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KCC-theory and its Applications to Coral Reef Modelling

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RAFAEL DOS SANTOS CAVALCANTI

KCC-THEORY AND ITS APPLICATIONS TO CORAL REEF MODELLING

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I dedicated this work to my nephews: Geovana Heloisa, Heitor Rafael and Jean Valentin.

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ABSTRACT

The systems of second order differential equations (SODE) have played a very important role in the study of physics and biological models, in particular, the Volterra-Hamilton system is one of the most useful SODE in ecological problems. We develop the required background of Finsler geometry and classical equation of ecological models to study some aspects of the trajectories of a Volterra-Hamilton system and other subject called semispray. Some geometrical invariants, called KCC-invariants, there are five, are computed to study aspects of the solution trajectories of a semispray. We use Volterra-Hamilton systems theory and their associated cost functional to study the population dynamics and productive processes of coral reefs together with their symbiont algae in recovery from bleaching and show that the cost of production remains the same after the process. The KCC-theory geometrical invariants are determined for the model proposed to describe the renewed symbiotic interaction between coral and algae.

Keywords: Finsler spaces; KCC-theory; Volterra-Hamilton; production stability, coral reef.

RESUMO

As equações diferenciais de segunda ordem (SODE) têm desempenhado um importantissímo papel do estudo de modelos físicos e biológicos, em particular, o sistema de Volterra-Hamilton é um dos SODE mais usados em problemas ecológicos. Desenvolvemos os assuntos necessários de geometria Finsler e quações clássicas de modelos ecológicos afim de esturdarmos alguns aspectos das trajetórias que são soluções de um sistema de Volterra-Hamilton e outro objeto chamado de semispray. Alguns invariantes geométricos, chamados de invariantes KCC, são cinco, são calculados para estudar aspectos das trajetótias soluções de um semispray. Usamos a teoria dos sistemas de Volterra-Hamilton e seus funcionais de custo para estudar a dinamica populacional e o processo de produção de um recife de cora, junto com suas algas simbióticas, em recuperação de branqueamento, mostrar que o custo de produção permanece o mesmo após o processo. A teoria KCC com seus invariantes geométricos são determinantes para o modelo proposto afim de descrever a interação simbiótica renovada entre as algas e os corais.

Palavras-chaves: espaços de Finsler; teoria KCC; Volterra-Hamilton; estabilidade de produção; recife de corais.

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1 INTRODUCTION

There is no doubt that increasing seawater temperature leads to coral bleaching (MARSHALL, 2006). This process occurs when corals are stressed by changes in environmental conditions such as temperature, light, or nutrients, leading to the expelling of the symbiotic algae which lives in the coral's tissues, causing it to turn white. So global warming causes coral bleaching as increasing local seawater temperature stresses symbiotic algae (commonly called *zooxanthellae*) in hermatypic coral (*reef-building*) (GLYNN, 1991) which leads to a breakdown in the symbiotic relationship between the coral animal and its zooxanthellae. This kind of seaweed has been severely affected by global warming around the world (BAKER, 2003; BAKER, 1997). It is important to note that an individual coral is compound by thousand or even million of polyps which are animals of a few millimeters thick. The symbiotic algae living within the polyp makes energy from sunlight; they share that energy with polyp in exchange for a comfortable environment and their interaction produce $CaCO_3$ for the reef building. If there are no symbionts, the polyp run out of energy and dies within a few weeks or months, causing the coral to appear white or "bleached" (SAMMARCO; STRYCHAR, 2013), unless they take more symbionts among those algae that naturally floating in the water around the coral barrier. Some coral reefs have been observed to recover from bleaching in appropriate situations (T.R.MCCLANAHAN, 2000; EDMUNDS; CARPENTER, 2001). In order to model this recovering we suppose that before bleaching each polyp contains symbiotic algae living inside in a stable symbiotic relationship, and that there exist different kinds of algae outside the polyp, which benefit from the coral but do not influence it (*commensal*) some of which are possibly better adapted to higher seawater temperatures. The recent paper (SCHARFENSTEIN et al., 2022) shows that most coral species is supposed to acquire new symbionts from the environment to have an adaptation to increases occurs following bleaching.

Techniques of Finsler geometry has been used to solve many problems of dynamical system and ecology (ANTONELLI; INGARDEN; MATSUMOTO, 1993; ANTONELLI P. L.; RUTZ, 2003; BOEHMER; HARKO; SABAU, 2012). So, our first approach in this work is to describes the population and production dynamics between a specie of coral $\#1$ and a specie of algae $\#2$ living in a coral reef before the bleaching occurs and after the bleaching recovery it will be supposed to happens. The main results used to treat the models involved in this work will be obtained from the KCC-theory (ANTONELLI, 2003a; ANTONELLI; BUCATARU, 2001a).

The principal model equations used here is the famous Volterra-Hamilton (VH) equations. We show that a system of equation (VH) has a semispray form.

The production given by a simple symbiotic relation is a non optimal production in the sense of Euler-Lagrange, so, we consider a Finsler functional that generates geodesics which describes an optimal production relation and its trajectories are stable in Jacobi's sense. For a given Finsler functional $F = F(x, dx)$, we can consider it to be a cost functional of production and treat about it in dimension 2. We provide a curvature in dimension 2, called Gaussian curvature, associated to the geodesics of F, which determines the stability of these geodesic equations. Thus, to dribble some "problems" caused by the simple form of the first approach, we will consider the system obtained by the Euler-Lagrange equations of F to model our problem.

2 BACKGROUND OF FINSLER GEOMETRY

At this time, we need to present convenient concepts and results for our aim. So, every definition and proposition can be found in at least one of the references(ANTONELLI; INGARDEN; MATSUMOTO, 1993; MIRON, 2015; ANTONELLI, 2003a). In almost all cases the author has made an appropriated modification on the concepts and results to make the reading easier. We begin with a general and necessary concept:

Suppose M is a Hausdorff and second-countable topological space. We say that M is a smooth manifold of dimension n if:

(1) Exist a smooth structure $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ such that $\phi_\alpha : U_\alpha \subseteq M \to \mathbb{R}^n$ is an homeomorphism and U_α is open in M;

(2) $M = \bigcup_\alpha U_\alpha$, for all $(U_\alpha, \phi_\alpha) \in \mathcal{A}$;

(3) For all α, β, with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \emptyset$ the sets $\phi_\alpha(U_\alpha)$ and $\phi_\beta(U_\beta)$ are open in $\mathbb{R}^n$ and $\phi_\alpha^{-1} \circ \phi_\beta$ are smooth maps.

We require $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ to be maximal with respect to conditions (1) and (2) above. We call this maximal smooth structure by maximal smooth atlas or just smooth atlas. The pair (U_α, ϕ_α) is called chart of the atlas $\mathcal{A}$.

Example 2.1. *The most simple and important examples of a smooth manifold are the Euclidean Spaces. For each nonnegative integer* n*, the Euclidean Spaces is a smooth n-manifold determined by the single chart* $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$*. We call the atlas formed by this single chart the standart atlas on* $\mathbb{R}^n$*.*

Let M be an $n-$dimensional smooth manifold and $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ a smooth atlas in M. For each local chart (U, ϕ) in $p \in U \subset M$ we denote by (x^i) as local coordinates induced by ϕ such that $\phi(p) = (x^i(p)) \in \mathbb{R}^n$.

If (U_α, ϕ_α) and (U_β, ϕ_β) are any two charts in $p \in M$ with local coordinates (x^i) and $(\widetilde{x}^i)$, respectively, the transition maps $\phi_\alpha^{-1} \circ \phi_\beta$ mentioned in the last item of the above definition of an n-dimentional is an isomorphism, that is $\det\left(\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right) \neq 0$.

Let M be an n-dimensional smooth manifold and $p \in M$ a fixed point. The literature contains some different definitions of tangent space, for example (KUNHEL, 2002), although it is usual to think about the tangent space at point $p \in M$ as an n-dimensional vector space whose elements are "directional vectors". We denote the tangent space to M at p by T_pM. The disjoint union of all tangent spaces at all points of M is called the *tangent bundle*e of M, which is still a manifold, and denoted by $TM = \coprod_{p\in M} T_pM$.

