# A BAYESIAN APPROACH FOR MODELING STOCHASTIC DETERIORATION

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A comissão examinadora, composta pelos professores abaixo, sob a presidência do(a) primeiro(a), considera o candidato RODRIGO BERNARDO DA SILVA **APROVADO.**

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# Abstract

Deterioration modeling has been in the vanguard of Bayesian reliability analyses. The most known approaches found in literature for this purpose evaluate the behaviour of the reliability measure over time in the light of only empirical data. In the context of reliability analysis, these approaches are of limited applicability given that one is often dealing with situations characterized by scarce empirical data. Inspired on Bayesian strategies that aggregate empirical data and expert opinions when modeling time-independent reliability measures, this work proposes a framework for dealing with time-dependent reliability. The proposed methodology encapsulates known Bayesian approaches such as Bayesian methods for combining data and expert estimates and time-dependent Bayesian models, proposed by Droguett et al (2004) and Atwood et al (2003), respectivelly. Furthermore, this work promotes improvements upon them in order to find a more realistic model for describing the deterioration process of a given component or system. The cases to be discussed are the ones typically encountered by the reliability practitioner (via simulation): run-time data for failure rate and the amount of deterioration assessment, demand-based data for failure probability assessment, and expert-based evidence for failure rate, amount of deterioration and failure probability analysis. These case studies show that the use of expert information can lead to a reduction of the uncertainty on the reliability measures distributions, particularly in situations where few or no failures have been observed.

**Keywords:** Bayesian Inference; Empirical data; Experts' opinions; Deterioration; Counting Processes; Non-Homogeneous Poisson Processes.

# Resumo

A modelagem de deterioração tem estado na vanguarda das análises Bayesianas de confiabilidade. As abordagens mais conhecidas encontradas na literatura para este propósito avaliam o comportamento da medida de confiabilidade ao longo do tempo à luz dos dados empíricos, apenas. No contexto de engenharia de confiabilidade, essas abordagens têm aplicabilidade limitada uma vez que frequentemente lida-se com situações caracterizadas pela escassez de dados empíricos. Inspirado em estratégias Bayesianas que agregam dados empíricos e opiniões de especialistas na modelagem de medidas de confiabilidade não-dependentes do tempo, este trabalho propõe uma metodologia para lidar com confiabilidade dependente do tempo. A metodologia proposta encapsula conhecidas abordagens Bayesianas, como métodos Bayesianos para combinar dados empíricos e opiniões de especialistas e modelos Bayesianos indexados no tempo, promovendo melhorias sobre eles a fim de encontrar um modelo mais realista para descrever o processo de deterioração de um determinado componente ou sistema. Os casos a serem discutidos são os tipicamente encontrados na prática de confiabilidade (por meio de simulação): avaliação dos dados sobre tempo de execução para taxas de falha e a quantidade de deterioração, dados com base na demanda para probabilidade de falha; e opiniões de especialistas para análise da taxa de falha, quantidade de deterioração e probabilidade de falha. Estes estudos de caso mostram que o uso de informações especializadas pode levar a uma redução da incerteza sobre distribuições de medidas de confiabilidade, especialmente em situações em que poucas ou nenhuma falha é observada.

**Keywords:** Inferência Bayesiana; Dados empíricos; Opiniões de especialistas; Deterioração; Processos de contagem; Processos de Poisson não-homogêneos.

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# Chapter 1

# Introduction

## 1.1 Opening Remarks

This work is inserted in the context of Reliability Engineering that deals with evaluation of the probability that a system will adequately perform its intended purpose for a given period of time under stated environmental conditions, Singpurwalla (2006)[62].

Reliability analysis is a set of techniques aimed to suport decision making that helps managers in pursuit of ensuring satisfactory implementation of functions of the items on a given system, considering its limitations, its wear and factors that influence their performance such as environmental and operating conditions, Firmino and Droguett (2004)[25].

The application of techniques of reliability engineering provides cost reduction, optimization of production process, establishment of a more accurate preventive maintenance, reduction of the waiting time for *setup* operations, lower levels of stocks, increase of the useful life of plants production, among other advantages, Moura (2006)[52].

This work addresses two main aspects: time-dependent reliability analysis for modeling stochastic deterioration and Bayesian approaches for combining data and expert opinions in order to infer about the deterioration model's parameters. Time-dependent reliability analysis has been required for supporting strategic planning and maintenance optimization in engineering infrastructure systems in general, Noortwijk (2009)[69]. The use of this type of analysis is an important aid, particularly in applications in which the available time for testing a new product or component is restricted and the traditional life-tests provide few or no failures.

Some works have proposed the use of deterioration measurements and deterioration models as an alternative to the usual reliability analyses. Nicolai et al (2004)[54], for instance, focuses on the deterioration process of the organic coating layer that protects steel structures from corrosion. Straub (2009)[66] have proposed a generic framework for stochastic modeling of deterioration processes based on dynamic Bayesian networks. His framework is demonstrated and investigated through two applications to probabilistic modeling of fatigue crack growth. The use of deterioration measures for assessing component (or systems) lifetime was addressed in the early work of Gertsbakh and Kordonsky (1969)[30] who used a sample path approach to assess reliability for a simple linear deterioration path with random slope and intercept. Later, Lu and Meeker (1993)[46] estimated the failure time distribution of a unit that fails when the cumulative damage exceeds a given threshold value. Their parametric statistical model allows for a bootstrap method to generate confidence intervals around the failure time distribution, but does not necessarily allow for a closed-form expression of the distribution. More recently, Meeker and Escobar (1998) [47] provide a useful summary of degradation models emphasizing the use of linear models with assumed log-normal deterioration rates. In such case, the full lifetime distribution can be computed analytically. Kelly and Smith (2009)[42] present a recent overview of the main concepts for performing time-dependent Bayesian reliability analyses. Supported by the handbook of parameter estimation for probabilistic risk assessment of the US Nuclear Regulatory Commission (NUREG/CR-6823), Atwood et al (2003) [6], they present various methods based on generalized linear models, log-linear and power-law functions to model the underlying deterioration via stochastic process. In this perspective, time trends are studied by evaluating the behavior of reliability measures such as failure rate or probability of failure on demand, in subsequent periods (whether in terms of time or number of demands) of equal length, according to subjacent homogeneous Poisson, or Binomial (for the number of failures on demand), process for each period.

On the other hand, Bayesian inference aims to summarize available information about unknown parameters that define statistical models through the specification of probability density functions. 'Unknown parameters that define statistical models' refers to things like failure probabilities or mean system lifetimes; they are the parameters of interest.

'Available information' normally comes in the form of test data, experience with related systems, and engineering judgment. 'Probability density functions' occur in four flavors: prior densities, sampling densities or likelihood functions, posterior densities, and predictive densities, Hamada et al (2008)[36].

Applications of Bayesian frameworks can be encountered, for instance, in Droguett et al (2006)[19] where they present a Bayesian approach for population variability analysis involving the use of non-conjugate variability models that works over a continuous, rather than the discretized, variability model parameter space. Røed and Aven (2009)[60] present and discuss several Bayesian approaches for the establishment of a warning criterion to disclose significant deterioration. The Norwegian petroleum industry is the starting point for their paper, but the study is relevant for other application areas as well. Straub (2009)[66] proposes a generic framework for stochastic modeling of deterioration processes based on dynamic Bayesian networks. His framework facilitates computationally efficient and robust reliability analysis. While this work is about applications of risk assessment to technology, it should be noted that there have also been many applications of Bayesian inference in behavioral science [33], finance [31], human health [45], process control [13], and ecological risk assessment [17]. These other applications are mentioned to indicate the degree to which Bayesian inference is being used in the wider community, including other branches of risk assessment.

## 1.2 Justification and Contribution

The reduction of profits and damage to image and reputation because of unavailability is one of the largest concerns of organizations today. According to Galvani (2003)[28], for instance, current regulatory legislation determines that part of the revenue of a transmission power company is variable and dependent on the availability of its assets and / or facilities. Then, the smaller the frequency of failures and downtime the greater the revenue allocated to the transmitting company.

One of the main motivations of this work is the attempt to develop a viable and more precise methodology to model stochastic deterioration using all the available information such as run-time data and expert estimates. Hence, it is proposed a Bayesian approach

for incorporating failure data and expert's opinions into a time-dependent framework via a stochastic process. This latter characteristic will be treated through non-homogeneous Poisson (Binomial for the number of failures on demand) process for discrete observations. For the case where it is assumed a continuous process, it will be assumed the gamma process for describing the stochastic deterioration, a choice that is justified as follows.

Among several stochastic processes able to model deterioration of a system, such as Brownian motion with drift (also called Gaussian or Wiener process) and the compound Poisson process, the gamma process is best suited to model behavior of the monotonic cumulative deterioration in a given time $t$. This is because Brownian motion with drift is a stochastic process with independent, real-valued increments and decrements having a normal distribution and it is able to model the exchange value of shares and the movement of small particles in fluids and air. The main characteristic of this process (in the context of structural reliability) is that a structure's resistance alternate increases and decreases, similar to the exchange value of a share. For this reason, the Brownian motion is inadequate in modeling deterioration which is monotone. On the other hand, the compound Poisson process is a stochastic process with independent and identically distributed jumps which occur according to a Poisson process, Ferguson and Klass (1972)[24]. Like the compound Poisson process, the gamma process is a jump process. According to Singpurwalla and Wilson (1998) [64], the main difference between these two jump processes is as follows. Compound Poisson processes have a finite number of jumps in finite time intervals, whereas gamma processes have an infinite number of jumps in finite time intervals. The former are suitable for modelling usage such as damage due to sporadic shocks and the latter are suitable for describing gradual damage by continuous use.

Therefore, the gamma process is suitable to model gradual damage monotonically accumulating over time in a sequence of tiny increments, such as wear, fatigue, corrosion, crack growth, erosion, consumption, creep, swell, degrading health index amongst others. Noortwijk (2007) [71] presents a method to combine the two stochastic processes of deteriorating resistance and fluctuating load for computing the time-dependent reliability of a structural component. The deterioration process is modelled as a gamma process with identical scale parameter. However, their method depends strongly on empirical

data which is a limitation of the approach when data is scarce or missing. Bérenguer et al (2003)[9] have studied a condition-based maintenance policy based on a control limit structure for a continuously deteriorating monitored system. The mathematical model is based on the regenerative properties of the maintained system state and has been elaborated in the special case of a deterioration modeled by a gamma stochastic process. As in Noortwijk's work, their methodology does not make use of expert estimates resulting in a lower precision of results when data is scarce. Castanier et al (2003)[12] have investigated the problem of inspecting and maintaining a system subject to a assumed stochastic gamma-based gradual deterioration. Again, their approach neglects the effect of scarcity of data in final results. In order to circumvent these limitations, the expected contribution of this work is to propose a methodology that makes possible the combined use of run-time data (e.g., failures and operational time) and expert opinions about the monotonic process of deterioration through gamma process, resulting in a more realistic reliability behaviour model.

## 1.3 Objectives

### 1.3.1 General Objective

Develop a methodology for evaluating reliability behaviour of a system under deterioration via stochastic process and Bayesian inference for estimating deterioration model's parameters based on empirical data and expert opinions.

### 1.3.2 Specific Objectives

- Literature review of the main properties of reliability theory, Bayesian inference and stochastic processes as Markovian and gamma processes;
- A survey of the application of integrated treatment of exposure data and estimates in reliability measures analysis;
- Assessment of reliability measures distributions over time based on exposure data and estimates from various sources such as engineering judgments and reliability

databases via Non-homogeneous Poisson process (NHPP)-based Bayesian model, for discrete observations, and gamma-based Bayesian process for the case where continuous data (amount of deterioration) are observed.

- Implement computacional algorithms that allow decision makers deal with the proposed model.
- Development of examples of application via simulation.

## 1.4 Organization of the Work

The rest of the work is organized as follows. Chapter 2 covers the key concepts that form the theoretical basis of this work. Chapter 3 provides the proposed model based on integration between stochastic process and Bayesian inference for discrete and continuous observations. In Chapter 4, three case studies are presented in order to evaluate the performance of the models and Chapter 5 concludes the work.

# Chapter 2

# Theoretical Basis

The purpose of this chapter is to introduce the main concepts of reliability, illustrate the Bayesian philosophy and some of its consequences and develop the main definitions of stochastic processes concepts.

## 2.1 Reliability Concepts

### 2.1.1 Reliability Function

Reliability is the ability of an item to perform a required function, under given environmental and operational conditions and for a stated period of time, ISO 8402[39].

- The term 'item' is used here to denote any component, subsystem, or system that can be considered as an entity.
- A required function may be a single function or a combination of functions that is necessary to provide a specified service.
- All technical items (components, subsystems, systems) are designed to perform one or more (required) functions. Some of these functions are active and some functions are passive. Complex systems (e.g., an automobile) usually have a wide range of required functions. To assess the reliability (e.g., of an automobile), we must first specify the required function(s) we are considering.

- For a hardware item to be reliable, it must do more than meet an initial factory performance or quality specification-it must operate satisfactorily for a specified period of time in the actual application for which it is intended.

The reliability function of an item is defined by

$$R(t) = 1 - F(t) = P(T > t), \text{ for } t > 0. \tag{2.1}$$

where $F(t)$ is the cumulative distribution function of the failure distribution. In other words, $F(t)$ is the probability that a failure ocurrs before $t$.

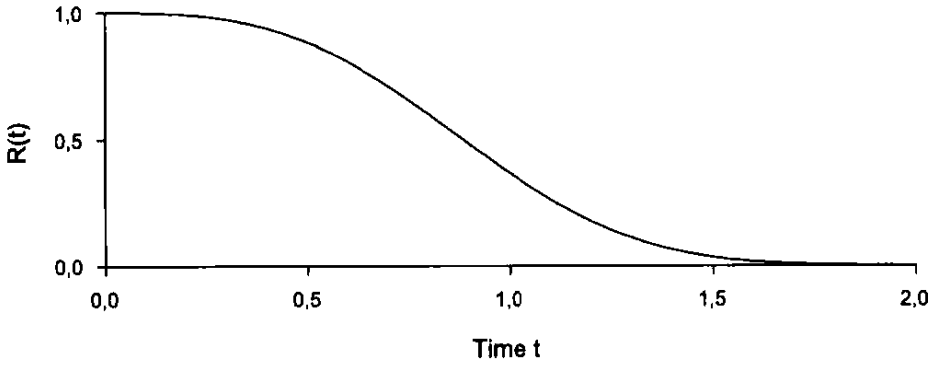



Figure 2.1: The reliability (survival) function $R(t)$.

or equivalently

$$R(t) = 1 - \int_0^t f(u)du = \int_t^\infty f(u)du \tag{2.2}$$

where $T$ is the time to failure of an item, i.e., the time elapsing from when the item is put into operation until it fails for the first time.

### 2.1.2 Failure Rate Function

The probability that an item will fail in the time interval $(t, t + \Delta t]$ when we know that the item is functioning at time $t$ is given by

$$P(t < T \leq t + \Delta t | T > t) = \frac{P(t < T \leq t + \Delta t)}{P(T > t)} = \frac{F(t + \Delta t) - F(t)}{R(t)} \tag{2.3}$$

where $F(t)$ is the cumulative distribution function of $T$.
By dividing this probability by the length of the time interval, $\Delta t$, and letting $\Delta t \to 0$,

we get the failure rate function $\lambda(t)$ of the item

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t | T > t)}{\Delta t} \tag{2.4}$$

$$= \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \frac{1}{R(t)} = \frac{f(t)}{R(t)} \tag{2.5}$$

The reliability function $R(t)$ can be obtained from the failure rate function through the relationship explicited in Equation 2.6:

$$R(t) = \exp\left[-\int_0^t \pi(\tau) d\tau\right] \tag{2.6}$$

### 2.1.3 Repairable X Unrepairable Systems

When modeling failure time data, there is a distinction that needs to be made between one-time-use and multiple-time-use (or repairable) systems. When a one-time-use system fails, we simply replace it with a new system of the same type. A light bulb is an example of a one-time-use system. To assess the reliability of a one-time-use system, testing a sample of these systems and treating their failure times as a random sample from an appropriate distribution usually suffices. A distinguishing feature of repairable system models is that they allow for the reliability growth or decay of a system. For example, consider a complex computer system made up of many subsystems. When the computer fails because a cooling fan fails, the fan can be replaced and the computer system restored to full operation. The fan's failure may perhaps have affected the reliability of other components, however, and so the computer system may have experienced reliability decay.

## 2.2 Bayesian Inference

Lifetime or repair models have one or more unknown parameters. The classical statistical approach considers these parameters as fixed but unknown constants to be estimated using sample data taken randomly from the population of interest. A confidence interval for an unknown parameter is really a frequency statement about the likelihood that numbers calculated from a sample capture the true parameter. Strictly speaking, one cannot make probability statements about the true parameter since it is fixed, not random.

The Bayesian approach, on the other hand, treats these population model parameters as random, not fixed, quantities. Before looking at the current data, we use old information, or even subjective judgments, to construct a prior distribution model for these parameters. This model expresses our starting assessment about how likely various values of the unknown parameters are. We then make use of the current data (via Baye's formula) to revise this starting assessment, deriving what is called the posterior distribution model for the population model parameters. Parameter estimates, along with confidence intervals (known as credibility intervals), are calculated directly from the posterior distribution. Credibility intervals are legitimate probability statements about the unknown parameters, since these parameters now are considered random, not fixed.

### 2.2.1 Basic Concepts

The central tool required for the application of Bayesian methods is Bayes' theorem.

**Theorem 2.2.1** *(Bayes'Theorem) Let $B_1, B_2, \ldots$ be mutually exclusive and exhaustive events contained in a sample space $S$, that is*

$$P(\bigcup_{i=1}^{\infty} B_i) = 1$$

$$B_i \cap B_j = \emptyset, \text{ for } i \neq j$$

$$P(B_i) > 0, \text{ for each } i$$

*and let $A$ be an event in $S$ such that $P(A) > 0$. Then for each $k$,*

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_j P(A|B_j)P(B_j)} \tag{2.7}$$

Bayes' theorem was named after the Reverend Thomas Bayes (1702-1761), who used the theorem in the fundamental paper, *An Essay toward Solving a Problem in the Doctrine of Chances*, which was published in 1763.

