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DATA ENVELOPMENT ANALYSIS AND GAME THEORY:
literature review and models to support resource sharing in efficiency measurement contexts

Recife
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Thesis presented to the Graduate Program in Management Engineering at the Federal University of Pernambuco, as a partial requirement for obtaining the title of Doctor of Philosophy (Ph.D.) in Management Engineering.

Research area: Operational Research.

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To my mother, Francisca, and my father, Calisto, who inspire me.

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ABSTRACT

Organizations have continually sought alternatives to improve their production systems to achieve competitive advantages and increase market share. Globalized competition and more conscious and demanding customers have demanded better performance. In this context, the measurement of efficiency is presented as an essential assessment for organizations to identify bottlenecks and implement improvements. Among the techniques available in the literature, Data Envelopment Analysis (DEA) is one of the most relevant, having a great diversity of modeling and economic sectors in which evaluations have already been carried out. However, there are still limitations to be mitigated so that investigation can adequately reflect the investigated realities. Thus, this study proposes the integration of DEA with Game Theory to adequately portray the realities of cooperation and non-cooperation existing within organizations. For this, network and dynamic modeling are used to portray aspects related to temporal impacts and the internal structures of companies. These models are combined with cooperative and non-cooperative Game Theory approaches to discuss the allocation of shared resources within these structures and their respective impacts on the efficiencies of processes and organizations. Given the significant number of results generated, a two-dimensional representation of dynamic models is also proposed to help the decision-making process through the representation of these efficiency frontiers. The relevance of the research is related to three main aspects: first, the study carries out an in-depth discussion of the integrated literature on DEA and Game Theory, identifying the pattern of development, main research fronts, and gaps to be discussed in future developments. Second, it also fills a gap in the literature by proposing the introduction of models that adequately reflect the allocation of resource in internal networks. It also provides an alternative to evaluate and improve this usage in cooperative and non-cooperative scenarios. In addition to the two proposed models, the thesis also performs the application of the models proposed for the context of Brazilian public higher education, aiming to demonstrate the applicability of the proposed models in a relevant case for the country's development.

Keywords: data envelopment analysis; game theory; efficiency; shared resources; higher education.

RESUMO

As organizações têm buscado alternativas para realizar aprimoramento em seus sistemas produtivos visando alcançar vantagens competitivas e aumentar o market-share. A concorrência globalizada e clientes mais conscientes e exigentes tem exigido melhor performance. Nesse contexto, a mensuração de eficiência se apresenta como avaliação indispensável para que as organizações possam identificar gargalos e implementar melhorias. Dentre as técnicas disponível na literatura, a Análise Envoltória de Dados (DEA) consiste em uma das mais relevantes, possuindo uma grande diversidade de modelagens e setores econômicos nas quais avaliações já foram realizadas. Contudo, ainda existem limitações a serem mitigadas para que as investigações possam refletir adequadamente as realidades investigadas. Desse modo, este estudo propõe a integração de DEA com Teoria dos Jogos visando retratar adequadamente realidades de cooperação e não-cooperação existentes dentro das organizações. Para isso, são utilizadas modelagens em rede e dinâmicas para retratar aspectos relativos aos impactos temporais e as estruturas internas das empresas. Essas modelagens são combinadas com abordagens cooperativas e não cooperativas de Teoria dos Jogos para discutir alocação de recursos compartilhados dentro dessas estruturas e seus respectivos impactos nas eficiências dos processos e das organizações. Tendo em vista a quantidade significativa de resultados gerados, também é proposta uma representação bi-dimensional de modelos dinâmicos com o intuito de auxiliar o processo decisório por intermédio da representação dessas fronteiras de eficiência. A relevância da pesquisa está inicialmente em preencher uma lacuna da literatura, quanto à introdução da melhoria e aumento na utilização de recursos nas organizações e da proposição de modelagens que reflitam adequadamente as realidades investigadas. Além dos dois modelos propostos, o estudo realiza uma discussão aprofundada da literatura integrada de DEA e Teoria dos Jogos, identificando as principais frentes de pesquisa futura e lacunas a serem discutidas. A tese realiza a aplicação das modelagens propostas para o contexto de ensino superior público brasileiro, visando demostrar a aplicabilidade das modelagens propostas em um caso real e relevante para o desenvolvimento do país.

Palavras-chave: análise envoltória de dados; teoria dos jogos; compartilhamento de recursos; ensino superior.

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LIST OF ACRONYMS

BCC	(Banker, Charnes e Cooper) Classic Model with variable returns to scale
CCR	(Charnes, Cooper e Rhodes) Classic Model with constant returns to scale
COLS	Corrected Ordinary Least Square
CRS	Constant Returns to Scale
DDEA	Dynamic Data Envelopment Analysis
DEA	Data Envelopment Analysis
DMU	Decision Making Units
DNDEA	Dynamic model with network structure
DNSBM	Dynamic SBM with network structure
GT	Game Theory
NDEA	Network Data Envelopment Analysis
SBM	Slacks-Based Measure
SFA	Stochastic Frontier Analysis
VRS	Variable Returns to Scale

CONTENTS

1 INTRODUCTION

1.1 THE DEA-GT CONTEXTUALISATION

Companies are continually seeking alternatives to improve their production systems in order to obtain competitive advantages and increase market share. Demanding customers and a greater number of competitors demand increasingly better performance, both from their products and their processes.

There is a growing interdisciplinary interest in performance in all its manifestations, this interest reflects on the very rapidly development of the efficiency and productivity analysis using frontier estimation methodologies field in the last four decades (DARAIO *et al.*, 2020; LAMPE; HILGERS, 2015).

Management by performance gains relevance to best utilize restricted resources and to sustain competitiveness in the private sector or to increase value for money, making government and policy more result-oriented and these features promoted growth in managerial interest in performance has been mirrored in the development of actual performance management practices and academic devotion (LAMPE; HILGERS, 2015).

In the context of efficiency measurement, parametric and non-parametric approaches have been developed. Among the non-parametric alternatives, Data Envelopment Analysis (DEA) is one of the prominent non-parametric performance measurement approaches for calculating the efficiency scores of a set of homogeneous decision-making units (DMUs) in the presence of multiple inputs and outputs (EMROUZNEJAD; YANG, 2018).

Emrouznejad and Yang (2018) and Camanho et al. (2023), in their literature surveys, confirms the continuous growth of papers published covering a wide range of production activities and this interest has also extended to policy issues of great significance.

The decision-making process has become highly complex, and hybrid models constitute a systematic approach. The literature on Data Envelopment Analysis (DEA) shows the same trend: a broad range of applications has been combined with other techniques such as Game Theory (GT).

Identifying efficient decision-making units (DMUs) is to recognize the best strategy to promote an individual DMU's achievement at the expense of competing members (HAO; WEI; YAN, 2000). In addition to strategy recognition, the assumption that the DMU chooses the best set of weights to maximize its efficiency without any concern regarding the impact on the efficiency of other DMUs may not always hold. Furthermore, it is essential to note that there might be situations in which DMUs can cooperate to obtain better efficiency values or try to minimize the efficiency of others.

These statements are related to the essence of Game Theory, which aims to determine the choice of a decision-maker in which his or her individual choice depends on the choice of others. In addition, the combination of DEA and GT is a more natural source for analyzing competitive situations. Furthermore, it is beneficial to reveal additional structural insights into DEA-efficient production surfaces and provide broader practical information (HAO; WEI; YAN, 2000).

The combination of DEA and GT has been recognized as relevant in the DEA literature in several studies. Banker (1980) proposition is highlighted as one of the primary studies of the global path of DEA development, contributing to the investigation of the internal structures of DMUs. In the context of network DEA models, GT can improve the relationship representation between the internal structures of a DMU (HALKOS; TZEREMES; KOURTZIDIS, 2014; KAO, 2014). Cross-efficiency consists in one other leading research topic and game cross-efficiency-based models have gained relevance (LIU; LU; LU, 2016). GT is also highlighted as one of the leading techniques when using decision theory within the DEA scope (NEPOMUCENO; COSTA; DARAIO, 2020)

Given the proximity of points and the individual capabilities of both techniques to assist in the decision-making process, this study initially aims to observe how this literary branch has developed, what its state of the art is and its future trends. Furthermore, knowing the technical capacity of DEA in measuring relative efficiency and Game Theory to deal with conflicts or to stimulate cooperation through fair allocation of benefits, the study proposes to combine the two techniques to explore the potential benefits of collaboration in cases of sharing resources, as well as measuring whether DMUs are being efficient in these aspects.

1.2 JUSTIFICATION AND RELEVANCE

The central concept of strategic benchmarking is resource management efficiency, which ultimately results in profitability. However, little is known about performance measurement from resource-based perspectives (KWEH et al., 2024). When common resources are pooled and shared, the performance of partners is improved due to the complementarity of resources (AN et al., 2019a).

Sharing of common resources, such as labor, capital, information, demand, and knowledge, among different entities is usual, but resource sharing also widely exists among multiple linked stages of a single network system, such as a bank or a university (AN et al., 2019b).

Camanho et al. (2023) points out that there is a focus of the academic community on research analyzing cost efficiency, while revenue, resource and profit efficiency are relatively understudied. As pointed out by An et al. (2019), many scholars have studied the potential gains derived from resource sharing. However, these approaches do not discuss the impact of resource sharing in a dynamic framework of economic efficiency assessments.

In this context, this study aims to address literature gaps on providing new decision models to support better decision-making processes in the context of resource sharing. Resources are scarce and must be used efficiently and effectively. In the current context of growing concerns about environmental issues and the impacts caused by productive activities on the environment, the efficient use of all resources becomes even more essential.

For this reason, the analysis of efficiency and resource allocation is configured as a powerful resource to be explored by the most diverse productive sectors, by the Academy, by the public administration to measure, communicate and propose improvement goals, thus reducing inefficiencies and misuse of resources.

In this sense, this thesis will address the previously mentioned issues with the aid of DEA and Game Theory. This choice resides on the fact that the combination of the techniques has been applied in relevant management contexts in distinct industries, such as universities (SHI et al., 2020b), energy (ZHU et al., 2022), banks (AN *et al.*, 2021), iron and steel enterprises (WANG et al., 2021). It is also relevant to mention that the thesis aims to promote a literature discussion, the development of new models and their use in the presence of public resources.

Specifically, regarding the scope of this work, some of the models will be applied to Brazilian federal universities, which are responsible for a large part of the research and innovation generated in Brazil (ANDIFES, 2017). In addition to the importance of this field, the report on investments in research and development in the world carried out by the United Nations Educational, Scientific, and Cultural Organization (UNESCO) considering 2014 to 2018, shows that the budget reduction of the Ministry of Science, Technology and Innovations (MCTI) in the same period was around 50% (UNESCO, 2021). Considering 2012 to 2021, the reduction corresponds to 84% (from R$11.5 billion to R$1.8 billion, in inflation-adjusted values).

Despite the reduction, the report indicates the continuous growth of scientific production. As pointed out by the UNESCO report, the increase in publications over the last years indicates that Brazilian research is resilient. However, resilience also has its limits.

In addition to proposing models to assist in decision making, the current study investigates a sector in which the allocation of resources is particularly relevant as these resources are financed with public funds and directly impact the country's scientific production. Through this application, the relevance and applicability of the study becomes even clearer, also contributing to the literature since, as verified by Camanho et al. (2023), studies aimed at resource efficiency are not yet widely discussed.

To this end, the impacts of the study can be divided into three pillars: social, economic-financial and environmental. The main social contribution of developing efficiency analyzes for production systems consists of identifying the current situation and providing quantifiable goals and reference units so that they can become efficient. This context becomes even more relevant for cases where the units investigated are managed and/or financed with public resources. For these situations, transparency in the use of resources and the efficiency of public spending are aspects that are increasingly being demanded from governments. Thus, the thesis illustrates in chapter four, an applied case of Brazilian federal universities. These institutions are financed with public resources and are of great relevance to society and innovation in the country.

Still in the social sphere, public policies based on the results of the efficiency analysis will make it possible to provide better services to the population and the efficient use of resources will probably increase the scope of the services provided. It is also worth highlighting that although the analyzes developed are focused on the educational segment, the modeling can be applied to other areas of activity.

From an economic-financial point of view, all models were developed to maximize the efficiency of the productive systems investigated. Regarding this aim, the objective is to maximize the output produced or reduce the resources used by the production system. Regardless of which approach is implemented, resources are used more efficiently. Furthermore, through goals, the models provide quantifiable results of how much these variables must be reduced or increased for organizations to become efficient. Furthermore, by investigating cases in which the evaluated units are financed with public resources, the study contributes to society as it provides guidance for the best use of public funds.

In economic-financial terms, one of the models developed directly addresses increases in profits, which come from internal cooperation for better use of resources. This investigation is also capable of providing positive impacts in the economic-financial field.

In the environmental sphere, there are some contributions. As previously mentioned, the approaches developed aim to measure efficiency. By providing quantifiable goals for organizations to use to become efficient, their resources will be used efficiently, eliminating waste. From this perspective, as the proposed methodology can be applied in several different areas, the study helps to reduce waste, ensuring that only necessary quantities are actually used.

Specifically using the model presented in Chapter 3 of the thesis, the study is able to prove that equal resources used by different internal processes must be grouped to reduce the quantities needed by organizations, thus improving their profits. With this result, it is possible to prove to organizations that concern with environmental issues, that is, the efficient use of resources, is also fundamental to improving financial results.

Finally, it is important to mention that although none of the models have been directly implemented for investigations related to environmental aspects, all modeling is flexible to encompass this type of assessment and the results verified in chapter two attest to the applicability and relevance of models such as those developed in the current thesis for decision-making in environmental and sustainability contexts.

1.3 RESEARCH OBJECTIVES

This subsection aims to list the main and secondary objectives of the thesis.

1.3.1 Main objective

The central objective of this study is to explore the state of the art of efficiency measurements using Data Envelopment Analysis and Game Theory and to formulate decision models using such techniques to evaluate efficiency, to promote better resource usage and to assist strategic decision making.

1.3.2 Secondary objectives

However, in order to achieve the general objective, some specific objectives are proposed.

a) Develop a literature review of hybrid DEA and GT approaches to understand the state-of-the-art on the field;
b) Identify gaps in this branch of literature and propose alternatives to address them;
c) Propose the incorporation of temporal dynamics in network approaches to investigate resource sharing;

d) Combine dynamic models with network structure with game theory approaches to quantify and fairly allocate the benefits arising from cooperation;
e) Use leader-follower game assumptions to investigate efficiency decomposition and aid in a strategical decision-making process;
f) Propose and apply a bi-dimensional representation of the dynamic efficiency frontier in order to provide a straightforward frontier and a measurable distance of non-efficient units to the frontier.

1.4 THE THESIS STRUCTURE

The thesis is structured in 6 Chapters, as displayed in Figure 1, and described as follows. The first chapter, Introduction, presents the motivation and relevance for the development of the thesis. Main objectives and secondary goals are also presented in this Section.

Chapter 2, Bibliometric Review, covers the conceptual foundation of Data Envelopment Analysis and Game Theory. In addition to the concepts, the chapter presents a classification based on nine categories to systematize the 119 papers considered in the sample used to develop the literature review of hybrid DEA and GT approaches. Citation networks and category analysis made it possible to map the state of the art. The analysis will provide valuable insights for academics and practitioners regarding research trends, core publications, as well as gaps to be explored in this area of knowledge.

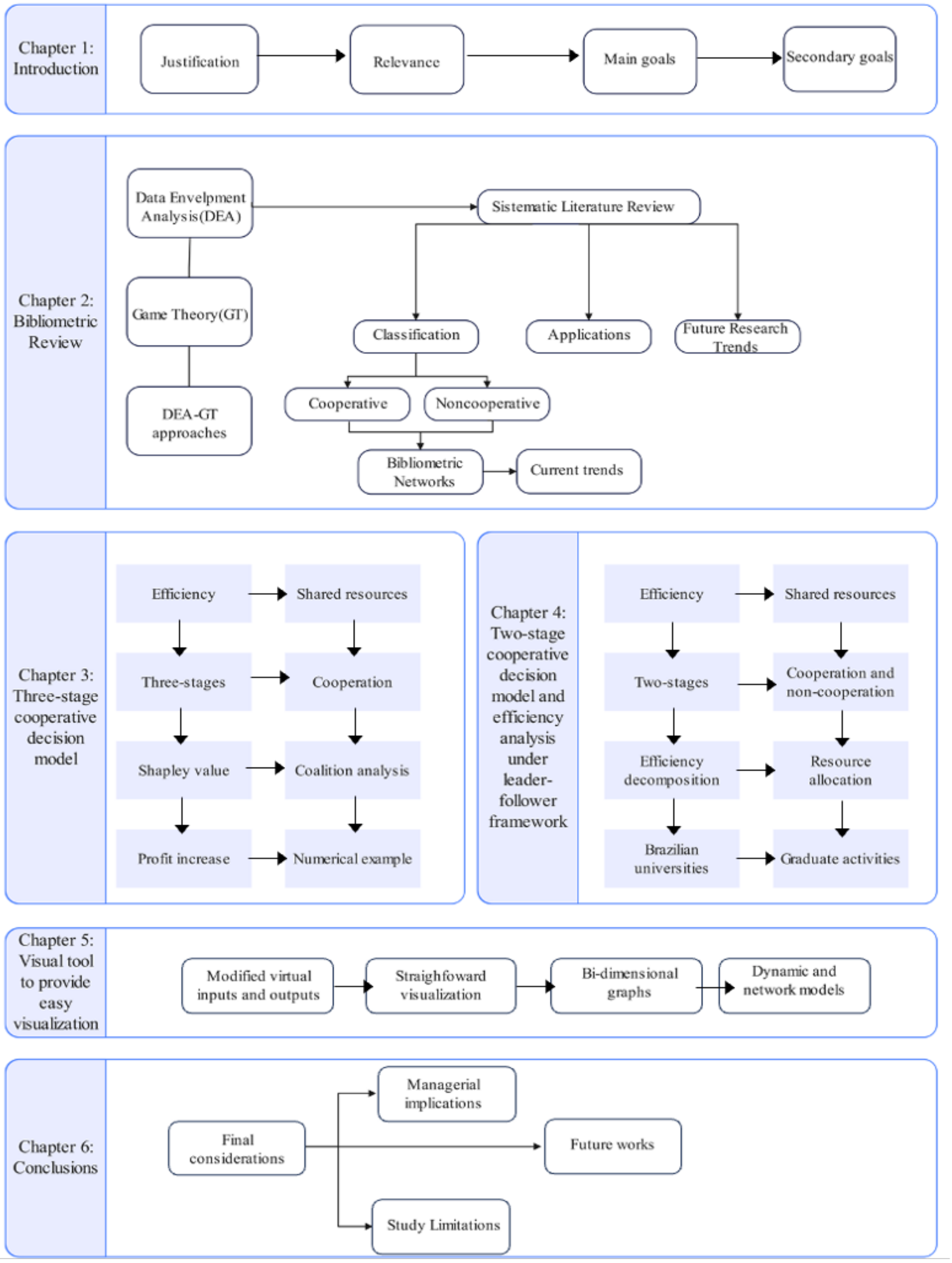
Chapter 3 details a new proposed model using a dynamic model with a DEA network structure and the cooperative game theory approach, Shapley value, to verify how sharing resources within an internal network can benefit from better use of resources. resources. Three different linear programming models are proposed to calculate the profits before and after the collaboration and the Shapley value is then applied to distribute the benefits arising from the collaboration. Through a numerical application, the super additivity of the model is confirmed, as well as the potential benefits of pooling resources within an organization. The work detailed in this chapter was recently published in the form of a research paper (TORRES, L.; RAMOS, 2024a).

Chapter 4, two-stage cooperative decision model and efficiency analysis under leader-follower framework, substantiates the assumptions of a new dynamic cooperative model with two stages which share resources. In addition to considering internal structure and temporal aspects, the study proposes an efficiency decomposition based on the leader-follower game to verify efficiency variations for cases where one of the stages needs to be prioritized. This modeling provides decision makers with a set of relevant information for better decision-making regarding the allocation of network resources, as well as the impacts on efficiency in cases of prioritization. Through an application with a set of data from Brazilian federal universities, it was possible to demonstrate the relevance and applicability of the method. The work detailed in this chapter was recently published in the form of a research paper (TORRES; RAMOS, 2024b)

In the fifth chapter, the author addresses the design and architecture of a simple visualization tool for dynamic Data Envelopment Analysis models results. The representation of efficiency frontiers has been the subject of discussion in the literature since the technique was proposed. However, for cases of multiple inputs and outputs, current visualization approaches become difficult to understand. Knowing that discussions of dynamic frontiers are scarce in the literature, the chapter presents a simple and direct two-dimensional representation for visualizing the different levels of efficiency provided by these models. In order to prove the ease of visualization of the technique, the data and efficiency results from the previous chapter are used.

Chapter 6, Conclusions, the final considerations are elucidated and the managerial and literature contributions are listed. Finally, discussions regarding future propositions to extend the discussions presented in this work are presented.

Figure 1 - Thesis structure



Source: The Author (2024).

2 DATA ENVELOPMENT ANALYSIS AND GAME THEORY: WHERE ARE WE? WHERE ARE WE GOING?

This chapter introduces the main topics related to Data Envelopment Analysis and Game Theory. Also, it demonstrates in detail the trajectory of the technique's combinations, identify seminal articles, provide literature gaps, main areas of application, and detect future trends to discuss the current state-of-the-art. This discussion aims to contextualize the theoretical background that will be used in the next chapters.

2.1 AN INTRODUCTION TO THE CONTEXT OF DATA ENVELOPMENT ANALYSIS AND GAME THEORY

The decision-making process has become highly complex, and hybrid models constitute a systematic approach. The literature on Data Envelopment Analysis (DEA) shows the same trend: a broad range of applications has been combined with other techniques such as Game Theory (GT).

Banker (1980) was the first to examine this relationship. Identifying efficient decision-making units (DMUs) is to recognize the best strategy to promote an individual DMU's achievement at the expense of competing members (HAO; WEI; YAN, 2000). In addition to strategy recognition, the assumption that the DMU chooses the best set of weights to maximize its efficiency without any concern regarding the impact on the efficiency of other DMUs may not always hold. Furthermore, it is essential to note that there might be situations in which DMUs can cooperate to obtain better efficiency values or try to minimize the efficiency of others.

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The combination of DEA and GT has been recognized as relevant in the DEA literature in several studies. Banker (1980) proposition is highlighted as one of the primary studies of the global path of DEA development, contributing to the investigation of the internal structures of DMUs. In the context of network DEA models, GT can improve the relationship representation between the internal structures of a DMU (HALKOS; TZEREMES; KOURTZIDIS, 2014; KAO, 2014). Cross-efficiency consists in one other leading research topic and game cross-efficiency-based models have gained relevance (LIU; LU; LU, 2016). GT is also highlighted as one of the leading techniques when using decision theory within the DEA scope (NEPOMUCENO; COSTA; DARAIO, 2020)

Generally, the DEA field has been developing at a fast rate over the last four decades, and searches in relevant databases have returned over 20,000 published papers. Thus, it is difficult to comprehend the development of the field without guidance from survey-type studies (LIU; LU; LU, 2016). Several authors have conducted surveys to map and organize knowledge in the DEA field (DARAIO et al., 2019; EMROUZNEJAD; YANG, 2018; LAMPE; HILGERS, 2015; LIU; LU; LU, 2016; LIU, et al., 2013a, b; NEPOMUCENO; COSTA; DARAIO, 2020).

These studies highlight the diversity of applications, growth of practical works, combination of DEA with other techniques, and advancement of DEA models to portray DMUs more accurately. These findings illustrate the ability of the method to solve real problems and its flexibility to be used in several segments.

Considering these results and the importance of combined approaches to improving DEA performance, this study reviews DEA and GT combinations in depth. We highlight the technical benefits of this hybrid approach, and we discuss the main applications in economic areas, and it is possible to understand how the results can be improved. This study fills a gap in the literature because, to the author's knowledge, a joint analysis of DEA and GT has not been conducted. The most relevant contributions of this study are summarized as follows.

- A systematic review was developed considering works from 1980 to 2020;
- Bibliometric information was systematized through the investigation of 119 articles;
- We developed citation networks to understand the field's patterns of development;
- The most used DEA and GT approaches are highlighted, as well as the main objectives for combining the two techniques;
- The main areas of application and other techniques used along with DEA and GT are discussed to facilitate the perception of trends and prevailing approaches;
- We identified eleven gaps in the literature, which can represent future research directions.

This chapter is organized into six subsections. The second and third subsection presents a brief contextualization of DEA and GT. Next, we discuss some methodological aspects of the adopted research procedures. The following subsection details the processes used in this study, and the sixth subsection presents an in-depth analysis of the results. Finally, the conclusions are presented.

2.1.1 Data Envelopment Analysis

Data Envelopment Analysis (DEA) is a non-parametric technique used to determine the efficiency of decision-making units (DMUs). Cook and Seiford (2009) point out that DMUs located at the efficiency frontier have the best practices. Such a configuration makes benchmarking possible.

Introduced by Charnes, Cooper and Rhodes (1978), DEA admits the use of multiple inputs and outputs. Models can be oriented both to outputs (the purpose is to increase the outputs when there are no changes in inputs) and to inputs (that aims to reduce inputs, keeping the outputs unchanged).

The seminal work of Charnes, Cooper and Rhodes (1978) advocates a constant return on the scale model. A constant return of scale (CRS) occurs when any change in the input causes a proportional variation in the products. This model is called CCR (Charnes, Cooper and Rhodes) in honor to its authors. On the other hand, the configuration of variable returns of scale (VRS) proposed by (BANKER; CHARNES; COOPER, 1984) emerged as a progress of the first model, in which it is established that the variation in the outputs is not necessarily proportional to the variation in the inputs. This is the BCC (Banker, Charnes and Cooper) model.

These DEA approaches are considered "black boxes". A system glimpsed without many details, encompassing only their respective inputs and products. To reflect the greater discrimination of the analyzed situations, new models have emerged to meet such needs. These models include network models (KAO, 2009; TONE; TSUTSUI, 2009a), dynamic model (FÄRE; GROSSKOPF, 1997; NEMOTO; GOTO, 1999; TONE; TSUTSUI, 2010) and dynamic models with network structures (TONE; TSUTSUI, 2014a).

2.1.2 Game Theory

Game theory is a field of applied mathematics that studies the strategic behavior of rational agents. In other words, game theory is a collection of analytical tools that can be used to make optimal choices in interactional and decision-making problems (SOHRABI; AZGOMI, 2020).

The first propositions related to game theory date back to the 18th century, but more robust mathematical formulations were initially proposed by Zermelo (1913). The development of game theory has made significant contributions to the works of Nash (1950, 1951,1953) and Von Neumann and Morgenstern (1944). These address the relationship between game theory and economics, non-cooperative games, and bargaining theory.

Players, strategies, and payoffs are among the main concepts of Game Theory. All of these were defined for all the games. The players consist of the parties involved in the game, that is, the decision makers. Each player has a set of options, called strategies. There is also a payoff related to each player's pure strategy profile. There is a point, called the breakeven point, that none of the players want to leave, as any change will not increase the payoffs and may even decrease payoffs.

Games can be classified as static and dynamic, cooperative and non-cooperative, and zero-sum and non-zero sum. When there is simultaneity in players' decisions, the games are called static; when not, they are said to be dynamic. Games are classified as non-cooperative in cases where the player seeks better results in isolation, focusing on individual strategies. However, if different groups of players come together in coalitions to benefit from this union, the games are said to be cooperative.

In addition to these denominations, we can also highlight the cases in which the total value of the game remains constant; that is, the loss of one player induces the gain of another. In these cases, a zero-sum game is said to occur. In cases where this value does not remain constant, that is, the sum of the players' payoffs differs from zero, it is said to be a non-zero-sum game.

2.2 METHODOLOGICAL NOTES

Literature reviews are essential for developing and accumulating scientific knowledge and conducting surveys on previously published materials (DARAIO *et al.*, 2020). The knowledge acquired through this process can be used by researchers, users, policymakers, and a wide range of people (PAGE *et al.*, 2021). This method is particularly relevant for mapping the main topics studied, providing a complete view of the existing knowledge from the studies on the subject analyzed, and identifying possible gaps and opportunities for future studies (HENRIQUES *et al.*, 2020).

We conduct a literature review based on these characteristics and the objectives of this study. To achieve this, we followed the approach of Lage Junior and Godinho Filho (2010), already used in bibliographic reviews in the DEA field, such as Mariano et al. (2015) and Henriques et al. (2020), which have been published in high-impact journals.

In addition to the adequacy of the approach to the context, the structured steps proposed by the authors ensured the replicability of the procedure. Furthermore, the development of a classification system for articles complies with the standards for data extraction from the papers included in the sample (features and information such as details and methods applied in the studies) as discussed in Tranfield et al. (2003).

However, some choices were made before the article search (Table 1). The main research question considered in this study was the investigation of studies in which DEA and GT were used jointly, aiming to understand the motivations and benefits arising from such a combination, the different ways in which techniques can be combined, and the most researched areas of application.

Only papers published in journals were eligible for inclusion in the study. We excluded books, conference papers, and articles that used DEA and GT but did not integrate them (this exclusion can be a limitation of the research, but it is relevant to mention that the number of studies about those categories was not high).

Considering this methodology, it is essential to mention that we only considered Thomson Reuters Web of Science (WOS) and Scopus as data sources in this systematic review. They do not contain all published papers but cover many high-impact articles, thus comprising two essential databases worldwide.

Table 1 - Methodological considerations for the review

Research Question	Identify all existing DEA and Game Theory studies in which the techniques were combined. How can Game Theory be used to improve DEA performance? In how many economic sectors have these been proposed? Identify existing gaps and develop recommendations of perceived opportunities for further research.
Eligibility criteria	We include only papers published in journals, so we exclude books, conference papers, and articles that used DEA and GT but did not integrate them.
Explicit methodology	A systematic review on Scopus and Web of Science
Systematic search	The queries run on the database are described in Table 2 and reported in a PRISMA flow chart detailed in Figure 2.

Systematic presentation and synthesis	To obtain an in-depth comprehension, we considered a classification system regarding the main characteristics of the studies considering DEA and GT specificities. The main outcomes of the research are reported in Table 4, Table 5, Table 6, Figure 6, Figure 8 and Figure 10.

Source: The Author (2024).

The method proposed by Lage Junior & Godinho Filho (2010) involves five steps (Table 2). Searches were performed using a carefully chosen set of keywords to select papers. A meticulous selection of keywords is crucial for research because it directly affects the paper sample. Further discussion of the systematic analysis is presented in the following section.

Table 2 - Five steps of the methodological procedure

Step	Description
1	Assessing the articles published in significant databases, using a set of pre-established keywords
2	Screening the articles found by reading their abstracts
3	Developing a classification and an analysis system that can represent all dimensions of the object researched
4	Building the profile of scientific production and critical outcomes identified in each article, based on the previously developed classification system
5	Analyzing the gaps as well as the opportunities and challenges that may guide future research on the topic

Source: The Author (2024).

2.3 SYSTEMATIC SEARCH

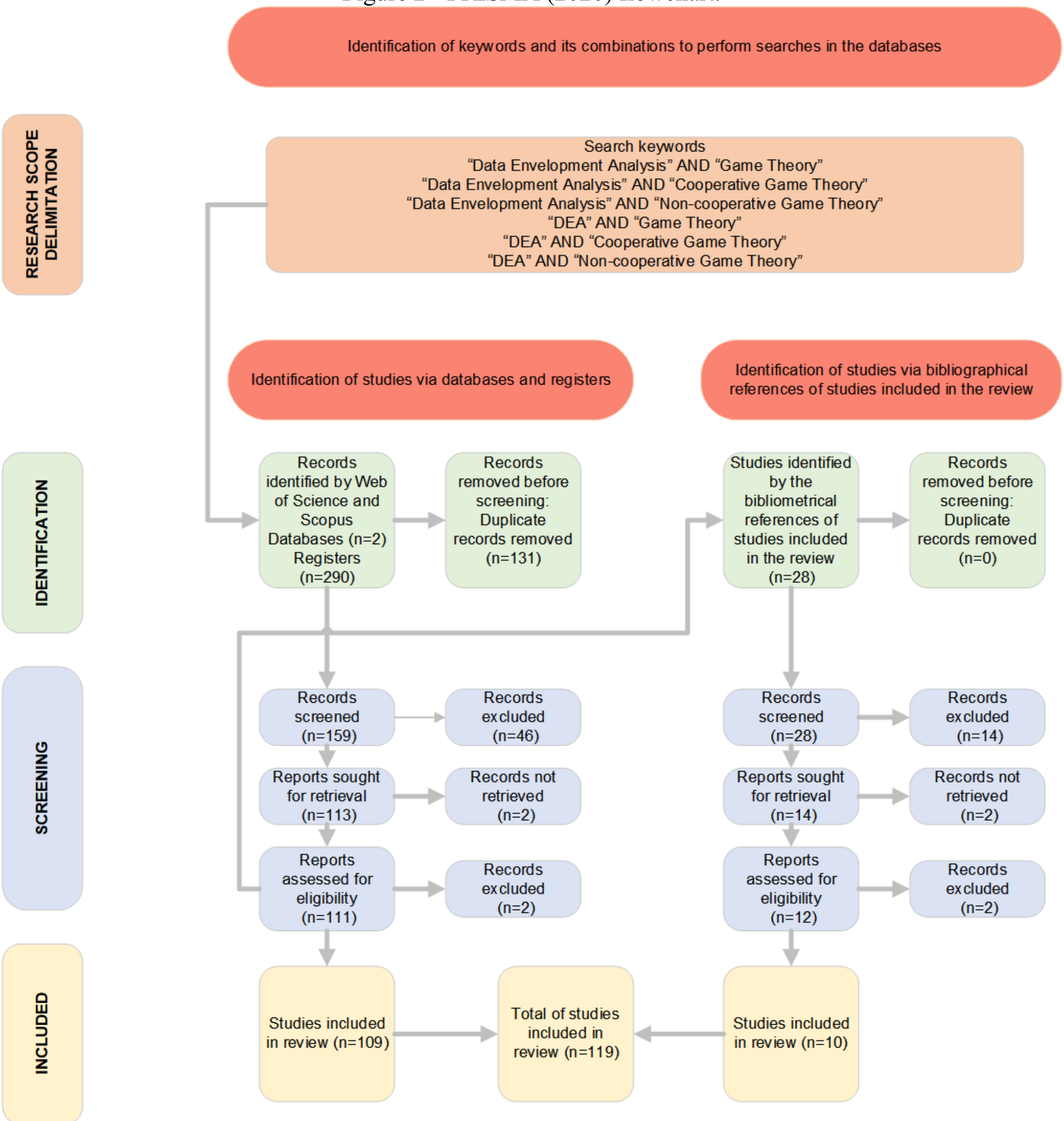
This activity developed shortly after sample collection. The initial sample considered by the authors was obtained in March 2021. Articles were screened by reading their abstracts, titles, and keywords. It is essential to highlight the referenced period because there is a strong possibility that novel papers have been developed and published since then.

As previously mentioned, the search was executed on the Scopus and WOS systems owing to their importance and coverage. This study covered the period from 1980 to December 2020, which corresponds to the first article identified by the survey until the most recent year.

Page et al. (2021) state that authors of systematic reviews should prepare a transparent, complete, and accurate account of why the review was conducted, what they did, and what they found. To clarify the process that resulted in the sample used in this study, the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA 2020) flowchart was developed to describe the results obtained by searching the selected databases and keywords (Step 1).

Figure 2 presents a description of the keywords and their respective combinations used to obtain the initial sample of the studies and shows that the initial analysis identified 290 studies. After verifying duplicate studies in Scopus and WOS, 131 papers were removed. Subsequently, we screened 159 articles using a filter to ensure that they were within the scope of this research. After analyzing the titles, keywords, and abstracts, we excluded 46 papers that did not meet the research objectives. We had a total of 113 studies, but we could not find the complete text for two of them. Subsequently, 111 articles were read to ensure that they fit the scope of this review. After this last step, we removed two other studies because they did not pertain to the research objective and the final sample contained 109 papers.

Figure 2 - PRISMA (2020) flowchart.



Source: The Author (2024).

To ensure a reliable sample, we consulted the bibliographical references of the papers and identified 28 studies that could fit the scope. After reading them, we verified that some studies were cited throughout the literature reviews but were not found by the search engines of the databases. After reading their abstracts, titles, and keywords, 14 papers did not address issues pertinent to this review, and two of those papers were not eligible for retrieval. After reading the full text of the 12 papers, two did not fit the scope of the current research. Therefore, we added ten more articles by verifying bibliographical references. Figure 2 shows all the steps of sample formation, with 119 articles.

After defining the sample, we propose systematization to categorize the paper (Step 3). The systematization aims to obtain a broad comprehension of the papers' main aspects and identify gaps in this literature branch. We then develop a classification with nine categories covering essential topics related to DEA and Game Theory literature. Each article was categorized according to its characteristics and results.

2.3.1 Categories

Category 1 indicates the type of study developed: "theoretical," "application-centered," and "theoretical-practical" studies (LIU et al., 2013a). Theoretical studies have focused on models and mathematical relationships without associating them with empirical data. The use of data contributed only to the simulation of the model results. In application-centered studies, the authors employ the developed models in real situations, whereas the theoretical-practical model proposes innovations in the existing models. Liu et al. (2013a) affirmed that this type of classification identifies the overall usefulness of developed methodologies and provides information on how each model is applied. The application indicates the trend in methodology adoption and thus helps users of the methods to catch up with the latest technology and comprehend the development pattern of the literature.

Category 2 addresses the types of game approaches and is classified into three levels: cooperative, non-cooperative, and cooperative/non-cooperative. The discussion in literature justifies and supports the selection of this category. Kao (2014) and Halkos et al. (2014) highlighted game models in the context of network data envelopment analysis (DEA). By contrast, Liu et al. (2013b) and Liu et al. (2016) discussed the presence of this combined approach in the main path of DEA development and cross-efficiency models, respectively. These authors referred to cooperative and noncooperative situations.

Category 3 analyzes the research objectives, clarifies the purpose of each paper, and describes how the techniques provide methodological support to achieve its objectives. This category responds to the main objective of the research because after comprehending the purpose of applying the combination, it is possible to identify the DEA and GT approaches used to do so, with which guidelines can be proposed and gaps can be identified. It was possible to cluster the most relevant and recurrent objectives after classifying all the papers.

Category 4 pertains to game specificity. The literature on GT presents different cooperative and noncooperative games: the simplest model, Cournot-Nash (static, non-cooperative), the Stackelberg model (dynamic, non-cooperative) until cooperative models, bargaining models, and those that utilize the Shapley value. This category is fundamental to comprehending and completing the results obtained in Category 2.

Categories 5, 6, and 7 address aspects related to the DEA method. We highlight the DEA model used, the number of DMUs, the number of variables (inputs, outputs, and intermediate variables), and the number of stages. This information and the research purpose (Category 3) provided relevant insights. Henriques et al. (2020) discussed that although many surveys exist, there is still no unanimity regarding the fundamental aspects of a DEA study, such as which DEA model and orientation should be adopted and how to select variables. The use of Categories 5, 6, and 7 can shed light on which DEA framework is more appropriate for a specific purpose in the context of DEA and GT.

Category 8 refers to the application area of the study that does not apply to theoretical studies. Liu et al. (2013a) state that investigating applications in the DEA context could provide the proportion of applied papers, significant applications, trends in each approach, and development trajectories for each application area. Nepomuceno et al. (2020) stated that mapping economic applications are relevant for identifying fields that have not been explored and present a high potential for exploration. Daraio et al. (2020) noted that this type of evaluation specifies how the distribution of papers has evolved in different areas. The applications also compare the results generated by various proposed approaches, demonstrating the advantages and disadvantages of the models (KAO, 2014).

Category 9 concerns studies that combined DEA and GT with other approaches. As decision-making has become increasingly complex, hybrid models constitute a systematic approach, and integration of techniques has been applied to refine the results. This analysis investigates the purpose of such combinations and whether there is a trend in combining DEA and GT with other techniques.

The following section presents the results and gaps obtained in Steps 4 and 5. It is essential to highlight that the selection of the nine categories is directly related to the main goal of the research and aims to answer the research questions in Table 1. We strongly believe that the information provided by this categorization is sufficient to achieve the goals of this study.

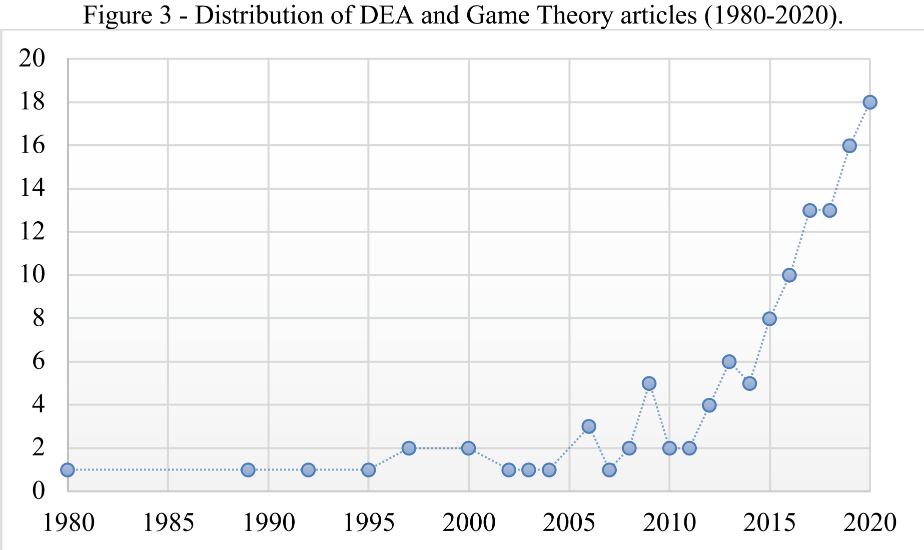
2.4 RESULTS OF THE LITERATURE ANALYSIS

It would be interesting to begin with bibliometric analysis. Next, the results obtained using our classification are discussed. This organization provides a clear panorama of the state of the art, gaps, and future trends in new developments.

2.4.1 Bibliometric Analysis

Figure 3 shows the annual number of publications, and it is possible to observe the growth of papers over the past 40 years that combine DEA and GT. This increase illustrates the popularity of both techniques and their consolidation in literature. We observed no significant developments in the 1980s and the 1990s, and only seven more papers have been published in the two decades. This low number can be related to the development of the DEA literature.

Figure 3 - Distribution of DEA and Game Theory articles (1980-2020).



Source: The Author (2024)

Liu et al. (2013a) stated that most of the studies published in the 1980s and the 1990s were purely methodological. Relevant models such as BCC (BANKER; CHARNES; COOPER, 1984), cross-efficiency (SEXTON; SILKMAN; HOGAN, 1986), and super-efficiency (ANDERSEN; PETERSEN, 1993) were developed at this period. These studies laid the mathematical foundations for DEA and allowed for the advances needed for the broader application of the technique.

Between 2000 and 2015, the number of publications slightly increased. The increase in publications was consistent and is confirmed by the number of publications to the present date, especially the many recent papers in the last five years considered in this research (65,54%).

This indicates that it is a dynamic research area. One of the factors that have aided the development of new methods and publications is the expansion of tools, software, and computational power, which assists in obtaining solutions for more complex models such as nonlinear models.

After verifying the distribution of publications, we analyzed the journals in which publications occurred. Table 3 highlights the eight journals that published the most influential papers. The top five journals represented approximately 45% of all papers. This result aligns with the findings of Emrouznejad and Yang (2018) who verified a high number of DEA publications in the same journals. This is reasonable because the theoretical aspects and most applications fall within Management Science and Operational Research (MS/OR) covered by these journals (EMROUZNEJAD; YANG, 2018).

Table 3 - Total number of papers by journals (1980-2020)

Journals	Number of Papers	Percentage
European Journal of Operational Research	21	17,6%
Omega (United Kingdom)	11	9,2%
Expert Systems with Applications	10	8,4%
Annals of Operations Research	7	5,9%
Journal of the Operational Research Society	5	4,2%
Computers and Industrial Engineering	4	3,4%
Sustainability (Switzerland)	3	2,5%
Others	58	48,7%

Source: The Author (2024)

However, the journals “Computers and Industrial Engineering” and “Sustainability” have different scopes. The first addresses issues related to industrial engineering, focusing on problems that can be solved with a computational aid. The second journal was dedicated to environmental, cultural, economic, and social sustainability. Sustainability is related to a similar trend in the DEA literature; environmental issues constitute a prominent DEA research field (EMROUZNEJAD; YANG, 2018; LIU; LU; LU, 2016).

Using the VOSviewer tool, we investigated the keywords used in these studies (ECK, VAN; WALTMAN, 2010). VOSviewer found 509 different keywords, but we only included keywords with more than five occurrences in the network. Forty-six keywords met the threshold. Each point displayed in Figure 4 represents one of the keywords included in the analysis, and the color is related to the time when the keyword was used with a higher incidence.

We observe that “data envelopment analysis”, “DEA”, “efficiency”, “model”, and “performance” are the most used keywords, and similar results are present in the DEA literature (DARAIO et al., 2020; EMROUZNEJAD; YANG, 2018).

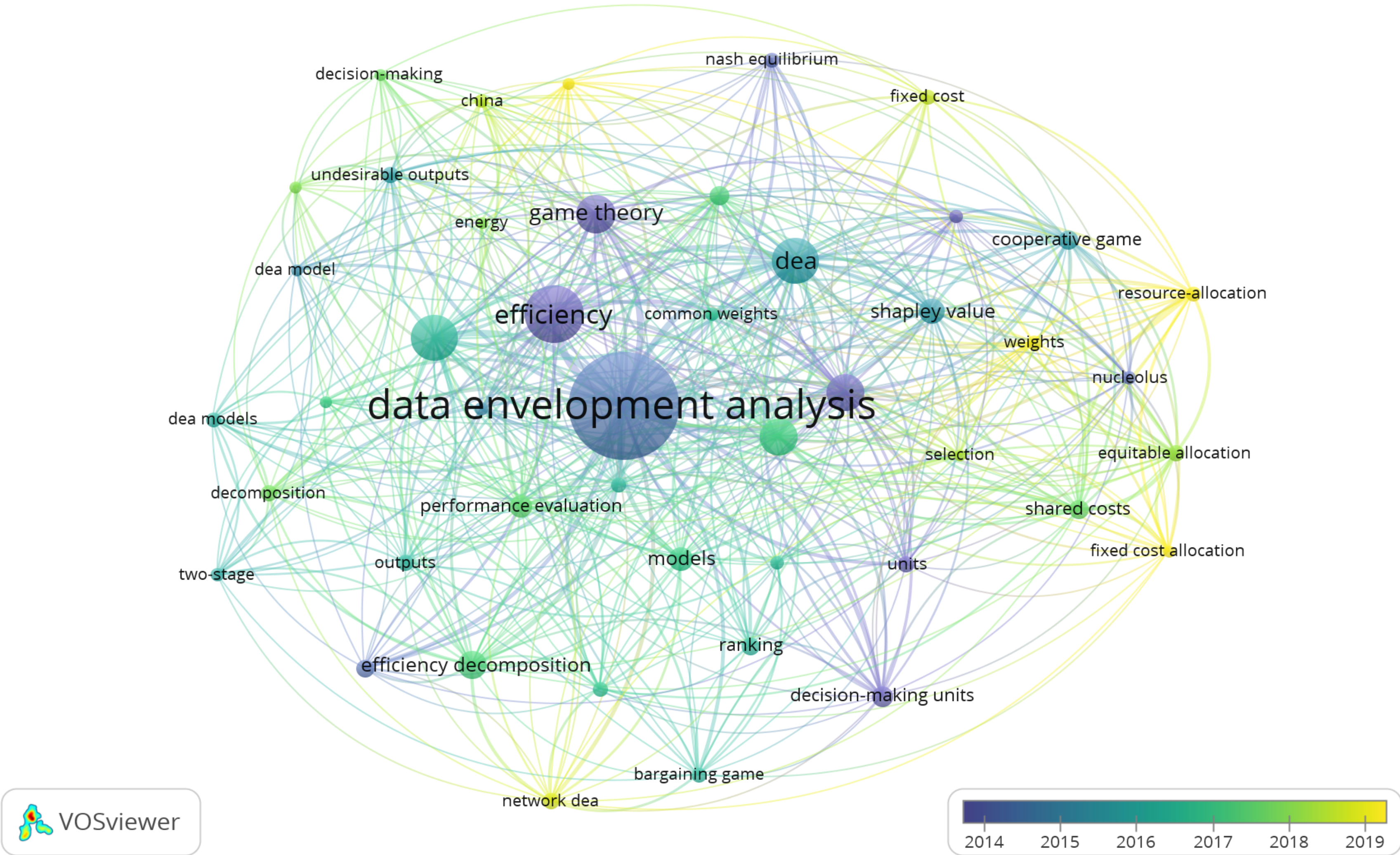
Following these six keywords, “Game Theory”, “Shapley Value”, and “cooperative games” are the most used terms. This finding indicates that Game Theory is used more to address specific issues in the DEA literature than imagined.

Other keywords address DEA models, such as “cross-efficiency”, “network”, and “two-stage”. It is also noticeable that a similar pattern is verified in Game Theory, as many keywords refer to GT approaches, such as “Shapley Value”, “Nash equilibrium”, “bargaining games”, and “nucleolus”. The remaining keywords address objectives or specific DEA issues (equitable allocation, resources, cost allocation, ranking, undesirable outputs, performance evaluation, and efficiency decomposition). The last set of keywords is essential because it can explain the reasons and purposes that led authors to combine DEA and Game Theory. Figure 4 also shows the relationship between keywords and periods:

- before 2015, the most used keywords were related to “DEA,” “Game Theory”, “Nash equilibrium”, “Shapley Value”, and “Cross-efficiency”.
- between 2016 and 2018, “efficiency decomposition”, “ranking”, “allocation”, “bargaining game”, “shared costs”, “energy”, and “equitable allocation”.
- after 2018, we observed another shift in the keywords: “network DEA”, “environmental efficiency”, “resource allocation”, “fixed costs”, and “weights” gained importance.

A partial alignment was verified between the DEA literature trends and the DEA-GT field. This verification confirmed the dynamism of the research field and the advancement of its use on new fronts. It is also possible to affirm that a late set of keywords is a hotspot and indicates current research trends. Energy and environmental issues have been highlighted as relevant areas of application. Efficiency decomposition, network, and two-stage models are also related to the DEA methodological trends (LIU; LU; LU, 2016).

Figure 4 - Network for DEA and GT keywords



Source: The Author (2024).

2.4.2 Type of analysis

We now discuss the second level of results applying the classification presented in Section 2.3.1. First, to provide an overview of literature development, we analyzed the types of articles and created a citation network to highlight the most relevant articles. Subsequently, we focus on the type of game applied and segregate the studies to deepen the analysis and identify gaps.

Regarding Category 1, there were 25 theoretical papers and 94 theoretical-practical papers in the sample. This result indicates that studies that develop applied methods predominate when referring to DEA and GT fields. These findings suggest that advances have been made and verified in practical cases to validate their applicability. Thus, the first gap emerges.

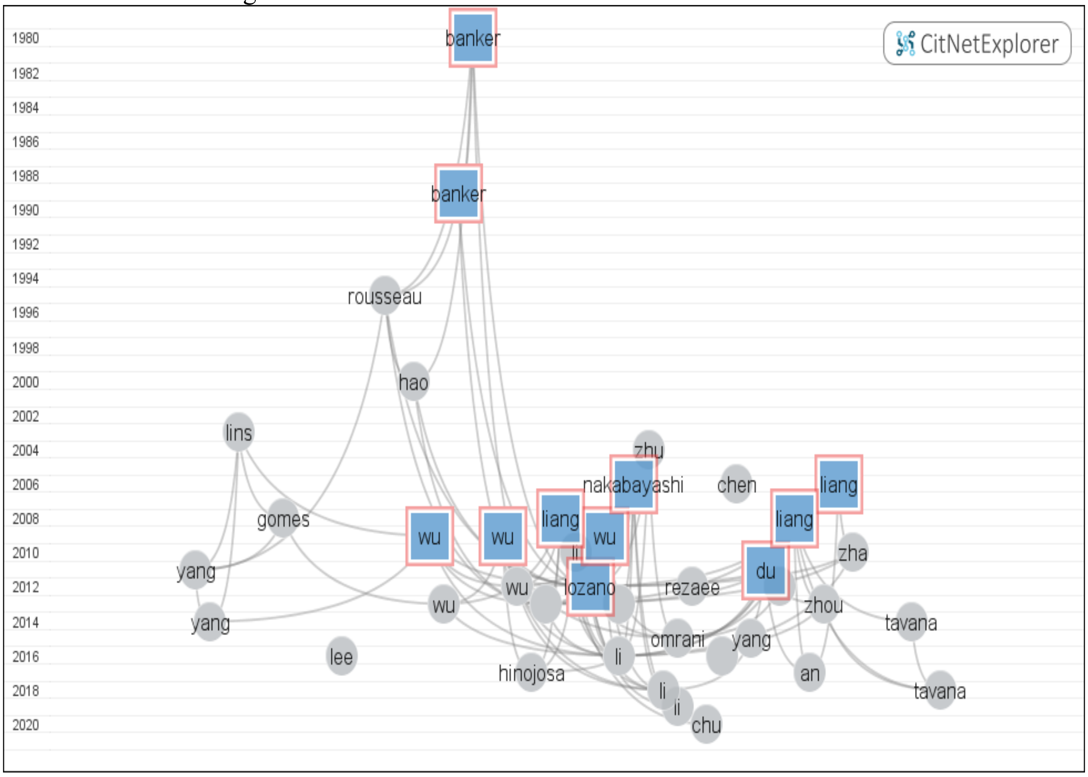
Gap$_1$: The DEA and GT fields lack "application-centered" studies. Applying theoretical and theoretical-practical propositions already developed in different fields allows us to compare their accomplishments and generate guidelines concerning their adaptability to other realities.

A partial alignment was verified between the DEA literature trends and the DEA-GT field. This verification confirmed the dynamism of the research field and the advancement of its use on new fronts. It is also possible to affirm that a late set of keywords is a hotspot and indicates current research trends. Energy and environmental issues have been highlighted as relevant areas of application. Efficiency decomposition, network, and two-stage models are also related to the DEA methodological trends (LIU; LU; LU, 2016).

A citation network was selected to understand the development pattern in the field. This allowed us to identify how the area grew and the studies that most affected the development of the others. In addition, the network structure can provide insights into knowledge dissemination. Furthermore, the network details the interactions between studies, and paper citations were used as connectors. CitNetExplorer was used to plot the citation networks because the number of articles included in the sample was significant.

The choice of citations to analyze studies is justified because citations in academic articles contain rich information on how knowledge disseminates, and they have long been used to evaluate the level of contribution a scientist makes to the practice of science (Liu et al., 2013b). Another advantage of citation analysis is that it does not represent the opinion of any single expert but combines the judgment of many experts in a field (LAMPE; HILGERS, 2015).

Figure 5 - Citation network for DEA and GT literature



Source: The Author (2024).

Given the many citation relations, CitNetExplorer utilizes the concept of transitive reduction to analyze the links between citations and plot the network image. Figure 5 shows the GT and DEA networks. Publications that are close to each other in a citation network tend to be positioned close to each other in a horizontal dimension (ECK, VAN; WALTMAN, 2014). It is important to note that CitNetExplorer does not include all studies in the network; only the 40 most relevant studies. Each node represents a paper and it receives the first author's name.

An analysis of the citation network and types of publications allowed us to conclude that theoretical studies were initially proposed. However, these studies do not address real applications because their primary purpose is to mathematically prove the existence of relationships between DEA and Game Theory.

Figure 5 shows that the discussion regarding the relationship between DEA and Game Theory was initially proposed by Banker (1980). The author introduced a correspondence between CCR propositions to measure the efficiency of DMUs and game-theoretical models and considered cases with only one input and multiple outputs in a two-person zero-sum game.

Banker's (1980) initial proposition was expanded in different ways by Banker et al. (1989), Sengupta (1992), Rousseau and Semple (1995), Semple (1997), Hao et al. (2000a) and Hao et al. (2000b) using non-cooperative approaches to develop the relationship between DEA and Game Theory.

Theoretical studies have increasingly analyzed the assumptions in which DEA models are based on and the relations of different DEA models with Game Theory. This first moment reflects the development of a theoretical foundation and establishment of mathematical relations between the theoretical assumptions of DEA and GT with the exclusive use of non-cooperative approaches. The main reason behind this choice relates to the core of DEA: the DMU under evaluation will have an incentive to focus on inputs and outputs that yield the highest possible scores for its behavior (BANKER, 1980). To do this, the DMU must evaluate its input and output values by comparing them with those of the other DMUs in the reference set. This type of choice is non-cooperative and explains the natural combinations with non-cooperative games.

One concern in mapping the state-of-the-art literature is defining the most critical contributions to the analysis; quantitative and qualitative measures can perform such tasks (NEPOMUCENO; COSTA; DARAIO, 2020). The recognition of the most relevant studies makes it possible to raise evidence regarding the main contributions of this combination (DEA and GT) and the most used models. CitNetExplorer was used to identify the relevant studies. The internal citation score comprises the selected metric to rank the studies and represents the number of citations of a publication within the citation network being analyzed (VAN ECK; WALTMAN, 2014).

We highlight the ten most relevant papers (11 because of a tie in the last position). They had a blue square shape, as shown in Figure 5. The most referenced in the network are Banker (1980), Banker et al. (1989), Liang et al. (2006), Nakabayashi and Tone (2006) Liang et al. (2008), Liang, Wu, et al. (2008), Wu, Liang and Yang (2009), Wu, Liang and Chen (2009), Wu, Liang, Yang, et al. (2009), Du et al. (2011) and Lozano (2012).

Banker (1980) and Banker et al. (1989) were the first in this segment. Nakabayashi and Tone (2006) were among the first to address the combination of DEA and GT from the cooperative perspective. The authors analyze a new scheme for allocating or imputing benefits to players, a situation that they categorize as an egoist`s dilemma.

Liang et al. (2006) proposed a model for analyzing two-stage processes. Liang et al. (2008) examine the cross-efficiency concept under the condition that the cross-efficiency of other DMUs does not deteriorate. Liang et al. (2008) also address two-stage processes and offer alternatives to deal with efficiency decomposition from a leader-follower non-cooperative perspective.

Wu, Liang and Yang (2009) consider a cooperative approach using the Shapley value to mitigate problems with average cross-efficiency scores. Wu, Liang and Chen (2009) proposed an iterative procedure based on a non-cooperative cross-efficiency game combined with assurance regions to investigate the context of the Olympic games. Wu, Liang, Yang, et al. (2009) researched the aspects of cross-efficiency by developing a Nash bargaining game model to measure DMU performance.