Consider the natural projection $\pi : TM \to M$ defined in a natural way: if $v \in T_pM$ then $\pi(p, v) = p$. It is obviously that $\pi^{-1}(p) = T_pM, \forall p \in M$. To be more rigorous the *tangent bundle* is the triple (TM, π, M). For a local chat $(U, \phi = (x^i))$ on M, we denote by $(\pi^{-1}(U), \Phi = (x^i, y^i))$ the induced local chat on TM. Sometimes is necessary to consider the *slit tangent bundle* $\widetilde{TM} \equiv \coprod_{p\in M} \{T_pM \setminus O_p\} = TM \setminus \mathbb{O}$, where each O_p is null vector of T_pM and $\mathbb{O} : M \to TM$ is the null section.

Proposition 2.1. *The tangent bundle TM is a $2n-$dimensional smooth manifold with the natural smooth atlas $\tilde{\mathcal{A}} = \left\{\left(\pi^{-1}(U_\alpha), \tilde{\phi_\alpha}\right)\right\}$ where $\tilde{\phi_\alpha}(p, v) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n)$.*

This proposition above will be useful in the definition of positively homogeneous functions. Looking ahead we make the convenient substitution $(v^i) = (y^i)$.

Definition 2.1. *A vector field in M is a continuous map $X : M \to TM$ with the property that $\pi \circ X = \mathrm{Id}_M$. One can look to X as a section of the map $\pi : TM \to M$.*

We denote by $\mathcal{X}(M)$ the set of all vector field of M and $\mathcal{F}(M)$ the set of all real smooth functions defined in M. These sets are both vector spaces over $\mathbb{R}$.

According to (LEE, 2000) we define the Cotangent Space at p, denoted by T_p^*M, to be the dual space to T_pM:

$$T_p^*M = (T_pM)^* .$$

Elements of T_p^*M are costumary called tangent covectors at p, or just covectors at p. It is costumary to call tangent vectors $(v \in T_pM)$ contravariant vectors and tangent convectors as covariant vectors. Hence, $T_p^*M = \{\omega_p : T_pM \longrightarrow \mathbb{R},\ \omega_p \text{ is linear}\}$. We consider the union

$$T^*M = \bigcup_{p\in M} T_p^*M$$

of cotangent spaces, which has a differentiable structure of C^∞-class and dimension $2n$.

Since at point p on a manifold M we have defined a tangent space T_pM one can use these tangent space to define tensors. Then, as for every p there is a tensor, we can let p varies on the manifold to define a tensor field.

An 1-*form* on the manifold M can be defined as a smooth map $\omega : M \longrightarrow T^*M$ such that $\pi^* \circ \omega = \mathrm{Id}_M$, where the canonical submersion $\pi^* : T^*M \to M$ is defined by $\pi^*(\omega) = q$

if and only if $\omega \in T_q^*M$. The set of 1-forms on M is denoted by $\Lambda^1(M)$. One can also define $\Lambda^1(M)$ in terms of $\mathcal{X}(M)$ as $\Lambda^1(M) = \{\omega : \mathcal{X}(M) \longrightarrow \mathcal{F}(M), \omega$ is $\mathcal{F}(M)$-linear $\}$. In the same sense, q-forms are q-$\mathcal{F}(M)$-linear ans skew-symmetric map:

$$\omega : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{q\text{-times}} \longrightarrow \mathcal{F}(M).$$

The set of all q-forms is denoted by $\Lambda^q(M)$.

Definition 2.2. *A tensor field of (r,s)-type is an $\mathcal{F}(M)$-linear map*

$$T : \underbrace{\Lambda^1(M) \times \cdots \times \Lambda^1(M)}_{r\text{-times}} \times \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{s\text{-times}} \longrightarrow \mathcal{F}(M).$$

This is a good moment to introduce an important convention that will be exhaustively used throughout this work. Because of the big number of summations that would be used in the following definitions, results and computations, makes necessary to use the Einstein Summation Convention, that is, if the same index name (such as i in the expression $\Gamma = x^i \xi_i$) appears twice in any term, once as an upper index and once as a lower index (or vice versa), that term is understood to be summed over all possible values of that index.

Consider the map $h_r : TM \to TM$ defined by $h_r(x, y) = (x, ry)$. This map is called dilatation of ratio r and is very important to define one of the most important definitions and theorems on this work:

Definition 2.3. *Let $f \in \mathcal{F}(\widetilde{TM})$ be continuous on the null section $\mathbb{O} : M \to TM$. We say that f is positively homogeneous (or just p-homogeneous) in y^i of degree $k \in \mathbb{Z}$ if $f \circ h_r = r^k f$, for all $r > 0$.*

Proposition 2.2. *If f is p-homogeneous of degree k in y^i then any partial derivative of f with respect to y^j is p-homogeneous of degree $k - 1$.*

Proof. Suppose f p-homogeneous of degree k in y^i. By definition $f(x, ry) = r^k f(x, y)$. Taking partial derivative with respect to y^j in both side we have $r\dfrac{\partial f}{\partial y^j} = r^k \dfrac{\partial f}{\partial y^j}$. Since $r > 0$ the result follows. □

Although this definition makes clear the understanding of p-homogeneous functions the famous following Euler's Theorem turns easier the verification to determine if f is positively homogeneous:

Theorem 2.1 (Euler). *If $f \in \mathcal{F}(\widetilde{TM})$ be continuous on the null section $\mathbb{O} : M \to TM$ is p-homogeneous of degree k in y^i if and only if $\frac{\partial f}{\partial y^i} y^i = kf(x,y)$*

Proof. Let us take f a p-homogeneous function of degree k in y^i. By definition we have the relation $f(x, ry) = r^k f(x,y)$, for any positive real number r. Let $g(r) = f(x, ry)$. Then we get $f(x, ry) = g(r) = r^k f(x,y)$. Such as, differentiating both sides with respect to r, the chain rule give us:

$$\frac{\partial f}{du^i}\frac{du^i}{dr} = g'(r) = kr^{k-1} f(x,y),$$

where $u^i = ry^i$. Setting $r = 1$, one can see that

$$\frac{\partial f}{\partial y^i} y^i = kf(x,y).$$

□

Corolary 2.1.
- *Linearity: if f and g are p-homogeneous of degree k then $af + bg$ is p-homogeneous of degree k.*
- *if f is p-homogeneous of degree k_1 in y^i and g is p-homogeneous of degree k_2 then $f.g$ is p-homogeneous of degree $(k_1 + k_2)$ in y^i.*

Definition 2.4. *A vector field $X \in \mathcal{X}(TM)$ is said to be p-homogeneous of degree k in y^i if*

$$X \circ h_r = r^{k-1} h_r \circ X.$$

Definition 2.5. *A Finsler space is a pair $\mathbb{F}^n = (M, F(x,y))$ where $F : TM \to \mathbb{R}$ is a function satisfying:*

i) F is a smooth function in $\widetilde{TM}$ and continuous on null section $\mathbb{O} : TM \to M$;

ii) F is positive definite in $\widetilde{TM}$;

iii) F is p-homogeneous of degree 1 in y^i, that is $F(x, ry) = rF(x,y), \forall r \in (0, +\infty)$;

iv) The hessian matrix of F^2, whose entries are

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

is positive definite in $\widetilde{TM}$. We call g_{ij} the fundamental tensor.

Remark 2.1. *If F is a function satisfying the above definition, we call F to be a Finslerian Function, that is, if the pair (M, F) is a Finsler space we say that F is a Finslerian function.*

Definition 2.6. *A Finsler space $\mathbb{F}^n = (M, F(x,y))$ is a Riemannian space if $g_{ij}(x,y)$ is independent of y.*

By assumption F is p-homogeneous of degree 1 in y^i, then F^2 is p-homogeneous of degree 2 in y^i. According to Proposition (2.2) is easy to verify that $g_{ij}(x,y)$ is p-homogeneous of degre 0 in y^i. This implies that $\frac{\partial g_{ij}}{\partial y^r}y^r = 0$ by definition. Since F^2 is p-homogeneous of degree 2, Theorem (2.1) guarants that $\frac{\partial F^2}{\partial y^i}y^i = 2F^2$. Applying the same Theorem one more time we get $F^2(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}y^i y^j = g_{ij}(x,y)y^i y^j$. One important example is the Riemannian metric $g_{ij}(x)$ who depends only on the points of the manifold. So if (M, g_{ij}) is a Riemannian manifold, the function $F(x,y) = \sqrt{g_{ij}(x)y^i y^j}$ defines a Finsler space (M, F). This is the most recurrent space used in applications because of the nature of problems. In our case we consider $2-$dimensional spaces like that.