In the non-Bayesian setup, let $X$ be a random variable with probability density function $f(x, \theta)$, $\theta \in \Omega$, where $\Omega$ is parametric space. According to the Bayesian point of view, $\theta$ is interpreted as a realization of a random variable $\Theta$ in $\Omega$ with some density $f_\Theta(\theta)$. The density $f_\Theta(\theta)$ expresses what one thinks (believes) about the occurring value

of $\Theta$, before any observation has been taken, that is, *a priori*. $f_\Theta(\theta)$ is called the *prior density* of $\Theta$. $f(x,\theta)$ is then read as the conditional density of $X$ , given $\Theta = \theta$, and rewritten as $f_{X|\Theta}(x|\theta)$.

With this interpretation, the joint density of $X$ and $\Theta$, $f_{X,\Theta}(x,\theta)$ ,is given by

$$f_{X,\Theta}(x,\theta) = f_{X|\Theta}(x|\theta) f_\Theta(\theta) \tag{2.8}$$

Proceeding on this basis, the marginal density of $X$ , $f_X(x)$ is

$$\begin{aligned} f_X(x) &= \int_\Omega f_{X,\Theta}(x,\Theta) d\theta \\ &= \int_\Omega f_{X|\Theta}(x|\theta) f_\Theta(\theta) d\theta \end{aligned} \tag{2.9}$$

The conditional density of $\Theta$, given $X = x$, becomes

$$f_{\Theta|X}(\theta|x) = \frac{f_{X,\Theta}(x,\theta)}{f_X(x)} \tag{2.10}$$

or

$$f_{\Theta|X}(\theta|x) = \frac{f_{X|\Theta}(x|\theta) f_\Theta(\theta)}{f_X(x)} \tag{2.11}$$

which is seen to be a simple form of Bayes's theorem 2.2.1.

By $f_{\Theta|X}(\theta|x)$ we express our belief concerning the distribution of $\Theta$ after having observed $X = x$, that is, *a posteriori*, and $f_{\Theta|X}(\theta|x)$ is therefore called the *posterior density* of $\Theta$. Note that when $X$ is observed, $f_X(x)$ occurs in 2.11 as a constant. Hence $f_{\Theta|X}(\theta|x)$ is always proportional to $f_{X|\Theta}(x|\theta) f_\Theta(\theta)$, which we write as:

$$f_{\Theta|X}(\theta|x) \propto f_{X|\Theta}(x|\theta) f_\Theta(\theta) \tag{2.12}$$

The Bayesian approach may be characterized as an updating process of our information about the parameter $\theta$. First, a probability density for $\Theta$ is assigned before any observations of $X$ is taken. Then, as soon as the first $X$ is observed and becomes available, the prior distribution of $\Theta$ is updated to the posteriori distribution of $\Theta$, given $X = x$. The observed value of $X$ has therefore changed our belief regarding the value of $\Theta$. This process may be repeated. In the next step our posterior distribution of $\Theta$, given $X = x$, is chosen as the new prior distribution, another $X$ is being observed, and one is lead to a second posterior distribution, and so on.

### 2.2.2 Choice of Prior Distribution

#### Conjugate Families of Distributions

First, we state a useful definition:

**Definition 2.2.1** *A parametric family $\mathfrak{P}$ of distributions $f_\Theta(\theta)$ is said to be closed in sampling with respect to a family $\mathfrak{F}$ of distributions $f_{X|\Theta}(x|\theta)$ if*

$$f_\Theta(\theta) \in \mathfrak{P} \text{ and } f_{X|\Theta}(x|\theta) \in \mathfrak{F} \Rightarrow f_{\Theta|X}(\theta|x) \in \mathfrak{P} \tag{2.13}$$

*In that case $\mathfrak{P}$ is also said to be a conjugate family to $\mathfrak{F}$ or, for short, conjugate to $\mathfrak{F}$.*

**Theorem 2.2.2** *The family of gamma distributions $(\alpha, \beta)$ is conjugate to the family of exponential distributions.*

The above theorem can be verified through the following example.

**Example**: Let $\mathfrak{F}$ be the family of exponential distributions defined by the probability density function

$$f_{T|\Lambda}(t|\lambda) = \lambda e^{-\lambda t}, \text{ for } t > 0 \tag{2.14}$$

Let us show that the class $\mathfrak{P}$ of gamma distributions

$$f_\Lambda(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \text{ for } \lambda > 0 \tag{2.15}$$

is conjugate to $\mathfrak{F}$. That is, we have to show that the corresponding $f_{\Lambda|T}(\lambda|t)$ is a gamma distribution. In this case

$$\begin{aligned} f_{T,\Lambda}(t,\lambda) &= \lambda e^{-\lambda t} \cdot \frac{\beta}{\Gamma(\alpha)} (\beta\lambda)^{\alpha-1} e^{-\beta\lambda} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^\alpha e^{-\lambda(\beta+t)}, \text{ for } t > 0, \ \lambda > 0 \end{aligned} \tag{2.16}$$

Furthermore

$$\begin{aligned} f_T(t) &= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^\alpha e^{-\lambda(\beta+t)} d\lambda \\ &= \frac{\alpha\beta^\alpha}{(\beta+t)^{\alpha+1}} \text{ for } t > 0 \end{aligned} \tag{2.17}$$

Hence

$$\begin{aligned} f_{\Lambda|T}(\lambda|t) &= \frac{\beta^\alpha(\lambda^\alpha e^{-\lambda(\beta+t)})/\Gamma(\alpha)}{\alpha\beta^\alpha/(\beta+t)^{\alpha+1}} \\ &= \frac{(\beta+t)^{\alpha+1}}{\Gamma(\alpha+1)} \lambda^\alpha e^{-\lambda(\beta+t)} \end{aligned} \tag{2.18}$$

which is a gamma density with parameters $\alpha + 1$ and $\beta + t$.

Clearly the assumption of a gamma density as a prior distribution in connection with an exponential distribution is mathematically convenient. Our intention with this prior distribution, however, is to express a priori knowledge of $\lambda$, and we raise the question whether or not this purpose is taken care of by using the gamma density as prior. The answer to this question is that the gamma distribution $(\alpha, \beta)$ is a very flexible distribution. It may take on a wide variety of shapes through varying the parameters $\alpha$ and $\beta$. Almost any conceivable shape of the density of $\Lambda$ can essentially be obtained by proper choice of $\alpha$ and $\beta$.

**Theorem 2.2.3** *The family of gamma distributions $(\alpha, \beta)$ is conjugate to the family of Poisson distributions.*

**Example**: Let $\mathfrak{F}$ be the family of Poisson distributions defined by the probability mass function

$$P(N(t) = n|\Lambda = \lambda) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \text{ for } n = 0, 1, 2, \ldots \tag{2.19}$$

where $N(t)$ can be interpreted as the number of valve failures during an accumulated time $t$ in service.

The marginal distribution of $N(t)$ then becomes

$$\begin{aligned} P(N(t) = n) &= \int_0^\infty P(N(t) = n|\Lambda = \lambda) f_\Lambda(\lambda) d\lambda \\ &= \int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda \\ &= \frac{\beta^\alpha t^n}{\Gamma(\alpha) n!} \int_0^\infty \lambda^{\alpha+n-1} e^{-(\beta+t)\lambda} d\lambda \\ &= \frac{\beta^\alpha t^n}{\Gamma(\alpha) n!} \frac{\Gamma(\alpha + n)}{(\beta + t)^{n+\alpha}} \end{aligned} \tag{2.20}$$

where $\lambda$ represents a realization of a random variable $\Lambda$ with the gamma prior density, say the failure rate of the valves.

By combining 2.15, 2.19 and 2.20 we get the posterior density of $\Lambda$, given $N(t) = n$,

$$f_{\Lambda|N(t)}(\lambda|n) = \frac{(\beta + t)^{\alpha+n}}{\Gamma(\alpha + n)} \lambda^{\alpha+n-1} e^{-(\beta+t)\lambda} \tag{2.21}$$

which is recognized as the gamma distribution with parameters $(\alpha + n)$ and $(\beta + t)$.

**Theorem 2.2.4** *The family of beta distributions* $(r, s)$ *is conjugate to the family of binomial distributions.*

Proof: Let $\mathfrak{F}$ be the binomial distribution defined by

$$f_{X|\Theta}(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x},\ x = 0, 1, 2, \ldots, n,\ 0 \leq \theta \leq 1 \tag{2.22}$$

Let us show that the class $\mathfrak{P}$ of beta distributions, defined by the density

$$f_\Theta(\theta) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{r-1}(1-\theta)^{s-1},\ 0 \leq \theta \leq 1,\ r > 0,\ s > 0 \tag{2.23}$$

is a conjugate family to $\mathfrak{F}$.

To prove this statement, we have to show that the corresponding density $f_{\Theta|X}(\theta|x)$ represents a beta distribution. To do so, we make use of 2.12:

$$\begin{aligned} f_{\Theta|X}(\theta|x) \quad &\propto \quad \binom{n}{x}\theta^x(1-\theta)^{n-x} \\ &\quad \cdot\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{r-1}(1-\theta)^{s-1},\ 0 \leq \theta \leq 1 \end{aligned} \tag{2.24}$$

Put differently,

$$f_{\Theta|X}(\theta|x) \propto \theta^{x+r-1}(1-\theta)^{n-x+s-1}$$

. Hence

$$f_{\Theta|X}(\theta|x) = k(x)\theta^{(x+r)-1}(1-\theta)^{(n-x+s)-1},\ 0 \leq \theta \leq 1 \tag{2.25}$$

Knowing 2.25 to be a probability density, the "constant" $k(x)$ has to be

$$k(x) = \frac{\Gamma(n+r+s)}{\Gamma(x+r)\Gamma(n-x+s)} \tag{2.26}$$

Equation 2.26 introduced into 2.25, gives the beta density with parameters $(x + r$ , and $(n - x + s)$.

The assumption of a beta prior in connection with a binomial distribution is mathematically convenient. Furthermore the beta distribution is a very flexible distribution. Its density can take on a wide variety of shapes by proper choice of $r$ and $s$. Note that by choosing $r = 1$, $s = 1$, the beta density represents a uniform density over $[0, 1]$ which corresponds to a no priori preference for any $\theta$ in $[0, 1]$. (In this case we have a distribution for $\theta$.)

**Noninformative Prior Distribution**

**Example**:

A certain type of valves is assumed to have constant (unknown) failure rate $\lambda$, where $\lambda$ represents a realization of a random variable $\Lambda$. Let $T_1, T_2, \ldots, T_n$ denote the observed lifetimes of $n$ such valves. Assume $T_1, T_2, \ldots, T_n$ to be independent and identically distributed, conditional on $\lambda$, with density

$$f_{T_i|\Lambda}(t_i|\lambda) = \lambda e^{-\lambda t_i}, \text{ for } t_i > 0, \ \lambda > 0$$

When there is no prior information available about the true value of $\Lambda$, a noninformative prior distribution seems appropriate.

A noninformative prior distribution is characterized by giving no preference to any of the possible parameter values. Hence, if the possible parameter values constitute a finite interval, the parameter is assumed to be uniformly distributed over that interval. Hence a noninformative prior distribution for the failure rate $\Lambda$ in our case may, for example, be given by the uniform density

$$f_\Lambda(\lambda) = \begin{cases} 1/M & \text{for } 0 \leq \lambda \leq M \\ 0 & \text{otherwise} \end{cases} \tag{2.27}$$

where $M$ is taken to be a very large number, say $10^{10}$. Then the posterior density of $\Lambda$, given the data $D$, is approximately

$$\begin{aligned} f_{\Lambda|T_1,T_2,\ldots,T_n}(\lambda|t_1,t_2,\ldots,t_n) &\propto L(\lambda|D) f_\Lambda(\lambda) \\ f_{\Lambda|T_1,T_2,\ldots,T_n}(\lambda|t_1,t_2,\ldots,t_n) &\propto \lambda^n e^{-\lambda \sum_{i=1}^n t_i} \cdot \frac{1}{M} \\ f_{\Lambda|T_1,T_2,\ldots,T_n}(\lambda|t_1,t_2,\ldots,t_n) &\propto \lambda^n e^{-\lambda \sum_{i=1}^n t_i} \end{aligned} \tag{2.28}$$

where $L(\lambda|D)$ is the likelihood function of the distibution in question.

To become a proper density, the right-hand side of 2.28 must be multiplied by

$$(\sum_{i=1}^n t_i)^{n+1}/\Gamma(n+1)$$

Hence

$$f_{\Lambda|T_1,T_2,\ldots,T_n}(\lambda|t_1,t_2,\ldots,t_n) = \frac{(\sum_{i=1}^n t_i)^{n+1}}{\Gamma(n+1)} \lambda^n e^{-\lambda \sum_{i=1}^n t_i} \text{ for } \lambda > 0 \tag{2.29}$$

which we recognize as the density of a gamma distribution with parameters $(n+1)$ and $\sum_{i=1}^n t_i$.

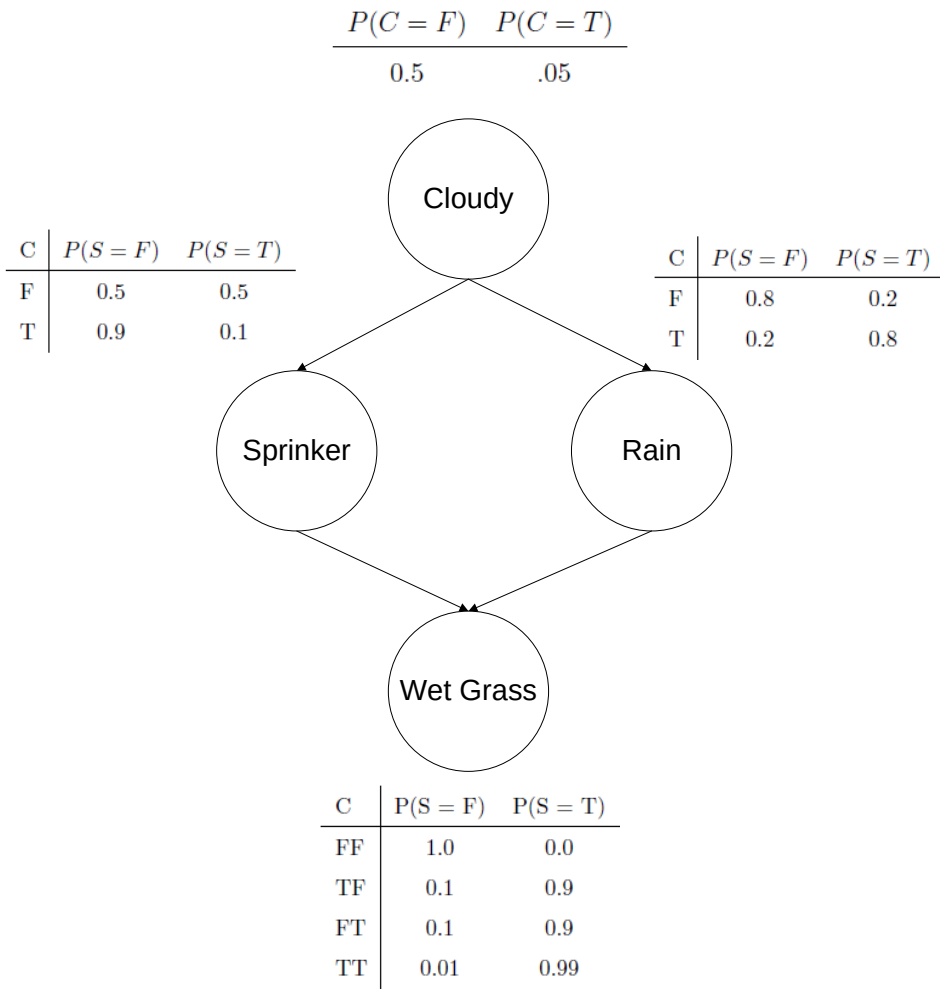
### 2.2.3 Bayesian Networks

Bayesian networks (BNs) are graphical structures for representing the probabilistic relationships among a large number of variables and doing probabilistic inference with those variables. More specifically, according to Torres and Succar (1998)[68], Bayesian networks are directed acyclic graphs (DAG), see Figure 2.2, in which the nodes represent propositions (or variables), the arcs signify direct dependencies between the linked propositions and the strength of these dependencies are quantified by conditional probabilities. That the graph is acyclic means that no cycles are allowed, i.e. following the arcs from one node to another in the way they are directed, there should be no possibility to come back to a node that is already visited. For example, if there is an arc from node $n_2$ to node $n_1$, there should be no arc from node $n_1$ to node $n_2$. Such graphical structures, known also as influence diagrams or belief networks, are used for representing expert knowledge. The graph represents a set of random variables and its dependency and independency relations. It is used to estimate the posterior probability of unknown variables given other variables (evidence), through a process known as probabilistic reasoning. This generates recommendations or conclusions about a particular problem, and can be used for explanation, the process of communicating the relevant information to user.

When the graph is used for diagnosis it is called a probabilistic expert system. In this context, the reasoning is based on dependency relations: fault-symptoms, cause-effects, hypothesis-evidence. Every fault and symptom is modeled by random variables with a finite range of possible values. A graph is constructed with a node for each variable and it has an edge (arc) from one node, which represents a fault to another node which represents a symptom, e.g. the symptom C is dependent on A in Figure 2.2.

