \in \Omega}} v(z) \geq u(y_0) \quad \forall y_0 \in \partial\Omega \right\},$$

and the *Poisson modification* of u in Ω is given by

$$P(u,\Omega)(x) := \begin{cases} u_\Omega(x) & \text{if } x \in \Omega \\ u(x) & \text{if } x \in \Omega' \setminus \Omega. \end{cases}$$

The Poisson modification carries the idea of local smoothing of an $\mathcal{A}$-superharmonic function in a regular set. This is the content of the next theorem [Chlebicka and Zatorska-Goldstein 2022, Theorem 3]

Theorem Y. *Let $u \in \mathcal{S}_{\mathcal{A}}(\Omega')$ and let $\Omega \subset \Omega'$ an open subset compactly contained in Ω' with $\overline{\Omega}$ regular. Then*

($\mathbf{i}$) $P(u,\Omega) \in \mathcal{S}_{\mathcal{A}}(\Omega')$,
($\mathbf{ii}$) $P(u,\Omega)$ *is $\mathcal{A}$-harmonic in Ω,*
($\mathbf{iii}$) $P(u,\Omega) \leq u$ *in Ω'.*

To illustrate the property of Ω is regular, we bring the proof of Theorem Y.

Proof. The proof is adapted from [Heinonen, Kilpeläinen and Martio 2006, Lemma 7.14]. Let Ω be an open set with $\overline{\Omega}$ regular. Notice that $u_\Omega \leq u$ in Ω, since u is an $\mathcal{A}$-superharmonic function in Ω and, by the lower-semicontinuity, clearly

$$\varliminf_{\substack{z \to y_0 \\ z \in \Omega}} u(z) \geq u(y_0) \quad \forall y_0 \in \partial\Omega.$$

Hence $P(u,\Omega) \leq u$ in Ω'.

Next, fix an increasing sequence $\{\varphi_i\} \subset C^\infty(\mathbb{R}^n)$ which converges pointwise to u in $\overline{\Omega}$. Let $u_i \in W^{1,G}(\Omega) \cap C(\overline{\Omega})$ the unique solution to (A.0.10) with $w = \varphi_i$, i.e. $u_i = \varphi_i$ on $\partial\Omega$ for all $i \geq 1$. Being $\overline{\Omega}$ is regular, the sequence $\{u_i\}$ is increasing by comparison principle (Definition 2.3.15 ($\mathbf{iii}$)), since for all $i \geq 1$, it follows

$$u_{i+1} = \varphi_{i+1} \geq \varphi_i = u_i \text{ on } \partial\Omega \text{ (pointwise)} \implies u_{i+1} \geq u_i \text{ in } \Omega.$$

Setting $\overline{u}(x) = \lim_i u_i(x)$ for $x \in \Omega$, from Harnack's conergence theorem [Chlebicka and Zatorska-Goldstein 2022, Corollary 4.9], $\overline{u}$ is an $\mathcal{A}$-harmonic function in Ω. By Maximum principle (Lemma W $(\mathbf{b})$), we have that $u_i \leq u$ in Ω for all $i \geq 1$, since by definition of φ_i

$$\sup_{\Omega} u_i = \sup_{\partial\Omega} u_i = \max_{\partial\Omega} \varphi_i \leq u \quad \forall i.$$

This implies that $\overline{u} \leq u$ in Ω. Fix $y_0 \in \partial\Omega$ and let $\{z_j\} \subset \Omega$ be a sequence converging to y_0. Then by continuity of u_i (for all i)

$$\lim_{j\to\infty} \lim_{i\to\infty} u_i(z_j) = \lim_{i\to\infty} u_i(y_0),$$

whence

$$\varliminf_{\substack{z\to y_0 \\ z\in\Omega}} \overline{u}(z) \geq \lim_{i\to\infty} u_i(y_0) = \lim_{i\to\infty} \varphi_i(y_0) = u(y_0) \quad \forall y_0 \in \Omega.$$

Consequently, $\overline{u} \geq P(u, \Omega)$ in Ω.

On the other hand, being u_i $\mathcal{A}$-harmonic in Ω, from comparson principle again, it follows

$$u_i \leq P(u, \Omega) \quad \text{in} \quad \Omega \quad \forall i \geq 1.$$

Then $\overline{u} \leq P(u, \Omega)$ in Ω, which gives $\overline{u} = P(u, \Omega)$ in Ω. This implies that $P(u, \Omega)$ $\mathcal{A}$-harmonic in Ω and is lower semicontinuous. Using [Chlebicka and Zatorska-Goldstein 2022, Lemma 4.3], $P(u, \Omega)$ is also an $\mathcal{A}$-superharmonic in Ω'. □

We need the following result regarding Sobolev functions [Harjulehto, Hästö and Toivanen 2017, Lem. 3.5].

Lemma Z. *If $u \in W^{1,G}(\Omega)$ with $\operatorname{supp} u \subset \Omega$, then $u \in W_0^{1,G}(\Omega)$.*

We are now in a position to prove the estimates in (2.3.18). We start by proving the upper bound in (2.3.18). By Remark A.0.1, we suppose u is a continuous bounded $\mathcal{A}$-supersolution in $B_{2R} = B(x_0, 2R)$. To the proof of the upper bound, we may also reduce to a simpler case. In light of the Poisson modification, we will modify u to be a $\mathcal{A}$-harmonic function in a countable union of disjoint annuli shrinking to the reference point x_0. For this purpose, we use the Poisson Modification of u over a family of annuli. The crucial fact is that the corresponding measure in each annulus concentrates on the boundary of the particular annulus, but in a controllable way since the measure corresponding to the new solution belongs to $\left(W^{1,G}(B_R)\right)^*$.