Du et al. (2011) address a two-stage model's efficiency measurement and decomposition. The authors investigated the role of the intermediate variable by considering a cooperative Nash bargaining game between the stages. Lozano (2012) introduced a cooperative DEA game to promote horizontal cooperation among DMUs to share information. This information sharing can provide a more extensive possibility production set, and consequently, a more precise efficiency estimation.

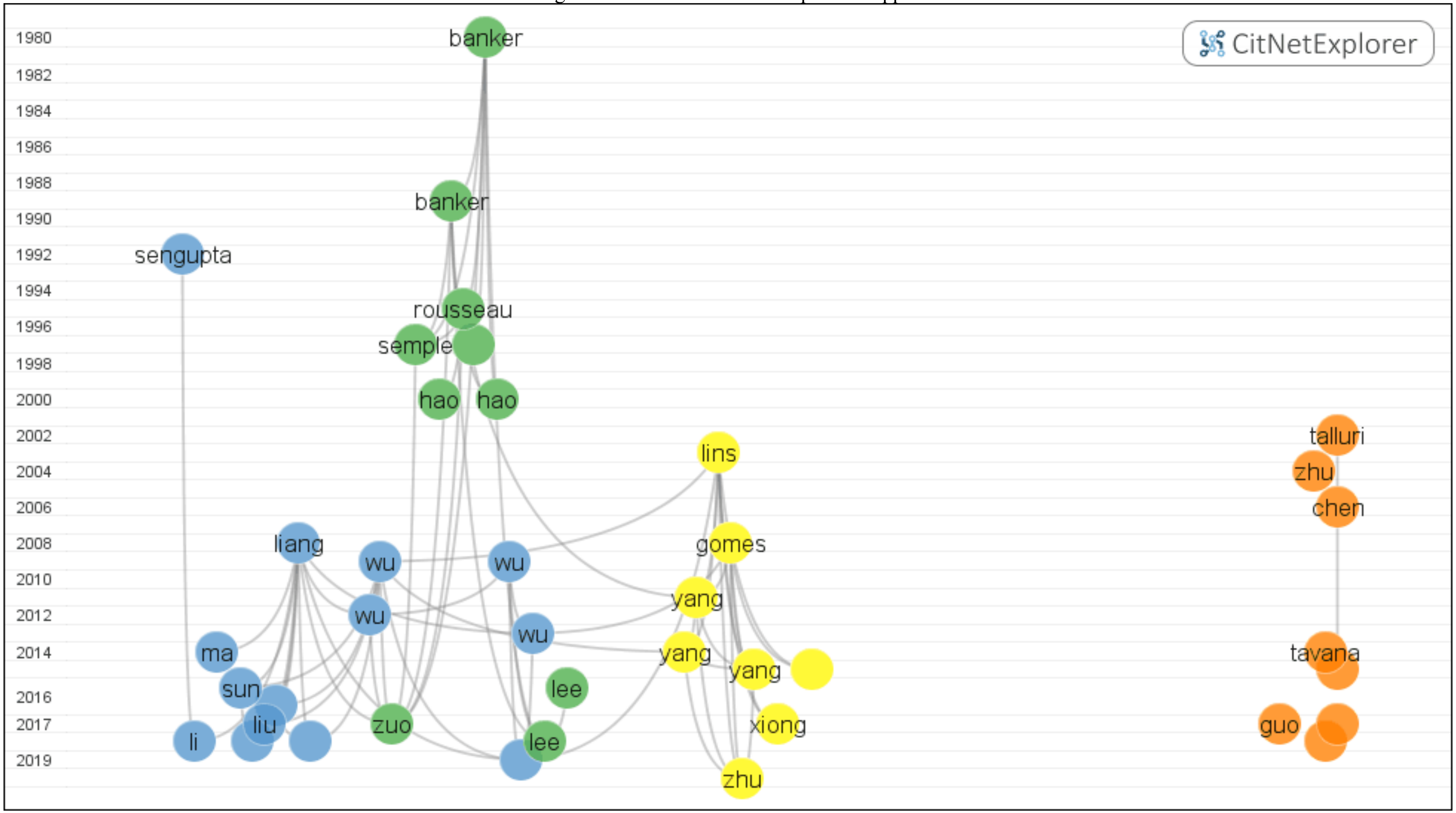
Category 2 discusses the game approach: fifty-one studies use cooperative games, whereas forty-six studies apply non-cooperative approaches, and twenty-two consider both game types. This information allowed us to detail the second stage of development of the literary branch identified in Category 1. The first moment in the literature covers the period from 1980 to 2005. We identified only non-cooperative approaches (12 papers), and these investigations started with a purely theoretical character.

In the last years of this period, the beginning of theoretical-practical propositions can be seen, deepening the discussion of the proposed methods in applied contexts, such as supply chains, Olympic games, and sales.

This second moment can be subdivided into two: i) between 2006 and 2015, studies using cooperative and non-cooperative approaches, and the number of cooperative publications growing more intensely, surpassing the number of non-cooperatives; and ii) between 2016 and 2020, we have an intensified growth of all types of publications. However, cooperatives remain predominant and show more intense growth than others. Thus, the following question arises.

> **Gap$_2$**: Why do researchers use and explore more cooperative approaches? One possible answer is related to the investigation of the internal structure of DMUs. The internal stages of a DMU are part of the same entity. In this sense, decision-makers aim to obtain the best system performance to the detriment of one stage. The cooperative assumption is more readily accepted in this context than the non-cooperative one. This hypothesis lacks testing but could be verified in future studies.

Figure 6 - Network for non-cooperative approaches



Source: The Author (2024).

Table 4 - Systematization of non-cooperative studies

Paper	Type	Objective	Game Theory Approach	DEA Approach	DMU	I*	O*	IM*	S*
Banker (1980)	T	Propose relations between families of games and linear program	Zero-sum game	CCR	-	-	-	-	-
Banker et al. (1989)	T	Propose relations between families of games and linear program	Zero-sum game	CCR	-	-	-	-	
Sengupta (1992)	T	Generalize the scope of DEA applicability	Zero-sum game	CCR	-	-	-	-	-
Rousseau and Semple (1995)	T	Development of a game that is connected with CCR model	Two-person game	CCR	69	5	3	-	-
Rousseau and Semple (1997)	T	Dominant Competitive Factors for distinguishing exceptional aspects of individual performance	Two-person game	CCR	69	5	3	-	-
Semple (1997)	T	Insert constraints on the strategies available to each of the players	Two-person game	CCR	-	-	-	-	-
Hao et al. (2000a)	T	Propose a more robust correspondence between the game family and the DEA family	Two-person game	Cone-ratio DEA	-	-	-	-	-
Hao et al. (2000b)	T	Propose a more robust correspondence between the game family and the DEA family	Two-person game	Generalized DEA	-	-	-	-	-
Talluri and Baker (2002)	T/P	Restrict weight flexibility	Two-person game	CCR	18	2	4	-	-
Lins et al. (2003)	T/P	Efficiency evaluation	Zero-sum game	BCC	80	2	1	-	-
Zhu (2004)	T/P	Establishes the linkage between buyer-seller game models and DEA	Two-person game	CCR	12	4	5	-	-
Chen et al. (2006)	T/P	Efficiency evaluation	Bargaining model/ Nash equilibrium	CCR	10	4	2	2	2
Gomes and Lins (2008)	T/P	Efficiency evaluation	Zero-sum game	CCR with undesirable output	64	1	3	-	-
Liang et al. (2008)	T/P	Propose a cross efficiency game	Nash equilibrium	Cross efficiency	37	1	5	-	-
Wu et al. (2009)	T/P	Propose a cross efficiency game	Bargaining model/Nash equilibrium	CCR/Cross efficiency	37	1	5	-	-
Wu, Liang and Chen (2009)	T/P	Propose a cross efficiency game	Nash equilibrium	Cross efficiency/BCC	78	2	3	-	-
Yang et al. (2011)	T/P	Efficiency measurement and efficiency frontier	Zero-sum game	Fixed sum output DEA	79	4	1	-	-
Wu and Liang (2012)	T	Use DEA as MCDM tool	Nash equilibrium	Cross efficiency	6	4	4	-	-
Wu et al. (2013)	T/P	Resource allocation	Bargaining model	BCC with undesirable output	15	3	1	-	-
Tavana and Khalili-Damghani (2014)	T/P	Solve fuzzy NDEA model	Leader-follower	NDEA	20	4	4	4	2
Ma et al. (2014)	T/P	Efficiency decomposition	Leader-follower	NDEA/ Cross efficiency	30	3	2	2	2

Yang et al. (2014)	T/P	Efficiency measurement and efficiency frontier	Zero-sum game	Fixed sum output DEA	18	2	2	-	-
Ding et al. (2015)	T/P	Efficiency in the presence of dual-role factors	Zero-sum game	CRS with dual role factors	18	2	2	-	-
Chiu et al. (2015)	T/P	Resource allocation	Zero-sum game	Non-radial SBM DEA	24	1	3	-	-
Yang et al. (2015)	T/P	Efficiency evaluation and efficiency frontier	Zero-sum game	Fixed sum output DEA	85	2	3	-	-
Lee (2016)	T/P	Address issues in imperfect markets	Nash equilibrium	DDF DEA	2	2	2	-	-
Machado, Mello and Costa Roboredo (2016)	T/P	Benchmark	Leader-follower	Cross efficiency	61	1	2	-	-
Sun et al. (2016)	T/P	Resource allocation/ sharing resources	Leader-follower	Cross efficiency	30	5	7	-	-
Shafiee (2017)	T	Efficiency evaluation and decomposition under uncertainty	Rough Stackelberg model	NDEA	8	2	2	2	2
Xiong et al. (2017)	T/P	Resource allocation/ sharing resources	Zero-sum game	CCR	30	6	1	-	-
Guo and Zhu (2017)	T	Efficiency decomposition	Leader-follower	NDEA	10	8	1	4	2
Liu et al. (2017)	T/P	Efficiency evaluation	Minmax	Cross efficiency	7	3	3	-	-
Zuo and Guan (2017)	T/P	Efficiency measurement and decomposition	Centralized model	Parallel DEA model	3	6	5	-	-
Costa, Meza and Roboredo (2018)	T/P	Increase DEA discrimination	Nash equilibrium	Cross efficiency	54	3	1	-	-
Lee (2018)	T/P	Use Mixed strategy to create indexes	Mixed strategies in a Nash Cournot	DDF DEA	30	3	2	-	-
Essid, Ganouati and Vigeant (2018)	T/P	Weight multiplicity	Nash equilibrium	Cross efficiency	30	2	2	-	-
Song et al. (2018)	T/P	Efficiency evaluation and decomposition	Leader-follower	NDEA with undesirable outputs	66	3	7	2	2
Li et al. (2018)	T/P	Resource allocation	Leader-follower	NDEA	24/50	2/4	2/4	2	2
Sun et al. (2019)	T/P	Efficiency evaluation and efficiency frontier	Leader-follower	NDEA	4	5	2	4	2
Lee (2019)	T/P	Resource allocation	Nash equilibrium	DEA with fixed and variable input under CRS and NIRS	33	3	2	-	-
Shi (2019)	T/P	Efficiency measurement and decomposition	Leader-follower	NDEA	30	4	6	3	2
Li et al. (2020)	T/P	Resource allocation	Nash equilibrium	CCR	13	2	1	-	-
Shi et al. (2020a)	T/P	Efficiency evaluation and decomposition under uncertainty	Stackelberg game/Leader-follower	Parallel NDEA	52	5	5	-	2
Orkcu, Ozsoy and Orkcu (2020)	T/P	Ranking efficient DMU`s	Minimax regret criterion	Cross efficiency	37	1	5	-	-

Zhu et al. (2020a)	T/P	Efficiency measurement and efficiency frontier	Zero-sum game	Fixed sum output DEA	30	3	2	-	-
Zhu et al. (2020b)	T	Efficiency measurement and efficiency frontier	Zero-sum game	Fixed sum output DEA	6	1	2	-	-

T = Theoretical, P = Practical, T/P = Theoretical-practical, I = Input, O = Output, IM = intermediate measure (which can represent links or carryover), S = Stages

Source: The Author (2024).

Categories 3–7 were analyzed and the game approach was used to segregate the studies. Table 4, Table 5 and Table 6 detail the research objective, the game's specificity, the DEA model, the number of DMUs, and the variables used in each study. Figure 6, Figure 8 and Figure 10 show the citation networks for non-cooperative, cooperative, and papers using both approaches, respectively.

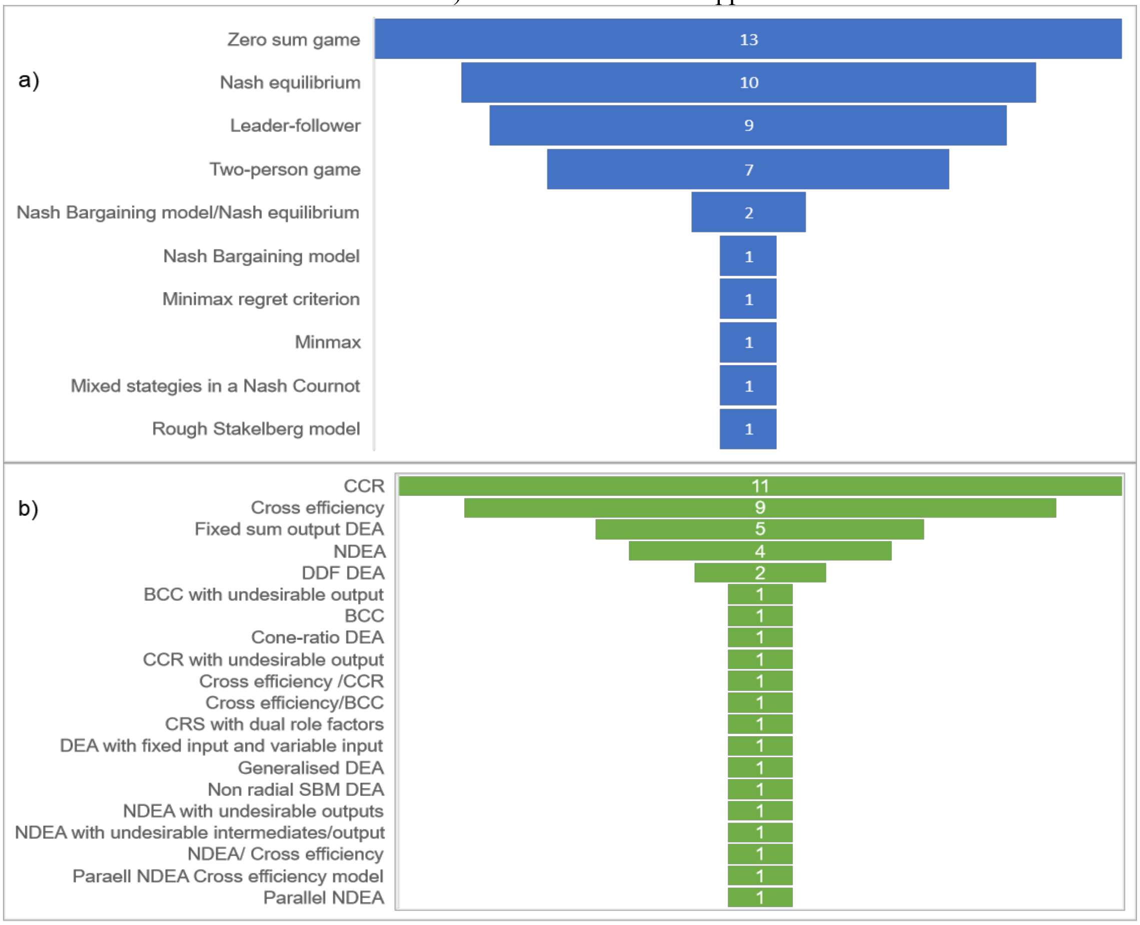
The first level of discussion refers to non-cooperative approaches (Table 4). Concerning Category 3, we can see many different objectives for developing DEA and GT joint systems. This diversity reaffirms the ability of this combination to mitigate DEA limitations and address various problems. Emphasis is placed on proposing mathematical relations between DEA and GT, efficiency measurement and decomposition, construction of efficiency frontiers, and presenting cross-efficiency games.

Category 4 discussed the game models used in these studies. Figure 7 shows all models used for both GT and DEA. Part a) of Figure 7 displays the GT approach, whereas Part b) displays the DEA approach. With the aid of Part a), it is possible to ascertain the predominance of studies considering zero-sum games (13 studies), leader-follower cases (9 studies), and the identification of Nash equilibrium (10 studies).

Category 5 discusses the type of DEA modeling in the studies, and Part B of Figure 7 shows the predominance of studies considering the CCR (11 studies), cross-efficiency (9 studies), fixed sum outputs (5 studies), and NDEA models (4 studies). It is also essential to note that some studies apply more than one of these models or variations to address a specific problem; therefore, their usage is even higher if they consider such details. For example, in NDEA models, specific NDEAs are designed to address undesirable outputs and intermediate measures. Some propositions address parallel network structures and combinations of the NDEA and cross-efficiency. Such cases increase the number of previously discussed points and reaffirm the research intensity in the abovementioned areas.

Categories 6 and 7 detail the quantities of the DMUs and variables used in the studies. Table 4 shows that only two were used in the case of stages, and the number of intermediate measures ranged from two to four. The number of inputs went from one to eight and outputs from one to seven, whereas the number of DMUs varied from 2 to 85.

Figure 7 - Distribution of DEA and GT model for non-cooperative studies a) Distribution for GT approach b) Distribution for DEA approaches



Source: The Author (2024).

The analysis of game types, DEA modeling, and the network shown in Figure 6 provides us with an even broader panorama to understand these studies. After analyzing the articles and their systematization, it is possible to state four main objectives for combining DEA with non-cooperative games: i) the development of mathematical parallels between DEA and Game Theory models; ii) the DEA models that consider cases with a fixed sum of outputs and build reliable efficiency frontiers; iii) the upgrade of cross-efficiency models to improve DEA discrimination, and iv) the address of efficiency measurement and decomposition in network DEA models.

The first objective was verified with greater intensity in DEA and GT studies. The papers in this group are highlighted in green in Figure 6. In most studies, classical DEA models have been combined with zero-sum or two-person games to develop various approaches. state that proximity between techniques is justified by the inherently competitive nature of DEA when performing an efficiency measurement that promotes an individual DMU's achievement at the expense of other competing members (HAO; WEI; YAN, 2000). This identification is equivalent to determining an optimal production strategy. Several studies have investigated these relationships and expanded Banker's (1980) considerations using new models and approaches.

Banker et al. (1989) address two shortcomings: virtual output to reduce the multiple output case back to the single output case and the possibility of non-zero slack presence. Sengupta (1992) proposed a fuzzy game-theoretic formulation in the context of DEA models to generalize the scope of DEA applicability as an input contain noise.

Rousseau and Semple (1995) rigorously connect the two-person zero-sum game and CCR model, and the approach does not require expected payoffs or payoff matrices. Semple (1997), Hao, Wei and Yan (2000a) and Hao, Wei and Yan (2000b) used the framework of Rousseau and Semple (1995). Semple (1997) proposed using polyhedral cone constraints to avoid unreasonable combinations of multipliers.

Hao et al. (2000a) investigated the relationship between a generalized DEA and two-person zero-sum finite game with closed convex cone constraints. Hao et al. (2000b) combined a generalized DEA model and convex cones to represent game constraints. More recently, Lee (2016, 2018, 2019) developed mathematical parallels between DEA and GT by inserting the characteristics of imperfectly competitive markets into efficiency measurements.

The second objective is to adapt DEA modeling to specific cases where the total outputs are fixed. These studies are highlighted in yellow in Figure 6. Such cases are similar to a zero-sum game in which a player wins whatever is lost by one or more others (LINS et al., 2003). This proximity justifies the development of models for this case. Classical DEA models assume that the total output supply is expandable. However, DMUs often compete for limited resources, and the only way for a firm to grow is to obtain market share from competitors.

Studies by Gomes and Lins (2008), Yang et al. (2011), and Xiong et al. (2017) have advanced the methodology proposed by Lins et al. (2003). First, we extend these considerations to undesirable outputs. The second extends fixed-sum outputs to multiple dimensions and analyzes the minimum output improvement required to become technically efficient under CRS and VRS conditions. The third considers multiple inputs and outputs to define the allocation quotas.

Yang et al. (2014, 2015), Zhu, Song, et al. (2020) and Zhu, Li, et al. (2020) propose the construction of efficiency frontiers for cases with of outputs. The first two studies used only DEA and GT. The last two combined DEA with other techniques (extended secondary goal approach and minimum degree of satisfaction) to improve the proposed model.

The third verified objective was to enhance the cross-efficiency models to improve DEA discrimination; the studies are shown in blue in Figure 6. Each DMU selects a set of weights that maximize its efficiency in the DEA. In cross-efficiency models, these weights are also used to calculate the efficiency of other DMUs. However, because of the possibility of weight multiplicity, depending on which of the alternate optimal solutions to the linear DEA programs is used, it may be possible to improve the DMUs (cross-efficiency) performance rating, but generally only by worsening the ratings of others (LIANG *et al.*, 2008).

In this context, it is possible to verify that there is competition among DMUs in choosing the set of weights that maximizes their efficiency score. The literature portrays this competition through a Nash bargaining game and proposes new models to ensure that the results match the Nash equilibrium.

Liang, Wu, et al. (2008) propose an approach in which each DMU corresponds to a player in a game. Cross-efficiency scores may be viewed as payoffs, and each DMU attempts to maximize its payoff under the condition that the cross-efficiency of other DMUs does not deteriorate.

The work of Wu, Liang, Yang, et al. (2009) also deals with the issues of weight multiplicity in cross-efficiency. The authors proposed a game in which each DMU is a player, and the bargaining solution can be obtained using the classical Nash bargaining game model. They selected this model because the efficiency score obtained can be accepted as a fair evaluation as it is a Pareto solution (WU, LIANG, YANG, et al., 2009)

Other studies continued these two trends, extending the considerations to methodological as returns to scale (WU, LIANG AND CHEN, 2009), output orientation (COSTA, MEZA; ROBOREDO, 2018), and better aggregation of the scores obtained in cross-efficiency (LI et al., 2018). It was also verified that these methods were used in other contexts, such as applying multi criteria decision-making (WU AND LIANG, 2012), allocating resources (Sun et al., 2016), fully ranking DMUs (ORKCU; OZSOY; ORKCU., 2020), and combining GT, DEA, and cluster analysis to provide more realistic benchmarks (MACHADO; MELLO; COSTA ROBOREDO, 2016).

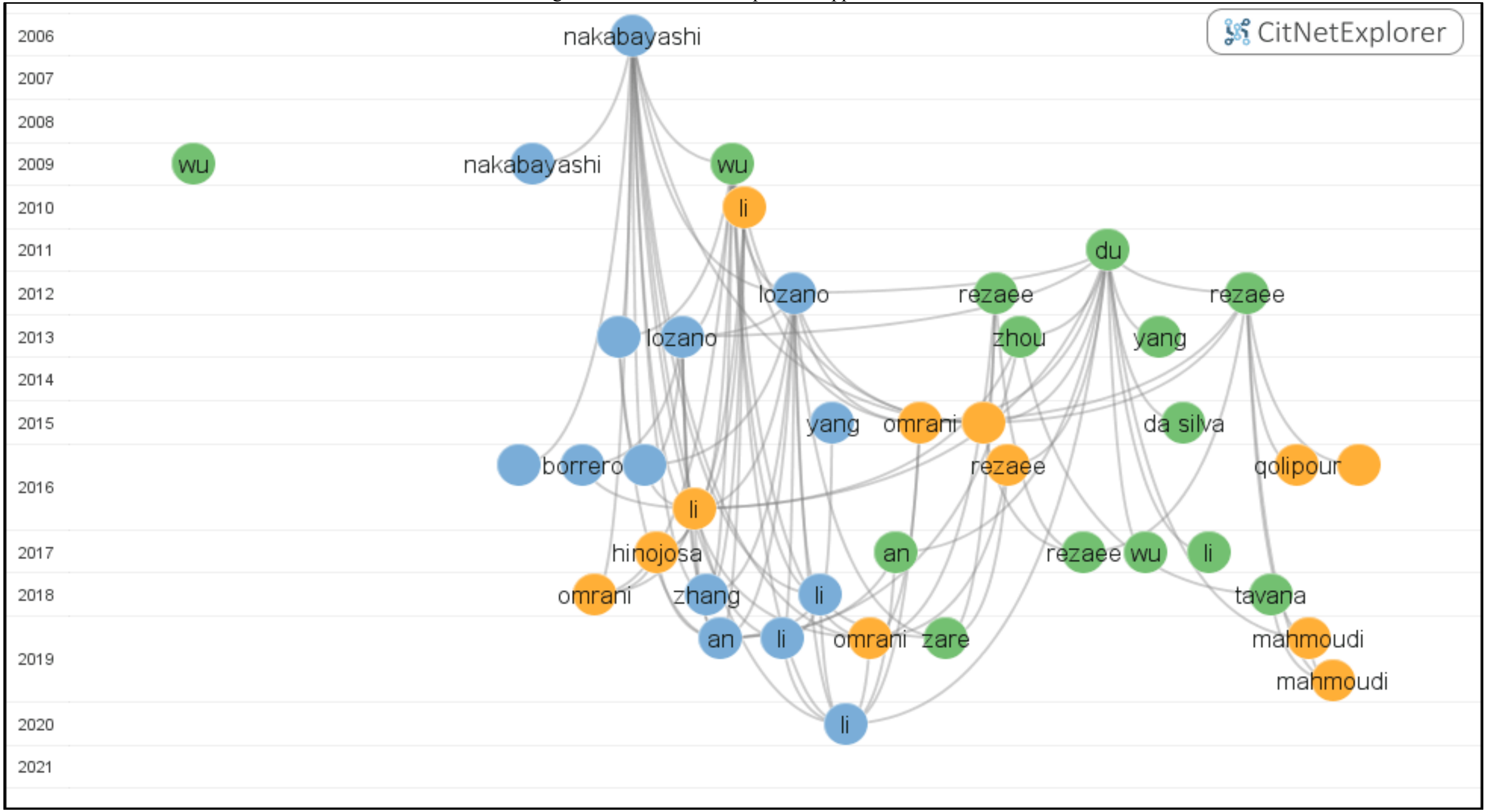
The fourth objective was to calculate the efficiency score for the DMUs and decompose them among the stages in the NDEA models. Classic DEA models envision the DMU structure as a black box, meaning that the internal structure of the DMUs is disregarded. Network models consider that a system's overall efficiency consists of a composition of different processes (sub-DMUs or SDMUs) connected by intermediate products internally consumed by the system (LOZANO, 2016). Unlike classical models, network models do not have a standard form, as this depends on the network's structure, with two basic network typologies: series and parallel (KAO, 2009).

These intermediate variables are the primary factors in the combination of the DEA and GT. These correspond to the output from the first stage and input from the next stage. Thus, the first stage aimed to increase the intermediate variable, whereas the second stage aimed to reduce it. The non-cooperative game approach is adequate to address this issue because of this conflict. In addition, DEA can incur the problem of multiple optimal weights, which can bring a particular issue to NDEA models: efficiency decomposition. The stages within the DMU prefer a weight set that favors efficiency and generates competition among the DMU stages. Decomposition is another justification for using GT.

These studies are highlighted in orange in Figure 6. Not all studies use NDEA models; however, some address the supply chain efficiency issues. Investigations of supply chains usually discuss their stages, justifying the inclusion of these studies in the same group. In the non-cooperative approach, only two-stage cases were verified, and the leader-follower game was the most used. The choice for the leader-follower configuration considers that one stage holds manipulative power and acts as a leader; thus, its efficiency is prioritized in the decomposition.

In this context, methodological advances have been presented in the literature for two-stage configurations, such as shared inputs between stages (GUO; ZHU, 2017), undesirable outputs, SBM modeling (SONG et al., 2018), parallel models with uncertainties (SHI et al., 2020), insertion of uncertainties with fuzzy theory (TAVANA; KHALILI-DAMGHANI, 2014), Rough Set Theory (SHAFIEE, 2017), specific applications for environmental efficiency (SHI, 2019), and circular economic systems (SUN et al., 2019).Thus, the gap resulting from classifications 4 and 5 in non-cooperative studies is as follows:

Figure 8 - Network for cooperative approaches



Source: The Author (2024).

Table 5 - Systematization of cooperative studies

Paper	Type	Objective	Game Theory Approach	DEA Approach	DMU	I*	O*	IM*	S*
Nakabayashi and Tone (2006)	T	Allocating or imputing benefits	Shapley value, core, and nucleolus	CCR	-	-	-	-	-
Wu, Liang, and Yang (2009)	T	Propose a cross efficiency game	Shapley value	Cross efficiency	5	3	2	-	
Nakabayashi, Sahoo and Tone (2009)	T	Resource allocation	Shapley value and nucleolus	CCR	-	-	-	-	-
Wu, Liang and Zha (2009)	T	Preference voting and aggregation	Nash equilibrium	Cross efficiency	-	-	-	-	-
Li and Liang (2010)	T	Importance of variables in DEA	Shapley value	Radial DEA	8	6	2	-	-
Du et al. (2011)	T/P	Efficiency measurement and decomposition	Nash Bargaining model	NDEA	30	2	2	2	2
Rezaee, Moini and Asgari (2012)	T/P	Efficiency measurement and propose a Compose index	Bargaining model	BCC	45	4	6	-	-
Lozano (2012)	T/P	Resource allocation/ sharing resources	Shapley value, nucleolus, and τ-value	BCC	12	2	2	-	-
Rezaee, Moini, and Makui (2012)	T/P	Efficiency measurement and propose a Compose index	Nash Bargaining model	Cross efficiency	24	8	3	-	-
Lozano (2013a)	T	Select the best partner for a horizontal cooperation	Shapley value	Minimum input cost DEA	12	2	2	-	-
Lozano (2013b)	T	Shared resources	Owen set solution, Shapley value, nucleolus, and τ-value	DEA Game	20	1	1	-	-
Zhou et al. (2013)	T/P	Efficiency decomposition	Nash Bargaining model	NDEA	10	3	2	2	2
Yang and Morita (2013)	T/P	Efficiency under multiple perspectives	Nash Bargaining model	CCR	65	4	1	-	-
Yang and Zhang (2015)	T/P	Resource allocation	Modified Shapley value	CCR	12	3	2	-	-
Omrani et al.(2015)	T/P	Increase DEA discrimination	Nash Bargaining model	CCR	37	6	8	-	-
Rezaee (2015)	T/P	Increase DEA discrimination	Shapley value	MODEA	20	13	3	-	-
Silva, Miranda and Martins (2015)	T/P	Define production strategies	Nash Bargaining model	Fuzzy CCR	30	3	2	-	-
Wu et al. (2016)	T/P	Resource allocation	Shapley value	MILP DEA VRS	8/28	4	2/3	-	-
Li et al. (2016)	T/P	Efficient DMU`s evaluation	Shapley value	Super efficiency	14	3	3	-	-
Borrero, Hinojosa and Mármol (2016)	T	Extend concept of production games	Owen set solution	DEA game	-	-	-	-	-
Mostafaeipour, Qolipour and Mohammadi(2016)	T/P	Distinguish the relationships between decision-making components and criteria	Nash Bargaining model	Cross efficiency	14	4	4	-	-
Qolipour et al. (2016)	T/P	Distinguish the relationships between decision-making components and criteria	Nash Bargaining model	Cross efficiency	6	3	3	-	-
Rezaee, Izadbakhsh and Yousefi (2016)	T/P	Increase DEA discrimination and combine efficiency scores	Nash Bargaining model	Cross efficiency	46	6	1	-	-
Peng and Cui (2016)	T/P	Resource allocation	Nucleolus	CCR	4	3	3	-	-

Rezaee and Shokry (2017)	T/P	Multi-level efficiency measurement	Nash Bargaining model	Cross efficiency	17	8	5	-	-
Hinojosa et al. (2017)	T/P	Efficient DMU`s evaluation	Shapley value	CCR	14	3	2	-	-
An et al. (2017)	T/P	Fairness to define intermediate product target	Nash Bargaining model	NDEA	24	2	2	2	2
Wu et al. (2017)	T/P	Efficiency decomposition	Nash Bargaining model	NDEA with undesirable outputs	30	6	3	1	2
Li (2017)	T/P	Efficiency measurement and decomposition	Leader-follower	NDEA	24/30	2/3	2/3	2	2
Omrani, Shafaat and Emrouznejad (2018)	T/P	Ranking efficient DMU`s	Core and Shapley value	Cross efficiency	30	3	4	-	-
Amirkhan et al. (2018)	T/P	Efficiency decomposition	Nash Bargaining model	NDEA	4	3	1	4	3
Li, Zhu, and Liang (2018)	T/P	Resource allocation	Shapley value	Cross efficiency	18	3	3	-	-
Zhang et al. (2018)	T/P	Resource allocation	Nucleolus and Shapley Value	Cross efficiency	4	4	3	-	-
Tavana et al. (2018)	T/P	Efficiency measurement and decomposition under uncertainty	Nash Bargaining model	NDEA	60	3	3	1	2
Li et al. (2019)	T/P	Resource allocation	Nucleolus	CCR	10	4	2	-	-
Mahmoudi et al. (2019a)	T/P	Efficiency measurement	Nash Bargaining model	NDEA	8	1	1	1	3
Mahmoudi et al. (2019b)	T/P	Conflicts between stages and insufficient number of DMU	Nash Bargaining model	NDEA	30	9	4	8	5
Zare et al. (2019)	T/P	Resource allocation	Centralized model	BCC/ Cross efficiency	17	14	7	-	-
Mahmoudi et al. (2019c)	T/P	Increase DEA discrimination	Nash Bargaining model	Cross efficiency	24	8	3	-	-
An et al. (2019)	T	Resource allocation	Shapley value	NDEA	-	-	-	-	-
Omrani, Shafaat and Alizadeh (2019)	T/P	Efficient DMU`s evaluation	Shapley value	Cross efficiency	31	5	4	-	-
Wei et al. (2019)	T/P	Efficiency measurement	Nash Bargaining model	Cross DDF efficiency	28	4	2	-	-
Mousavi-nasab, Safari and Hafezalkotob (2019)	T	Resource allocation	Nash Bargaining model	CCR, Cross efficiency, Additive Model	10	1	3	-	-
Yousefi, Rezaee and Solimanpur (2019)	T/P	Efficiency measurement	Nash Bargaining model/Nash equilibrium	Two-stage Simultaneous DEA	10	1	2	-	2
Contreras and Lozano (2020)	T/P	Resource allocation	Bargaining model with Nash, Kalai-Smorodinsky, Egalitarian and Utilitarian solutions	Centralized DEA	47	3	3	-	-
Omrani et al. (2020)	T/P	Efficient DMU`s evaluation	Shapley value	Cross efficiency	17	1	17	-	-
Li et al. (2020)	T/P	Resource allocation/ target setting	Nucleolus	DDF DEA	31	3	2	-	-
Omrani, Fahimi and Mahmoodi (2020)	T/P	Compose index and improve DEA discrimination	Shapley value	Cross efficiency	10	1	68	-	-
Meng, Wu and Chu (2020)	T/P	Resource/ cost allocation	Shapley value	CCR	5	2	1	-	-
Ding et al. (2020)	T/P	Efficiency measurement	Centralized	NDEA	37	5	2	3	2

An, Wand and Shi(2020)	T/P	Resource/ cost allocation	Nucleolus	NDEA	27	3	2	3	2

T = theoretical, P = practical, T/P = theoretical-practical, I = input, O = output, IM = intermediate measure (which can represent links or carryovers according to the DEA model), S = stages

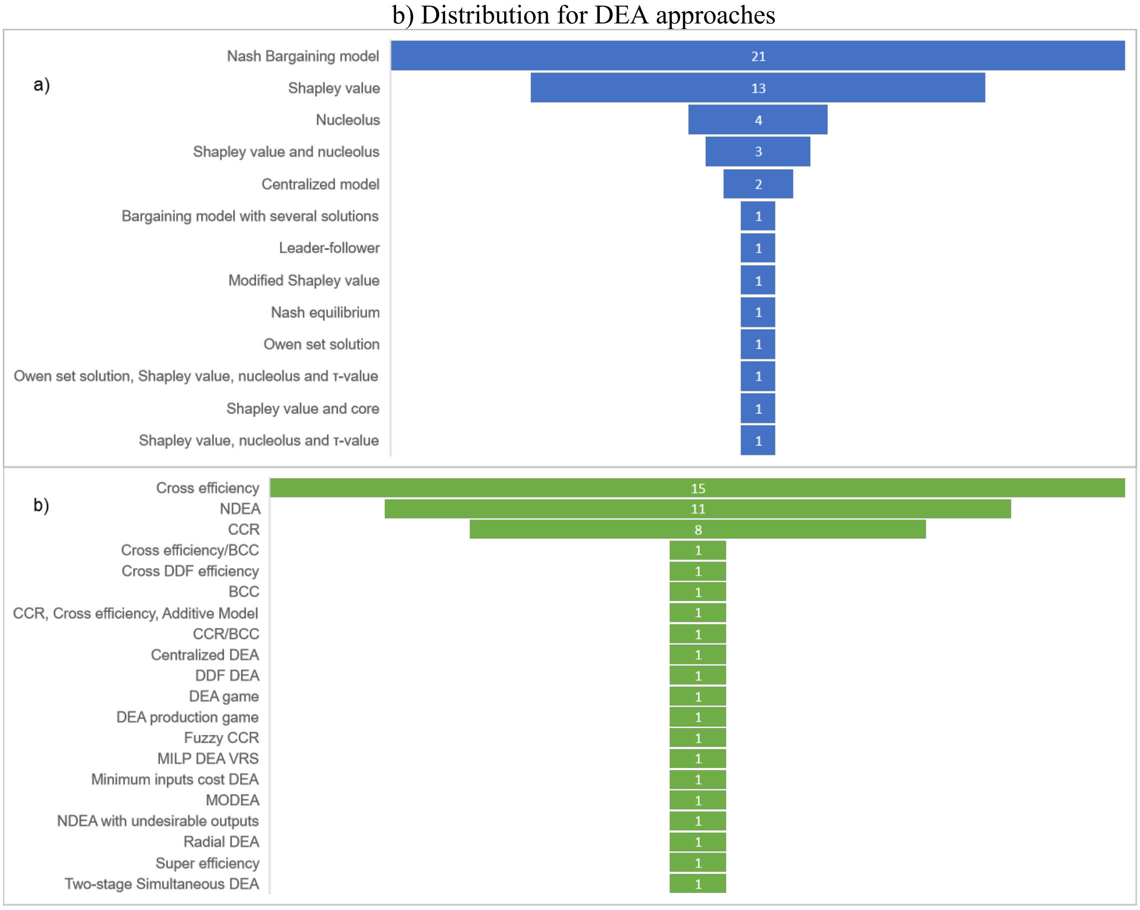
Source: The Author (2024).

Gap3: Given that an increase in the number of stages can generate incentives for collaboration or promote fiercer competition between stages, studies that consider more than two stages are necessary to find evidence regarding the impact of more stages on efficiency.

Cases of cooperative approaches (Table 5) were also analyzed. Concerning Category 3, we can see many different objectives for developing DEA and GT approaches. This feature is similar to that verified in noncooperative studies, but the main goals for the combination are slightly different. The issues of efficiency measurement and decomposition were identical. We also confirm the aim of investigating resource sharing and propose alternatives to improve DEA discrimination, costs, and resource allocation.

Category 4 discussed the game models used in these studies. Figure 9 shows the diversity of the models used in these studies. It is possible to verify in Part a), the predominance of studies considering Nash bargaining games (21 studies), the use of the Shapley value (13 studies), and the combination of allocation methods such as the Shapley value, nucleolus, and core (five studies).

Figure 9 - Distribution of DEA and GT model for cooperative studies a) Distribution for GT approach b) Distribution for DEA approaches



Source: The Author (2024).

Category 5 discusses the DEA modeling employed in these studies. Various models were used, as shown in Figure 9 Part b). There was a predominance of studies that considered cross-efficiency (15 studies), NDEA (11 studies), and CCR (eight studies).

Categories 6 and 7 detail the quantities of DMUs and the variables adopted in the studies. The number of inputs ranges from 1 to 14 and outputs from 1 to 68, whereas the number of DMUs varies from 4 to 65. In the cases where stages were considered, Table 5 shows that stage numbers ranged from 2 to 5, and intermediate measures ranged from 1 to 8.

After analyzing the articles and their systematization, it is possible to state three main objectives for combining DEA with cooperative games. The first consists of developing models to deal with efficiency measures and decomposing the network DEA models. The second relates to developing DEA models that consider cases where costs and resources are shared or allocated within the DEA framework. Third, we propose alternatives to improve DEA discrimination.

Given that efficiency measurement is the main objective of DEA, many related studies are expected. In a cooperative framework, Wu, Liang and Yang (2009) model stands out. Wu, Liang, and Yang (2009) proposed a cooperative approach to cross-efficiency assessment. The authors base their considerations on Nakabayashi and Tone (2006) proposition of obtaining standard weights associated with the imputation of the Shapley value and cores.

Following Wu, Liang, & Yang (2009), we verified several studies that focused on the particularities of the NDEA models. The reasons for investigating the NDEA models are similar to those for the non-cooperative case. The difference lies in the assumption that the stages cooperate to achieve better system efficiency. This could be one of the reasons for the high number of Nash-bargaining models, as shown in Figure 9.

Du et al. (2011) and Zhou et al. (2013) investigated the efficiency of a two-stage system. Both consider the stages as players and the Nash-bargaining game between them. These studies addressed the efficiency of the system and its decomposition into stage efficiency. An et al. (2017) focused on setting a target for the intermediate measures of a two-stage system.

Wu et al. (2017) and Ding et al. (2020) analyzed two-stage systems with a particular characteristic. Both studies investigated situations in which recyclable products are present. Therefore, a circular relationship exists between these stages. Ding et al. (2020) also proposed a dynamic Malmquist model to evaluate the efficiency over time periods.

Other developments have also been verified, such as exploring how to combine two different efficiency measures in a unified structure (JAHANGOSHAI REZAEE; MOINI; MAKUI, 2012), a combination of cluster techniques and two-stage bargaining games to provide proper performance evaluation (JAHANGOSHAI REZAEE; MOINI; HAJI-ALI ASGARI spective DEA models (YANG and MORITA, 2013), fuzzy two-stage bargaining DEA model (TAVANA *et al.*, 2018) and a directional distance function cross-bargaining efficiency model (WEI et al., 2019).

Unlike noncooperative models, studies using network models consider more than two stages. Amirkhan et al. (2018) considered a three-player Nash-bargaining as three-stage systems. Rezaee and Shokry (2017) combined DEA, GT, and Balanced Scorecard (BSC) to investigate a four-series stage. In the proposed model, each division represents a BSC perspective: finance, customers, internal processes, learning, and growth. Given the above, the following gaps emerge:

> **Gap$_4$**: How is the efficiency affected by the type of game approach chosen? The use of different models allows for a comparison of their results and provides a broader comprehension of the situations in which each model is more suitable. This gap is valid for both cooperative and non-cooperative approaches, and this verification becomes even more relevant as a more significant number of stages are considered for analysis.
>
> **Gap$_5$**: Studies that consider dynamic performance measurement, such as Ding et al. (2020) are essential for verifying the impact of cooperation and competition over time.

The second objective verified was the development of models to share or allocate costs and resources within the DEA framework. Most of these studies use an imputation method to allocate costs and resources, as shown in Figure 9. This choice is related to the purpose of cooperative game theory, which is to analyze whether incentives for cooperation are present or to allocate the payoff (gain or cost) of a game (ZARE *et al.*, 2019). Therefore, assigning the total benefits obtained from cooperation among all partners is crucial for motivating participants to achieve the maximum benefits. Nucleolus and Shapley values are highlighted as the most used imputation tools in Figure 9.

Concerning cost allocation, Li, Zhu, and Liang (2018), Zhang et al. (2018), Li et al. (2019) and Meng et al. (2020) discussed cost allocation among the DMUs in the reference set while An et al. (2020) discussed it under the internal structure of the DMU in a two-stage framework and also considered a cooperative relationship between the DMUs.

Most analyses have focused on the resource allocation between DMUs. An et al. (2019) study differs in that it considers shared resources in the internal structure of a DMU. The authors evaluated a serial system with three stages, in which there is the possibility of resource sharing.

They also proved that sharing benefits both the stages and DMUs. A pattern similar to that of the cost allocation was verified for the resource allocation.

Unlike other studies that used imputation methods for resource allocation, (CONTRERAS; LOZANO, 2020) proposed a combination of a centralized model and Nash bargaining game to allocate additional resources. The authors discuss four bargaining solutions: Nash, Kalai-Smorodinsky, egalitarian, and utilitarian. The results show that the different bargaining solutions have similar output and utility targets, although they suggest different resource allocations. In addition, slight input alterations can significantly modify the utility targets. Considering the small number of studies that investigated solutions other than Nash equilibrium, the following gap was identified:

> **$\mathbf{Gap_6}$:** As most of the literature on DEA and GT focuses on obtaining the Nash equilibrium to identify a solution for the proposed game or ensure a unique solution, there is a lack of studies seeking to investigate and compare other solutions, such as Kalai-Smorodinsky, egalitarian, and utilitarian.

Similar to other techniques, the DEA has certain limitations. There is the possibility of multiple efficient DMUs and the need for specific proportions of DMUs and inputs/outputs to ensure discrimination of the results. Thus, it is possible to verify the proposed model to mitigate the problems mentioned above. Li and Liang (2010) proposed an efficiency change ratio (ECR) to analyze the impact of each variable on efficiency. They used the ECR to develop a characteristic function and determine the importance of the variables with the aid of the Shapley value.

Approaches for ranking efficient DMUs have also been proposed. Hinojosa et al. (2017) addressed this topic by considering the variation in the efficiency of inefficient DMUs when they were removed from the observation group. The Shapley value was then applied to rank the DMUs.

Omrani et al. (2018) proposed a full ranking of efficient DMUs to ensure homogeneity. The authors proposed the creation of clusters to ensure that the DMUs are homogeneous; then, a Nash bargaining game is implemented to evaluate the DMUs of each cluster, and the efficient ones are ranked with the aid of the Shapley value. Omrani, Amini, et al. (2020) simplified the approach of Nakabayashi and Tone (2006) to avoid solving all linear programming models to rank DMUs fully.

Discussions regarding the relationship between the number of DMUs and the number of inputs/outputs are also present in this literature field. There is a rule of thumb that the number of inputs and outputs should be less than one-third of the number of units (FRIEDMAN;

SINUANY-STERN, 1998). In the real world, policymakers and managers should analyze the performance of existing DMUs; however, in many industries, the number of existing DMUs is not sufficient to fulfill this DEA condition (MAHMOUDI; EMROUZNEJAD; *et al.*, 2019).

Rezaee (2015) suggested the creation of several input categories and proposed multi-objective DEA to generate multiple frontiers. Under a cooperative assumption, the Shapley value was applied to obtain fair weights for the importance of each object.

Mahmoudi, Emrouznejad, and Rasti-Barzoki (2019) evaluated the insufficient number of DMUs in the context of network models. The authors considered each input category and stage as a player in the Nash bargaining game. The game DEA model maximizes the distance between each player's efficiency scores and corresponding breakdown points.

Omrani et al. (2015) and Mahmoudi, Emrouznejad, Khosroshahi, et al. (2019) combined DEA, GT, and Principal Component Analysis (PCA) to reduce variable dimensions when the number of DMUs is insufficient. Omrani et al. (2015) proposed reducing the variables using PCA and bargaining game models to discriminate between DMUs. Mahmoudi, Emrouznejad, Khosroshahi, et al. (2019) combined cluster analysis, PCA, Nash bargaining game, Shannon entropy, and TOPSIS to evaluate the performance of DMUs when inputs are classified into two groups, and an insufficient number of DMUs is present.

Omrani et al. (2020) considered a different case from the one previously mentioned. They dealt with many indicators and used DEA to obtain a single index. Traditional models are not appropriate because of their large, the model of Wu, Liang, & Yang (2009) was used to obtain the weights for the indicators divided into categories and get a single index. Given the analysis of studies aimed at improving DEA discrimination, the following gaps have emerged:

> **Gap_7**: Most of the articles in the literature on DEA and GT focus on enhancing DEA discrimination in "black-box" cases. There is a gap in the literature to investigate if the "thumb rule" of black-box models is valid to ensure discretion in more recent models. Only Mahmoudi, Emrouznejad and Rasti-Barzoki (2019) proposed verifying such issues using two-stage models.

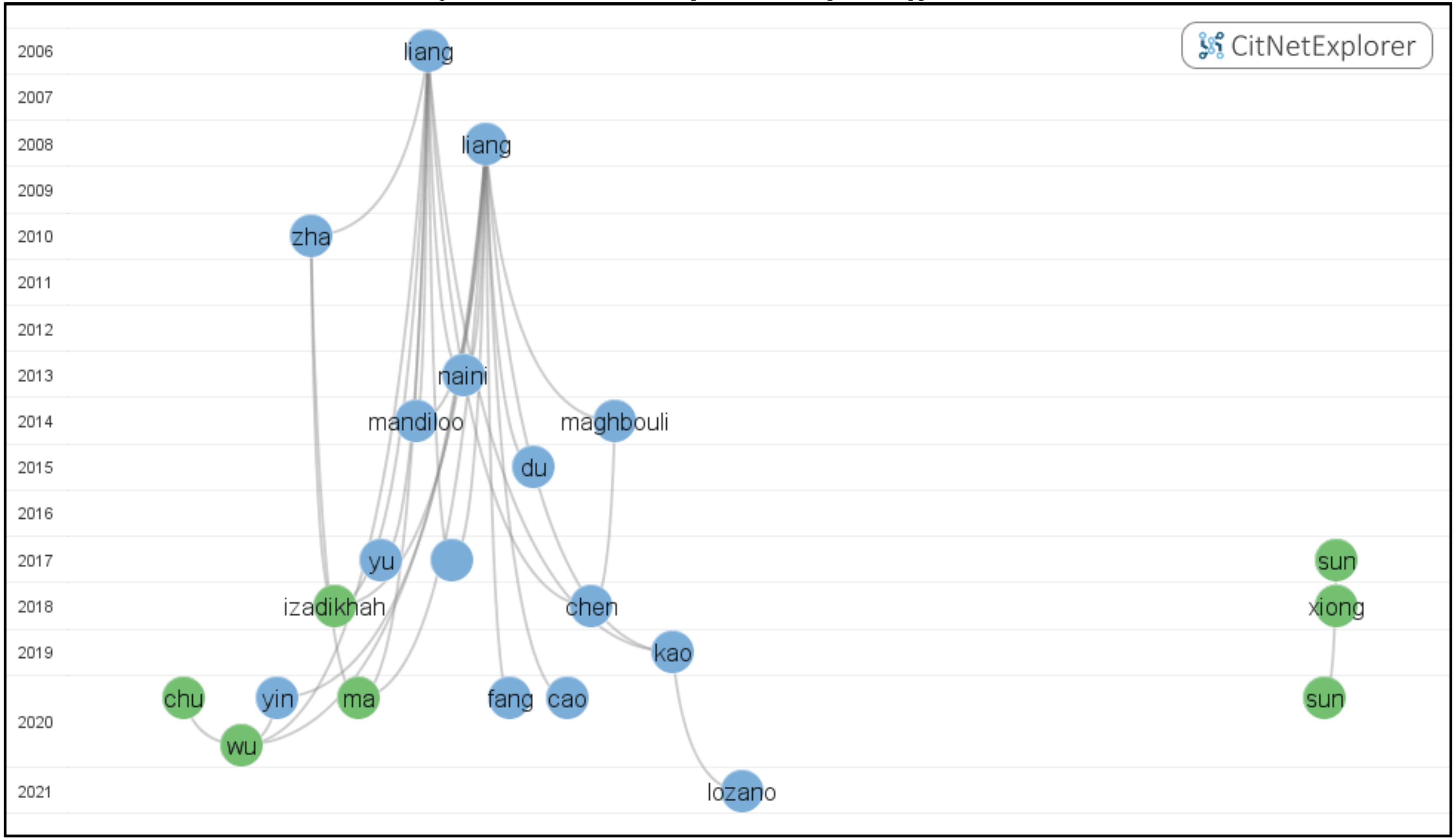
Based on the previous discussion, another point is unrelated to the specific objective. Figure 9 shows a high incidence of studies using the Nash-Bargain model. In this game model, studies have considered that the DMU will withdraw from the game if it obtains optimal efficiency scores lower than the breakdown points, representing the minimum achievable efficiency for the stages (MAHMOUDI; EMROUZNEJAD; *et al.*, 2019).

Different methods have been proposed in the literature to obtain breakdown points. Du et al. (2011) suggested using the efficiencies obtained from Liang, Cook, et al. (2008) leader-follower model as an alternative for breakdown points. They also proposed the construction of the least ideal DMU for each stage and used its DEA efficiency score as the breakdown point. Mahmoudi, Emrouznejad, Khosroshahi, et al. (2019) considered the following three positions: the manager's decision, the efficiency of the previous period obtained with a dynamic model, and creating a virtual DMU with enormous input and smallest output. The authors used cross-efficiency to define the breakdown points. We consider the following gap related to the breakdown points.

> **Gap$_8$**: Owing to the diversity of alternatives for obtaining breakdown points, their role as a parameter for the models, and the limited sensitivity analysis for only a few combinations of their options, it is possible to state that the investigation of the impact of breakdown points in the efficiency measurement when adopting Game approaches consists of a gap.

After identifying the main objectives and the approaches used, it is possible to understand why the groups of studies are closer in the cooperative citation network. Many studies employed similar game and DEA models to address interrelated objectives, which can explain a more intricate citation network.

Figure 10 - Network for non-cooperative and cooperative approaches



Source: The Author (2024).

Table 6 - Systematization of cooperative and non-cooperative studies

Paper	Type	Objective	Game Theory Approach	DEA Approach	DMU	I*	O*	IM*	S*
Liang et al. (2006)	T	Efficiency evaluation	Centralized model/Leader-follower	CCR	10	4	2	3	-
Liang, Cook and Zhu (2008)	T/P	Efficiency measurement and decomposition	Centralized model/Leader-follower	NDEA	30	2	3	2	2
Zha and Liang (2010)	T/P	Efficiency decomposition and resource allocation	Leader-follower	NDEA	30	3	3	2	2
Naini et al. (2013)	T/P	Efficiency decomposition	Leader-follower	NDEA	35/20	2/3	4	2	3
Mahdiloo et al. (2014)	T/P	Integrate different efficiency scores	Centralized model/Leader-follower	CCR with undesirable output	30	4	6	-	-
Maghbouli, Amirteimoori and Kordrostami (2014)	T/P	Address issues of undesirable intermediate measures in NDEA	Centralized model/Leader-follower	NDEA	39	6	4	1	2
Du et al. (2015)	T/P	Efficiency decomposition	Centralized model/Leader-follower	Parallel NDEA	34	4	3	-	4
Sun et al. (2017)	T/P	Resource allocation	Centralized and individual model	CCR	30	2	2	-	-
Esfandiari et al. (2017)	T/P	Efficiency measurement and decomposition under uncertainty	Centralized model/Leader-follower	NDEA	20	4	4	4	2
Yu and Su (2017)	T/P	Efficiency measurement and decomposition	Centralized model/Leader-follower	NDEA	4	3	1	1	2
Xiong et al. (2018)	T/P	Resource allocation	Centralized model/Leader-follower	Parallel NDEA	30	8	4	2	2
Izadikhah et al. (2018)	T/P	Shared resources	Centralized model/Leader-follower	NDEA	15	5	3	2	2
Chen et al. (2018)	T/P	Efficiency measurement and decomposition	Centralized model/Leader-follower	NDEA with undesirable outputs	30	5	2	1	2
Kao (2019)	T/P	Efficiency decomposition	Combined intermediate measure/Free intermediate measure	NDEA	10	3	2	2	3
Cao, Ma and Muren (2020)	T/P	Classify DMU`s in groups and according to the inclination for cooperation	Strike Degree	Generalized DEA	9	4	3	-	-
Wu et al. (2020)	T/P	Resource allocation/ target setting	Leader-follower/Nash Bargaining model	SBM NDEA	30	4	4	3	2
Fang (2020)	T/P	Efficiency decomposition and weight priority	Centralized model/Leader-follower	NDEA	24	2	2	2	2
Chu et al. (2020)	T/P	Resource allocation	Leader-follower/Nash Bargaining model	NDEA	27	3	3	2	2
Lozano and Khezri (2020)	T	Efficiency evaluation	Combined intermediate measure/Free intermediate measure	NDEA	24	1	1	1	2
Ma et al. (2020)	T/P	Shared resources	Additive Centralized/Leader-follower	CCR/NDEA	27	3	2	1	2

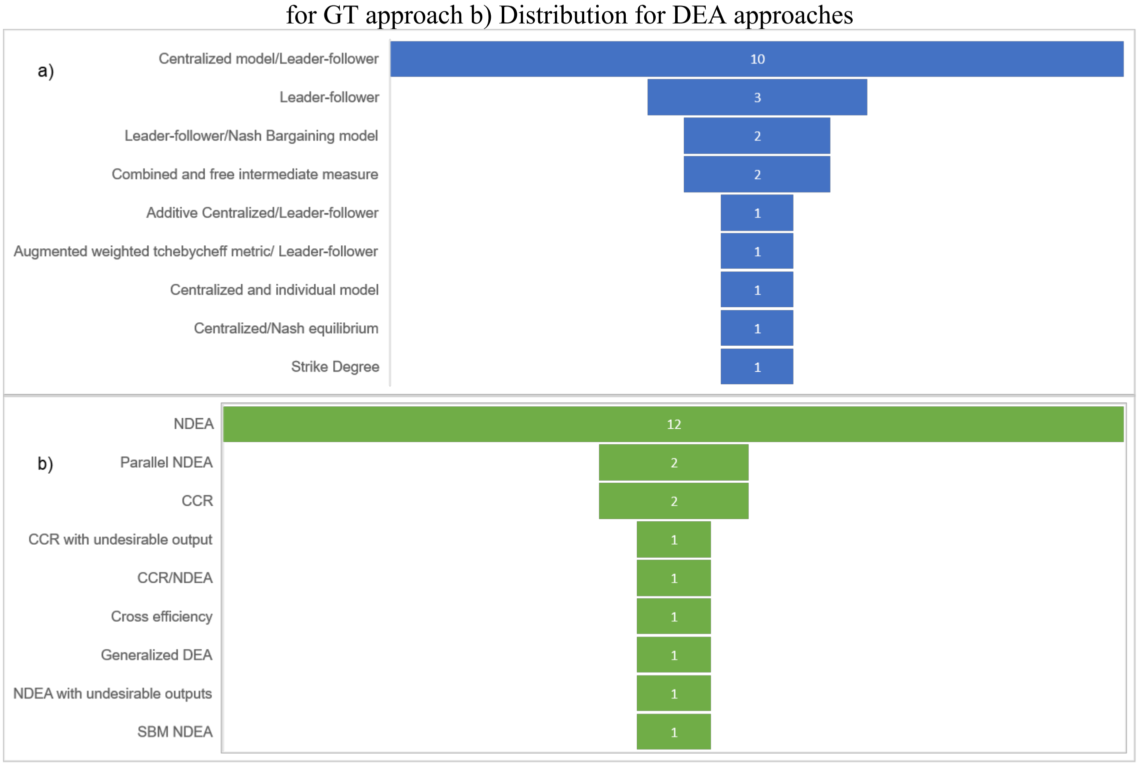
Yin et al. (2020)	T/P	Efficiency measurement and decomposition	Augmented weighted Tchebycheff metric/ Leader-follower	NDEA	68	4	2	2	2
Yaya et al. (2020)	T/P	Efficiency evaluation	Centralized Nash/equilibrium	Cross Efficiency	31	3	3	-	-

T = theoretical, P = practical, T/P = theoretical-practical, I = input, O = output, IM = intermediate measure (which can represent links or carryovers according to the DEA model), S = stages

Source: The Author (2024).