Remark 2.2. *When M is supposed to be a Finslerian manifold, that is, $(M, \mathbb{F})$ is a Finsler space we call a tensor as in Definition* (2.2) *by a Finsler tensor.*

The tensor of Cartan $\mathcal{C}_{ijk}$ is a p-homogeneous function of degree -1 defined in terms of derivatives of F^2 as:

$$\mathcal{C}_{ijk} := \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}.$$

Theorem 2.2. *A Finsler space $\mathbb{F}^n = (M, F(x,y))$ is Riemannian if and only if $\frac{\partial g_{ij}}{\partial y^k} = 0$.*

Corolary 2.2. *A Finsler space $\mathbb{F}^n = (M, F(x,y))$ is Riemannian if and only if the Cartan tensor $\mathcal{C}_{ijk}$ vanishes.*

Proof. Since $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$, one can see that $2\mathcal{C}_{ijk} = \frac{\partial g_{ij}}{\partial y^k}$. Thus, by the above theorem we can conclude the result. □

Definition 2.7. *A smooth parametrized curve in M is a smooth map $\gamma : [a,b] \to U \subset M$. We denote by $\widetilde{\gamma} : [a,b] \to \pi^{-1}(U) \subset \widetilde{TM}$ the natural lift of γ.*

For each $t \in [a,b]$, $\gamma(t) = (x^i(t)) \in U$ and $\widetilde{\gamma}(t) = ((x^i(t)), (y^i(t))) \in \pi^1(U)$, where $(y^i(t)) = \left(\frac{dx^i}{dt}\right) = \frac{dx}{dt}$ is the velocity vector. For our propose we will assume all curves to be oriented with increasing parameter. Given a smooth parametrized curve γ the function $F(x,y)$ provides us a length notion as follow:

Definition 2.8. *For a smooth parametrized curve $\gamma : [a,b] \to U \subset M$ having the fixed endpoints $\gamma(a)$ and $\gamma(b)$ we define its length as $L(\gamma) = \int_a^b F\left(x, \frac{dx}{dt}\right) dt$.*

The length $L(\gamma)$ is well-defined since given different induced coordinates in TM we obtain the same value for the integral.

Theorem 2.3 (Carathéodory). *The integral $L(\gamma)$ is independent of choice of parameter.*

This is just an adaptation of Carathéodory Theorem (ANTONELLI; INGARDEN; MATSUMOTO, 1993). The original one requires $F(x,y)$ to be p-homogeneous of degree 1 in y and γ oriented. These conditions are already satisfied from our theory. As a consequence of this result we will be apt to investigate a canonical parameter with such peculiarities as follows.

Corolary 2.3. *Given a smooth curve $\gamma : [a,b] \to M$ in a Finsler space $\mathbb{F}^n = (M, F(x,y))$ there is a canonical parameter s such that $F\left(x(s), \dfrac{dx(s)}{ds}\right) = 1$.*

Proof. Fix $t_0 \in [a,b]$ and consider the so called arc length $s(t) = \displaystyle\int_{t_0}^{t} F\left(x(u), \frac{dx(u)}{dt}\right) du$. By the fundamental Theorem of calculus we know that $s(t)$ is a differentiable function and $\dfrac{ds}{dt} = F\left(x(t), \dfrac{dx(t)}{dt}\right) > 0$. Thus $s(t)$ is invertible by inverse Theorem function with inverse $t = t(s)$. Note that by homogeneity we have $F\left(x(t(s)), \dfrac{dx(t(s))}{ds}\right) = F\left(x, \dfrac{dx}{dt}(dt/ds)\right) = 1$. So we obtain the result. □

Definition 2.9. *The curve $\gamma = \gamma_0$ is extremal of $L(\gamma)$ if γ satisfy: for every smooth family $\{\gamma_\tau\}$ of curves $\gamma_\tau : [a,b] \to M$ such that $\gamma_\tau(a) = \gamma_\tau(b)$ then $\left.\dfrac{dL\left[\gamma_\tau\right]}{d\tau}\right|_{\tau=0} = 0$*

The Theorem (2.3) allow us to suppose that all curves γ_τ have their extremities fixed in the above definition.

Theorem 2.4. *The curve γ is an extremal for $L(\gamma)$ if γ satisfy the Euler-Lagrange equation:*

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial y}\right) = 0, \quad y = \frac{dx}{dt} \tag{2.1}$$

Proof. Let $\eta(t) := \eta(x(t))$ be a contravariant vector field along γ such that $\eta(a) = 0 = \eta(b)$. In order to obtain γ as an extremal for the function $L(\gamma)$ we define the *ε-family* for γ by $x^i_\varepsilon(t) = x^i(t) + \varepsilon \cdot \eta^i(t)$ where $\gamma_\varepsilon(t) = (x^i_\varepsilon(t))$, $\gamma(t) = (x^i(t))$ and also $|\varepsilon|$ is sufficiently small.

So consider the arc length of γ_ε:

$$L\left(\gamma_\epsilon\right) = \int_a^b F\left(x(t) + \varepsilon\eta(t), \frac{dx}{dt}(t) + \epsilon\frac{d\eta}{dt}(t)\right) dt,$$

and suppose $\left.\dfrac{dL\left(\gamma_\varepsilon\right)}{d\varepsilon}\right|_{\varepsilon=0} = 0.$

Differentiating under the integral sign and since F is limited along the the lift σ_ε we obtain:

$$\begin{aligned} \left.\frac{dL(\gamma_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} &= \int_a^b \left.\frac{d}{d\varepsilon}\left[F\left(x_\varepsilon(t), y_\varepsilon(t)\right)\right]\right|_{\varepsilon=0} dt \\ &= \int_a^b \left(\frac{\partial F}{\partial x_\varepsilon}\eta + \frac{\partial F}{\partial y_\varepsilon}\dot{\eta}\right) dt \end{aligned}$$

where $y_\varepsilon = \dfrac{dx_\varepsilon}{dt}$ and $\dot{\eta} = \dfrac{d\eta}{dt}$.

Note that

$$\int_a^b \left(\frac{\partial F}{\partial y_\varepsilon}\dot{\eta}\right) dt = \left.\frac{\partial F}{\partial y_\varepsilon}\eta\right|_a^b - \int_a^b \eta \cdot \frac{d}{dt}\left(\frac{\partial F}{\partial y_\varepsilon}\right) dt = -\int_a^b \eta \cdot \frac{d}{dt}\left(\frac{\partial F}{\partial y_\varepsilon}\right) dt$$

because of integration-by-parts and the condition $\eta(a) = 0 = \eta(b)$. Now, evaluating the expression $\left.\dfrac{dL(\gamma_\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = 0$ at $\varepsilon = 0$ we have:

$$\int_a^b \left[\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial y}\right)\right] \eta(t) dt = 0$$

and the *Euler-Lagrange* equation is a consequence of the arbitrariness of $\eta(t)$. □

For our purpose it is convenient consider the coordinate form of *Euler-Lagrange* equation

$$\frac{\partial F}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) = 0, \quad y^i = \frac{dx^i}{dt}. \tag{2.2}$$

Example 2.2. *Let* $F(x, y, t) = \frac{1}{2}e^{\lambda t}y^2$, $y = \dfrac{dx}{dt}$. *Let us compute the derivatives of* F *in the Euler Lagrange equations. Easily we can see that* $\dfrac{\partial F}{\partial x} = 0$ *while* $\dfrac{\partial F}{\partial y} = e^{\lambda t}y$. *So, applying Euler Lagrange equation we get:*

$$\begin{aligned} 0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial y}\right) &= 0 - \frac{d}{dt}(e^{\lambda t}y) \\ &= -\lambda e^{\lambda t}y - e^{\lambda t}\frac{dy}{dt} \\ &= -e^{\lambda t}\left(\lambda y + \frac{dy}{dt}\right). \end{aligned}$$

The Euler Lagrage equation is therefore

$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} = 0.$$

The solution of this ODE known as Gompertz curve has a biological interpretation that will be teased in the next chapters.