To be more precise, let $R_k = 2^{1-k}R$ and $B_k = 2^{1-k}B_R = B(x_0, R_k)$, $k = 0, 1, 2, \ldots$. Consider the union of annuli

$$\Omega = \bigcup_{k=1}^{\infty} \frac{3}{2} B_k \setminus \overline{B}_k.$$

By definition, Ω is regular, and we can consider $v := P(u,\Omega)$. From Theorem Y, $v = u$ in $B_{2R}\setminus\Omega$, $v \in \mathcal{S}_{\mathcal{A}}(B_{2R})$ and v is $\mathcal{A}$-harmonic in Ω, that is

$$-\operatorname{div}\Big(\mathcal{A}(x,\nabla v)\Big) = 0 \quad \text{in} \quad \Omega.$$

Notice that v is continuous by the assumed continuity of u, whence v is also $\mathcal{A}$-supersolution in B_{2R}. Consequently, there exists $\mu_v \in M^+(B_{2R})$ such that

$$-\operatorname{div}\Big(\mathcal{A}(x,\nabla v)\Big) = \mu_v \quad \text{in} \quad B_{2R}. \tag{A.0.11}$$

Proof of the upper bound in Theorem 2.3.19. The basic idea is to introduce comparison solutions with zero boundary values and measures given by μ_v. Recall $\mu = \mu_u$. We begin by establishing that

$$\mu_v(B_k) = \mu(B_k) \quad \forall k \geq 0. \tag{A.0.12}$$

This is a consequence of the inner regularity. Indeed, let $\varphi \in C_c^\infty(B_k)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on a compact set K satisfying

$$\frac{3}{2}\overline{B}_{k+1} \subset K \subset B_k.$$

Note that $u = v$ in $\operatorname{supp}\nabla\varphi \subset B_k \setminus (3/2\overline{B}_{k+1}) \nsubseteq \Omega$. From this, we deduce by testing φ in (A.0.1) and in (A.0.11) that

$$\begin{aligned}\int_{B_k} \varphi \,\mathrm{d}\mu_v &= \int_{B_k} \mathcal{A}(x,\nabla v)\cdot\nabla\varphi\,\mathrm{d}x \\ &= \int_{B_k} \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = \int_{B_k}\varphi\,\mathrm{d}\mu.\end{aligned}$$

Consequently, $\mu_v(K) = \mu(K)$ and, by exhausting B_k with such K, the inner regularity of these measures yields (A.0.12).

Next, notice that μ_v belongs to $\Big(W_0^{1,G}(4/3B_{k+1})\Big)^*$ for all $k \geq 0$, since

$$\mu_v \in \Big(W^{1,G}_{\mathrm{loc}}(B_{2R})\Big)^* \hookrightarrow \left(W_0^{1,G}\Big(\frac{4}{3}B_{k+1}\Big)\right)^* \quad \forall k \geq 0.$$

From Theorem Q, there exists $v_k \in W_0^{1,G}(4/3B_{k+1})$ satisfying

$$-\operatorname{div}\Big(\mathcal{A}(x,\nabla v_k)\Big) = \mu_v \quad \text{in} \quad \frac{4}{3}B_{k+1}. \tag{A.0.13}$$

Setting $\mu_k = \mu_{v_k}$, one has

$$\mu_k\Big(\frac{4}{3}B_{k+1}\Big) = \mu_v\Big(\frac{4}{3}B_{k+1}\Big) \quad \forall k \geq 0. \tag{A.0.14}$$

In light of Lemma P, we may assume $v_k \geq 0$ (almost everywhere). Since $4/3B_{k+1} \setminus \overline{B}_{k+1} \subset 3/2B_{k+1} \setminus \overline{B}_{k+1}$ and v is $\mathcal{A}$-harmonic in $3/2B_{k+1} \setminus \overline{B}_{k+1}$, it follows from (A.0.14) that v_k is $\mathcal{A}$-harmonic in $4/3B_{k+1} \setminus \overline{B}_{k+1}$, whence v_k takes continuously zero boundary value on $\partial(4/3B_{k+1})$. Moreover, $v - \max_{\partial(4/3B_{k+1})} v \leq 0$ on $\partial(4/3B_{k+1})$. From this,

$$\Big(v - \max_{\partial\frac{4}{3}B_{k+1}} v - v_k\Big)_+ = 0 \quad \text{on} \quad \partial\frac{4}{3}B_{k+1}.$$

This means that $\operatorname{supp}\Big(v - \max_{\partial(4/3B_{k+1})} v - v_k\Big)_+ \subset 4/3B_{k+1}$ and, by Lemma Z,

$$\Big(v - \max_{\partial\frac{4}{3}B_{k+1}} v - v_k\Big)_+ \in W_0^{1,G}\Big(\frac{4}{3}B_{k+1}\Big).$$

A subtraction of equations of v and v_k, (A.0.11) and (A.0.13) respectively, with the previous test function, gives

$$\begin{aligned} 0 &= \int_{\frac{4}{3}B_{k+1}} \Big(v - \max_{\partial\frac{4}{3}B_{k+1}} v - v_k\Big)_+ \,\mathrm{d}\mu_k - \int_{\frac{4}{3}B_{k+1}} \Big(v - \max_{\partial\frac{4}{3}B_{k+1}} v - v_k\Big)_+ \,\mathrm{d}\mu_v \\ &= \int_{\frac{4}{3}B_{k+1}} \mathcal{A}(x, \nabla v_k)\cdot\nabla\Big(v - \max_{\partial\frac{4}{3}B_{k+1}} v - v_k\Big)_+ \,\mathrm{d}x - \int_{\frac{4}{3}B_{k+1}} \mathcal{A}(x, \nabla v)\cdot\nabla\Big(v - \max_{\partial\frac{4}{3}B_{k+1}} v - v_k\Big)_+ \,\mathrm{d}x. \end{aligned}$$