Several applications of Bayesian networks in the context of reliability analysis can be found in the literature as Faber et al. (2002)[23], Friis-Hansen (2004)[27] and Langseth and Portinale (2007)[44]. Few researchers have applied BNs in the context of deterioration modeling. Friis-Hansen (2001)[26] studied the application of a BN for deterioration modeling and inspection planning by means of an example considering fatigue crack growth; Montes-Iturrizaga et al. (2009)[48] used a BN for optimizing inspection efforts for offshore structures subject to multiple deterioration mechanisms; Attoh-Okine and Bowers

(2006)[5] presented an empirical model of bridge deterioration using BNs. However, in contrast to these previous works, which made use of the BN capabilities mainly for modeling the system aspects or for optimizing inspection and maintenance decision, the present work focuses on the deterioration modeling. One example of a simple Bayesian network can be seen in the figure below.




| $P(C=F)$ | $P(C=T)$ |
|---|---|
| 0.5 | .05 |

| C | $P(S=F)$ | $P(S=T)$ |
|---|---|---|
| F | 0.5 | 0.5 |
| T | 0.9 | 0.1 |

| C | $P(S=F)$ | $P(S=T)$ |
|---|---|---|
| F | 0.8 | 0.2 |
| T | 0.2 | 0.8 |

| C | P(S = F) | P(S = T) |
|---|---|---|
| FF | 1.0 | 0.0 |
| TF | 0.1 | 0.9 |
| FT | 0.1 | 0.9 |
| TT | 0.01 | 0.99 |

Figure 2.2: A simple Bayesian network example.

One can see that the event 'grass is wet' ($W$ =true) has two possible causes: either the water sprinker is on ($S$ =true) or it is raining ($R$ =true). The strength of this relationship is shown in the table in Figure 2.2. For example, one can see that

$$P(W = \text{true}|S = \text{true}, R = \text{false}) = 0.9$$

(second row), and hence,

$$P(W = \text{false}|S = \text{true}, R = \text{false}) = 1 - 0.9 = 0.1$$

since each row must sum to one. Since the $C$ node has no parents, its CPT specifies the prior probability that it is cloudy (in this case, 0.5).

The simplest conditional independence relationship encoded in a Bayesian network can be stated as follows: a node is independent of its ancestors given its parents, where the ancestor/parent relationship is with respect to some fixed topological ordering of the nodes.

By the chain rule of probability, the joint probability of all the nodes in the graph above is

$$P(C, S, R, W) = P(C)P(S|C)P(R|C, S)P(W|C, S, R)$$

By using conditional independence relationships, we can rewrite this as

$$P(C, S, R, W) = P(C)P(S|C)P(R|C)P(W|S, R)$$

where we were allowed to simplify the third term because $R$ is independent of $S$ given its parent $C$, and the last term because $W$ is independent of $C$ given its parents $S$ and $R$.

The most common task we wish to solve using Bayesian networks is probabilistic inference. For example, consider the water sprinkler network, and suppose we observe the fact that the grass is wet. There are two possible causes for this: either it is raining, or the sprinkler is on. Which is more likely? One can use Bayes' rule to compute the posterior probability of each explanation (where $0 \equiv false$ and $1 \equiv true$).

$$\begin{aligned} P(S = 1|W = 1) &= \frac{P(S = 1, W = 1)}{P(W = 1)} = \frac{\sum_{c,r} P(C = c, S = 1, R = r, W = 1)}{P(W = 1)} = \frac{0.2781}{0.6471} \\ P(R = 1|W = 1) &= \frac{P(R = 1, W = 1)}{P(W = 1)} = \frac{\sum_{c,s} P(C = c, S = s, R = 1, W = 1)}{P(W = 1)} = \frac{0.4581}{0.6471} \end{aligned}$$

where

$$P(W = 1) = \sum_{c,r,s} P(C = c, S = s, R = r, W = 1) = 0.6471$$

is a normalizing constant, equal to the probability (likelihood) of the data. So one can see that it is more likely that the grass is wet because it is raining: the likelihood ratio is $0.7079/0.4298 = 1.647$.

In the above example, notice that the two causes 'compete' to 'explain' the observed data. Hence $S$ and $R$ become conditionally dependent given that their common child, $W$, is observed, even though they are marginally independent. For example, suppose the grass is wet, but that we also know that it is raining. Then the posterior probability that the sprinkler is on goes down:

$$P(S = 1|W = 1, R = 1) = 0.1945$$

### 2.2.4 Bayesian models for combining data and expert estimates

A mere use of the Bayesian or a use of background information does not necessarily imply an adherence to the Bayesian paradigm. A deeper characterization of the Bayesian paradigm is the exploitation and systematic use of the concept of subjective or personal probability, Singpurwalla (2006)[62]. In this sense, in several approaches for inferring about a given reliability measure based on run-time or demand-based data, opinions regarding hazard functions or probabilities of failure may also be available and constitute another valuable source of information, especially in situations where scarcity of empirical data prevails.

Let $\Lambda$ be the reliability measure of interest, as summarized by [20], the assessment of the uncertainty about $\Lambda$ can based on the following types of information:

- Type $E_0$: The expert's prior state of knowledge such as engineering experience on the system or component;
- Type $E_1$: Exposure data (run-time or demand-based data) from operating experience with similar systems;
- Type $E_2$: Estimates or distributions about the reliability measure $\Lambda$ from various sources such as reliability databases and engineering judgments.

The third source of information rises from the previous ones and usually consists of opinions from experts with engineering knowledge about the design, manufacturing, and operation of the system to empirical estimates based on observed performance of the same class of systems in similar applications. Alternatively, a distribution can be

provided expressing the range of values for the reliability measure with mean, mode or median representing the best estimate of the expert.

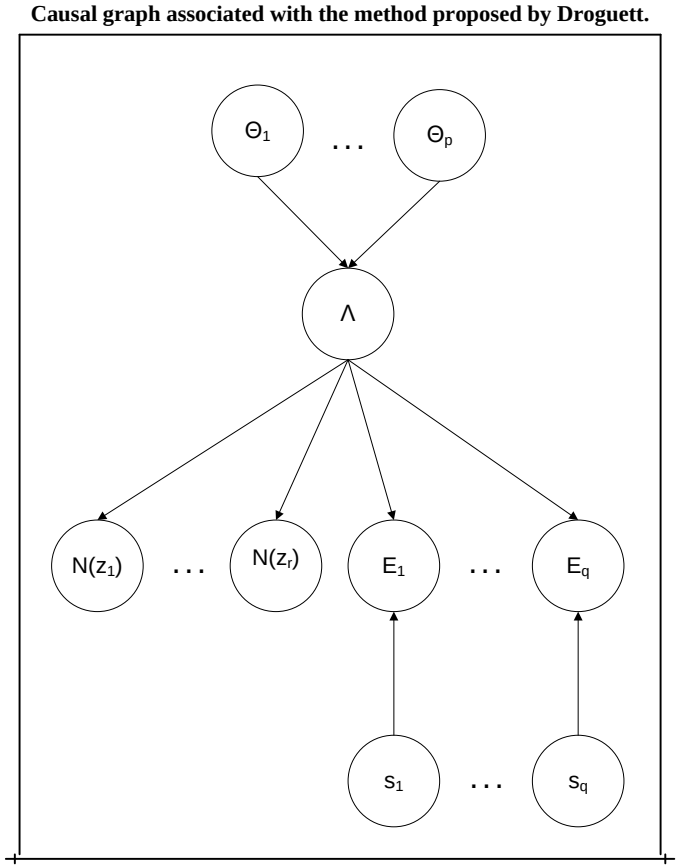



Figure 2.3: Directed acyclic graph related to a Bayesian model for aggregating data and experts' estimates

Figure 2.3 shows the Bayesian network associated to the methodology proposed by Droguett et al (2004)[20], where $\Theta_1, \ldots, \Theta_p$ represent the hyperparameters (parents) that govern the reliability metric (failure rate) $\Lambda$, which, in turn, governs exposure data and expert estimates, represented by $N_{(z_1)}, \ldots, N_{(z_r)}$ and $E_1, \ldots, E_q$, respectivelly. One can note that the quantities $s_1, \ldots, s_q$ represent the uncertainty about experts opinions.

Applications of experts' estimates can be encountered, for instance, in Campodonico and Singpurwalla (1995)[11], where they propose a Bayesian approach for predicting the number of failures in a piece of software using, in a formal manner, expert knowledge on software testing, as for example, published information on the empirical experiences of other researchers. Noortwijk et al. (1992)[70] proposed a comprehensive method for

the use of expert opinion for obtaining lifetime distributions required for maintenance optimization. They use a Bayesian methodology with the Dirichlet distribution as the prior distribution for the elicited discrete lifetime distribution. Bedford et al. (2006)[8] reviews the role of expert judgment to support reliability assessments within the systems engineering design process. Nicolai et al. (2007)[55] fitted a gamma process on deterioration data of coating systems protecting steel structures against corrosion. Due to the lack of inspection data, they had to estimate a non-stationary gamma process with expected power-law deterioration on the basis of expert judgement. The expert data were given by percentiles of the times at which three different deterioration levels were exceeded. Mosleh and Apostolakis (1985)[51] addressed the problem of population variability by means of a Bayesian methodology for assessing curves based on both exposure data and expert estimates information. The methodology is however limited because it decoupled exposure data and expert estimates, i.e. for single systems, it allows only one type of information to be used. Furthermore, applications of the methodology were carried out using a discretized parameter space. Thus, as in Kaplan (1983)[40], their methodology was extended by Droguett et al. (2004)[20] which surpasses the limitations of the Mosleh and Apostolakis (1985)[51] formalism. In particular, they concern a capability to aggregate both exposure data and experts' estimates, but allowing for decoupled and coupled situations, i.e. it is possible to handle both types of evidence when they are available for the same system.

## 2.3 Stochastic Processes

A stochastic process $X(t),\ t \in T$ is a collection of random variables. That is, for each $t \in T$, $X(t)$ is a random variable. The index $t$ is often interpreted as time and, as a result, we refer to $X(t)$ as the state of the process at time $t$. The set $T$ is called the index set of the process. When $T$ is a countable set the stochastic process is said to be a discrete-time process. If $T$ is an interval of the real line, the stochastic process is said to be a continuous-time process, Ross (1993)[59].

The state space of a stochastic process is defined as the set of all possible values that the random variables $X(t)$ can assume. Thus, a stochastic process is a family of random

variables that describes the evolution through time of some (physical) process.

### 2.3.1 Counting Processes

Consider a repairable system that is put into operation at time $t = 0$. The first failure (event) of the system will occur at time $S_1$ . When the system has failed, it will be replaced or restored to a functioning state. The repair time is assumed to be so short that it may be neglected. The second failure will occur at time $S_2$ and so on. We thus get a sequence of failure times $S_1, S_2, \ldots$. Let $T_i$ be the time between failure $i-1$ and failure $i$ for $i = 1, 2, \ldots$, where $S_0$ is taken to be 0. $T_i$, will be called the *interoccurrence time* $i$ for $i = 1, 2, \ldots$. $T_i$ may also be called the time between failures, and the interarrival time. In general, counting processes are used to model sequences of events.

The sequence of interoccurrence times, $T_1, T_2, \ldots$ will generally not be independent and identically distributed-unless the system is replaced upon failure or restored to an 'as good as new' condition, and the environmental and operational conditions remain constant throughout the whole period. A precise definition of a counting process is given below, Ross (1996)[58].

**Definition 2.3.1** *A stochastic process* $\{N(t), t \geq 0\}$ *is said to be a counting process if* $N(t)$ *satisfies:*

1. $N(t) \geq 0$.
2. $N(t)$ *is integer valued.*
3. *If* $s < t$, *then* $N(s) \leq N(t)$.
4. *For* $s < t$, $[N(t) - N(s)]$ *represents the number of failures that have occurred in the interval* $(s, t]$.

A counting process $\{N(t), t \geq 0\}$ may alternatively be represented by the sequence of failure (calendar) times $S_1, S_2, \ldots$, or by the sequence of interoccurrence times $T_1, T_2, \ldots$. The three representations contain the same information about the counting process.

The main four types of counting processes are:

1. Homogeneous Poisson processes (HPP)

2. Renewal processes

3. Non-homogeneous Poisson processes (NHPP)

4. Imperfect repair processes

While renewal processes are an important class of models, they are inappropriate to use in situations where reliability growth (or decay) may occur. Consequently, we turn our attention to a broad class of models, which allows the possibility of reliability growth or decay. To begin, let us define an important type of counting process called a *Poisson process*, which encapsulates HPP and NHPP processes.

**Definition 2.3.2** *A counting process $N(t)$ is a Poisson process if*

1. $N(0) = 0$.

2. *For any $a < b \leq c < d$, the random variables $N(a, b]$ and $N(c, d]$ are independent. That is, counts in nonoverlapping intervals are independent, where $N(a, b] = N(b) - N(a)$.*

3. *A function $\lambda(\cdot)$, called the intensity function, exists as defined by*

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(N(t, t + \Delta t] = 1)}{\Delta t} \tag{2.30}$$

4. *There are no simultaneous failures, expressed as*

$$\lim_{\Delta t \to 0} \frac{P(N(t, t + \Delta t] \geq 2)}{\Delta t} = 0$$

A consequence of these four conditions presented in the Poisson process definition is that

$$P[N(t) = x] = \frac{\Lambda(t)^x \exp[-\Lambda(t)]}{x!} \tag{2.31}$$

where

$$\Lambda(t) = \int_0^t \lambda(z)dz \tag{2.32}$$

The probability statement in Equation 2.31 implies that N(a, b] has a Poisson distribution with parameter $\int_a^b \lambda(z)dz$. In other words,

$$\mathrm{E}(N(a, b]) = \mathrm{Var}(N(a, b]) = \int_a^b \lambda(z)dz = \Lambda(b) - \Lambda(a).$$

One performance measure of Poisson processes is the *rate of occurrence of failures* (ROCOF), defined as

$$\frac{d}{dt}\mathrm{E}\left[N(t)\right]$$

for differentiable $\mathrm{E}\left[N(t)\right]$. It turns out that the ROCOF and intensity function in Equation 2.30 are equal when the probability of simultaneous failures is zero (i.e., Definition 2.3.2, point 4). Consequently, when the intensity function $\lambda(t)$ given in Equation 2.30 is large, we expect many failures in a time interval, and if $\lambda(t)$ is small, we expect few failures in a time interval. A useful characterization of the failure times from a Poisson process is as follows:

$$\Lambda(T_i) - \Lambda(T_{i-1}) \sim Exponential(1). \tag{2.33}$$

That is, $\Lambda(T_i) - \Lambda(T_{i-1})$, $i = 1, \ldots, n$, (with $T_0 = 0$) are i.i.d $Exponential(1)$. Using Equation 2.33, we can express $T_i$ in terms of $T_{i-1}$ as

$$T_i \sim \Lambda^{-1}\left[\Lambda(T_{i-1}) + Exponential(1)\right], \tag{2.34}$$

which suggests how to simulate failure times from a Poisson process, where $\Lambda^{-1}(\cdot)$ denotes the inverse of $\Lambda(\cdot)$ for $\Lambda(\cdot)$ defined in Equation 2.32. Note that for specific models, $\Lambda^{-1}(\cdot)$, which involves an integral, simplifies to a function that is easily invertible.

Next, we consider each type of Poisson process in turn.

### Homogeneous Poisson Process

The homogeneous Poisson process was cited in section 1.2. The HPP may be defined in a number of different ways. Two alternative definitions of the HPP are presented in the following to illustrate different features of the HPP. The first of the two definitions are from Ross (1996)[58].

**Definition 2.3.3** *The counting process* $\{N(t), t \geq 0\}$ *is said to be an HPP having rate* $\lambda$*, for* $\lambda > 0$*, if*

1. $N(0) = 0$.

2. *The process has independent increments.*

3. *The number of events in any interval of length $t$ is Poisson distributed with mean $\lambda t$. That is, for all $s, t > 0$,*

$$P(N(t+s) - N(s) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \text{ for } n = 0, 1, 2, \ldots. \tag{2.35}$$

Note that it follows from property 3 that an HPP has stationary increments and also that $E(N(t)) = \lambda t$, which explains why $\lambda$ is called the rate of the process.

**Definition 2.3.4** *The counting process $\{N(t), t \geq 0\}$ is said to be an HPP having rate $\lambda$, for $\lambda > 0$, if*

1. $N(0) = 0$.
2. *The process has stationary and independent increments.*
3. $P(N(\Delta t) = 1) = \lambda \Delta t + o(\Delta t)$.
4. $P(N(\Delta t) \geq 2) = o(\Delta t)$.

### Compound HPPs

Consider an HPP $\{N(t), t \geq 0\}$ with rate $\lambda$. A random variable $V_i$ is associated to failure event $i$ , for $i = 1, 2, \ldots$. The variable $V_i$ may, for example, be the consequence (economic loss) associated to failure $i$. The variables $V_1, V_2, \ldots$ are assumed to be independent with common distribution function

$$F_V(v) = P(V \leq v)$$

The variables $V_1, V_2, \ldots$ are further assumed to be independent of $N(t)$. The cumulative consequence at time $t$ is

$$Z(t) = \sum_{i=1}^{N(t)} V_i \quad \text{for } t \geq 0 \tag{2.36}$$

The process $\{Z(t), t \geq 0\}$ is called a compound Poisson process. Compound Poisson processes are discussed, for example, by Ross (1996)[58] and Taylor and Karlin (1984)[67]. The same model is called a cumulative damage model by Barlow and Proschan (1975)[7].

**Non-Homogeneous Poisson Process**

**Definition 2.3.5** *A counting process* $\{N(t), t \geq 0\}$ *is a non-homogeneous (or non-stationary) Poisson process with rate function* $w(t)$ *for* $t \geq 0$*, if*

1. $N(0) = 0$.
2. $\{N(t), t \geq 0\}$ *has independent increments.*
3. $P(N(t+\Delta t) - N(t) \geq 2) = o(\Delta t)$*, which means that the system will not experience more than one failure at the same time.*
4. $P(N(t+\Delta t) - N(t) = 1) = w(t)\Delta t + o(\Delta t)$.