Figure 10 and Table 6 present the studies that considered cooperative and non-cooperative approaches. With the aid of Table 6, we verified fewer purposes than isolated approaches for Category 3. The focus was on efficiency measurement, efficiency decomposition, resource sharing, cost, and resource allocation. Category 4 discussed the game models used in these studies. The models used are shown in Figure 11 Part a). It is possible to verify the predominance of studies that combined the centralized model and the leader-follower game (10 studies). It is also highlighted that most other studies have used one of the previously mentioned methods.

Figure 11 - Distribution of DEA and GT model for cooperative and noncooperative studies a) Distribution for GT approach b) Distribution for DEA approaches



Source: The Author (2024).

Category 5 discusses the DEA modeling employed in these studies. Distinct models were used, as shown Figure 11 Part b). There was a predominance of studies that considered the NDEA (12 studies). Other studies have considered variations in the NDEA or used CCR, generalized, or cross-efficiency models.

The papers involved one to eight inputs and one to six outputs in a set of twenty-two studies that used both cooperative and non-cooperative frameworks. In contrast, the number of DMUs analyzed varied considerably, from 4 to 68, while intermediate variables ranged from 1 to 4. It is also important to note that 12 of the 15 studies that applied a NDEA model considered two-stage cases.

The analysis of game types, DEA modeling, and citation networks is shown in Figure 9, allowing us to infer the presence of two main groups of objectives. The first is the measurement and decomposition of efficiency, and the second is related to the allocation of costs and resources.

It is important to note that the topics discussed using both approaches have previously been discussed using isolated models. However, in cases in which both models are used, there are propositions to compare the results of cooperative and non-cooperative situations (KAO, 2019), refine the results (ZHA; LIANG, 2010) and assess the same problem from a distinct perspective (SUN et al., 2017).

The use of network models and the predominance of two-stage models followed a previously verified trend. The first verified objective was similar to isolated cooperative and non-cooperative approaches. Liang et al. (2006) and Liang, Cook, et al. (2008) must be highlighted within this context. Studies on this group are shown in blue in Figure 10.

Liang et al. (2006) presents a pioneer proposition because it is one of the first to discuss the impact of an intermediate variable on the efficiency of a two-stage model. The authors employed the CCR model to investigate cooperative and non-cooperative situations to consider the stages of a supply chain. In this sense, the authors propose to model the non-cooperative case by considering the leader-follower model, whereas a centralized model represents a cooperative one.

In the leader-follower structure, the leader is first evaluated, and then the follower is considered using information related to the leader's efficiency. In cooperative design, joint efficiency, modeled as the average of the seller's and buyer's efficiency scores, is maximized, and both stages are evaluated simultaneously (LIANG *et al.*, 2006). Owing to this characteristic, the approach is centralized because the aim is to improve the system's performance.

Liang, Cook, et al. (2008) proposed an extension of Liang et al. (2006) and discussed the relationships between the results of the centralized, leader-follower, and CCR models. The authors mathematically proved that the efficiency of cooperative and non-cooperative systems corresponds to the product of the efficiencies of the stages. Thus, it provides a way to measure system and stage efficiency.

Following these propositions, Liang, Cook and Zhu (2008) was expanded to consider certain aspects, uncertainty in discrete data (ESFANDIARI *et al.*, 2017), consideration of uncertainties through fuzzy models (YU & SU, 2017), and intermediate products and exogenous outputs in both stages (CHEN et al., 2018).

Gap$_9$: Given the difficulty in some cases to determine if the situation under analysis is cooperative or non-cooperative, more studies comparing different cooperative and non-cooperative models in the DEA-GT literature are necessary.

Coooperative and noncooperative approaches were also combined to investigate parallel network structures. Naini et al. (2013) considered the following three stages: two in parallel but in series with the third. The authors considered cooperation between parallel stages while competing with one in series to analyze the network structure.

Du et al. (2015) proposed a general model for purely parallel systems, whereas Xiong et al. (2018) considered a bidirectional interactive parallel system for resource allocation. The authors developed a centralized view for all DMUs and a leader-follower model between the internal stages.

The second objective was to investigate the issues related to cost and resource allocation. These studies are highlighted in green in Figure 10. Sun et al. (2017) proposed a model to allocate emission permits among companies, and Xiong et al. (2018) allocated resources among the parallel internal stages of DMUs. Chu et al. (2020) addressed fixed cost allocation in two-stage systems, Wu et al. (2020) investigated the proposition of pollution targets. Izadikhah et al. (2018) and Ma et al. (2020) investigated shared resources among stages in a two-stage system.

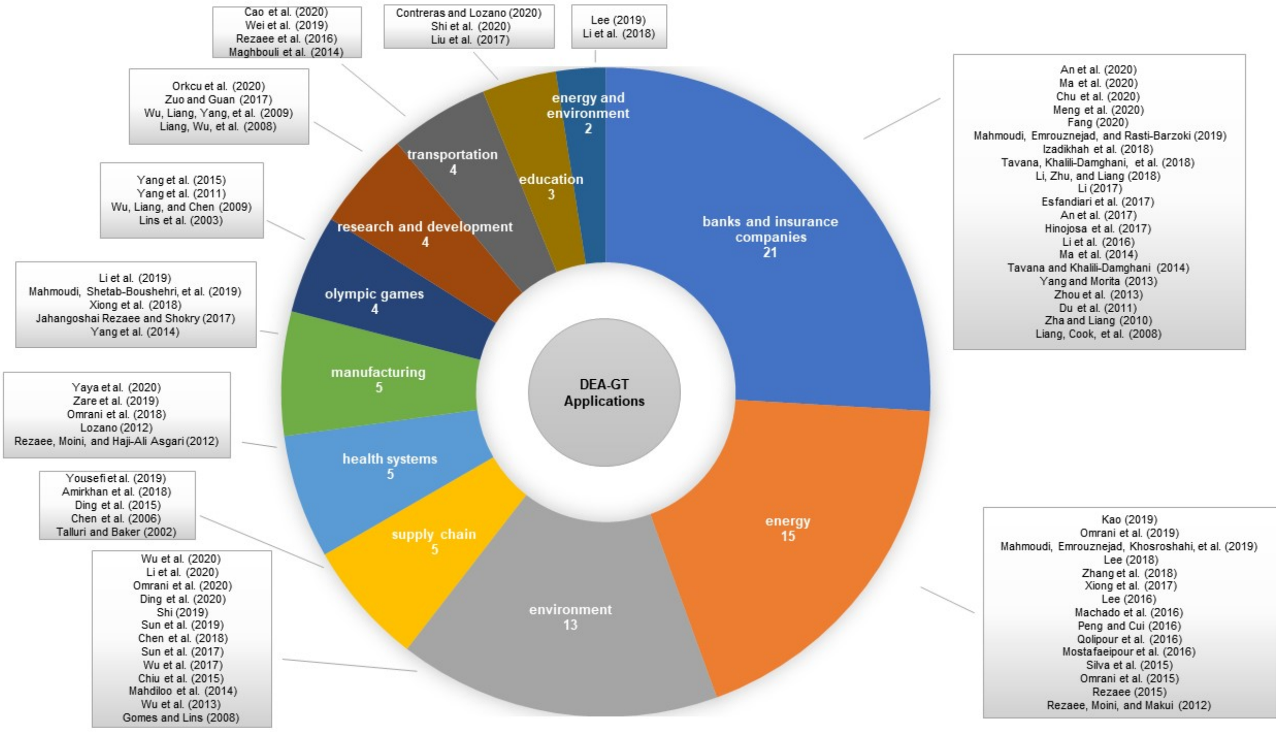
Gap$_{10}$: As most of the articles in the literature focus on investigating parallel and two-stage structures, there is a lack of studies that consider more complex structures, such as hybrid structures (with parallel and in series simultaneously) and the impacts of intermediate variables under cooperative and non-cooperative hypotheses.

2.4.3 Development of the DEA-GT applications

This subsection describes DEA-GT applications. Figure 12 shows the areas of the literature and their respective papers. It is essential to highlight that this subsection covers ninety-four papers. As discussed in Category 1, some studies are theoretical and do not consider their applications. Therefore, we don’t include them.

Almost 79% of the sample considered some type of application, and the high number of papers with applications is a trend verified in DEA literature (EMROUZNEJAD AND YANG, 2018; LIU et al., 2013a). However, in the DEA and GT segments, studies are mainly of the “theoretical-practical” type, indicating that the focus lies on the proposition of new methodologies or generating improvements in some aspects of existing ones.

Figure 12 - The most published real-world applications areas of DEA-GT approaches



Source: The Author (2024).

We identify over 20 areas, and Figure 12 shows the 11 areas with the highest number of publications. The main areas investigated in the DEA-GT field were banks, insurance companies, energy companies, and the environment. In addition, we have studies about ports, supply chain, environment, cities, sales, investments, multinational companies, R&D, forests, banks and energy, peer-to-peer platforms, hotels, portfolio, and paper industries.

Banks are among the most widely studied data envelopment analysis (DEA) and GT methods. This synopsis depicts a framework of partial alignment with DEA's single-application literature. This was confirmed by comparing the rankings obtained in the present study with those obtained by Liu et al. (2013a), who highlighted the following areas: banks, energy, health, supply chains, agriculture, farming, and telecommunications.

The other results are consistent with Liu et al. (2016) and Emrouznejad and Yang (2018). The authors identified energy as a rising area, which has been confirmed by the amount of research, many of which have been very recent. These studies also addressed the necessity and rise of research in the environmental sector, as confirmed by the current research results.

Some studies have addressed these issues in more than one area. This fact and the diversity of the identified areas indicate that DEA-GT methods are flexible and adaptable in different contexts. This finding corroborates the relevance of this segment in the literature. It is important to note that the two-stage models and the extension of the DEA-GT models combined with other techniques stand out in the main application areas despite their different application purposes.

2.4.4 Other techniques

In addition to the methodological discussions previously covered in this study, Classification 9 investigated the use of other techniques combined with DEA and GT. We chose to address this topic because of its relevance in developing robust methodologies to mitigate further problems. This also verifies that specific trends were noticeable. Figure 13 shows the details of the verified approaches.

Figure 13 - Other techniques applied with DEA and GT

Source: The Author (2024).

Thirty-four studies (28.5% of the sample) employed other approaches combined with DEA and GT. The distribution of these studies is as follows: 18 cooperatives, 12 non-cooperatives, and three that considered both approaches. The results indicated that fuzzy theory (six studies), Directional Distance Function (four studies), Balanced Scorecard (three studies), Assurance regions (three studies), and TOPSIS (three studies) were the most used techniques. This finding indicates that the combinations of other techniques have different purposes.

Sengupta (1992) was one of the first to use the Fuzzy Theory in the DEA context. The author developed a tolerance approach that inserted fuzziness into the DEA model because input-output vectors contain noise elements. The tolerance levels for both the objective function and constraint violations were considered.

Tavana and Khalili-Damghani (2014), Yu and Su (2017), and Tavana et al. (2018) used the fuzzy theory to investigate two-stage systems. The authors applied the Fuzzy theory to propose models that adequately deal with imprecise data. Tavana and Khalili-Damghani (2014) and Tavana et al. (2018) proposed approaches to investigate banks, whereas Yu and Su (2017) addressed issues regarding the carbon footprint. Shi et al. (2020) also considered the imprecision in the input and output data. The authors propose an alternative approach to address this issue in parallel systems.

On the other hand, Silva et al. (2015) investigated production strategies to decide the most profitable products to insert in the portfolio and identify portfolio products that are more sensitive to the occurrence of uncertainty.

We can infer that the use of Fuzzy Theory in the context of this study is directly related to data imprecision. This issue is mainly addressed in cases that investigate the internal structure of the DMUs. In real-world problems, the values of the variables (inputs, outputs, and intermediate measures) can be imprecise or vague. Ambiguous evaluations may result from unquantifiable, incomplete, and unobtainable information (TAVANA; KHALILI-DAMGHANI, 2014). This explains the proposition of various fuzzy methods to address the impreciseness and ambiguity of DEA (HATAMI-MARBINI; EMROUZNEJAD; TAVANA, 2011).

The second technique is the Function (DDF). DDF was proposed to estimate productivity and efficiency. These estimations can be performed under either parametric or nonparametric conditions. Determining the orientation for projecting an inefficient firm onto the frontier may significantly affect efficiency estimation results. Therefore, it is crucial to select an appropriate direction to measure the distance from an inefficient DMU to the frontier (LEE, 2016; WEI et al., 2019).

Regarding this issue, Lee (2016) considered Nash equilibrium as a direction for proposing Nash profit efficiency and its decomposition to investigate changes in market structures in imperfectly competitive markets. Lee (2018) extended the previous approach by considering a mixed-strategy Nash equilibrium to address uncertain competition in the same market structure.

In contrast to the previous two studies, Wei et al. (2019) proposed the use of DDF to improve cross-efficiency results. The authors considered a cross-bargaining game approach to direction selection. Each pair of inefficient DMUs in the group determines a common direction by bargaining to make the evaluation results more acceptable.

Li et al. (2020) applied the DDF to consider the free disposability of inputs and outputs. Both desirable and undesirable outputs were presented in a context requiring allocation during their development.

The third technique is the balanced scorecard (BSC). The BSC is a conceptual framework for translating strategic objectives into efficiency measures from four perspectives: financial, customer, internal process, and developmental. Mostafaeipour et al. (2016) and Qolipour et al. (2016) proposed a combination of DEA, GT, and BSC to properly incorporate environmental assessment into efficiency measures to perform a technical-economic evaluation to select sites for energy generation.

Rezaee and Shokry (2017) addressed different aspects of the BSC. In their proposition, each stage corresponds to a BSC perspective. The authors affirm that analysis based on BSC may encounter problems identifying inefficiency in resource usage and that BSC can be a valuable framework for organizing input/outputs in DEA models.

The assurance region method is identified as the fourth technique. This technique addresses weight flexibility in DEA. The flexibility of weights can be perceived as an advantage because, if a DMU is found to be relatively inefficient, it cannot be argued that the weighting structure used did not fairly represent the values of that DMU. However, on the other hand, low values for some output/input to exclude them from the assessment of the target DMU may not accurately reflect its performance (DYSON; THANASSOULIS, 1988).

Nakabayashi and Tone (2006) and Nakabayashi et al. (2009) applied assurance regions to incorporate preferences regarding criteria and determine the lower and upper bounds of the ratio of weights. (YANG; LIANG, 2015) used assurance regions to ensure that the weights of medals in Olympic games are related to the importance of each type of medal.

The last highlighted technique was TOPSIS. TOPSIS is a technique for order preference based on the similarity to an ideal solution. This multi-criteria decision analysis method is used to compare a set of alternatives, select the best option, calculate the geometric distance between each alternative, and sort them (OMRANI; FAHIMI; MAHMOODI, 2020). Mousavi-Nasab et al. (2019) and Omrani et al. (2020) applied TOPSIS to compare and validate the results of DEA and GT approaches. The choice of TOPSIS concerning other techniques is justified by the fact that TOPSIS is the best, most rational, and most popular method compared to other MCDM techniques in a selection problem (MOUSAVI-NASAB; SAFARI; HAFEZALKOTOB, 2019).

Mahmoudi et al. (2019) combined several techniques. For example, cluster analysis, PCA, Shannon entropy, and TOPSIS can rank DMUs fully when their numbers are insufficient.

TOPSIS was applied in the final stages of the procedure to classify DMUs based on the relative closeness coefficient of this technique.

Based on the above, it is possible to verify that using some of the highlighted techniques constitutes a natural extension of the literature to deal with known problems in DEA but has not yet been fully explored in new models or contexts. For example, using fuzzy theory to investigate uncertainties in two-stage network models and assurance regions to incorporate bounds and preferences into weights supports this statement.

By contrast, propositions considering DDF, BSC, and TOPSIS represent alternative approaches to performing alternative analyses, such as selecting inputs and outputs, validating the combination of DEA and GT, and ranking efficient DMUs. Thus, combining DEA and GT with other techniques allows us to address problems beyond the direct measure of efficiency.

> **Gap$_{11}$**: Combining DEA with GT and other techniques makes it impossible to use standard tools developed to solve DEA models. Thus, free platforms where these models are available to potential users can help disseminate knowledge and practice these techniques.

2.5 CONCLUSION REMARKS AND DISCUSSIONS

The evolution of the DEA and Game Theory field was analyzed in the current chapter by a systematic review of the literature. A considerable number of published articles, including significant breakthroughs in theory and a great diversity of papers on DEA applications, both in the public and private sectors were found and represent a clear opportunity to apply models designed to solve real problems and enrich the DEA literature (EMROUZNEJAD; YANG, 2018; LIU et al., 2013a; MARIZ; ALMEIDA; ALOISE, 2018).

119 papers indexed in Scopus and Web of Science were screened in nine created categories, ranging from the type of study, DEA, and game theory models to their application in different market niches.

The results showed that 79% of the studies were practical or theoretical-practical. Compared to other reviews such as Liu et al. (2013a) and Emrouznejad and Yang (2018), there is a unity of results highlighting theoretical-practical publications, indicating a focus on validating models, a wide range of applications, and rapid growth. It is also discovered that the beginning of this literature branch is marked by studies analyzing how game propositions can achieve similar results to DEA. Later, the techniques were combined to mitigate DEA limitations and better represent the DMUs.

We filled a void in the DEA literature by surveying DEA and GT applications because to the author's knowledge an in-depth investigation considering cooperative and non-cooperative studies had not yet been proposed. Our review addresses researches in the DEA and GT domains and explores the existing knowledge.

We divide our analysis based on the implemented game approach. Studies have found that the most frequent objectives are efficiency measurement and decomposition, resource and cost allocation, and alternatives to improve DEA performance. We verified the predominance of the NDEA, cross-efficiency, and classical models by considering DEA models. In contrast, the GT field must highlight Nash bargaining games, leader-follower games, zero-sum games, and imputation techniques.

The research identified several areas of applications, banks and insurance companies, energy, and the environment as the most investigated areas. It is possible to notice that more cooperative approaches were applied. However, it is essential to note the growth of studies that contrapose cooperative and noncooperative approaches.

Regarding other techniques combined with GT and DEA, we verify that clustering algorithms, fuzzy theory, assurance region, principal component analysis, directional distance function, and balanced scorecard were the most used. This amplitude of techniques indicates a search for superior methods aiming to represent the objects of study more faithfully. Consequently, it improves management considerations and enhances the decision-making processes.

Nevertheless, the interpretation of the survey results should consider the following limitations. The sample used did not constitute a universe. The Web of Science and Scopus constitute two significant databases, but they do not cover all studies published to date. However, the selected papers represented this area because of their quality, publication journals, and depth of analysis.

This area remains in development and has been recognized in the literature as an alternative for conducting more complex analyses. We expect the present systematization to allow future developments and a better understanding of this research field's theoretical and practical aspects. One important aspect that deserves attention is the direct comparison between the exclusive use of DEA and the hybrid approaches. This aspect has been overlooked and also aids in highlighting the relevance of the combined propositions.

The proposed classification and the division of analysis allowed the identification of gaps related to the DEA and GT literature. We highlight eleven gaps, and we present some propositions for future research based on the gaps highlighted in the study:

Application-centered studies: This type of study can provide insights into the model's performance in different research areas and provide a more in-depth analysis of sectors that have not yet been explored.

Comparison of cooperative and non-cooperative models: Studies that consider both approaches can provide comparisons and indicate the impact on the efficiency score, situations, and areas in which these approaches are more suitable.

The proposition of more complex network models: Development of models that consider a higher number of stages and more complex network structures is necessary. They allow the identification of the impact of intermediate variables. They can provide directions for solving more complex models and using other techniques to obtain unique solutions. They are likely to be nonlinear and cannot always assure globally optimal solutions.

Dynamic performance measurement: The effect of time variation on efficiency is discussed in the literature. This dynamic aspect naturally leads to repeated games to observe players' strategies in each period under analysis. The Malmquist index, dynamic DEA models, dynamic DEA models with a network structure, and window analysis are alternatives to perform such an analysis combined with Game Theory.

Game solutions: Most studies have aimed to obtain the Nash equilibrium as a solution for the game. However, this is not the only alternative available in the literature. Getting different solutions and presenting them to decision-makers to compare and analyze which one they prefer can provide insights into which type of solution is more suitable for a specific situation.

Increase in DEA discrimination: There is a gap in the literature concerning the relationship between the number of variables used in the model and the number of investigated DMUs to ensure the discrimination of the DEA results. Therefore, discussing a "thumb-rule" for network, dynamic and dynamic network models is very important to the DEA field.

Breakdown points: Because breakdown points are a parameter for several DEA-GT models, a sensitivity analysis is fundamental, and with the development of propositions that consider a higher number of stages, an in-depth discussion of breakdown points becomes even more critical.

Internal DMU's hypothesis: In the current literature, the papers adopted the hypothesis of cooperation or non-cooperation between the internal stages of the DMUs. An in-depth discussion to contemplate cooperation and competition scenarios between the DMUs and their internal stages consists of a possible research extension.

Considering the previous discussion, the thesis aims to propose models to address the Gaps 3 and 4. Dynamic models, the insertion of network aspects into efficiency measures and the presence of shared inputs in the networks will be discussed in depth in the following chapters (Chapter 3 and 4).

3 RESOURCE ALLOCATION WITH SHAPLEY VALUE AND NUCLEOLUS IN A DYNAMIC NETWORK SETTING

This chapter aims to explore the discussion regarding resource sharing in an internal network and measure the potential benefits of resource sharing in this type of system. A Data Envelopment Analysis and Cooperative Game Theory combined approach is developed to aid in quantifying the benefits obtained from this sharing and propose a fair division of them to promote cooperation.

3.1 CONTEXTUALISATION

The sharing of resources - labor, machinery, capital, and raw materials - is common among partners in supply chains. This type of sharing can also be present among multiple linked stages of a single network system, such as a bank or a university (AN; WEN; CHU; et al., 2019; AN; WEN; DING; et al., 2019).

It is also relevant to notice that when we pool and share common resources, it improves the performance, efficiency, and profits due to the complementarity of resources (AN; WEN; DING et al., 2019). In addition to complementarity, this sharing can promote synergy; the merger between these resources can create more than before.

Several approaches are available in the literature to measure resource allocation performance, such as linear programming models, game-theoretical techniques, Stochastic Frontier Approach, and Data Envelopment Analysis. Specifically, Charnes et al. (1978) initially proposed Data Envelopment Analysis (DEA), and it consists of a non-parametric method for measuring the efficiency of decision-making units (DMUs) contemplating multiple inputs and outputs.

However, in many situations, the internal processes that define the DMU are more complex, and the assumption that all inputs impact all outputs should be abandoned (IMANIRAD; COOK; ZHU, 2013). Shared flows, multi-levels, and network models are among the DEA models that allow consideration of the internal structure of the DMUs (CASTELLI; PESENTI; UKOVICH, 2010).

The development of models with network structures stands out among the most relevant research fronts in the DEA literature (LIU; LU; LU, 2016). Network DEA models (NDEA) aim to account for divisional efficiencies and the overall efficiency in a unified framework (TONE; TSUTSUI, 2009b).

Management by performance gains relevance to best utilize restricted resources ancan consider three basic types in the internal structure of the DMUs in the NDEA model: series, parallel, and mixed (KAO, 2009).

Observe that in NDEA, there is an assumption that all inputs affect all outputs and intermediate outputs of the respective stage. However, in the literature on network modeling, some studies analyze different assumptions from those previously mentioned. It is possible to identify investigations concerning shared inputs between stages (MA; QI; DENG, 2020), shared outputs (LI et al., 2016), common inputs (AVILÉS-SACOTO et al. 2020), and also partial inputs to partial outputs (IMANIRAD *et al.*, 2015).

The diversity of considerations demonstrates the need for flexibility in modeling to represent and evaluate the network's specificities accurately. The complementarity of resources and synergy can explain the increasing number of publications relating to network models and resource sharing.

Dynamic DEA models with network structure (DNDEA) also check aspects related to the internal processes of DMUs and observes the dynamic change of the period efficiency and the dynamic change of the divisional efficiency of DMUs (TONE; TSUTSUI, 2014). The development of this research front contemplates approaches of different specificities, such as fuzzy inputs and outputs (SOLTANZADEH; OMRANI, Hashem, 2018), super SBI-efficiency (MORENO; LOZANO, Sebastián, 2018), non-homogeneous DMUs (YAN et al. 2019), and common weights (GHARAKHANI *et al.*, 2018). However, the analysis of resource sharing in this modeling is still developing in the literature.

In the literature, we can find few practical examples of resource sharing in DNDEA. Chao et al. (2018) proposed a two-stage situation to analyze the performance of container shipping developed a model to deal with uncertainty in banking systems. The authors considered a three-stage proposition with shared resources (employee salaries and fixed assets) and aimed to find optimal proportions of the shared resources. An et al. (2020) presented a verification concerning the Chinese high-tech industry in a framework that shares inputs and outputs between stages.

This chapter develops a DNDEA model that considers shared resources among the three stages of the network structure over consecutive periods. We contemplate a cooperative situation between the stages since they belong to the same DMU. Also, fair allocation of benefits is essential to promote cooperation between stages, mainly in an internal network structure because the production of one stage may affect the production of other stages (AN;

WEN; DING.; et al., 2019). Therefore, these two factors justify the choice for cooperative games approaches.

The proposition develops a DNDEA model to calculate optimal profits for each DMU before and after resource sharing. Then, payoff allocation methods are used to obtain its distribution between the DMU's stages. Different methods are present in the literature to perform this procedure, Shapley value (SHAPLEY, 1953) and Nucleolus (SCHMEIDLER, 1969) are the selected ones in this study.

The base of these methods is the concept of fairness, which is perceived differently by them. The Shapley Value considers each player's average contributions in different coalitions. The Nucleolus finds the degree of dissatisfaction of one coalition when a particular allocation is realized (LOZANO et al., 2016). Therefore, we used both to compare its results.

The current development promotes three main methodological contributions. First, we investigate the resource sharing benefits in a DNDEA framework, which distinguishes it from others since most papers that applied DEA to investigate resource sharing used network models but disregarded the temporal impacts on efficiency. Second, this study proposition differs from others because it does not consider optimal proportions but assumes that the amount of shared input is known for each stage. Third, this development also fills a gap for the conjoint applications of DEA and Game Theory models because a three-stage system is analyzed, and allocation rules are applied to allocate fairly the benefits obtained due to resource sharing. Kao (2014) points out that game-theoretical DEA models are limited to just two players, requiring developments for multiple players in specific network structures. The confirmation that resource sharing can yield profit enhancement when the full-time period is considered presents an alternative source for profit increase. To our knowledge, using cooperative game theory concepts to fairly allocate potential profits obtained by pooling resources in the DNDEA framework is a pioneer proposition in the literature.

The chapter is organized in subsections as follows. Studies and relevant theoretical discussion regarding DNDEA, resource sharing and combined approaches of DEA and cooperative Game Theory are presented in subsection 3.2 and 3.3. Subsection 3.4 explains the proposed method. Then, a numerical example and its results are presented in Subsection 3.5 and final considerations are draft in Subsection 3.6.

3.2 THEORECTICAL BACKGROUND IN NETWORK AND DYNAMIC DEA MODELS

Network and dynamic DEA modelings have been present in the literature for a certain period and have received prominence due to the insights they provide. However, the expansion of dynamic considerations for network systems is a challenging topic to be explored (AVKIRAN; MCCRYSTAL, 2013; KAO, 2014).

The network modeling consists of the theoretical basis necessary for analyzing the internal structure of the DMUs (LOZANO, 2017), whereas the dynamic model provides an understanding of the relationships between periods through carry-over variables (KAWAGUCHI; TONE; TSUTSUI, 2014; MARIZ; ALMEIDA; ALOISE, 2018).

DNDEA models can adequately represent the reality of DMUs through multiple dynamic stages connected by network structure links in each period analysis and this structure involves interacting with a finite number of static models (FÄRE et al., 2014; CHAO et al., 2015). The juxtaposition of these models allows conducting a thorough analysis by observing the change in the efficiency of the period, the dynamic modification of the divisional efficiency of the DMUs, possible improvements and efficiency estimates arising from a more comprehensive analysis where interactions between periods and between divisions are considered (TONE; TSUTSUI, 2014b).

The different DNDEA models found in the literature make it possible to open the black box of traditional DEA models and the considerations of the internal heterogeneous organizations of the DMUs, in which the divisions are mutually connected by link variables and by the internal exchange of intermediate products (KAWAGUCHI; TONE; TSUTSUI, 2014; KHUSHALANI; OZCAN, 2017).

The study of Chen (2009) consists in a pioneer proposition and incorporates the dynamic effects in the network structure to initially calculate the efficiency of the SDMU's and then the system's efficiency. This approach is called efficiency measure ψ, representing the minimum requirement for aggregate inputs concerning the final aggregate production in the period (CHEN, 2009).

After this study, it is possible to verify some theoretical developments, such as Liu et al. (2011), Chen (2012), and Li and Wang (2015). In addition to theoretical studies, different theoretical-practical propositions are verified in distinct application segments: health (AVKIRAN; MCCRYSTAL, 2013, 2014; GAVUROVA; KOCISOVA; SOPKO, 2021; LOBO et al., 2016; SEE; HAMZAH; YU, 2021), banks (CHAO; YU; WU, 2015; FUKUYAMA; WEBER, 2015, 2016; KWEH et al., 2018), transportation (BAI-CHEN; YING; QIAN-QIAN, 2012; LOSA et al., 2020; OMRANI; SOLTANZADEH, 2016; SOLTANZADEH; OMRANI, 2018), education (FUKUYAMA; WEBER; XIA, 2016a; TRAN, C. D. T. T.; VILLANO, R. A., 2021; TRAN, C. T. T. D., 2021) and energy (LI, Lin *et al.*, 2016; TONE; TSUTSUI, 2014b; YOU; JIE, 2016).

The diversity of theoretical and practical studies also allows observing the proposition of new models based upon different foundations of the DEA literature. Avkiran and Mccrystal (2013) used the concept of range-adjusted measure. Tone and Tsutsui (2014) applied slack-based measures (SBM) in their development. Omrani and Soltanzadeh (2016) considered the properties of relational models of Kao (2009) and Kao (2013). Moreno and Lozano (2018) developed a super slack-based inefficiency model. Kalantary and Farzipoor Saen (2019) proposed an inverse SBM model. More recently, an additive model nested within a slacks-based measure (SBM)(CHANG; TONE; WU, 2021) and the employment of directional distances in a multiplicative model (LIN; LIU, 2021; ZHANG et al., 2021) represent new alternatives in the DNDEA framework.

In addition to models based on different DEA concepts, there are also developments proposing alternatives to mitigate DEA limitations or to address specificities: models to deal with uncertainty in the inputs (SOLTANZADEH AND OMRANI, 2018), for sharing inputs (CHAO; YU; HSIEH, 2018), for non-homogeneous DMUs (YAN et al., 2019), and using common weights for efficiency measuring (GHARAKHANI et al., 2019).

This range of modeling, applications, and specificities demonstrates that, although recent, this branch of literature has received much attention and prominence among practitioners and researchers. One of the probable reasons for this is the set of information provided by the models. They identify the inefficiencies more precisely and provide metrics (slacks and projections) to improve efficiency.

Also, it is possible to point out some developments concerning shared inputs in the DNDEA framework. Yu et al. (2016) consider sharing inputs between stages in the context of multi-activities for evaluating bus traffic. Chao et al. (2018) are based on modeling by Yu et al. (2016) to consider multiple shared inputs to evaluate shipping container companies. On the other hand, An et al. (2020) expand the previous considerations when considering simultaneously sharing inputs and outputs and the lagged effects of consumption of inputs in a production system.

Despite the diversity of investigations and applications, studies covering the theme of resource sharing in DNDEA modeling are still scarce in the literature. We aim to mitigate this gap by combining a DNDEA model with cooperative games, an unprecedented literature proposal.

3.3 RESOURCE SHARING

Resource sharing may bring potential gains, and it exists not only among independent entities but also within a network structure system with multiple stages (AN; WEN; DING; et al., 2019). Beasley (1995) was a precursor in the investigation of shared resources. The author considered chemistry and physics departments in the United Kingdom and the common resources used by the departments' research and teaching activities. The study proposes a non-linear model based on DEA to measure the department's efficiency considering the two activities together.

Cook et al. (2000) investigated how to perform multi-component efficiency measurement for Canadian banks. The authors considered that bank branches divided their resources between two main processes (services and sales) and proposed a model for the agency's efficiency to be measured given all the activities taken in place. Cook and Hababou (2001) extended this model through an additive DEA model and goal programming combination. This juxtaposition aims to consider non-volume-related activities and obtain the optimal division of shared resources. Jahanshahloo et al. (2004) perform an analysis similar to Cook and Hababou (2001) when considering multiple components to measure efficiency. All parts are involved in producing some outputs and also consider non-discretionary factors.

These studies demonstrate the relevance of considering the internal aspects of the DMU for the accurate measurement of efficiency. This need propels the development of new models. Among these, network models initially proposed by Färe and Grosskopf (2000) must be highlighted.

There is a diversity of models with a network structure that contemplates inputs shared between the stages. Zha and Liang (2010) consider two serial stages that allocate inputs freely. The authors implement a non-cooperative game theory approach to define the limits of stage efficiencies. In a later stage, a heuristic solves the cooperative model.

Chen et al. (2010) also verify the sharing of inputs in two-stage serial network models and propose a new approach to address intermediate products' issues and share non-divisible inputs adequately. The authors state that models such as Cook and Hababou (2001) would not be appropriate because the DMU does not have an internal network structure. Also, new models are required since adaptations of other network models, such as Kao and Hwang (2008), would make the approach highly non-linear. Consequently, there is no guarantee of a globally optimal solution.

Ma (2015) extends the development of Cook and Hababou (2001) to simultaneously consider shared resources and the portion of the intermediate product used by the next stage. Bian et al. (2015) carry out investigations similar to those of Ma (2015), focusing on allocating optimal proportions of shared inputs in systems with parallel stages.

Wu et al. (2017) investigated the total-factor energy efficiency of Chinese industries with two types of inputs, energetic and non-energetic. The study verifies the sharing of non-energy inputs between the energy use and pollution treatment stages to identify the optimal allocation. Their model also considers particularities, such as specific inputs in the second stage and specific and undesirable outputs.

An, Wen, Ding, et al. (2019) and An, Wen, Chu, et al. (2019) assess resource sharing in three-stage serial network models. An, Wen, Ding, et al. (2019) focuses on benefits allocation obtained by the network stages when sharing resources. An, Wen, Chu, et al. (2019) deepens into the decomposition of profit inefficiency for a three-stage serial system. The authors decompose the overall profit inefficiency into the product of technical profit inefficiency, resource sharing profit inefficiency, and free allocation profit inefficiency.

It is important to note that the studies by An, Wen, Ding, et al. (2019) and An, Wen, Chu, et al. (2019) differ from the others previously mentioned, since they do not calculate optimal proportions for the allocation of these resources to each of the stages. Their studies assume that the quantities of inputs for each stage are known and can have benefit from pooling these inputs together.

As previously mentioned, investigations of resource sharing for DNDEA models are still nascent in the literature. We can see some developments in Yu et al. (2016), Chao et al. (2018), Zhou et al. (2019), and An et al. (2020). Yu et al. (2016) developed a multi-activity dynamic model with a network structure. The model-based its considerations on the propositions of Tone and Tsutsui (2014) and Yu and Lin (2008). Shared inputs flow between two of the three investigated stages in the network. The object of analysis is 20 bus transit firms in Taiwan over three years.

Chao et al. (2018) proposed a model to investigate shipping companies in three years. The authors applied a model to assess the efficiency of thirteen of the largest companies in the sector: two divisions, operational and marketing, share expenses, and employees. The authors point out that the modeling measures the companies' efficiency stages efficiency while also providing the optimal proportions for allocating the shared resources.

Zhou et al. (2019) developed a model to deal with uncertainty in the banking industry. The authors considered a three-stage proposition with shared resources between all of them. The study is based on the hypothesis that the proposed model is solved by discovering optimal proportions of the shared resources. To deal with uncertainty, the authors combined an SBM DNDEA proposition with Fuzzy Theory to investigate the three stages, capital organization, capital allocation, and capital profitability.

An et al. (2020) presented a verification of the Chinese high-tech industry in a three year-analysis. The framework considers simultaneously shared inputs, shared outputs, and the lagged effects of input consumption in a production system. We consider two stages, technology research and development and technology digestion and absorption, to investigate 29 Chinese regions.

The current proposal investigates resource-sharing issues differently from the previous DNDEA proposals. We aim to combine cooperative aspects of game theory to allocate the benefits of increasing DMU's profit by sharing resources within the DMU's internal network structure. Through this proposition, we developed a distinct DNDEA model that investigates the increase in profit and evaluates all possible collaborations between the stages, in addition to calculating and allocating these benefits using the Shapley value and Nucleolus.

3.4 MODEL STRUCTURE

3.4.1 A framework for resource sharing in a Dynamic model with a three-stage network

Table 7 details the indexes, the parameters, and the decision variables of the model, while Figure 14 details the network structure considered in the study.

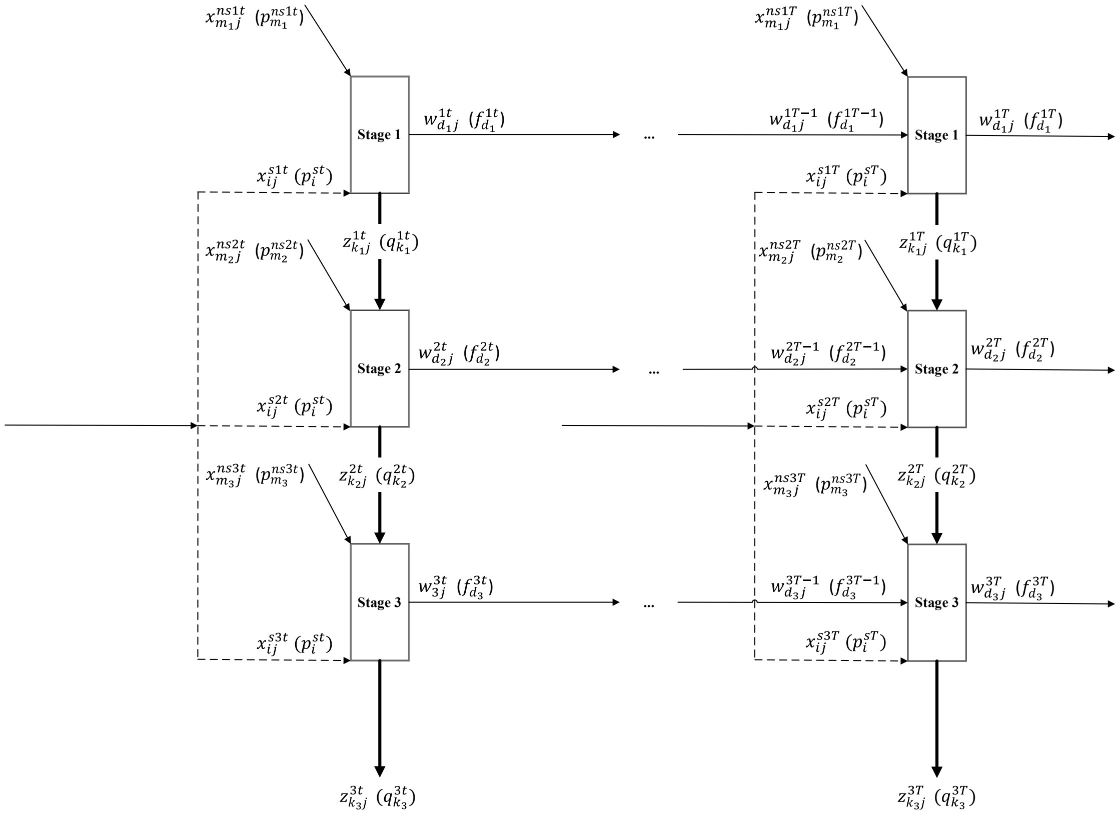
Table 7 - Indexes, parameters, and variables of the model

Indexes	
$j = 1, \ldots, n$	Index of the *j*th DMU index;
$t = 1, \ldots, T$	Index of the *t*th period;
$l = 1, \ldots, L$	Index of *l*th stage;
$i = 1, \ldots, I$	Index of the *i*th shared input between stages;
$m_l = 1, \ldots, M_l$	Index of the *m*th specif input of stage l;
$k_l = 1, \ldots, K_l$	Index of the *k*th output produced by the stage l;
$d_l = 1, \ldots, D_l$	Index of the *d*th carry-over connecting stage *l* between periods;
Parameters	
$p_i^{st}/p_{m_l}^{nslt}/q_{k_l}^{lt}/f_{d_l}^{lt}$	Unit price of the *i*th shared input i in period *t*/ the m*th* specif input of stage l in period *t* / the k*th* output of the stage *l* in period *t* / the d*th* carry-over connecting stage *l* between consecutive periods *t* and *t+1*;
$x_{ij}^{slt}/x_{m_lj}^{nslt}/z_{k_lj}^{lt}/w_{d_lj}^{lt}$	The *i*th shared input of DMU *j* for stage l in period t/ the m_l specif input of DMU *j* for stage l in period t/ the k*th* output of DMU j for stage l/ the d*th* carry-over connecting stage *l* between consecutive periods;
S	coalition
Variables	

$\rho_j^{lt}/\lambda_j^{lt}$	Index of multiplier variable corresponding to the stage l in period t of DMU j before/after resource sharing;
$x_{ij}^{\sim slt}/x_{m_l j}^{\sim nslt}/z_{k_l j}^{\sim lt}/w_{d_l j}^{\sim lt}$	The index of ith shared input of DMU j for stage l in period t/ the m_l specif input of DMU j for stage l in period t/ the kth output of DMU j for stage l/ the dth carry-over connecting stage l between consecutive periods in the optimal situation before resource sharing;
$x_{ij}^{*slt}/x_{m_l j}^{*nslt}/z_{k_l j}^{*lt}/w_{d_l j}^{*lt}$	The index of ith shared input of DMU j for stage l in period t/ the m_l specif input of DMU j for stage l in period t/ the kth output of DMU j for stage l/ the dth carry-over connecting stage l between consecutive periods in the optimal situation after resource sharing.

Source: The Author (2024).

Figure 14 - A proposed dynamic model with a three-stage network structure



Source: The Author (2024).

3.4.2 Analysis of the pre-collaboration between stage

A three-stage serial structure considering multiple periods is considered. In DNDEA framework, each stage has specific inputs ($x_{m_l j}^{nslt}$), and they can pool together similar shared inputs denoted as x_{ij}^{slt} . Stages are observed from period 1 to T. Carry-over ($w_{d_l j}^{lt}$) variables are responsible for connecting two consecutive periods. The different stages produce outputs consumed by the following stages (outputs are links ($z_{k_l j}^{lt}$)); in the last stage, we have the system's output.

It is considered that enterprises pursue profit maximization rather than cost minimization or revenue maximization (AN, WEN, DING, et al., 2019). Model (1) obtains the profit before collaborating between stages (P_{pre}). This first model will be the benchmark for the next step, and it must be solved *n* times.

The objective function maximizes the system's profits considering all DMU stages and all periods under evaluation. Constraints (1.1) to (1.6) ensure that the results are within the production possibility set. They are presented in a generic way to allow different types of links and carry-overs, as presented in the definition of the production set of Tone and Tsutsui (2014). The authors developed this generic representation to consider four categories of carry-overs (good, bad, fixed, and free) and links (inputs, outputs, free and non-discretionary).

Model (1) is used to obtain the profit before the collaboration between stages (P_{pre}). This first model will be the benchmark for the next step, and it must be solved n times.

$$\max\ P_{preo} = \sum_{t=1}^{T}\sum_{k_3=1}^{K_3} q_{k_3}^{3t} z_{k_3o}^{\sim 3t} + \sum_{l\epsilon\{1,2,3\}}\sum_{d_l}^{D_l} f_{d_l}^{lT} w_{d_lo}^{\sim lT} - \sum_{t=1}^{T}\sum_{l\epsilon\{1,2,3\}}\sum_{i=1}^{I} p_i^{st} x_{io}^{\sim slt} - \sum_{t=1}^{T}\sum_{l\epsilon\{1,2,3\}}\sum_{m_l=1}^{M_l} p_{m_l}^{nslt} x_{m_lo}^{\sim nslt}$$

Subject to:

$x_{io}^{\sim slt} \geq \sum_{j=1}^{n} \rho_j^{lt} x_{ij}^{slt};\ i = 1,\dots,I;\ l = 1,\dots,L;\ t = 1,\dots,T$ (1.1)

$x_{m_lo}^{\sim nslt} \geq \sum_{j=1}^{n} \rho_j^{lt} x_{m_lj}^{nslt};\ m_l = 1,\dots,M_l;\ l = 1,\dots,L;\ t = 1,\dots,T$ (1.2)

$z_{k_lo}^{\sim lt} \geq=\leq \sum_{j=1}^{n} \rho_j^{lt} z_{k_lj}^{lt};\ k_l = 1,\dots,K_l;\ l = 1,\dots,L;\ t = 1,\dots,T$ (1.3)

$z_{k_{(l-1)}o}^{\sim (l-1)t} \geq=\leq \sum_{j=1}^{n} \rho_j^{lt} z_{k_{(l-1)}j}^{(l-1)t};\ k_{(l-1)} = 1,\dots,K_{(l-1)};\ l = 2,\dots,L;\ t = 1,\dots,T$ (1.4)

$w_{d_lo}^{\sim lt} \geq= \leq \sum_{j=1}^{n} \rho_j^{lt} w_{d_lj}^{lt};\ d_l = 1,\dots,D_l\ l = 1,\dots,L;\ t = 1,\dots,T-1$ (1.5)

$w_{d_lo}^{\sim lt-1} \geq=\leq \sum_{j=1}^{n} \rho_j^{lt} w_{d_lj}^{lt-1};d_l = 1,\dots,D_l\ l = 1,\dots,L;\ t = 1,\dots,T-1$ (1.6)

$x_{io}^{\sim slt} \leq x_{io}^{slt};\ i = 1,\dots,I;\ l = 1,\dots,L;\ t = 1,\dots,T;$ (1.7)

$x_{m_lo}^{\sim nslt} \leq x_{m_lo}^{nslt};\ m_l = 1,\dots,M_l;\ l = 1,\dots,L;\ t = 1,\dots,T;$ (1.8)

$z_{k_lj}^{\sim lt} \geq z_{k_lj}^{lt};\ k_l = 1,\dots,K_l;\ l = 1,\dots,L;\ t = 1,\dots,T;$ (1.9)

$w_{d_lj}^{\sim lt} \geq w_{d_lj}^{lt};\ d_l = 1,\dots,D_l;\ l = 1,\dots,L;\ t = 1,\dots,T;$ (1.10)

$\sum_{j=1}^{n} \rho_j^{lt} = 1;\ l = 1,\dots,L;\ t = 1,\dots,T;$ (1.11)

$\rho_j^{lt}, x_{ij}^{\sim slt}, x_{m_lj}^{\sim nslt}, z_{k_lj}^{\sim lt}, w_{d_lj}^{\sim lt} \geq 0;\ l = 1,\dots,L;\ t = 1,\dots,T;$ (1.12)

This general proposition considers the different types of carry-overs and links and provides flexibility to portray different scenarios, extending the methods to several research fronts. The restrictions (1.3) and (1.4) refer to the links: the first contemplates the link performance as outputs of division k in period t. At the same time, (1.4) represents the performance as an input in the subsequent division in period t. Restrictions (1.5) and (1.6) refer to carry-overs: the first refers to the role as carry-overs from period t, (1.6) as carry-overs to period t + 1.

Constraints (1.7) and (1.8) ensure, respectively, that the values of the shared and exclusive inputs do not exceed the initial values. Constraints (1.9) and (1.10) ensure that the output levels of the stages (outputs and carry-overs) are not lower than the current ones. The constraint (1.11) assumes variable returns to scale and can be removed from the model to account for constant returns to scale. Finally, (1.12) ensures the non-negativity of the values.

3.4.3 Post collaboration and coalitions

We use a combination of DEA and cooperative games to analyze the benefits of collaboration between the stages. Each of the three stages consists of a player, and defining three players makes it possible to identify possible forms of cooperation between the stages. These arrangements are called coalitions (S).

Then, we observe coalitions of one player ({1},{2},{3}), two players ({1,2}, {1,3}, {2,3}), and the grand coalition {1,2,3}. To analyze the gains arising from the sharing of resources between the stages, the characteristic function $[v(S)]$ is defined as the difference between the profits obtained by each DMU after and before the resource sharing considering the different coalitions created by the stages $[v(S) = P_{post}^{S} - P_{pre}]$.

The single-player coalitions represent the initial situation in which the players act in isolation without sharing resources. Since the characteristic function measures the difference between the profits from resource sharing and the stages working alone, the coalition of one player does not obtain any benefit: therefore, $v(1) = v(2) = v(3) = 0$.

In the case of two-player coalitions, we can see that there are three possible combinations S = ({1,2}, {1,3}, {2,3}). Model (2) illustrates how to calculate profit for the coalition {1,2}. The procedure for the remaining coalitions ({1,3}, {2,3}) is similar.

$$\max\ P_{post}^{[1,2]} = \sum_{t=1}^{T}\sum_{k_3=1}^{K_3} q_{k_3}^{3} z_{k_3o}^{*3t} + \sum_{l\epsilon\{1,2,3\}}\sum_{d_l}^{D_l} f_{d_l}^{lT} w_{d_lo}^{*lT} - \sum_{t=1}^{T}\sum_{l\epsilon\{1,2,3\}}\sum_{i=1}^{I} p_i^{st} x_{io}^{*slt}$$
$$- \sum_{t=1}^{T}\sum_{l\epsilon\{1,2,3\}}\sum_{m_l=1}^{M_l} p_{m_l}^{nslt} x_{m_lo}^{*nslt}$$

Subject to:

$x_{io}^{*slt} \geq \sum_{j=1}^{n} \lambda_j^{lt} x_{m_lj}^{\sim slt};\ i = 1, \dots, I;\ l = 1, \dots, L;\ t = 1, \dots, T$ (2.1)

$x_{m_lo}^{*nslt} \geq \sum_{j=1}^{n} \lambda_j^{lt} x_{m_lj}^{\sim nslt};\ m_l = 1, \dots, M_l;\ l = 1, \dots, L;\ t = 1, \dots, T$ (2.2)

$z_{k_lo}^{*lt} \geq=\leq \sum_{j=1}^{n} \lambda_j^{lt} z_{k_lj}^{\sim lt};\ k_l = 1, \dots, K_l;\ l = 1, \dots, L;\ t = 1, \dots, T$(2.3)

$z_{k_{l-1}o}^{*l-1t} \geq=\leq \sum_{j=1}^{n} \lambda_j^{lt} z_{k_{l-1}j}^{\sim l-1t}; k_l = 1, \ldots, K_l; l = 2, \ldots, L; t = 1, \ldots, T$ (2.4)

$w_{d_l j}^{*lt} \geq=\leq \sum_{j=1}^{n} \lambda_j^{lt} w_{d_l j}^{\sim lt}; d_l = 1, \ldots, D_l; l = 1, \ldots, L; t = 1, \ldots, T-1$ (2.5)

$w_{d_l j}^{*lt-1} \geq=\leq \sum_{j=1}^{n} \lambda_j^{lt} w_{d_l j}^{\sim lt-1}; d_l = 1, \ldots, D_l; l = 1, \ldots, L; t = 1, \ldots, T-1$ (2.6).

$\sum_{l\epsilon\{1,2\}} x_{io}^{*slt} \leq \sum_{l\epsilon\{1,2\}} x_{io}^{\sim slt}$ $i = 1, \ldots, I; t = 1, \ldots, T$ (2.7)

$x_{io}^{*slt} \leq x_{io}^{\sim slt}; i = 1, \ldots, I; t = 1, \ldots, T; l = 3$ (2.8)

$x_{m_l o}^{*nslt} \leq x_{m_l o}^{\sim nslt}; m_l = 1, \ldots, M_l; l = 1, \ldots, L; t = 1, \ldots, T$ (2.9)

$\sum_{j=1}^{n} \lambda_j^{lt} = 1$ $l = 1, \ldots, L; t = 1, \ldots, T$; (2.10)

$\lambda_j^{lt}, x_{ij}^{*slt}, x_{m_l j}^{*nslt}, z_{k_l j}^{*lt}, w_{d_l j}^{*lt} \geq 0$ $l = 1, \ldots, L; t = 1, \ldots, T$; (2.11)

This model uses the optimal values obtained from Model 1 ($x_{m_l j}^{\sim slt}, x_{m_l j}^{\sim nslt}, z_{k_l j}^{\sim lt}, w_{d_l j}^{\sim lt}$) to identify the Model's (2) optimal values of the decision variables. Similar to Model (1), restrictions (2.1) to (2.6) ensure that the values of the decision variables are in the possibility production set. Constraint (2.7) demonstrates that stages 1 and 2 group their resources so that the total amount of shared resources is less significant than the current one.

Constraint (2.8) assumes that the values of shared inputs in stage 3 do not exceed the initial values; we adopt a similar assumption to calculate optimal profits from other coalitions. Constraint (2.9) ensures that exclusive inputs' values do not exceed the initial values. Constraints (2.10) and (2.11) indicate variable returns to scale and non-negativity of the variables.

Through the resolution of Model (2), it becomes possible to obtain the post-collaboration profit of this coalition and calculate $v(12)$ using the expression (3). The procedure to get $v(12)$ must also be applied to obtain the values of $P_{post}^{[1,3]}$, $P_{post}^{[2,3]}$, and consequently, $v(13)$ and $v(23)$.

$$v(12) = P_{post}^{[1,2]} - \left(\sum_{t=1}^{T} \sum_{k_3=1}^{K_3} q_{k_3}^{3t} z_{k_3 o}^{\sim 3t} + \sum_{l\epsilon\{1,2,3\}} \sum_{d_l}^{D_l} f_{d_l}^{lT} w_{d_l o}^{\sim lT} - \sum_{t=1}^{T} \sum_{l\epsilon\{1,2,3\}} \sum_{i=1}^{I} p_i^{st} x_{io}^{\sim nslt} - \sum_{t=1}^{T} \sum_{l\epsilon\{1,2,3\}} \sum_{m_l=1}^{M_l} p_{m_l}^{nslt} x_{m_l o}^{\sim nslt} \right) = P_{post}^{[1,2]} - P_{preo} \quad (3)$$

Model (4) allows us to obtain the profit values for the grand coalition, that is, for the cases where all stages cooperate S = {1,2,3}. The main distinction between Model (2) and (4) considers all shared inputs as a single amount. Constraint (4.7) aims to ensure that the total of inputs shared after the collaboration will be lower than in cases where it does not occur. The other constraints of Model (4) are identical to Model (2).

$$\max\ P_{post}^{[1,2,3]} = \sum_{t=1}^{T}\sum_{k_3=1}^{K_3} q_{k_3}^{3} z_{k_3 o}^{*3t} + \sum_{l\epsilon\{1,2,3\}}\sum_{d_l}^{D_l} f_{d_l}^{lT} w_{d_l o}^{*lT} - \sum_{t=1}^{T}\sum_{l\epsilon\{1,2,3\}}\sum_{i=1}^{I} p_i^{st} x_{io}^{*slt} - \sum_{t=1}^{T}\sum_{l\epsilon\{1,2,3\}}\sum_{m_l=1}^{M_l} p_{m_l}^{nslt} x_{m_l o}^{\sim nslt}$$

Subject to:

$x_{io}^{*slt} \geq \sum_{j=1}^{n} \lambda_j^{lt} x_{m_l j}^{\sim slt};\ i = 1, \dots, I;\ l = 1, \dots, L;\ t = 1, \dots, T$ (4.1)

$x_{m_l o}^{*nslt} \geq \sum_{j=1}^{n} \lambda_j^{lt} x_{m_l j}^{\sim nslt};\ m_l = 1, \dots, M_l;\ l = 1, \dots, L;\ t = 1, \dots, T$ (4.2)

$z_{k_l o}^{*lt} \geq=\leq \sum_{j=1}^{n} \lambda_j^{lt} z_{k_l j}^{\sim lt};\ k_l = 1, \dots, K_l;\ l = 1, \dots, L;\ t = 1, \dots, T$ (4.3)

$z_{k_{l-1} o}^{*l-1t} \geq=\leq \sum_{j=1}^{n} \lambda_j^{lt} z_{k_{l-1} j}^{\sim l-1t};\ k_l = 1, \dots, K_l;\ l = 2, \dots, L;\ t = 1, \dots, T$ (4.4)

$w_{d_l j}^{*lt} \geq=\leq \sum_{j=1}^{n} \lambda_j^{lt} w_{d_l j}^{\sim lt};\ d_l = 1, \dots, D_l\ l = 1, \dots, L;\ t = 1, \dots, T-1$ (4.5)

$w_{d_l j}^{*lt-1} \geq=\leq \sum_{j=1}^{n} \lambda_j^{lt} w_{d_l j}^{\sim lt-1};d_l = 1, \dots, D_l\ l = 1, \dots, L;\ t = 1, \dots, T-1$ (4.6)

$\sum_{l\in\{1,2,3\}} x_{io}^{*slt} \leq \sum_{l\in\{1,2,3\}} x_{io}^{\sim slt};\ i = 1, \dots, I;\ t = 1, \dots, T$ (4.7)

$x_{m_l o}^{*nslt} \leq\ x_{m_l o}^{\sim nslt}\ m_l = 1, \dots, M_l;\ l = 1, \dots, L;\ t = 1, \dots, T;$ (4.8)

$\sum_{j=1}^{n} \lambda_j^{lt} = 1\ l = 1, \dots, L;\ t = 1, \dots, T;$ (4.9)

$\lambda_j^{lt}, x_{ij}^{*slt}, x_{m_l j}^{*nslt}, z_{k_l j}^{*lt}, w_{d_l j}^{*lt} \geq 0\ l = 1, \dots, L;\ t = 1, \dots, T;$ (4.10)

Through the resolution of Model (4), it becomes possible to obtain the post-collaboration profit of this coalition and calculate $v(123)$ using the expression (5), which details the characteristic function of the grand coalition.

$$v(123) = P_{post}^{[1,2,3]} - \left(\sum_{t=1}^{T}\sum_{k_3=1}^{K_3} q_{k_3}^{3t} z_{k_3 o}^{\sim 3t} + \sum_{l\epsilon\{1,2,3\}}\sum_{d_l}^{D_l} f_{d_l}^{lT} w_{d_l o}^{\sim lT} - \sum_{t=1}^{T}\sum_{l\epsilon\{1,2,3\}}\sum_{i=1}^{I} p_i^{st} x_{io}^{\sim slt} - \sum_{t=1}^{T}\sum_{l\epsilon\{1,2,3\}}\sum_{m_l=1}^{M_l} p_{m_l}^{nslt} x_{m_l o}^{\sim nslt} \right) = P_{post}^{[1,2,3]} - P_{preo} \quad (5)$$

Also, it is possible to explore cases with and without carry-overs in the period zero $(w_{d_l o}^{*l0}, w_{d_l o}^{\sim l0})$. In situations where the initial carry-overs are present, we add the term $\sum_{l\epsilon\{1,2,3\}}\sum_{d_l}^{D_l} f_{d_l}^{l0} w_{d_l o}^{\sim l0}$ to the objective function, and we insert the constraint $w_{d_l o}^{\sim l0} \geq=\leq$

$\sum_{j=1}^{n} \rho_j^{l1} w_{d_l j}^{l0}$ $(d_l = 1, \dots, D_l\ l = 1, \dots, L)$ in Model (1), Model (2) and Model (4). The type of signal used in the constraint depends on the type of carry-over present in the situation. For more details regarding the PPS of DNDEA SBM models refers to Tone and Tsutsui (2014).