On account of $\operatorname{supp}\nabla\Big(v - \max_{\partial(4/3B_{k+1})} v - v_k\Big)_+ \subset 4/3B_{k+1} \cap \{v - \max_{\partial(4/3B_{k+1})} v \geq v_k\}$, we have from the previous equality

$$\begin{aligned} 0 &= \int_{\frac{4}{3}B_{k+1}\cap\{v - \max_{\partial(4/3B_{k+1})} v \geq v_k\}} \mathcal{A}(x, \nabla v_k)\cdot\nabla(v - v_k)\,\mathrm{d}x \\ &\quad - \int_{\frac{4}{3}B_{k+1}\cap\{v - \max_{\partial(4/3B_{k+1})} v \geq v_k\}} \mathcal{A}(x, \nabla v_k)\cdot\nabla(v - v_k)\,\mathrm{d}x \\ &= \int_{\frac{4}{3}B_{k+1}\cap\{v - \max_{\partial(4/3B_{k+1})} v \geq v_k\}} \Big(\mathcal{A}(x, \nabla v_k) - \mathcal{A}(x, \nabla v)\Big)\cdot\nabla(v - v_k)\,\mathrm{d}x \leq 0, \end{aligned}$$

where the last inequality is due the monotonicity of $\mathcal{A}$ in (4.4.6). Accordingly,

$$\nabla\Big(v - \max_{\partial(4/3B_{k+1})} v - v_k\Big)_+ = 0 \quad \text{in} \quad \frac{4}{3}B_{k+1},$$

whence

$$v_k \geq v - \max_{\partial\frac{4}{3}B_{k+1}} v \quad \text{in} \quad \frac{4}{3}B_{k+1}. \tag{A.0.15}$$

Note that $3/2B_{k+2} \setminus \overline{B}_{k+2} \subset 4/3B_{k+1}$. By (A.0.14), we have $\mu_k(3/2B_{k+2} \setminus \overline{B}_{k+2}) = \mu_v(3/2B_{k+2} \setminus \overline{B}_{k+2}) = 0$, since v is $\mathcal{A}$-harmonic in $3/2B_{k+2} \setminus \overline{B}_{k+2}$. From this, v_k is $\mathcal{A}$-harmonic in $3/2B_{k+2} \setminus \overline{B}_{k+2}$. Using Harnack's inequality (A.0.8), there exists a constant $c_1 = c_1(n, p, q, \alpha, \beta) > 0$ such that

$$\max_{\partial\frac{4}{3}B_{k+2}} v_k \leq c_1 \min_{\partial\frac{4}{3}B_{k+2}} v_k. \tag{A.0.16}$$

We will consider two cases. First, assume that $\min_{\partial(4/3)B_{k+2}} v_k = 0$. From (A.0.16), $\max_{\partial(4/3B_{k+2})} v_k = 0$, and by (A.0.15)

$$\max_{\partial\frac{4}{3}B_{k+2}} v - \max_{\partial\frac{4}{3}B_{k+1}} v \leq 0. \tag{A.0.17}$$

Next, suppose that $\min_{\partial(4/3)B_{k+2}} v_k > 0$. For $k = 0, 1, \ldots$, we set

$$w_k(x) = \min\left\{v_k(x), \min_{\partial\frac{4}{3}B_{k+2}} v_k\right\} \quad \forall x \in \frac{4}{3}B_{k+1}.$$

From [Chlebicka and Zatorska-Goldstein 2022, Corollary 4.2], $w_k \in \mathcal{S}_{\mathcal{A}}(4/3B_{k+1})$. We claim that

$$\mu_{w_k}\left(\frac{4}{3}B_{k+1}\right) = \mu_v\left(\frac{4}{3}B_{k+1}\right) \quad \forall k \geq 0. \tag{A.0.18}$$

Indeed, if we prove that $\mu_{w_k}(4/3B_{k+1}) = \mu_k(4/3B_{k+1})$ for all $k \geq 0$, the assertion follows by (A.0.14). By the inner regularity,

$$\mu_{w_k}\left(\frac{4}{3}B_{k+1}\right) = \sup \mu_{w_k}(K),$$

where the supremum is taken over all compact sets $K \subset 4/3B_{k+1}$. Let $\varphi_K \in C_c^\infty(4/3B_{k+1})$ such that $0 \leq \varphi_K \leq 1$ and $\varphi_K = 1$ on K. We may suppose that $4/3B_{k+2} \subset K$. By continuity of v_k, $v_k \leq \min_{\partial(4/3B_{k+2})} v_k$ in a neighborhood V of $\partial(4/3B_{k+1})$, since $v_k = 0$ on $\partial(4/3B_{k+1})$. It follows that $w_k = v_k$ in V and $\operatorname{supp}\nabla\varphi_K \subset V$. By taking the supremum in all compact sets K with $4/3B_{k+1} \setminus K \subset V$, we arrive at

$$\begin{aligned}
\mu_{w_k}\left(\frac{4}{3}B_{k+1}\right) &= \sup \mu_{w_k}(K) = \sup \int_{\frac{4}{3}B_{k+1}} \varphi_K \,\mathrm{d}\mu_{w_k} \\
&= \sup \int_{\frac{4}{3}B_{k+1}} \mathcal{A}(x, \nabla w_k) \cdot \nabla\varphi_K \,\mathrm{d}x \\
&= \sup \int_{\frac{4}{3}B_{k+1}} \mathcal{A}(x, \nabla v_k) \cdot \nabla\varphi_K \,\mathrm{d}x = \sup \int_{\frac{4}{3}B_{k+1}} \varphi_K \,\mathrm{d}\mu_k = \mu_k\left(\frac{4}{3}B_{k+1}\right).
\end{aligned}$$