Definition 2.10. *A curve* $\gamma : t \in [a, b] \mapsto \gamma(t) = (x^i(t))$ *is called a geodesic if it is solution of Euler-Lagrange equation.*

To be a little more formal than the previous example and make clear the above definition we present another example:

Example 2.3. *Let* $\alpha > 0$*. Consider the function* $F(x, y, t) = \frac{1}{2} \cdot e^{2\alpha x - \lambda t} \cdot y^2$*,* $y = \dfrac{dx}{dt}$*. According to Theorem (2.1) it is easy to see that* F *is p-homogeneous of degree 2 in* y *and satisfies the other conditions to* $(M, F(x, y))$ *be a Finsler space. Clearly,* $\dfrac{\partial F}{\partial x} = \alpha e^{2\alpha x - \lambda t} \cdot y^2$ *and* $\dfrac{\partial F}{\partial y} = e^{2\alpha x - \lambda t} \cdot y$*. The Euler-Lagrange equation becomes:*

$$
\begin{aligned}
0 &= \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial y}\right) \\
&= \alpha e^{2\alpha x - \lambda t} \cdot y^2 - \frac{d}{dt}\left(e^{2\alpha x - \lambda t} \cdot y\right) \\
&= \alpha e^{2\alpha x - \lambda t} y^2 - \left[\frac{\partial}{\partial x}\left(e^{2\alpha x - \lambda t} y\right)\right]\frac{dx}{dt} - \left[\frac{\partial}{\partial y}\left(e^{2\alpha x - \lambda t} y\right)\right]\frac{dy}{dt} - \left[\frac{\partial}{\partial t}\left(e^{2\alpha x - \lambda t} y\right)\right] \\
&= \alpha e^{2\alpha x - \lambda t} y^2 - \left[2\alpha e^{2\alpha x - \lambda t} y\right]\frac{dx}{dt} - \left[e^{2\alpha x - \lambda t}\right]\frac{dy}{dt} + \lambda e^{2\alpha x - \lambda t} y \\
&= e^{2\alpha x - \lambda t}(\alpha y^2 - 2\alpha y^2 - \frac{dy}{dt} + \lambda y).
\end{aligned}
$$

The Euler-Lagrange equation is therefore the second order differential equation:

$$
\frac{d^2 x}{dt^2} + \alpha\left(\frac{dx}{dt}\right)^2 - \lambda\frac{dx}{dt} = 0.
$$

The solutions of the equations obtained in the last two examples plays a very important role in some mathematics models, for example in biological problems as we will see throughout this text. As these curves are solution of Euler-Lagrange equation they are the geodesics of its respective Finsler space; it is not coincidence they have the form of a second order differential equation, the next proposition describe it very well.

Proposition 2.3. *The geodesisc of a Finsler space* $\mathbb{F}^n = (M, F(x, y))$ *are solution of*

$$
\frac{d^2 x^i}{ds^2} + \gamma^i_{jk}\left(x, \frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,
$$

where γ^i_{jk} *are the Christoffel symbols of the fundamental tensor* g_{ij}*:*

$$
\gamma^i_{jk} := \frac{1}{2} g^{ir}\left(\frac{\partial g_{rk}}{\partial x^j} + \frac{\partial g_{jr}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^{r'}}\right).
$$

Proof. Remarking that $F^2 = g_{ij} y^i y^j$, one can compute the following derivatives by the chain rule.

$$
\frac{\partial F}{\partial x^i} = \frac{1}{2F}\frac{\partial}{\partial x^i}(g_{ij}(x, y) y^i y^j) = \frac{1}{2F}\frac{\partial F^2}{\partial x^i},
$$

$$\frac{\partial F}{\partial y^i} = \frac{1}{2F}\frac{\partial}{\partial y^i}(g_{ij}(x,y)y^i y^j) = \frac{1}{2F}\frac{\partial F^2}{\partial y^i}.$$

Suppose that $\gamma(t) = (x^i(t))$ satisfies the *Euler-Lagrange* equations (2.2). So, for this curve we have:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial F^2}{\partial y^i}\right) - \frac{\partial F^2}{\partial x^i} &= \frac{d}{dt}\left(2F\frac{\partial F}{\partial y^i}\right) - 2F\frac{\partial F}{\partial x^i}\\ &= 2\frac{dF}{dt}\frac{\partial F}{\partial y^i} + 2F\frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) - 2F\frac{\partial F}{\partial x^i}\\ &= 2\frac{dF}{dt}\frac{\partial F}{\partial y^i} + 2F\left(\frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i}\right)\\ &= 2\frac{dF}{dt}\frac{\partial F}{\partial y^i}.\end{aligned}$$

Therefore we obtain the equation:

$$\frac{d}{dt}\left(\frac{\partial F^2}{\partial y^i}\right) - \frac{\partial F^2}{\partial x^i} = 2\frac{dF}{dt}\frac{\partial F}{\partial y^i}, \quad y^i = \frac{dx^i}{dt}.$$

Since corolary (2.3) ensures that the canonical parametrization gives constant Finslerian speed, the above equation is equivalent to:

$$\frac{d}{ds}\left(\frac{\partial F^2}{\partial y^i}\right) - \frac{\partial F^2}{\partial x^i} = 0, \tag{2.3}$$

with $x^i = x^i(s)$ and $y^i = \dfrac{dx^i}{ds}$. For now, a convenient change in the indices of g_{ij} when necessary will leads us to obtain what is expected. All below here in this proof is considering x and y as functions of s. By the chain rule we have:

$$\begin{aligned}\frac{d}{ds}\left(\frac{\partial F^2}{\partial y^r}\right) &= \frac{\partial F^2}{\partial y^r \partial x^k}y^k + \frac{\partial F^2}{\partial y^r \partial y^i}\frac{d^2x^i}{ds^2}\\ &= \frac{\partial F^2}{\partial y^r \partial x^k}y^k + 2g_{ir}\frac{d^2x^r}{ds^2},\end{aligned}$$

where g_{ij} is the fundamental tensor given by the metric, which has g^{ij} as its inverse.

The equations (2.3) can be rewrite as follow:

$$\begin{aligned}0 &= \frac{d}{ds}\left(\frac{\partial F^2}{\partial y^r}\right) - \frac{\partial F^2}{\partial x^r}\\ &= \frac{\partial F^2}{\partial y^r \partial x^k}y^k + 2g_{ir}\frac{d^2x^i}{ds^2} - \frac{\partial F^2}{\partial x^r}\\ &= 2g_{ir}\left\{\frac{1}{2}g^{ir}\frac{\partial F^2}{\partial y^r \partial x^k}y^k - \frac{1}{2}g^{ir}\frac{\partial F^2}{\partial x^r} + \frac{d^2x^j}{ds^2}\right\},\end{aligned}$$

which implies:

$$\frac{d^2x^i}{ds^2} + \frac{1}{2}g^{ir}\left(\frac{\partial F^2}{\partial y^r \partial x^k}y^k - \frac{\partial F^2}{\partial x^r}\right) = 0.$$

There is one more step to conclude the proof. Let us do it. Note that

$$\begin{aligned}
\frac{\partial F^2}{\partial y^r \partial x^k} y^k &= \frac{\partial}{\partial x^k}\left(\frac{\partial g_{ij}}{\partial y^r} y^i y^j + g_{rj} y^j + g_{ir} y^i\right) y^k \\
&= \frac{\partial}{\partial x^k}\left(\frac{\partial g_{ij}}{\partial y^r} y^i y^j + g_{rj} y^j + g_{jr} y^j\right) y^k \\
&= \left(\frac{\partial g_{ij}}{\partial y^r \partial x^k} y^i y^j + \frac{\partial g_{rj}}{\partial x^k} y^j + \frac{\partial g_{jr}}{\partial x^k} y^j\right) y^k \\
&= \left(\frac{\partial g_{rj}}{\partial x^k} + \frac{\partial g_{jr}}{\partial x^k}\right) y^j y^k,
\end{aligned}$$

this is sufficient to show that $\frac{\partial F^2}{\partial y^r \partial x^k} y^k - \frac{\partial F^2}{\partial x^r} = \left(\frac{\partial g_{rk}}{\partial x^j} + \frac{\partial g_{jr}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^r}\right) \frac{dx^j}{ds}\frac{dx^k}{ds}$. The equivalence between (2.1) and (2.3) ensures that solution of Euler-Lagrange equations are also solution of

$$\frac{d^2 x^i}{ds^2} + \gamma^i_{jk}\left(x, \frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

□

Many of biological interactions are supposed to have a production associated to each individual in their dynamics. The cost functional of this dynamics of production is usually Finslerian function depending on production and its variation, so, sometimes will be convenient express the geodesics associated to this functional.