The basic 'parameter' of the NHPP is the rate of occurence of failure (ROCOF) function $w(t)$. This function is also called the peril rate of the NHPP. The cumulative rate of the process is

$$W(t) = \int_0^t w(u)du \tag{2.37}$$

This definition of course covers the situation in which the rate is a function of some observed explanatory variable that is a function of the time $t$.

It is important to note that the NHPP model does not require stationary increments. This means that failures may be more likely to occur at certain times than others, and hence the interoccurrence times are generally neither independent nor identically distributed. Consequently, statistical techniques based on the assumption that the data are independent and identically distributed cannot be applied to an NHPP.

The NHPP is often used to model trends in the interoccurrence times, that is, improving (*happy*) or deteriorating (*sad*) systems. It seems intuitive that a happy system will have a decreasing ROCOF function, while a sad system will have an increasing ROCOF function. Several studies of failure data from practical systems have concluded that the NHPP was an adequate model, and that the systems that were studied approximately satisfied the properties of the NHPP listed in Definition 2.3.5.

Now consider a system consisting of a large number of components. Suppose that a critical component fails and causes a system failure and that this component is immediately replaced by a component of the same type, thus causing a negligible system

downtime. Since only a small fraction of the system is replaced, it seems natural to assume that the systems's reliability after the repair essentially is the same as immediately before the failure. In other words, the assumption of minimal repair is a realistic approximation. When an NHPP is used to model a repairable system, the system is treated as a *black box* in that there is no concern about how the system 'looks inside'.

Consider an NHPP with ROCOF $w(t)$, and suppose that failures occur at times $S_1, S_2, \ldots$. An illustration of $w(t)$ is shown in Figure 2.4.

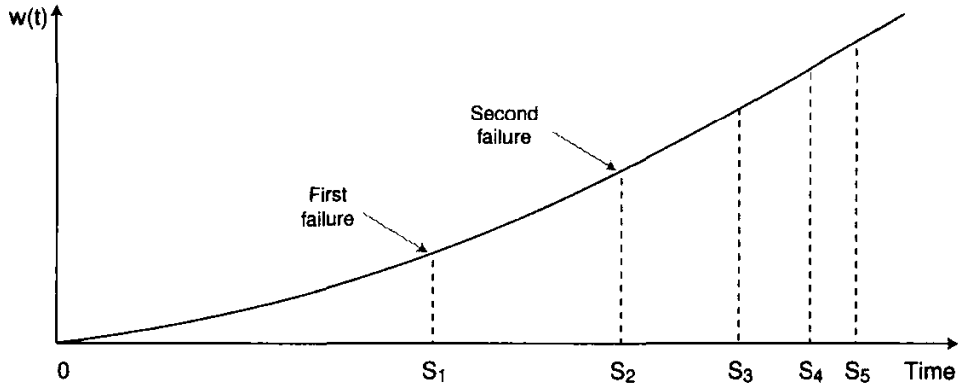



Figure 2.4: The ROCOF $w(t)$ of an NHPP and random failure times.

From the definition of the NHPP it is straightforward to verify, see [58] that the number of failures in the interval $(0, t]$ is Poisson distributed:

$$P(N(t) = n) = \frac{[W(t)]^n}{n!} e^{-W(t)}, \text{ for } n = 0, 1, 2, \ldots \tag{2.38}$$

The mean number of failures in $(0, t]$ is therefore

$$E(N(t)) = W(t)$$

and the variance is $Var(N(t)) = W(t)$. The cumulative rate $W(t)$ of the process is therefore the mean number of failures in the interval $(0, t]$ and is sometimes called the mean value function of the process. When $n$ is large, $P(N(t) \leq n)$ may be determined by normal approximation:

$$\begin{aligned} P(N(t) \leq n) &= P\left(\frac{N(t) - W(t)}{\sqrt{W(t)}} \leq \frac{n - W(t)}{\sqrt{W(t)}}\right) \\ &\approx \Phi\left(\frac{n - W(t)}{\sqrt{W(t)}}\right) \end{aligned} \tag{2.39}$$

From 2.38 it follows that the number of failures in the interval $(\upsilon, r+\upsilon]$ is Poisson distributed:

$$P(N(t+\upsilon)-N(\upsilon)=n) = \frac{[W(t+\upsilon)-W(\upsilon)]^n}{n!}\exp\{-[W(t+\upsilon)-W(\upsilon)]\}$$
$$\text{for} \quad n=0,1,2,\ldots$$

and that the mean number of failures in the interval $(\upsilon, t+\upsilon]$ is

$$E(N(t+\upsilon)-N(\upsilon)) = W(t+\upsilon)-W(\upsilon) = \int_{\upsilon}^{t+\upsilon} w(u)du \tag{2.40}$$

The probability of no failure in the interval $(t_1, t_2)$ is

$$P(N(t_2)-N(t_1)=0) = \exp\left\{-\int_{t_1}^{t_2} w(t)dt\right\} \tag{2.41}$$

Let $S_n$ denote the time until failure $n$ for $n=0,1,2,\ldots$, where $S_0=0$. The distribution of $S$, is given by:

$$P(S_n > t) = P(N(t) \leq n-1) = \sum_{k=0}^{n-1} \frac{W(t)^k}{k!} e^{-W(t)} \tag{2.42}$$

When $W(t)$ is small, this probability may be determined from standard tables of the Poisson distribution. When $W(t)$ is large, the probability may be determined by normal approximation, see 2.39:

$$\begin{aligned} P(S_n > t) &= P(N(t) \leq n-1) \\ &\approx \Phi\left(\frac{n-1-W(t)}{\sqrt{W(t)}}\right) \end{aligned} \tag{2.43}$$

### 2.3.2 Markov Processes

Markovian processes are one of the most important stochastic process for modeling dynamic systems. In the context of Reliability Engineering these models are used basicaly when failures trees, tree events or block diagrams are insufficient to functional representation of a complex system. Applications of Markov processes in reliability can be seen in Moura et al. [53].

In this work, we assume Markovian processes with continuous and finite state space and continuous time $t > 0$. A stochastic process $\{X(t), t > 0\}$ is said to be a Markov process if it satisfies the Markovian property:

$$P(X(t) = j|X(\upsilon) = i, X(u) = x(u)) = P(X(t) = j|X(\upsilon) = i) \tag{2.44}$$

for all $x(u), 0 \leq u < \upsilon$, see Howard [37].

The Markovian property, also known as 'lack of memory', means that when the current state of the process is known the future development is independent of the past.

Graphically, the Markovian process is represented by a chain, defined as Markov chain. This chain is formed by nodes which represent the possible states that the process can take, and arrows which represent the transition between states. Such transitions may represent failure or repair rates.

Probabilistically, a Markov process in time is characterized by a vector of initial probabilities $P(X(t_0) = k)$, $k = 0, 1, \ldots, n$ and by transition probabilities between states

$$p_{ij}(\upsilon, t) = P(X(t) = j|X(\upsilon) = i) \tag{2.45}$$

for $0 \leq \upsilon \leq t$ and $i, j = 0, 1, \ldots, n$, where $n$ is the number of states of the Markovian process.

The transitions probabilities of a Markovian process satisfy the Chapman-Kolmogorov equation, see [37], which establish $\forall i, \forall j \in X$,

$$p_{ij}(\upsilon, t) = \sum_{k=0}^{n} p_{ik}(\upsilon, u) p_{kj}(u, t) \tag{2.46}$$

for $0 \leq \upsilon < u < t$. In practice, however, the use of Equation 2.46 results in a system of $n^2$ equations, where $n$ is the number of states of the Markov process. In general, the transition probabilities are obtained from a system of differential equations as follows.

In fact, it can be showed that for each $i$ and $j$, with $i \neq j$, there is a continuous and non negative function $\lambda_{ij}(t)$, known as the transition rate from state $i$ to state $j$, defined as

$$\lambda_{ij}(t) = \frac{\partial p_{ij}(\nu, t)}{\partial t} \mid_{\nu=t} = \lim_{\Delta t \to 0} \frac{p_{ij}(t, t + \Delta t)}{\Delta t} \tag{2.47}$$

Therefore, the transition probabilities and the transition rates are related as follows:

$$p_{ij}(t, t + \Delta t) = \lambda_{ij}(t)\Delta t + o(\Delta t) \tag{2.48}$$

for $i \neq j$ and $o(\Delta t)$ is the 'little oh' defined as a function of $\Delta t$ that tends to zero faster than $\Delta t$, Ross [58].

Replacing $t$ for $t + \Delta t$ in Equation 2.46, we have

$$p_{ij}(\nu, t + \Delta t) = \sum_{k=0}^{n} p_{ik}(\nu, u) p_{kj}(u, t + \Delta t)$$

Thus

$$p_{ij}(\nu, t+\Delta t) - p_{ij}(\nu, t) = \sum_{k=0, k\neq j}^{n} p_{ik}(\nu, u) p_{kj}(u, t+\Delta t) - [1 - p_{jj}(u, t + \Delta t)]\, p_{ij}(\nu, u) \quad (2.49)$$

Dividing both sides by $\Delta t$ and considering the limit $\Delta t \to 0$ and $u \to t$, we obtain the Differential Equations of Kolmogorov. For $0 \leq \nu < t$ and $i, j \in X$:

$$\frac{\partial p_{ij}(\nu, t)}{\partial t} = \sum_{k=0, k\neq j}^{n} p_{ik}(\nu, t) \lambda_{kj}(t) - p_{ij}(\nu, t) \lambda_{jj}(t) \quad (2.50)$$

It can be shown that the differential equation for the inconditional state probability $P_j(t)$ is given by:

$$\frac{dP_j(t)}{dt} = \sum_{i=0, i\neq j}^{n} P_i(t) \lambda_{ij}(t) + P_j(t) \lambda_{jj}(t) \quad (2.51)$$

There are various methods of resolutions of the system composed by the $n$ equations from the form of Equation 2.51. Among which one can cite the Laplace transformed, the method of Runge-Kutta of 4th or 5th order or the utilization of additional variables, see Oliveira et al. [56].

The Equation 2.51 becomes more simple when it comes to the homogeneous Markov process. In this type of Markov process, the transition rates $\lambda_{ij}(t)$ are constant in time, which means that the distribution of residence time $T_j$ in the state $j$ follows an Exponential distribution, with parameter $\lambda_{ij}(t) = \lambda$. This type of process is not recommended for modeling systems that suffer aging, since that the transition rates (failure or repair) are considered constant over time, meaning that the system does not suffer the effects of processes of deterioration.

As an alternative to modeling systems suffering aging from homogeneous Markov processes, one can cite the approach of modeling from the concept of missions with multiple stages, see [53].

On the other hand, in the non-homogeneous Markov process, the transition rates depend on the time of procedure, also known as the time schedule. Therefore, the aging process of the systems can be better modeled since the transition rates vary over time. It is possible that through this Markov process, the residence time $T_j$ in a state $j$ has an arbitrary distribution, as Weibull or Lognormal.

### 2.3.3 Gamma Process

#### Definition of non-stationary gamma process

In words, a gamma process is a stochastic process with independent, non-negative increments (e.g., the increments of crest-level decline of a dike) having a gamma distribution with an identical scale parameter. The main advantage of modelling deterioration processes through gamma processes is that the required mathematical calculations are relatively straightforward.

In mathematical terms, a non-stationary gamma process is defined as follows. Recall that a random quantity $X$ has a gamma distribution with shape parameter $v > 0$ and scale parameter $u > 0$ if its probability density function is given by

$$G(x|v,u) = \frac{u^v}{\Gamma(v)} x^{v-1} e^{-ux} I_{0,\infty}(x), \tag{2.52}$$

where $I_A(x) = 1$ for $x \in A$ and $I_A(x) = 0$ for $x \notin A$, $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function for $a > 0$. Furthermore, let $v(t)$ be a non-decreasing, right-continuous, real-valued function for $t \geq 0$, with $v(t) \equiv 0$. The gamma process with shape function $v(t) > 0$ and scale parameter $u > 0$ is a continuous-time stochastic process $\{X(t), t \geq 0\}$ with the following properties:

1. $X(0) = 0$ with probability one;
2. $X(\tau) - X(t) \sim G(v(\tau) - v(t), u)$ for all $\tau > t \geq 0$;
3. $X(t)$ has independent increments.

**Definition of stationary gamma process**

When the expected deterioration is linear over time, it is convenient to rewrite the probability density function of $X(t)$ in $f_{X(t)}(x) = G(x|v(t), u)$ using the following reparameterisation:

$$f_{X(t)}(x) = G(x|[\mu^2 t]/\sigma^2, \mu/\sigma^2) \tag{2.53}$$

for $\mu, \sigma > 0$ with $\mathrm{E}(X(t)) = \mu t$ and $\mathrm{Var}(X(t)) = \sigma^2 t$. A gamma process with $\mu = \sigma = 1$ is called a standardised gamma process. Due to the stationarity of the above stochastic deterioration process, both the mean value and the variance of the deterioration are linear in time. By using

$$P\{X(t) \geq y\} = \int_{x=y}^{\infty} f_{X(t)} dx = \frac{\Gamma(v(t), yu)}{\Gamma(v(t))}$$

the lifetime distribution can then be rewritten as

$$P\{X(t) \geq y\} = \int_{x=y}^{\infty} f_{X(t)} dx = \frac{\Gamma([\mu^2 t]/\sigma^2, [yu]/\sigma^2)}{\Gamma([\mu^2 t]/\sigma^2)} \tag{2.54}$$

Applications of gamma processes can be found, for instance, in Abdel-Hameed (1975)[1] who was the first to propose the gamma process as a proper model for deterioration occurring randonly in time. Other examples of the application of gamma processes are Nicolai et al (2004)[54] where they focus on the deterioration process of the organic coating layer that protects steel structures from corrosion.

### 2.3.4 Time-dependent Bayesian models

In probabilistic risk analysis, an important issue is to model the reliability measure of interest as a function of time; for example, the hazard function $\lambda(t)$. This time-dependent behavior has motivated the development of several approaches. A generalized linear model (GLM) has often been used to model the time trend in $\lambda$, [42]. Various link functions can be used in such a model, but a standard choice for the probability of failure on demand ($p$) and hazard function ($\lambda$), as suggested by Atwood et al. (2003)[6], is the logit($p$) and log($\lambda$), respectively.

Due to practical and theoretical difficulties in modeling time trends, much past work was devoted to repairable systems under the assumption that repairs restore the system

to a 'same as new' condition (renewal process). Huang (2003)[38], for instance, reviews several plausible approaches for dealing with deterioration in repairable systems and develops a possible structural design of decision support systems by considering the sensitivity analysis as well as the optimal prior and posterior decisions. Attardi and Pulcini (2005)[4] propose a new model to describe the failure pattern of complex repairable systems subjected to reliability deterioration with the operating time, and showing a finite bound for the intensity function. Their model can be viewed as a dynamic power-law process, whose shape parameter ranges from 2 to 1 as the system age increases, converging asymptotically to the homogeneous Poisson process. For other examples the reader may refer to Armstrong (2002)[3], Bloch-Mercier (2002)[10] and Weckman et al (2001)[72].

On the other hand, a minority of papers has addressed the minimum repair which means that the system is recovered to an 'as bad as old' state just after a repair action. Yet, most of these works deal with qualitative analysis and frequentist estimation. Guo and Love (1992)[35] propose an imperfectly repair model which effectively includes imperfect repair and bad-as-old regimes that have appeared separately in reliability engineering literature. One of the most cited papers in the area, Kijima (1989)[43] presents stochastic models of the operation in time of repairable systems which is maintained by a general repair and analyzes the related stochastic process. In his paper, [43] represents a general repair as a sequence of random variables $A_n$ taking values between 0 and 1, where $A_n$ denotes the degree of the $n$th repair. For the extremal values 0 and 1, $A_n = 1$ means a minimal repair and $A_n = 0$ a perfect repair, two models are constructed depending on how the repair affects the virtual process.

When the renewal process is governed by a memoryless distribution, e. g. Exponential or Geometric, the reliability measure is independent of the exposure time period and the number of failures in time (or number of demands) $t$, $\mathrm{N}(t)$, is characterized as a HPP, generally modelled via a Poisson (for $\lambda$) or a Binomial (for $p$) distribution. Under an HPP, the expected number of failures in $t$ is given by $\lambda t$. Instead, relaxing the assumption of constant $\lambda$ gives rise to the NHPP, where the number of failures in $t$ is still Poisson or Binomial-distributed, but the expected number of failures in any given time (or number of demands) period, $[t_1, t_2]$, is given by $\int_{t_1}^{t_2} \lambda(t)dt$. If $\lambda(t)$ increases over time, the component is aging or wearing out; otherwise, the component is experiencing reliability growth such

as in the first phase of the bath-tube curve.

The functional form of $\lambda(t)$ is usually specified via parametric modeling. For the sake of illustration, common forms for $\lambda(t)$ include the power-law model ($\lambda(t) = (b/a)(t/a)^{b-1}$) and log-linear functions ($\lambda(t) = e^{a+bt}$) whilst for $p(t)$ the logit function ($p(t) = (e^{a+bt})/(1+e^{a+bt})$) can be mentioned. For practical purposes, it is usual to evaluate the observed number of failures in non-overlapping consecutive time intervals (or number of demands) of equal size ($(t_0, t_1], (t_1, t_2], \ldots, (t_{k-1}, t_k]$), here simply denoted by $(t_1, \ldots, t_k)$, where $\lambda(t_i)$ is considered constant within $[t_{i-1}, t_i], i = 1, 2, \ldots, k$. Authors like Atwood et al. (2003)[6] and Kelly and Smith (2009)[42] resort to this kind of approach. Figure 2 illustrates the causal topology related to this methodology, where the dotted and solid directed arcs imply in deterministic and probabilistic causal relationships, respectively. The parametric form of $\lambda(t)$ is deterministically defined by instantiating the hyper-parameters $(\Theta_1, \ldots, \Theta_v)$, i.e., $\lambda t$ is parametrically modeled via a pre-set function which in turns depends on the hyper-parameters.