Accordingly, it follows from (A.0.18) and (4.4.6) that

$$\begin{aligned}
\left(\min_{\partial\frac{4}{3}B_{k+2}} v_k\right)\mu_v\left(\frac{4}{3}B_{k+1}\right) &= \int_{\frac{4}{3}B_{k+1}} \left(\min_{\partial\frac{4}{3}B_{k+2}} v_k\right) \mathrm{d}\mu_v \geq \int_{\frac{4}{3}B_{k+1}} w_k \,\mathrm{d}\mu_v \\
&= \int_{\frac{4}{3}B_{k+1}} w_k \,\mathrm{d}\mu_{w_k} = \int_{\frac{4}{3}B_{k+1}} \mathcal{A}(x, \nabla w_k) \cdot \nabla w_k \,\mathrm{d}x \\
&\geq \alpha \int_{\frac{4}{3}B_{k+1}} G(|\nabla w_k|) \,\mathrm{d}x.
\end{aligned}$$

Combining the Modular Poincaré inequality (Lemma O) with (2.3.5) in the previous estimate, we obtain

$$\Big(\min_{\partial\frac{4}{3}B_{k+2}} v_k\Big)\mu_v\Big(\frac{4}{3}B_{k+1}\Big) \geq c_2 \int_{\frac{4}{3}B_{k+1}} G\Big(\frac{w_k}{R_{k+1}}\Big)\,\mathrm{d}x \geq c_3 \int_{\frac{4}{3}B_{k+1}} G\Big(\frac{w_k}{R_k}\Big)\,\mathrm{d}x$$
$$\geq \int_{\frac{4}{3}B_{k+2}} G\Big(\frac{w_k}{R_k}\Big)\,\mathrm{d}x \geq \int_{\frac{4}{3}B_{k+2}} G\Big(\frac{\min_{\partial\frac{4}{3}B_{k+2}} v_k}{R_k}\Big)\,\mathrm{d}x$$
$$= c_4 R_k^n\, G\Big(\frac{\min_{\partial\frac{4}{3}B_{k+2}} v_k}{R_k}\Big),$$

where c_2, c_3 and c_4 are positive constants depending only on n and p, q, α, β. Consequently, by (2.3.3)

$$\frac{\mu_v\Big(\frac{4}{3}B_{k+1}\Big)}{R_k^{n-1}} \geq c_4\Big(\frac{\min_{\partial\frac{4}{3}B_{k+2}} v_k}{R_k}\Big)^{-1} G\Big(\frac{\min_{\partial\frac{4}{3}B_{k+2}} v_k}{R_k}\Big) \geq c_4\, g\Big(\frac{\min_{\partial\frac{4}{3}B_{k+2}} v_k}{R_k}\Big),$$

where $c_4 = c_4(n, p, q, \alpha, \beta) > 0$. Since $\min_{\partial(4/3B_{k+2})} v_k > 0$, (A.0.16) leads to

$$\max_{\partial\frac{4}{3}B_{k+2}} v_k \leq c_5\, R_k\, g^{-1}\Big(\frac{\mu_v\Big(\frac{4}{3}B_{k+1}\Big)}{R_k^{n-1}}\Big),$$

where $c_5 = c_5(n, p, q, \alpha, \beta) > 0$ is obtained from combining c_4 with (2.3.6). By (A.0.15), the preceding inequality gives

$$\max_{\partial\frac{4}{3}B_{k+2}} v - \max_{\partial\frac{4}{3}B_{k+1}} v \leq c_5\, R_k\, g^{-1}\Big(\frac{\mu_v\Big(\frac{4}{3}B_{k+1}\Big)}{R_k^{n-1}}\Big). \tag{A.0.19}$$

Thus, in all cases, by summing up (A.0.17) and (A.0.19) in $k = 2, 3, \ldots$, we deduce from (A.0.8) that

$$\overline{\lim_{k\to\infty}} \max_{\partial\frac{4}{3}B_{k+2}} v \leq \max_{\partial\frac{4}{3}B_3} v + c_5 \sum_{k=2}^{\infty} R_k\, g^{-1}\Big(\frac{\mu_v\Big(\frac{4}{3}B_{k+1}\Big)}{R_k^{n-1}}\Big)$$
$$\leq c_6 \min_{\partial\frac{4}{3}B_3} v + c_5 \sum_{k=2}^{\infty} R_k\, g^{-1}\Big(\frac{\mu_v\Big(\frac{4}{3}B_{k+1}\Big)}{R_k^{n-1}}\Big), \tag{A.0.20}$$

since v is $\mathcal{A}$-harmonic in $3/2B_3 \setminus \overline{B}_3$. Recall that $v \leq u$ in B_{2R}. From this, combining the weak Harnack inequality (A.0.6) with Minimum Principle (Lemma W (**a**)), we obtain

$$\min_{\partial\frac{4}{3}B_3} v \leq \inf_{\partial\frac{4}{3}B_3} u \leq \Big(\fint_{\frac{4}{3}B_3} u^{s_0}\,\mathrm{d}x\Big)^{\frac{1}{s_0}}$$
$$\leq c_7\Big(\fint_{B_{2R}} u^{s_0}\,\mathrm{d}x\Big)^{\frac{1}{s_0}} \leq c_8 \inf_{B_R} u,$$

where $c_7 = c_7(n) > 0$ and $c_8 = c_8(n, p, q, \alpha, \beta) > 0$. Using this in (A.0.20), there exists $c_9 = c_9(n, p, q, \alpha, \beta) > 0$ such that

$$\overline{\lim_{k\to\infty}} \max_{\partial\frac{4}{3}B_{k+2}} v \leq c_9\Big(\inf_{B_R} u + \sum_{k=2}^{\infty} R_k\, g^{-1}\Big(\frac{\mu_v\Big(\frac{4}{3}B_{k+1}\Big)}{R_k^{n-1}}\Big)\Big). \tag{A.0.21}$$