To give an example of the above description of how are the geodesic equation in a Finsler space we resort to our geodesic definition.

For example, consider the Finsler space $\mathbb{F} = (\mathbb{R}\backslash\{0\}, F)$, where

$$F(x, y) = \frac{1}{2} e^{\lambda t} y^2, \qquad y = \frac{dx}{dt}.$$

We have already computed its Euler-Lagrange equations in Example 2.2. So, the geodesics of $\mathbb{F}$ are solution of the second order differential equation:

$$\frac{d^2 x}{dt^2} + \lambda \frac{dx}{dt} = 0.$$

On the other hand the Finsler space $\widetilde{\mathbb{F}} = (\mathbb{R}^*, F)$, with F as in Example 2.3 has a quick difference on its geodesics because Euler-Lagrange equation for this function F has the form:

$$\frac{d^2 x}{dt^2} + \alpha \left(\frac{dx}{dt}\right)^2 - \lambda \frac{dx}{dt} = 0,$$

and hence, by definition, the geodesics of $\widetilde{\mathbb{F}} = (\mathbb{R}^*, F)$ are solution of the above equation.

3 KCC-THEORY

3.1 SEMISPRAYS

Throughout this chapter, we recall some basic definitions and statements about the geometric aspects of a system of second order differential equations. We make use of the principal work on this subject (ANTONELLI, 2003a).

Fix an n-dimensional manifold M.

Definition 3.1. *A vector field $X \in \mathcal{X}(\widetilde{TM})$ is called a semispray if*

$$JX = \Gamma,$$

where $J : \mathcal{X}(TM) \to \mathcal{X}(TM)$, defined by $J = \dfrac{\partial}{\partial y^\imath} \otimes dx^\imath$, which is globally defined on TM and is called the almost tangent structure (or vertical endomorphism of TM) and the Liouville vector field $\Gamma = y^i \dfrac{\partial}{\partial y^i}$.

Remark 3.1. *Let $X \in \mathcal{X}(\widetilde{TM})$. Write $X = X^i \dfrac{\partial}{\partial x^i} + Y^i \dfrac{\partial}{\partial y^i}$. We can see that:*

$$\begin{cases} J\left(\dfrac{\partial}{\partial x^i}\right) = \dfrac{\partial}{\partial y^i}, \\ J\left(\dfrac{\partial}{\partial y^i}\right) = 0. \end{cases}$$

Thus, $JX = X^i \dfrac{\partial}{\partial y^i}$ and we say that X is a semispray if: $X^i \dfrac{\partial}{\partial y^i} = y^i \dfrac{\partial}{\partial y^i}$.

Remark 3.2. *A important case of the above definition is when X is said to be a p-homogeneous vector field of degree 2 in y^i. When this happens we say that X is a spray.*

For a given semispray X the property holds:

Proposition 3.1. *For a neighborhood $\pi^{-1}(U)$ of a induced local chart a semispray X can be written as*

$$X = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}.$$

Proof. In fact, it is known that a vector field X is uniquely written as $X = a^i(x, y) \dfrac{\partial}{\partial x^i} + b^i(x, y) \dfrac{\partial}{\partial y^i}$. Assuming X to be a semispray we have $a^i(x, y) \dfrac{\partial}{\partial y^i} = y^i \dfrac{\partial}{\partial y^i}$. Hence $a^i(x, y) = y^i$. Taking $b^i(x, y) = -2G^i(x, y)$ we obtain the desired local coordinated form for X. □

A set of functions $G^i(x, y)$ is called the *local coefficients* of semispray X. These functions are necessary and sufficient to determine a unique semispray. The local coefficients of a semispray X play a very important role on the characterization of sprays.

Proposition 3.2. *A semispray X is a spray if and only if its local coefficients $G^i(x, y)$ are homogeneous functions of degree 2 in y^i.*

Proof. As y^i is p-homogeneous of degree 1 in y^i and $\dfrac{\partial}{\partial x^i}$ is p homogeneous of degree 1 in y^i follows that $y^i \dfrac{\partial}{\partial x^i}$ is p-homogeneous of degree 2 in y^i. If we set $X = y^i \dfrac{\partial}{\partial x^i} - 2G^i(x, y)\dfrac{\partial}{\partial y^i}$ then X is p-homogeneous of degree 2 in y^i if and only if $G^i(x, y)$ is p-homogeneous of degree 2 in y^i because $\dfrac{\partial}{\partial y^i}$ is p-homogeneous of degree 0 in y^i. □

On a induced local chart $(\pi^{-1}(U), \Phi = (x^i, y^i))$ the paths integral of a semispray $X \in \mathcal{X}(\pi^{-1}(U))$ are of form

$$\frac{dx^i}{dt} = y^i, \quad \frac{dy^i}{dt} = -2G^i(x, y), \tag{3.1}$$

where $G^i(x, y)$ are the local coefficients of X. With this description about the paths integral of X we can say that on a coordinated neighborhood $U \subset M$ these curves are solution of

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \tag{3.2}$$

Remark 3.3. *We call a differential equation as* (3.2) *by SODE.*

Therefore, for a given semispray $X \in \mathcal{X}(\widetilde{TM})$ with local coefficients $G^i(x, y)$ there is a equivalence from X to a SODE as (3.2), for every coordinated neighborhood $U \subset M$. This motives us to state the following definition:

Definition 3.2. *The curves $\gamma : t \in [a, b] \mapsto \gamma(t) = (x^i(t)) \in U \subset M$ which are solution of* (3.2) *are called path of the semispray X.*

Instead of be referring to semispray as a vector field $X \in \mathcal{X}(\widetilde{TM})$ we state that a semispray is a set of second order differential equation of the form (3.2). The easier example of semispray is to consider straight line equations.

Let M be the Euclidian n-dimensional space $\mathbb{R}^n$ with its canonical metric $g_{ij} = \delta_{ij}$. Denote by X_0 the point $(x_0^1, x_0^2, \dots, x_0^n) \in \mathbb{R}^n$. The straight line $X + tV$, $V = (v_0, \dots, v_n) \in \mathbb{R}^n$, has the simple parametrization $x^i = x_0^i + tv^i$. It is clear that these equations satisfies:

$$\frac{d^2x^i}{dt^2} = 0.$$

There is no news that straight lines are simple examples of geodesics curves in the Euclidian Spaces. Although, this example is just a motivation to note that if $\mathbb{F}^n = (M, F(x,y))$ is a Finsler space and if we take $2G^i(x,y) = \gamma^i_{jk}\left(x, \frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds}$, whit γ^i_{jk} being the Christoffel symbols, the equations $\frac{d^2x^i}{ds^2} - 2G^i(x,y) = 0$ gives a semispray whose solutions are the geodesics of the finsler space $\mathbb{F}^n$.