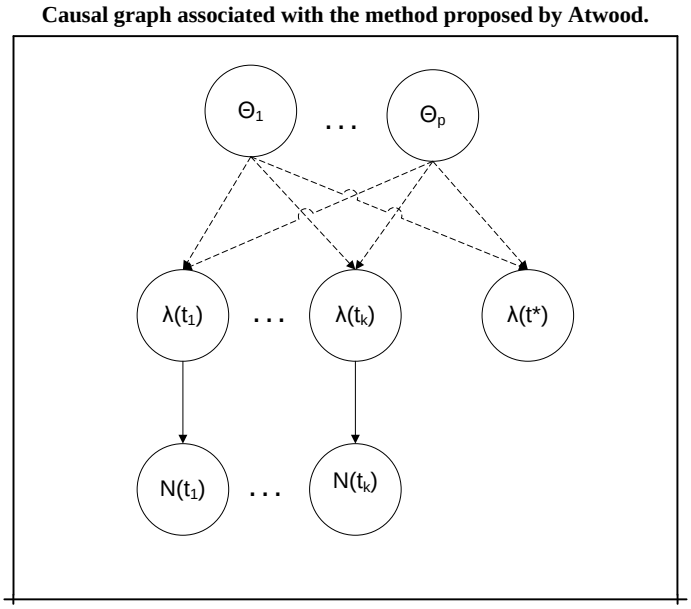



Figure 2.5: Directed acyclic graph related to a NHPP-based Bayesian model

Considering $\lambda(t)$ a failure rate, one has a Poisson distribution underlying the probability mechanism of the number of failures in $t$ time units, $\mathrm{N}(t)$. Thus, if $\Theta_1$ and $\Theta_2$ are associated with parameters $a$ and $b$ of the power-law function, respectively, and the support of the posterior distribution of $\Theta_2$ (conditioned on a failure data set of cardinality $k$) is around 1, one can conclude that the failure rate is time-independent, for instance. Instead, if $\lambda(t)$ represents the probability of failure on demand, $p(t)$, one has a Binomial distribution modelling the number of failures in $t$ demands, $\mathrm{N}(t)$. Anyway, since the empirical data is introduced into the model it becomes possible to predict the behaviour of the reliability measure within a given exposure time period, $t* > t_k$.

One of the main criticisms against generalized linear models arises from the fact that it is applied only to cases where there is enough empirical data. The Bayesian paradigm enables a natural procedure for resorting to subjective data in these cases. In other words, it would be possible to incorporate not only empirical data, but also experts' opinions about the performance of the system and, in particular, its deterioration process as well.

# Chapter 3

# Proposed Bayesian Stochastic Process for Deterioration Modeling

The aim of this chapter is to develop the proposed methodology for dealing with stochastic deterioration using all available information such as run-time data and expert opinions. First, the Bayesian analytic framework for describing deterioration modeling is presented and next the numerical solution is approached through Markov Chain Monte Carlo methods via Bayesian Networks.

## 3.1 The Proposed Framework

Reliability measures, such as failure rate, probability of failure or amount of deterioration of a component or system are often used as characterization of our expectations regarding the system's ability to perform its intended function. As discussed by Singpurwalla (2006)[62], reliability measures are not objective entities, but rather are expressions of our personal uncertainty about the system's failure behavior over time. These expressions can be formulated for individual systems within their particular applications even when the system or application is unique.

Most of Bayesian approaches for reliability analysis neglect the effect of time on the reliability measure of a component or system. Thus, the aim of this work is to propose a framework for modeling deterioration processes in the light of both run-time or demand-based (for modeling probability of failure on demand) data and experts' estimates.

Indeed, it is assumed that the distribution underlying the deterioration process of a given system can be described by a member of a given family of parametric distribution models. In situations where the true distribution is uncertain, this uncertainty is expressed in the form of a probability distribution over the members of that family of models.

If one uses $\varphi(x(t)) = \varphi(x(t)|\theta_1, \ldots, \theta_r)$ to denote the parametric distribution of the random variable that describes the reliability measure over time, $X(t)$, i.e., the stochastic process underlying the reliability measure behaviour over time; then a probability distribution $\pi(\theta_1, \ldots, \theta_r)$ over the parameters of the model can be used to describe the uncertainty about the reliability measure distribution. The estimated reliability measure density is then taken as

$$\hat{p}(x(t)) = \int \cdots \int_{\theta_1,\ldots,\theta_r} \varphi(x(t)|\theta_1, \ldots, \theta_r)\pi(\theta_1, \ldots, \theta_r)d\theta_1, \ldots, d\theta_r \tag{3.1}$$

The estimated density function of the reliability measure over time therefore consists of a weighted mix of distributions of the chosen model, as opposed to being formed by a single 'best' distribution chosen from the set of distributions possible within the definition of the model, e.g. a Maximum Likelihood estimator.

Reliability variability distributions are used as 'generic' prior distributions in time-specific Bayesian assessment procedures [62], which is justified according to the discussion in Deely and Lindley (1981)[18]. Let $\pi(\theta_1, \ldots, \theta_r|E_0(t), E_1(t), E_2(t))$ be a probability distribution over the parameters of the variability model $\varphi(x(t)|\theta_1, \ldots, \theta_r)$ over time, conditional on $E_0(t)$, $E_1(t)$ and $E_2(t)$, where $E_0(t)$, $E_1(t)$ and $E_2(t)$ are the evidence sources defined in Chapter 2 and also vary over time. In special, the information type $E_2(t)$ (expert estimates) varies over time according to the expert's profile and may be more or less reliable throughout the process. Then the time-specific distribution over $X(t)$ conditional on $E_0(t)$, $E_1(t)$, $E_2(t)$ and system-specific evidence $E_3(t)$ for time $t$ is given by

$$\pi(x(t)|\underline{E}(t)) = \frac{P(E_3(t)|x(t), E_0(t), E_1(t), E_2(t)) \int_{\underline{\theta}} \varphi(x(t)|\underline{\theta})\pi(\underline{\theta}|E_0(t), E_1(t), E_2(t))d\underline{\theta}}{\int_x P(E_3(t)|x(t), E_0(t), E_1(t), E_2(t)) \int_{\underline{\theta}} \varphi(x(t)|\underline{\theta})\pi(\underline{\theta}|E_0(t), E_1(t), E_2(t))d\underline{\theta}dx(t)} \tag{3.2}$$

where $\underline{E}(t) = \{E_0(t), E_1(t), E_2(t), E_3(t)\}$ and $\underline{\theta} = \{\theta_1, \ldots, \theta_r\}$. The posterior distribution

of the parameters which model the reliability measure uncertainty based on information types $E_0(t), E_1(t), E_2(t)$ is developed by applying Bayes' theorem

$$\pi(\underline{\theta}|E_0(t), E_1(t), E_2(t)) = \frac{P(E_1(t), E_2(t)|\underline{\theta}, E_0(t))\pi_0(\underline{\theta}|E_0(t))}{\int_{\underline{\theta}} P(E_1(t), E_2(t)|\underline{\theta}, E_0(t))\pi_0(\underline{\theta}|E_0(t))d\underline{\theta}} \tag{3.3}$$

where $P(E_1(t), E_2(t)|\underline{\theta}, E_0(t))$ is the likelihood of the information, and $\pi_0(\underline{\theta}|E_0(t))$ is prior probability distribution on $\underline{\theta}$. Assuming that exposure data $E_1(t)$ and sources estimates $E_2(t)$ are independent, the likelihood function becomes $P(E_1(t), E_2(t)|\underline{\theta}, E_0(t)) = P(E_1(t)|\underline{\theta}, E_0(t))P(E_2(t)|\underline{\theta}, E_0(t))$ where the first member of the right side is the likelihood of the exposure data, and the second is the likelihood of the estimates information. Moreover, writing this equation in terms of the likelihoods of the information for each system we have

$$P(E_1(t), E_2(t)|\underline{\theta}, E_0(t)) = \prod_{i=1}^{n} P(E_{1i}(t)|\underline{\theta}, E_0(t))P(E_{2i}(t)|\underline{\theta}, E_0(t)) \tag{3.4}$$

where $P(E_{1i}(t)|\underline{\theta}, E_0(t))$ and $P(E_{2i}(t)|\underline{\theta}, E_0(t))$ is the probability of observing evidence $E_{1i}(t)$ and $E_{2i}(t)$, respectively, for the $i$th out of $n$ systems assuming that the set of parameters of the reliability variability curve is $\underline{\theta}$.

According to our model, $X(t)$ is distributed according to $\varphi(x(t)|\underline{\theta})$, with $\underline{\theta}$ also being unknown. Therefore, it is calculated the probability of observing the information $E_1(t)$ and $E_2(t)$ by allowing the reliability measure to assume all possible values, i.e. by averaging $P(E_{1i}(t), E_{2i}(t)|\underline{\theta}, E_0(t))$ over the distribution of $X(t)$

$$P(E_{1i}(t), E_{2i}(t)|\underline{\theta}, E_0(t)) = \int_{x(t)} P(E_{1i}(t)|\underline{\theta}, E_0(t))P(E_{2i}(t)|\underline{\theta}, E_0(t))\varphi(x(t)|\underline{\theta})dx(t) \tag{3.5}$$

which can be substituted in Equation 3.4 to obtain the likelihood function of the total information. By writing the likelihood as in Equation 3.5, the estimate and the exposure data are coupled. In other words, the source's estimate of the reliability measure is for the same system for which exposure data have been observed. Obviously, two special cases of Equation 3.5 arise when only one type of information is available for the $i$th time. In fact, in case of exposure data only, the likelihood is $\int_{x(t)} P(E_{1i}(t)|\underline{\theta}, E_0(t))\varphi(x(t)|\underline{\theta})dx(t)$, and it becames $\int_{x(t)} P(E_{2i}(t)|\underline{\theta}, E_0(t))\varphi(x(t)|\underline{\theta})dx(t)$ when just estimate information is available for the $i$th time.

It is important to note that the likelihood as written in Equation 3.4 also allows for the decoupled treatment of information types $E_1(t)$ and $E_2(t)$.

## 3.2 The Numerical Implementation of the Proposed Procedure

The likelihood functions and prior distributions have been incorporated in a Bayesian inference procedure in which the posterior density $\pi(\underline{\theta}|E)$ is computed. The Bayesian inference is performed using a Markov Chain Monte Carlo Method (MCMC), which allows samples to be generated from a continuous unnormalized density [15]. The MCMC method, which is frequently applied to Bayesian Inference problems [32], results in a sample set $\Theta = \{\underline{\theta}_1, \ldots, \underline{\theta}_m\}$, representing the posterior density over the parameters of the reliability measure distribution model $\varphi(x(t)|\underline{\theta})$, where $\underline{\theta} = \{\theta_1, \ldots, \theta_r\}$.

Given $\Theta$, the estimated reliability measure density is computed as

$$\hat{p}(x(t)) = \int_{\underline{\theta}} \varphi(x(t)|\underline{\theta}) d\underline{\theta} = \frac{1}{m} \sum_{i=1}^{m} \varphi(x(t)|\underline{\theta}_i) \tag{3.6}$$

The corresponding mean and variance are computed as

$$\hat{\mu}_{x(t)} = \int_{x(t)} x(t)\hat{p}(x(t))dx(t) = \frac{1}{m} \sum_{i=1}^{m} \mu(\underline{\theta}_i) \tag{3.7}$$

$$\hat{\sigma}^2_{x(t)} = \int_{x(t)} (x(t) - \hat{\mu}_{x(t)})^2 \hat{p}(x(t))dx(t) = \frac{1}{m} \sum_{i=1}^{m} \sigma^2(\underline{\theta}_i) \tag{3.8}$$

where $\mu(\underline{\theta})$ and $\sigma(\underline{\theta})$ are the mean and variance of $\varphi(x(t)|\underline{\theta})$.

Furthermore, the generated results include uncertainty bounds of the cumulative reliability measure density

$$P(x) = \int_0^x p(x)dx \tag{3.9}$$

in the form of $z$-percentiles $P_z(x)$, defined as

$$P(P(x) < P_z) = \frac{z}{100} \tag{3.10}$$

The $z$-percentiles are determined by finding the value $\hat{P}_z(x)$ for which a for a fraction $z/100$ of the samples $\underline{\theta}_i \in \Theta$

$$\int_0^x \varphi(x|\underline{\theta}_i)dx < \hat{P}_z(x) \tag{3.11}$$

These bounds provide the analyst with a basis to assess the uncertainty associated with the estimated reliability measure behaviour distribution.

In the next section, the proposed framework is specialized to deal with two cases: time-dependent failure rate and time-dependent probability of failure on demand. In both cases, the underlying stochastic process is assumed as a non-homogeneous Poisson process.

## 3.3 Bayesian Non-Homogeneous Poisson Process Model

The presented case is based on discrete observations under the hypothesis that the occurrence of failures is a non-homogeneous Poisson process (NHPP) with a power-law intensity function. According to the value of the power law, the rate of occurrence of failures is a decreasing, constant, or increasing function of time, respectively. The computational implementation makes use of a Bayesian network with the proposed Bayesian procedure underlying it.

Indeed, for computational implementation, the first conceptual step in the formulation of the model should be the construction of a directed graph representing the deterioration model. As mentioned in Chapter 2, such a graph represents all quantities as nodes in a directed graph in which arrows between nodes represent directed influences. Figure 3.1 shows the causal graph of the integrated framework taking into account $p$ run-time data sources and $q$ experts for inferring about the deterioration of the system according to the proposed model. Note that a solid arrow indicates a stochastic dependency, while a dashed arrow indicates a deterministic function. Thus, in Figure 3.1 dotted directed arcs represent the deterministic causality from hyperparameters $(\Theta_1, \ldots, \Theta_r, S)$ to the expected value of $\Lambda(t)$, $m(t)$, and solid directed arcs reflect the probabilistic causality among variables.

The proposed Bayesian approach for discrete observations of stochastic deterioration proceeds as follows. First, it is specified the hyperprior joint distribution $\pi(\theta_1, \ldots, \theta_r, s)$

for the $r+1$ hyperparameters in the box in Figure 3.1. Here, it is assumed independence of the hyperparameters, therefore

$$\pi(\theta_1, \ldots, \theta_r, s) = \prod_{i=1}^{r+1} \pi(\theta_i) \tag{3.12}$$

where $\theta_{r+1} = s$.

One must choose hyperpriors that are diffuse enough to contain the true values of the hyperparameters. That means the analyst must have some vague idea of the true values of the hyperparameters. In principle, this should come from experience with other, similar data sets. As suggested by Guida et al. (1989) [34], a uniform distribution over a reasonable range is a good first choice. The same argument is used in the case studies in Chapter 4. Inference on the hyperparameters $(\Theta_1, \ldots, \Theta_r, S)$ and the reliability measure vector (e.g. failure rate or probability of failure on demand) $\Lambda(t)$, requires to obtain Monte Carlo samples from the joint posterior, say $g(\Theta_1, \ldots, \Theta_r, S, \Lambda(t)|n(t), e(t))$, where the data vector is composed by information types $E_1$: $n(t) = (n_1(t), \ldots, n_p(t))$; and $E_2$: $e(t) = (e_1(t), \ldots, e_q(t))$. Formalizing, one has

- Type $E_1$ (exposure evidence): number of failures and exposure (number of demands). For example, $\{n_i(t),\ i = 1, \ldots, p\}$ where $n_i(t)$ is the number of failures in exposure time $t$ of the $i$th system, and $p$ is the total number of similar systems.

- Type $E_2$ (sources' estimates): it comes in the form of estimates of possible values of a reliability measure. The evidence is then in the form of $\left\{(\lambda_j^*(t), \sigma_j),\ j = 1, \ldots, q\right\}$ where $\lambda_j^*$ is the estimate provided by the $j$th source at time $t$, and $\sigma_j$ is the logarithmic standard deviation of representing the uncertainty of source $j$, which represents the analyst's subjective measure of confidence in the $j$th expert. The quality of the information provided by experts will depend on your knowledge state about the process itself.

The letter $g$ is used here to denote both prior and posterior densities. These samples are generated, and then summary statistics from these samples are used to obtain the desired estimates. In order to calculate samples from this joint posterior the analyst must successively sample from the full conditional distributions. That is, he must successively sample from the conditional distribution of each stochastic node given all the

other stochastic nodes in the graph. Conditional independence is expeditiously exploited in directed graphs in order to simplify these full conditional distributions.

The hyperparameter $S$ denotes a dispersion parameter in such a way that $[\Lambda(t)|m(t), S = s]$ follows a given distribution. For instance, if $\Lambda(t)$ is a failure rate, then $[\Lambda(t)|m(t), S = s]$ follows a Lognormal distribution $\text{LN}(\log m(t), s)$ based on Droguett et al (2004)[20]. On the other hand, in the case of $\Lambda(t)$ being a probability of failure on demand, then $[\Lambda(t)|m(t), S = s]$ can be a Beta or a truncated Lognormal distribution. Furthermore, if one considers $S \to 0$ then $\Lambda(t)$ converges towards $m(t)$ and, similarly to the model exposed in Atwood et al (2003)[6], the shape of the reliability measure becomes deterministically defined by instantiating the hyperparameters. Otherwise, if one involves uncertainty over $S$, then the possibility of introducing prior analyst's knowledge about $\lambda(t)$ is enabled, just as made in the methodology introduced by Droguett et al (2004)[20].

As mentioned above, the function $m(t)$ represents the expected value of $\Lambda(t)$. The functional form of $m(t)$ must be specified in order to proceed the parametric analysis. Common forms for $m(t)$ include the power-law process,

$$m(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1} \tag{3.13}$$

the log-linear model,

$$m(t) = \exp\{\alpha + \beta t\} \tag{3.14}$$

and the linear model,

$$m(t) = \alpha + \beta t \tag{3.15}$$

The power-law process is mathematically convenient and has been the subject of some past analyses, Noortwijk (2009)[69]. Furthermore, it subsumes the constant model ($\alpha = 1$) and the linear model ($\alpha = 2$). However, any of these functions above can describe the monotonic character of the process.