On the other hand, by definition of Poisson modification, $u = v$ in $B_k \setminus (3/2\overline{B}_{k+1}) \subset B_{2R} \setminus \Omega$ for all $k \geq 2$. Due to continuity of u and v, we have

$$u(x_0) = \lim_{k\to\infty} \min_{B_k \setminus \frac{3}{2}\overline{B}_{k+1}} u = \lim_{k\to\infty} \min_{B_k \setminus \frac{3}{2}\overline{B}_{k+1}} v \leq \overline{\lim_{k\to\infty}} \max_{\partial \frac{4}{3} B_{k+2}} v.$$

A combination of (A.0.21) with the previous inequality, yields

$$u(x_0) \leq c_9 \Big(\inf_{B_R} u + \sum_{k=2}^{\infty} R_k\, g^{-1}\Big(\frac{\mu_v\big(\frac{4}{3}B_{k+1}\big)}{R_k^{n-1}} \Big) \Big).$$

According to (A.0.12) and reminding of $R_k = 2^{1-k}R$ for $k \geq 0$, we estimate the preceding series as follows:

$$\begin{aligned}
\sum_{k=2}^{\infty} R_k\, g^{-1}\Big(\frac{\mu_v\big(\frac{4}{3}B_{k+1}\big)}{R_k^{n-1}} \Big) &= \sum_{k=2}^{\infty} R_k\, g^{-1}\Big(\frac{\mu\big(\frac{4}{3}B_{k+1}\big)}{R_k^{n-1}} \Big) \\
&\leq \sum_{k=2}^{\infty} R_k\, g^{-1}\Big(\frac{\mu\big(\frac{4}{3}B_{k+1}\big)}{R_k^{n-1}} \Big) \\
&= \sum_{k=2}^{\infty} (R_{k-1} - R_k)\, g^{-1}\Big(\frac{\mu\big(\frac{2}{3}B_{k}\big)}{R_k^{n-1}} \Big) \\
&\leq c_{10} \sum_{k=2}^{\infty} (R_{k-1} - R_k)\, g^{-1}\Big(\frac{\mu(B_k)}{R_{k-1}^{n-1}} \Big) \\
&= c_{10} \sum_{k=2}^{\infty} \int_{R_k}^{R_{k-1}} g^{-1}\Big(\frac{\mu(B(x_0,R_k))}{R_{k-1}^{n-1}} \Big)\, \mathrm{d}x \\
&\leq c_{10} \sum_{k=2}^{\infty} \int_{R_k}^{R_{k-1}} g^{-1}\Big(\frac{\mu(B(x_0,s))}{s^{n-1}} \Big)\, \mathrm{d}x \leq c_{10}\, \mathbf{W}_G^R \mu(x_0),
\end{aligned}$$

where $c_{10} = c_{10}(p, q, \alpha, \beta) > 0$. This completes the proof of the upper bound in (2.3.18) by taking $C_2 = c_9 \max\{1,\, c_{10}\}$. $\square$

We next prove the lower bound. Observe that here we do not need to use the Poisson modification of u.

Proof of the lower bound in Theorem 2.3.19. For $k = 0, 1, 2, \ldots$, let $\eta_k \in C_c^\infty(B_k)$ satisfying $0 \leq \eta_k \leq 1$, $\operatorname{supp} \eta_k \subset 5/4B_{k+1}$ and $\eta_k = 1$ in B_{k+1}. We set $\mu_k = \eta_k \mu$. Notice that $\mu_k \in \Big(W_0^{1,G}(B_k)\Big)^*$ and $\mu_k(B_{k+1}) = \mu(B_{k+1})$. Using Theorem Q, there exists $u_k \in W_0^{1,G}(B_k)$ satisfying

$$-\mathrm{div}\Big(\mathcal{A}(x, \nabla u_k)\Big) = \mu_k \quad \text{in} \quad B_k. \tag{A.0.22}$$

On account of the $\operatorname{supp} \eta_k \subset 5/4B_{k+1}$, one has

$$u_k \quad \text{is } \mathcal{A}\text{-harmonic in} \quad B_k \setminus \frac{5}{4}\overline{B}_{k+1},$$

and $u_k = 0$ continuously on ∂B_k. Since $(-u + \min_{\partial B_k} u)_+ = 0$ on ∂B_k, it follows that $(u_k - u + \min_{\partial B_k} u)_+ = 0$ on ∂B_k, whence $\operatorname{supp}(u_k - u + \min_{\partial B_k} u)_+ \subset B_k$ and, by Lemma Z,

$$\Big(u_k - u + \min_{\partial B_k} u\Big)_+ \in W_0^{1,G}(B_k).$$

A subtraction of equations of u and u_k, (A.0.1) and (A.0.22), respectively, with the preceding test function gives

$$\begin{aligned} 0 &\leq \int_{B_k} \Big(u_k - u + \min_{\partial B_k} u\Big)_+ \,\mathrm{d}\mu - \int_{B_k} \Big(u_k - u + \min_{\partial B_k} u\Big)_+ \,\mathrm{d}\mu_k \\ &= \int_{B_k \cap \{u - \min_{\partial B_k} u \leq u_k\}} \Big(\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_k)\Big) \cdot \nabla(u_k - u)\,\mathrm{d}x \leq 0, \end{aligned}$$

where in the last inequality was used the monotonicity of $\mathcal{A}$ (4.4.6). From this, $\nabla(u_k - u + \min_{\partial B_k} u)_+ = 0$ in B_k, and consequently

$$u_k \leq u - \min_{\partial B_k} u \quad \text{in} \quad B_k. \tag{A.0.23}$$

Let $\varphi \in C_c^\infty(B_k)$ be such that

$$\begin{cases} 0 \leq \varphi \leq 1 & \text{in} \quad B_k, \\ \varphi = 1 \quad \text{in} \quad \dfrac{2}{3}B_k, & |\nabla\varphi| \leq \dfrac{c}{R_k}. \end{cases} \tag{A.0.24}$$