3.4.4 Payoff allocation using Shapley value and Nucleolus

Expression 6 details the mathematical formulation for calculating the Shapley value: $s = |S|$ represents the number of players in coalition S, and $n = |N|$ comprises the number of players in the grand coalition. $\emptyset_l(v)$ corresponds to the profit of stage l, which increased due to resource sharing. We calculate for all stages in all coalitions of all DMUs, and the Shapley value will identify the corresponding allocation for each stage participating in the coalition as to their contribution to increasing the system's profit.

$$\emptyset_l(v) = \sum_{S \subseteq N \backslash l} \frac{s!(n-s-1)!}{n!} (v(S \cup \{l\} - v(S)) \quad (6)$$

To calculate the Nucleolus, we must define the excess of a coalition S concerning an allocation $x \in IR^n$. Numerically, the excess can be expressed by:

$$e(S, x) = v(S) - \sum_{i \in S} x_i \quad (7)$$

This expression portrays a degree of dissatisfaction with a coalition S associated with the allocation x. To proceed with the Nucleolus calculation, we define a vector of ordered excesses $\theta(x) \epsilon\, IR^{2^N}$, which represents the excess of the coalitions in 2^N arranged in increasing magnitude, as shown in (8).

$$\theta(x) = \left(e(S^1, x), \dots, e\left(S^{2^n-2}, x\right)\right), e(S^1, x) \geq \dots \geq e\left(S^{2^n-2}, x\right) \quad (8)$$

It must also be considered the lexicographic ordering of the vectors $\theta(x)$, i.e., $\theta(x) > \theta(y)_L$ if $\exists k \in \{1, \dots, 2^n - 2\}$ such as $e\left(S^i, x\right) > e\left(S^i, y\right)(i = 1, \dots, k-1)$ and $e(S^k, x) > e(S^k, y)$. The imputation that minimizes this vector of non-increasing ordered excesses, according to the lexicographic order within the set of imputations, consists of the Nucleolus [24]. Considering the full imputation of the game (N, x) equal to X, then the Nucleolus is defined as described in (9).

$$\mu(X) = \{x \in X\ /\ \theta(x) \leq \theta(y)_L\ \ \forall y \in X\} \quad (9)$$

3.5 NUMERICAL EXAMPLE

The numerical example considers ten DMUs and evaluate the three stages of these DMUs over two time periods, with two shared inputs, one exclusive input, one intermediate product, and one carry-over for each stage.

It is important to note that there must be at least one common input in each stage to verify the possibility of sharing resources and at least one intermediate product between the stages. At least one of the stages must have a carry-over to carry out a dynamic evaluation.

We generated all the values randomly in Excel 2020. Unit prices are from 1 to 11, while inputs, carry-overs, and intermediate products range from 1 to 31. The selected range of values ensures the random generation of values that will not impact results and follows the same guidelines adopted by An, Wen, Ding, et al. (2019).

The prices of intermediate products generated in the first and second stages are zero because they are consumed internally. In addition, the price of shared inputs was considered the same at all stages. The data obtained are in Table 8.

Regarding carry-overs, fixed and free carry-overs, both proposed by Tone and Tsutsui (2014), were considered. These intermediate products follow the propositions of An, Wen, Ding, et al. (2019), which is in line with the free link case of Kao (2009), Tone and Tsutsui, 2009) and Tone and Tsutsui (2014). The free link consideration is appropriate since the DNDEA models assess the internal structure of the DMU, and it is reasonable to say that DMU generally has control over its internal network structure and production.

The choice to investigate two types of carry-over resides on the possibility that carry-overs represent aspects of infrastructure or productive resources, which may remain fixed for some time or require government permission to change. Thus, the analysis of fixed and free carry-overs allows investigation of the impacts of resource sharing in scenarios of total autonomy of the DMU (free carry-over) or partial autonomy (fixed carry-over). The investigation regarding the super additivity of the model lies in two cases: with free carry-overs and with fixed carry-overs.

Table 8 - Numerical data

t=1	DMU	1	2	3	4	5	6	7	8	9	10	Unit price
Stage 1	I=2	18.49	11.42	11.18	5.56	1.86	13.74	5.48	21.79	9.06	3.01	9.99
		1.53	3.91	11.05	2.29	21.63	1.36	1.02	1.15	7.10	9.11	8.58
	M=1	15.95	7.26	10.11	2.85	8.44	1.53	12.90	3.54	6.91	12.34	1.56
	K=1	23.23	17.36	26.98	1.28	12.93	13.29	10.68	20.68	20.71	8.59	0
	C=1	18.91	21.81	13.70	21.54	19.58	20.87	11.97	10.30	27.70	21.84	10.8
Stage 2	I=2	1.68	2.64	11.29	3.97	18.63	2.17	7.35	1.23	2.81	2.61	9.99

		6.58	12.00	2.22	8.39	4.25	10.06	14.52	1.23	16.89	9.62	8.58
	M=1	1.02	2.13	3.87	2.35	1.64	1.12	2.06	1.81	1.32	1.49	9.12
	K=1	14.62	16.91	18.63	6.62	16.96	12.85	16.64	11.94	20.01	13.57	0
	C=1	28.51	13.77	7.70	22.37	15.82	22.53	3.16	23.58	14.63	21.97	6.29
Stage 3	I=2	2.13	17.10	1.28	5.40	3.01	19.08	4.17	1.01	4.22	6.72	9.99
		10.10	3.93	9.40	1.44	2.99	2.20	1.56	18.11	1.02	2.35	8.58
	M=1	4.10	1.15	3.42	3.26	1.46	2.32	2.29	1.09	1.55	2.41	2.52
	K=1	27.49	30.18	30.91	16.38	29.55	25.26	26.23	29.75	26.99	25.96	8.83
	C=1	1.04	23.46	12.42	3.32	12.67	29,62	22.34	13.30	6.88	24.74	8.73
t=2	DMU	1	2	3	4	5	6	7	8	9	10	Unit price
Stage 1	I=2	11.94	7.29	25.28	2.09	20.78	26.41	22.70	8.59	10.03	26.50	8.8
		4.89	30.69	6.13	2.58	22.67	12.02	7.35	8.25	4.61	27.63	8.39
	M=1	17.57	14.04	28.81	29.01	22.96	28.59	3.72	11.87	30.48	20.35	1.84
	K=1	8.39	4.98	9.67	1.18	17.5	13.86	25.64	18.83	26.13	1.11	0
	C=1	8.52	23.22	26.73	26.71	5.12	1.91	3.90	21.40	16.44	20.52	5.05
Stage 2	I=2	21.30	13.26	16.05	29.10	19.11	7.24	30.54	30.34	16.93	28.41	8.8
		30.01	15.08	14.53	24.70	29.58	21.23	19.42	11.96	27.53	6.30	8.39
	M=1	4.16	25.72	3.49	28.40	4.18	18.73	23.48	2.16	12.55	8.86	7.32
	K=1	15.83	15.86	9.14	8.66	21.51	25.13	7.78	13.62	6.59	26.67	0
	C=1	24.31	25.60	10.23	22.51	14.51	24.27	4.72	23.14	2.35	7.28	4.20
Stage 3	I=2	30.03	22.52	19.62	4.71	14.20	20.77	26.30	24.07	2.91	28.97	8.8
		30.48	13.06	17.31	22.21	11.21	27.25	25.71	23.98	8.05	18.67	8.39
	M=1	4.65	5.02	27.26	21.91	12.51	18.44	4.94	25.56	13.19	28.10	1.95
	K=1	14.43	26.70	29.43	25.12	26.82	11.04	1.15	29.26	26.11	16.95	10.91
	C=1	15.00	20.76	28.94	16.62	8.74	5.63	7.00	10.51	20.76	13.26	9.69

Source: The Author (2024).

The findings and their implications should be discussed in the broadest context possible. Authors should discuss the results and how they can be interpreted from the perspective of previous studies and of the working hypotheses. Future research directions may also be highlighted.

3.5.1 Characteristic function

Models (1), (2), and (4) allows to obtain the objective function values of the numerical example detailed in Table 8. Initially, Model (1) brings the profit values for each DMU before sharing resources. Afterward, Models (2) and (4) calculate the coalition profits and get each DMU's characteristic function values. Table 9 displays the results for the two cases.

In Table 9, $v(1)$ to $v(123)$ represent the increase in profit gained from sharing resources between the stages of the DMU. Thus, the DMU is efficient when we verify null values because there is no profit increase even with sharing resources.

Table 9 - Characteristic functions values of the two cases

Case 1: Fixed carry-overs										
Characteristic function	DMU1	DMU2	DMU3	DMU4	DMU5	DMU6	DMU7	DMU8	DMU9	DMU10
$v(1)$	0,00	0,00	0,00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$v(2)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$v(3)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$v(12)$	162.29	204.46	228.36	433.21	766.91	804.75	307.40	0.00	200.69	0.00
$v(13)$	0.00	0.00	0.00	233.16	766.91	0.00	307.40	0.04	309.86	0.00
$v(23)$	0.00	389.88	0.00	254.81	766.91	850.96	307.40	0.00	168.36	0.00
$\boldsymbol{v(123)}$	**221.21**	**389.88**	**228.36**	**460.67**	**766.91**	**850.96**	**307.40**	**76.38**	**350.23**	**0.00**
$v(12) - v(1) - v(2)$	162.29	204.46	228.36	433.21	766.91	804.75	307.40	0.00	200.69	0.00
$v(13) - v(1) - v(3)$	0.00	0.00	0.00	233.16	766.91	0.00	307.40	0.04	309.86	0.00
$v(23) - v(2) - v(3)$	0.00	389.88	0.00	254.81	766.91	850.96	307.40	0.00	168.36	0.00
$v(123) - v(1) - v(2) - v(3)$	221.21	389.88	228.36	460.67	766.91	850.96	307.40	76.38	350.23	0.00
$v(123) - v(12) - v(3)$	58.92	185.42	0.00	27.46	0.00	46.20	0.00	76.38	149.54	0.00
$v(123) - v(23) - v(1)$	221.21	0.00	228.36	205.86	0.00	0.00	0.00	76.38	181.87	0.00
$v(123) - v(13) - v(2)$	221.21	389.88	228.36	227.51	0.00	850.96	0.00	76.34	40.38	0.00
Case 2: Free carry-overs										
Characteristic function	DMU1	DMU2	DMU3	DMU4	DMU5	DMU6	DMU7	DMU8	DMU9	DMU10
$v(1)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$v(2)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$v(3)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$v(12)$	125.97	259.95	346.04	282.81	295.22	138.22	820.61	408.94	93.34	20.04
$v(13)$	0.00	1.10	477.91	0.00	356.45	138.22	0.00	408.94	170.68	42.04
$v(23)$	0.00	1.10	491.43	0.00	356.45	138.22	862.95	408.94	171.01	20.04
$\boldsymbol{v(123)}$	**125.97**	**259.95**	**491.43**	**282.81**	**532.88**	**138.22**	**862.95**	**408.94**	**171.01**	**45.73**
$v(12) - v(1) - v(2)$	125.97	259.95	346.04	282.81	295.22	138.22	820.61	408.94	93.34	20.04
$v(13) - v(1) - v(3)$	0.00	1.10	477.91	0.00	356.45	138.22	0.00	408.94	170.68	42.04
$v(23) - v(2) - v(3)$	0.00	1.10	491.43	0.00	356.45	138.22	862.95	408.94	171.01	20.04
$v(123) - v(1) - v(2) - v(3)$	125.97	259.95	491.43	282.81	532.88	138.22	862.95	408.94	171.01	45.73
$v(123) - v(12) - v(3)$	0.00	0.00	145.39	0.00	237.65	0.00	42.34	0.00	77.67	25.69
$v(123) - v(23) - v(1)$	125.97	258.85	0.00	282.81	176.43	0.00	0.00	0.00	0.00	25.69
$v(123) - v(13) - v(2)$	125.97	258.85	13.53	282.81	176.43	0.00	862.95	0.00	0.34	3.69

Source: The Author (2024).

It is observed that only DMU 10 in case 1 was efficient. In all cases, many non-zero values exist for the coalitions $v(12)$, $v(13)$, $v(23)$, and $v(123)$, indicating that sharing resources between DMU stages over consecutive periods can increase system profits.

This analysis shows that for all investigated scenarios, the proposed game is super-additive $[v(S \cup T) - v(S) - v(T) \geq 0 \ \forall S, T \in 2^N, and\ (S \cap T) = \emptyset]$. It is possible to prove this with the following example. Consider $S = \{1,2\}\ and\ T = \{3\}$; we need to verify that $v(123) \geq v(12) + v(3)$. From the discussion of section 3.2, $v(3) = 0$, then $v(123) \geq$

$v(12) + v(3)$ becomes $v(123) \geq v(12)$. In Section 3.2, we define $v(S) = P_{post}^{S} - P_{pre}$, and since P_{pre} is equal for both cases, it is necessary to prove that $P_{post}^{123} \geq P_{post}^{12}$.

The optimal solution for model (2) is $\left(\lambda_j^{*lt}, x_{ij}^{**slt}, x_{m_l j}^{**nslt}, z_{k_l j}^{**lt}, w_{d_l j}^{**lt}, \forall l = \{1,2,3\}, \forall t\right)$. For each t, $\sum_{l\in\{1,2\}} x_{i0}^{**slt} \leq \sum_{l\in\{1,2\}} x_{i0}^{*slt}$ and $x_{i0}^{**s3t} \leq x_{i0}^{*s3t}$, so we infer that in each period $\sum_{l\in\{1,2\}} x_{i0}^{**slt} + x_{i0}^{**s3t} < \sum_{l\in\{1,2,3\}} x_{i0}^{*slt}$. Conjointly with constraint (4.8), and if it is valid for each period, it will also be true for the entire horizon. The remaining constraints in Model (4) are similar to Model (2), so it is possible to conclude that the optimal solution for Model (2) is feasible in model (4). It is also true that the objective function of Model (4) is the highest profit achievable by the system. Then, we can prove $v(123) \geq v(12)$.

The discussion is also valid for the other coalitions; therefore, we can prove the super additivity for the model. For all the cases, the coalitions' characteristic function values minus the individual stages' values are in Table 9. These results align with Moreno and Lozano (2018) verifying that sharing resources can increase profits. It is important to note that even in cases where there is no total autonomy of the DMUs, the DMUs are limited in a certain way, and the model returns super-additive results. The proposed approach reinforces that this practice, over time, provides positive results, both for the stages and the system.

3.5.2 Payoff allocation

In this study, we use the Shapley value and Nucleolus to allocate the benefits arising from cooperation between the stages of the DMUs. We can notice that the sum of each column of Tables 5 and 6 corresponds to the grand coalition value illustrated in demonstrating that the result obtained by the system corresponds to the sum of the results obtained by the stages.

In Table 10, it is possible to verify that some stages have zero profit. This result can portray two situations, and the first corresponds to cases similar to DMU 10 in case 1. In this circumstance, the DMU is efficient, and sharing resources between the stages does not benefit the system.

Table 10 - Shapley values of the two cases

Case 1: Fixed carry-overs										
Shapley Value	DMU 1	DMU 2	DMU 3	DMU 4	DMU 5	DMU 6	DMU 7	DMU 8	DMU 9	DMU 10
Stage 1	100.79	34.08	114.18	179.68	255.64	134.13	102.47	25.47	145.72	0.00
Stage 2	100.79	229.02	114.18	190.51	255.64	559.60	102.47	25.45	74.97	0.00
Stage 3	19.64	126.79	0.00	90.48	255.64	157.23	102.47	25.47	129.55	0.00
Case 2: Free carry-overs										
Shapley Value	DMU 1	DMU 2	DMU 3	DMU 4	DMU 5	DMU 6	DMU 7	DMU 8	DMU 9	DMU 10
Stage 1	62.98	129.79	137.33	141.40	167.42	46.07	136.77	136.31	44.00	18.91

Stage 2	62.98	129.79	144.09	141.40	167.42	46.07	568.24	136.31	44.17	7.91
Stage 3	0.00	0.37	210.02	0.00	198.03	46.07	157.94	136.31	82.84	18.91

Source: The Author (2024).

The other situation is illustrated by DMU 1 in case 2 and DMU 3 in case 1. It is important to remember that the Shapley value measures the average marginal benefit that the player adds to the coalition's profit. The stage's addition does not provide any additional increase for cases like this. It is possible to verify this fact by observing that $v(123)$ is equal to $v(12)$. For these cases, entering the third stage when forming the grand coalition does not increase profit. Thus, the null result of the Shapley value was expected and is in line with the definition of the allocation method and conceptual background.

Table 11 displays the Nucleolus values. Initially, we can observe that the Shapley value and the Nucleolus allocations are the same for some cases. The first consists of efficient DMUs, since all coalitions have characteristic functions with a value equal to zero.

Table 11 - The Nucleolus of the stages in the two scenarios

Case 1: Fixed carry-overs										
Nucleolus	DMU 1	DMU 2	DMU 3	DMU 4	DMU 5	DMU 6	DMU 7	DMU 8	DMU 9	DMU 10
Stage 1	95.88	0.00	114.18	205.81	255.64	0.00	102.47	25.47	174.69	0.00
Stage 2	95.88	297.17	114.18	227.46	255.64	827.86	102.47	25.47	33.19	0.00
Stage 3	29.46	92.71	0.00	27.41	255.64	23.11	102.47	25.47	142.36	0.00
Case 2: Free carry-overs										
Nucleolus	DMU 1	DMU 2	DMU 3	DMU 4	DMU 5	DMU 6	DMU 7	DMU 8	DMU 9	DMU 10
Stage 1	62.99	129.98	110.84	141.41	157.22	46.07	0.00	136.31	31.00	21.94
Stage 2	62.99	129.98	124.36	141.41	157.22	46.07	841.78	136.31	31.33	1.85
Stage 3	0.00	0.00	256.23	0.00	218.45	46.07	21.17	136.31	108.67	21.94

Source: The Author (2024).

The second case consists of situations where all two-player coalitions and the grand coalition have the same value. For these cases, $v(123) = v(12) = v(13) = v(23)$, minimizing dissatisfaction is assigning values equal to the players, which must correspond to one-third of $v(123)$. Otherwise, the lexicographical order in the excess vector would be inverted, and this allocation would be different, not corresponding to the definition of the Nucleolus. We verify this situation for DMUs 5 and 7 in case 1.

The third case occurs when one of the two-player coalitions has a value equal to the grand coalition, and the others have zero value, such as DMU 3 in case 1. For these two cases, $v(123) = v(12)$ and $v(13) = v(23) = 0$. Thus, to satisfy the conditions of the nucleolus definition, the grand coalition's value must be divided equally between the parties that make up the coalition of two players of equal value.

The differences can be explained by how each method defines fairness for the other allocations. The nucleolus-based allocation plan first addresses the least happy coalition; intuitively, favoring the least coalition is a generous philosophy and will cause less resistance to organizations' implementation of the allocation plan (LI et al., 2019). On the other hand, the Shapley value considers the marginal contribution of each player to the coalition.

In the specific profit case, the payoff allocation results give managers a direction to prioritize efforts to obtain the best possible outcome. For the current issue, the discussion of these allocations takes place in a fictitious example. Shapley value would be better accepted, thus promoting a better incentive for collaboration. For real situations, the decision-maker knowing in depth the analyzed context, will choose which best meets their needs.

3.6 CONCLUSION REMARKS AND DISCUSSIONS

Resource sharing consists of a way to improve organizational performance. The current study developed an integrated proposition of DNDEA and cooperative approaches of Game Theory, namely Shapley's Value and Nucleolus, to explore resource sharing in a three-stage network structure considering multiple periods. Joint developments of DEA and Game Theory have been in the literature since the 1980s. However, combining the techniques to investigate the benefits of sharing resources in internal network structures is scarce when contemplating the temporal aspects.

Thus, the current study fills a gap in the literature when considering resource sharing with DNDEA models. Few studies address this theme, and most identify optimal proportions for sharing resources. Our proposition differs because it assumes that the stages have a known quantity of resources and that pooling them together will benefit the DMUs.

With the aid of Shapley value and Nucleolus, the allocation of benefits arising from sharing is carried out. Therefore, we develop models to calculate pre- and post-collaboration profits between the stages of DMUs. A numerical example containing 10 DMUs acting over two time periods validates the developed proposition.

The results indicate the benefits of resource sharing over time through the super additivity verified in the characteristic functions. Using Shapley value and Nucleolus, it is possible to allocate the benefits obtained based on the marginal contributions of each stage, providing incentives to motivate and maintain cooperation between the stages of the organization.

The results obtained with the Shapley value would be better accepted for the developed approach, thus promoting a greater incentive for cooperation. In addition, the greater simplicity of the intuition on the Shapley value and the greater ease in explaining their achievements may be more attractive to managers when compared to Nucleolus. However, these hypotheses require practical validation.

In addition to contributing to the theoretical developments of DEA and Game Theory, this study provides advances by considering a case with three stages perceived as players since there is a preponderance in the literature of cases with only two players.

Several directions can advance the current development. Initially, we highlight the application in a real case, considering that access to actual data was a limitation. The analysis with more stages is another direction to check. Simultaneously investigating the cooperation between the internal structures of the DMUs and between the DMUs can be an alternative to obtain even better results. Finally, verifying non-cooperative aspects in the network structure considering multiple periods can contribute to the literature.

The next chapter presents a conceptual model that incorporates some of these suggestions. Both chapters offer models to aid in better resource usage and increasing performance. Although the models use different DEA and Game Theory assumptions, the aim of the thesis is not to compare the results but to broaden the range of actual cases that can use such methodologies to have a better decision-making process.

4 DYNAMIC DEA WITH SHARED INPUTS: INSIGHTS FROM COOPERATIVE AND NON-COOPERATIVE FRAMEWORKS

This chapter detail the proposition of a dynamic model with a two-stage network structure with shared inputs between the stages. The proposed model considers a centralized approach in which the divisions cooperate to obtain the best system performance. A weighted additive method combines the two individual stages and all the periods under evaluation. Then, under a leader-follower framework, a new set of additive models are used to investigate the efficiency decomposition when one of the stages is prioritized. Jointly with the model in Chapter 3, it answers the question, 'How to assist an organization to increase performance by exploring resource complementary?'.

4.1 CONTEXTUALISATION

Universities represent a driving force of science and knowledge in the countries, it is crucial for providing a skilled and expert workforce in the job market (FOLADI; SOLIMANPUR; JAHANGOSHAI REZAEE, 2020). Evaluating their results is a complex process due to the existence of different indicators to obtain an overview of system performance (STUMBRIENE; CAMANHO, Ana S.; JAKAITIENE, 2020). Understanding how to increase the universities' performance is challenging for governments, leading operators, and funders (LEE, B. L.; JOHNES, J., 2022; MONCAYO–MARTÍNEZ; RAMÍREZ–NAFARRATE; HERNÁNDEZ–BALDERRAMA, 2020). The last decades have shown a worldwide trend to implement exercises about evaluation and a comparison of various estimation methods (MOORE; COATES; CROUCHER, 2019).

In Brazil, it is possible to verify the same trend. In 2004, Law 10.861/2004 instituted the National Higher Education Assessment System (SINAES). SINAES aims to improve the results of Brazilian higher education system and consists of three main components: the evaluation of institutions, courses, and student performance. The Anísio Teixeira National Institute for Educational Research and Studies (INEP) annually performs a census for the higher education sector, collecting data about the three dimensions to obtain indicators used in SINAES to assess and accredit courses and institutions.

The Ministry of Education uses four leading indicators relating to the quality of higher education. The grade of senior students in Enade (ES) and the Difference Indicator Score between Observed and Expected Performance (IDD) evaluate senior undergraduate students.

The Preliminary Course Concept (CPC) and General Index of Courses (IGC) refer to the evaluation of undergraduate courses and higher education institutions (HEIs). The evaluation processes are coordinated and supervised by the National Commission for the Evaluation of Higher Education (CONAES), while the operation is the responsibility of INEP (PEREIRA, C.; ARAÚJO; MACHADO, M. De L., 2015). On the other hand, the evaluation of graduate programs is performed by the Coordination for the Improvement of Higher Education Personnel (CAPES) which receive a score ranging from one to seven.

Despite successive improvements and changes made over the last decade to improve these assessments, there are still several criticisms in the literature regarding these indicators. In general, it is possible to verify problems regarding the weighting of the considered criteria. Technical notes issued by the government do not justify the choice of weights, and minor weight variations can significantly change the results (BITTENCOURT *et al.*, 2010; ZANELLA; OLIVEIRA, 2021). Another criticism relates to using the same criteria for courses in different areas, in different types of institutions, and for different regions of the country (IKUTA, 2016).

In addition to these issues, another point deserves attention. There are individual and in-depth assessments for undergraduate and graduate courses. However, the indicator relating to higher education institutions (IGC) aggregates information from all undergraduate and graduate courses at these HEIs without any distinction between these levels of education. This aggregation does not allow proposing improvement targets for any of the two stages. Therefore, it fails to allow the development of action plans to bring improvements to these institutions in a practical way.

Brazilian postgraduate courses are responsible for a large part of the research and innovation generated in the country (ANDIFES, 2017). Despite its importance, the report on investments in research and development in the world carried out by the United Nations Educational, Scientific and Cultural Organization (UNESCO) considering 2014 to 2018, shows that the budget reduction of the Ministry of Science, Technology and Innovations (MCTI) in the same period was around 50% (UNESCO, 2021). If more recently data is considered (2012 to 2021), the reduction corresponds to 84% (from R$ 11.5 billion to R$ 1.8 billion, in inflation-adjusted values). Despite the drastic budget reduction, the same report points out that scientific production continued to grow.

Due to the absence of a governmental procedure to provide a global panorama of graduate courses in Brazilian HEIs and due to the importance of these activities for national science, the current study is focused on proposing an evaluation model for Brazilian universities regarding graduate activities performance.

Education institutions are multi-product organizations: they produce teaching, research, and third mission (the last reflecting universities' engagement with society) (JOHNES, J., 2015). In addition, several institutions can be not-for-profits, making inappropriate conventional performance measures. Besides being multi-product organizations, the educational process usually takes several years. Investigating productivity changes across time is necessary to comprehend whether universities have improved, stagnated, or regressed their performance (PARTEKA; WOLSZCZAK-DERLACZ, 2013).

Therefore, in the literature, we have distinct operational research techniques to perform efficiency estimates, such as Ordinary Least Squares regression, multilevel modeling (MLM), Multiple-criteria decision analysis (MCDA), Data Envelopment Analysis (DEA), and Stochastic Frontier Analysis (SFA) (JOHNES, J., 2015). The presence of multiple inputs and outputs show that DEA is an instructive tool in the educational context (THANASSOULIS *et al.*, 2016a). Among the most discussed types of evaluations within the DEA applications, we have cost efficiency, technical efficiency, research performance, administrative services evaluation, university rankings, assessing academics on teaching and research activities, and student performance (THANASSOULIS *et al.*, 2016b).

The seminal work of (CHARNES; COOPER, W. W.; RHODES, 1978) fostered the development of a literary branch with more than 20,000 articles indexed in databases such as Web of Science and Scopus. This study investigates the Follow Through program in the USA, initiating the application in the educational area. The survey of (LIU, J. S. *et al.*, 2013a) and more recently (EMROUZNEJAD; YANG, Guo Liang, 2018) highlights the importance of this research front and its impact on the development of applied DEA studies. Surveys regarding educational applications, such as (JOHNES, J., 2015; JOHNES, J.; PORTELA, M.; THANASSOULIS, 2017; THANASSOULIS *et al.*, 2016b; WITTE, DE; LÓPEZ-TORRES, 2017), detail the broad range of investigations and literature gaps in the field. Inquiries regarding higher education and comparisons among different educational systems are significant to comprehend which features allow for better performance.

Since educational processes possess a multi-period feature, suitable models are required to adequality portrays the situation. Malmquist Index, Dynamic DEA (DDEA), and Dynamic DEA models with network structure (DNDEA) represent the DEA alternatives available to incorporate temporal aspects into efficiency measures. Several works have addressed DEA and university evaluation, considering dynamic (EMROUZNEJAD; YANG, Guo Liang, 2018; KUMAR; THAKUR, 2019; PARTEKA; WOLSZCZAK-DERLACZ, 2013; XIONG, X. *et al.*, 2022) or dynamic network aspects (COSSANI *et al.*, 2022; FUKUYAMA; WEBER; XIA, 2016b; TRAN, C. D. T. T.; VILLANO, R. A., 2018; TRAN, C. T. T. D., 2021; TRAN, C.-D.; VILLANO, R., 2021).

However, most investigations visualize universities without internal processes or consider teaching and research disregarding their existence at undergraduate and graduate levels. In the case of Brazilian universities (FRANÇA, DE; FIGUEIREDO, DE; LAPA, DOS, 2010; HAMMES JUNIOR; FLACH; MATTOS, L. K., 2020; TAVARES; ANGULO-MEZA; SANT'ANNA, 2021; WANKE *et al.*, 2022; ZOGHBI; ROCHA; MATTOS, E., 2013), these approaches discussed distinct aspects, including cost efficiency, pertinent variables to Brazilian reality, or aggregated values to compare public and private institutions.

In this paper, we will bridge this knowledge gap by proposing an innovative way of dealing with teaching and research activities at the graduate level to provide an aggregated view of the graduate programs in Brazilian HEIs. It is an DNDEA-based approach contemplating the formative and scientific production processes. Among the distinct available models, we propose a new two-stage DNDEA model which share inputs in α and (1-α) parcels between the stages. Therefore, this study differentiates from previous ones since we propose a centralized two-stage DNDEA modeling with resource sharing to maximize system efficiency. After this initial proposition, we consider a leader-follower case to investigate the efficiency decomposition of the stages. In this sense, we expand the considerations of (CHEN, Yao *et al.*, 2010) and (TOLOO; EMROUZNEJAD; MORENO, 2017).

It is important to verify that national assessments include a large amount of data to be analyzed. In the Brazilian case, partial evaluations for 2021 indicate the existence of 27711, 1054, 829 and 37 graduate programs in federal, state, private and municipal institutions, respectively. Therefore, an analysis of such dimensions requires significant effort on the committees and evaluation teams. It is also noteworthy that both in the Brazilian case and in international assessments, the commissions are multidisciplinary and not always all members are familiar with mathematical programming models.

Considering such particularities, we develop a bi-dimensional representation to visually display the efficiency frontier and the DMUs' positions concerning the frontier. Since DNDEA models provide several efficiency levels, modified virtual inputs and outputs constitute the selected tool used to represent all the different efficiency scores obtained with the DNDEA model. The use of visual representations for these models provides a better comprehension, given that the similar nomenclature for the different types of efficiency can represent an obstacle for decision-makers to understand the results.

The bi-dimensional representation summarizes this information in a simple and straightforward way. This can help decision-makers who need to make faster and more accurate decisions. Because in a world where data is increasingly abundant, clarity and simplifications can be very valuable (TORRES; REIS; SOARES DE MELLO, J. C. C. B., 2022). This tool allows direct efforts, helps persuade managers and policymakers about the validity of the results, and translates recommendations into actions (OZCAN *et al.*, 2010).

The work provides four main contributions. The first corresponds to the proposition of a procedure to investigate HEIs performance regarding Brazilian graduate activities and its process structure in a dynamic manner. Therefore, we provide a tool for the Brazilian government to obtain an overview of universities graduate activities, which will allow to develop specific action plans to improve performance.

Second, we investigate resource-sharing in a DNDEA framework, which distinguishes it from others because we simultaneously consider the network structure and temporal impacts on efficiency. Analyzing shared inputs is necessary, as students and teachers divide their workloads between both processes. In this way, the analysis of the allocation of these resources, that is, whether it is being efficient or not, can benefit the performance of the HEIs.

This analysis is vital given federal government spending. Approximately half of the Brazilian postgraduate courses are developed in public universities. Data from 2020 indicate that the federal government spent 23 billion in federal universities to finance personnel and charges in the same year. In addition, it is worth mentioning that Brazilian research agencies such as Coordination for the Improvement of Higher Education Personnel (CAPES) and the National Council for Scientific and Technological Development (CNPq) finance scholarships for masters and doctoral students in these institutions. Therefore, this analysis helps in the best use of public resources.

Third, this development also fills a gap for the conjoint applications of DEA and Game Theory models, because we consider cooperation scenarios among the stages and use the leader-follower framework to investigate efficiency decomposition in the network. We also demonstrate that the model can easily be adapted to contemplate situations without resource sharing and with exogenous inputs. Fourth, we developed a new framework to visually represent the efficiency frontier and the DMUs position in a simple, but effective way for all the efficiency types provided by the DNDEA model.

The following section detail the Brazilian context regarding graduate activities and some DEA applications in this context. Section 3 discusses the studies developed in the context of

DNDEA framework. Section 4 details the DNDEA models and the bi-dimensional representation procedure. Section 5 presents the results, and Section 6 concludes.

4.2 MODEL STRUCTURE

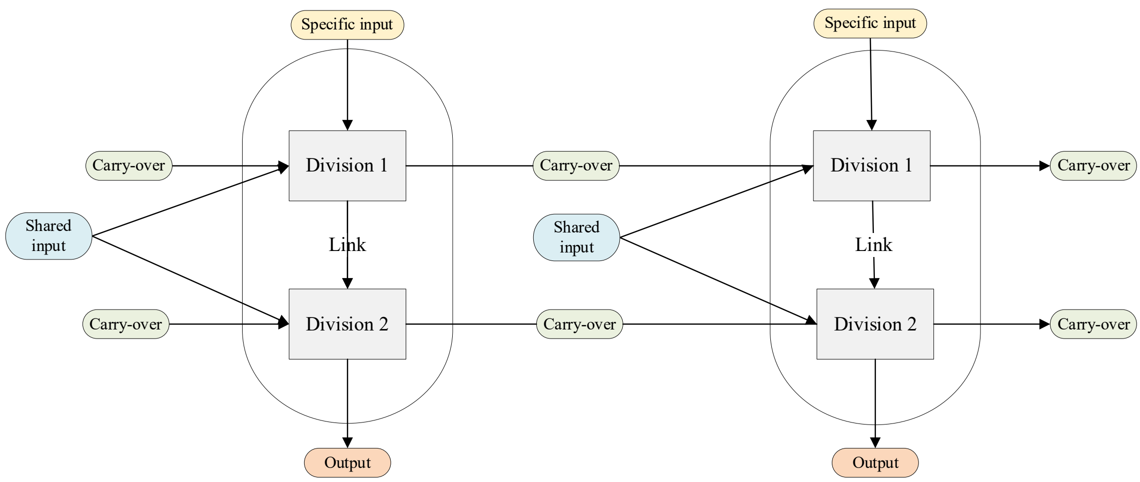
4.2.1 Model Outline and Assumptions

The developed model aims to investigate resource sharing in a two-stage network model and to measure efficiency in a dynamic manner. The framework considered to develop our model is displayed in Figure 15.

We considered the presence of shared and specific inputs. However specific inputs are present only in the first stage. The following models are designed to deal with shared inputs among the two stages. Therefore, a parcel α_{pj} of the shared inputs p is consumed by the first division, while $(1-\alpha_{pj})$ is used by the second. It is also relevant to notice that all resources used in Division 1 are used to produce links and carry-overs, therefore exogenous outputs are not considered for this division.

Following similar assumption, there are no exogenous inputs entering the second division. It is also considered that all intermediate measures produced by the first stage is consumed by the second. With these assumptions in mind, we proposed two distinct frameworks to investigate the referred context, a cooperative and a non-cooperative one.

Figure 15 - Dynamic framework for a two-stage system



Source: The Author (2024).

In Table 12, the indexes, parameters and variables of the model are presented. Using these notations, a centralized relational DNDEA model and a leader-follower form of the DNDEA model are presented.

Table 12 - Indexes, parameters and variables of the model

Indexes	
$j = 1, \ldots, n$	Index for *j*th DMU;
$t = 1, \ldots, T$	Index for *t*th period;
$k = 1, \ldots, K$	Index for *k*th division;
$i = 1, \ldots, m$	Index for *i*th specific input;
$p = 1, \ldots, P$	Index for *p*th input shared between the divisions;
$r = 1, \ldots, s$	Index for *s*th output;
$d = 1, \ldots, D$	Index for *d*th link;
$l = 1, \ldots, L$	Index for *l*th carry-over;
Parameters	
$x_{ij}^{(t)}$	*i*th specific input of DMU *j* in division 1 in period *t;*
$x_{pj}^{(t)}$	*p*th shared input of DMU *j* between divisions 1 and 2 at period *t;*
$y_{rj}^{(t)}$	*r*th output of DMU *j* at division *2* at period *t;*
$z_{dj}^{(t)}$	*d*th link of DMU *j* leaving division 1 to division 2 at period *t*;
$c_{lj}^{(t,k)}$	*l*th carry-over at DMU *j* in division *k* that connects period *t* to the next one;
Variables	
$\propto_{pj}$	The proportion of the shared input of DMU *j* that will be used by division *1*;
$v_i^*, v_p^*, u_{r,}^* w_l^*, f_d^*$	The optimal weights attached to specific input, shared inputs, outputs, carry-overs and links respectively;

Source: The Author (2024).

4.2.2 Centralized Approach

For the two-stage system illustrated by Figure 15, the stages of an observed DMU can be evaluated considering constant returns to scale by Model (9) and (10) in each period.

$$E_j^{(t,1)} = max \frac{\sum_{l \in l^1} f_l c_{lo}^{(t,1)} + \sum_{d=1}^{D} w_d z_{do}^{(t)}}{\sum_{l \in l^1} f_l c_{lo}^{(t-1,1)} + \sum_{p=1}^{P} \alpha_{pj} v_p x_{po}^{(t)} + \sum_{i=1}^{m} v_i x_{io}^{(t)}}$$

$$\text{s.t.} \quad \frac{\sum_{l \in l^1} f_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}}{\sum_{l \in l^1} f_l c_{lj}^{(t-1,1)} + \sum_{p=1}^{P} \alpha_{pj} v_p x_{pj}^{(t)} + \sum_{i=1}^{m} v_i x_{ij}^{(t)}} \leq 1 \ (j = 1, \ldots, n)$$

$$L_{pj}^1 \leq \alpha_{pj} \leq L_{pj}^2$$

$$v_i, w_l, f_d, v_p, \geq \varepsilon; \ \ i = 1, \ldots, m; l = 1, \ldots, L; d = 1, \ldots, D; \ p = 1, \ldots, P \qquad (9)$$

$$E_j^{(t,2)} = max \frac{\sum_{r=1}^{s} u_r y_{ro}^{(t)} + \sum_{l \in l^2} f_l c_{lo}^{(t,2)}}{\sum_{l \in l^2} f_l c_{lo}^{(t-1,2)} + \sum_{p=1}^{P} (1 - \alpha_{pj}) v_p x_{po}^{(t)} + \sum_{d=1}^{D} w_d z_{do}^{(t)}}$$

$$\text{s.t} \frac{\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{l \in l^2} f_l c_{lj}^{(t,2)}}{\sum_{l \in l^2} f_l c_{lj}^{(t-1,2)} + \sum_{p=1}^{P} (1-\alpha_{pj}) v_p x_{pj}^{(t)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}} \leq 1 \ (j = 1, \ldots, n)$$

$$L_{pj}^1 \leq \alpha_{pj} \leq L_{pj}^2$$

$$v_i, u_{r,} w_l, f_d, v_p \geq \varepsilon; \ \ i = 1, \ldots, m; r = 1, \ldots, s; l = 1, \ldots, L; d = 1, \ldots, D; \ p = 1, \ldots, P \qquad (10)$$

Therefore, similar to Kao & Hwang (2008) assumption of the centralized model, we considered the same weights for the variables in all periods. We proposed a weighted average of stage 1 and 2 for each period as displayed in (11).

$$E_j^{(t,sys)} = w_1^t E_j^{(t,1)} + w_2^t E_j^{(t,2)} \quad (11)$$

In order to define w_1^t and w_2^t, the consideration of Chen et al. (2010) was selected. The authors discussed that the proportion of total resources devoted to each stage presents one reasonable choice of weight to reflect the relative size of a stage. It is important to note that in dynamic models with network structures, carry-overs and links play a dual role. Carry-overs represent both the output of one period and an input of the following one, while links consist of outputs from the first stage and inputs from the second. Therefore, we define w_1^t and w_2^t in (12) and (13).

$$w_1^t = \frac{\sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{p=1}^P \alpha_{pj} v_p x_{pj}^{(t)} + \sum_{l=1}^{l_1} f_l c_{lj}^{(t-1,1)}}{\sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{p=1}^P v_p x_{pj}^{(t)} + \sum_{d=1}^D w_d z_{dj}^{(t)} + \sum_{k=1}^K \sum_{l=1}^L f_l c_{lj}^{(t-1,k)}} \quad (12)$$

$$w_2^t = \frac{\sum_{p=1}^P \left(1-\alpha_{pj}\right) v_p x_{pj}^{(t)} + \sum_{d=1}^D w_d z_{dj}^{(t)} + \sum_{l=1}^{l_2} f_l c_{lj}^{(t-1,2)}}{\sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{p=1}^P v_p x_{pj}^{(t)} + \sum_{d=1}^D w_d z_{dj}^{(t)} + \sum_{k=1}^K \sum_{l=1}^L f_l c_o^{(t-1,k)}} \quad (13)$$

In (12) and (13), $\sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{p=1}^P v_p x_{pj}^{(t)} + \sum_{d=1}^D w_d z_{dj}^{(t)} + \sum_{k=1}^K \sum_{l=1}^L f_l c_o^{(t-1,k)}$ represents the total amount of resources (inputs) used by the stages in a period t. On the other hand, $\sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{p=1}^P \alpha_{pj} v_p x_{pj}^{(t)} + \sum_{l=1}^{l_1} f_l c_{lj}^{(t-1,1)}$ and $\sum_{p=1}^P \left(1-\alpha_{pj}\right) v_p x_{pj}^{(t)} + \sum_{d=1}^D w_d z_{dj}^{(t)} + \sum_{l=1}^{l_2} f_l c_{lj}^{(t-1,2)}$ indicates the resource size of stage 1 and 2, respectively. Therefore, the system efficiency in each period is detailed in (14).

$$E_j^{(t,sys)} = \frac{\sum_{r=1}^s u_r y_{rj}^{(t)} + \sum_{d=1}^D w_d z_{dj}^{(t)} + \sum_{k=1}^K \sum_{l=1}^L f_l c_{lj}^{(t,k)}}{\sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{p=1}^P v_p x_{pj}^{(t)} + \sum_{d=1}^D w_d z_{dj}^{(t)} + \sum_{k=1}^K \sum_{l=1}^L f_l c_{lj}^{(t-1,k)}} \quad (14)$$

We also considered that the overall efficiency is a weighted average of the system efficiency in each period. The proportion of total resources devoted to each period presents the choice to reflect the relative size of the period. Therefore, we define w^t in (15).

$$w^t = \frac{\sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{p=1}^P v_p x_{pj}^{(t)} + \sum_{k=1}^K \sum_{l=1}^L f_l c_{lj}^{(t-1,k)} + \sum_{d=1}^D w_d z_{dj}^{(t)}}{\sum_{t=1}^T \sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{t=1}^T \sum_{p=1}^P v_p x_{pj}^{(t)} + \sum_{t=1}^T \sum_{k=1}^K \sum_{l=1}^L f_l c_{lj}^{(t-1,k)} + \sum_{t=1}^T \sum_{d=1}^D w_d z_{dj}^{(t)}} \quad (15)$$

In (7), $\sum_{t=1}^T \sum_{i=1}^m v_i x_{ij}^{(t)} + \sum_{t=1}^T \sum_{p=1}^P v_p x_{pj}^{(t)} + \sum_{t=1}^T \sum_{k=1}^K \sum_{l=1}^L f_l c_{lj}^{(t-1,k)} + \sum_{t=1}^T \sum_{d=1}^D w_d z_{dj}^{(t)}$ represents the total amount of resources (inputs) used in all time frame considered. On the other hand, indicates the resource size of each period t. Therefore, the overall system efficiency is detailed in (16).

$$E_j^{(sys)} = \frac{\sum_{t=1}^{T}\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{t=1}^{T}\sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t,k)}}{\sum_{t=1}^{T}\sum_{i=1}^{m} v_i x_{ij}^{(t)} + \sum_{t=1}^{T}\sum_{p=1}^{P} v_p x_{pj}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t-1,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D} w_d z_{dj}^{(t)}} \quad (16)$$

Thus, under CRS, the overall efficiency score can be evaluated by solving the following fractional program as presented in Model (9).

$$\theta_o^* = Max \frac{\sum_{t=1}^{T}\sum_{r=1}^{s} u_r y_{ro}^{(t)} + \sum_{t=1}^{T}\sum_{d=1}^{D} w_d z_{do}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lo}^{(t,k)}}{\sum_{t=1}^{T}\sum_{i=1}^{m} v_i x_{io}^{(t)} + \sum_{t=1}^{T}\sum_{p=1}^{P} v_p x_{po}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lo}^{(t-1,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D} w_d z_{do}^{(t)}}$$

$$\text{s.t } \frac{\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t,k)}}{\sum_{i=1}^{m} v_i x_{ij}^{(t)} + \sum_{p=1}^{P} v_p x_{pj}^{(t)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t-1,k)}} \le 1 \; (j = 1, \dots, n; t = 1, \dots, T)$$

$$\frac{\sum_{l\in l^1} f_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}}{\sum_{l\in l^1} f_l c_{lj}^{(t-1,1)} + \sum_{p=1}^{P} \alpha_{pj} v_p x_{pj}^{(t)} + \sum_{i=1}^{m} v_i x_{ij}^{(t)}} \le 1 \; (j = 1, \dots, n; t = 1, \dots, T)$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{l\in l^2} f_l c_{lj}^{(t,2)}}{\sum_{l\in l^2} f_l c_{lj}^{(t-1,2)} + \sum_{p=1}^{P} (1-\alpha_{pj}) v_p x_{pj}^{(t)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}} \le 1 \; (j = 1, \dots, n; t = 1, \dots, T)$$

$$L_{pj}^1 \le \alpha_{pj} \le L_{pj}^2$$

$$v_i, u_r, w_l, f_d, v_p \ge \varepsilon; \; i = 1, \dots, m; r = 1, \dots, s; l = 1, \dots, L; d = 1, \dots, D; \; p = 1, \dots, P \quad (17)$$

With the aid of the Charnes–Cooper transformation, the fractional program proposed in Model (9) can be converted into Model (18).

$$\theta_o^* = max \sum_{t=1}^{T}\sum_{r=1}^{s} \mu_r y_{ro}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lo}^{(t,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D} \mu_d z_{do}^{(t)}$$

$$\sum_{t=1}^{T}\sum_{i=1}^{m} v_i x_{io}^{(t)} + \sum_{t=1}^{T}\sum_{p=1}^{P} v_p x_{po}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lo}^{(t-1,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D} \mu_d z_{do}^{(t)} = 1$$

$$\sum_{r=1}^{s} \mu_r y_{rj}^{(t)} + \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lj}^{(t,k)} - \sum_{i=1}^{m} v_i x_{ij}^{(t)} - \sum_{p=1}^{P} v_p x_{pj}^{(t)} - \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} - \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lj}^{(t-1,k)}$$

$$\le 0 \; (j = 1, \dots, n; t = 1, \dots, T)$$

$$\sum_{l\in l^1} \gamma_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} - \sum_{i=1}^{m} v_i x_{ij}^{(t)} - \sum_{p=1}^{P} \alpha_{pj} v_p x_{pj}^{(t)} - \sum_{l\in l^1} \gamma_l c_{lj}^{(t-1,1)} \le 0 \; (j = 1, \dots, n; t = 1, \dots, T)$$

$$\sum_{r=1}^{s} \mu_r y_{rj}^{(t)} + \sum_{l\in l^2} \gamma_l c_{lj}^{(t,2)} - \sum_{l\in l^2} \gamma_l c_{lj}^{(t-1,2)} - \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} - \sum_{p=1}^{P} (1-\alpha_{pj}) v_p x_{pj}^{(t)}$$

$$\le 0 \; (j = 1, \dots, n; t = 1, \dots, T)$$

$$L_{pj}^1 \le \alpha_{pj} \le L_{pj}^2$$

$$\nu_i, \nu_p, \mu_r, \gamma_l, \mu_d \geq \varepsilon;\ i = 1, \dots, m; r = 1, \dots, s; l = 1, \dots, L; d = 1, \dots, D;\ p = 1, \dots, P \qquad (18)$$

Model (18) is non-linear since $\alpha_{pj}\nu_p$ is present in the constrains related to stage efficiency. It is possible to obtain a linear model considering that $\beta_{pj} = \alpha_{pj}\nu_p$ ($p = 1, \dots, P, j = 1, \dots, n$). After this substitution, Model (10) can be converted into Model (19).

$$\theta_o^* = max \sum_{t=1}^{T}\sum_{r=1}^{s} \mu_r y_{ro}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lo}^{(t,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D} \mu_d z_{do}^{(t)}$$

$$\sum_{t=1}^{T}\sum_{i=1}^{m} \nu_i x_{io}^{(t)} + \sum_{t=1}^{T}\sum_{p=1}^{P} \nu_p x_{po}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lo}^{(t-1,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D} \mu_d z_{do}^{(t)} = 1$$

$$\sum_{r=1}^{s} \mu_r y_{rj}^{(t)} + \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lj}^{(t,k)} - \sum_{i=1}^{m} \nu_i x_{ij}^{(t)} - \sum_{p=1}^{P} \nu_p x_{pj}^{(t)} - \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} - \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lj}^{(t-1,k)}$$
$$\leq 0\ \ (j = 1, \dots, n; t = 1, \dots, T)$$

$$\sum_{l \in l^1} \gamma_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} - \sum_{i=1}^{m} \nu_i x_{ij}^{(t)} - \sum_{p=1}^{P} \beta_p x_{pj}^{(t)} - \sum_{l \in l^1} \gamma_l c_{lj}^{(t-1,1)} \leq 0\ \ (j = 1, \dots, n; t = 1, \dots, T)$$

$$\sum_{r=1}^{s} \mu_r y_{rj}^{(t)} + \sum_{l \in l^2} \gamma_l c_{lj}^{(t,2)} - \sum_{l \in l^2} \gamma_l c_{lj}^{(t-1,2)} - \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} - \sum_{p=1}^{P} (\nu_p - \beta_{pj}) \nu_p x_{pj}^{(t)}$$
$$\leq 0\ \ (j = 1, \dots, n; t = 1, \dots, T)$$

$$\nu_p L_{pj}^1 \leq \beta_{pj} \leq \nu_p L_{pj}^2$$

$$\nu_i, \nu_p, \mu_r, \gamma_l, \mu_d \geq \varepsilon;\ i = 1, \dots, m; r = 1, \dots, s; l = 1, \dots, L; d = 1, \dots, D;\ p = 1, \dots, P \qquad (19)$$

4.2.3 Efficiency decomposition

After solving Model (19), it is possible to obtain all efficiency scores discussed previously, namely: process efficiency, system efficiency and overall efficiency. Still, it is possible for Model (19) to present alternative optimal solutions. This multiplicity implicates in that the efficiency decomposition may not be unique. To investigate this, a leader-follower approach was adopted. This type of analysis has been employed in several DEA studies, such as Kao and Hwang (2008), Liang et al., (2008) and Li et al., (2018).