In the model described in Figure 3.1, $\Lambda(t)$ represents the time-dependent reliability measure in study. This quantity can be the failure rate or the probability of failure on demand. Thus, if one considers a set of discrete count data $y(t)$, varying on time, which it is desired to estimate their mean $\lambda(t)$, assuming a Poisson distribution with mean $\lambda(t)$ for the data, one must write

$$y_j(t) \sim \text{Poisson}(\lambda(t)), \text{ for } j = 1, \ldots, \eta. \tag{3.16}$$

In this case, the parameter of interest is the Poisson rate $\lambda(t)$ and the likelihood is given by

$$L(\lambda(t)) = \prod_{j=1}^{\eta} \frac{e^{-\lambda(t)}\,[\lambda(t)]^{y_j}}{y_j!} = \frac{e^{-n\lambda(t)}\,[\lambda(t)]^{\sum_{j=1}^{\eta} y_i}}{\prod_{j=1}^{\eta} y_j!} \tag{3.17}$$

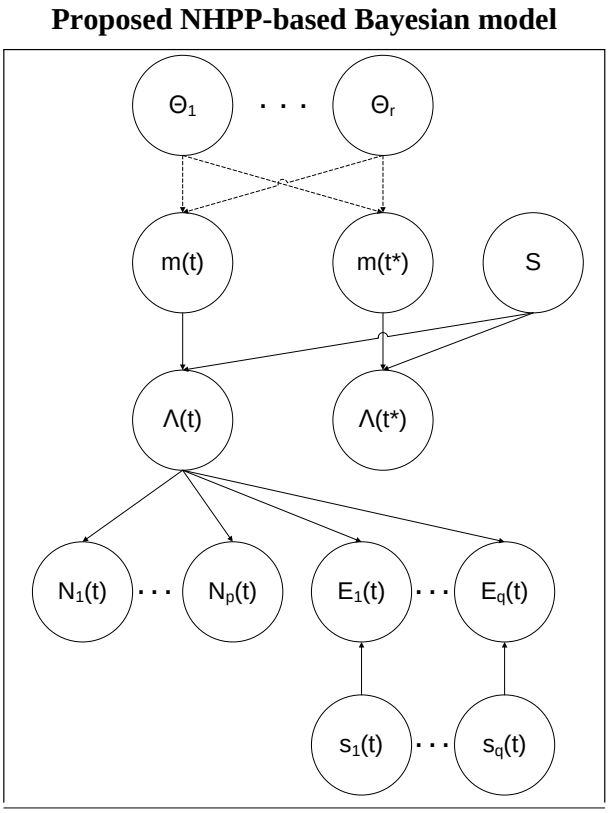



Figure 3.1: Directed acyclic graph related to the NHPP-based Bayesian model for aggregating diverse data sources and capturing time trend.

Note that, given the set of empirical and subjective data, it is possible to compute an aggregated uncertainty distribution over $\Lambda$ in a given mission time (or number of demands), $t^*$. Where $m(t^*)$ in Figure 3.1 represents the expected value of $\Lambda$ in the mission time $t^*$. Furthermore, the hyperparameter $S$ represents the uncertainty over $\Lambda(t)$ and $\Lambda(t^*)$

On the other hand, if one considers a set of binomial data $y_i$ that expresses the number of successes over $N_i$ attempts, then $y_i \sim \text{Binomial}(N_i, \lambda(t))$, resulting in a likelihood given by

$$
\begin{aligned}
L(\lambda(t)) &= \prod_{j=1}^{n} \left\{ \binom{N_j}{y_j} [\lambda(t)]^{y_j} [1-\lambda(t)]^{N_j - y_j} \right\} \\
&= \prod_{j=1}^{n} \binom{N_j}{y_j} [\lambda(t)]^{\sum_{j=1}^{n} y_j} [1-\lambda(t)]^{\sum_{n}^{j=1} N_j - \sum_{n}^{j=1} y_j} \\
&= \prod_{j=1}^{n} \binom{N_j}{y_j} [\lambda(t)]^{n\bar{y}} [1-\lambda(t)]^{N - n\bar{y}} \qquad (3.18)
\end{aligned}
$$

where $N = \sum_{j=1}^{n} N_j$ is the total number of the Bernoulli experiments in the sample and $\bar{y} = \frac{1}{n} \sum_{j=1}^{n} y_j$.

## 3.4 Bayesian Gamma Process Model

The presented case is based on continuous observations under the hypothesis that the stochastic process underlying the amount of deterioration is a gamma process, where the shape parameter is a time-dependent function, decreasing, constant or increasing depending on the nature of associated function; and the scale parameter is fixed.

Although we can approximate a gamma process with a limit of a compound Poisson process, Dufresne et al. (1991)[21] and Singpurwalla and Wilson (1998)[64] claim that it is not efficient to model a gamma process in such a way. This is because there are infinitely many jumps in each finite time interval. Thus, the proposed Bayesian gamma process model aims to construct a more realistic description of the monotonic increasing and positive process underlying the time-dependent deterioration mechanism.

Indeed, the Bayesian network presented in this Section is based on the framework of Section 3.1 and assumes that the reliability measure in study is the amount of deterioration at a given time $X(t)$. Thus,

$$
\varphi(x(t)|v(t),\alpha) = \frac{\alpha^{v(t)}}{\Gamma(v(t))} [x(t)]^{v(t)-1} e^{-\alpha x(t)} I_{0,\infty}(x(t)) \qquad (3.19)
$$

where $v(t)$ is a increasing, right-continuous, real-valued function for $t \geq 0$, with $v(t) \equiv 0$ and $\alpha > 0$ is the scale parameter, similarly to Equation 2.52, is the reliability measure distribution that describes the amount of deterioration over time. Therefore, Equation 3.19 represents the stochastic process underlying the deterioration process of a component

or system.

Using the same methodology as the one used in the previous model, the first step is the construction of a directed graph representing the influences among the variables. Figure 3.2 shows the causal graph of an integrated framework, for $p$ run-time data sources and $q$ experts providing estimates about the expected deterioration $m(t)$ at time $t$.

The proposed Gamma-based Bayesian approach for the stochastic deterioration of a component or system proceeds as follows. First, it is specified hyperprior distributions for the hyperparameters $\Theta_1, \ldots, \Theta_r$ and $\alpha$ in the box in Figure 3.2. In the same way as in the previous model, one must choose hyperpriors that are diffuse enough to contain the true values of the hyperparameters. That means the analyst must have some vague idea of the true values of the hyperparameters. Usually, no prior information is available. In this case, it is strongly recommended to specify a prior that will not influence the posterior distribution and 'let the data speak for themselves'.

As defined in Equation 3.19, the quantity $v(t)$ is the shape parameter of the gamma process and its functional form must be specified in order to try to respond as the expected deterioration increases over time. Accordingly Noortwijk (2009)[69], empirical studies show that the expected deterioration at time $t$ is often proportional to a power-law function. That means if $X(t)$ represents the cumulative amount of deterioration at time $t$ then:

$$E(X(t)) = \frac{v(t)}{\alpha} = \frac{ct^{\beta}}{\alpha} = ct^{\beta} \propto t^{\beta} \tag{3.20}$$

for some physical constants $c > 0$ and $\beta > 0$ and $\alpha$ is the scale parameter of the gamma process. With some effort, there is often engineering knowledge available about the shape of the expected deterioration in terms of the parameter $\beta$ in Equation 3.20. Some examples of expected deterioration according to a power-law are the expected deterioration of concrete due to corrosion of reinforcement (linear: $\beta = 1$; [22]), sulphate attack (parabolic: $\beta = 2$; [22]), diffusion-controlled ageing (square root: $\beta = 0.5$; [22]) and creep ($\beta = 1/8$; [16]).

As showed in the causal graph in Figure 3.2, $m(t)$ is governed by the quantity $v(t)$ and in turn influences the experts' opinions. An important advantage when setting $m(t)$ as the expected value of the related gamma process is that the experts will be elicited

about the average amount of deterioration at time $t$. Sometimes the experts prefer to think in terms of an $X$ percent increase or decrease around some central tendency value, but they might also find useful to think in terms of multiplicative factors around the central tendency values.

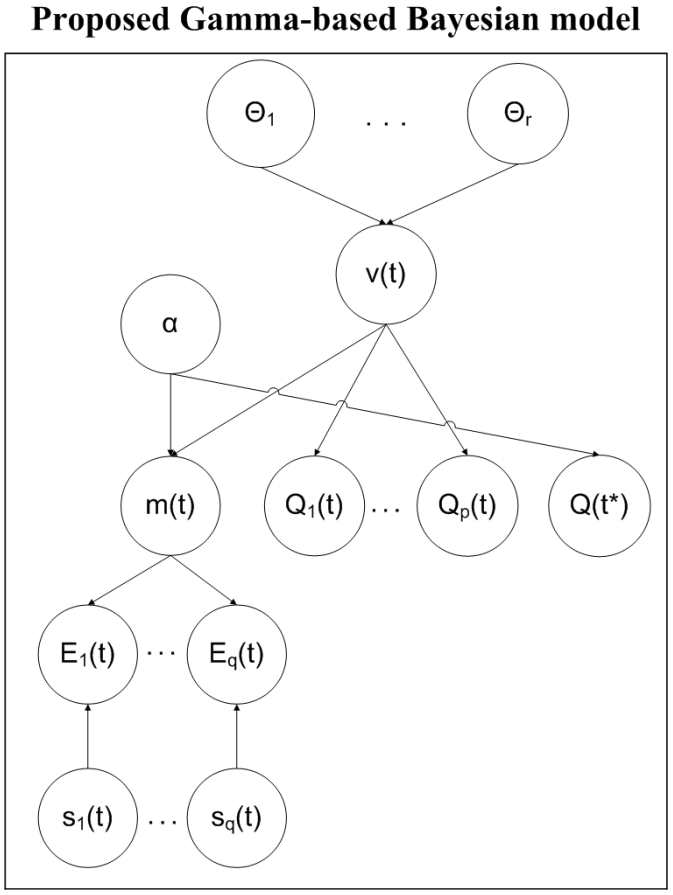



Figure 3.2: Directed acyclic graph related to a gamma process Bayesian model for aggregating diverse data sources and capturing time trend.

Concerning the available evidence for every time period bin, one may have a set of run-time data related to a number of similar components operating under the same conditions, $Q_j(t)$, as well as experts estimates, $E_l(t)$. Formalizing,

- Type $E_1$ (exposure evidence): amount of deterioration at time $t$. For example, $\{Q_i(t),\ i = 1, \ldots, p\}$ where $Q(t)$ is the cumulative of deterioration of the $i$th system at time $t$ and $p$ is the total number of similar systems.

- Type $E_2$ (source's estimates): it comes in the form of estimates of the expected

amount of deterioration of the component or system on a given time. The evidence is then in the form of $\{(m_j(t), s_j),\ j = 1, \ldots, q\}$ where $m_j(t)$ is the estimate provided by the $j$th source at time $t$ and $s_j$ is the logarithmic standard deviation of representing the uncertainty of source $j$, which represents the analyst's subjective measure of confidence in the $j$th expert.

Note that quantity $s_j$ is assigned by the analyst according to the credibility he attributes to the $j$th expert at time $t$ in such a way that $[E_l(t)|m(t_j), s_j]$ follows a given distribution. The quality of the information provided by the experts will depend on their knowledge about the process itself. This means that it is possible to discriminate the credibility on the experts as a function of time, just as in the previous case, in Section 3.3.

In the next chapter, the proposed framework is illustrated by means of some examples of application via simulation.

# Chapter 4

# Case Studies

In this chapter, three case studies are addressed via simulation. The first one concerns the problem of predicting the behavior of the failure rate of given equipment and the second one aims at inferring about the probability of failure on demand of a component both in the light of run-time data and experts' opinions. Finally, the third one aims at modeling the amount of deterioration of a component or system based on exposure data and expert estimates, both obtained from simulation procedures. The manipulation of the resulting Bayesian models is performed by applying Markov chain Monte Carlo algorithms methods. In particular, the approaches embedded into WinBUGS package (see Spiegelhalter et al. (2003)[65]) are adopted.

## 4.1 Time-Dependent Failure Rate

For the sake of illustration, assume the data in Table 4.1 from three equipments and two experts are available during six consecutive time bins of a particular type set of components or systems. The parameter of interest is $\lambda(t)$, the time-dependent intensity of the Poisson process that describes the number of failures, $n_j$, in exposure time $t$:

$$\mathrm{P}\left[N_j(t+h) - N_j(t) = n_j\right] = \frac{e^{-\lambda(t)}[\lambda(t)]^{n_j}}{n_j!},\ n_j = 0, 1, \ldots \tag{4.1}$$

where $N_j(t+h) - N_j(t)$ is the number of events in time interval $(t, t+h]$.

The prior belief is that $\lambda(t)$ varies over time in such a way that it increases or decreases randomly representing, thus, the reliability decay (or growth) of the component or system

in study. Table 4.1 exhibits the data set obtained by simulating $N_j(t) \sim \text{Poisson}(\lambda(t))$ and $E_l(t) \sim \text{Lognormal}(\log \lambda(t), s_l^2(t))$ distributions for representing the run-time data and experts' estimates, respectively. The deterioration process, in turn, was mimicked as governed by the power-law function with $\alpha = 1$ and $\beta = 2$. The main purpose of the modeling is, therefore, to predict the failure rate for 7th time interval.

| period (t) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $n_1(t)$ | 2 | 8 | 9 | 4 | 14 | 12 |
| $n_2(t)$ | 7 | 9 | 10 | 11 | 8 | 15 |
| $n_3(t)$ | 2 | 5 | 12 | 6 | 11 | 13 |
| $e_1(t)$ | 1.74 | 4.61 | 6.93 | 7.30 | 9.15 | 9.79 |
| $e_2(t)$ | 2.82 | 3.11 | 4.65 | 5.54 | 8.62 | 11.42 |

Table 4.1: Run-time data - $n_j(t)$ - and experts' estimates - $e_l(t)$ - simulated from a Poisson($\lambda(t)$) and a Lognormal($\log \lambda(t)$, $s_l^2(t)$), in this order, where $\lambda(t) = 2t$ (the power-law function with $\alpha = 1$ and $\beta = 2$)

Note that relatively little work has been done on Bayesian analysis of a power-law process. Notable references are Guida et al. (1989)[34] and Chen (2004)[14]. A reason for the dearth of work in this area may be the relative intractability of the Bayesian approach. As noted by [34], "[Bayesian procedures] are computationally much more onerous than the corresponding maximum likelihood ones, since they in general require a numerical integration." This problem has been obviated by the advent of Markov chain Monte Carlo (MCMC) techniques and software for implementing such approaches.

### 4.1.1 Proceedings

In this case study, the vector of hyperparameters is defined as $\Theta = (\text{A}, \text{B})^T$ and, independent, diffuse priors were used for them. Thus, based on the available data source (via simulation), the resulting Bayesian model can be decomposed by

$$\text{A} \sim \text{U}(1 \times 10^{-5}, 4), \;\; \text{B} \sim \text{U}(1 \times 10^{-5}, 4)$$

and the quantity $S$ is assumed following a uniform distribution $\text{U}(1 \times 10^{-5}, 5)$.

The functional form of the mean function $m(t)$ is assumed as in Equation 3.13 (with $\alpha = A$ and $\beta = B$). This means that $m(t)$ describes the non-decreasing process of aging through the power-law process in the illustration analysis. Furthermore, the time-dependent reliability measure is assumed to follow a lognormal distribution with parameters $\log m(t)$ and $s^2$, that is:

$$[\lambda(t)|m(t), s] \sim \text{Lognormal}(\log m(t), s^2) \tag{4.2}$$

where $\lambda(t)$ represents the time-dependent failure rate and $s^2$ is the variance of the related normal distribution.

About the available evidence, for the case of studying the failure rate, given a value for $\Lambda(t_i)$, $\lambda(t_i)$, $[N_j(t_i)|\lambda(t_i)] \sim \text{Poisson}(\lambda(t_i))$ whilst if the subject of analysis is the probability of failure on demand $[N_j(t_i)|\lambda(t_i)] \sim \text{Binomial}(t_i, \lambda(t_i))$. Anyway, regardless of working with failure rates or probability of failure on demand, $[E_l(t_i)|\lambda(t_i), s_l^2] \sim \text{Lognormal}(\log \lambda(t_i), s_l^2)$, where $s_l$ is assigned, as defined above, by analyst according to the credibility he attributes to the $l$th expert into $i$th period.

The assumption of a lognormal distribution for the sources estimates (experts' opinions) is derived from the literature where several authors assume the lognormal as the distribution that describes the variability of the different sources. In fact, there is evidence to suggest that the lognormal assumption for the experts' estimates is reasonably realistic. Droguett et al (2004)[20], for example, follow the multiplicative error model proposed by Mosleh (1992)[49] where the likelihood function of the $l$th source estimate can be written in terms of a lognormal distribution. Pulkkinen and Leden (1996)[57] describe a risk index model for the evaluation of traffic safety incorporating expert judgments into the analysis by a Bayesian additive or multiplicative error model, in which the error is lognormally distributed. For other works related to the use of lognormal for describing the behavior of the sources estimates, the reader may refer to Shirazi and Mosleh (2009)[61], Mosleh and Apostolakis (1983)[50] and Alzbutas (2009)[2].

### Analysis in WinBUGS

In this work, the WinBUGS package is used to illustrate how the proposed Bayesian technique can be used for parametric statistical modeling of time-dependent component

reliability. Briefly, the WinBUGS (the MS Windows operating system version of BUGS: Bayesian Analysis Using Gibbs Sampling), see [65], is a versatile package that has been designed to carry out MCMC computations for a wide variety of Bayesian models. It is used to perform the MCMC sampling from the joint posterior distribution of $(\Theta_1, \ldots, \Theta_p, S)$ and to obtain marginal distributions and summary statistics.