Observe that $\operatorname{supp}\nabla\varphi \subset B_k \backslash (2/3\overline{B}_k) \subset B_k \backslash (5/4\overline{B}_{k+1})$. Hence u_k is $\mathcal{A}$-harmonic in $\operatorname{supp}\nabla\varphi$. By Maximum and Minimum Principles (Lemma W), we obtain respectively

$$u_k(x) = \min\Big\{u_k(x), \max_{\partial\frac{2}{3}B_k} u_k\Big\} \quad \forall x \in \operatorname{supp}\nabla\varphi, \tag{A.0.25}$$

$$\min_{\partial\frac{2}{3}B_k} u_k \leq \min\Big\{u_k(x), \max_{\partial\frac{2}{3}B_k} u_k\Big\} \quad \text{in} \quad \forall x \in \frac{2}{3}B_k. \tag{A.0.26}$$

We will consider two cases as in the proof of the upper bound. First assume $\min_{\partial B_{k+1}} u_k > 0$. By the weak Harnack inequality (A.0.6), this positivity implies

$$\min_{\partial\frac{2}{3}B_k} u_k \geq \frac{1}{c_1}\bigg(\fint_{\frac{2}{3}B_k} u_k^{s_0}\,\mathrm{d}x\bigg)^{\frac{1}{s_0}} \geq \Big(\frac{4}{3}\Big)^n \frac{1}{c_1}\bigg(\fint_{B_{k+1}} u_k^{s_0}\,\mathrm{d}x\bigg)^{\frac{1}{s_0}} \geq \frac{1}{c_2}\min_{\partial B_{k+1}} u_k > 0,$$

where $c_1 > 0$ and $c_2 > 0$ are constants depending only on n, p, q, α, β. Using the previous inequality and taking into account φ given in (A.0.24), we compute by (A.0.26)

$$\begin{aligned} \Big(\min_{\partial\frac{2}{3}B_k} u_k\Big)\mu(B_{k+1}) &= \int_{B_{k+1}} \min_{\partial\frac{2}{3}B_k} u_k\,\mathrm{d}\mu = \int_{B_{k+1}} \min_{\partial\frac{2}{3}B_k} u_k\,\mathrm{d}\mu_k \\ &= \int_{B_{k+1}} \min_{\partial\frac{2}{3}B_k} u_k\,\varphi^q\,\mathrm{d}\mu_k \leq \int_{\frac{2}{3}B_k} \min_{\partial\frac{2}{3}B_k} u_k\,\varphi^q\,\mathrm{d}\mu_k \\ &\leq \int_{\frac{2}{3}B_k} \min\Big\{u_k, \max_{\partial\frac{2}{3}B_k} u_k\Big\}\varphi^q\,\mathrm{d}\mu_k \leq \int_{B_k} \min\Big\{u_k, \max_{\partial\frac{2}{3}B_k} u_k\Big\}\varphi^q\,\mathrm{d}\mu_k. \end{aligned} \tag{A.0.27}$$

From Lemma Z, $\phi := \min\Big\{u_k,\, \max_{\partial(2/3B_k)} u_k\Big\}\varphi^q \in W_0^{1,G}(B_k)$, whence testing ϕ in (A.0.22) and using (A.0.27), it follows

$$\begin{aligned}\Big(\min_{\partial\frac{2}{3}B_k} u_k\Big)\mu(B_{k+1}) &\le \int_{B_k} \mathcal{A}(x,\nabla u_k)\nabla\cdot\Big(\min\Big\{u_k,\,\max_{\partial\frac{2}{3}B_k} u_k\Big\}\varphi^q\Big)\mathrm{d}x\\ &= \int_{B_k} \mathcal{A}(x,\nabla u_k)\cdot\Big(\varphi^q\,\nabla\min\Big\{u_k,\,\max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big)\mathrm{d}x\\ &\quad + q\int_{B_k} \mathcal{A}(x,\nabla u_k)\cdot\Big(\min\Big\{u_k,\,\max_{\partial\frac{2}{3}B_k} u_k\Big\}\varphi^{q-1}\,\nabla\varphi\Big)\mathrm{d}x\\ &=: I_1 + I_2 \end{aligned} \tag{A.0.28}$$

Since $\nabla\min\Big\{u_k,\,\max_{\partial\frac{2}{3}B_k} u_k\Big\} = 0$ in $B_k \cap \{u_k > \max_{\partial(2/3)B_k} u_k\}$, combining Cauchy-Schawrz inequality with (2.3.3) and (4.4.6), we estimate I_1 as follows

$$\begin{aligned} I_1 &= \int_{B_k\cap\{u_k\le\max_{\partial\frac{2}{3}B_k} u_k\}} \mathcal{A}(x,\nabla u_k)\cdot\Big(\varphi^q\,\nabla\min\Big\{u_k,\,\max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big)\mathrm{d}x\\ &= \int_{B_k\cap\{u_k\le\max_{\partial\frac{2}{3}B_k} u_k\}} \mathcal{A}(x,\nabla u_k)\cdot\nabla u_k\,\varphi^q\,\mathrm{d}x\\ &\le q\beta\int_{B_k\cap\{u_k\le\max_{\partial\frac{2}{3}B_k} u_k\}} G(|\nabla u_k|)\,\varphi^q\,\mathrm{d}x\\ &\le q\beta\int_{B_k\cap\{u_k\le\max_{\partial\frac{2}{3}B_k} u_k\}} G\Big(\Big|\nabla\min\Big\{u_k,\,\max_{\partial\frac{2}{3}B_k} u_k\Big|\Big)\,\varphi^q\,\mathrm{d}x\\ &\le q\beta\int_{B_k} G\Big(\Big|\nabla\min\Big\{u_k,\,\max_{\partial\frac{2}{3}B_k} u_k\Big|\Big)\,\varphi^q\,\mathrm{d}x. \end{aligned} \tag{A.0.29}$$