A similar framework of Kao and Hwang (2008) and Chen et al. (2010) in which the first division has its efficiency maximized while the overall efficiency is maintained at the level identified with the aid of Model (16) was used. Let $\nu_i^*, \nu_{p,}^*, \mu_{r,}^* \gamma_l^*, \mu_d^*$ be the optimal weights, while θ_o^*, $\theta_o^{(t,sys)*}$, $\theta_o^{(t,1)*}$ and $\theta_o^{(2,sys)*}$ represents the optimal overall, optimal system efficiency by period, division 1 and division 2 at period t efficiency θ_o^* of an observed DMU$_o$. Suppose that the focus is on the maximization of the first stage, while maintaining the system by period and overall score, we have:

$$\theta_o^{(t,1)} = max \frac{\sum_{l\in l^1} f_l c_{lo}^{(t,1)} + \sum_{d=1}^{D} w_d z_{do}^{(t)}}{\sum_{i=1}^{m} v_i x_{io}^{(t)} + \sum_{l\in l^1} f_l c_{lo}^{(t-1,1)} + \sum_{p=1}^{P} \alpha_{pj} v_p x_{po}^{(t)}}$$

s.t. $$\frac{\sum_{l\in l^1} f_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}}{\sum_{i=1}^{m} v_i x_{ij}^{(t)} + \sum_{p=1}^{P} \alpha_{pj} v_p x_{pj}^{(t)} + \sum_{l\in l^1} f_l c_{lj}^{(t-1,1)}} \le 1 \ (j = 1, \dots, n)$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{l\in l^2} f_l c_{lj}^{(t,2)}}{\sum_{l\in l^2} f_l c_{lj}^{(t-1,2)} + \sum_{p=1}^{P} (1-\alpha_{pj}) v_p x_{pj}^{(t)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}} \le 1 \ (j = 1, \dots, n)$$

$$\frac{\sum_{r=1}^{s} u_r y_{ro}^{(t)} + \sum_{d=1}^{D} w_d z_{do}^{(t)} + \sum_{k=1}^{K} \sum_{l=1}^{L} f_l c_{lo}^{(t,k)}}{\sum_{i=1}^{m} v_i x_{io}^{(t)} + \sum_{p=1}^{P} v_p x_{po}^{(t)} + \sum_{d=1}^{D} w_d z_{do}^{(t)} + \sum_{k=1}^{K} \sum_{l=1}^{L} f_l c_{lo}^{(t-1,k)}} = \theta_o^{(t,sys)*}$$

$$\frac{\sum_{t=1}^{T} \sum_{r=1}^{s} u_r y_{ro}^{(t)} + \sum_{t=1}^{T} \sum_{d=1}^{D} w_d z_{do}^{(t)} + \sum_{t=1}^{T} \sum_{k=1}^{K} \sum_{l=1}^{L} f_l c_{lo}^{(t,k)}}{\sum_{t=1}^{T} \sum_{i=1}^{m} v_i x_{io}^{(t)} + \sum_{t=1}^{T} \sum_{p=1}^{P} v_p x_{po}^{(t)} + \sum_{t=1}^{T} \sum_{k=1}^{K} \sum_{l=1}^{L} f_l c_{lo}^{(t-1,k)} + \sum_{t=1}^{T} \sum_{d=1}^{D} w_d z_{do}^{(t)}} = \theta_o^*$$

$$w_1^{t*} * \frac{\sum_{l\in l^1} f_l c_{lo}^{(t,1)} + \sum_{d=1}^{D} w_d z_{do}^{(t)}}{\sum_{l\in l^1} f_l c_{lo}^{(t-1,1)} + \sum_{p=1}^{P} \alpha_{pj} v_p x_{po}^{(t)} + \sum_{i=1}^{m} v_i x_{io}^{(t)}} \le \theta_o^{(t,sys)*}$$

$$L_{pj}^1 \le \alpha_{pj} \le L_{pj}^2$$

$$v_i, u_r, w_l, f_d, v_p \ge \varepsilon; \ i = 1, \dots, m; r = 1, \dots, s; l = 1, \dots, L; d = 1, \dots, D; \ p = 1, \dots, P \quad (20)$$

Model (20) can be converted in a linear programming as displayed in Model (21).

$$\theta_o^{(t,1)*} = max \sum_{l\in l^1} \gamma_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} \mu_d z_{dj}^{(t)}$$

$$\sum_{i=1}^{m} v_i x_{io}^{(t)} + \sum_{p=1}^{P} \beta_p x_{po}^{(t)} + \sum_{l\in l^1} \gamma_l c_{lo}^{(t-1,1)} = 1$$

$$\sum_{l\in l^1} \gamma_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} - \sum_{i=1}^{m} v_i x_{io}^{(t)} - \sum_{p=1}^{P} \beta_p x_{pj}^{(t)} - \sum_{l\in l^1} \gamma_l c_{lj}^{(t-1,1)} \le 0 \ \ (j = 1, \dots, n)$$

$$\sum_{r=1}^{s} \mu_r y_{rj}^{(t)} + \sum_{l\in l^2} \gamma_l c_{lj}^{(t,2)} - \sum_{l\in l^2} \gamma_l c_{lj}^{(t-1,2)} - \sum_{p=1}^{P} (v_p - \beta_{pj}) v_p x_{pj}^{(t)} - \sum_{d=1}^{D} \mu_d z_{dj}^{(t)} \le 0 \ \ (j = 1, \dots, n)$$

$$\sum_{r=1}^{s} \mu_r y_{ro}^{(t)} + \sum_{d=1}^{D} \mu_d z_{do}^{(t)} + \sum_{k=1}^{K} \sum_{l=1}^{L} \gamma_l c_{lo}^{(t,k)} - \theta_o^{(t,sys)*} \left(\sum_{i=1}^{m} v_i x_{io}^{(t)} + \sum_{p=1}^{P} v_p x_{po}^{(t)} + \sum_{d=1}^{D} \mu_d z_{do}^{(t)} + \sum_{k=1}^{K} \sum_{l=1}^{L} \gamma_l c_{lo}^{(t-1,k)} \right) \le 0$$

$$\sum_{t=1}^{T}\sum_{r=1}^{s}\mu_r y_{ro}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{do}^{(t)} - \theta_o^*\left(\sum_{t=1}^{T}\sum_{i=1}^{m}\nu_i x_{io}^{(t)} + \sum_{t=1}^{T}\sum_{p=1}^{P}\nu_p x_{po}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t-1,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{do}^{(t)}\right) \leq 0$$

$$w_1^{t*} * \left(\sum_{l\in l^1}\gamma_l c_{lj}^{(t,1)} + \sum_{d=1}^{D}\mu_d z_{dj}^{(t)}\right) \leq \theta_o^{(t,sys)*}$$

$$\nu_p L_{pj}^1 \leq \beta_{pj} \leq \nu_p L_{pj}^2$$

$$\nu_i, \nu_p, \mu_r, \gamma_l, \mu_d \geq \varepsilon;\ i = 1, \dots, m; r = 1, \dots, s; l = 1, \dots, L; d = 1, \dots, D;\ p = 1, \dots, P \qquad (21)$$

As previously discussed, the system efficiency is a weighted average of the stages, therefore is possible to obtain the efficiency of the second stage as $\theta_o^{(t,2)} = \frac{\theta_o^{(t,sys)*} - w_1^{t*}\theta_o^{(t,1)*}}{w_2^{t*}}$. It is important to highlight that $\theta_o^{(t,sys)*}$, w_1^{t*} and w_2^{t*} are obtained with the optimal solution of Model (19) and $\theta_o^{(t,1)*}$ indicates that the efficiency of the Stage 1 was prioritized and optimized first. The same hypotheses can be used to investigate Stage 2 efficiency, as shown in Model (22).

$$\theta_o^{(t,2)*} = max \sum_{r=1}^{s}\mu_r y_{ro}^{(t)} + \sum_{l\in l^2}\gamma_l c_{lo}^{(t,2)}$$

$$\sum_{l\in l^2}\gamma_l c_{lo}^{(t-1,2)} - \sum_{d=1}^{D}\mu_d z_{do}^{(t)} - \sum_{p=1}^{P}(\nu_p - \beta_{po})\nu_p x_{po}^{(t)} = 1$$

$$\sum_{l\in l^1}\gamma_l c_{lj}^{(t,1)} + \sum_{d=1}^{D}\mu_d z_{dj}^{(t)} - \sum_{i=1}^{m}\nu_i x_{io}^{(t)} - \sum_{p=1}^{P}\beta_p x_{pj}^{(t)} - \sum_{l\in l^1}\gamma_l c_{lj}^{(t-1,1)} \leq 0\ \ (j = 1, \dots, n)$$

$$\sum_{r=1}^{s}\mu_r y_{rj}^{(t)} + \sum_{l\in l^2}\gamma_l c_{lj}^{(t,2)} - \sum_{l\in l^2}\gamma_l c_{lj}^{(t-1,2)} - \sum_{p=1}^{P}(\nu_p - \beta_{pj})\nu_p x_{pj}^{(t)} - \sum_{d=1}^{D}\mu_d z_{dj}^{(t)} \leq 0\ \ (j = 1, \dots, n)$$

$$\sum_{r=1}^{s}\mu_r y_{ro}^{(t)} + \sum_{d=1}^{D}\mu_d z_{do}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t,k)} - \theta_o^{(t,sys)*}\left(\sum_{i=1}^{m}\nu_i x_{io}^{(t)} + \sum_{p=1}^{P}\nu_p x_{po}^{(t)} + \sum_{d=1}^{D}\mu_d z_{do}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t-1,k)}\right) \leq 0$$

$$\sum_{t=1}^{T}\sum_{r=1}^{s}\mu_r y_{ro}^{(t)}+\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t,k)}+\sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{dj}^{(t)} - \theta_o^*\left(\sum_{t=1}^{T}\sum_{i=1}^{m}\nu_i x_{io}^{(t)}+\sum_{t=1}^{T}\sum_{p=1}^{P}\nu_p x_{po}^{(t)}+\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t-1,k)}+\sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{do}^{(t)}\right) \le 0$$

$$w_2^{t*} * \left(\sum_{r=1}^{s}\mu_r y_{ro}^{(t)}+\sum_{l\in l^2}\gamma_l c_{lo}^{(t,2)}\right) \le \theta_o^{(t,sys)*}$$

$$\nu_p L_{pj}^1 \le \beta_{pj} \le \nu_p L_{pj}^2$$

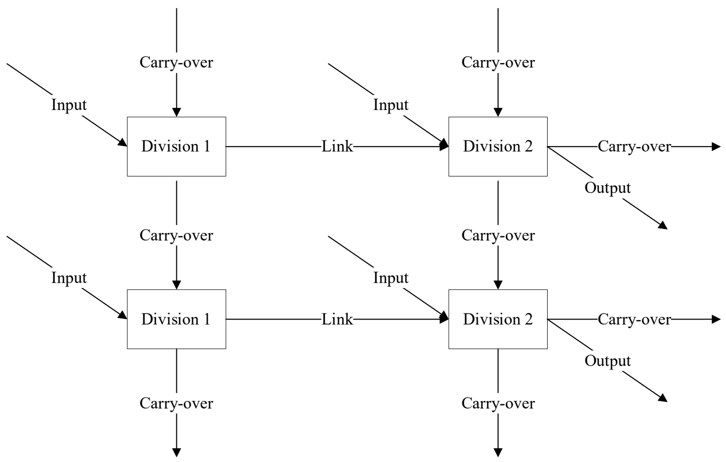
$$\nu_i,\nu_p,\mu_r,\gamma_l,\mu_d \ge \varepsilon;\ i=1,\dots,m; r=1,\dots,s; l=1,\dots,L; d=1,\dots,D;\ p=1,\dots,P \quad (22)$$

It is possible to obtain the efficiency of the first stage as $\theta_o^{(t,1)}=\frac{\theta_o^{(t,sys)*}-w_2^{t*}\theta_o^{(t,2)*}}{w_1^{t*}}$. It is important to mention that the proposed models and evaluation must be used for each period t under analysis. If $\theta_o^{(t,1)}=\theta_o^{(t,1)*}$ or, $\theta_o^{(t,2)}=\theta_o^{(t,2)*}$, there is a unique decomposition.

4.3 A GENERALIZATION OF THE DNDEA MODEL FOR CASES WITHOU SHARED RESOURCES

This subsection details a model that does not consider shared resources and allows for exogenous inputs in the second division, as illustrated in Figure 16. In this framework, $x_{ij}^{(t,k)}$ refers to the *ith* specific input of DMU *j* in division *k* in period *t*. The same hypothesis discussed in the previous are applied, and the system's efficiency consider a weighted average of Division 1 and 2 for each period as displayed in (23).

Figure 16 - Two-stage dynamic DEA framework



Source: The Author (2024).

$$E_j^{(t,sys)} = w_1^t * \frac{\sum_{l\in l^1} f_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}}{\sum_{l\in l^1} f_l c_{lj}^{(t-1,1)} + \sum_{m\in m^1} v_i x_{ij}^{(t,1)}} + w_2^t * \frac{\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{l\in l^2} f_l c_{lj}^{(t,2)}}{\sum_{l\in l^2} f_l c_{lj}^{(t-1,2)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{m\in m^2} v_i x_{ij}^{(t,2)}} \quad (23)$$

Where $w_1^t + w_2^t = 1$ and are defined as follows:

$$w_1^t = \frac{\sum_{l\in l^1} f_l c_{lj}^{(t-1,1)} + \sum_{m\in m^1} v_i x_{ij}^{(t,1)}}{\sum_{m\in m^1} v_i x_{ij}^{(t,1)} + \sum_{m\in m^2} v_i x_{ij}^{(t,2)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t-1,k)}} \ and$$

$$w_2^t = \frac{\sum_{l\in l^2} f_l c_{lj}^{(t-1,2)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{m\in m^2} v_i x_{ij}^{(t,2)}}{\sum_{m\in m^1} v_i x_{ij}^{(t,1)} + \sum_{m\in m^2} v_i x_{ij}^{(t,2)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t-1,k)}} \quad (24)$$

Therefore, the system efficiency in each period is detailed in (25).

$$E_j^{(t,sys)} = \frac{\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t,k)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}}{\sum_{k=1}^{K}\sum_{i=1}^{m} v_i x_{ij}^{(t,k)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t-1,k)}} \quad (25)$$

We also considered that the overall efficiency is a weighted average of the system efficiency in each period. Therefore, we define w^t in (26).

$$w^t = \frac{\sum_{k=1}^{K}\sum_{i=1}^{m} v_i x_{ij}^{(t,k)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t-1,k)}}{\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{i=1}^{m} v_i x_{ij}^{(t,k)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t-1,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D} w_d z_{dj}^{(t)}} \quad (26)$$

Considering that the overall efficiency is a weighted average of period efficiency, the overall efficiency score of the two-stage process for DMU_o can be evaluated by solving the following fractional program (27).

$$\theta_o^* = Max \frac{\sum_{t=1}^{T}\sum_{r=1}^{s} u_r y_{ro}^{(t)} + \sum_{t=1}^{T}\sum_{d=1}^{D} w_d z_{do}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lo}^{(t,k)}}{\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{i=1}^{m} v_i x_{io}^{(t,k)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lo}^{(t-1,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D} w_d z_{do}^{(t)}}$$

$$\text{s.t} \quad \frac{\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t,k)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}}{\sum_{k=1}^{K}\sum_{i=1}^{m} v_i x_{ij}^{(t,k)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L} f_l c_{lj}^{(t-1,k)}} \le 1 \ (j = 1, \dots, n; t = 1, \dots, T)$$

$$\frac{\sum_{l\in l^1} f_l c_{lj}^{(t,1)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)}}{\sum_{l\in l^1} f_l c_{lj}^{(t-1,1)} + \sum_{m\in m^1} v_i x_{ij}^{(t,1)}} \le 1 \ (j = 1, \dots, n; t = 1, \dots, T)$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{l\in l^2} f_l c_{lj}^{(t,2)}}{\sum_{l\in l^2} f_l c_{lj}^{(t-1,2)} + \sum_{d=1}^{D} w_d z_{dj}^{(t)} + \sum_{m\in m^2} v_i x_{ij}^{(t,2)}} \le 1 \ (j = 1, \dots, n; t = 1, \dots, T)$$

$$v_i, u_{r,} w_l, f_d, v_p \ge \varepsilon; \ \ i = 1, \dots, m; r = 1, \dots, s; l = 1, \dots, L; d = 1, \dots, D; \ p = 1, \dots, P \quad (28)$$

With the aid of the Charnes–Cooper transformation, the fractional program proposed in Model (28) can be converted into Model (29).

$$\theta_o^* = max \sum_{t=1}^{T}\sum_{r=1}^{s} \mu_r y_{ro}^{(t)} + \sum_{t=1}^{T}\sum_{d=1}^{D} \mu_d z_{do}^{(t)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L} \gamma_l c_{lo}^{(t,k)}$$

$$\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{i=1}^{m}\nu_i x_{io}^{(t,k)} + \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t-1,k)} + \sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{do}^{(t)} = 1$$

$$\sum_{r=1}^{s}\mu_r y_{rj}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lj}^{(t,k)} + \sum_{d=1}^{D}\mu_d z_{dj}^{(t)} - \sum_{k=1}^{K}\sum_{i=1}^{m}\nu_i x_{ij}^{(t,k)} - \sum_{d=1}^{D}\mu_d z_{dj}^{(t)} - \sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lj}^{(t-1,k)}$$
$$\leq 0 \ \ (j = 1,\dots,n; t = 1,\dots,T)$$

$$\sum_{l\in l^1}\gamma_l c_{lj}^{(t,1)} + \sum_{d=1}^{D}\mu_d z_{dj}^{(t)} - \sum_{l\in l^1}\gamma_l c_{lj}^{(t-1,1)} - \sum_{m\in m^1}\nu_i x_{ij}^{(t,1)} \leq 0 \ \ (j = 1,\dots,n; t = 1,\dots,T)$$

$$\sum_{r=1}^{s}\mu_r y_{rj}^{(t)} + \sum_{l\in l^2}\gamma_l c_{lj}^{(t,2)} - \sum_{l\in l^2}\gamma_l c_{lj}^{(t-1,2)} - \sum_{d=1}^{D}\mu_d z_{dj}^{(t)} - \sum_{m\in m^2}\nu_i x_{ij}^{(t,2)} \leq 0 \ \ (j = 1,\dots,n; t = 1,\dots,T)$$

$$\nu_i, \nu_p, \mu_r, \gamma_l, \mu_d \geq \varepsilon; \ i = 1,\dots,m; r = 1,\dots,s; l = 1,\dots,L; d = 1,\dots,D; \ p = 1,\dots,P \qquad (29)$$

After solving Model (29), it is possible to obtain all efficiency scores discussed previously, namely: process efficiency, system efficiency and overall efficiency. We proceed with efficiency decomposition similar to the procedure described in Section 3. The first division has its efficiency maximized while the overall efficiency is maintained at the level identified with the aid of Model (22).

Let $\nu_i^*, \nu_{p,}^*, \mu_{r,}^*\gamma_l^*, \mu_d^*$ be the optimal weights, while θ_o^*, $\theta_o^{(t,sys)*}$, $\theta_o^{(t,1)*}$ and $\theta_o^{(2,sys)*}$ represents the optimal overall, optimal system efficiency by period, Division 1 and Division 2 at period *t* efficiency θ_o^* of an observed DMU$_o$. Suppose the focus lies on the maximization of the first stage, while maintaining the system by period and overall score, we have:

$$\theta_o^{(t,1)*} = max\sum_{l\in l^1}\gamma_l c_{lo}^{(t,1)} + \sum_{d=1}^{D}\mu_d z_{do}^{(t)}$$

$$\sum_{m\in m^1}\nu_i x_{io}^{(t,1)} + \sum_{l\in l^1}\gamma_l c_{lo}^{(t-1,1)} = 1$$

$$\sum_{l\in l^1}\gamma_l c_{lj}^{(t,1)} + \sum_{d=1}^{D}\mu_d z_{dj}^{(t)} - \sum_{m\in m^1}\nu_i x_{ij}^{(t,1)} - \sum_{l\in l^1}\gamma_l c_{lj}^{(t-1,1)} \leq 0 \ \ (j = 1,\dots,n)$$

$$\sum_{r=1}^{s}\mu_r y_{rj}^{(t)} + \sum_{l\in l^2}\gamma_l c_{lj}^{(t,2)} - \sum_{l\in l^2}\gamma_l c_{lj}^{(t-1,2)} - \sum_{m\in m^2}\nu_i x_{ij}^{(t,2)} - \sum_{d=1}^{D}\mu_d z_{dj}^{(t)} \leq 0 \ \ (j = 1,\dots,n)$$

$$\sum_{r=1}^{s}\mu_r y_{ro}^{(t)} + \sum_{d=1}^{D}\mu_d z_{do}^{(t)} + \sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t,k)}$$
$$- \theta_o^{(t,sys)*}\left(\sum_{k=1}^{K}\sum_{i=1}^{m}\nu_i x_{io}^{(t,k)} - \sum_{d=1}^{D}\mu_d z_{do}^{(t)} - \sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t-1,k)}\right) \leq 0$$

$$\sum_{t=1}^{T}\sum_{r=1}^{s}\mu_r y_{ro}^{(t)}+\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t,k)}+\sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{do}^{(t)}$$
$$-\theta_o^*\left(\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{i=1}^{m}\nu_i x_{io}^{(t,k)}+\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t-1,k)}+\sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{do}^{(t)}\right)\leq 0$$

$$w_1^{t*}*\left(\sum_{l\in l^1}\gamma_l c_{lo}^{(t,1)}+\sum_{d=1}^{D}\mu_d z_{do}^{(t)}\right)\leq\theta_o^{(t,sys)*}$$

$$\nu_i,\nu_p,\mu_r,\gamma_l,\mu_d\geq\varepsilon;\ i=1,\dots,m;r=1,\dots,s;l=1,\dots,L;d=1,\dots,D;\ p=1,\dots,P \quad (30)$$

As previously discussed, the system efficiency is a weighted average of the stages, therefore is possible to obtain the efficiency of the second stage as $\theta_o^{(t,2)}=\frac{\theta_o^{(t,sys)*}-w_1^{t*}\theta_o^{(t,1)*}}{w_2^{t*}}$. The same hypotheses can be used to investigate Stage 2 efficiency, as shown in Model (31).

$$\theta_o^{(t,2)*}=max\sum_{r=1}^{s}\mu_r y_{ro}^{(t)}+\sum_{l\in l^2}\gamma_l c_{lo}^{(t,2)}$$

$$\sum_{m\in m^2}\nu_i x_{io}^{(t,2)}+\sum_{l\in l^2}\gamma_l c_{lo}^{(t-1,2)}+\sum_{d=1}^{D}\mu_d z_{do}^{(t)}=1$$

$$\sum_{l\in l^1}\gamma_l c_{lj}^{(t,1)}+\sum_{d=1}^{D}\mu_d z_{dj}^{(t)}-\sum_{m\in m^1}\nu_i x_{ij}^{(t,1)}+\sum_{l\in l^1}\gamma_l c_{lj}^{(t-1,1)}\leq 0\ \ (j=1,\dots,n)$$

$$\sum_{r=1}^{s}\mu_r y_{rj}^{(t)}+\sum_{l\in l^2}\gamma_l c_{lj}^{(t,2)}-\sum_{l\in l^2}\gamma_l c_{lj}^{(t-1,2)}-\sum_{m\in m^2}\nu_i x_{ij}^{(t,2)}-\sum_{d=1}^{D}\mu_d z_{dj}^{(t)}\leq 0\ \ (j=1,\dots,n)$$
$$\leq 0\ \ (j=1,\dots,n)$$

$$\sum_{r=1}^{s}\mu_r y_{ro}^{(t)}+\sum_{d=1}^{D}\mu_d z_{do}^{(t)}+\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t,k)}$$
$$-\theta_o^{(t,sys)*}\left(\sum_{k=1}^{K}\sum_{i=1}^{m}\nu_i x_{io}^{(t,k)}-\sum_{d=1}^{D}\mu_d z_{do}^{(t)}-\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t-1,k)}\right)\leq 0$$

$$\sum_{t=1}^{T}\sum_{r=1}^{s}\mu_r y_{ro}^{(t)}+\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t,k)}+\sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{do}^{(t)}$$
$$-\theta_o^*\left(\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{i=1}^{m}\nu_i x_{io}^{(t,k)}+\sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{l=1}^{L}\gamma_l c_{lo}^{(t-1,k)}+\sum_{t=1}^{T}\sum_{d=1}^{D}\mu_d z_{do}^{(t)}\right)\leq 0$$

$$w_2^{t*}*\left(\sum_{r=1}^{s}\mu_r y_{ro}^{(t)}+\sum_{l\in l^2}\gamma_l c_{lo}^{(t,2)}\right)\leq\theta_o^{(t,sys)*}$$

$$\nu_i,\nu_p,\mu_r,\gamma_l,\mu_d\geq\varepsilon;\ i=1,\dots,m;r=1,\dots,s;l=1,\dots,L;d=1,\dots,D;\ p=1,\dots,P \quad (31)$$

It is possible to obtain the efficiency of the first stage as $\theta_o^{(t,1)} = \frac{\theta_o^{(t,sys)*} - w_2^{t*}\theta_o^{(t,2)*}}{w_1^{t*}}$. It is important to mention that the proposed models and evaluation must be used for each period t under analysis. If $\theta_o^{(t,1)} = \theta_o^{(t,1)*}$ or, $\theta_o^{(t,2)} = \theta_o^{(t,2)*}$, there is a unique decomposition.

4.4 EXPLORING THE CASE STUDY

In Brazil, there are four distinct groups of Higher Education Institutions (HEIs): universities, university centers, faculties, and federal institutes. We can classify HEIs into four administrative categories: federal, state, municipal, and private.

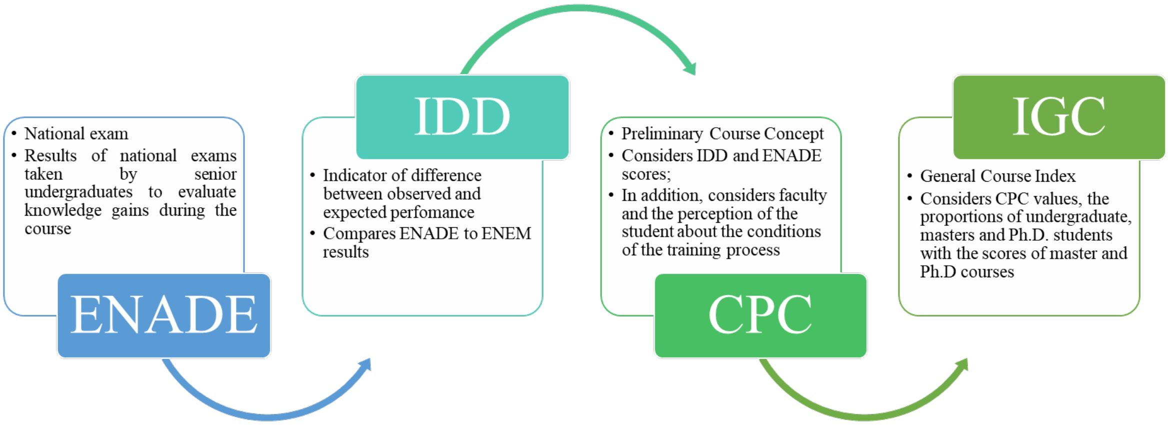
In 2004, Law 10.861/2004 created the National Higher Education Assessment System (SINAES) to improve the results of Brazilian higher education. SINAES consists of three main components: the evaluation of institutions, courses, and student performance. The Anísio Teixeira National Institute for Educational Research and Studies (INEP) annually performs a census for the higher education sector, collecting data about students, courses, and universities to obtain indicators used in SINAES to assess and accredit courses and institutions.

According to Normative Ordinance n. 550 (BRAZIL, 2007), SINAES is composed of six quality metrics: Institutional evaluation (AVALIES), course evaluation (ACG), General Index of Courses (IGC), Preliminary Concept of Courses (CPC), Indicator of the Difference between Observed and Expected Performances (IDD) and the National Student Performance Examination (ENADE). The last four converge in their results, but have little communication with the first two and only these four have their results released annually: ENADE, since 2004, while CPC and IGC since 2007 and IDD since 2014.

Considering all the above, it is possible to affirm that SINAES is a complex process involving different time periods and multiple tools, and it also enables the production, dissemination and management of indicators and information (such as census of higher education, teachers ‘records, record of educational institutions and courses) for Brazilian HEIs (BRUNSTEIN *et al.*, 2015).

In the case of universities, the General Course Index (IGC) is the quality indicator used to rank and guide universities' evaluation. It considers metrics of the quality of all undergraduate, master, and doctoral courses at an HEI, aggregating them all into one indicator. However, it is necessary to provide targets or projections of how each educational level should improve to enhance the institution's performance.

Figure 17 - Description of Brazilian quality indicators



Source: The Author (2024).

As shown in Figure 17, the indicators directly impact each other. However, it is important to mention that the CPC mainly uses metrics aggregated by factors that are not clearly discussed and justified in the technical note issued by the government. For this reason and for the fact that there is no indicator to aggregate and show a global overview of postgraduate activities, the current work proposes a method to limit this gap.

The implementation of graduate studies in Brazil took place through the standards defined by Report CFE 977 of 1965. National discussions are taking place to reformulate the evaluation process of graduate programs in Brazil. Coordination for the Improvement of Higher Education Personnel (CAPES) evaluates graduate programs concerning the National Graduate Plan (PNPG) guidelines. Currently, the seventh PNPG is in effect, but we are using data for the period (2019-2020) contemplated by the sixth plan. Thus, our results can help in this discussion and foster the evaluation for the seventh plan, which will still happen

For the sixth plan, we had political and economic crises (2011-2020). After 2015 and the impeachment of then-president Dilma Roussef, there was a reduction in federal government transfers to higher education, with budget cuts in science and technology. The scenario becomes even worse after 2019 with the contingency of part of the budget directed to discretionary spending by federal universities, including payment of academic grants and research inputs.

Our sample concerns the Federal Universities because: i) they are responsible for more than half of the country's master's and doctoral courses and students and produce most of the national science (ANDIFES, 2017); ii) they use public funds to finance their activities, and iii) they represent a set of more homogeneous institutions. (FRANÇA, DE; FIGUEIREDO, DE; LAPA, DOS, 2010) investigated the impact of information asymmetry on organizational efficiency using data about Brazilian undergraduate courses. Miranda, Gramani and Andrade

(2012) used DEA and SFA to investigate the efficiency of undergraduate business administration courses. Zoghbi, Rocha and Mattos (2013) applied ordinary least squares and SFA to investigate differences in private and public Brazilian universities performance. Hammes Junior and Mattos (2020) addresses the efficiency of public expenditure in federal universities.

The previous studies differ because they simultaneously considered financial efficiency and how to allocate public resources among undergraduate and graduate activities. However, the discussions disregard time effects and are redundant since they used quality metrics as outputs, and the referred indicators contemplate in their composition the same inputs used in the evaluation. More recently, Wanke et al. (2022) used a hybrid multi-criteria decision-making approach to investigate educational institution performance between 2014 and 2017. The authors verified that HEIs did not improve significantly in the considered time frame.

None of those studies analyzed the case of Brazilian graduate programs in depth. In addition, investigations contemplating the productivity changes of these institutions are even scarcer. Thus, considering these aspects and the current discussion on evaluating graduate programs and universities, a DNDEA model was implemented to help the discussions. As previously mentioned, universities present a multi-activity framework. In this study, we focus on the graduate activities in federal universities, directly responsible for a significant part of Brazilian research.

The literature presents several methodological developments of DEA; dynamic and network models are among the most recent DEA research fronts (LIU; LU; LU, 2016). The choice of dynamic models to evaluate the graduate process is because the activities repeat from period to period, and the outcomes of one period can impact the following. Dynamic modeling considers consecutive periods and changes in efficiency between these periods. When several periods with inter-relations are involved, the overall efficiency must be measured dynamically, considering the inter-relationship between consecutive periods; otherwise, the resulting efficiency measures will be misleading (KAO, 2013b). On the other hand, network models regard the internal structure of DMUs to measure efficiency, revealing the transformation process and accounting for divisional and overall efficiency in a unified way. Therefore, the total efficiency of DMUs is the main objective which involves divisional efficiencies as its components (TONE; TSUTSUI, 2009b).

Our DNDEA model considers two stages: the formative process and the scientific production process. In the formative process, universities employ resources to train students: faculty and enrolled students are views as inputs. The number of programs available in a university represents the carry-over variable, while master dissertations and Ph.D. thesis correspond to the intermediate factor linking the stages. Variable dropout reflects that some students do not finish their master's or Ph.D. training. This variable is an undesirable output and requires treatment to be used in the DEA framework. We subtracted values from a large number, ensuring the results were isotonic as discussed by Dyson et al. (2001).

The second stage (the scientific production process) converts the products of the formative process into research products: dissertations and thesis correspond to the research developed, representing the basis for generating papers and patents. The publications considered are in the SCOPUS database.

According to data released by the 2020 Higher Education Census, there are 68 federal universities in Brazil. The reduction in the number of universities analyzed was due to a lack of data on one or more variables, mainly in patents and Ph.D. Thesis. Consequently, our sample contains 32 universities, with data from 2019 to 2020. The selected time frame aims to evaluate the most recent available data and obtain a glimpse of the COVID-19 pandemic's impact on graduate activities. Table 13 presents the descriptive statistics of the data.

Table 13 - Descriptive statistics of data

Variable	Category	Average	SD	Minimum	Maximum
Formative process					
Faculty (number)	Input	1033,42	700,22	235	2913
Enrollments (number)	Input	2833,43	2289,97	321	9163
Programs (number)	Carry-over	100,94	71,71	9	321
Dropouts (number)	Output	40,35	23,49	7	89
Ph.D. Thesis (number)	Link	41,95	23,53	9	91
Master Dissertations (number)	Link	219,78	218,70	8	954
Scientific production process		584,60	408,58	84	1786
Publications (number)	Output	3019,22	2228,38	326	10400
Patents (number)	Output	52,32	51,55	1	210

Source: The Author (2024).

Because of the relevance of DNDEA models and the large amount of information generated, the current study proposes a bi-dimensional representation of the DNDEA model of Omrani and Soltanzadeh (2016). The following section presents a brief overview of frontier representation alternatives and how our approach diverges from them.

The application of DEA in the educational context goes back to the beginning of applied studies using the technique. The discussions employing DEA includes analysis in distinct education levels and for distinct types of investigations. (LIU et al., 2013a) indicates that there are two main paths when analyzing DEA development in the education field: higher education and basic education. In the context of higher education, Johnes (2015) details a broad range of topics covered with DEA studies, such as: university efficiency, efficiency of individual academic departments or programs within an institution, central administration or services across universities. (JOHNES; PORTELA; THANASSOULIS, 2017) also highlights using student ratings to assess performance in tertiary education, while (THANASSOULIS *et al.*, 2016b) details a new range of investigations such as cost efficiency, technical efficiency, research performance, rankings, personal and teaching evaluations in the higher education with DEA.

This breadth of investigations points out that higher education has a multi-activity framework, as well as there is a need to analyze educational issues over time in dynamic settings (JOHNES; PORTELA; THANASSOULIS, 2017; THANASSOULIS et al., 2016b).

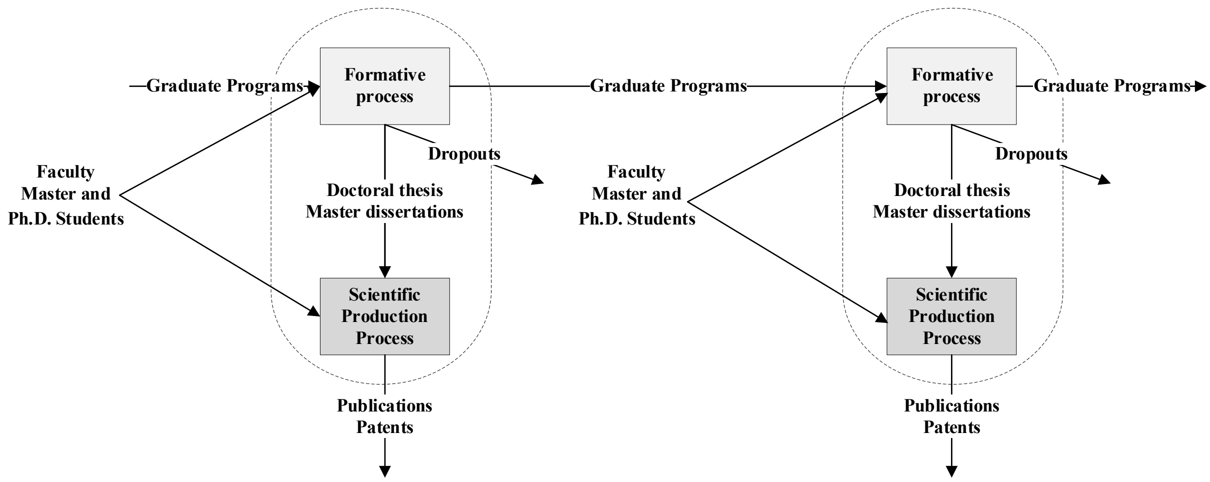
4.5 RESULTS AND DISCUSSIONS

First, we present the DNDEA efficiencies of the 32 federal universities with the aid of the bidimensional representation. Then, we discuss the efficiency decomposition under the leader-follower assumption with the procedure detailed in Section 4.2.

4.5.1 DNDEA efficiency results

The framework and variables in the proposition for investigation of graduate activities are displayed in Figure 18. We applied the developed DNDEA model discussed in Section 3 to investigate graduate activities in Brazilian Federal universities.

Figure 18 - Two-stage dynamic DEA model with shared inputs for graduate activities



Source: The Author (2024).

The choice of Federal institution resides in their responsibility for the most significant part of Brazilian research. Because public resources fund them, students do not pay tuition fees, and some also receive grants to finance the research from Brazilian research agencies. DNDEA models are especially pertinent to this case since the activities repeat from period to period, the outcomes of one period can impact the following one, and the network structure of the model accurately portrays the internal structure of the graduate process in these universities.

The proposed framework considers two divisions. The first represents the formative process, while the second corresponds to scientific production. In the formative process, universities employ resources to train students, and their education should generate products. Our proposal considers the faculty, master, and Ph.D. enrolled students as inputs. The number of programs available in a university represents the carry-over variable, while master dissertations and Ph.D. thesis correspond to the intermediate factor linking the stages. It is considered that some students do not finish their master's or Ph.D. training, which is also contemplated in the model with the dropout variable. This variable is undesirable and must undergo treatment before its use in modeling. The values were considered as inputs in the first stage, as one of the possible treatments, as indicated by Dyson et al. (2001).

However, faculty and students divide their workloads between both processes. Thus, entirely allocating these inputs to the first stage would be inappropriate and penalize its efficiency. In this way, these inputs are shared between the stages. Therefore, the second stage (the scientific production process) converts the products of the formative process using a part of the students and faculty workload into research products. In this sense, the dissertations and thesis correspond to the research developed, representing the basis for generating papers and patents. The publications considered are indexed in the SCOPUS database.

These variables were selected due to their relevance to the national reality. These variables are already used for the individual evaluation of programs, and most of them are also used in international literature and university rankings. We emphasize that the choice to use DEA aims to mitigate one of the main criticisms verified among the government's already-used indicators. Brazil is a country with very different regions in socio-economic and demographic terms. This national characteristic is reflected in the universities' very different missions and objectives. Therefore, the flexibility of the weights to weight the criteria is essential so that each university has the autonomy to reflect these characteristics and that the final result is not questioned, claiming that the weighting of the criteria benefited some to the detriment of others.

According to data released by the 2020 Higher Education Census, there are 68 federal universities in Brazil. The reduction in the number of universities analyzed was due to a lack of data on one or more variables, mainly in patents and Ph.D. Thesis. The sample contains 32 universities. Data correspond to the years 2019 and 2020. The selected time frame aims to evaluate the most recent available data and obtain a glimpse of the COVID-19 pandemic's impact on graduate activities. Reports generated by CAPES correspond to the data source used. As previously mentioned, CAPES is responsible for evaluating and consolidating information regarding individual graduate activities in Brazil.

First, we applied Model (31), considering 0.40 and 0.70 as lower and upper bounds for both shared inputs. Table 14 displays the descriptive statistics for all the efficiency results and has the overall efficiency in the second column. Columns three and four report the system efficiency, while five to eight present the process efficiencies for 2019 and 2020.

Table 14 - Descriptive results of the efficiencies

	$E^{(sys)}$	$E^{(1,sys)}$	$E^{(2,sys)}$	$E^{(1,1)}$	$E^{(1,2)}$	$E^{(2,1)}$	$E^{(2,2)}$
Mean	80,97%	79,59%	82,75%	88,63%	65,55%	81,87%	82,14%
S.D	5,07%	7,13%	5,89%	7,74%	11,23%	7,68%	10,97%
Max	89,57%	92,31%	94,47%	100%	89,29%	97,91%	100%
Min	68,94%	66,40%	66,77%	71,62%	46,42%	64,45%	58,66%

Source: The Author (2024).

The average overall efficiency of the considered period is 80,97%. When observing the periods, 2019 obtained an average result of 79,59%, while 2020 returned 82,75%. When analyzing the average division values, it is possible to verify that 2020 returned higher efficiency scores, and the increase in performance in scientific production can explain such results. The training process showed an efficiency decline of 6,76% (88,63% in 2019 to 81,87% in 2020). 26 of the 32 DMUs showed reduced efficiency when comparing the periods. On the other hand, there was an increase of 16,59% in efficiency (65,55% in 2019 to 82,14% in 2020) in the scientific production process. 30 of the 32 DMUs displayed increased performance.

Considering DNDEA scores, federal universities could increase their efficiency in a network structure of the formative process and scientific production by approximately 19,03%. The scores in Table 14 indicate that, on average, the training process had better results than the scientific production process before the COVID pandemic. However, in 2020, the average values are closer (81,87% and 82,14%), but with better results for the scientific production process.

The number of publications explains the better performance of the scientific production process in 2020. When comparing 2019 with 2020, there is a reduction in theses and dissertations numbers for more than 90% of the DMUs. However, the number of publications grew for all DMUs, and approximately 60% of DMUs also saw increased patent numbers.

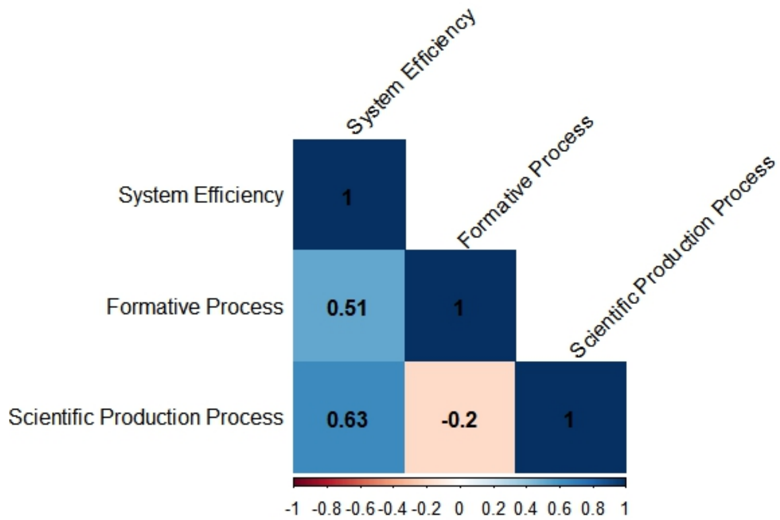
The performance fluctuations in 2020 may also be related to the COVID-19 pandemic. Teaching activities were suspended for several periods in Brazilian HEI, which corresponded for most of the year. During this interval, research activities and, consequently, publications derived from these researches have continued remotely. In addition, the significant impacts of the pandemic on the most diverse areas of knowledge and the need for quick responses stimulated the development of a high amount of research, as seen in special COVID-specific discussion sections at scientific events and special issues in various journals.

However, the verified impact on teaching activities was negative. Learning in remote teaching requires a learning curve for both students and teachers. It is also worth noting that, unfortunately, access to the internet with the minimum conditions necessary to participate in activities was a problem for some of the students, with classes being one of the activities most affected by these issues, directly impacting teaching and learning.

Figure 18 shows Spearman's correlation coefficients among the efficiencies of the formative process, scientific production process, and system efficiency over the two periods. The values indicate that the system efficiency strongly correlates to the formative and scientific production processes with a correlation value of 0,51 and 0,63 at the 5% significance level.

The correlation analysis indicates that the scientific production process presents a higher correlation with the system's performance. The scientific process plays an indispensable role in disseminating the research produced in the university to the academic community and society. It is important to note that in the period before the pandemic, the performance of this stage was significantly lower than the training process. These results indicate that the investigation of more recent data is necessary to verify if the increase in performance remains or if the difficulties verified in 2019 persist, indicating a significant difficulty in disseminating the produced knowledge beyond the university.

Figure 19 - Correlation matrix for the overall and divisional efficiencies 2019–2020



Source: The Author (2024).

The investigation of more recent data is indispensable because funds directed to graduate activities in Brazil have been reduced drastically over the last decade. As pointed out by the UNESCO report, the increase in publications over the last years indicates that Brazilian research is resilient. However, resilience also has its limits. Therefore, it is relevant to understand if the lower performance in the scientific production verified in 2019 can be related to difficulties in research funding. This topic becomes even more critical in the context of the migration of several journals to the open-access format, consequently increasing publishing costs. The increasing costs in a scenario of successive cuts in public funds can negatively impact the number of publications in Brazilian public universities.

The correlation between formative process and scientific production is negative. This result supports the previous discussion of the possible difficulty of transforming knowledge in products. Given that the thesis and dissertations consist of second-stage inputs, there must be an effort to increase them in order to obtain better results for this process. However, although most universities increased their performance in 2020 in this process, there is still room for improvement.

4.5.2 Efficiency decomposition

Tables 15 and 16 present the efficiency decomposition results. The first one portrays the case when the first stage is prioritized, while the second views Stage 2 as a leader. Besides efficiency values, these tables also present the optimal proportions of each shared input for all years under investigation.

Table 15 - Results Stage 1 as leader

University	2019				2020			
	α_1	α_2	Formative Process	Scientific Production	α_1	α_2	Formative Process	Scientific Production
UFSCPA	0,4	0,7	0,7166	0,7764	0,4	0,7	0,9977	0,8574
UFMS	0,5568	0,7	0,9306	0,8258	0,7	0,7	0,7998	0,9886
UFRR	0,7	0,7	0,9094	0,6362	0,7	0,7	0,7246	0,5801
UFS	0,4	0,7	0,8657	0,6422	0,4	0,7	0,9577	0,6762
UNIPAMPA	0,7	0,7	0,9499	0,5514	0,4	0,7	0,7910	0,8292
UFPI	0,7	0,7	0,9198	0,5797	0,7	0,7	0,8995	0,7453
UNB	0,4	0,7	0,8717	0,5572	0,4	0,7	0,9974	0,6817
UFBA	0,4	0,7	0,7460	0,5461	0,6336	0,7	0,8010	0,6830
UFGD	0,7	0,7	0,9944	0,8355	0,4	0,7	0,8032	0,8027
UFPB	0,7	0,7	0,9228	0,5548	0,7	0,7	0,9023	0,6311
UFAL	0,6656	0,7	0,9706	0,5901	0,4	0,7	0,7573	0,8913
UNIFAL-MG	0,4	0,7	0,9480	0,1355	0,7	0,7	0,8724	0,7413
UFCG	0,7	0,7	0,8816	0,6751	0,7	0,7	0,7834	0,9708
UFG	0,4	0,7	0,8554	0,6269	0,7	0,7	0,7910	0,7938
UNIFEI	0,7	0,7	0,8366	0,5961	0,7	0,7	0,8161	0,7159
UFJF	0,4	0,7	0,9264	0,4152	0,4	0,7	0,9408	0,6337
UFLA	0,7	0,7	1,0000	0,7904	0,4	0,7	0,8531	0,8424

UFMT	0,7	0,7	0,8043	0,7364	0,4	0,7	1,0000	0,6308
UFMG	0,7	0,7	1,0000	0,5257	0,7	0,7	1,0000	0,7070
UFOP	0,6509	0,7	0,8807	0,5704	0,7	0,7	0,7628	0,7570
UFPEL	0,4	0,7	0,8794	0,7999	0,7	0,7	0,8049	0,9767
UFPE	0,5274	0,7	0,9008	0,5106	0,5134	0,7	0,8330	0,6958
UNIR	0,7	0,7	1,0000	0,4534	0,4	0,7	0,7727	0,9889
UFSC	0,4870	0,7	0,8994	0,6225	0,7	0,7	0,7860	0,7930
UFSM	0,4	0,7	0,9463	0,7150	0,7	0,7	0,8406	0,8968
UFSCAR	0,4	0,7	0,9112	0,6035	0,5090	0,7	0,8438	0,7832
UFSJ	0,7	0,7	0,8750	0,6844	0,4	0,7	0,8847	0,7779
UNIFESP	0,4333	0,7	1,0000	0,6712	0,7	0,7	0,8443	0,7208
UFU	0,7	0,7	0,9061	0,5626	0,7	0,7	0,8471	0,6934
UFV	0,4	0,7	1,0000	0,7490	0,7	0,7	0,9008	0,9294
UFABC	0,5236	0,7	0,7777	0,5634	0,7	0,7	0,6450	0,7038
UFAC	0,4	0,7	0,9816	0,8712	0,7	0,7	0,7862	0,8795

Source: The Author (2024).

When analyzing the efficiency decomposition of the processes, it is possible to verify that the decomposition was unique only when one of the stages is considered efficient. This is the case of UFLA, UNIR, UNIFESP, and UFV in the formative process in 2019. This situation was also observed in UFMS, UFCG, UFPEL, and UNIR in 2020 for the scientific production process.

In Table 15, it is possible to identify that the number of efficient DMUs remains the same in 2019. However, in 2020, two DMUs became efficient when the first stage was the leader. In contrast, nine and seven are deemed efficient in 2019 and 2020, respectively, when the second stage becomes the leader, as displayed in Table 16.

It is also relevant to observe that when the first stage is prioritized, the allocation of students is even maintained or enlarged in 2020 for the majority of the DMUs. However, the same pattern is not verified for professors. On the other hand, the pattern verified for the second stage is similar for both years. The majority of DMUs are inclined to maintain or reduce both students' and professors' workloads when compared to the initial DNDEA results.

Table 16 - Results Stage 2 as leader

University	2019				2020			
	α_1	α_2	Formative Process	Scientific Production	α_1	α_2	Formative Process	Scientific Production
UFSCPA	0,59	0,70	0,5912	1,0000	0,4	0,7	0,9176	0,9893
UFMS	0,70	0,40	0,7304	1,0000	0,4	0,7	0,7853	1,0000
UFRR	0,70	0,70	0,8977	0,6770	0,7	0,7	0,7202	0,6163
UFS	0,40	0,70	0,7322	0,9045	0,4	0,4	0,8908	0,8533
UNIPAMPA	0,40	0,65	0,8310	0,8645	0,7	0,7	0,7282	1,0000
UFPI	0,40	0,48	0,7981	0,9292	0,4	0,7	0,8402	0,9195
UNB	0,40	0,70	0,6281	0,8130	0,4	0,7	0,8682	0,8284
UFBA	0,68	0,70	0,6388	0,7242	0,7	0,7	0,7522	0,8049
UFGD	0,40	0,40	0,8125	1,0000	0,4	0,4	0,7197	0,8671
UFPB	0,40	0,70	0,7222	0,8717	0,4	0,7	0,8048	0,8324
UFAL	0,58	0,70	0,7489	0,7960	0,7	0,4	0,7122	0,9278
UNIFAL-MG	0,40	0,40	0,8322	0,8273	0,4	0,4	0,8652	0,7917
UFCG	0,40	0,70	0,4090	1,0000	0,7	0,4	0,7457	1,0000

UFG	0,46	0,70	0,6540	0,8084	0,4	0,7	0,7579	0,8288
UNIFEI	0,70	0,70	0,7453	0,7454	0,7	0,7	0,7653	0,8300
UFJF	0,40	0,41	0,6915	0,7451	0,4	0,7	0,8770	0,7760
UFLA	0,40	0,70	0,7940	1,0000	0,4	0,7	0,7926	0,9125
UFMT	0,40	0,70	0,7087	0,8601	0,4	0,4	0,9526	0,7281
UFMG	0,40	0,70	0,6752	0,8794	0,4	0,7	0,8633	0,8591
UFOP	0,56	0,40	0,7271	0,8240	0,4	0,4475	0,7266	0,8430
UFPEL	0,40	0,70	0,6185	1,0000	0,4	0,7	0,7780	1,0000
UFPE	0,51	0,70	0,6346	0,8076	0,4	0,7	0,7370	0,8086
UNIR	0,40	0,70	0,8743	1,0000	0,7	0,7	0,7708	1,0000
UFSC	0,48	0,70	0,6164	0,8595	0,4	0,7	0,6870	0,8866
UFSM	0,45	0,70	0,6562	0,9396	0,4	0,7	0,7501	0,9821
UFSCAR	0,46	0,70	0,7065	0,7923	0,4	0,4	0,8137	0,8126
UFSJ	0,70	0,40	0,6396	1,0000	0,7	0,4	0,7095	0,9457
UNIFESP	0,69	0,70	0,9070	0,8719	0,4	0,7	0,8199	0,8000
UFU	0,67	0,70	0,7159	0,8100	0,4	0,4	0,7775	0,7981
UFV	0,40	0,70	0,7055	1,0000	0,4	0,7	0,8229	1,0000
UFABC	0,70	0,70	0,6485	0,8026	0,7	0,7	0,6338	0,7217
UFAC	0,70	0,70	0,7803	1,0000	0,7	0,4	0,6407	1,0000

Source: The Author (2024).

Efficiency decomposition analysis allows universities to evaluate different scenarios and consider the impact of prioritizing the performance of one process over another. In addition, the model used provides individual answers for each university, as well as the proportions of resource allocation for the investigated cases.

4.6 FINAL CONSIDERATIONS

Universities are essential for social and economic development. Public funds used in these institutions have stimulated the development of proposals for evaluation. DEA has stood out in the field of efficiency measurements in education, with the application of models in distinct areas, such as primary education, secondary schools, teachers, students, research, and teaching.

Educational processes usually span several consecutive periods. Therefore, it is adequate to use models considering the temporal effects on efficiency. We also consider that there is a network structure when analyzing the processes of graduate activities.

Thus, in this paper, there is a proposition of a two-stage dynamic network model that considers shared inputs among the stages. First, we propose a centralized approach that maximizes the efficiency of the system considering all periods and stages under investigation. The overall efficiency is obtained with a weighted sum of the periods and processes efficiency. In this initial view, the approach considers that all stages cooperate and acts in unity to obtain the best possible results considering the entire time frame evaluated.

Considering resource sharing between the stages makes it possible to represent the context of graduate activities more accurately. Nevertheless, the proposed DNDEA use is broader than the educational context and can be applied to others where the stages share common resources. Also, Appendix A points out that our approach can easily be adapted to cases without shared inputs and consider exogenous inputs in the second division of the DMU. After this initial analysis, we investigated the efficiency uniqueness of the centralized DNDEA with a decomposition based on a leader-follower approach. In this framework, we investigated the cases where the first stage takes priority and the situations where the second stage is the leader.

Results indicates that the DNDEA model is more suitable for analyzing universities. We verify an increase in system efficiency from 2019 to 2020. Results indicate that the COVID pandemic impacted the formative and scientific production processes differently. We also evaluated if there were significant performance differences when considering the five Brazilian macro-regions. No significant disparities were found when analyzing the statistical tests.

The formative and scientific production process results inversed patterns in 2019 and 2020. Before the pandemic, the formative process performed better, but the scientific production process obtained superior results in 2020. Correlation analyses between the efficiency scores highlight that the scientific production process significantly impacts the system's results. However, cuts in national budgets earmarked for education and research have been negatively impacting the performance of this activity. Furthermore, it is relevant to map and understand the main difficulties in the formative process because scientific production directly depends on the products generated by it.

The empirical results allow for ranking universities, aid in graduate activities' improvements, and support the development of public policies to enhance Brazilian research results. Our findings require further studies. First, a thorough analysis is necessary to investigate more recent data for both processes to verify whether the superior performance of the scientific production remains. The graduate activities have been resilient throughout a decade of successive budget cuts. However, it is essential to mention that this resilience is not unlimited.

Second, the investigations did not consider undergraduate activities, and they represent a significant part of federal universities operating processes and expenses. It is also relevant to mention that no indicator evaluates undergraduate activities in an aggregate manner to rank the universities. Thus, this extension represents a relevant contribution due to the importance of federal universities to society. Third, our study did not include quality metrics of graduate activities' products. Therefore, further studies should add variables that reflect quality. We can use the classification of publications considering journal impact factor or quartile to segregate this variable and provide more thorough evaluations.

Although federal universities are highly relevant to Brazilian research, the investigation of private, state, and municipal institutions should also be considered to assess the performance of graduate activities. The absence of these HEIs represents the main limitation of this research.

Besides the empirical contributions to the Brazilian HEIs, this paper provides two main methodological contributions. The first relates to a new framework to investigate two-stage systems in a dynamic setting with shared resources between the stages. The second relates to the discussion of efficiency decomposition to verify the uniqueness of the efficiency scores provided by the DNDEA model.

We concluded that methodological extensions of this work are also possible. Initially, it is important to highlight that the current study evaluates the efficiency decomposition after a cooperative evaluation considering collaboration between the stages. However, analyzing this context from a non-cooperative perspective is interesting to assess real cases in which cooperation cannot be guaranteed. Moreover, modifications of the current model using non-radial measures and its extension to multiple stages are also extremely valuable in improving the applicability range of the model.

5 BI-DIMENSIONAL REPRESENTATION OF DYNAMIC FRONTIERS

This chapter details a new approach to present the distinct levels of results provided by Dynamic DEA models. The graphical representation of the Data Envelopment Analysis efficiency frontier has aroused interest since the first propositions related to the technique. However, proposals to consider multiple inputs and outputs in more recent DEA models, such as dynamic models, remain incipient in the literature. Modified virtual inputs and virtual outputs allows to obtain a bi-dimensional frontier representation of the relational dynamic models. The proposition provides a framework to represent distinct types of efficiency and facilitate the comprehension of results and aid in a better decision making.

5.1 CONTEXTUALISATION

Charnes et al. (1978) initially proposed Data Envelopment Analysis to measure the efficiency of a homogeneous set of decision-making units (DMUs). Despite this pioneering work, a significant part of the theoretical foundation necessary for the development of DEA comes from the proposition of Farrell (1957), which represents the efficiency frontier and the distances of inefficient DMUs to the frontier.

For cases with few variables, specifically two inputs and one output or the inverse, the two-dimensional representation becomes simple. Each ax corresponds to one ratio of input/output or vice versa. However, for cases that consider multiple variables, this type of structure becomes unfeasible.

With the methodological advances in DEA literature, it is possible to verify the propositions of new alternatives to visualize these situations. However, problems such as more complex visualization with the increase of variables and DMUs, the use of transformed DEA models, which are difficult to interpret, and lack of explicit representation of the efficient frontier make their use difficult (BANA E COSTA; SOARES DE MELLO, J. C. C. B.; ANGULO MEZA, 2016).

The methodological development of DEA includes the proposition of new models. Liu et al. (2016) state that dynamic and network models are among the most recent DEA research fronts. Dynamic modeling differs from classic models because it considers consecutive periods and changes in efficiency between these periods. On the other hand, network models consider the internal structure of DMUs to measure efficiency.

Because of the relevance of these models, the current study proposes Bana e Costa et al. (2016) as a two-dimensional representation for the relational dynamic model of Kao (2013), the relational dynamic model with the network structure of Omrani and Soltanzadeh (2016) and for the model presented in Chapter 4. Modified virtual inputs and modified virtual outputs to represent the different levels of efficiency obtained with both models.

The use of visual representations for these models becomes even more relevant because of the large amount of information generated by their use. Furthermore, the similar nomenclature for the different types of efficiency can represent an obstacle for decision-makers to understand the results. This tool is useful in persuading managers and policymakers about the validity of the results and recommendations and greatly aids in translating recommendations into actions (Ozcan et al. 2010).

The next section discusses the studies developed in the context of DEA graphical representations. Subsection 5.3 presents the dynamic models for which a representation is developed. Subsection 5.4 introduces the proposed two-dimensional representation of the DNDEA model. The following subsection presents a discussion pertinent to the case. After that, results and final considerations of the study are drafted.

5.2 A BRIEF BACKGROUND ON DEA FRONTIER REPRESENTATION

The graphic representation of DMUs consists of a powerful support tool for decision-makers, which allows to ascertain how far the DMUs are from the efficient frontier or to look for concentrations of DMUs in some areas in the graph (BANA E COSTA; SOARES DE MELLO, J. C. C. B.; ANGULO MEZA, 2016).

This statement begins with Farrell (1957) that presents different isoquants to discuss the efficient frontier when the production function is known and estimate an efficient production function from observations of the inputs and outputs for some firms.

In their seminal paper, Charnes et al. (1978) considered two inputs and one output. They normalized each input with the respective output value, and plot this information in a bi-dimensional graph. The same idea can be applied for the case with one input and two outputs. However, this procedure was limited to three variables.

Belton & Vickers (1993) proposed a visual interactive DEA (VIDEA) consisting of an extension of the multiple criteria analysis model proposed previously by the same authors. They considered a multiple criteria hierarchical model to adapt the DEA model into an aggregate measure of input and output, which was used to plot a two-dimensional graph.

El-Mahgary and Lahdelma (1995) discussed alternatives to present DEA results. The authors proposed a set of two-dimensional charts to make its presentation to the managerial community more easily. The authors compared efficiencies with individual factors, the impacts of virtual outputs and the use of reference units to better comprehend the performance of inefficient DMU's.

Porembski et al. (2005) developed a combination of DEA and Sammon's Mapping. The developed method allows to visualize the efficiency and the reference relations identified with the use of DEA. The authors highlighted that several questions can be answered directly from the observations of the two-dimensional images obtained, such as: which DMUs are efficient and which are not; which DMUs exhibits influence on efficiency scores of other DMUs and how strong is the influence of a specific reference unit on an inefficient DMU.

Adler & Raveh (2008) proposed the use of Co-Plot, using the ratio of outputs to inputs rather than the original DEA results, stating that efficient DMUs are around the ring sector, but there is no efficient frontier in their proposition. Appa et al. (2010) proposed a bi-dimensional representation using one input and four outputs and considered normalization to adapt the CCR results and a defined efficiency frontier.

In the literature, software designed to visually represent DEA results has also been found. Ozcan et al. (2010) introduced the Interactive Data Envelopment Analysis Laboratory (IDEAL) as a tool to plot 3-D frontiers. Although 3-D graphs are helpful to see the results, the software is limited to three variables, and visualization becomes more challenging with the increase in the number of DMUs. Akçay et al. (2012) proposed the SmartDEA. The framework behind the software lies in the combination of DEA and data mining to develop a general decision support system (DSS) framework to analyze the results of basic DEA models.

Bana e Costa et al. (2016) proposed a more general approach to a bidimensional representation of CCR and BCC models. The authors used weight normalization based on the development of Belton & Vickers (1993) to obtain the modified virtual inputs and outputs. These metrics are then plotted on a graph for each DMU with an efficient frontier. The main advantage of this proposition lies in its simplicity: no modifications to the original model are required, the frontier is defined and easily obtained, the distance of the DMUs is obtainable, and visualization is easy even with a large number of DMUs.