The WinBUGS software package uses the underlying DAG model to express the joint posterior distribution as a product of conditional distributions, the network approach allows for almost 'math-free' inference, freeing the analyst from the need to write out Bayes' Theorem, which can quickly become quite complex as the number of parameters in a problem increases. It works primarily with three files, each one containing information relevant to the problem. The file usually named 'model' is where all the parameters of the model and their prior distributions are specified, the likelihood, the hierarchy and relationships between parameters. The second file, 'data' is where we specify the observed data, which will make up the likelihood. And, finally, a file called 'initial' specifies initial values for the parameters of interest.

The WinBUGS script for time-dependent failure rate model can be seen in Table A.1.

### 4.1.2 Results

Considering 100000 iterations of WinBUGS, after a burn-in period of 20000 samples, Figure 4.1 shows the posterior uncertainty distribution over the scale (A) and shape (B) parameters of the power-law intensity function in the light of the simulated data.

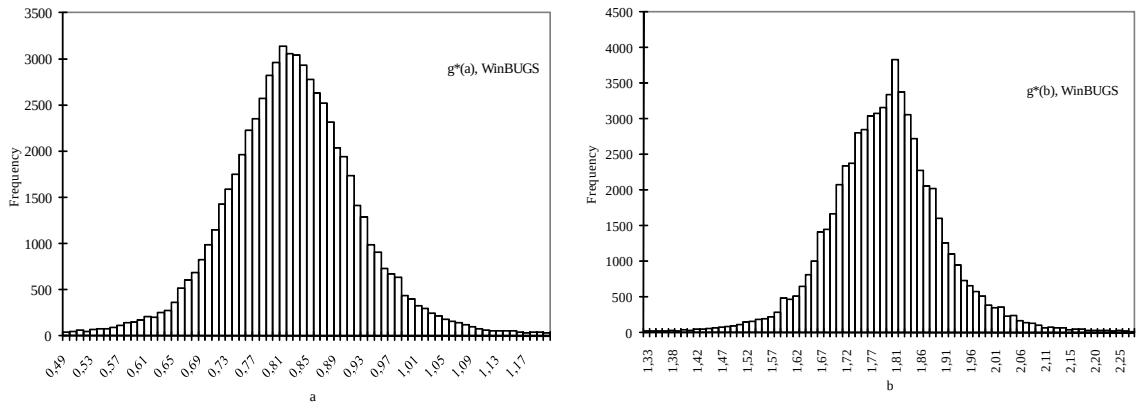



Figure 4.1: Posterior distributions for scale ($A$) and shape ($B$) parameters of the power-law intensity function in the light of the data presented in Table 4.1.

At first glance, one could perceive the reasonable skewness of such posteriors in relation to the respective true values ($\alpha = 1$, $\beta = 2$). Such skewness is partially explained by the randomness on which run-time data was generated, leading to a non-deterministic increase of the number of failures over time. Anyway, the respective 95% percentile intervals for $a$ and $b$ contain the true values - [0.613, 1.041] and [1.561, 2.035], respectively. On the other hand, Figure 4.2 exhibits dispersion and location measures of the posterior distributions related to the failure rate as time evolves. It can be seen the increasing tendency of the failure rate uncertainty but for the period $t = 4$, also explained by the random sample that composes the run-time data. Thus, it is expected a failure rate around 12.290 for period 7 (the true value is 14.0) and the probability of $\lambda(7)$ being into [8.046, 17.520] equals 95%, see Table 4.2.

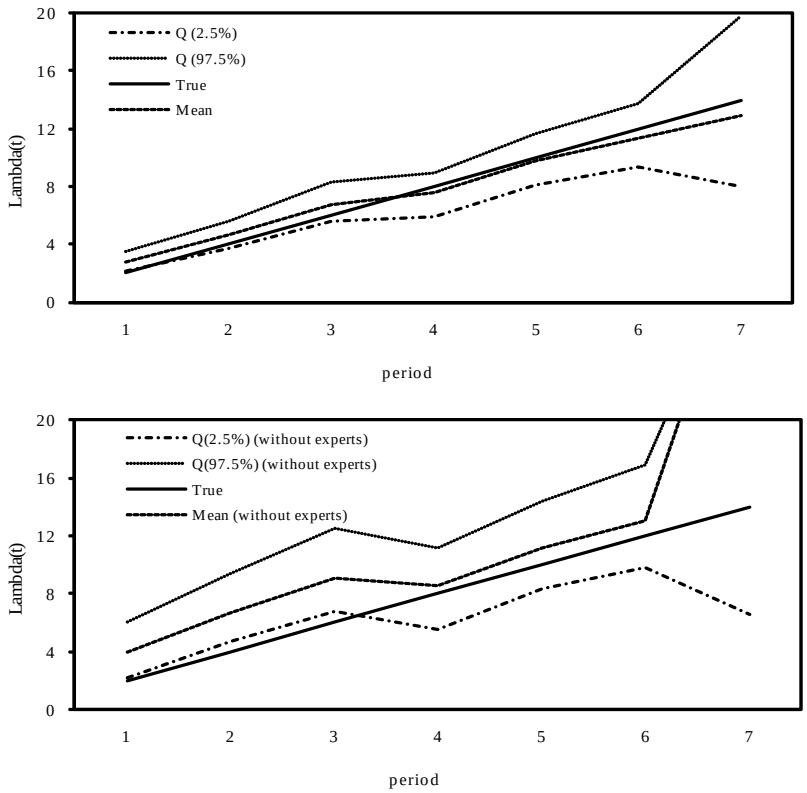



Figure 4.2: Statistical measures (mean, 2.5% and 97.5% quantiles) of the posterior distribution of the failure rate as a function of $t = 1, \ldots, 7$, $\lambda(t)$, according to the data presented in Table 4.1

This example show that by considering the uncertainty bounds around the estimated

distribution, it is in fact possible to distinguish between regions of the distribution where we believe to have a good estimate and those where we have a poor estimate. The amount of uncertainty is a reflection of the evidential strength of the available data. As shown in the example, engineering judgments can be used to supplement data, in case few or no failures were observed.

Table 4.2 reports summary statistics of the MCMC sample from the joint posterior distribution. These summary statistics are provided by WinBUGS and include the mean, standard deviation, median, lower 5% and upper 95% quantiles and the number of samples.

| t | node | mean | sd | 2.5% | median | 97.5% | TRUE | start | sample |
|---|---|---|---|---|---|---|---|---|---|
| | A | 0.824 | 0.1111 | 0.613 | 0.822 | 1.041 | | 20000 | 80000 |
| | B | 1.793 | 0.1246 | 1.561 | 1.792 | 2.035 | | 20000 | 80000 |
| | S | 0.108 | 0.1198 | 0.004 | 0.079 | 0.393 | | 20000 | 80000 |
| 1 | $\lambda[1]$ | 2.630 | 0.2741 | 2.140 | 2.615 | 3.210 | 2 | 20000 | 80000 |
| 2 | $\lambda[2]$ | 4.264 | 0.359 | 3.527 | 4.273 | 4.944 | 4 | 20000 | 80000 |
| 3 | $\lambda[3]$ | 6.248 | 0.4867 | 5.412 | 6.197 | 7.359 | 6 | 20000 | 80000 |
| 4 | $\lambda[4]$ | 7.218 | 0.6115 | 5.918 | 7.262 | 8.316 | 8 | 20000 | 80000 |
| 5 | $\lambda[5]$ | 9.298 | 0.7301 | 8.010 | 9.248 | 10.890 | 10 | 20000 | 80000 |
| 6 | $\lambda[6]$ | 10.740 | 0.8717 | 9.163 | 10.680 | 12.610 | 12 | 20000 | 80000 |
| 7 | $\lambda[7^*]$ | 12.290 | 39.560 | 8.046 | 11.850 | 17.520 | 14 | 20000 | 80000 |

Table 4.2: WinBUGS output for the time-dependent failure rate data: posterior statistics

Note that $\lambda[7^*]$ is the given mission time for the simulated case. Therefore, as noted earlier, the expected failure rate for given mission time $\lambda[7^*]$ is around 12.290 and the probability of $\lambda[7^*]$ being into $[8.046, 17.520]$ equals 95%. Furthermore, it is important report that the 95% posterior credible intervals for all $\lambda[i]$ that quantify the relative performance of a current period in comparison to the previous one.

## 4.2 Time-Dependent Probability of Failure on Demand

Similarly to the previous case, assume the data in Table 4.2 from three equipments and two experts are available during six consecutive time bins of a particular type set of components or systems. However, in this case the parameter of interest is $\lambda(t) = p(t)$, the time-dependent probability of failure on demand that describes the number of failures, $n_j$, in exposure time $t$.

In this case, the prior belief is that $\mathrm{p}(t)$ varies over time, increasing or decreasing at random. This implies that the reliability measure in study is experiencing reliability growth, or decay, depending on the behavior of $\mathrm{p}(t)$.

Table 4.2 exhibits the data set obtained by simulating $N_j(t) \sim \text{Binomial}(30, \mathrm{p}(t))$ and $E_l(t) \sim \text{Lognormal}(\log \lambda(t), s_l^2(t))$ distributions for representing the run-time data and experts' estimates, respectively. The deterioration process, in turn, was mimicked as governed by the logistic function with $a = -9$ and $b = 1.5$. The main purpose of the modeling is, therefore, to predict the probability of failure on demand for 7th time interval.

| period(t) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $n_1(t)$ | 0 | 0 | 1 | 2 | 7 | 16 |
| $n_2(t)$ | 0 | 0 | 0 | 2 | 6 | 19 |
| $n_3(t)$ | 0 | 0 | 0 | 0 | 5 | 12 |
| $e_1(t)$ | 0.0002 | 0.0032 | 0.0062 | 0.1025 | 0.1872 | 0.6293 |
| $e_2(t)$ | 0.0005 | 0.0011 | 0.0113 | 0.0386 | 0.2278 | 0.5359 |

Table 4.3: Run-time data - $n_j(t)$ - and experts' estimates - $e_l(t)$ - simulated from a Binomial(30, $\mathrm{p}(t)$) and a Lognormal($\log p(t)$, $s_l^2(t)$), in this order, where $p(t) = e^{-9+1.5t}/(1 + e^{-9+1.5t})$ (the logistic function with $a = -9$ and $b = 1.5$)

The logistic function occurs often in describing certain kinds of growth. These functions, like exponential functions, grow quickly at first, but because of restrictions that place limits on the size of the underlying population, eventually grow more slowly and then level off.

### 4.2.1 Proceedings

The vector of hyperparameters is defined as $\Theta = (\mathrm{A}, \mathrm{B})^T$, and, as the early example, independent, diffuse priors were attributed for them. Therefore, based on the available data source (via simulation), the resulting Bayesian model can be decomposed by

$$\mathrm{A} \sim \mathrm{U}(-15, 0),\ \mathrm{B} \sim \mathrm{U}(0, 4)$$

The quantity $S$ is assumed to follow a uniform distribution $\mathrm{U}(1 \times 10^{-5}, 5)$; and the functional form of the mean function is defined as

$$\mathrm{p}(t|a, b) = (e^{a+bt})/(1 + e^{a+bt}),$$

the logistic function. Note that $b = 0$ in this model corresponds to no trend, that is, there is no aging of the component or system. If p is increasing over time, then we will have $b > 0$. The time-dependent reliability measure in study is in turn the probability of failure on demand, following a truncated lognormal distribution, that is

$$[\Lambda(t)|p(t), s] \sim \mathrm{Lognormal}(\log \mathrm{p}(t), s^2) \tag{4.3}$$

The number of failures on demand is described by a Binomial distribution $[N_j(t)|\mathrm{p}(t)] \sim \mathrm{Binomial}(30, \mathrm{p}(t))$, and, as mentioned before, $[E_l|\mathrm{p}(t), s_l(t)] \sim \mathrm{Lognormal}(\log \mathrm{p}(t), s_l^2(t))$, where $j = 1, 2, 3$ and $l = 1, 2$.

#### Analysis in WinBUGS

As the previous case, the WinBUGS package was used to illustrate how the proposed Bayesian approach can be used for parametric statistical modeling of time-dependent component or system reliability.

The model for the current case study was run with 20000 burn-in iterations, followed by 80000 iterations for each chain to estimate parameter values. The problem is very similar in structure to the one above, with the exception that the aleatory model is now a binomial distribution.

The WinBUGS script for the time-dependent probability of failure can be seen in Table A.2.

### 4.2.2 Results

Figure 4.3 shows posterior uncertainty distribution over the parameters $(A)$ and $(B)$ of the logistic intensity function in the light of the simulated data. As in the previous case, the data was generated randomly, leading to a non-deterministic increase of the number of failures per demand over time. The values of $(A)$ and $(B)$ are included in $[-10.130, -9.027]$ and $[1.503, 1.787]$, respectively, with probability equals 0.95.

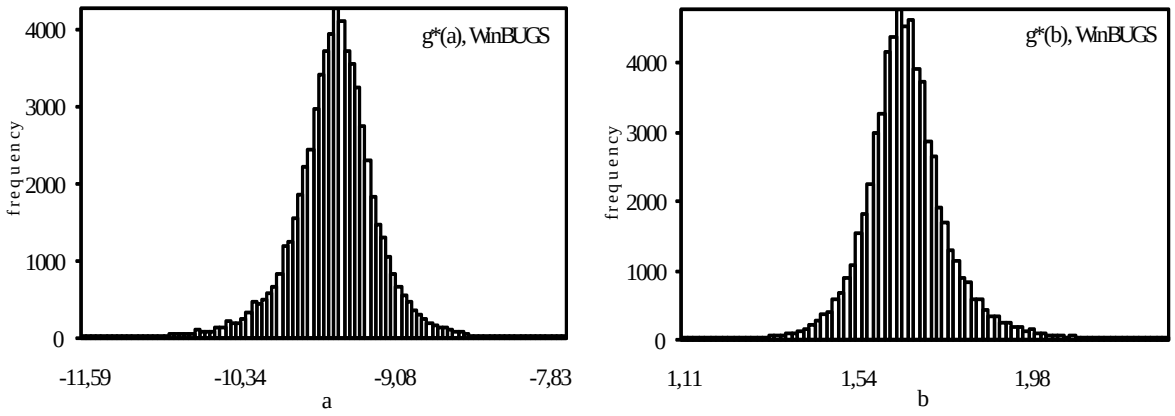



Figure 4.3: Posterior distributions for scale $(A)$ and shape $(B)$ parameters of the logistic intensity function in the light of the data presented in Table 4.2.

On the other hand, Figure 4.4 exhibits dispersion and location measures of the posterior distributions related to the probability of failure as time evolves. As one can see, Figure 4.4 clearly shows the increasing trend of the probability of failure uncertainty in the period sequence. Thus, it is expected a probability of failure around 0.84 for period $7^*$ (the true value is 0.82) and the probability of $P(7^*)$ being into [0.446, 1.000] equals 0.95.

Wider uncertainty bounds are shown in Figure 4.4 for the case in which absence of experts' opinions is observed. As in the previous case, the amount of uncertainty is a reflection of the evidential strength of the available data. Thus, engineering judgments can be used to supplement data, for cases in which few or no failures were observed.

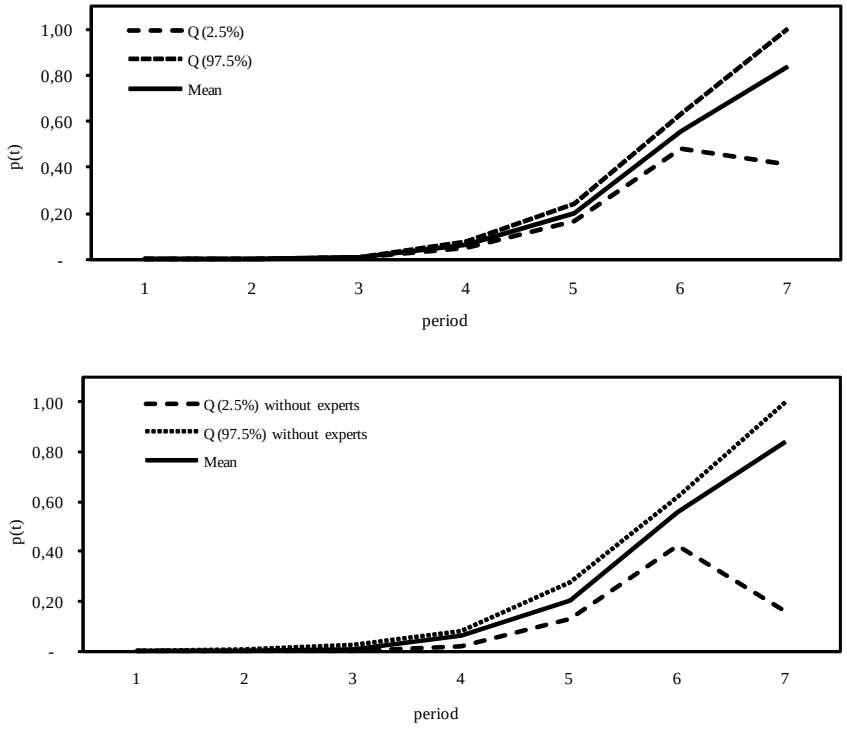



Figure 4.4: Statistical measures (mean, 2.5% and 97.5% quantiles) of the posterior distribution of the probability of failure as a function of $t = 1, \ldots, 7$, $\mathrm{p}(t)$, according to the data presented in Table 4.2

Table 4.4 shows summary statistics from the posterior samples such as the mean, variance and quantiles of the sample. As one can note, all true values are within the confidence limits, showing the increasing trend of the probability of failures over time.

| t | node | mean | sd | 2,50% | median | 97,50% | | start | sample |
|---|---|---|---|---|---|---|---|---|---|
| | A | -9,586 | 0,411 | -10,440 | -9,564 | -8,861 | | 20000 | 80000 |
| | B | 1,663 | 0,135 | 1,446 | 1,652 | 1,939 | true p | 20000 | 80000 |
| 1 | $P$[1] | 4E-04 | 5E-05 | 3E-04 | 4E-04 | 5E-04 | 5E-04 | 20000 | 80000 |
| 2 | $P$[2] | 0,002 | 1E-04 | 0,001 | 0,002 | 0,002 | 0,002 | 20000 | 80000 |
| 3 | $P$[3] | 0,009 | 9E-04 | 0,007 | 0,009 | 0,011 | 0,010 | 20000 | 80000 |
| 4 | $P$[4] | 0,062 | 0,007 | 0,048 | 0,061 | 0,077 | 0,049 | 20000 | 80000 |
| 5 | $P$[5] | 0,202 | 0,020 | 0,165 | 0,202 | 0,243 | 0,182 | 20000 | 80000 |
| 6 | $P$[6] | 0,559 | 0,038 | 0,484 | 0,559 | 0,634 | 0,500 | 20000 | 80000 |
| 7 | $P$[7*] | 0,840 | 0,169 | 0,415 | 0,876 | 1 | 0,818 | 20000 | 80000 |

Table 4.4: WinBUGS output for the time-dependent probability of failure data: posterior statistics

## 4.3 Time-Dependent Amount of Deterioration

The stochastic gamma process model presented in this work takes into account all available data (run-time data and expert's estimates) and temporal variability associated with a deterioration process that increases risk of failure of an aging component or system. Therefore, as in the previous models, assume the data in Table 4.5 from three equipments and two experts are available during six consecutive time bins of a particular type set of components or systems. In this case the parameter of interest is $x(t)$, the time-dependent amount of deterioration at exposure time $t$:

$$f_{X(t)}(x|v(t),\alpha) = \frac{\alpha^{v(t)}}{\Gamma\left(v(t)\right)} x^{v(t)-1} e^{-\alpha x} I_{(0,\infty)}(x), \tag{4.4}$$

where $I_A(x) = 1$ for $x \in A$ and $I_A(x) = 0$ for $x \notin A$, and $\Gamma(a) = \int_{z=0}^{\infty} z^{a-1}e^{-z}dz$ is the gamma function for $a > 0$.