3.2 KCC-INVARIANTS

There is no doubt that many of physical and biological problems may be modelled by a differential equation, or to be more precise, by a SODE as (3.2). Although the equations in (3.2) represents a set of analytical objects there are some due geometrical questions. Kosambi was pioneer in the proceed to find geometric invariants of a semispray in his titled paper "Parallelism and Path-Spaces" (KOSAMBI, 1933). A few years later Kosambi publish another work about this same subject (KOSAMBI, 1935) and theses works had the intuit to find the geometric invariants of (3.2) under nonsingular C^∞ coordinate transformations of type

$$\begin{aligned} \bar{x}^i &= f^i\left(x^1, \ldots, x^n\right), \quad i = 1, \ldots, n \\ \bar{t} &= t. \end{aligned} \tag{3.3}$$

On the other hand in the same year of the publication (KOSAMBI, 1933) Cartan was also working on this subject but with a little detail in the form of coordinate transformation (CARTAN, 1933) . This topic received more information when the geometer Chern published the paper "Sur la géometrie d'un système d'equations differentialles du second ordre" (CHERN, 1939). Throughout the works of Cartan and Chern we can find geometrical invariants of (3.2) under transformations of type

$$\begin{aligned} \bar{x}^i &= f^i\left(x^1, \ldots, x^n, t\right), \quad i = 1, \ldots, n \\ \bar{t} &= t \end{aligned}$$

The KCC-theory is associated to these two form of find geometrical invariants of a second order differential equations, there are five KCC-invariants in fact. It is obvious the motivation to call this theory "KCC" because the authors Kosambi, Cartan and Chern.

The theory developed by Kosambi is called KKC-theory of type (A) while the theory developed by Cartan and Chern is called KCC-theory of type (B). These invariants was exhaustively applied by Peter Antonelli on his works, for example (ANTONELLI; BUCATARU, 2001b). The use of KCC-theory of type (A) has been proved powerful in biological models. The aim of our work is treat a biological problem about the bleaching coral reef.

As many applications many need informations about the behavior of the trajectories of the system (3.2) in a vicinity of a point $(x^i(t_0))$, where for simplicity one can take $t_0 = 0$, we will introduce a concept of instability for solution trajectories of the semispray (3.2). For the following definition we will assume that M is a real-smooth manifold in some Euclidian space.

Definition 3.3. *Let $\gamma(t) = (x^i(t)) \in U \subset M$ be a path in the semispray X. If any other path with initial conditions close enough at $t = t_0$ remains close to $\gamma(t)$ for all $t > t_0$, we say that $\gamma(t)$ is a trajectory Jacobi stable. We define (3.2) to be Jacobi stable if all its solutions are Jacobi stable. Otherwise, we say that (3.2) is Jacobi unstable.*

The above definition has been shown to be very useful in many applications, because makes necessary to consider the Euclidian space as the universe in question (BOEHMER; HARKO; SABAU, 2012; ABOLGHASEM, 2013; HARKO; PANTARAGPHONG; SABAU, 2015).

Definition 3.4. *Let $\xi^i(t)$ be the contravariants components of a vector field $\xi(t)$ defined along a path $\gamma(t) = (x^i(t)) \in U \subset M$ of the semispray (3.2). The KCC-covariant differential is defined by*

$$\frac{\mathbb{D}\xi^i}{dt} := \frac{d\xi^i}{dt} + G^i_j\left(x, \frac{dx}{dt}\right)\xi^j \tag{3.4}$$

where $G^i_j := \dfrac{\partial G^i}{\partial y^j}$ is a set of functions determined by the semispray $X \in \mathcal{X}(\widetilde{TM})$.

If $(\pi^{-1}(U), \Phi = (x^i, y^i))$ e $(\pi^{-1}(V), \Psi = (\tilde{x}^i, \tilde{y}^i))$ are induced local charts in TM then it is known that the coordinates (x^i, y^i) and $(\tilde{x}^i, \tilde{y}^i)$ are related by the coordinate change formula in TM:

$$\begin{cases} \tilde{x}^i = \tilde{x}^i\left(x^1, \ldots, x^n\right), \quad \det\left(\partial \tilde{x}^i / \partial x^j\right) \neq 0, \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j} y^j. \end{cases} \tag{3.5}$$

For all $u \in TM$ we denote by T_uTM the tangent space to TM at u. Denote by $\left\{ \left.\dfrac{\partial}{\partial x^i}\right|_u, \left.\dfrac{\partial}{\partial y^i}\right|_u \right\}$ the natural base induced by the local chart $(\pi^{-1}(U), \Phi = (x^i, y^i))$ for T_uTM.

Using the coordinate change mentioned in (3.5) the natural base for T_uTM, just written, changes as follow:

$$\begin{cases} \left.\dfrac{\partial}{\partial x^i}\right|_u = \dfrac{\partial \widetilde{x}^j}{\partial x^i}(u)\left.\dfrac{\partial}{\partial \widetilde{x}^j}\right|_u + \dfrac{\partial \widetilde{y}^j}{\partial x^i}(u)\left.\dfrac{\partial}{\partial \widetilde{y}^j}\right|_u, \\ \left.\dfrac{\partial}{\partial y^i}\right|_u = \dfrac{\partial \widetilde{x}^j}{\partial x^i}(u)\left.\dfrac{\partial}{\partial \widetilde{y}^j}\right|_u. \end{cases} \tag{3.6}$$

Theses coordinate changes will allow us to investigate some geometrical invariants under nonsingular coordinate transformation as in (3.3). The first one to be investigated is the KCC-covariante derivative defined in (3.4). For that consider the following result:

Proposition 3.3. *Let $G^i(x, y)$ be the local coefficients of a semispray $X \in \mathcal{X}(\widetilde{TM})$. Under coordinate changes (3.5) in TM, the functions G^i_j change as:*

$$\frac{\partial \widetilde{x}^j}{\partial x^k} G^k_i = \widetilde{G}^j_k \frac{\partial \widetilde{x}^k}{\partial x^i} + \frac{\partial \widetilde{y}^j}{\partial x^i}. \tag{3.7}$$

Here, one has to observe that if $\widetilde{x}^i = \widetilde{x}^i\left(x^j\right)$ is a local coordinate change, the contravariant components of $\xi^i(t)$ transform as:

$$\widetilde{\xi}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}\xi^j.$$

Claim 1. *The KCC-covariant differential is invariant under local coordiante transformation $\widetilde{x}^i = \widetilde{x}^i\left(x^j\right)$ in M.*

Proof. This can be observed from evaluating $\dfrac{\partial \widetilde{x}^i}{\partial x^j}\dfrac{\mathbb{D}\xi^j}{dt}$:

$$\begin{aligned} \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{\mathbb{D}\xi^j}{dt} &= \frac{\partial \widetilde{x}^i}{\partial x^j}\left(\frac{d\xi^j}{dt} + G^j_k\left(x, \frac{dx}{dt}\right)\xi^j\right) \\ &= \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{d\xi^j}{dt} + \frac{\partial \widetilde{x}^i}{\partial x^j}G^j_k\left(x, \frac{dx}{dt}\right)\xi^j \\ &= \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{d\xi^j}{dt} + \left(\widetilde{G}^i_j\frac{\partial \widetilde{x}^j}{\partial x^k} + \frac{\partial \widetilde{y}^i}{\partial x^k}\right)\xi^j. \\ &= \frac{d\widetilde{\xi}^i}{dt} + \widetilde{G}^j_k\left(\widetilde{x}, \frac{d\widetilde{x}}{dt}\right)\widetilde{\xi}^j \\ &= \frac{\mathbb{D}\widetilde{\xi}^i}{dt} \end{aligned}$$

Note that (3.3) was used in the third equality in the above computation. □

This conclusion has an important consequence on what we are interested. Since we proved that $\dfrac{\mathbb{D}\widetilde{\xi}^i}{dt} = \dfrac{\partial \widetilde{x}^i}{\partial x^j}\dfrac{\mathbb{D}\xi^j}{dt}$, this shows that under local coordinate changes $\widetilde{x}^i = \widetilde{x}^i\left(x^j\right)$ in M the

KCC-covariant differential preserves tangent vectors, i.e. if v is a tangent vector along $\gamma(t)$ then $\frac{\mathbb{D}v}{dt}$ is still a tangent vector.