Torres (2017) extended the approach of Bana e Costa et al. (2016) to a relational network DEA model Kao (2009). The authors combined virtual inputs and outputs with modified virtual inputs and outputs to represent the overall efficiency and sub-process efficiency. The model can handle multiple inputs, outputs, and intermediate measures, but it is limited to two stages.

Assunção (2018) developed an extension of Bana e Costa et al. (2016) focused on the dynamic approach of Kao (2013). The author based its considerations on the use of virtual outputs and inputs to represent divisional efficiency and proposed an average of virtual inputs and outputs to represent the global efficiency of DMUs in a two-dimensional approach.

The current study is closely related to the propositions of Assunção (2018) and Bana e Costa et al. (2016). In contrast to Assunção (2018), we propose using different types of modified virtual outputs and inputs to represent each efficiency level provided by the DNDEA model. For each efficiency type, distinct modified virtual inputs and outputs are used to the detriment of using the average values of a single variety of modified virtual inputs and outputs. This choice is made to avoid information loss, and because the overall efficiency does not correspond to the average of divisional efficiencies.

It is also important to highlight that this study limits its propositions to a dynamic framework with a network structure. To the author's knowledge, we are the first to propose a bidimensional representation of the frontier for dynamic models with a network structure. We are also the first to deepen the discussion of bidimensional representation for all efficiency types measured by this type of modeling.

It is necessary to mention that the dynamic models with network structure simultaneously consider the internal network structure of the DMU and the temporal effects on the efficiency of this network. This configuration provides different levels of information, ranging from global efficiency to divisional efficiency by period. In this sense, as the number of stages or periods considered in the analysis increases, the volume of information increases significantly. This large amount of information can make it challenging to understand the results. Thus, the proposition of a visual tool to understand all levels of the results provided is of paramount importance to be understood and used well by the decision-maker, which is another significant contribution of the current study.

5.3 DYNAMIC DEA MODELS

This section details the two dynamic models considered in this study. The first corresponds to the dynamic relational proposition of Kao (2013), while the second corresponds to the dynamic model with network structure of Omrani and Soltanzadeh (2016).

5.3.1 The relational dynamic model of Kao (2013)

In DEA literature, it is found a dominance of static models, in which there is the assumption of consumption and production in the same period of time (FALLAH-FINI; TRIANTIS; JOHNSON, 2014). In the real world, when in a period, the inter-relations between inputs, outputs, and the current situation of the units are dependent on previous periods. As such, traditional DEA models cannot measure the efficiency of a DMU appropriately and the resulting efficiency scores could be misleading (FOLADI; SOLIMANPUR; JAHANGOSHAI REZAEE, 2020).

The term “dynamic DEA” means using DEA models to describe the inter-relationships between individual periods and using the associated solution methods to calculate the relative efficiencies for a set of multi-period DMUs (KAO, 2013b).

Dynamic models measure efficiency in an aggregate perspective, in which the link variable connects the periods (MARIZ; ALMEIDA; ALOISE, 2018). These link variables are usually referred in the literature as carry-over variables. The main difference between static models and dynamic ones is the presence of carry-over variables, and these variables can be classified into four categories: desirable(good); undesirable (bad); discretionary(free) and non-discretionary (fixed) (TONE; TSUTSUI, 2010).

When several periods with inter-relations are involved, the overall efficiency must be measured in a dynamic manner, taking into account the inter-relationship between consecutive periods, because the resulting efficiency measures will be misleading (Kao, 2013). The temporal interdependence between the periods can be attributed to one or more combinations of five factors associated to the dynamic aspects of production: production delays, inventory, capital or quasi-fixed factors, cost adjustments and incremental improvements or learning models (FALLAH-FINI; TRIANTIS; JOHNSON, 2014).

The study of Färe and Grosskopf (1996) introduced the first dynamic aspects of production and formalized the connections between activities with the aid of intermediate variables in DEA literature. Färe and Grosskopf (1996), Nemoto and Goto (1999) and Tone and Tsutsui (2010) represents the structuring models in this literature field (MARIZ; ALMEIDA; ALOISE, 2018).

The proposition of Kao (2013) is used in this study and consists in an extension of the relational network model of Kao (2009). This model contributes the development of measures that resulted from a radial model with aggregated constraints (MARIZ; ALMEIDA; ALOISE, 2018)

A careful investigation makes it possible to identify that the dynamic environment reported by Kao (2013) represents an overlap of a series system and a parallel one. Horizontally, it describes a series system, in which each process consumes intermediate products from the previous stage to produce the same output for consumption in the later stage. Vertically, it corresponds to a parallel system, in which each process absorbs the same exogenous inputs, providing the same exogenous outputs (KAO, 2014).

In this efficiency assessment, it is considered a set of n DMU's that uses m inputs to produce s outputs over T periods. These periods are connected with flows named carry-overs ($z_{dj}^{(t)}$). Let's denote $x_{ij}^{(t)}$ as input i of DMU j in period t, $y_{rj}^{(t)}$ as the output r of DMU j in period t, and $z_{dj}^{(t)}$ the carry-over d of DMU j that leaves period t and enters in t+1. It is also considered a flow entering period one ($z_{dj}^{(0)}$). Model (32) details the proposition of Kao (2013). Let's also consider $x_{ij} = \sum_{t=1}^{T} x_{ij}^{(t)}$ and $y_{rj} = \sum_{t=1}^{T} y_{rj}^{(t)}$.

$$E_o = \max \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} f_d z_{do}^{(T)}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} f_d z_{do}^{(t_0)} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} + \sum_{d=1}^{D} f_d z_{dj}^{(T)} - \left(\sum_{i=1}^{m} v_i x_{ij} + \sum_{d=1}^{D} f_d z_{dj}^{(0)}\right) \le 0, (j = 1, \dots, n)$$

$$\sum_{r=1}^{s} u_r y_{rj}^{(t)} + \sum_{d=1}^{D} f_d z_{dj}^{(t)} - \left(\sum_{i=1}^{m} v_i x_{ij}^{(t)} + \sum_{d=1}^{D} f_d z_{dj}^{(t-1)}\right) \le 0, (j = 1, \dots, n; t = 1, \dots, T)$$

$$v_i, u_{r,} w_f \ge \varepsilon, i = 1, \dots, m; r = 1, \dots, s; f = 1, \dots, g \qquad (32)$$

With the optimal solution (u_r^*, f_d^*, v_i^*), the proposition calculates overall efficiency and period efficiency as shown in Expressions (33) and (34).

$$E_O^{(sys)} = \frac{\sum_{r=1}^{s} u_r^* y_{ro} + \sum_{d=1}^{D} f_d^* z_{do}^{(T)}}{\sum_{i=1}^{m} v_i^* x_{io} + \sum_{d=1}^{D} f_d^* z_{do}^{(t_0)}} \qquad (33)$$

$$E_O^{(t,sys)} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}^{(t)} + \sum_{d=1}^{D} f_d^* z_{do}^{(t)}}{\sum_{i=1}^{m} v_i^* x_{io}^{(t)} + \sum_{d=1}^{D} f_d^* z_{do}^{(t-1)}} \qquad (34)$$

5.3.2 The relational dynamic model with network structure of Omrani and Soltanzadeh (2016)

The network and dynamic modeling of Data Envelopment Analysis have been presented in the literature for a certain period and have received prominence because of the insights they provide (Liu et al., 2016). However, the expansion of dynamic considerations for network systems is a challenging topic and presents itself as a more recent development in the literature (AVKIRAN; MCCRYSTAL, 2013; KAO, 2014).

Combining both models allows a more precise identification of possible improvements and efficiency estimates arising from a more comprehensive analysis, where interactions between periods and between divisions are considered (AVKIRAN; MCCRYSTAL, 2014).

Network modeling consists of the theoretical basis necessary for the analysis of the internal structure of DMUs (LOZANO, Sebastián, 2017), whereas the dynamic model provides an understanding of the relationships between periods through the use of carry-over variables (KAWAGUCHI; TONE; TSUTSUI, 2014; MARIZ; ALMEIDA; ALOISE, 2018).

Through the juxtaposition of these models, it is possible to carry out a thorough analysis by observing the change in the efficiency of the period and the dynamic modification of the divisional efficiency of the DMUs (TONE; TSUTSUI, 2014c). This structure consists of the interaction of a finite number of static models (FÄRE, Rolf; GROSSKOPF, Shawna; WHITTAKER, 2014).

This framework allows for consideration of the internal heterogeneous organizations of the DMUs, in which the divisions are mutually connected by link variables and by the internal exchange of intermediate products (KAWAGUCHI; TONE; TSUTSUI, 2014; KHUSHALANI; OZCAN, 2017).

It is noted that there are developments proposing alternatives that aim to mitigate DEA limitations or to address specificities such as models to deal with uncertainty in the inputs (SOLTANZADEH; OMRANI, Hashem, 2018), for sharing inputs (CHAO, S. L.; YU, Ming Miin; HSIEH, 2018), for non-homogeneous (YAN, Q. *et al.*, 2019), and using common weights for efficiency measurement (GHARAKHANI *et al.*, 2018).

This range of models demonstrates that, although recently, this branch of literature has received much attention from practitioners and researchers. One of the probable reasons for this is the information provided by them. They can identify inefficiencies more precisely and provide metrics (slack and projection) to improve efficiency.

In the current context, we considered the DNDEA developed by Omrani and Soltanzadeh (2016). This modelling combines the network relational proposition of Kao (2009) and the dynamic modeling of Kao (2013). Relational modeling considers the same weight for the link and carry-over variables in the analysis interval. This aspect mathematically simplifies the linear programming. Omrani & Soltanzadeh (2016) assume that n DMUs consist of k divisions over t periods. Let m_k and r_k be the number of inputs and outputs of division k. Denote lk as the number of links, d_k, as the number of carry-overs, $x_{ij}^{(t,k)}$ is the input ith of DMU *j* for division *k* in period *t*, $y_{rj}^{(t,k)}$ is the output *rth* of DMU *j* for division *k* in period *t*, $c_{lj}^{(t,k)}$ is the linking intermediate *lth* of DMU *j* from division *k* to subsequent division in period t and $z_{dj}^{(t,k)}$ is the carry-over *dth* of DMU *j*, at division *k* from period *t* to the next period. Let`s also consider $x_{io} = \sum_t^T \sum_k^K x_{io}^{(t,k)}$ and $y_{ro} = \sum_t^T \sum_k^K y_{ro}^{(t,k)}$. Model (35) details the linear programming of Omrani and Soltanzadeh (2016) which provides overall efficiency.

$$E_O^{(sys)} = \sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{K}\sum_{d=1}^{D} f_d z_{do}^{(T,k)}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{k=1}^{K}\sum_{d=1}^{D} f_d z_{do}^{(t_0,k)} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{K}\sum_{d=1}^{D} f_d z_{dj}^{(T,k)} - \left(\sum_{i=1}^{m} v_i x_{ij} + \sum_{k=1}^{K}\sum_{d=1}^{D} f_d z_{dj}^{(t_0,k)}\right) \leq 0 \ (j = 1, \dots, n)$$

$for\ division\ k = 1$

$$\sum_{r\in r^1} u_r y_{rj}^{(t,1)} + \sum_{l\in l^1} w_l c_{lj}^{(t,1)} + \sum_{d\in d^1} f_d z_{dj}^{(t,1)} - \left(\sum_{i\in i^1} v_i x_{ij}^{(t,1)} + \sum_{d\in d^1} f_d z_{dj}^{(t-1,1)}\right) \leq 0$$

$$(j = 1, \dots, n; t = 1, \dots, T)$$

$for\ division\ k = 2\ to\ k - 1$

$$\sum_{r\in r^k} u_r y_{rj}^{(t,k)} + \sum_{l\in l^k} w_l c_{lj}^{(t,k)} + \sum_{d\in d^k} f_d z_{dj}^{(t,k)} - \left(\sum_{i\in i^k} v_i x_{ij}^{(t,k)} + \sum_{l\in l^k} w_l c_{lj}^{(t,k-1)} + \sum_{d\in d^k} f_d z_{dj}^{(t-1,k)}\right) \leq 0$$

$$(j = 1, \dots, n; t = 1, \dots, T)$$

$for\ division\ k = K$

$$\sum_{r\in r^k} u_r y_{rj}^{(T,k)} + \sum_{l\in l^k} w_l c_{lj}^{(T,k)} + \sum_{d\in d^k} f_d z_{dj}^{(T,k)} - \left(\sum_{i\in i^k} v_i x_{ij}^{(t,k)} + \sum_{l\in l^k} w_l c_{lj}^{(t,k-1)} + \sum_{d\in d^k} f_d z_{dj}^{(t-1,k)}\right) \leq 0 \ \ (j = 1, \dots, n; t = 1, \dots, T)$$

$$v_i, u_{r,} w_l, f_d \geq \varepsilon, i = 1, \dots, m; r = 1, \dots, s; l = 1, \dots, L; d = 1, \dots, D \tag{35}$$

With the optimal solution (u_r^*, f_d^*, v_i^*, w_l^*), the overall efficiency is calculated as shown in Equation (36), the period efficiency as shown in Equation (37), and process efficiency, as shown in Equations (38), (39), and (40).

$$E_O^{(sys)} = \frac{\sum_{r=1}^{s} u_r^* y_{ro} + \sum_{k=1}^{K}\sum_{d=1}^{D} f_d^* z_{do}^{(T,k)}}{\sum_{i=1}^{m} v_i^* x_{io} + \sum_{k=1}^{K}\sum_{d=1}^{D} f_d^* z_{do}^{(t_0,k)}} \tag{36}$$

$$E_O^{(t,sys)} = \frac{\sum_{k=1}^{K}\sum_{r}^{s} u_r^* y_{ro}^{(t,k)} + \sum_{k=1}^{K}\sum_{d=1}^{D} f_d^* z_{do}^{(t,k)}}{\sum_{k=1}^{K}\sum_{i=1}^{m} v_i^* x_{io}^{(t,k)} + \sum_{k=1}^{K}\sum_{d=1}^{D} f_d^* z_{do}^{(t-1,k)}} \tag{37}$$

$$E_O^{(t,1)} = \frac{\sum_{r\in r^1} u_r^* y_{ro}^{(t,1)} + \sum_{l\in l^1} w_l^* c_{lo}^{(t,1)} + \sum_{d\in d^1} f_d^* z_{do}^{(t,1)}}{\sum_{i\in i^1} v_i^* x_{io}^{(t,1)} + \sum_{d\in d^1} f_d^* z_{do}^{(t-1,1)}} \tag{38}$$

$$E_O^{(t,k)} = \frac{\sum_{r\in r^k} u_r^* y_{ro}^{(t,k)} + \sum_{l\in l^k} w_l^* c_{lo}^{(t,k)} + \sum_{d\in d^k} f_d^* z_{do}^{(t,k)}}{\sum_{i\in i^k} v_i^* x_{io}^{(t,k)} + \sum_{l\in l^k} w_l^* c_{lo}^{(t,k-1)} + \sum_{d\in d^k} f_d^* z_{do}^{(t-1,k)}} \tag{39}$$

$$E_O^{(t,k)} = \frac{\sum_{r\in r^K} u_r^* y_{ro}^{(t,k)} + \sum_{d\in d^K} f_d^* z_{do}^{(t,K)}}{\sum_{i\in i^K} v_i^* x_{io}^{(t,K)} + \sum_{l\in l^K} w_l^* c_{lo}^{(t,K-1)} + \sum_{d\in d^K} f_d^* z_{do}^{(t-1,K)}} \tag{40}$$

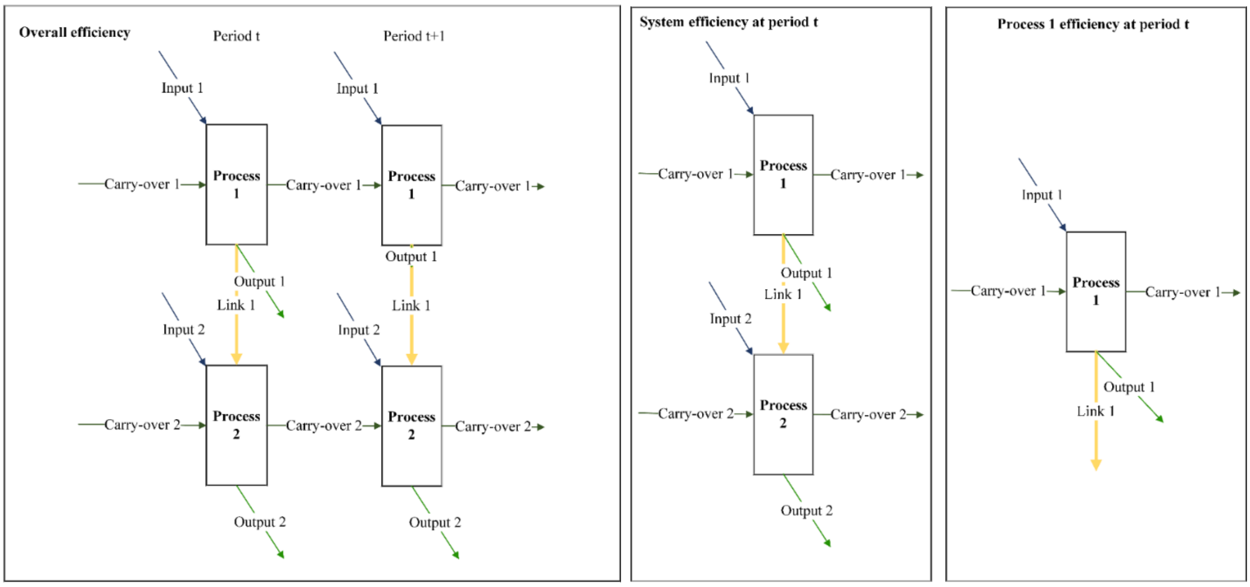
5.4 MODEL STRUCTURE

In this section, we detail the approach developed to represent the efficient frontier of DDEA and DNDEA models. To obtain the bi-dimensional representation, virtual inputs and outputs will be used. The main issue regarding the use of such metrics is the constraint that states that the virtual input or virtual output equals 1 which is added to linearize the mathematical model. So, in a virtual input versus virtual output plot all DMUs would be located on the same vertical straight line and such a graphical representation would be meaningless (Bana e Costa et al. 2016).

In order to bypass this limitation, we follow the proposition of Bana e Costa et al. (2016). The authors introduced a constraint that limits the sum of inputs weights to be equal to 1. In the case of dynamic models, it is necessary to add a parcel related to the carry-over since this variable also represents an input of the system. It is possible to use the results of Model (32), (35) and (19) to avoid running the altered proposed model with the additional constraint.

The approach developed for the DDEA and DNDEA model are similar. The difference lies on the number of virtual inputs and virtual outputs, since DNDEA models provides a higher number of efficiency measures. Dynamic models measure overall and system`s efficiency at each period. On the other hand, dynamic models with network structure provide overall, system`s efficiency at each period and process efficiency at each period. The distinct DNDEA efficiency levels are displayed in Figure 20.

Figure 20 - Types of efficiencies in DNDEA models



Source: The Author (2024).

It is possible to use the results of models (32), (35) and (19) to avoid running the altered proposed model with the additional constraint. The discussed models are input-oriented, and the approach is easily adaptable to the output-orientation. The weights of the models will be divided by the total sum of the input and carry-over weights of the DMU of reference for the first two models, while for Model (19), it corresponds to the sum of specific inputs, shared inputs, carry-overs and links.

Table 17 details each step of the approach for the bi-dimensional representation. The first and second columns provides the details for each step, while columns three to five presents the mathematical aspects for each one of the considered dynamic models. It is also relevant to mention that the algebraic manipulations used preserves the results obtained with each model. These proofs are available in Appendix A for all the considered models.

Table 17 - Step-by-step procedure for bi-dimensional representation

		Model of Kao (2013)	Model of Omrani and Soltanzadeh (2016)	Model (19) displayed in Chapter 4.
	Consider	v'_{ij}, u'_{rj}, and f'_{dj} to be the modified weights of the input *i*, output *r* and carry-over *d* of DMU *j*	v'_{ij}, u'_{rj}, f'_{dj} and w'_{lj} be the modified weights of the input *i*, output *r*, carry-over *d*, and link *l* of DMU *j*	v'_{ij},v'_{pj}; β'_{pj}, γ'_{lj}, μ'_{dj} and μ'_{rj} to be the modified weights of the specific input *i*, shared input *p*, carry-over *l*, link *d* and output *r* of DMU *j*
	Step Description			
01	Run the input-oriented model for each DMU *j*	Model (32)	Model (35)	Model (19)
02	Calculate the S_j which corresponds to the sum of all input and carry-over weights for each DMU *j*	$S_j = \sum_{i=1}^{m} v_{ij} + \sum_{d=1}^{D} f_{dj}$ (41)	$S_j = \sum_{i=1}^{m} v_{ij} + \sum_{d=1}^{D} f_{dj}$ (49)	$S_j = \sum_{i=1}^{m} v_{ij} + \sum_{p=1}^{P} v_{pj} + \sum_{l=1}^{L} \gamma_{lj} + \sum_{d=1}^{D} \mu_{dj}$ (62)
03	Calculate the modified weights	$v'_{ij} = \frac{v_{ij}}{S_j}$ (42) $u'_{rj} = \frac{u_{rj}}{S_j}$ (43) $f'_{dj} = \frac{f_{dj}}{S_j}$ (44)	$v'_{ij} = \frac{v_{ij}}{S_j}$ (50) $u'_{rj} = \frac{u_{rj}}{S_j}$ (51) $f'_{dj} = \frac{f_{dj}}{S_j}$ (52) $w'_{lj} = \frac{w_{lj}}{S_j}$ (53)	$v'_{ij} = \frac{v_{ij}}{S_j}$ (63) $v'_{pj} = \frac{v_{pj}}{S_j}$ (64) $\beta'_{pj} = \frac{\beta_{pj}}{S_j}$ (65) $\gamma'_{lj} = \frac{\gamma_{lj}}{S_j}$ (66) $\mu'_{dj} = \frac{\mu_{dj}}{S_j}$ (67) $\mu'_{rj} = \frac{\mu_{rj}}{S_j}$ (68)
04	Calculate the modified overall system virtual input and output for each DMU *j*.	$I_j'^{(sys)} = \sum_{i=1}^{m} v'_{ij} x_{ij} + \sum_{d}^{D} f'_{dj} z_{dj}^{(t_o,k)}$ (45) $O_j'^{(sys)} = \sum_{r=1}^{s} u'_{rj} y_{rj} + \sum_{d}^{D} f'_{dj} z_{dj}^{(T,k)}$ (46)	$I_j'^{(sys)} = \sum_{i=1}^{m} v'_{ij} x_{ij} + \sum_{t=1}^{T} \sum_{d}^{D} f'_{dj} z_{dj}^{(t_o,k)}$ (54) $O_j'^{(sys)} = \sum_{r=1}^{s} u'_{rj} y_{rj} + \sum_{t=1}^{T} \sum_{d}^{D} f'_{dj} z_{dj}^{(T,k)}$ (55)	$I_j'^{(sys)} = \sum_{i=1}^{m} v'_{ij} x_{ij} + \sum_{p=1}^{P} v'_{pj} x_{pj} + \sum_{d=1}^{D} \mu'_{dj} z_{dj} + \sum_{t=1}^{T} \sum_{k=1}^{K} \sum_{l=1}^{L} \gamma'_{lj} c_{lj}^{(t-1,k)}$ (75) $O_j'^{(sys)} = \sum_{r=1}^{s} \mu'_{rj} y_{rj} + \sum_{d=1}^{D} \mu'_{dj} z_{dj} + \sum_{t=1}^{T} \sum_{k=1}^{K} \sum_{l=1}^{L} \gamma'_{lj} c_{lj}^{(t,k)}$ (76)

05	Calculate the modified system period of the virtual input and output for each DMU *j*.	$I'^{(t,sys)}_j = \sum_{i=1}^{m} v'_{ij} x^{(t)}_{ij} + \sum_{d=1}^{D} f'_{dj} z^{(t-1)}_{dj}$ (47) $O'^{(t,sys)}_j = \sum_{r=1}^{s} u'_{rj} y^{(t)}_{rj} + \sum_{d=1}^{D} f'_{dj} z^{(t)}_{dj}$ (48)	$I'^{(t,sys)}_j = \sum_{k=1}^{K}\sum_{i=1}^{m} v'_{ij} x^{(t,k)}_{ij} + \sum_{k=1}^{K}\sum_{d=1}^{D} f'_{dj} z^{(t-1,k)}_{dj}$ (56) $O'^{(t,sys)}_j = \sum_{k=1}^{K}\sum_{r=1}^{s} u'_{rj} y^{(t,k)}_{rj} + \sum_{k=1}^{K}\sum_{d=1}^{D} f'_{dj} z^{(t,k)}_{dj}$ (57)	$I'^{(t,sys)}_j = \sum_{i=1}^{m} v'_{ij} x^{(t)}_{ij} + \sum_{p=1}^{P} v'_{pj} x^{(t)}_{pj} + \sum_{d=1}^{D} \mu'_{dj} z^{(t)}_{dj} + \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma'_{lj} c^{(t-1,k)}_{lj}$ (75) $O'^{(t,sys)}_j = \sum_{r=1}^{s} \mu'_{rj} y^{(t)}_{rj} + \sum_{d=1}^{D} \mu'_{dj} z^{(t)}_{dj} + \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma'_{lj} c^{(t,k)}_{lj}$ (76)
06	Calculate the process-modified virtual input and output for each DMU *j*.	-	$I'^{(t,k)}_j = \sum_{i \in i^k} v'_{ij} x^{(t,k)}_{ij} + \sum_{l \in l^k} w'_{lj} c^{(t,k-1)}_{lj} + \sum_{d \in d^k} f'_{dj} z^{(t-1,k)}_{dj}$ (58) $O'^{(t,k)}_j = \sum_{r \in r^k} u'_{rj} y^{(t,k)}_{rj} + \sum_{l \in l^k} w'_{lj} c^{(t,k)}_{lj} + \sum_{d \in d^k} f'_{dj} z^{(t,k)}_{dj}$ (59) (58) and (59) will be shortened in the first and last stages of the network. This occurs because there is no link entering the first stage and exiting the last one. Thus, they become (59) and (60) for the first and last stages. $I'^{(t,1)}_j = \sum_{i \in i^1} v'_{ij} x^{(t,1)}_{ij} + \sum_{d \in d^1} f'_{dj} z^{(t-1,1)}_{dj} \; for\ k = 1$ (60) $O'^{(t,K)}_j = \sum_{r \in r^k} u'_{rj} y^{(t,K)}_{rj} + \sum_{d \in d^k} f'_{dj} z^{(t,K)}_{dj} \; for\ k = K$ (61)	$I'^{(t,sys)}_j = \sum_{i=1}^{m} v'_{ij} x^{(t)}_{ij} + \sum_{p=1}^{P} v'_{pj} x^{(t)}_{pj} + \sum_{d=1}^{D} \mu'_{dj} z^{(t)}_{dj} + \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma'_{lj} c^{(t-1,k)}_{lj}$ (75) $O'^{(t,sys)}_j = \sum_{r=1}^{s} \mu'_{rj} y^{(t)}_{rj} + \sum_{d=1}^{D} \mu'_{dj} z^{(t)}_{dj} + \sum_{k=1}^{K}\sum_{l=1}^{L} \gamma'_{lj} c^{(t,k)}_{lj}$ (76)
07	Use the modified virtual input $I'^{(sys)}_j$ in the x-axis and the modified virtual output O'^{sys}_j in the y-axis in a bi-dimensional graph for each DMU j for overall efficiency.			
08	Use the modified virtual input $I'^{(t,sys)}_j$ in the x-axis and the modified virtual output $O'^{(t,sys)}_j$ in the y-axis in a bi-dimensional graph for each DMU j for system efficiency.			
09	Use the modified virtual input $I'^{(t,k)}_j$ in the x-axis and the modified virtual output $O'^{(t,k)}_j$ in the y-axis in a bi-dimensional graph for each DMU j for process efficiency.			
10	Draw the 45° line representing the efficient frontier where efficient DMU presents $I'^{(t,k)}_j = O'^{(t,k)}_j$, $I'^{(t,sys)}_j = O'^{(t,sys)}_j$ and $I'^{(sys)}_j = O'^{(sys)}_j$ for process efficiency, system efficiency in each period, and overall system efficiency			

*	For output-oriented models, the approach alters only in S_j calculation	$S_j = \sum_{s=1}^{r} u_{rj} + \sum_{d=1}^{D} f_{dj}$ (77)	$S_j = \sum_{r=1}^{s} u_{rj} + \sum_{d=1}^{D} f_{dj}$ (78)	$S_j = \sum_{r=1}^{s} \mu_{rj} + \sum_{l=1}^{L} \gamma_{lj} + \sum_{d=1}^{D} \mu_{dj}$ (79)

Source: The Author (2024).

5.5 EXPLORING THE CASE STUDY

It is possible to verify some studies in the literature using DDEA and DNDEA models to investigate efficiency in universities. Emrouznejad and Thanassoulis (2005) propose a dynamic DEA model to deal with inter-temporal dependence by using input-output paths mapped out by operating units over time to assess them. The proposed approach was applied to a sample of British universities to assess their responsibility for delivering knowledge. Kumar and Thakur (2019) proposed a new approach based on DDEA to obtain an objective university ranking, and data from India was used to validate the procedure. Xiong et al. (2022) developed a dynamic model to deal with the fixed-sum output to investigate research resources in 31 Chinese province-level regions.

Fukuyama et al. (2016) investigated 25 universities during 1990-2005 concerning nanobiotechnology knowledge production with a dynamic network DEA model and spillover effects. Tran and Villano (2018) and Tran (2021) also applied DNDEA models to investigate Vietnamese and Australian universities. The first study considered financial and academic divisions to explore 116 public colleges from 2011-2013. The second investigates the teaching-industry linkage of 12 education fields over three years in Australia. Cossani et al. (2022) applied a DNDEA model to look into teaching and research activities with shared resources in 33 Chilean universities from 2013 to 2016.

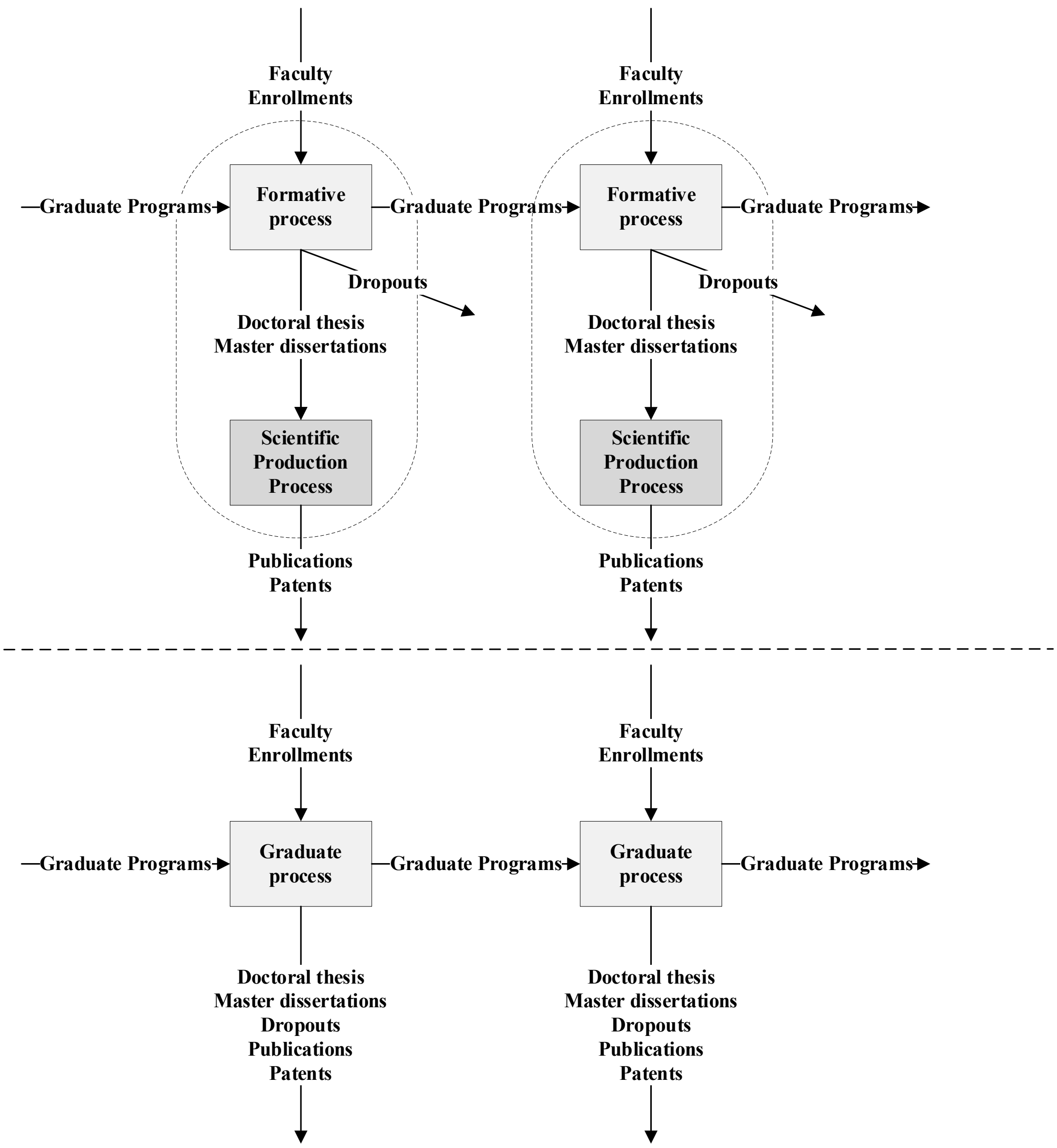
It is relevant to mention the DDEA and DNDEA models present themselves as appropriate for investigating universities since their activities occur across several periods. Since they are multi-product institutions, they require distinct processes to archive their goals, as observed in the previous studies.

Brazilian federal universities are a part of the Brazilian heritage, contribute to society on several different fronts, are responsible for more than half of the country's master's and doctoral courses and students, and produce most of the national science (ANDIFES, 2017).

Due to the importance of these institutions, the use of public funds to finance such activities, and since they represent a set of more homogeneous institutions, the sample was limited to federal universities.

Figure 21 illustrates the structure and variables in DDEA and DNDEA models proposed. As previously mentioned, universities present a multi-activity framework. In this study, the focus corresponds to the graduate activities in Federal universities, since they are directly responsible for a significant part of Brazilian's research.

Figure 21 - DDEA and DNDEA frameworks



Source: The Author (2024).

The choice for dynamic models to evaluate graduate process is based on the fact that the activities repeat from period to period and the outcomes of one period can impact the following one. The selection of a DNDEA model also portrays another real characteristic, since the internal structure of universities corresponds to a network.

In the DNDEA framework, we considered two processes. The first stage in the network corresponds to the formative process and the second one corresponds to scientific production process. In the formative process, universities employ resources to train students, and their education should generate products. In our proposal, we consider that the faculty and enrolled students as the inputs. The number of programs available in a university represents the carry-over variable, while master dissertations and Ph.D. thesis corresponds to the intermediate factor linking the stages. It is also considered that some students do not finish their master's or Ph.D. training, which is also contemplated in the model, the dropout variable

The undesirable output underwent treatment prior to its use in modeling. Values were subtracted from a large number. This transformation ensures that the results are isotonic is regarded as an undesirable output (DYSON *et al.*, 2001).

The second stage (the scientific production process) converts the products of the formative process into research products. In this sense, we considered that the dissertations and thesis correspond to the research developed and it represents the basis to generate papers and patents. The publications considered are indexed in the SCOPUS database.

The eight variables selected to represent this activity and their descriptive statistics are in Table 18. The choice of these variables is based on their previous use in the literature, university rankings, and data availability. Reports generated by the Coordination for the Improvement of Higher Education Personnel (CAPES) correspond to the data source of data used. CAPES is responsible for evaluating and consolidating information regarding graduate activities in Brazil.

Table 18 - Descriptive statistic of data

Variable	Category	Average	SD	Minimum	Maximum
Formative process					
Faculty (number)	Input	1114,7	787,7	186	3689
Enrollments (number)	Input	3092,5	2624,8	313	12050
Programs (number)	Carry-over	44,0	26,8	7	125
Dropouts (number)	Output	129,3	110,5	9	612
Ph.D. Thesis (number)	Link	241,5	249,7	5	1092
Master Dissertations (number)	Link	645,2	454,6	74	1853
Scientific production process					
Publications (number)	Output	3089,4	2418,7	294	11915
Patents (number)	Output	46,8	44,2	1	210

Source: The Author (2024).

According to data released by the 2020 Higher Education Census, there are 68 federal universities in Brazil. The reduction in the number of universities analyzed was due to lack of data on one or more variables, mainly in patents and Ph.D. Thesis. The sample contains 51 universities, data corresponds to the years of 2018 to 2020. Therefore, the sample accounts for 75% of total federal universities in Brazil. The time frame selected aims to evaluate the most recent available data and also to obtain a glimpse of the COVID-19 pandemic impact on graduate activities.

5.6 RESULTS AND DISCUSSSION

In order to better portray the analyzes, the results are divided into two parts. First, the dynamic efficiencies of the 51 federal universities obtained with DDEA and the DNDEA models are calculated. Then, the weights obtained with both models are used in the step-by-step procedure detailed in Subsection 5.3 to obtain the bi-dimensional representation of the results.

5.6.1 DDEA and DNDEA results

To demonstrate and compare the ability of DDEA and DNDEA models, the distinct levels of results are displayed in Figure 22. Regarding the results provided by the models, the overall and system efficiency in each period can be directly compared between the DDEA and DNDEA models.

The overall and system average efficiencies for DDEA and DNDEA are displayed in Table 19. It is possible to observe that, in average, the overall efficiency for DDEA is 86,07%, while for DNDEA a lower value of 53,15% is observed. Regarding the system's average efficiency in each year, DDEA results are also higher than DNDEA ones for all years investigated. The pattern repeats for the individual values as displayed in Figure 22.

Table 19 - Statistical description of overall and system's efficiency

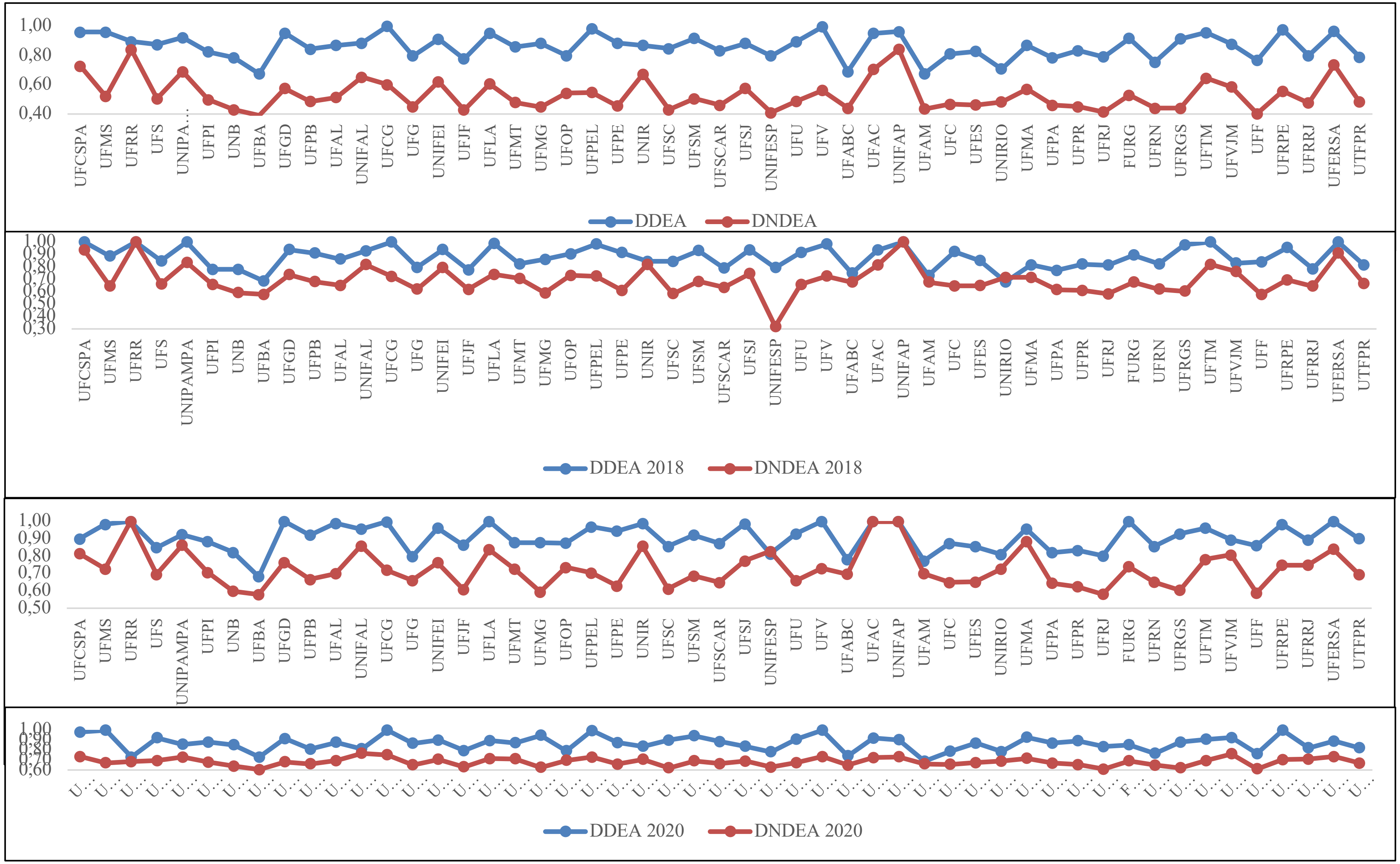
	DDEA				DNDEA			
	Overall	2018	2019	2020	Overall	2018	2019	2020
Mean	86,07%	87,53%	90,60%	86,81%	53,15%	69,58%	72,72%	68,24%
SD	8,41%	8,90%	7,67%	7,62%	10,90%	11,63%	10,66%	4,08%
Maximum	99,88%	100%	100%	100%	84,09%	99,93%	99,88%	76,70%
Minimum	67,35%	67,93%	68,11%	68,83%	39,02%	32,03%	57,89%	60,41%

Source: The Author (2024).

Therefore, it is possible to infer that the DNDEA can provide more discrimination between the universities' efficiency than DDEA results. The analysis of the number of efficient DMU's also corroborates the greater power of discrimination of the DNDEA model. In the evaluation with the DDEA model, 7 DMUs are considered efficient in 2018 and 2019, and four obtain maximum results in 2020. While in the DNDEA model, no DMU is deemed efficient.

Despite the differences in values, it is possible to verify that both models demonstrate the same pattern regarding average system performance. There is a slight increase in efficiency from 2018 to 2019 (3,07% for DDEA and 3,14% for DNDEA) and a reduction from 2019 to 2020 (3,14% for DDEA and 4,47% for DNDEA).

Figure 22 - DDEA and DNDEA system efficiency scores



Source: The Author (2024).

This reduction was expected due to the impact of the COVID-19 pandemic and its effects on the learning process to an unknown reality of remote learning. In the DDEA results, almost 63% of the sample presented a decrease in performance, while in DNDEA, 60% of the sample performed worse in 2020.

Since, DNDEA results presents higher discrimination among the results, the analyzes were deepened for this modeling. Considering DNDEA scores, federal universities could potentially increase, on average, their efficiency in a network structure of formative process and scientific production by approximately 46,8%.

As previously mentioned, in addition to the overall and system efficiencies in each period, the DNDEA modeling also provides the process efficiencies. Table 20 details these efficiencies for the three years investigated.

Table 20 - Statistical description of process efficiency

	Formative Process			Scientific Production		
	2018	2019	2020	2018	2019	2020
Mean	81,35%	84,60%	75,16%	39,48%	41,00%	58,22%
SD	8,85%	6,99%	4,23%	11,33%	10,94%	15,56%
Maximum	100%	100%	85%	88,649%	93,647%	100%
Minimum	44,89%	69,92%	65,07%	17,30%	16,68%	21,08%

Source: The Author (2024).

Table 20 indicate that, on average, the training process has better results than the scientific production process. It is also observed that this process has a similar pattern to system efficiency, increasing by 3,25%, from 81,35% in 2018 to 84,60% in 2019, then declining by 9,44% to 75,16% in 2020. On the other hand, the scientific production process presents different results from the previous stage. There is an average growth in all periods evaluated, the first increase corresponds to 1,52%, while the second and the highest one observed is 17,22%.

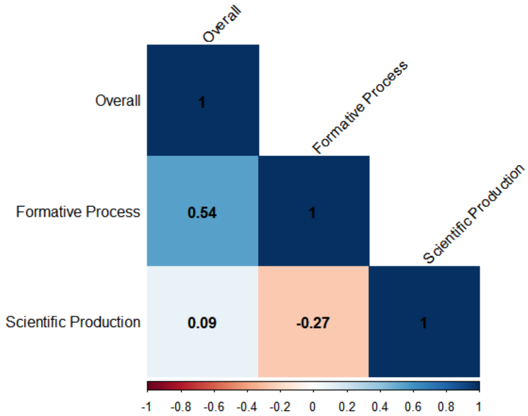
The process efficiency scores indicate that the university's graduate programs performed worst in the scientific production process in the entire time frame. This result suggests greater difficulty in converting the academic products generated by the programs (theses and dissertations) into publications and patents. Thus, efforts should be directed toward this activity. Despite the better results of the first process, there is still a significant room for improvement. In average 24,84%, considering 2020 results.

The increase in efficiency in 2020 may also be related to the COVID-19 pandemic. Teaching activities were suspended for different periods in universities. During this interval, research activities and, consequently, publications arising from these researches continued to be developed remotely. In addition, it is essential to mention that the emergency nature of the pandemic, the numerous impacts on the most diverse areas of knowledge, and the need for quick responses to deal with emerging problems stimulated the development of a high amount of research.

This context was reflected in special COVID-specific discussion sections at scientific events and special issues in various journals. Considering the 51 universities in the sample, only one has no increase in the publication's variable. And when comparing the 2019 and 2020 values of this variable, there was an increase of 26758 publications. However, whether this increase can be attributed to the influence of the COVID pandemic is out of our consideration because deterministic and causal relationships in this context are difficult to identify. On the other hand, it is a possibility to be deeply investigated.

The Spearman's correlation coefficients among the efficiencies of the formative process, scientific production process and the system efficiency over the three periods are presented in Figure 23. The values in the correlation matrix indicate that the system efficiency is strongly correlated to the formative process with a correlation value of 0,54 at the 5% level of significance.

Figure 23 - Correlation analysis



Source: The Author (2024).

In spite of the fact that scientific production process impact less in the system's performance than the previous process, it plays an indispensable role of disseminating the research produced in the university with the academic community and the society. It is important to note that in periods before the pandemic, the performance of this stage was significantly lower than the training process.

This result indicates a significant difficulty in disseminating the knowledge produced beyond the university. A critical analysis by the programs of the relevance of the research developed is necessary, or if the reduced number of publications and patents is due to difficulties in research funding. This second topic becomes even more critical in the context of the migration of several journals to the open access format, consequently increasing publishing costs. The increasing costs cannot continuously be financed by graduate students and their programs in a scenario of successive cuts in public funds destined for Brazilian public universities.

It is observed that the correlation between formative process and scientific production is negative and significant at the 5% level. This result corroborates the previous discussion of the difficulty of disseminating research products in these universities. Because the inputs of the second stage consist solely and exclusively of the products of the first, there must be an effort to increase publications and patents on the same scale. However, it is observed that universities have not achieved this feat.

5.6.2 Bi-dimensional representation

Following the procedure described in Sections 5.4.1.2 and 5.4.2.3, the first step requires running DDEA and DNDEA models presented in Section 3. The values obtained with the Step-by-step approach are detailed in Appendix A and B. Appendix A details the DDEA results and their graphics, and Appendix B describes the values for DNDEA results. In this subsection, we present the graphs and the empirical findings obtained with their aid.

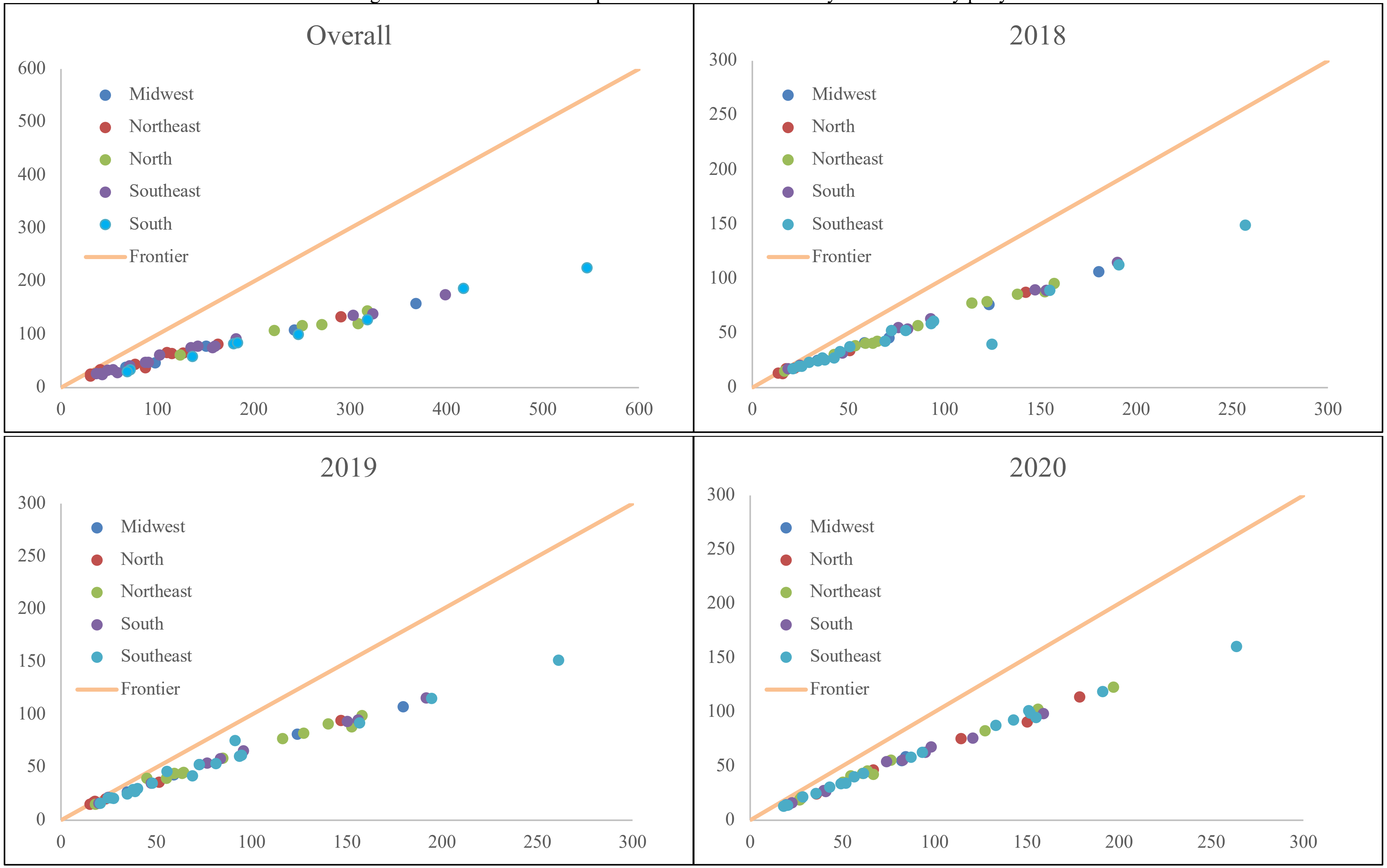
Figure 24 displays the frontier for the overall DNDEA efficiency and the system's efficiency for all years in the time frame. The green line leaving the origin (0,0) corresponds to the efficiency frontier in the proposed bi-dimensional representation.

The different colors in the graphs relate to the five Brazilian macro-regions. The choice to highlight the macro-regions relates to the significant social and economic discrepancies among them and previous literature findings that indicate the impact of the DMU's location on the efficiency score.

The graphics presented in Figure 24 indicate that for the global efficiency and the system efficiencies per year, there are no major discrepancies between the Brazilian macro-regions regarding the efficiency scores.

To confirm this finding, the Kruskal-Wallis non-parametric test was used to verify if there are differences among the efficiency scores median of the regions. Table 21 present the test results for the overall efficiency and system's efficiency for all years. At a 5% and 10% significant level, it is possible to infer that there are no differences in the median of the Brazilian macro regions.

Figure 24 - Bi-dimensional representation of overall and system efficiency per year



Source: The Author (2024).

Table 21 - Kruskal-Wallis's test results for overall and system efficiencies

Efficiency	Degrees of freedom	Chi-square	p-value
Overall	4	5,2676	0,2609
2018	4	6,5564	0,1613
2019	4	7,6205	0,1065
2020	4	2,0721	0,7225

Source: The Author (2024).

Although the differences between the medians of the macro-regions are not statistically significant, the northern region had the highest average value, having the two most efficient DMUs when considering the overall efficiency and the system efficiency in 2018 and 2019.

UNIFAP and UFRR have the lowest values of all inputs in the sample and the lowest values of undesirable output. However, the number of patents of both does not correspond to the smallest of the sample. For 2018 and 2019, both present four and five times more than the minimum value for all universities, and this pattern does not repeat in 2020. There is a reduction in the number of patents, as they do not represent the lowest values in all inputs.

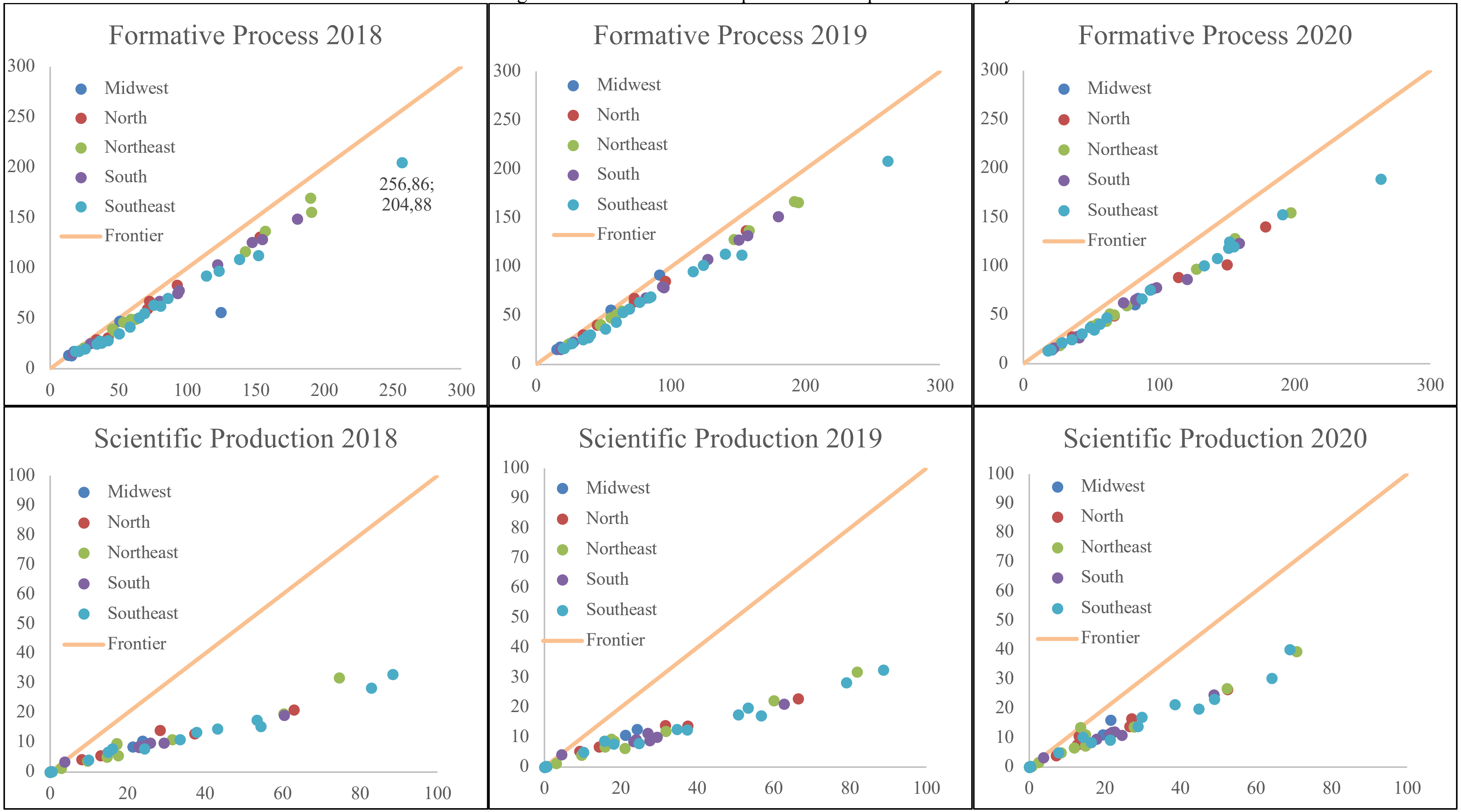
There is also partial alignment in the worst-performing DMUs. UFBA, UFF, and UFRJ are at the ranking's end considering the overall efficiency and the system efficiencies for the three years. UFRJ has the most extensive inputs in the sample. However, the performance in the patent variable is much lower than in universities of similar size. UFMG, for instance, has 20.98% fewer professors, 23.96% fewer enrollments, and 30.4% fewer programs.

Even with significantly lower numbers, it produces 45% more patents and only 12% fewer publications. These values reflected the situation in 2020. However, similar patterns are verified for the other two years.

The cases of UFBA and UFF have similar behavior to UFRJ. Considering the selected inputs, they alternate between the sixth and seventh highest values in the sample. However, the numbers related to theses and dissertations are lower than universities with similar inputs such as UFSC. Despite similar inputs, UFF's results are better than UFBA's, thus justifying UFBA's last place in the global ranking and in the years 2019 and 2020.