Being u_k an $\mathcal{A}$-supersolution in B_k, $\min\Big\{u_k, \max_{\partial(2/3B_k)} u_k\Big\}$ is also an $\mathcal{A}$-supersolution in B_k, thence

$$w_k := \max_{\partial\frac{2}{3}B_k} u_k - \min\Big\{u_k, \max_{\partial\frac{2}{3}B_k} u_k\Big\} \quad \text{is an } \mathcal{A}\text{-subsolution in } B_k,$$

and it is nonnegative by definition. Applying Caccioppoli estimate (Lemma V) to w_k in B_k and taking into account (A.0.24),

$$\begin{aligned}\int_{B_k} G\Big(\Big|\nabla\min\Big\{u_k,\,\max_{\partial\frac{2}{3}B_k} u_k\Big|\Big)\,\varphi^q\,\mathrm{d}x &\le \int_{B_k} G(|\nabla w_k|)\,\varphi^q\,\mathrm{d}x\\ &\le c_3\int_{B_k} G(w_k\,|\nabla\varphi|)\,\mathrm{d}x\\ &\le c_3\int_{B_k\cap\,\mathrm{supp}\nabla\varphi} G\Big(\frac{c\,w_k}{R_k}\Big)\,\mathrm{d}x\\ &\le c_4\int_{B_k\cap\,\mathrm{supp}\nabla\varphi} G\Big(\frac{\max_{\partial\frac{2}{3}B_k} u_k}{R_k}\Big)\,\mathrm{d}x\\ &\le c_4\int_{B_k\setminus(5/4\overline{B}_{k+1})} G\Big(\frac{\max_{\partial\frac{2}{3}B_k} u_k}{R_k}\Big)\,\mathrm{d}x, \end{aligned}$$

where $c_3,\ c_4$ depending only n and p,q,α,β. Recall that u_k is $\mathcal{A}$-harmonic in $B_k\setminus(5/4\overline{B}_{k+1})$. By (A.0.9),

$$\max_{\partial\frac{2}{3}B_k} u_k \le c_5 \min_{\partial\frac{2}{3}B_k} u_k.$$

Combining this with (A.0.29), we obtain

$$\begin{aligned} I_1 &\le c_6 \int_{B_k\setminus(5/4\overline{B}_{k+1})} G\Big(\frac{c_5 \min_{\partial\frac{2}{3}B_k} u_k}{R_k}\Big)\,\mathrm{d}x \\ &\le c_6 \int_{B_k} G\Big(\frac{c_5 \min_{\partial\frac{2}{3}B_k} u_k}{R_k}\Big)\,\mathrm{d}x \le c_7\, R_k^n\, G\Big(\frac{\min_{\partial\frac{2}{3}B_k} u_k}{R_k}\Big). \end{aligned} \tag{A.0.30}$$

Next, using the Cauchy-Schwarz inequality, (4.4.6) and (A.0.25), one has

$$\begin{aligned} I_2 &\le c_8 \int_{B_k\cap\operatorname{supp}\nabla\varphi} g(|\nabla u_k|)\Big|\min\Big\{u_k,\ \max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big|\varphi^{q-1}\,|\nabla\varphi|\,\mathrm{d}x \\ &= c_8 \int_{\operatorname{supp}\nabla\varphi} g\Big(\Big|\nabla\min\Big\{u_k,\ \max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big|\Big)\Big|\min\Big\{u_k,\ \max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big|\varphi^{q-1}\,|\nabla\varphi|\,\mathrm{d}x \\ &\le c_8 \int_{\operatorname{supp}\nabla\varphi} g\Big(\Big|\nabla\min\Big\{u_k,\ \max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big|\Big)\varphi^{q-1}\ \max_{\partial\frac{2}{3}B_k} u_k\,|\nabla\varphi|\,\mathrm{d}x \\ &\le c_8 \int_{\operatorname{supp}\nabla\varphi} g\Big(\Big|\nabla\Big(\max_{\partial\frac{2}{3}B_k} u_k - \min\Big\{u_k,\ \max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big)\Big|\Big)\varphi^{q-1}\ \max_{\partial\frac{2}{3}B_k} u_k\,|\nabla\varphi|\,\mathrm{d}x. \end{aligned}$$

Since $0\le\varphi\le1$, $G^*(\varphi^{q-1}t)\le c_8\,\varphi^q G^*(t)$ for all $t\ge0$ by (2.3.7). From this, with the aid of Caccioppoli estimate (Lemma V), a combination of Young's inequality (2.3.1) and (2.3.9) gives

$$\begin{aligned} I_2 &\le c_8 \int_{\operatorname{supp}\nabla\varphi} G\Big(\max_{\partial\frac{2}{3}B_k} u_k\,|\nabla\varphi|\Big)\,\mathrm{d}x \\ &\quad + c_8 \int_{\operatorname{supp}\nabla\varphi} G^*\Big(g\Big(\Big|\nabla\Big(\max_{\partial\frac{2}{3}B_k} u_k - \min\Big\{u_k,\ \max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big)\Big|\Big)\varphi^{q-1}\Big)\,\mathrm{d}x \\ &\le c_8 \int_{\operatorname{supp}\nabla\varphi} G\Big(\max_{\partial\frac{2}{3}B_k} u_k\,|\nabla\varphi|\Big)\,\mathrm{d}x \\ &\quad + c_{10} \int_{\operatorname{supp}\nabla\varphi} G\Big(\Big|\nabla\Big(\max_{\partial\frac{2}{3}B_k} u_k - \min\Big\{u_k,\ \max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big)\Big|\Big)\varphi^{q}\,\mathrm{d}x \\ &\le c_8 \int_{\operatorname{supp}\nabla\varphi} G\Big(\max_{\partial\frac{2}{3}B_k} u_k\,|\nabla\varphi|\Big)\,\mathrm{d}x \\ &\qquad + c_{11} \int_{\operatorname{supp}\nabla\varphi} G\Big(\Big(\max_{\partial\frac{2}{3}B_k} u_k - \min\Big\{u_k,\ \max_{\partial\frac{2}{3}B_k} u_k\Big\}\Big)\,|\nabla\varphi|\Big)\,\mathrm{d}x \\ &\le c_{12} \int_{\operatorname{supp}\nabla\varphi} G\Big(\max_{\partial\frac{2}{3}B_k} u_k\,|\nabla\varphi|\Big)\,\mathrm{d}x \le c_{13}\, R_k^n\, G\Big(\frac{\min_{\partial\frac{2}{3}B_k} u_k}{R_k}\Big), \end{aligned} \tag{A.0.31}$$