In this case, the assumption is that the amount of deterioration accumulated over time is monotonically non-decreasing. Thus, the shape parameter of related gamma process is defined as a power-law process, just as in the time-dependent failure rate model.

| period (t) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $q_1$ | 1.085 | 3.042 | 4.618 | 7.218 | 10.250 | 10.910 |
| $q_2$ | 0.617 | 2.837 | 6.508 | 6.943 | 10.340 | 12.280 |
| $q_3$ | 4.110 | 5.675 | 6.503 | 10.070 | 7.564 | 8.852 |
| $e_1$ | 1.903 | 3.445 | 5.116 | 7.360 | 11.620 | 13.690 |
| $e_2$ | 1.904 | 4.586 | 5.431 | 6.183 | 9.520 | 10.820 |

Table 4.5: Run-time data - $q_j(t)$ - and experts' estimates - $e_l(t)$ - simulated from Gamma$(v(t), \alpha)$, where $\alpha = 1$, and a Lognormal$(\log m(t),\ s_l^2(t))$, in this order, where $v(t) = 2t$ (the power-law function with a = 1 and b = 2)

### 4.3.1 Proceedings

Both hyperparameters A and B are assigned an uniform hyper prior distribution $U(0.4, 4)$, while the hyperparameter $\alpha$ is assigned an independent uniform $U(1 \times 10^{-5}, 5)$ hyperprior distribution.

The shape parameter of the associated gamma process $v(t)$ is defined as in Equation 3.13 (with $\alpha = a$ and $\beta = b$). This means that $v(t)$ describes the non-decreasing process of aging through the power-law process. Here, mean function $m(t)$ is defined as the expected value of the cumulative amount of deterioration at time $t$:

$$E(X(t)) = \frac{v(t)}{\alpha} \tag{4.5}$$

as suggested by Noortwijk (2009)[69] and explained in the previous chapter.

Concerning the available evidence, for the case of studying the average amount of deterioration of a given component or system, given a value, $m(t)$, $[E_l(t_i)|m(t_i), s_l] \sim$ Lognormal$(\log m(t_i), s_l^2)$. In the same way, as the shape parameter $v(t)$ governs run-time information, $[Q_j(t)|v(t), \alpha] \sim$ Gamma$(v(t), \alpha)$.

#### Analysis in WinBUGS

The WinBUGS package was used to illustrate how the proposed Bayesian approach can be used for parametric statistical modeling of time-dependent component or system reliability. The model for the current case study was run with 20000 burn-in iterations, followed by 80001 iterations for each chain to estimate parameter values.

The proposed method for describing stochastic deterioration was performed to all examples on Microsoft Seven Professional in a Personal Computer Intel T2060 + 1.60GHz, 1GB of RAM.

The WinBUGS script for the gamma process-based Bayesian model can be seen in Table A.3.

### 4.3.2 Results

Figure 4.5 shows posterior uncertainty distribution over the parameters $(A)$ and $(B)$ of the power-law shape function in the light of the simulated data. As in the previous model, the data was generated randomly, leading to a non-deterministic increasing of the amount of deterioration over time. The values of $(a)$ and $(b)$ are included in $[0.494, 1.013]$ and $[1.899, 2.157]$, respectively, with probability equals 0.95.

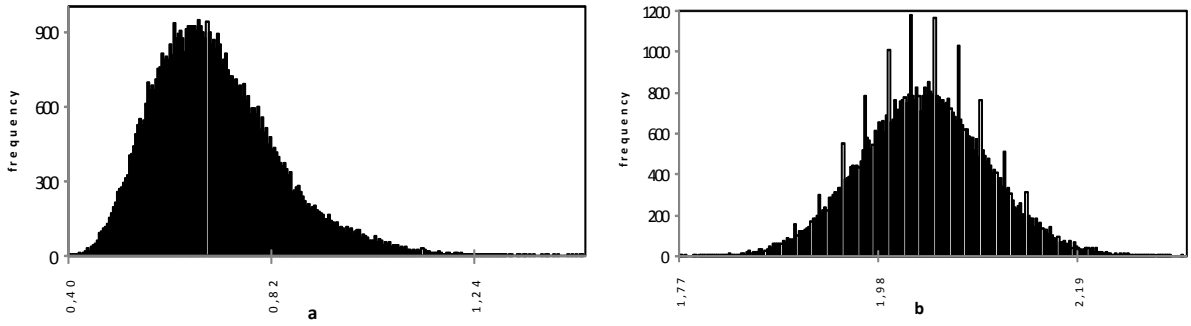



Figure 4.5: Posterior distributions for scale $(A)$ and shape $(B)$ parameters of the power-law shape function in the light of the data presented in Table 4.5.

On the other hand, Figure 4.6 exhibits dispersion and location measures of the posterior distributions related to the amount of deterioration as time evolves. As one can note, Figure 4.6 clearly shows the increasing trend of the deterioration uncertainty in the period sequence.

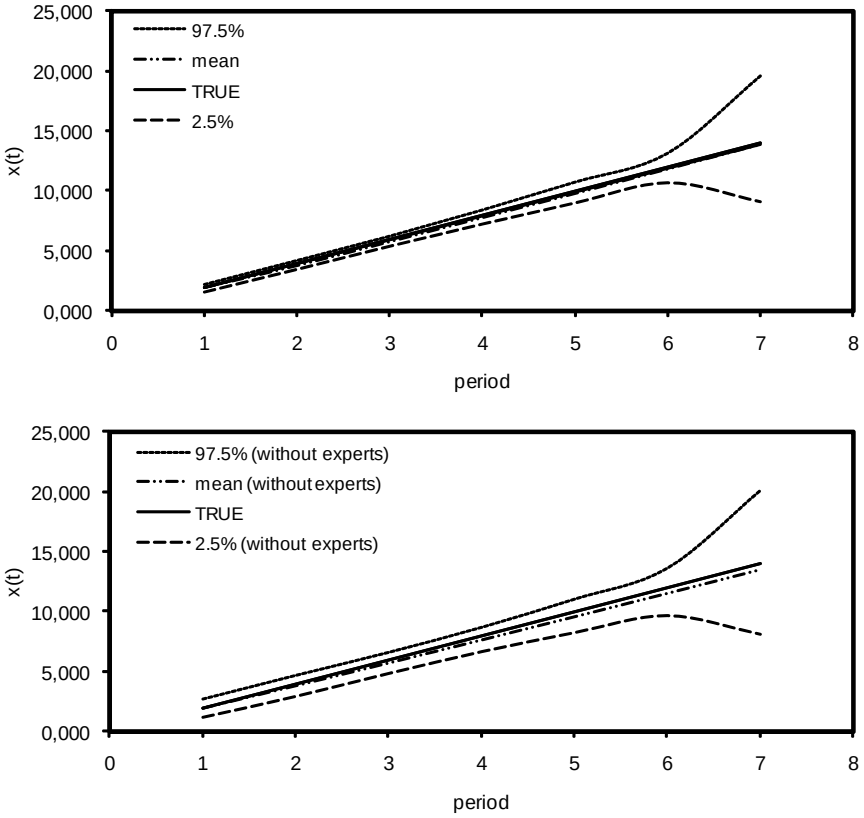



Figure 4.6: Statistical measures (mean, 2.5% and 97.5% quantiles) of the posterior distribution of the failure rate as a function of $t = 1, 2, \ldots, 7$, $\lambda(t)$, according to the data presented in Table 4.5

Note the wider uncertainty bounds for the case where is verified absence of experts'estimates. It suggests a lack of precision when absence of experts' estimates are observed.

Table 4.6 provides estimates of the posterior mean, standard deviation, and quantiles (including the median) for the given generated sample. The total number of iterations (generated sample size) and the number of iterations that the generated sample started (hence the burnin period) are also provided.

| t | node | mean | sd | 2.5% | median | 97.5% | | start | sample |
|---|---|---|---|---|---|---|---|---|---|
| | A | 0.700 | 0.131 | 0.494 | 0.682 | 1.013 | | 20000 | 80000 |
| | B | 2.027 | 0.066 | 1.899 | 2.026 | 2.157 | | 20000 | 80000 |
| | $\alpha$ | 2.425 | 0.800 | 1.093 | 2.346 | 4.210 | true x | 20000 | 80000 |
| 1 | X[1] | 1.884 | 0.166 | 1.574 | 1.878 | 2.224 | 2 | 20000 | 80000 |
| 2 | X[2] | 3.828 | 0.194 | 3.457 | 3.824 | 4.221 | 4 | 20000 | 80000 |
| 3 | X[3] | 5.802 | 0.225 | 5.376 | 5.797 | 6.256 | 6 | 20000 | 80000 |
| 4 | X[4] | 7.796 | 0.311 | 7.212 | 7.788 | 8.431 | 8 | 20000 | 80000 |
| 5 | X[5] | 9.806 | 0.456 | 8.955 | 9.793 | 10.750 | 10 | 20000 | 80000 |
| 6 | X[6] | 11.83 | 0.642 | 10.630 | 11.810 | 13.160 | 12 | 20000 | 80000 |
| 7 | X | 13.86 | 2.676 | 9.064 | 13.690 | 19.590 | 14 | 20000 | 80000 |

Table 4.6: WinBUGS output for the amount of deterioration data: posterior statistics

From these results we can infer that the expected amount of deterioration is equal to 13.86 in period 7. We may also report the 95% posterior credible intervals for all X[$i$] that quantify the relative performance of a current period in comparison to the previous one. As the previous models, all intervals lie around the reference value of true, which indicates a increasing trend of the amount of deterioration over the period studied.

# Chapter 5

# Concluding Remarks

According to Kelly and Smith (2009)[42], in the scientific and engineering communities, we rely on mathematical models of reality, both deterministic and random. These models contain parameters - whose values are estimated from information - of which data are a subset. Uncertain parameters (in the epistemic sense) are inputs to the models used to infer the values of future observations, leading to an increase in scientific knowledge. Further, these parameters may be known to high precision and thus have little associated epistemic uncertainty (e.g.,the speed of light), or they may be imprecisely known and therefore subject to large epistemic uncertainties (e.g., probability of failure of a component). The advent of MCMC-based sampling methods, coupled with easy-to-use software and powerful computers, allows us to encode information via Bayes' Theorem for a large variety of problems, domains, model types, data sets, and complications.

In this sense, the models presented in this work represent an extension of the formalisms found in Atwood et al (2003) [6] and Droguett et al (2004) [20]. The proposed extension allows for the assessment of reliability measures distributions over time based not only on exposure data but also on estimates from various sources such as engineering judgments and reliability databases about a reliability characteristic of similar systems. Furthermore, the proposed models are capable to deal with different types of reliability characteristics, such as failure rate, probability of failure on demand or the amount of deterioration in a given time. It has been shown that the use of estimates information can lead to a reduction of the uncertainty on the reliability measures distributions, particularly in situations where few or no failures have been observed.

Concerning the experts' opinions, the proposed models are capable to handle with different types of experts opinions (according to experts' profiles). Furthermore, it was stated that data was viewed as an observable quantity. This statement implies a temporal constraint on the collection of data: data are collected prior to the present moment. While it is possible to speak about data being collected in the future, such data do not exist until that future time when they are observed. If experts are asked to predict what might happen in the future - for example to predict how many component failures we can expect to see over the next 10 years - this is not a data collection activity. Instead, these opinions were viewed as information, not data, since they have not yet been observed. However, it would be incorrect to assume that such information elicited from experts has less value than available data or that this information cannot be used for quantitative inference. Thus, information, in all its various forms, is just as useful as data when performing inference and may have more organizational value than data, depending on the type, quality, and quantity of the data.

For discrete observations, this work presents a NHPP-based Bayesian model for aggregating diverse data sources and capturing time trend, and, when assuming continuous observations, the stochastic process of deterioration is modelled as a gamma process having independent gamma-distributed increments with identical scale parameter. As the former, the proposed Bayesian gamma process model captures temporal variability associated with the evolution of deterioration, as well as adding various sources of information.

Applications to simulated data was also presented. WinBUGS generated confidence intervals from the posterior predictive distribution, averaging over the joint posterior distribution of hyperparameters. Similarly, it was also used to predict the reliability measure for 7th time interval.

The proposed Bayesian analysis approach was illustrated by means of three examples. For each example, a reliability characteristic (failure rate, probability of failure on demand and the amount of deterioration on a given time) was assessed (via simulation) with and without the presence of experts' estimates. From the results, it was observed the need of addtional information in order to suplement data and increase estimates precision. Thus, as illustrated in all examples, the approach allows for the explicit quantification of the

uncertainty in a reliability measure analysis, which is a valuable aid in the process of decision making under uncertainty.

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# Appendix A

# WinBUGS scripts for the proposed Bayesian models

## A.1 Time-depedent failure rate

```
model
{
A∼ dunif(0.00001, 4)
B∼dunif(0.00001, 4)
S∼dunif(0.00001, 5)
for (t in 1 : 6) {
m[t] < − (B/A)*pow(t/A, B-1)      #power-law function
ln m [t]< − log(m[t])
tau[t] < − 1/(S*S)
L[t] ∼ dlnorm(ln m[t], tau[t])
N1[t] ∼ dpois(L[t])
N2[t] ∼ dpois(L[t])
N3[t] ∼ dpois(L[t])
tau1[t]< −10*t
tau2[t]< −70-tau1[t]
ln L[t] < − log(L[t])
E1[t] ∼ dlnorm(ln L[t], tau1[t])
E2[t] ∼ dlnorm(ln L[t], tau2[t])
}
ln m < − log((B/A)*pow(7/A, B-1))
tau < − 1/(S*S)
L ∼ dlnorm(ln m, tau)
}
```

Table A.1: WinBUGS model specification code for the time-dependent failure rate model

## A.2 Time-dependent probability of failure on demand

```
model
{
A ~ dunif(-15, 0)
B ~ dunif(0,4)
S ~ dunif(0.00001, 5)
for(i in 1:6){
p[i] <- exp(A + B*i) / (1 + exp(A + B*i))
log.p[i] <- log(p[i])
tau[i] <- 1/(S*S)
aux[i] ~ dlnorm(ln.p[i], tau[i])
P[i] <- min(1, aux[i])     #truncated lognormal distribution
N1[i] ~ dbin(P[i],30)
N2[i] ~ dbin(P[i],30)
N3[i] ~ dbin(P[i],30)
tau1[i] <- 0.2 + i
tau2[i]<- 7.4 + tau1[i]
E1[i] ~ dlnorm(ln.p[i], tau1[i])
E2[i] ~ dlnorm(ln.p[i], tau2[i])
}
ln.p. <- A + B * 7 - log(1+ exp(A+B*7))
tau. <- 1/(S*S)
P. ~ dlnorm(ln.p., tau.)
}
```

Table A.2: WinBUGS model specification code for the time-dependent probability of failure model

## A.3 Gamma Bayesian model

```
model
{
A ∼ dunif(0.4, 4)
B ∼ dunif(0.4, 4)
alpha ∼ dunif(0.00001, 5)
for(t in 1 : 6){
v[t] < − (B/A)*pow(t/A, B-1)
Q1[t] ∼ dgamma(v[t], alpha)
Q2[t] ∼ dgamma(v[t], alpha)
Q3[t] ∼ dgamma(v[t], alpha)
X[t] < − v[t]/alpha
log X[t] < − log(X[t])
tau1[t] < − 10*t
tau2[t] < − 70-tau1[t]
E1[t] ∼ dlnorm(log X[t] , tau1[t])
E2[t] ∼ dlnorm(log X[t] , tau2[t])
}
v < − (B/A)*pow(7/A, B-1)
X ∼ dgamma( v, alpha)
}
```

Table A.3: WinBUGS model specification code for the gamma-based Bayesian model