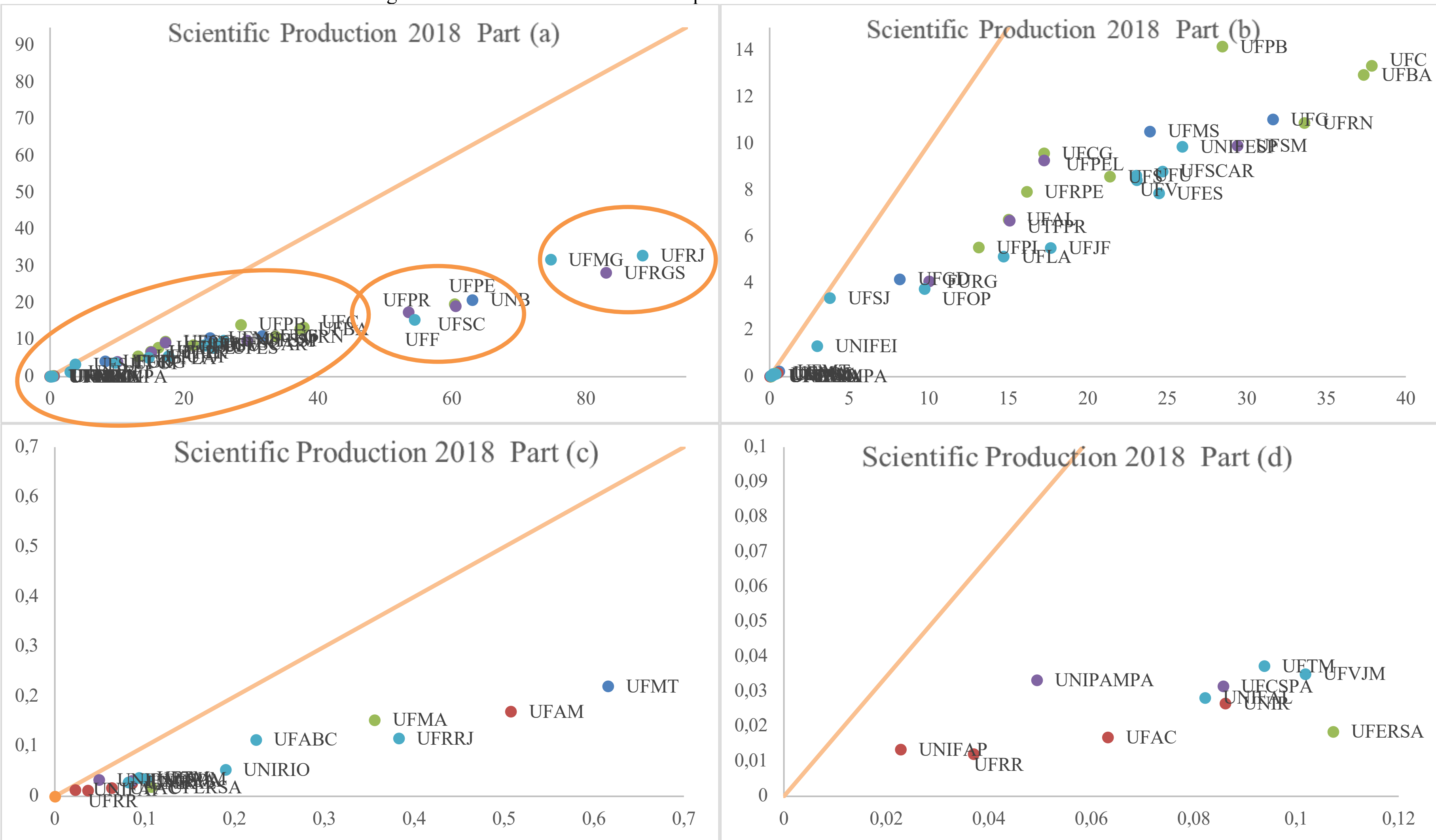
We further examine the bi-dimensional representation of process efficiencies. Figure 25 presents the graphics for both processes and for all years of the sample. The results indicates a reduction in 2020. This reduction was expected due to the impact of the COVID-19 pandemic and its effects on the learning process to an unknown reality of remote learning. In the DDEA results, almost 63% of the sample presented a decrease in performance, while in DNDEA, 60% of the sample performed worse in 2020.

Figure 25 - Bi-dimensional representation of process efficiency



Source: The Author (2024).

Figure 26 - Detailed bi-dimensional representation of Scientific Production in 2018



Source: The Author (2024).

Since, DNDEA results presents higher discrimination among the results, the analyzes were deepened for this modeling. Considering DNDEA scores, federal universities could potentially increase, on average, their efficiency in a network structure of formative process and scientific production by approximately 46,8%.

As previously mentioned, in addition to the overall and system efficiencies in each period, the DNDEA modeling also provides the process efficiencies. Table 20 details these efficiencies for the three years investigated.

Table 20 showed that, on average, the performance of the formative process is superior to that of scientific production. This finding is easily verifiable when observing that most DMUs are further away from the efficiency frontier when compared to the formative process graphic.

In addition to visually identifying inferior results for the training process, the graphic visualization also shows us a lower number of efficient DMUs and few DMUs with relatively high efficiencies. Two efficient DMUs were identified for the training process in 2018, five in 2019, and none in 2020. On the other hand, in the scientific production process, efficient DMUs were not identified in 2018 and 2019, and in 2020, three were considered efficient.

Regarding the absence of efficient DMUs in 2018 and 2019, only one DMU each year obtained scores higher than 70%. In 2020, despite the performance improvement, only 10 DMUs achieved similar values. This information corroborates the initial perceptions of significant difficulties in scientific production and the attention needed to improve this process.

It was also possible to observe that the same university, UFSJ, obtained the highest efficiency values for 2018 and 2019. The more outstanding patent production justifies the better performance compared to DMUs of similar size. In 2020, this university remained among the best in the ranking; however, it was not considered efficient. UNIPAMPA, UFCG, and UNIFAP are deemed efficient in 2020. UNIPAMPA can produce more publications and patents with a relatively small number of dissertations and thesis. UFCG must be highlighted because of its patent product. UNIFAP presents the smallest input values but can obtain the sixth lowest position in the sample regarding this variable.

When observing the efficient DMUs, it is relevant to highlight that the production of patents is a common factor among the universities considered efficient. Thus, universities such as UFCG, which with intermediate quantities of theses and dissertations managed to produce the second largest volume of the sample, must be observed so that good practices can be identified and replicated to improve the performance of the others.

Considering the training process, there is a partial alignment with the system ranking per year considering the same DMUs in first positions in 2018 and 2019. In 2020, UFV, UFPR and UFPE were placed at the top of the ranking.

It is possible to observe that there are no significant discrepancies between the Brazilian macro-regions regarding the efficiency scores, similarly to what was verified for overall and system efficiency per year. Kruskal-Wallis's teste was also used to verify these differences. Table 22 present the test results for the overall efficiency and system's efficiency for all years. At a 5% and 10% significant level, it is possible to infer that there are no differences in the median of the Brazilian macro regions regarding the efficiencies of both processes.

Table 22 - Kruskal-Wallis's test results for process efficiencies

Efficiency	Degrees of freedom	Chi-square	p-value
Formative Process in 2018	4	4,5773	0,3335
Formative Process in 2019	4	1,6322	0,803
Formative Process in 2020	4	5,6436	0,2274
Scientific Production in 2018	4	3,9352	0,4148
Scientific Production in 2019	4	7,7578	0,1009
Scientific Production in 2020	4	7,1857	0,1264

Source: The Author (2024).

Although it does not represent the main objective of the analysis, the value of the DMU's efficiency is easily obtained by observing the graph. The example in Figure 6 quickly illustrates this statement. For the highlighted DMU, its virtual output corresponds to 204.88, whereas the virtual input corresponds to 256.86. Therefore, the efficiency corresponds to 0.7976 (204.88/256.86).

Observing the graphs obtained with the bi-dimensional representation makes it possible to notice the formation of university clusters. We selected the scientific production graph in 2018 to illustrate this feature. In Figure 25, we represent the graphic in different scales, as the virtual inputs and outputs have different dimensions.

Part (a) of the figure presents a graphic with all the DMUs. Initially, three groups stand out. UFRJ, UFMG, and UFRGS demonstrate the highest values of the inputs considered in the stage. The second group is composed of UFPE, UFPR, UFSC, UFF, and UNB. These five DMUs present the highest values of Theses and Dissertations after the values of the universities of the first identified group. When checking these DMUs, there is an absence of northern universities, and a greater concentration of universities in the South and Southeast.

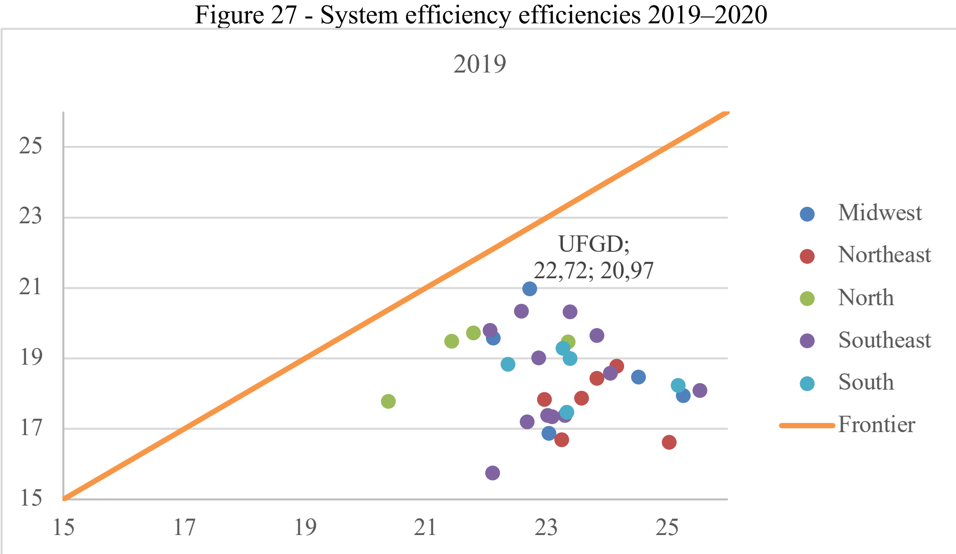
To better understand the third cluster observed in the graph, a cutout is made focusing on these universities. These values are represented in parts B, C and D of Figure 26. When analyzing Part B of Figure 26, there is still a contraction near the origin of the graph, thus, Part C and D detail these DMUs.

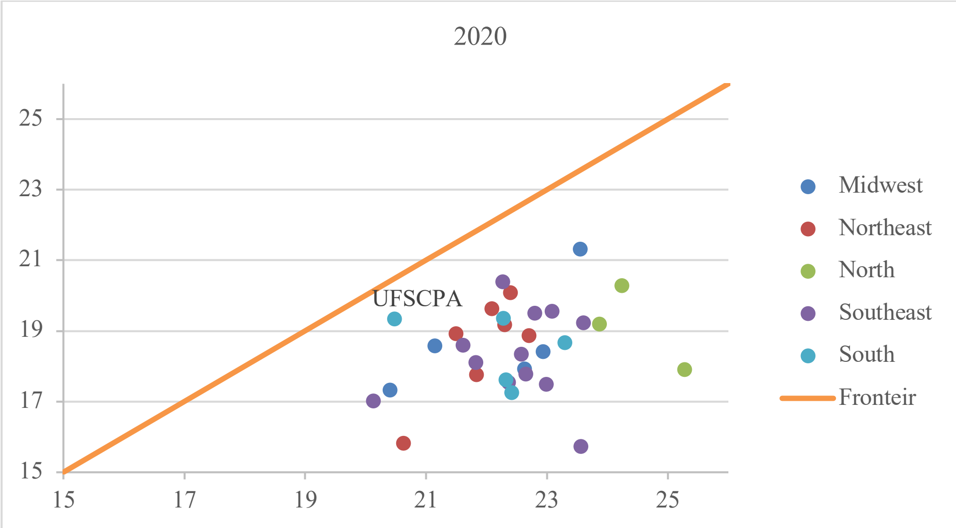
Due to DEA's flexibility to obtain weights in order to maximize results, the use of weights and, consequently, modified inputs and outputs can raise questions. However, in the current case, the two-dimensional representation can provide managers with a starting point for the creation of size metrics or class generation for DMUs in order to obtain information to verify if the size of universities can impact universities efficiency.

Since the number of periods for applying the DNDEA in Chapter 4 is different from the previous case and the efficiency results were discussed in the previous chapter, we present only the discussions related to the two-dimensional representation.

Following the procedure described in Section 5.4, the first step requires running the DNDEA model. In this subsection, we present the graphs and the empirical findings. Appendix B details the mathematical proof that the efficiency values are maintained with the bi-dimensional representation.

Figure 27 - System efficiency efficiencies 2019–2020





Source: The Author (2024).

Figure 27 displays the frontier for the system efficiency in 2019 and 2020. The green line leaving the origin (0,0) corresponds to the efficiency frontier in our bi-dimensional representation. The different colors in the graphs relate to the five Brazilian macro-regions.

The choice to highlight the macro-regions relates to the significant social and economic discrepancies among them and previous literature findings that indicate the impact of the DMUs location on the efficiency score.

The graphics in Figure 25 indicate that for the system efficiencies per year, there are no significant discrepancies between the Brazilian macro-regions. It is also noteworthy that no DMU obtained maximum performance in 2019 and 2020. This fact is also true for overall efficiency values. The results presented so far show the greater power of discrimination of the proposed DNDEA model.

The Kruskal-Wallis non-parametric test was used to investigate if the differences between the macro-regions are significant. Table 23 presents the test results for the overall efficiency and system's efficiency for all years. At a 5% and 10% significant level, it is possible to infer that there are no differences in the median of the Brazilian macro-regions.

Table 23 - Kruskal-Wallis's test results for overall and system efficiencies

Efficiency	Degrees of freedom	Chi-square	p-value
Overall	4	3,7004	0,4481
2019	4	7,2126	0,1251
2020	4	2,3201	0,6771

Source: The Author (2024).

We further examine the bi-dimensional representation of process efficiencies. The results in Figure 28 show no significant discrepancies between the Brazilian macro-regions for both processes. This finding is supported by Kruskal-Wallis's test results for process efficiencies in Table 24. At a 5% and 10% significant level, it is possible to infer that there are no differences in the median of the Brazilian macro-regions regarding process efficiencies.

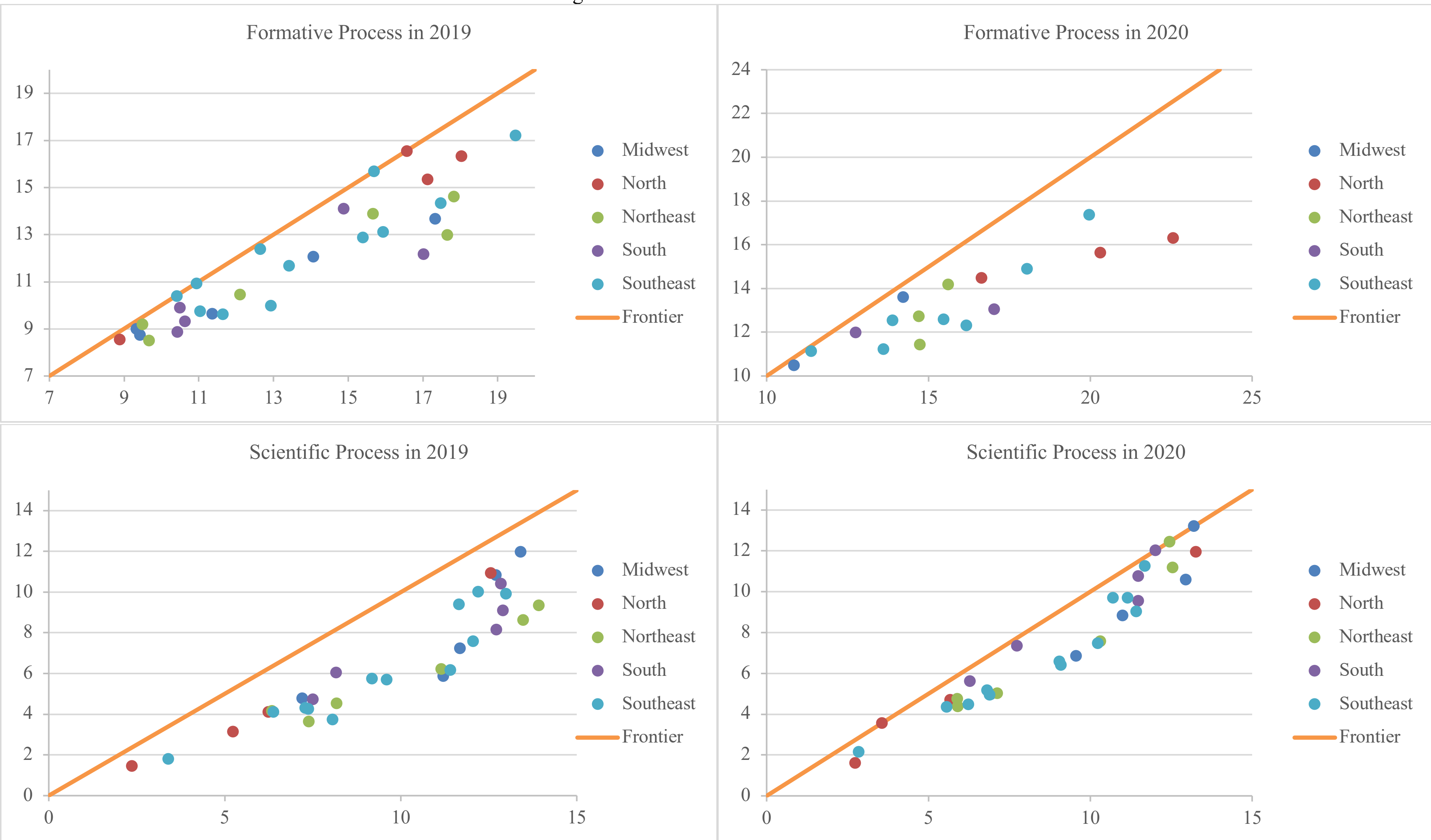
Table 24 - Kruskal-Wallis's test results for process efficiencies

Efficiency	Degrees of freedom	Chi-square	p-value
Formative Process in 2019	4	3,5366	0,4723
Formative Process in 2020	4	3,8443	0,4275
Scientific Production in 2019	4	4,7731	0,3114
Scientific Production in 2020	4	6,3686	0,1734

Source: The Author (2024).

However, unlike overall and systems efficiencies for which no DMU obtained maximum performance in 2019 and 2020, four DMUs were considered efficient in the training process in 2019, and four were considered efficient in the scientific process in 2020. It is also important to mention that the findings that the average performance of the formative process is superior to that of scientific production in 2019 and that the pattern reversed in 2020 is easily verifiable when observing that most DMUs are further away from the efficiency frontier in the respective graphics of Figure 28. The results presented so far show the greater power of discrimination of the proposed DNDEA model.

Figure 28 - Process efficiencies 2019–2020



Source: The Author (2024).

Table 25 - Results of the centralized model

University	θ^{*}	$\theta^{(1,sys)*}$	$\theta^{(2,sys)*}$	w^{1}	w^{2}	α_1	α_2	Formative Process $\theta^{(1,1)*}$	Scientific Production $\theta^{(1,2)*}$	w_1^1	w_2^1	Formative Process $\theta^{(2,1)*}$	Scientific Production $\theta^{(2,2)*}$	w_1^2	w_2^2
UFSCPA	0,823	0,724	0,945	0,551	0,449	0,40	0,70	0,716	0,739	0,676	0,324	0,941	0,950	0,622	0,378
UFMS	0,896	0,885	0,906	0,484	0,516	0,70	0,40	0,928	0,853	0,426	0,574	0,785	1,000	0,439	0,561
UFRR	0,782	0,872	0,709	0,446	0,554	0,70	0,70	0,906	0,610	0,885	0,115	0,724	0,587	0,892	0,108
UFS	0,826	0,777	0,881	0,529	0,471	0,40	0,59	0,821	0,655	0,738	0,262	0,909	0,805	0,726	0,274
UNIPAMPA	0,821	0,842	0,801	0,490	0,510	0,70	0,70	0,949	0,631	0,665	0,335	0,766	0,896	0,731	0,269
UFPI	0,846	0,833	0,860	0,512	0,488	0,40	0,70	0,897	0,658	0,733	0,267	0,871	0,828	0,746	0,254
UNB	0,772	0,710	0,850	0,553	0,447	0,40	0,70	0,858	0,525	0,557	0,443	0,967	0,716	0,532	0,468
UFBA	0,711	0,664	0,767	0,548	0,452	0,69	0,70	0,736	0,491	0,705	0,295	0,778	0,741	0,714	0,286
UFGD	0,863	0,923	0,803	0,498	0,502	0,70	0,40	0,966	0,893	0,410	0,590	0,784	0,818	0,435	0,565
UFPB	0,793	0,773	0,814	0,522	0,478	0,43	0,70	0,887	0,555	0,657	0,343	0,866	0,705	0,674	0,326
UFAL	0,804	0,777	0,831	0,503	0,497	0,70	0,40	0,969	0,641	0,413	0,587	0,757	0,891	0,448	0,552
UNIFAL-MG	0,844	0,831	0,856	0,501	0,499	0,70	0,55	0,884	0,528	0,852	0,148	0,871	0,754	0,876	0,124
UFCG	0,821	0,758	0,889	0,516	0,484	0,40	0,40	0,881	0,672	0,410	0,590	0,746	1,000	0,437	0,563
UFG	0,762	0,732	0,792	0,504	0,496	0,40	0,70	0,849	0,619	0,493	0,507	0,782	0,803	0,514	0,486
UNIFEI	0,765	0,745	0,785	0,510	0,490	0,70	0,70	0,823	0,577	0,684	0,316	0,815	0,718	0,692	0,308
UFJF	0,769	0,708	0,846	0,559	0,441	0,40	0,70	0,821	0,464	0,684	0,316	0,903	0,718	0,690	0,310
UFLA	0,874	0,900	0,848	0,495	0,505	0,40	0,70	1,000	0,807	0,484	0,516	0,798	0,906	0,537	0,463
UFMT	0,811	0,753	0,879	0,537	0,463	0,70	0,40	0,790	0,664	0,706	0,294	0,959	0,716	0,672	0,328
UFMG	0,814	0,772	0,861	0,527	0,473	0,70	0,70	0,981	0,541	0,526	0,474	0,979	0,730	0,526	0,474
UFOP	0,760	0,758	0,761	0,497	0,503	0,70	0,70	0,837	0,592	0,679	0,321	0,762	0,759	0,704	0,296
UFPEL	0,862	0,829	0,897	0,510	0,490	0,40	0,40	0,852	0,811	0,448	0,552	0,778	1,000	0,464	0,536
UFPE	0,743	0,718	0,770	0,509	0,491	0,50	0,70	0,865	0,557	0,521	0,479	0,800	0,735	0,540	0,460
UNIR	0,852	0,904	0,805	0,477	0,523	0,40	0,40	1,000	0,602	0,760	0,240	0,771	1,000	0,851	0,149
UFSC	0,769	0,749	0,790	0,511	0,489	0,40	0,70	0,877	0,642	0,455	0,545	0,744	0,833	0,486	0,514
UFSM	0,840	0,813	0,870	0,512	0,488	0,40	0,70	0,946	0,704	0,448	0,552	0,796	0,939	0,485	0,515
UFSCAR	0,782	0,751	0,813	0,506	0,494	0,40	0,70	0,885	0,629	0,478	0,522	0,836	0,790	0,494	0,506
UFSJ	0,827	0,824	0,830	0,522	0,478	0,70	0,40	0,827	0,821	0,488	0,512	0,789	0,870	0,489	0,511
UNIFESP	0,855	0,897	0,815	0,483	0,517	0,70	0,58	1,000	0,643	0,711	0,289	0,826	0,781	0,765	0,235
UFU	0,770	0,755	0,786	0,504	0,496	0,62	0,70	0,871	0,593	0,583	0,417	0,825	0,727	0,601	0,399
UFV	0,892	0,869	0,916	0,512	0,488	0,40	0,70	1,000	0,764	0,445	0,555	0,864	0,962	0,476	0,524
UFABC	0,689	0,713	0,668	0,484	0,516	0,70	0,70	0,774	0,626	0,585	0,415	0,644	0,705	0,614	0,386
UFAC	0,871	0,909	0,837	0,469	0,531	0,70	0,40	0,963	0,871	0,414	0,586	0,759	0,902	0,453	0,547

Source: The Author (2024).

Table 25 details the same results that are presented in Figure 27 and Figure 28. However, it is simpler to identify patterns and obtain a quicker understanding of the results. It is also noteworthy that both in the Brazilian case and international assessments, the commissions responsible for evaluations are multidisciplinary, and not all the members involved are always familiar with mathematical programming models.

The visual analysis will also aid in faster identification of DMU performance patterns and faster identification of the best-performing units. It can also help to verify the existence of performance discrepancies between geographical regions, and these checks are faster than analyzing large data tables. Although it does not represent the main objective of the analysis, the value of the DMU’s efficiency is easily obtained by observing the graph.

We can quickly obtain the value of the DMU's efficiency by observing Figure 27: for the highlighted DMU (UFGD), its virtual output corresponds to 22.72, whereas the virtual input corresponds to 20.97, and the efficiency corresponds to 0.9230 (20.97/22.72). Table 25 also shows that the proposed method makes ranking universities based on efficiency values possible. In addition to these values, the results related to the proportions of the allocation of resources shared between the stages and the importance of the stages reflected by the proportions of inputs are presented.

5.7 FINAL CONSIDERATIONS

The importance of universities for social and economic development and the public funds used in these institutions has stimulated the development of different proposals for evaluating the most different universities. DEA has stood out in the field of efficiency measurements in education, with the application of models in distinct areas, such as primary education, secondary schools, teachers, students, research, and teaching.

Bearing that educational training processes span several consecutive periods, the use of models that consider the temporal effects on efficiency is shown to be adequate. Thus, the current study uses dynamic DEA models to evaluate postgraduate activities in Brazilian federal universities.

We also consider that there is a network structure when analyzing the processes of graduate activities. Thus, comparisons were made between the results of two relational models: dynamic and dynamic with network structure. According to these results, DNDEA provides more discrimination when compared to DDEA. DNDEA results are close to reality because of the consideration of network structure in the model.

This paper also presents a new framework for a bi-dimensional representation of the efficiency frontier and the location of DMUs considering multiple inputs and outputs for the previously mentioned dynamic modeling. Considering this, a step-by-step procedure to represent the efficiency results of the DDEA model of Kao (2013) and the DNDEA model of Omrani and Soltanzadeh (2016) is presented.

A linearization of weights extending the proposition of Bana e Costa et al. (2016) was developed to obtain a bi-dimensional representation. The proposed linearization allows the calculation of a new set of weights, the modified weights. Then, these were used to obtain modified virtual inputs and modified virtual outputs. Based on these results, a two-dimensional representation is obtained.

The combination of the models and their bi-dimensional representation indicates that the DNDEA model is more suitable for the investigated reality. In addition, it was possible to verify an increase in efficiency in the first period examined, followed by a reduction. The impacts of the pandemic could explain the decrease in performance. The analysis of the processes corresponding to postgraduate activities indicates a national difficulty in disseminating research, and significant performance differences were not identified when considering the five Brazilian macro-regions.

Correlation analyses between the results highlight the greater impact of the training process on the results of the system. Furthermore, an increase in the performance of this stage needs to be carried out carefully so that the second stage can follow these improvements. A thorough analysis is necessary to investigate the main difficulties faced in the publication process in these universities. However, it is believed that cuts in national budgets earmarked for education and research may have negatively impacted the performance of this activity in 2018 and 2019.

The proposed approach's advantages are related to the simplicity of the method, which does not require the use of other techniques. The bi-dimensional choice provides graphs that are easily comprehended. It is also important to highlight that the dynamic models provide a more comprehensive range of information when compared with classical models such as CCR and BCC. In this sense, graphical representation offers an important way to deliver all the information to the decision-maker. This proposition is simple and efficient. However, its main disadvantage is the inability to identify reference DMUs for inefficient units.

The empirical results can be used to improve graduate activities and support the development of public policies to enhance Brazilian research results. Despite the small number of institutions in relation to the total number of higher education institutions present in Brazil, the sample consists of a representative set of institutions in the country's research activities, considering that they cover more than half of the country's graduate students.

Further studies should be conducted to supplement our findings. First, graduate activities represent the investigation focus, but it did not consider undergraduate activities that represent significant parts of federal universities operating processes.

Thus, more divisions, for instance, extension activities represent a relevant contribution to federal universities to society. Therefore, a more detailed investigation of the most pertinent divisions can provide significant results to Brazilian universities. Second, although federal universities are highly relevant to Brazilian research, the investigation of private, state, and municipal institutions should also be taken into account to assess the performance of graduate activities.

Third, our study did not include metrics that are proxies for the quality of graduate activities. Therefore, further studies should add variables that reflect quality. For instance, the classification of publications considering journal impact factor or quartile could be used to segregate this variable and provide more thorough evaluations. Considering the bidimensional development, this research can be extended to other DDEA and DNDEA models and relate the virtual outputs and inputs to measure and determine the DMU size.

6 CONCLUSION

6.1 FINAL CONSIDERATIONS AND MANAGERIAL IMPLICATIONS

This doctoral thesis was structured to present methodological and practical contributions to the fields of productivity and efficiency measurement concerning Data Envelopment Analysis and Game Theory. The purpose of combining such approaches is justified by the need to include in this measurement, potential conflicts or collaborations in the internal structures of the investigated agents. This consideration seeks to provide realistic metrics that help stakeholders in the decision-making process.

This thesis provides significant contributions as it reviews the state of the art in joint applications of the previously mentioned techniques. The second chapter is dedicated to present a systematic literature review based on the methodology proposed by Lage Junior and Godinho Filho (2010) to identify gaps and research trends in the literature.

To archive this purpose, English written works published between January 1980 and December 2020 indexed in Scopus and Web of Science were considered. After a thorough analysis, 119 studies included the final sample. Since the seminal discussions of Banker (1980), methodological and applied propositions contributed to cover several areas of knowledge and distinct managerial issues.

Bibliometric tools such as Vosviewer and CitNetExplorer were used to: identify the main topics discussed with both techniques, identify core publications, observe the development of this literary branch and highlight gaps and future research trends. The research proposes nine categories to systematize the studies. Through this systematization, it was possible to reach the following conclusions:

1. The predominance of applied studies;
2. The existence of three main combinations of DEA and GT: cooperative, non-cooperative approaches and the joint use of the previous two;
3. Literature development started with non-cooperative propositions; however cooperative approaches are currently the majority;
4. Network, cross-efficiency, and classical models are the most used DEA approaches
5. Nash-bargaining, leader-follower, zero-sum games, and imputation methods stands out in the GT context;

6. Banks and insurance companies, energy and environment are the main researched areas;
7. The results indicated that Fuzzy Theory, Balanced Scorecard, and TOPSIS were the most applied techniques in conjunction with DEA-GT.

In addition to the previously mentioned conclusions, the literature review identified eleven gaps, which consist of directions to be followed. Among these, it was primarily addressed the consideration of more complex networks, dynamic investigations, and the comparison between cooperative and non-cooperative approaches. The findings of this literature review are relevant and useful for scholars and practitioners because they can identify the available methods to address a wide range of managerial situations and highlights paths that need to be addressed to provide support for more complex and dynamic situations.

A crucial aspect verified by this initial development of the thesis is of particular interest of the field due to the hybrid applied methodologies. Since the models are becoming more complex, they demand a more robust computational support to solve the mathematical approaches. However, the absence of dedicated software or packages to solve the hybrid DEA-GT models in the same platform can represent an obstacle to the dissemination of this particular research field. This a particular topic is relevant to be addressed by future research.

The second major contribution from this work is the proposition of dynamic models which include the internal structure of the DMUs (DNDEA models) and also contemplate shared resources in this structure. The combination of DNDEA models with Game Theory makes it possible to address competition and cooperation inside the organizations and how these dynamics can and will impact their efficiency when shared resources are present among the internal stages. The new models are detailed in Chapters 3 and 4 and they provide relevant insights to managers to support the decision-making process.

The investigations of shared resources in internal networks were addressed under two distinct frameworks. First, the amount of resources used by each stage is known and under this supposition, the proposition investigates how much profit can increase under cooperation if these resources are pooled together. The second considers that we aim to find optimal proportions to allocate a fixed number of resources among the stages.

The efficiency evaluation considering shared resources, and the simultaneous contemplation of cooperation or competition between the stages and dynamic network framework allows managers to identify the inefficient processes, better resource usage and the identification of benchmarks.

The modeling presented in Chapter 3 allowed the simultaneous discussion of a network with a greater number of stages, the dynamic characteristic of the model and the use of imputation models to allocate the benefits arising from the sharing of resources between the stages of the network. Numerical example results confirm the potential for increasing profits through pooling of resources.

Chapter four also presents a new modeling. This consists of a new DNDEA model that also includes resource sharing. However, unlike Chapter 3, the model aims to measure efficiency and not profit. In addition to detailing a new DNDEA approach, the chapter also presents the discussion of efficiency decomposition and its relevance to show managers the impacts when there is a need to prioritize one of the stages of the network.

The developed model is applied for the evaluation of graduate courses in Brazilian universities. The results presented provide guidance to public managers on which processes need more attention. In addition, we also present the best allocation for resources shared between processes.

Finally, chapter five discusses a different topic from the previous ones, the visual representation of the efficiency frontier. Chapters 3 and 4 provide discussions of dynamic models with networked structures. These models have different levels of efficiency and the similar nomenclature can lead to difficulties in understanding by decision makers. Thus, the development presented in this chapter aims to help in the presentation of results, using a bi-dimensional representation. Modified virtual inputs and outputs are used to represent the results of all efficiencies levels provided by the DNDEA model.

6.2 FUTURE RESEARCH AND LIMITATIONS

This work developed a comprehensive analysis of both the DEA and GT literature and efficiency measurement applications in the education sector. As previously discussed, relevant databases were used, however it is not possible to confirm that our sample encompasses all studies, therefore this could be pointed out as a limitation. Regarding future research, the eleven gaps pointed out in Chapter 2 correspond to opportunities to expand the current considerations presented in this thesis.

In addition to discussions on how to advance hybrid DEA and Game Theory models, investigations in the Brazilian education segment show that these models can provide relevant insights for creating more suitable indicators for evaluating the performance and quality of Brazilian HEIs.

Specifically in this segment, the problems related to the definition of previously established weights for these evaluations is a potentially problematic characteristic due to the economic and social characteristics of Brazil. Therefore, the combination of DEA and GT to limit the wide flexibility of DEA, the use of imputation method or bargaining models to define these weights considering the possible conflicts between the interests of the HEIs consist of relevant discussions to extend the propositions already presented in that study.

As previously mentioned, the current study proposed new mathematical models combining DEA and Game Theory. As for the model presented in Chapter 3, the application of other imputation methods and the extension to multi-stage approaches correspond to the main discussions to be addressed in future works.

Regarding the modeling developed in Chapter 4, the discussion of the uniqueness of the solutions is the main topic to be investigated by future research. The consideration of a two-stage network can be perceived as a limitation. The construction of a framework that allows for multi-stage evaluation is a relevant research direction. In addition, the use of modeling in other areas will allow validating its implementation for other segments.

It is also relevant to note that for our models, there is an explicit assumption of cooperation for Chapter 3 and cooperation/non-cooperation for Chapter 4 model. Therefore, managers must know if this assumption hold, the approaches may not be useful if it does not. Concerning the applied field investigates, education, it is significant to highlight that studies to address the distinct missions and goals of the HEIs presents itself as an important discussion. The HEIs mission directly impacts on its numbers and some aspects regarding the services provided by these HEIs for the community and its impact on social and economic development have not been considered in the current research. Therefore, these considerations must be considered and consists as an important direction for future research.

Discussions on the use of visual tools to represent the efficiency frontier of dynamic and network models based on clearance or non-radial considerations represent an opportunity to expand the approach proposed here. Finally, it is important to highlight the current development of a technological product to detail and implement the bi-dimensional representation presented in Chapter 5.

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APPENDIX A - MATHEMATICAL PROOF AND STEP-BY-STEP PROCEDURE FOR THE DDEA MODEL

The modified virtual input for the system efficiency in each period will be addressed as $I'^{(t,sys)}_j$ while the modified virtual output for the same context as $O'^{(t,sys)}_j$. They are determined by (A1) and (A2), respectively.

$$I'^{(t,sys)}_j = \sum_{i=1}^{m} v'_{ij} x^{(t)}_{ij} + \sum_{d=1}^{D} f'_{dj} z^{(t-1)}_{dj} \quad (A1)$$

$$O'^{(t,sys)}_j = \sum_{r=1}^{s} u'_{rj} y^{(t)}_{rj} + \sum_{d=1}^{D} f'_{dj} z^{(t)}_{dj} \quad (A2)$$

$I'^{(t,sys)}_j$ and $O'^{(t,sys)}_j$ will be used to plot each DMU in a bi-dimensional representation. The normalization proposed considering the sum of input and carry-over weights will produce a representation of the frontier that corresponds to a straight line from the origin that bisects the quadrant as similarly verified in Bana e Costa et al. (2016).

This will occur because the virtual input is equal to the virtual outputs when the DMU is efficient. The efficiency of a DMU can be obtained dividing the modified virtual output by the modified virtual input. In (A3), there is proof that the algebraic manipulations used preserves the results obtained with model.

$$E^{(t,sys)}_O = \frac{\sum_r^s u^*_{ro} y^{(t,k)}_{ro} + \sum_d^D f^*_{do} z^{(t,k)}_{do}}{\sum_{i=1}^{m} v^*_{io} x^{(t,k)}_{io} + \sum_d^D f^*_{do} z^{(t-1,k)}_{do}} = \frac{\dfrac{\sum_r^s u^*_{ro} y^{(t,k)}_{ro} + \sum_d^D f^*_{do} z^{(t,k)}_{do}}{S_0}}{\dfrac{\sum_{i=1}^{m} v^*_{io} x^{(t,k)}_{io} + \sum_d^D f^*_{do} z^{(t-1,k)}_{do}}{S_o}}$$

$$= \frac{\sum_r^s \frac{u^*_{ro}}{S_0} y^{(t,k)}_{ro} + \sum_d^D \frac{f^*_{do}}{S_0} z^{(t,k)}_{do}}{\sum_{i=1}^{m} \frac{v^*_{io}}{S_o} x^{(t,k)}_{io} + \sum_d^D \frac{f^*_{do}}{S_o} z^{(t-1,k)}_{do}} = \frac{\sum_r^s u'_{ro} y^{(t,k)}_{ro} + \sum_d^D f'_{do} z^{(t,k)}_{do}}{\sum_{i=1}^{m} v'_{io} x^{(t,k)}_{io} + \sum_d^D f'_{do} z^{(t-1,k)}_{do}}$$

$$= \frac{O'^{(t,sys)}_o}{I'^{(t,sys)}_o} \quad (A3)$$

The process to obtain the representation of overall efficiency is similar to the efficiency in each period. The modified virtual input for the system efficiency in each period will be addressed as I'_j while the modified virtual output for the same context as O'_j. They are determined by (A4) and (A5), respectively.

$$I'^{(sys)}_j = \sum_{i=1}^{m} v'_{ij} x_{ij} + \sum_{d}^{D} f'_{dj} z^{(t_o,k)}_{dj} \quad (A4)$$

$$O'^{(sys)}_j = \sum_{r=1}^{s} u'_{rj} y_{rj} + \sum_{d}^{D} f'_{dj} z^{(T,k)}_{dj} \quad (A5)$$

$I'^{(sys)}_j$ and $O'^{(sys)}_j$ will be used to plot the DMU in a bi-dimensional representation.

The frontier obtained for the overall system efficiency will follow the same pattern detailed in 5.4.1.1 and can be obtained dividing the modified virtual output by the modified virtual input. In (A6), there is proof that the algebraic manipulations used preserves the results obtain with model (32).

$$E_O^{(sys)} = \frac{\sum_r^s u_{ro}^* y_{ro} + \sum_d^D f_{do}^* z_{do}^{(T,k)}}{\sum_{i=1}^m v_{io}^* x_{io} + \sum_d^D f_{do}^* z_{do}^{(t_o,k)}} = \frac{\frac{\sum_r^s u_{ro}^* y_{ro} + \sum_d^D f_{do}^* z_{do}^{(T,k)}}{S_0}}{\frac{\sum_{i=1}^m v_{io}^* x_{io} + \sum_d^D f_{do}^* z_{do}^{(t_o,k)}}{S_o}} = \frac{\sum_r^s \frac{u_{ro}^*}{S_0} y_{ro} + \sum_d^D \frac{f_{do}^*}{S_o} z_{do}^{(T,k)}}{\sum_{i=1}^m \frac{v_{io}^*}{S_o} x_{io}^{(t,k)} + \sum_d^D \frac{f_{do}^*}{S_o} z_{do}^{(t_o,k)}}$$

$$= \frac{\sum_r^s u'_{ro} y_{ro} + \sum_d^D f'_{do} z_{do}^{(T,k)}}{\sum_{i=1}^m v'_{io} x_{io} + \sum_d^D f'_{do} z_{do}^{(t_o,k)}} = \frac{O'^{(sys)}_o}{I'^{(sys)}_o} \quad (A6)$$

Following the procedure described in Section 5, the first step requires running the input-oriented dynamic model of Kao (2013). The results of this step are presented in Table A1.

Table A1 - Efficiency scores and weights of the DDEA model

DMU	Overall	2018	2019	2020	v1	v2	f1	u1	u2	u3	u4	u5
UFCSPA	0,959201	1	0,898909	0,981217	0,001362	0,000001	0,000001	0,00013	0,000365	0,000292	0,000246	0,001098
UFMS	0,959092	0,886383	0,982724	1	0,000001	0,000179	0,000001	0,000001	0,000128	0,000102	8,84E-05	0,000227
UFRR	0,893199	1	1	0,728644	0,000001	0,000966	0,003242	0,000245	0,000736	0,000377	0,000403	0,000001
UFS	0,873816	0,847403	0,849099	0,923753	0,000105	9,78E-05	0,000001	0,000001	0,000164	0,000131	6,28E-05	0,000152
UNIPAMPA	0,920539	1	0,923135	0,85843	4,39E-05	0,000598	0,000423	0,000105	0,000171	0,000136	0,000335	0,000721
UFPI	0,823086	0,779545	0,883349	0,881642	0,000152	8,83E-05	0,002311	0,000001	0,000148	0,000118	7,02E-05	0,000261
UNB	0,784781	0,779725	0,821995	0,853937	0,000122	0,000001	0,001049	0,000001	4,16E-05	3,33E-05	2,27E-05	0,000001
UFBA	0,67347	0,687389	0,681084	0,729236	6,69E-05	2,73E-05	0,00057	0,000001	8,92E-05	0,000001	3,12E-05	0,000001
UFGD	0,951615	0,939187	1	0,917064	0,000323	0,000217	0,000001	5,05E-05	0,000281	0,000225	0,000172	0,000364
UFPB	0,842131	0,911254	0,920401	0,809967	8,21E-05	3,65E-05	0,002487	0,000001	0,000166	0,000133	2,94E-06	0,000267
UFAL	0,867372	0,861662	0,988202	0,877469	0,000135	8,17E-05	0,006495	0,000001	0,000316	0,000252	0,000001	0,000619
UNIFAL	0,884051	0,928619	0,955422	0,813002	0,000001	0,000501	0,002553	5,08E-05	0,0005	0,0004	0,000209	0,00027
UFCG	0,998796	1	0,997314	1	0,00034	2,51E-05	0,005761	3,63E-05	0,000362	0,00029	0,000001	0,000582
UFG	0,796233	0,795686	0,796544	0,866782	9,5E-05	3,38E-05	0,000965	0,000001	5,3E-05	4,24E-05	3,92E-05	0,000001
UNIFEI	0,91125	0,937807	0,961646	0,901373	0,000933	0,000001	0,01103	0,000123	0,000791	0,000633	0,000001	0,001301
UFJF	0,775849	0,775513	0,863788	0,793926	8,77E-05	8,5E-05	0,002598	0,000001	0,000188	0,00015	4,17E-05	0,000001
UFLA	0,951726	0,989035	1	0,894965	0,000447	0,000001	0,002391	4,62E-05	0,000284	0,000001	8,76E-05	0,000001
UFMT	0,858619	0,8233	0,87777	0,873009	8,1E-05	0,00013	0,000001	0,000001	0,00025	0,0002	6,03E-05	7,26E-05
UFMG	0,881472	0,859155	0,877331	0,948807	0,000105	0,000001	0,000779	0,000001	7,58E-05	0,000001	1,99E-05	0,000117
UFOP	0,797747	0,903858	0,874808	0,792871	0,000223	0,000101	0,006814	2,35E-06	0,000454	0,000363	8,14E-06	0,000733
UFPEL	0,980134	0,983522	0,968015	0,995398	6,79E-05	0,000101	0,001466	0,000001	0,000337	0,000001	6,13E-05	0,00036
UFPE	0,883368	0,913896	0,944298	0,876443	0,000132	0,000001	0,002024	0,000001	0,000259	2,49E-05	0,000001	0,000248
UNIR	0,868384	0,842883	0,987531	0,837338	0,000451	0,000234	0,007023	5,91E-05	0,000472	0,000378	0,000177	0,000001
UFSC	0,845709	0,843127	0,856465	0,899794	0,000139	0,000001	0,001053	0,000001	0,000108	0,000001	2,73E-05	0,000001
UFSM	0,916739	0,929696	0,921134	0,944919	0,000237	0,000001	0,002052	0,000001	8,06E-05	6,45E-05	4,33E-05	0,000001
UFSCAR	0,830126	0,789595	0,872969	0,88501	0,000001	8,66E-05	0,00124	0,000001	0,000342	0,000001	3,52E-05	0,000001
UFSJ	0,88227	0,936433	0,985304	0,837855	0,000001	0,000278	0,009939	6,48E-05	1,25E-06	0,000001	9,17E-05	0,00179
UNIFESP	0,795824	0,792675	0,812739	0,782121	0,000001	8,39E-05	0,000001	0,000001	0,000306	0,000001	3,72E-05	0,000001
UFU	0,893934	0,917021	0,927893	0,910674	9,02E-05	5,99E-05	0,003468	0,000001	0,000298	0,000169	0,000001	0,000395
UFV	0,994181	0,982458	1	1	0,000375	0,000001	0,000001	3,58E-05	0,000101	8,06E-05	6,82E-05	0,000304
UFABC	0,689112	0,749678	0,779134	0,741005	0,000167	0,000107	0,005504	0,000001	6,55E-05	5,24E-05	9,61E-05	0,000491
UFAC	0,950792	0,933949	1	0,920767	8,74E-05	0,000544	0,000465	0,000102	0,00023	0,000184	0,000308	0,000001
UNIFAP	0,961659	1	1	0,902369	0,000263	0,000599	0,000001	0,000156	1,25E-06	0,000001	0,000409	0,001145
UFAM	0,675246	0,730862	0,772112	0,68828	9,76E-05	0,000103	0,003011	0,000001	0,000219	0,000176	4,93E-05	0,000129
UFC	0,809952	0,922868	0,873515	0,787929	0,000155	0,000001	0,002575	0,000001	0,000141	0,000113	0,000001	0,000228
UFES	0,827006	0,849629	0,855058	0,869446	6,98E-05	6,69E-05	0,002043	0,000001	0,000148	0,000118	3,31E-05	0,000001
UNIRIO	0,707343	0,679283	0,810209	0,785987	0,000001	0,000224	0,004544	0,000001	0,000782	0,000159	6,8E-05	0,000001
UFMA	0,868246	0,81468	0,956975	0,929397	0,000163	9,32E-05	0,00562	0,000001	3,82E-05	3,05E-05	8,95E-05	0,000537
UFPA	0,783417	0,771648	0,819349	0,869897	0,000157	0,000001	0,001354	0,000001	5,35E-05	4,28E-05	2,9E-05	0,000001
UFPR	0,831645	0,820831	0,833468	0,892622	7,75E-05	2,74E-05	0,000782	0,000001	4,3E-05	3,44E-05	3,19E-05	0,000001
UFRJ	0,789106	0,812509	0,801676	0,835296	8,15E-05	0,000001	0,00061	0,000001	6,24E-05	0,000001	1,63E-05	0,000001
FURG	0,915759	0,894486	1	0,853182	0,000169	0,000174	0,000001	5,43E-05	0,000564	0,000001	0,00012	0,000353

UFRN	0,753964	0,822587	0,855696	0,76661	6,96E-05	3,1E-05	0,00209	0,000001	0,000139	0,000111	2,82E-06	0,000226
UFRGS	0,91238	0,977101	0,92821	0,881565	0,000107	0,000001	0,00129	0,000001	0,000186	0,000001	1,02E-05	0,000001
UFTM	0,955977	1	0,960919	0,907633	0,000516	0,000342	0,000001	8,27E-05	0,000591	0,000473	0,000246	0,000369
UFVJM	0,876744	0,830511	0,893869	0,925147	0,000001	0,000406	0,001761	4,28E-05	0,000405	0,000324	0,000171	0,000001
UFF	0,766275	0,840461	0,860098	0,763737	7,24E-05	2,15E-05	0,001997	0,000001	0,000131	0,000104	0,000001	0,000001
UFRPE	0,976464	0,954792	0,982635	1	0,000112	0,00013	0,00218	0,000001	0,000474	0,000001	8,06E-05	0,000595
UFRRJ	0,79789	0,783214	0,891719	0,821729	0,00012	0,000118	0,003606	0,000001	0,000261	0,000209	5,74E-05	0,000001
UFERSA	0,963611	1	1	0,891972	0,000556	0,000383	0,000001	0,000215	0,002103	0,000001	0,000221	0,000001
UTFPR	0,788214	0,815228	0,90123	0,821918	0,000125	5,7E-05	0,003846	0,000001	0,000256	0,000205	4,41E-06	0,000412

Source: The Author (2024).

With Table A1 results, Step 2 was performed, the calculation of S_j. S_j allows to follow to Step 3, and obtain the modified weights (v'_1,v'_2,f'_1, u'_1, u'_2,u'_3,u'_4,u'_5). S_j values are in column 1 in Table A2, and modified weights are in columns 2 to 9.

Table A2 - Modified weights of the DDEA model

DMU	S_j	v'_1	v'_2	f'_1	u'_1	u'_2	u'_3	u'_4	u'_5
UFCSPA	0,001364	0,998533	0,000733	0,000733	0,095386	0,26744	0,213952	0,180561	0,805203
UFMS	0,000181	0,005534	0,988933	0,005534	0,005534	0,708194	0,566555	0,489148	1,258202
UFRR	0,004209	0,000238	0,229515	0,770247	0,058255	0,174858	0,08968	0,095703	0,000238
UFS	0,000204	0,515835	0,479266	0,004899	0,004899	0,803016	0,642413	0,307847	0,74285
UNIPAMPA	0,001065	0,041213	0,561829	0,396958	0,098413	0,160174	0,128139	0,314812	0,676853
UFPI	0,002551	0,0595	0,034635	0,905866	0,000392	0,057911	0,046329	0,027536	0,102274
UNB	0,001173	0,10421	0,000853	0,894937	0,000853	0,035496	0,028397	0,019368	0,000853
UFBA	0,000664	0,100722	0,041159	0,858119	0,001506	0,13432	0,001506	0,047021	0,001506
UFGD	0,000541	0,597172	0,40098	0,001848	0,093249	0,52009	0,416072	0,318226	0,672222
UFPB	0,002606	0,031521	0,014003	0,954476	0,000384	0,063584	0,050867	0,001128	0,102545
UFAL	0,006711	0,020158	0,012166	0,967675	0,000149	0,047016	0,037613	0,000149	0,092176
UNIFAL	0,003054	0,000327	0,16389	0,835783	0,016641	0,163726	0,130981	0,068324	0,088269
UFCG	0,006126	0,055521	0,004089	0,94039	0,005927	0,059128	0,047303	0,000163	0,094938
UFG	0,001093	0,086855	0,030875	0,882269	0,000915	0,04844	0,038752	0,035829	0,000915
UNIFEI	0,011964	0,078003	8,36E-05	0,921913	0,010244	0,066157	0,052925	8,36E-05	0,108732
UFJF	0,002771	0,031655	0,030659	0,937686	0,000361	0,067858	0,054286	0,015061	0,000361
UFLA	0,002839	0,157394	0,000352	0,842254	0,016286	0,099956	0,000352	0,030862	0,000352
UFMT	0,000212	0,381221	0,614073	0,004706	0,004706	1,17669	0,941352	0,284029	0,341528
UFMG	0,000885	0,118728	0,001131	0,880141	0,001131	0,08564	0,001131	0,022477	0,132839
UFOP	0,007138	0,031283	0,014107	0,95461	0,000329	0,06363	0,050904	0,001141	0,102631
UFPEL	0,001635	0,041559	0,061573	0,896868	0,000612	0,206408	0,000612	0,037492	0,220154
UFPE	0,002157	0,061231	0,000464	0,938306	0,000464	0,119904	0,011546	0,000464	0,114926
UNIR	0,007708	0,058552	0,030342	0,911106	0,007669	0,061222	0,048978	0,022914	0,00013
UFSC	0,001193	0,11655	0,000838	0,882611	0,000838	0,090278	0,000838	0,022903	0,000838
UFSM	0,002289	0,103391	0,000437	0,896172	0,000437	0,035227	0,028181	0,018932	0,000437
UFSCAR	0,001328	0,000753	0,065192	0,934055	0,000753	0,257755	0,000753	0,026477	0,000753
UFSJ	0,010218	9,79E-05	0,027206	0,972696	0,006344	0,000122	9,79E-05	0,008975	0,17519
UNIFESP	8,59E-05	0,011641	0,976718	0,011641	0,011641	3,558773	0,011641	0,432915	0,011641
UFU	0,003618	0,024925	0,016564	0,95851	0,000276	0,082442	0,046612	0,000276	0,109031
UFV	0,000377	0,994691	0,002655	0,002655	0,095024	0,267534	0,214027	0,18109	0,806655
UFABC	0,005777	0,028839	0,018493	0,952668	0,000173	0,01134	0,009072	0,016636	0,084909
UFAC	0,001097	0,079691	0,496343	0,423966	0,093392	0,209721	0,167777	0,2804	0,000912
UNIFAP	0,000863	0,30431	0,694531	0,001159	0,180991	0,001449	0,001159	0,473472	1,326958
UFAM	0,003212	0,030385	0,032176	0,937439	0,000311	0,068317	0,054653	0,015345	0,040105
UFC	0,002731	0,056657	0,000366	0,942977	0,000366	0,051567	0,041253	0,000366	0,083626
UFES	0,002179	0,032051	0,030678	0,937271	0,000459	0,06783	0,054264	0,01518	0,000459
UNIRIO	0,004769	0,00021	0,047004	0,952786	0,00021	0,163963	0,033342	0,014256	0,00021
UFMA	0,005877	0,027771	0,015855	0,956374	0,00017	0,006493	0,005195	0,015231	0,091313
UFPA	0,001512	0,103833	0,000661	0,895505	0,000661	0,035372	0,028298	0,019167	0,000661
UFPR	0,000887	0,087374	0,030917	0,881708	0,001128	0,048504	0,038803	0,035958	0,001128
UFRJ	0,000693	0,117611	0,001444	0,880945	0,001444	0,090091	0,001444	0,0235	0,001444
FURG	0,000344	0,490672	0,506423	0,002905	0,157796	1,637455	0,002905	0,348213	1,026392
UFRN	0,002191	0,031793	0,014128	0,954079	0,000456	0,063594	0,050875	0,001289	0,102961
UFRGS	0,001398	0,076713	0,000715	0,922572	0,000715	0,133323	0,000715	0,007268	0,000715
UFTM	0,000859	0,600964	0,397871	0,001164	0,096251	0,688065	0,550452	0,286566	0,429933
UFVJM	0,002168	0,000461	0,187396	0,812142	0,019746	0,186686	0,149349	0,078844	0,000461
UFF	0,002091	0,034631	0,010268	0,955101	0,000478	0,062458	0,049967	0,000478	0,000478
UFRPE	0,002422	0,046214	0,05386	0,899926	0,000413	0,195616	0,000413	0,033282	0,245631

UFRRJ	0,003844	0,031249	0,030639	0,938112	0,00026	0,067886	0,054309	0,014939	0,00026
UFERSA	0,00094	0,591364	0,407572	0,001063	0,228729	2,236661	0,001063	0,235084	0,001063
UTFPR	0,004027	0,030957	0,014155	0,954888	0,000248	0,063669	0,050935	0,001096	0,102209

Source: The Author (2024).

In Steps 4 and 5, the virtual inputs and outputs for overall system efficiency and the system efficiency in each period are displayed in Table A3, respectively.