If $\gamma(t) = (x^i(t))$ is a path in M then $\frac{dx}{dt} = \left(\frac{dx^i}{dt}\right)$ is a contravariant vector field. Assume $\gamma(t)$ is solution of a semispray as (3.2) the computation (3.4) for $\frac{dx}{dt}$ gives:

$$\begin{aligned}\frac{\mathbb{D}}{dt}\left(\frac{dx^i}{dt}\right) &= \frac{d^2x^i}{dt^2} + G^i_j\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}\\ &= -2G^i\left(x, \frac{dx}{dt}\right) + G^i_j\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}.\end{aligned}$$

This is a "practical" way to evaluate this derivative, and for that it will be necessary call $\mathcal{E}^i(x,y) = 2G^i(x,y) - G^i_j(x,y)\frac{dx^j}{dt}$. As we showed that $\frac{\mathbb{D}}{dt}$ is invariant under nonsingular C^∞ coordinate transformation of type (3.3) we have the contravariant vector field

$$\frac{\mathbb{D}}{dt}\left(\frac{dx^i}{dt}\right) = -\mathcal{E}^i(x,y)$$

defining the *first KCC-invariant* of (3.2), or the *deviation tensor* as described in (GRIFONE, 1972). One can prove that $\mathcal{E}^i(x,y) = 2G^i(x,y) - G^i_j(x,y)\frac{dx^i}{dt}$ is a (1,0)-type Finsler tensor field. When $\mathcal{E}^i(x,y)$ is not the null vector field, this represents an 'external force' associated to the second order differential equations (3.2).

Consider a semispray X. Any path of the semispray X is solution of the system (3.2). Consider $\gamma(t) = (x^i(t))$ a trajectory of which is solution of (3.2), and let vary it into nearby, to define a new trajectory $\tilde{\gamma}(t) = (\tilde{x}^i(t))$, ones according to:

$$\tilde{x}^i(t) := x^i(t) + \varepsilon\xi^i(t)$$

where ε denotes a scalar parameter value $|\varepsilon|$, and $\xi^i(t)$ are components of a contravariant vector field along $\gamma(t)$ with conditions $\xi^i(a) = \xi^i(b) = 0$. Suppose that $\tilde{\gamma}$ is also solution of the SODE (3.2). Now, using $\tilde{\gamma}$ with the semispray equations system we obtain:

$$\frac{d^2x^i}{dt^2} + \varepsilon\frac{d^2\xi^i}{dt^2} + 2\widetilde{G}^i\left(\tilde{x}, \frac{d\tilde{x}}{dt}\right) = \frac{d^2\tilde{x}^i}{dt^2} + 2\widetilde{G}^i\left(\tilde{x}, \frac{d\tilde{x}}{dt}\right) = 0, \tag{3.8}$$

assuming differentiability on the local coefficients of X, $G^i(x,y)$, we will express $\widetilde{G}^i(\tilde{x}, d\tilde{x}/dt)$ as a Taylor series of first order at $\varepsilon = 0$:

$$\widetilde{G}^i(\tilde{x}, d\tilde{x}/dt) = G^i(x, dx/dt) + \left[\frac{\partial\widetilde{G}^i(\tilde{x}, d\tilde{x}/dt)}{\partial x^j}\xi^j + \frac{\partial\widetilde{G}^i(\tilde{x}, d\tilde{x}/dt)}{\partial y^j}\frac{d\xi^j}{dt}\right]\varepsilon,$$

where $O\left(\varepsilon^2\right)$ This implies that (3.8) has a new face:

$$\begin{aligned}
0 &= \frac{d^2\widetilde{x}^i}{dt^2} + 2\widetilde{G}^i\left(\widetilde{x}, \frac{d\widetilde{x}}{dt}\right) \\
&= \varepsilon\frac{d^2\xi^i}{dt^2} + \left[\frac{d^2x^i}{dt^2} + 2G^i(x, dx/dt)\right] + 2\left[\frac{\partial\widetilde{G}^i(\widetilde{x}, d\widetilde{x}/dt)}{\partial x^j}\xi^j + \frac{\partial\widetilde{G}^i(\widetilde{x}, d\widetilde{x}/dt)}{\partial y^j}\frac{d\xi^j}{dt}\right]\varepsilon \\
&= \varepsilon\frac{d^2\xi^i}{dt^2} + 2\left[\frac{\partial\widetilde{G}^i(\widetilde{x}, d\widetilde{x}/dt)}{\partial x^j}\xi^j + \frac{\partial\widetilde{G}^i(\widetilde{x}, d\widetilde{x}/dt)}{\partial y^j}\frac{d\xi^j}{dt}\right]\varepsilon \\
&= \frac{d^2\xi^i}{dt^2} + 2\frac{\partial\widetilde{G}^i(\widetilde{x}, d\widetilde{x}/dt)}{\partial x^j}\xi^j + 2\frac{\partial\widetilde{G}^i(\widetilde{x}, d\widetilde{x}/dt)}{\partial y^j}\frac{d\xi^j}{dt}.
\end{aligned}$$

Note that $\widetilde{G}^i(\widetilde{x}, d\widetilde{x}/dt) = G^i(x+\varepsilon\xi, dx/dt+\varepsilon d\xi/dt)$ and letting $\varepsilon \to 0$ yields to the so-called *variational equations*:

$$\frac{d^2\xi^i}{dt^2} + 2\frac{\partial G^i}{\partial x^j}\xi^j + 2\frac{\partial G^i}{\partial y^j}\frac{d\xi^j}{dt} = 0. \tag{3.9}$$

There is an equivalent form of the variational equations (3.9) in terms of the KCC-covariant differential which will be necessary write it in that form. First, note that

$$\begin{aligned}
0 &= \frac{d^2\xi^i}{dt^2} + 2\frac{\partial G^i}{\partial x^j}\xi^j + 2\frac{\partial G^i}{\partial y^j}\frac{d\xi^j}{dt} \\
&= \left(\frac{d^2\xi^i}{dt^2} + 2G^i_j\frac{d\xi^j}{dt}\right) + 2\frac{\partial G^i}{\partial x^j}\xi^j.
\end{aligned} \tag{3.10}$$

Now, it is convenient to apply the KCC-covariant differential to itself:

$$\begin{aligned}
\frac{\mathbb{D}^2\xi^i}{dt^2} &= \frac{\mathbb{D}}{dt}\left(\frac{d\xi^i}{dt} + G^i_j\xi^j\right) \\
&= \frac{d}{dt}\left(\frac{d\xi^i}{dt} + G^i_j\xi^j\right) + G^i_r\left(\frac{d\xi^r}{dt} + G^r_j\xi^j\right) \\
&= \frac{d^2\xi^i}{dt^2} + \frac{d}{dt}\left(G^i_j\right)\xi^j + G^i_j\frac{d\xi^j}{dt} + G^i_r\frac{d\xi^r}{dt} + G^i_rG^r_j\xi^j \\
&= \frac{d^2\xi^i}{dt^2} + \left(\frac{\partial G^i_j}{\partial x^r}\frac{dx^r}{dt} + \frac{\partial G^i_j}{\partial y^r}\frac{dy^r}{dt}\right)\xi^j + 2G^i_j\frac{d\xi^j}{dt} + G^i_rG^r_j\xi^j.
\end{aligned} \tag{3.11}$$

From (3.11) we see that

$$\frac{d^2\xi^i}{dt^2} + 2G^i_j\frac{d\xi^j}{dt} = \frac{\mathbb{D}^2\xi^i}{dt^2} - \left(\frac{\partial G^i_j}{\partial x^r}\frac{dx^r}{dt} + \frac{\partial G^i_j}{\partial y^r}\frac{dy^r}{dt}\right) - G^i_rG^r_j\xi^j.$$

Then, equations in (3.10) becomes to:

$$\begin{aligned}
0 &= \frac{\mathbb{D}^2\xi^i}{dt^2} + 2\frac{\partial G^i}{\partial x^j}\xi^j - \left(\frac{\partial G^i_j}{\partial x^r}\frac{dx^r}{dt} + \frac{\partial G^i_j}{\partial y^r}\frac{dy^r}{dt}\right)\xi^j - G^i_rG^r_j\xi^j \\
&= \frac{\mathbb{D}^2\xi^i}{dt^2} + 2\frac{\partial G^i}{\partial x^j}\xi^j - \frac{\partial G^i_j}{\partial x^r}y^r\xi^j + 2\frac{\partial G^i_j}{\partial y^r}G^r\xi^j - G^i_rG^r_j\xi^j \\
&= \frac{\mathbb{D}^2\xi^i}{dt^2} + \left(2\frac{\partial G^i}{\partial x^j} - \frac{\partial G^i_j}{\partial x^r}y^r + 2\frac{\partial G^i_j}{\partial y^r}G^r - G^i_rG^r_j\right)\xi^j.
\end{aligned}$$