where the last inequality follows by the same method as in (A.0.30). Here $c_i>0$, $i=5,\dots,13$

are constants depending only on n and p, q, α, β. Applying (A.0.30) and (A.0.31) in (A.0.28),

$$\begin{aligned}\frac{\mu(B_{k+1})}{R_k^{n-1}} &\leq c_{14}\frac{R_k}{\min_{\partial\frac{2}{3}B_k} u_k} G\Big(\frac{\min_{\partial\frac{2}{3}B_k} u_k}{R_k}\Big)\\ &\leq c_{15}\, g\Big(\frac{\min_{\partial\frac{2}{3}B_k} u_k}{R_k}\Big).\end{aligned}$$

Accordingly, for all $k \geq 0$ with $\min_{\partial B_{k+1}} u_k > 0$, it holds

$$R_k g^{-1}\Big(\frac{\mu(B_{k+1})}{R_k^{n-1}}\Big) \leq c_{16} \min_{\partial\frac{2}{3}B_k} u_k,$$

where $c_i > 0$, $i = 14, 15, 16$, depend only on n and p, q, α, β. Since u is $\mathcal{A}$-superharmonic in B_k, we have

$$\min_{\partial\frac{2}{3}B_k} u = \min_{\frac{2}{3}B_k} u \leq \min_{B_{k+1}} u \leq \min_{\partial B_{k+1}} u.$$

Using this in (A.0.23), we deduce that

$$\min_{\partial\frac{2}{3}B_k} u_k \leq \min_{\partial B_{k+1}} u - \min_{\partial B_k} u.$$

Hence

$$R_k g^{-1}\Big(\frac{\mu(B_{k+1})}{R_k^{n-1}}\Big) \leq c_{16}\Big(\min_{\partial B_{k+1}} u - \min_{\partial B_k} u\Big). \tag{A.0.32}$$

Next, assume that $\min_{\partial B_{k_0}} u = 0$ for some $k_0 \geq 0$. The weak Harnack inequality (A.0.6) implies that $u_{k_0} = 0$ in B_{k_0}. From this, we infer that u is $\mathcal{A}$-harmonic in B_{k_0+1} since $\mu(B_{k_0+1}) = \mu_{k_0}(B_{k_0+1}) = 0$, whence

$$\mu(B_j) = 0 \quad \forall j \geq k_0 + 1. \tag{A.0.33}$$

By Minimum Principle (Lemma W **(a)**), $\min_{\partial B_{k_0+1}} u = \min_{B_{k_0+1}} u \leq u(x_0)$.

Thus, by summing up all cases, we concluded from (A.0.32) and (A.0.33) that

$$\begin{aligned}u(x_0) \geq u(x_0) - \min_{\partial B_0} u &\geq \lim_{k\to\infty}\Big(\min_{\partial B_{k+1}} u - \min_{\partial B_0} u\Big)\\ &= \sum_{k=0}^{\infty}\Big(\min_{\partial B_{k+1}} u - \min_{\partial B_k} u\Big) \geq \frac{1}{c_{16}}\sum_{k=0}^{\infty} R_k g^{-1}\Big(\frac{\mu(B_{k+1})}{R_k^{n-1}}\Big)\end{aligned}$$

Reminding of $R_k = 2^{1-k}R$ for $k \geq 0$, by (2.3.6), we deduce that

$$\begin{aligned}u(x_0) &\geq \frac{1}{c_{16}}\sum_{j=1}^{\infty} R_{j-1} g^{-1}\Big(\frac{\mu(B_j)}{R_{j-1}^{n-1}}\Big)\\ &= \frac{4}{c_{16}}\sum_{j=1}^{\infty} R_{j+1} g^{-1}\Big(\frac{\mu(B_j)}{4^{n-1}R_{j+1}^{n-1}}\Big) \geq c_{17}\sum_{j=1}^{\infty} R_{j+1} g^{-1}\Big(\frac{\mu(B_j)}{R_{j+1}^{n-1}}\Big)\end{aligned}$$

The estimate

$$\begin{aligned}
\mathbf{W}^R_G\mu(x_0) &= \int_0^R g^{-1}\Big(\frac{\mu(B(x_0,s))}{s^{n-1}}\Big)\,\mathrm{d}s \\
&= \sum_{k=1}^{\infty}\int_{R_{k+1}}^{R_k} g^{-1}\Big(\frac{\mu(B(x_0,s))}{s^{n-1}}\Big)\,\mathrm{d}s \le \sum_{k=1}^{\infty}\int_{R_{k+1}}^{R_k} g^{-1}\Big(\frac{\mu(B(x_0,R_k))}{R_{k+1}^{n-1}}\Big)\,\mathrm{d}s \\
&= \sum_{k=1}^{\infty}(R_k - R_{k+1})g^{-1}\Big(\frac{\mu(B_k)}{R_{k+1}^{n-1}}\Big) = \sum_{k=1}^{\infty} R_{k+1}g^{-1}\Big(\frac{\mu(B_k)}{R_{k+1}^{n-1}}\Big)
\end{aligned}$$

completes the proof of the lower bound in (2.3.18) by taking $C_1 = c_{17}$, which depends only on n and p, q, α, β. □