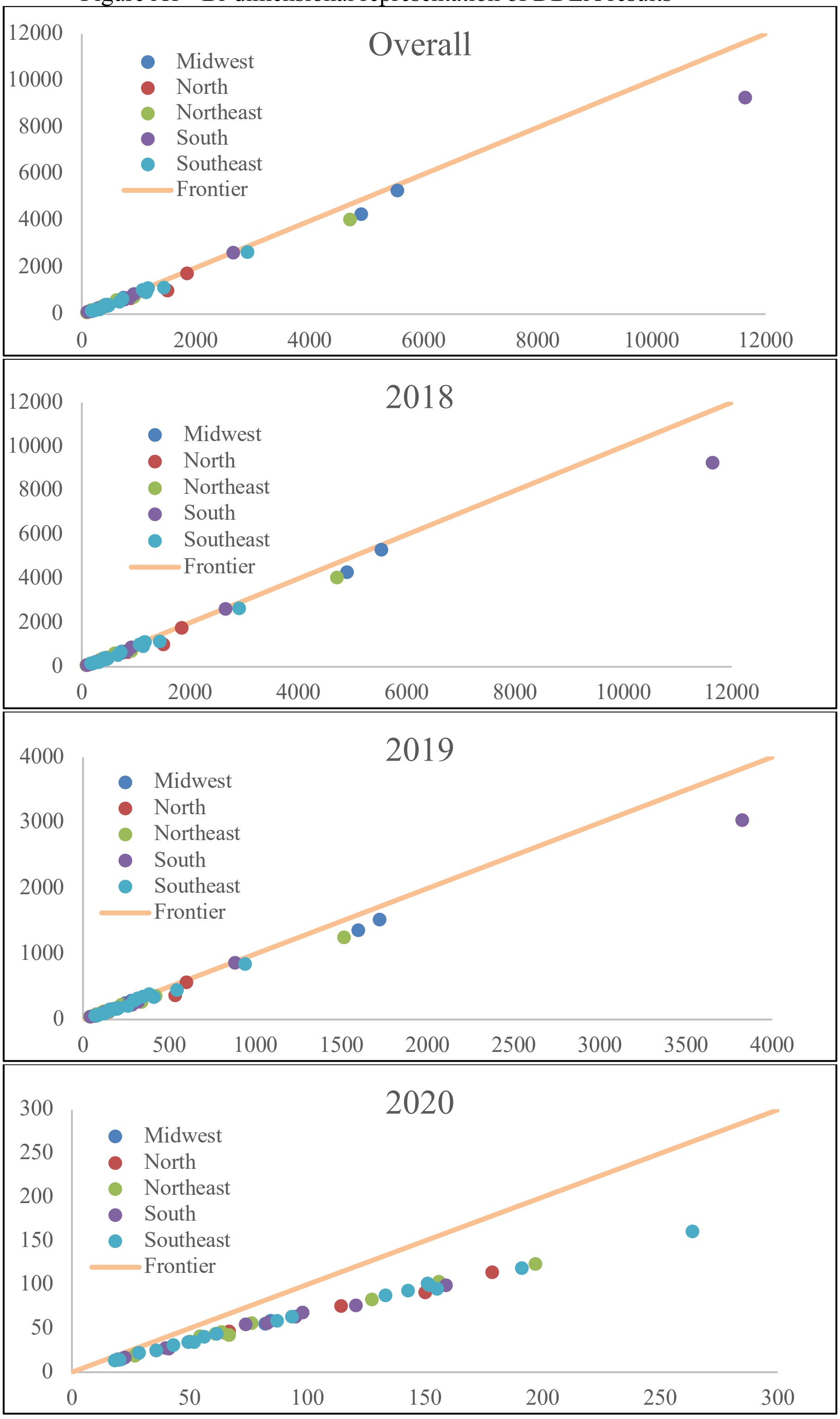
Table A3 - Virtual inputs and outputs of the DDEA model

DMU	$I'^{(sys)}_j$	$O'^{(sys)}_j$	$E^{(sys)}_j$	$I'^{(1,sys)}_j$	$O'^{(1,sys)}_j$	$E^{(1,sys)}_j$	$I'^{(2,sys)}_j$	$O'^{(2,sys)}_j$	$E^{(2,sys)}_O$	$I'^{(3,sys)}_j$	$O'^{(3,sys)}_j$	$E^{(3,sys)}_O$
UFCSPA	733,39	703,47	0,959201	245,10	245,10	1	252,10	226,62	0,898909	236,21	231,77	0,981217
UFMS	5533,69	5307,32	0,959092	1720,90	1525,38	0,886383	1785,58	1754,74	0,982724	2027,64	2027,64	1
UFRR	237,60	212,23	0,893199	77,29	77,29	1	79,12	79,12	1	93,52	68,14	0,728644
UFS	4899,35	4281,13	0,873816	1598,61	1354,66	0,847403	1641,87	1394,11	0,849099	1659,35	1532,83	0,923753
UNIPAMPA	938,98	864,37	0,920539	281,35	281,35	1	311,80	287,83	0,923135	357,75	307,10	0,85843
UFPI	392,05	322,69	0,823086	149,67	116,68	0,779545	151,46	133,79	0,883349	157,95	139,26	0,881642
UNB	852,84	669,29	0,784781	337,72	263,33	0,779725	337,44	277,37	0,821995	336,08	286,99	0,853937
UFBA	1505,62	1013,99	0,67347	534,29	367,27	0,687389	545,91	371,81	0,681084	555,85	405,35	0,729236
UFGD	1848,03	1758,62	0,951615	598,73	562,32	0,939187	610,24	610,24	1	639,13	586,13	0,917064
UFPB	383,77	323,18	0,842131	165,90	151,17	0,911254	169,85	156,33	0,920401	170,19	137,85	0,809967
UFAL	149,00	129,24	0,867372	70,58	60,82	0,861662	70,37	69,54	0,988202	74,81	65,65	0,877469
UNIFAL	327,44	289,47	0,884051	108,07	100,35	0,928619	115,12	109,99	0,955422	134,33	109,21	0,813002
UFCG	163,24	163,04	0,998796	71,50	71,50	1	73,17	72,98	0,997314	71,22	71,22	1
UFG	914,72	728,33	0,796233	339,02	269,75	0,795686	346,35	275,88	0,796544	350,22	303,57	0,866782
UNIFEI	83,58	76,17	0,91125	39,37	36,93	0,937807	36,66	35,26	0,961646	36,13	32,57	0,901373
UFJF	360,88	279,99	0,775849	143,32	111,15	0,775513	140,78	121,60	0,863788	143,36	113,82	0,793926
UFLA	352,19	335,19	0,951726	120,44	119,12	0,989035	141,42	141,42	1	149,29	133,61	0,894965
UFMT	4706,44	4041,04	0,858619	1514,39	1246,80	0,8233	1597,04	1401,83	0,87777	1595,39	1392,79	0,873009
UFMG	1130,56	996,55	0,88147	420,65	361,40	0,859155	429,64	376,94	0,877331	430,77	408,72	0,948807
UFOP	140,10	111,76	0,797747	61,64	55,72	0,903858	64,80	56,69	0,874808	69,02	54,72	0,792871
UFPEL	611,66	599,51	0,980134	221,40	217,75	0,983522	231,37	223,97	0,968015	239,61	238,51	0,995398
UFPE	463,52	409,46	0,883368	204,24	186,65	0,913896	206,24	194,75	0,944298	202,24	177,25	0,876443
UNIR	129,73	112,66	0,868384	46,25	38,99	0,842883	51,75	51,11	0,987531	56,32	47,16	0,837338
UFSC	838,38	709,03	0,845709	316,47	266,83	0,843127	326,44	279,58	0,856465	327,86	295,01	0,899794
UFSM	436,80	400,44	0,916739	174,57	162,30	0,929696	179,56	165,40	0,921134	180,35	170,42	0,944919
UFSCAR	753,07	625,14	0,830126	282,74	223,25	0,789595	282,23	246,38	0,872969	283,37	250,78	0,88501
UFSJ	97,87	86,35	0,88227	44,31	41,49	0,936433	46,92	46,23	0,985304	49,44	41,42	0,837855
UNIFESP	11641,04	9264,22	0,795824	3826,50	3033,17	0,792675	3898,76	3168,67	0,812739	3916,87	3063,47	0,782121
UFU	276,36	247,05	0,893934	118,55	108,71	0,917021	120,22	111,55	0,927893	120,99	110,18	0,910674
UFV	2654,63	2639,18	0,994181	880,65	865,20	0,982458	884,43	884,43	1	889,79	889,79	1
UFABC	173,09	119,28	0,689112	68,82	51,59	0,749678	72,32	56,35	0,779134	79,58	58,97	0,741005
UFAC	911,69	866,83	0,950792	268,38	250,65	0,933949	311,43	311,43	1	342,47	315,34	0,920767
UNIFAP	1159,04	1114,60	0,961659	319,14	319,14	1	384,75	384,75	1	455,16	410,73	0,902369
UFAM	311,35	210,24	0,675246	123,44	90,22	0,730862	125,92	97,22	0,772112	125,73	86,54	0,68828
UFC	366,14	296,55	0,809952	157,76	145,59	0,922868	165,59	144,65	0,873515	171,97	135,50	0,787929
UFES	458,87	379,49	0,827006	185,26	157,41	0,849629	188,35	161,05	0,855058	185,54	161,32	0,869446
UNIRIO	209,69	148,32	0,707343	83,66	56,83	0,679283	83,38	67,55	0,810209	87,43	68,72	0,785987
UFMA	170,17	147,75	0,868246	68,95	56,17	0,81468	75,99	72,72	0,956975	90,26	83,89	0,929397
UFPA	661,34	518,11	0,783417	258,76	199,67	0,771648	268,94	220,36	0,819349	273,34	237,78	0,869897
UFPR	1127,69	937,84	0,831645	408,97	335,69	0,820831	422,84	352,42	0,833468	429,91	383,74	0,892622
UFRJ	1443,79	1139,30	0,789106	545,20	442,98	0,812509	556,74	446,33	0,801676	557,68	465,83	0,835296
FURG	2904,92	2660,20	0,915759	939,27	840,16	0,894486	974,05	974,05	1	991,75	846,15	0,853182
UFRN	456,48	344,17	0,753964	197,79	162,70	0,822587	200,85	171,87	0,855696	206,68	158,44	0,76661
UFRGS	715,17	652,50	0,91238	293,43	286,71	0,977101	295,47	274,26	0,92821	293,26	258,53	0,881565
UFTM	1164,41	1113,15	0,955977	383,75	383,75	1	391,26	375,97	0,960919	389,42	353,45	0,907633
UFVJM	461,22	404,37	0,876744	157,23	130,58	0,830511	164,18	146,76	0,893869	170,68	157,90	0,925147
UFF	478,25	366,47	0,766275	208,41	175,16	0,840461	209,63	180,30	0,860098	208,25	159,05	0,763737
UFRPE	412,84	403,12	0,976464	153,83	146,88	0,954792	159,07	156,30	0,982635	160,23	160,23	1
UFRRJ	260,14	207,56	0,79789	104,49	81,84	0,783214	100,32	89,45	0,891719	106,93	87,87	0,821729
UFERSA	1063,47	1024,77	0,963611	347,90	347,90	1	357,37	357,37	1	358,22	319,52	0,891972
UTFPR	248,31	195,72	0,788214	109,02	88,87	0,815228	113,79	102,55	0,90123	119,08	97,88	0,821918

Source: The Author (2024).

Steps 6 and 7 correspond to the graphical representation of the results. Modified virtual inputs were plotted in the x-axis and modified virtual outputs in the y-axis. Efficient DMU's reside on the frontier. The graphs are display in Figure A1.

Figure A1 - Bi-dimensional representation of DDEA results



Source: The Author (2024).

APPENDIX B - MATHEMATICAL PROOF AND STEP-BY-STEP PROCEDURE FOR THE DNDEA MODEL

The efficiency of a given DMUo, $E_O^{(t,k)}$, is obtained by dividing the virtual output by the virtual input. In (61), there is proof that the value obtained by considering the modified virtual inputs and outputs is equal to the value of the unchanged virtual values.

$$E_O^{(t,k)} = \frac{\sum_{r\in r^k} u_{ro}^* y_{ro}^{(t,k)} + \sum_{l\in l^k} w_{lo}^* c_{lo}^{(t,k)} + \sum_{d\in d^k} f_{do}^* z_{do}^{(t,k)}}{\sum_{i\in i^k} v_i^* x_{io}^{(t,k)} + \sum_{l\in l^k} w_{lo}^* c_{lo}^{(t,k-1)} + \sum_{d\in d^k} f_{do}^* z_{do}^{(t-1,k)}} =$$

$$\frac{\frac{\sum_{r\in r^k} u_{ro}^* y_{ro}^{(t,k)} + \sum_{l\in l^k} w_{lo}^* c_{lo}^{(t,k)} + \sum_{d\in d^k} f_{do}^* z_{do}^{(t,k)}}{S_0}}{\frac{\sum_{i\in i^k} v_i^* x_{io}^{(t,k)} + \sum_{l\in l^k} w_{lo}^* c_{lo}^{(t,k-1)} + \sum_{d\in d^k} f_{do}^* z_{do}^{(t-1,k)}}{S_o}} = \frac{\sum_{r\in r^k} \frac{u_{ro}^*}{S_o} y_{ro}^{(t,k)} + \sum_{l\in l^k} \frac{w_{lo}^*}{S_o} c_{lo}^{(t,k)} + \sum_{d\in d^k} \frac{f_{do}^*}{S_o} z_{do}^{(t,k)}}{\sum_{i\in i^k} \frac{v_i^*}{S_o} x_{io}^{(t,k)} + \sum_{l\in l^k} \frac{w_{lo}^*}{S_o} c_{lo}^{(t,k-1)} + \sum_{d\in d^k} \frac{f_{do}^*}{S_o} z_{do}^{(t-1,k)}} =$$

$$\frac{\sum_{r\in r^k} u_{ro}' y_{ro}^{(t,k)} + \sum_{l\in l^k} w_{lo}' c_{lo}^{(t,k)} + \sum_{d\in d^k} f_{do}' z_{do}^{(t,k)}}{\sum_{i\in i^k} v_{io}' x_{io}^{(t,k)} + \sum_{l\in l^k} w_{lo}' c_{lo}^{(t,k-1)} + \sum_{d\in d^k} f_{do}' z_{do}^{(t-1,k)}} = \frac{O'^{(t,k)}_o}{I'^{(t,k)}_o} \quad \text{(A7)}$$

$E_O^{(t,sys)}$ is obtained by dividing the virtual output by the virtual input. In (A8), the value obtained by considering the modified virtual inputs and outputs is equal to the value of the unchanged virtual values.

$$E_O^{(t,sys)} = \frac{\sum_{k=1}^{K} \sum_{r}^{s} u_{ro}^* y_{ro}^{(t,k)} + \sum_{k=1}^{K} \sum_{d}^{D} f_{do}^* z_{do}^{(t,k)}}{\sum_{k=1}^{K} \sum_{i=1}^{m} v_{io}^* x_{io}^{(t,k)} + \sum_{k=1}^{K} \sum_{d}^{D} f_{do}^* z_{do}^{(t-1,k)}}$$

$$= \frac{\frac{\sum_{k=1}^{K} \sum_{r}^{s} u_{ro}^* y_{ro}^{(t,k)} + \sum_{k=1}^{K} \sum_{d}^{D} f_{do}^* z_{do}^{(t,k)}}{S_0}}{\frac{\sum_{k=1}^{K} \sum_{i=1}^{m} v_{io}^* x_{io}^{(t,k)} + \sum_{k=1}^{K} \sum_{d}^{D} f_{do}^* z_{do}^{(t-1,k)}}{S_o}}$$

$$= \frac{\sum_{k=1}^{K} \sum_{r}^{s} \frac{u_{ro}^*}{S_o} y_{ro}^{(t,k)} + \sum_{k=1}^{K} \sum_{d}^{D} \frac{f_{do}^*}{S_o} z_{do}^{(t,k)}}{\sum_{k=1}^{K} \sum_{i=1}^{m} \frac{v_{io}^*}{S_o} x_{io}^{(t,k)} + \sum_{k=1}^{K} \sum_{d}^{D} \frac{f_{do}^*}{S_o} z_{do}^{(t-1,k)}}$$

$$= \frac{\sum_{k=1}^{K} \sum_{r}^{s} u_{ro}' y_{ro}^{(t,k)} + \sum_{k=1}^{K} \sum_{d}^{D} f_{do}' z_{do}^{(t,k)}}{\sum_{k=1}^{K} \sum_{i=1}^{m} v_{io}' x_{io}^{(t,k)} + \sum_{k=1}^{K} \sum_{d}^{D} f_{do}' z_{do}^{(t-1,k)}} = \frac{O'^{(t,sys)}_o}{I'^{(t,sys)}_o} \quad \text{(A8)}$$

Similarly, we prove that the original values are equal to those obtained with the modified virtual inputs and outputs, as indicated in Equation (69).

$$E_O^{(sys)} = \frac{O_o^{(sys)}}{I_o^{(sys)}} = \frac{\sum_{r=1}^{s} u_{ro} y_{ro} + \sum_{t=1}^{T} \sum_{d}^{D} f_{do} z_{do}^{(T,k)}}{\sum_{i=1}^{m} v_{io} x_{io} + \sum_{t=1}^{T} \sum_{d}^{D} f_{do} z_{do}^{(t_o,k)}} = \frac{\frac{\sum_{r=1}^{s} u_r y_{ro} + \sum_{t=1}^{T} \sum_{d}^{D} f_{do} z_{do}^{(T,k)}}{S_o}}{\frac{\sum_{i=1}^{m} v_{io} x_{io} + \sum_{t=1}^{T} \sum_{d}^{D} f_{do} z_{do}^{(t_o,k)}}{S_o}}$$

$$= \frac{\sum_{r=1}^{s} \frac{u_{ro}}{S_o} y_{ro} + \sum_{t=1}^{T} \sum_{d}^{D} \frac{f_{do}}{S_o} z_{do}^{(T,k)}}{\sum_{i=1}^{m} \frac{v_{io}}{S_o} x_{io} + \sum_{t=1}^{T} \sum_{d}^{D} \frac{f_{do}}{S_o} z_{do}^{(t_o,k)}} = \frac{\sum_{r=1}^{s} u_{rj}' y_{rj} + \sum_{t=1}^{T} \sum_{d}^{D} f_{dj}' z_{dj}^{(T,k)}}{\sum_{i=1}^{m} v_{ij}' x_{ij} + \sum_{t=1}^{T} \sum_{d}^{D} f_{dj}' z_{dj}^{(t_o,k)}}$$

$$= \frac{O'^{(sys)}_o}{I'^{(sys)}_o} \quad \text{(A9)}$$

Following the procedure described in Chapter 5, the first step requires running the input-oriented dynamic model with network structure of Omrani and Soltanzadeh (2016). The results of this step are presented in Table B1 and Table B2.

Table B1 - Efficiency results of the DNDEA model

DMU	$E^{(sys)}$	$E^{(1,sys)}$	$E^{(2,sys)}$	$E^{(3,sys)}$	$E^{(1,1)}$	$E^{(2,1)}$	$E^{(1,2)}$	$E^{(2,2)}$	$E^{(1,3)}$	$E^{(2,3)}$
UFCSPA	72,66%	93,39%	81,48%	73,46%	93,68%	36,81%	81,77%	35,39%	73,79%	35,07%
UFMS	51,99%	64,34%	72,58%	67,07%	83,10%	44,09%	88,65%	51,90%	73,78%	74,35%
UFRR	83,77%	99,86%	99,88%	68,39%	100%	32,90%	100%	38,99%	68,45%	52,39%
UFS	50,43%	66,28%	69,47%	69,48%	81,15%	40,14%	81,80%	50,60%	79,46%	56,93%
UNIPAMPA	68,95%	83,57%	86,28%	72,82%	83,64%	67,50%	86,48%	41,58%	72,82%	100,00%
UFPI	49,72%	65,89%	70,62%	67,82%	77,56%	42,24%	82,37%	47,07%	75,62%	59,45%
UNB	42,92%	59,07%	59,89%	63,92%	82,42%	33,17%	84,10%	34,49%	78,45%	50,60%
UFBA	39,02%	57,72%	57,89%	60,41%	73,73%	34,70%	73,44%	36,70%	67,47%	60,89%
UFGD	57,43%	73,60%	76,45%	68,21%	85,36%	51,16%	87,64%	57,58%	77,22%	54,59%
UFPB	48,52%	68,11%	66,57%	66,16%	80,59%	49,84%	81,82%	43,90%	77,25%	52,29%
UFAL	51,36%	65,05%	69,84%	69,33%	78,23%	44,96%	85,84%	43,89%	73,24%	80,17%
UNIFAL	65,16%	81,67%	85,86%	76,70%	81,89%	34,37%	86,06%	44,85%	76,81%	65,17%
UFCG	60,01%	72,27%	71,90%	75,25%	86,55%	55,58%	86,52%	53,61%	75,25%	100%
UFG	45,01%	62,08%	66,01%	65,08%	78,75%	34,94%	81,88%	38,02%	76,19%	49,08%
UNIFEI	62,14%	79,25%	76,19%	70,66%	84,94%	43,95%	82,93%	40,57%	74,14%	62,47%
UFJF	42,84%	61,76%	60,87%	63,21%	79,32%	31,30%	82,25%	29,82%	74,70%	48,46%
UFLA	60,55%	73,94%	83,66%	71,54%	92,71%	35,03%	100%	42,65%	79,94%	55,33%
UFMT	47,95%	70,67%	72,45%	71,15%	71,35%	35,91%	73,10%	41,74%	71,73%	43,99%
UFMG	44,77%	59,06%	59,45%	62,53%	81,50%	42,64%	85,14%	38,94%	78,41%	55,76%
UFOP	54,23%	73,07%	73,49%	69,75%	86,10%	38,85%	85,33%	41,90%	76,69%	58,81%
UFPEL	54,75%	72,48%	70,39%	72,85%	82,95%	53,87%	82,71%	48,09%	77,87%	74,14%
UFPE	45,52%	60,91%	62,79%	65,91%	86,79%	32,66%	86,72%	37,04%	82,26%	51,24%
UNIR	67,23%	82,02%	85,75%	70,64%	82,30%	30,86%	86,04%	37,05%	70,69%	74,78%
UFSC	42,97%	58,41%	61,06%	62,07%	85,32%	31,83%	87,78%	33,58%	77,34%	50,37%
UFSM	50,54%	68,23%	68,62%	69,14%	89,23%	33,71%	89,02%	33,75%	79,45%	53,87%
UFSCAR	45,99%	63,38%	64,86%	66,26%	80,42%	35,74%	84,80%	32,22%	80,47%	44,82%
UFSJ	57,50%	74,57%	77,28%	68,87%	75,75%	88,65%	78,03%	93,65%	70,28%	85,01%
UNIFESP	40,89%	32,03%	82,76%	62,79%	44,89%	38,05%	100%	41,83%	71,52%	53,25%
UFU	48,67%	65,72%	65,87%	67,34%	83,71%	37,55%	83,89%	38,77%	79,30%	53,51%
UFV	56,10%	72,38%	72,96%	73,42%	92,58%	36,53%	93,35%	36,66%	84,58%	53,63%
UFABC	44,01%	67,64%	69,55%	64,83%	67,93%	50,85%	69,92%	46,54%	65,07%	58,86%
UFAC	70,40%	81,34%	99,86%	72,50%	81,63%	26,72%	100%	50,91%	72,59%	59,85%
UNIFAP	84,09%	99,93%	99,86%	73,12%	100%	58,69%	100%	47,37%	73,12%	100%
UFAM	43,51%	67,62%	69,94%	66,13%	68,28%	33,47%	70,69%	31,24%	66,45%	53,77%
UFC	46,69%	64,69%	64,89%	65,65%	84,75%	35,30%	84,52%	33,18%	75,26%	57,07%
UFES	46,15%	64,80%	65,20%	67,47%	82,38%	32,19%	82,90%	32,21%	80,57%	42,83%
UNIRIO	48,24%	71,50%	72,47%	69,02%	71,90%	28,19%	72,91%	30,15%	69,32%	43,54%
UFMA	56,96%	71,45%	88,54%	71,73%	71,93%	43,08%	88,93%	53,85%	71,84%	80,92%
UFPA	45,99%	61,62%	64,38%	66,98%	81,73%	33,83%	86,93%	34,69%	78,38%	55,37%
UFPR	45,02%	60,89%	62,38%	65,33%	85,27%	32,82%	84,66%	37,05%	82,27%	47,42%
UFRJ	41,35%	58,15%	58,08%	60,82%	79,76%	37,28%	79,60%	36,62%	71,74%	58,21%
FURG	52,79%	67,65%	73,92%	69,14%	80,29%	40,84%	84,87%	49,32%	75,02%	62,69%
UFRN	44,04%	62,05%	65,00%	64,98%	78,49%	32,42%	80,68%	36,49%	75,34%	48,54%
UFRGS	43,97%	60,40%	60,40%	62,16%	89,18%	34,08%	86,86%	35,77%	79,87%	47,26%
UFTM	64,47%	81,96%	78,14%	69,39%	82,22%	39,87%	78,42%	37,81%	69,56%	53,77%
UFVJM	58,49%	76,40%	80,76%	76,20%	76,66%	34,48%	81,05%	34,74%	76,35%	54,89%
UFF	40,17%	57,79%	58,84%	61,18%	82,92%	28,42%	84,01%	30,37%	77,26%	44,37%
UFRPE	55,56%	69,42%	74,74%	70,40%	83,34%	49,20%	86,33%	55,82%	76,93%	71,85%
UFRRJ	47,50%	64,48%	74,85%	70,82%	65,11%	30,37%	75,57%	30,36%	71,24%	45,19%
UFERSA	73,58%	91,17%	84,00%	73,34%	91,69%	17,30%	84,58%	16,68%	73,80%	21,08%
UTFPR	48,25%	66,66%	69,50%	66,89%	77,01%	44,54%	81,80%	42,97%	76,06%	51,37%

Source: The Author (2024).

Table B1 details the efficiency scores. The first column corresponds to the overall efficiency results. Columns two, three and four present the system's efficiency in each period, while the last six correspond to process efficiency. In Table B2, the weights of all the variables are presented.

Table B2 - Weights of the DNDEA model

DMU	Variables							
	v_1	v_2	f_1	w_1	w_2	u_1	u_2	u_3
UFCSPA	0,001005	0,000001	0,026162	0,00031	6,96E-05	2,21E-06	0,000001	1,35E-05
UFMS	1,69E-05	0,000129	0,006483	0,000001	0,000307	0,000245	2,64E-05	0,000368
UFRR	0,000467	0,000496	0,023475	0,000455	6,38E-05	2,24E-06	0,000001	0,000001
UFS	0,000203	1,45E-05	0,005922	2,83E-05	0,000197	0,000157	1,7E-05	0,000236
UNIPAMPA	0,000422	0,000206	0,019992	0,000258	4,87E-05	5,5E-06	0,000001	2,71E-05
UFPI	0,000282	0,000001	0,00779	3,8E-05	0,000215	0,000172	1,85E-05	0,000258
UNB	0,000103	0,000001	0,002609	0,000001	9,31E-05	7,45E-05	8,02E-06	0,000112
UFBA	0,000113	0,000001	0,003126	1,52E-05	8,76E-05	7,01E-05	7,55E-06	0,000105
UFGD	0,000492	3,5E-05	0,014347	6,84E-05	0,000477	0,000382	4,11E-05	0,000572
UFPB	0,000157	0,000001	0,004354	2,12E-05	0,000121	9,68E-05	1,04E-05	0,000145
UFAL	0,000261	1,86E-05	0,00761	3,63E-05	0,000253	0,000202	2,18E-05	0,000303
UNIFAL	0,00018	0,000208	0,023785	0,000125	4,87E-05	5,5E-06	0,000001	2,71E-05
UFCG	0,000372	0,000001	0,008724	5,56E-05	0,000298	0,000239	2,57E-05	0,000358
UFG	0,000144	0,000001	0,003976	1,94E-05	0,000111	8,86E-05	9,54E-06	0,000133
UNIFEI	0,000805	0,000001	0,017589	0,000223	0,000813	9,19E-05	1,67E-05	0,000454
UFJF	0,000256	0,000001	0,007064	3,45E-05	0,000195	0,000156	1,68E-05	0,000234
UFLA	0,000341	0,000001	0,00941	4,6E-05	0,000259	0,000207	2,23E-05	0,00031
UFMT	0,000146	3,08E-05	0,010019	3,33E-05	1,16E-05	9,29E-06	0,000001	1,39E-05
UFMG	9,04E-05	0,000001	0,0023	0,000001	8,22E-05	6,57E-05	7,08E-06	9,86E-05
UFOP	0,000399	0,000001	0,010995	5,37E-05	0,000302	0,000242	2,6E-05	0,000362
UFPEL	0,000246	0,000001	0,006783	3,31E-05	0,000187	0,00015	1,61E-05	0,000225
UFPE	0,000119	0,000001	0,00302	0,000001	0,000108	8,61E-05	9,27E-06	0,000129
UNIR	0,000799	0,000001	0,020865	0,000246	1,09E-05	8,68E-06	0,000001	0,000001
UFSC	0,000117	0,000001	0,002968	0,000001	0,000106	8,46E-05	9,11E-06	0,000127
UFSM	0,000192	0,000001	0,005311	2,59E-05	0,000147	0,000118	1,36E-05	0,000001
UFSCAR	0,00019	0,000001	0,005256	2,57E-05	0,000146	0,000116	1,25E-05	0,000175
UFSJ	0,000136	0,000177	0,013694	8,19E-05	0,000999	0,000113	2,05E-05	0,000558
UNIFESP	0,000142	0,000001	0,003918	1,91E-05	0,000109	8,73E-05	1,01E-05	0,000001
UFU	0,000218	0,000001	0,006023	2,94E-05	0,000167	0,000133	1,43E-05	0,0002
UFV	0,000259	0,000001	0,007141	3,49E-05	0,000197	0,000158	1,7E-05	0,000236
UFABC	0,000301	0,000001	0,014235	5,36E-05	1,16E-05	9,29E-06	0,000001	1,39E-05
UFAC	0,000243	0,000274	0,031723	0,000167	1,09E-05	8,68E-06	0,000001	0,000001
UNIFAP	0,001199	0,000001	0,031289	0,000371	6,38E-05	2,24E-06	0,000001	0,000001
UFAM	0,000237	0,000001	0,011167	4,21E-05	1,16E-05	9,29E-06	0,000001	1,39E-05
UFC	0,000139	0,000001	0,003849	1,88E-05	0,000107	8,58E-05	9,24E-06	0,000129
UFES	0,000195	0,000001	0,005376	2,62E-05	0,000149	0,000119	1,28E-05	0,000179
UNIRIO	0,000351	0,000001	0,016665	6,27E-05	1,09E-05	8,68E-06	0,000001	0,000001
UFMA	0,000185	3,85E-05	0,01272	4,22E-05	1,16E-05	9,29E-06	0,000001	1,39E-05
UFPA	0,000131	0,000001	0,00331	0,000001	0,000118	9,43E-05	1,02E-05	0,000141
UFPR	0,000125	0,000001	0,003168	0,000001	0,000113	9,03E-05	9,72E-06	0,000135
UFRJ	6,89E-05	0,000001	0,001761	0,000001	6,31E-05	5,05E-05	5,43E-06	7,57E-05
FURG	0,000386	0,000001	0,010655	5,2E-05	0,000293	0,000234	2,52E-05	0,000351
UFRN	0,000129	0,000001	0,003561	1,74E-05	9,94E-05	7,96E-05	8,57E-06	0,000119
UFRGS	9,48E-05	0,000001	0,002409	0,000001	8,6E-05	6,88E-05	7,41E-06	0,000103
UFTM	0,000902	0,000001	0,023501	0,000278	4,87E-05	5,5E-06	0,000001	2,71E-05
UFVJM	0,00033	6,71E-05	0,022799	7,56E-05	1,09E-05	8,68E-06	0,000001	0,000001
UFF	0,000119	0,000001	0,003024	0,000001	0,000108	8,62E-05	9,29E-06	0,000129
UFRPE	0,000287	2,04E-05	0,00836	3,99E-05	0,000278	0,000222	2,39E-05	0,000333
UFRRJ	0,000287	0,000001	0,013598	5,12E-05	1,09E-05	8,68E-06	0,000001	0,000001
UFERSA	0,001026	0,000001	0,026729	0,000317	6,38E-05	2,24E-06	0,000001	0,000001
UTFPR	0,000222	0,000001	0,006137	3E-05	0,00017	0,000136	1,46E-05	0,000203

Source: The Author (2024).

With the results detailed in Tables B1 and B2, it is possible to proceed to Step 2, the calculation of S_j. The S_j values are followed in Step 3, and the modified weights (v'_1,v'_2,f'_1,w'_1, w'_2, u'_1, u'_2, u'_3). S_j results are detailed in column 2 of Table B3, while the modified weights are shown in columns 3 to 10.

Table B3 - Modified weights of the DNDEA model

DMU	S_j	Modified weights							
		v'_1	v'_2	f'_1	w'_1	w'_2	u'_1	u'_2	u'_3
UFCSPA	0,027168	0,036979	3,68E-05	0,962985	0,011419	0,002561	8,15E-05	3,68E-05	0,00049626
UFMS	0,006629	0,002546	0,019466	0,977988	0,000151	0,046239	0,036991	0,003983	0,05547489
UFRR	0,024438	0,019099	0,020297	0,960604	0,018626	0,00261	9,17E-05	4,09E-05	4,092E-05
UFS	0,00614	0,033072	0,002356	0,964572	0,004602	0,032063	0,025651	0,002762	0,03846802
UNIPAMPA	0,02062	0,020453	0,009971	0,969576	0,012524	0,002359	0,000267	4,85E-05	0,0013166
UFPI	0,008073	0,034966	0,000124	0,96491	0,004712	0,026586	0,021268	0,00229	0,03189589
UNB	0,002713	0,037887	0,000369	0,961745	0,000369	0,034317	0,027454	0,002956	0,0411717
UFBA	0,00324	0,034809	0,000309	0,964882	0,004703	0,027039	0,021631	0,002329	0,03243982
UFGD	0,014874	0,033072	0,002356	0,964572	0,004602	0,032063	0,025651	0,002762	0,03846802
UFPB	0,004512	0,034883	0,000222	0,964895	0,004707	0,026825	0,02146	0,002311	0,03218369
UFAL	0,007889	0,033072	0,002356	0,964572	0,004602	0,032063	0,025651	0,002762	0,03846802
UNIFAL	0,024173	0,007443	0,008596	0,983961	0,005177	0,002013	0,000227	4,14E-05	0,00112308
UFCG	0,009097	0,0409	0,00011	0,95899	0,006107	0,0328	0,02624	0,002826	0,03935198
UFG	0,004121	0,034865	0,000243	0,964892	0,004706	0,026877	0,021502	0,002315	0,03224563
UNIFEI	0,018395	0,043749	5,44E-05	0,956197	0,01213	0,044201	0,004994	0,000909	0,02466506
UFJF	0,007321	0,034955	0,000137	0,964908	0,004711	0,026617	0,021293	0,002293	0,03193337
UFLA	0,009752	0,034984	0,000103	0,964913	0,004713	0,026533	0,021227	0,002286	0,0318331
UFMT	0,010195	0,014307	0,003024	0,982668	0,003267	0,001139	0,000911	9,81E-05	0,00136605
UFMG	0,002392	0,037807	0,000418	0,961775	0,000418	0,034353	0,027482	0,002959	0,04121484
UFOP	0,011395	0,034997	8,78E-05	0,964915	0,004713	0,026497	0,021198	0,002283	0,03178957
UFPEL	0,007029	0,034951	0,000142	0,964907	0,004711	0,026631	0,021305	0,002294	0,03195005
UFPE	0,00314	0,037967	0,000318	0,961714	0,000318	0,034281	0,027424	0,002953	0,04112794
UNIR	0,021664	0,036864	4,62E-05	0,96309	0,011362	0,000501	0,000401	4,62E-05	4,6159E-05
UFSC	0,003086	0,037958	0,000324	0,961718	0,000324	0,034285	0,027428	0,002953	0,04113282
UFSM	0,005504	0,034917	0,000182	0,964901	0,004709	0,026727	0,021382	0,002476	0,00018168
UFSCAR	0,005448	0,034916	0,000184	0,964901	0,004709	0,026732	0,021386	0,002303	0,03207165
UFSJ	0,014007	0,009728	0,01267	0,977601	0,005844	0,071338	0,00806	0,001466	0,03980822
UNIFESP	0,004061	0,034862	0,000246	0,964891	0,004706	0,026886	0,021509	0,00249	0,00024626
UFU	0,006242	0,034935	0,00016	0,964904	0,00471	0,026675	0,02134	0,002298	0,03200285
UFV	0,007401	0,034957	0,000135	0,964908	0,004711	0,026613	0,021291	0,002293	0,03192901
UFABC	0,014536	0,020678	6,88E-05	0,979253	0,003688	0,000799	0,000639	6,88E-05	0,00095812
UFAC	0,03224	0,007528	0,008513	0,983959	0,005186	0,000337	0,000269	3,1E-05	3,1017E-05
UNIFAP	0,03249	0,036916	3,08E-05	0,963054	0,011406	0,001963	6,9E-05	3,08E-05	3,0779E-05
UFAM	0,011405	0,020759	8,77E-05	0,979153	0,003694	0,001018	0,000814	8,77E-05	0,00122118
UFC	0,003989	0,034859	0,000251	0,964891	0,004705	0,026897	0,021517	0,002317	0,03226925
UFES	0,005572	0,034919	0,000179	0,964901	0,004709	0,026722	0,021378	0,002302	0,03205959
UNIRIO	0,017017	0,020611	5,88E-05	0,979331	0,003683	0,000638	0,00051	5,88E-05	5,8766E-05
UFMA	0,012944	0,014279	0,002976	0,982745	0,003264	0,000897	0,000717	7,73E-05	0,00107599
UFPA	0,003442	0,038012	0,000291	0,961697	0,000291	0,03426	0,027408	0,002951	0,04110364
UFPR	0,003294	0,037991	0,000304	0,961705	0,000304	0,03427	0,027416	0,002952	0,04111501
UFRJ	0,001831	0,037602	0,000546	0,961852	0,000546	0,034446	0,027557	0,002967	0,04132642
FURG	0,011043	0,034994	9,06E-05	0,964915	0,004713	0,026504	0,021203	0,002283	0,03179782
UFRN	0,00369	0,034841	0,000271	0,964888	0,004704	0,026947	0,021557	0,002321	0,03232903
UFRGS	0,002505	0,037838	0,000399	0,961763	0,000399	0,034339	0,027471	0,002958	0,04119832
UFTM	0,024405	0,036968	4,1E-05	0,962991	0,011402	0,001993	0,000225	4,1E-05	0,00111242
UFVJM	0,023196	0,014227	0,002891	0,982881	0,003259	0,000468	0,000374	4,31E-05	4,3112E-05
UFF	0,003145	0,037968	0,000318	0,961714	0,000318	0,03428	0,027424	0,002953	0,04112756
UFRPE	0,008667	0,033072	0,002356	0,964572	0,004602	0,032063	0,025651	0,002762	0,03846802
UFRRJ	0,013886	0,020662	7,2E-05	0,979266	0,003687	0,000781	0,000625	7,2E-05	7,2017E-05
UFERSA	0,027756	0,03695	3,6E-05	0,963014	0,011414	0,002298	8,08E-05	3,6E-05	3,6028E-05
UTFPR	0,00636	0,034938	0,000157	0,964905	0,00471	0,026667	0,021334	0,002297	0,03199414

Source: The Author (2024).

In Steps 4, 5, and 6, the virtual inputs and outputs for the process efficiency, system efficiency in each period, and overall system efficiency are displayed in Tables B4, B5 and B6.

Table B4 - Virtual inputs and outputs of the overall and system efficiency per period in the DNDEA model

DMU	Overall Efficiency			System efficiency by period								
	2018-2020			2018			2019			2020		
	$I'^{(sys)}_j$	$O'^{(sys)}_j$	$E^{(sys)}_j$	$I'^{(1,sys)}_j$	$O'^{(1,sys)}_j$	$E^{(1,sys)}_j$	$I'^{(2,sys)}_j$	$O'^{(2,sys)}_j$	$E^{(2,sys)}_O$	$I'^{(3,sys)}_j$	$O'^{(3,sys)}_j$	$E^{(3,sys)}_O$
UFCSPA	18,712	17,476	93,39%	18,712	17,476	93,39%	19,935	16,243	81,48%	19,347	14,213	73,46%
UFMS	71,237	45,831	64,34%	71,237	45,831	64,34%	72,688	52,757	72,58%	82,225	55,147	67,07%
UFRR	17,527	17,503	99,86%	17,527	17,503	99,86%	17,900	17,879	99,88%	20,862	14,267	68,39%
UFS	86,189	57,129	66,28%	86,189	57,129	66,28%	84,906	58,986	69,47%	84,381	58,629	69,48%
UNIPAMPA	24,855	20,771	83,57%	24,855	20,771	83,57%	24,945	21,523	86,28%	27,784	20,234	72,82%
UFPI	65,101	42,898	65,89%	65,101	42,898	65,89%	64,563	45,594	70,62%	65,606	44,492	67,82%
UNB	180,485	106,616	59,07%	180,485	106,616	59,07%	179,745	107,643	59,89%	178,633	114,179	63,92%
UFBA	152,351	87,936	57,72%	152,351	87,936	57,72%	152,706	88,404	57,89%	150,226	90,754	60,41%
UFGD	34,054	25,064	73,60%	34,054	25,064	73,60%	34,891	26,675	76,45%	35,907	24,494	68,21%
UFPB	114,335	77,869	68,11%	114,335	77,869	68,11%	116,392	77,477	66,57%	114,403	75,689	66,16%
UFAL	62,796	40,847	65,05%	62,796	40,847	65,05%	63,718	44,500	69,84%	66,793	46,305	69,33%
UNIFAL	24,249	19,805	81,67%	24,249	19,805	81,67%	24,805	21,298	85,86%	27,737	21,276	76,70%
UFCG	53,705	38,814	72,27%	53,705	38,814	72,27%	55,393	39,826	71,90%	54,528	41,033	75,25%
UFG	123,453	76,638	62,08%	123,453	76,638	62,08%	123,990	81,851	66,01%	127,408	82,916	65,08%
UNIFEI	29,555	23,423	79,25%	29,555	23,423	79,25%	27,594	21,023	76,19%	26,856	18,977	70,66%
UFJF	69,134	42,694	61,76%	69,134	42,694	61,76%	69,088	42,051	60,87%	66,885	42,276	63,21%
UFLA	50,926	37,652	73,94%	50,926	37,652	73,94%	55,592	46,511	83,66%	63,566	45,473	71,54%
UFMT	58,373	41,253	70,67%	58,373	41,253	70,67%	59,253	42,930	72,45%	61,036	43,430	71,15%
UFMG	191,020	112,814	59,06%	191,020	112,814	59,06%	194,566	115,668	59,45%	196,996	123,179	62,53%
UFOP	45,645	33,352	73,07%	45,645	33,352	73,07%	47,857	35,171	73,49%	50,219	35,029	69,75%
UFPEL	75,975	55,064	72,48%	75,975	55,064	72,48%	76,869	54,107	70,39%	76,261	55,556	72,85%
UFPE	157,235	95,765	60,91%	157,235	95,765	60,91%	158,101	99,264	62,79%	156,010	102,821	65,91%
UNIR	22,018	18,060	82,02%	22,018	18,060	82,02%	23,534	20,180	85,75%	26,610	18,796	70,64%
UFSC	153,320	89,559	58,41%	153,320	89,559	58,41%	155,898	95,188	61,06%	159,072	98,740	62,07%
UFSM	92,856	63,354	68,23%	92,856	63,354	68,23%	95,865	65,785	68,62%	98,130	67,850	69,14%
UFSCAR	93,145	59,034	63,38%	93,145	59,034	63,38%	93,807	60,839	64,86%	95,036	62,969	66,26%
UFSJ	36,512	27,227	74,57%	36,512	27,227	74,57%	37,982	29,353	77,28%	39,911	27,486	68,87%
UNIFESP	124,931	40,012	32,03%	124,931	40,012	32,03%	91,398	75,642	82,76%	120,631	75,749	62,79%
UFU	80,021	52,589	65,72%	80,021	52,589	65,72%	81,321	53,565	65,87%	82,804	55,763	67,34%
UFV	72,549	52,508	72,38%	72,549	52,508	72,38%	72,679	53,030	72,96%	73,838	54,214	73,42%
UFABC	37,755	25,537	67,64%	37,755	25,537	67,64%	38,929	27,074	69,55%	41,072	26,627	64,83%
UFAC	15,873	12,911	81,34%	15,873	12,911	81,34%	17,216	17,191	99,86%	22,527	16,333	72,50%
UNIFAP	13,619	13,610	99,93%	13,619	13,610	99,93%	15,430	15,410	99,86%	18,101	13,235	73,12%
UFAM	50,759	34,322	67,62%	50,759	34,322	67,62%	51,490	36,012	69,94%	52,016	34,397	66,13%
UFC	122,198	79,054	64,69%	122,198	79,054	64,69%	127,408	82,680	64,89%	133,267	87,495	65,65%
UFES	94,375	61,154	64,80%	94,375	61,154	64,80%	94,914	61,885	65,20%	93,426	63,035	67,47%
UNIRIO	34,308	24,530	71,50%	34,308	24,530	71,50%	34,782	25,206	72,47%	35,704	24,643	69,02%
UFMA	42,564	30,412	71,45%	42,564	30,412	71,45%	45,209	40,030	88,54%	56,311	40,393	71,73%
UFPA	142,558	87,841	61,62%	142,558	87,841	61,62%	146,932	94,600	64,38%	151,100	101,213	66,98%
UFPR	147,459	89,789	60,89%	147,459	89,789	60,89%	150,454	93,849	62,38%	151,873	99,213	65,33%
UFRJ	256,859	149,373	58,15%	256,859	149,373	58,15%	261,255	151,748	58,08%	263,661	160,364	60,82%
FURG	46,920	31,743	67,65%	46,920	31,743	67,65%	47,428	35,060	73,92%	49,278	34,070	69,14%
UFRN	138,241	85,775	62,05%	138,241	85,775	62,05%	140,354	91,233	65,00%	142,917	92,869	64,98%
UFRGS	190,259	114,921	60,40%	190,259	114,921	60,40%	191,775	115,826	60,40%	191,213	118,855	62,16%
UFTM	21,151	17,334	81,96%	21,151	17,334	81,96%	20,709	16,181	78,14%	20,302	14,088	69,39%
UFVJM	25,666	19,608	76,40%	25,666	19,608	76,40%	26,356	21,286	80,76%	28,440	21,671	76,20%
UFF	155,082	89,620	57,79%	155,082	89,620	57,79%	156,854	92,292	58,84%	155,124	94,905	61,18%
UFRPE	58,965	40,934	69,42%	58,965	40,934	69,42%	59,604	44,548	74,74%	61,437	43,251	70,40%
UFRRJ	42,560	27,443	64,48%	42,560	27,443	64,48%	40,175	30,070	74,85%	43,142	30,553	70,82%
UFERSA	16,926	15,431	91,17%	16,926	15,431	91,17%	18,255	15,335	84,00%	19,144	14,041	73,34%
UTFPR	80,830	53,883	66,66%	80,830	53,883	66,66%	83,720	58,183	69,50%	87,250	58,364	66,89%

Source: The Author (2024).

Table B5 - Virtual inputs and outputs of the process 1 efficiency in the DNDEA model

DMU	$I'^{(1,1)}$	$O'^{(1,1)}$	$E^{(1,1)}$	$I'^{(2,1)}$	$O'^{(2,1)}$	$E^{(2,1)}$	$I'^{(3,1)}$	$O'^{(3,1)}$	$E^{(3,1)}$
UFCSPA	18,7121	17,5302	93,68%	19,935	16,300	81,77%	19,347	14,277	73,79%
UFMS	71,2368	59,2012	83,10%	72,688	64,434	88,65%	82,225	60,662	73,78%
UFRR	17,5274	17,5274	100,00%	17,900	17,900	100,00%	20,862	14,281	68,45%
UFS	86,1894	69,9416	81,15%	84,906	69,450	81,80%	84,381	67,046	79,46%
UNIPAMPA	24,8546	20,7872	83,64%	24,945	21,573	86,48%	27,784	20,234	72,82%
UFPI	65,1014	50,4909	77,56%	64,563	53,184	82,37%	65,606	49,611	75,62%
UNB	180,4850	148,7624	82,42%	179,745	151,162	84,10%	178,633	140,131	78,45%
UFBA	152,3513	112,3292	73,73%	152,706	112,141	73,44%	150,226	101,358	67,47%
UFGD	34,0542	29,0672	85,36%	34,891	30,578	87,64%	35,907	27,726	77,22%
UFPB	114,3354	92,1448	80,59%	116,392	95,234	81,82%	114,403	88,376	77,25%
UFAL	62,7960	49,1236	78,23%	63,718	54,698	85,84%	66,793	48,916	73,24%
UNIFAL	24,2491	19,8586	81,89%	24,805	21,348	86,06%	27,737	21,305	76,81%
UFCG	53,7050	46,4808	86,55%	55,393	47,924	86,52%	54,528	41,033	75,25%
UFG	123,4535	97,2237	78,75%	123,990	101,518	81,88%	127,408	97,071	76,19%
UNIFEI	29,5549	25,1051	84,94%	27,594	22,883	82,93%	26,856	19,910	74,14%
UFJF	69,1340	54,8364	79,32%	69,088	56,828	82,25%	66,885	49,964	74,70%
UFLA	50,9257	47,2154	92,71%	55,592	55,592	100,00%	63,566	50,813	79,94%
UFMT	58,3734	41,6473	71,35%	59,253	43,315	73,10%	61,036	43,783	71,73%
UFMG	191,0200	155,6861	81,50%	194,566	165,652	85,14%	196,996	154,460	78,41%
UFOP	45,6452	39,3019	86,10%	47,857	40,839	85,33%	50,219	38,513	76,69%
UFPEL	75,9750	63,0228	82,95%	76,869	63,581	82,71%	76,261	59,388	77,87%
UFPE	157,2352	136,4614	86,79%	158,101	137,110	86,72%	156,010	128,338	82,26%
UNIR	22,0181	18,1198	82,30%	23,534	20,249	86,04%	26,610	18,812	70,69%
UFSC	153,3197	130,8186	85,32%	155,898	136,848	87,78%	159,072	123,020	77,34%
UFSM	92,8563	82,8592	89,23%	95,865	85,336	89,02%	98,130	77,968	79,45%
UFSCAR	93,1450	74,9036	80,42%	93,807	79,549	84,80%	95,036	76,472	80,47%
UFSJ	36,5124	27,6590	75,75%	37,982	29,636	78,03%	39,911	28,048	70,28%
UNIFESP	124,9305	56,0873	44,89%	91,398	91,398	100,00%	120,631	86,277	71,52%
UFU	80,0209	66,9822	83,71%	81,321	68,220	83,89%	82,804	65,664	79,30%
UFV	72,5492	67,1682	92,58%	72,679	67,846	93,35%	73,838	62,450	84,58%
UFABC	37,7553	25,6474	67,93%	38,929	27,220	69,92%	41,072	26,726	65,07%
UFAC	15,8735	12,9572	81,63%	17,216	17,216	100,00%	22,527	16,351	72,59%
UNIFAP	13,6193	13,6193	100,00%	15,430	15,430	100,00%	18,101	13,235	73,12%
UFAM	50,7591	34,6595	68,28%	51,490	36,399	70,69%	52,016	34,564	66,45%
UFC	122,1980	103,5579	84,75%	127,408	107,686	84,52%	133,267	100,291	75,26%
UFES	94,3749	77,7496	82,38%	94,914	78,685	82,90%	93,426	75,269	80,57%
UNIRIO	34,3085	24,6667	71,90%	34,782	25,359	72,91%	35,704	24,748	69,32%
UFMA	42,5644	30,6150	71,93%	45,209	40,202	88,93%	56,311	40,455	71,84%
UFPA	142,5579	116,5113	81,73%	146,932	127,734	86,93%	151,100	118,429	78,38%
UFPR	147,4588	125,7392	85,27%	150,454	127,375	84,66%	151,873	124,943	82,27%
UFRJ	256,8588	204,8833	79,76%	261,255	207,948	79,60%	263,661	189,162	71,74%
FURG	46,9203	37,6702	80,29%	47,428	40,250	84,87%	49,278	36,969	75,02%
UFRN	138,2412	108,5099	78,49%	140,354	113,235	80,68%	142,917	107,672	75,34%
UFRGS	190,2587	169,6662	89,18%	191,775	166,567	86,86%	191,213	152,728	79,87%
UFTM	21,1506	17,3908	82,22%	20,709	16,239	78,42%	20,302	14,123	69,56%
UFVJM	25,6656	19,6744	76,66%	26,356	21,362	81,05%	28,440	21,715	76,35%
UFF	155,0823	128,5962	82,92%	156,854	131,766	84,01%	155,124	119,846	77,26%
UFRPE	58,9653	49,1432	83,34%	59,604	51,456	86,33%	61,437	47,264	76,93%
UFRRJ	42,5597	27,7093	65,11%	40,175	30,360	75,57%	43,142	30,734	71,24%
UFERSA	16,9260	15,5198	91,69%	18,255	15,440	84,58%	19,144	14,129	73,80%
UTFPR	80,8300	62,2463	77,01%	83,720	68,485	81,80%	87,250	66,366	76,06%

Source: The Author (2024).

Steps 7, 8, 9, and 10 correspond to the graphical representation of the efficiency results. The modified virtual inputs are plotted on the x-axis, and the modified virtual outputs are plotted on the y-axis. As previously mentioned, efficient DMUs reside on the frontier; therefore, virtual inputs are equal to the virtual outputs. The graphs of the results obtained in steps 7 to 10 are display in Section 5.

As previously mentioned, the efficiency of the DMUs can be obtained by dividing the virtual output by the virtual input. The graph illustrates the positions of the DMUs concerning the efficiency frontier. The graph provides indirect information on the efficiency of each DMU.

Table B6 - Virtual inputs and outputs of the process 2 efficiency in the DNDEA model

DMU	$I'^{(1,2)}$	$O'^{(1,2)}$	$E^{(1,2)}$	$I'^{(2,2)}$	$O'^{(2,2)}$	$E^{(2,2)}$	$I'^{(3,2)}$	$O'^{(3,2)}$	$E^{(3,2)}$
UFCSPA	0,0858	0,0316	36,81%	0,0890	0,0315	35,39%	0,0981	0,0344	35,07%
UFMS	23,9147	10,5441	44,09%	24,2754	12,5980	51,90%	21,5011	15,9864	74,35%
UFRR	0,0371	0,0122	32,90%	0,0343	0,0134	38,99%	0,0286	0,0150	52,39%
UFS	21,4056	8,5930	40,14%	21,1811	10,7172	50,60%	19,5459	11,1281	56,93%
UNIPAMPA	0,0494	0,0333	67,50%	0,0858	0,0357	41,58%	0,0561	0,0561	100,00%
UFPI	13,1439	5,5514	42,24%	14,3402	6,7504	47,07%	12,6228	7,5045	59,45%
UNB	63,0610	20,9146	33,17%	66,4309	22,9117	34,49%	52,5325	26,5811	50,60%
UFBA	37,3570	12,9636	34,70%	37,4976	13,7611	36,70%	27,1146	16,5105	60,89%
UFGD	8,1954	4,1927	51,16%	9,2022	5,2984	57,58%	7,1181	3,8860	54,59%
UFPB	28,4618	14,1861	49,84%	31,6540	13,8970	43,90%	26,5947	13,9076	52,29%
UFAL	15,0378	6,7610	44,96%	18,1736	7,9765	43,89%	13,1653	10,5543	80,17%
UNIFAL	0,0822	0,0283	34,37%	0,0903	0,0405	44,85%	0,0827	0,0539	65,17%
UFCG	17,2595	9,5932	55,58%	17,4563	9,3589	53,61%	13,5400	13,5400	100,00%
UFG	31,6397	11,0543	34,94%	31,7310	12,0637	38,02%	27,7962	13,6416	49,08%
UNIFEI	3,0013	1,3190	43,95%	3,1304	1,2701	40,57%	2,4858	1,5529	62,47%
UFJF	17,6735	5,5316	31,30%	21,0539	6,2774	29,82%	14,9160	7,2280	48,46%
UFLA	14,7206	5,1572	35,03%	15,8350	6,7535	42,65%	11,9559	6,6157	55,33%
UFMT	0,6158	0,2211	35,91%	0,6613	0,2760	41,74%	0,6297	0,2770	43,99%
UFMG	74,7452	31,8734	42,64%	81,8563	31,8730	38,94%	70,7122	39,4314	55,76%
UFOP	9,7297	3,7799	38,85%	9,7562	4,0879	41,90%	8,4578	4,9743	58,81%
UFPEL	17,2514	9,2930	53,87%	18,2527	8,7785	48,09%	14,8173	10,9856	74,14%
UFPE	60,4366	19,7407	32,66%	60,1075	22,2620	37,04%	52,3327	26,8161	51,24%
UNIR	0,0861	0,0266	30,86%	0,1099	0,0407	37,05%	0,0616	0,0461	74,78%
UFSC	60,5261	19,2660	31,83%	62,7203	21,0601	33,58%	48,9241	24,6440	50,37%
UFSM	29,4215	9,9167	33,71%	29,5124	9,9607	33,75%	21,9325	11,8144	53,87%
UFSCAR	24,6951	8,8256	35,74%	27,6035	8,8937	32,22%	24,4705	10,9680	44,82%
UFSJ	3,8044	3,3725	88,65%	4,4452	4,1628	93,65%	3,7521	3,1898	85,01%
UNIFESP	25,9503	9,8745	38,05%	27,0848	11,3284	41,83%	22,5196	11,9907	53,25%
UFU	23,0469	8,6542	37,55%	23,9325	9,2775	38,77%	21,2971	11,3956	53,51%
UFV	23,1002	8,4396	36,53%	23,3930	8,5766	36,66%	17,7616	9,5258	53,63%
UFABC	0,2238	0,1138	50,85%	0,2730	0,1270	46,54%	0,2397	0,1411	58,86%
UFAC	0,0633	0,0169	26,72%	0,0501	0,0255	50,91%	0,0466	0,0279	59,85%
UNIFAP	0,0228	0,0134	58,69%	0,0398	0,0188	47,37%	0,0228	0,0228	100,00%
UFAM	0,5077	0,1699	33,47%	0,5621	0,1756	31,24%	0,3601	0,1936	53,77%
UFC	37,8706	13,3669	35,30%	37,4241	12,4176	33,18%	29,8070	17,0108	57,07%
UFES	24,4720	7,8765	32,19%	24,7820	7,9811	32,21%	21,3990	9,1645	42,83%
UNIRIO	0,1899	0,0535	28,19%	0,2199	0,0663	30,15%	0,1873	0,0816	43,54%
UFMA	0,3561	0,1534	43,08%	0,3734	0,2011	53,85%	0,3256	0,2634	80,92%
UFPA	43,3256	14,6551	33,83%	50,7326	17,5986	34,69%	38,5771	21,3609	55,37%
UFPR	53,5157	17,5650	32,82%	53,2621	19,7360	37,05%	48,9372	23,2078	47,42%
UFRJ	88,4986	32,9879	37,28%	88,6777	32,4776	36,62%	68,9058	40,1076	58,21%
FURG	10,0184	4,0911	40,84%	10,2411	5,0507	49,32%	7,7709	4,8713	62,69%
UFRN	33,6401	10,9054	32,42%	34,6425	12,6400	36,49%	28,7682	13,9655	48,54%
UFRGS	83,0528	28,3080	34,08%	79,0008	28,2597	35,77%	64,2281	30,3549	47,26%
UFTM	0,0938	0,0374	39,87%	0,0933	0,0353	37,81%	0,0752	0,0405	53,77%
UFVJM	0,1018	0,0351	34,48%	0,1174	0,0408	34,74%	0,0965	0,0529	54,89%
UFF	54,4507	15,4742	28,42%	56,6926	17,2181	30,37%	44,8317	19,8909	44,37%
UFRPE	16,1600	7,9507	49,20%	15,6342	8,7262	55,82%	14,2554	10,2424	71,85%
UFRRJ	0,3827	0,1162	30,37%	0,4160	0,1263	30,36%	0,3310	0,1496	45,19%
UFERSA	0,1072	0,0186	17,30%	0,1259	0,0210	16,68%	0,1118	0,0236	21,08%
UTFPR	15,0778	6,7149	44,54%	18,0645	7,7627	42,97%	16,4538	8,4517	51,37%

Source: The Author (2024).