The above equations may be rewrite as

$$\frac{\mathbb{D}^2\xi^i}{dt^2} + \mathcal{B}^i_j \xi^j = 0, \tag{3.12}$$

where

$$\mathcal{B}^i_j := 2\frac{\partial G^i}{\partial x^j} - \frac{\partial G^i_j}{\partial x^r} y^r + 2G^i_{jr}G^r - G^i_r G^r_j \tag{3.13}$$

is called the *Jacobi endomorphism* in (CRAMPIN M.; MARTINEZ, 1996) and $G^i_{jk} := \dfrac{\partial G^i_j}{\partial y^k}$. The interesting study about (3.13) appears in the works, cited in the beginning of this chapter, of *Cartan*, *Chern* and *Kosambi* in the search to find geometric invariants of a SODE. It can be proved that $\mathcal{B}^i_j$ is a (1,1)-type Finsler tensor field. The tensor $\mathcal{B}^i_j$ defines the second *KCC-invariant* of the SODE (3.2).

There is a very known equation obtained from (3.12) in the Riemannian case, the celebrated *Geodesic Deviation Equation of Jacobi*. To make sure that *Jacobi endomorphism* is a good name to $\mathcal{B}^i_j$ be called, we present an important theorem for Riemannian Geometry, where can be found in the book (LEE, 1979).

If for now we consider the second order differential equations (3.1) to be the geodesic equations in a Riemannian manifold M we have the following theorem:

Theorem 3.1 (The Jacobi Equation)**.** *Let γ be a geodesic and V a vector field along γ. If V is the variation field of a variation through geodesics, then V satisfies*

$$\frac{\mathbb{D}^2 V^i}{dt^2} + \mathcal{B}^i_j V^j = 0.$$

When system (3.2) describes the geodesic equations in either Riemannian or Finsler space, equation (3.12) is the usual Jacobi equation.

Now we quote a result that relates the *Jacobi endomorphism* with the solutions of the semispray X. The *second KCC-invariant* $\mathcal{B}^i_j$ gives information about the Jacobi instability of (3.2).

Proposition 3.4 ((BOEHMER; HARKO; SABAU, 2012))**.** *The trajectories of (3.2) are Jacobi stable if and only if the real part of the eigenvalues of the tensor $\mathcal{B}^i_j$ are strictly negative everywhere, and Jacobi unstable, otherwise.*

As mentioned previously there are five KCC-invariants of a SODE, but we showed only two of them. The *third KCC-invariant* is defined in terms of $\mathcal{B}^i_j$ and the *fourth KCC-invariant* in terms of the third:

$$\begin{cases} \mathcal{B}^i_{jk} := \dfrac{1}{3}\left(\dfrac{\partial \mathcal{B}^i_j}{\partial y^k} - \dfrac{\partial \mathcal{B}^i_k}{\partial y^j}\right), \\ \mathcal{B}^i_{jkl} := \dfrac{\partial \mathcal{B}^i_{jk}}{\partial y^l}. \end{cases}$$

The third invariant is interpreted as a *torsion tensor*, while the fourth is the *Riemann–Christoffel curvature tensor* as described in (ANTONELLI, 2003a). The fifth and last one *KCC-invariant* is called *Douglas tensor* and define by:

$$\mathcal{D}^i_{jkl} := \frac{\partial^2 G^i_j}{\partial y^k \partial y^l} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$$

Theorem 3.2 ((DOUGLAS, 1928))**.** *The Douglas tensor $\mathcal{D}^i_{jkl}$ vanish if and only if G^i is independent of y. Moreover, this condition is coordinate invariant.*

Proof. First of all, note that $\mathcal{D}^i_{jkl} = \dfrac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$ vanish if and only if $G^i = G^i(x, y)$ is independent of y. The proof that the condition is valid in every coordinate system can be found at the reference of this theorem. □

The main result of KCC-theory is the following:

Theorem 3.3 ((ANTONELLI, 2000))**.** *Two system of the form (3.2) on TM are equivalent relative to (3.3) if and only if the five KCC-invariants are equivalent. In particular, there exists coordinates $(\widetilde{x})$ in a coordinate neighborhood $\widetilde{U}$ of M for which the the $G^i(\bar{x}, d\bar{x}/ds)$ all vanish if and only if all KCC-invariants are zero.*

This theorem will be used in the next chapter to compare two sprays obtained from modeling a ecological problem. Its reference is indicated.

4 VOLTERRA-HAMILTON SYSTEM

The aim of this chapter is to describe the dynamics of two given species (or even a finite quantity $n > 2$) along time, which live in the same space even they have no interaction. To make clear our notation, denote by $\sum$ the population at a fixed location. Whenever we mention $\sum$ will be implicit that there is at least one individual of this species population for $t \in [0, T], T > 0$, otherwise we say that the population was extinct. Let $N(t) \geq 0$ be the density (or total number) of a population $\sum$ at time t. The density $N(t)$ will be assumed to be a continuous function for all time instant.

4.1 POPULATION DYNAMICS BY HUTCHINSON

For our purpose it is necessary to admit some conditions on the density function $N(t)$ to make easier the modelling of its dynamics. These conditions are known as Hutchinson Axioms (HUTCHINSON, 1978):

(1) There is no abrupt change in population variation, i.e., $\dfrac{dN}{dt} = f(N)$, where f is differentiable;

(2) Every organism arises from at least one other of its own kind. So if $N = 0$, then $\dfrac{dN}{dt} \equiv 0$;

(3) There is a finite upper bound on the number of organisms N that can utilize a given finite space or location.

With these axioms we can obtain various models about the specie given by $N(t)$.

First of all, axioms (1) and (2) allows us to expand $\dfrac{dN}{dt}$ around $N \equiv 0$ as:

$$f(N) = f'(0)N + f''(0)N^2 + \ldots \tag{4.1}$$

The first and easier model to think and obtain is when (4.1) is consider to be of first order. Then, this yields to:

$$\frac{dN}{dt} = f(N) = f'(0)N.$$

The solution of the above ODE is:

$$N(t) = e^{\lambda t + C} = N(0)e^{\lambda t}, \tag{4.2}$$

where $f'(0) = \lambda$. Note that given initial conditions $N(0) = N_0$, λ determines the population growth. Here, we suppose $\lambda > 0$ (if $\lambda < 0$, the population is dying out). The solution (4.2) is known as model of population growth and it is often called *Malthusian growth* (ANTONELLI; H.BRADBURY, 1996, page 1). Although this model of population growth has been exhaustively used, for example, in describing the bacteria growth, it does not satisfies the third Hutchinson axiom because $N(t)$ is not limited in this case:

$$\lim_{t\to+\infty} N(t) = \lim_{t\to+\infty} N_0 e^{\lambda t} = +\infty.$$

Since the expected location will have finite recurs the exponential growth will makes no sense to talk about the living of a population $\sum$ at this location because would be necessary infinite resources for every given specie, but this is not true in real life. So, in order to obtain an equation to model more real problems we will consider (4.1) to be of second order:

$$\frac{dN}{dt} = \lambda N - \frac{\lambda}{K}N^2, \tag{4.3}$$

where the constants λ is called the *intrinsic growth rate* and $K > 0$ is called the *carrying capacity* for $\sum$, i.e., K represents how much the resources are available for the species. Let us solve (4.3) for $N \neq K$:

Note that $\dfrac{dN}{dt} = \lambda N - \dfrac{\lambda}{K}N = \dfrac{\lambda N}{K}(K-N)$. Which imply $\dfrac{dN}{N(K-N)} = \dfrac{\lambda}{K}dt$. So, integrating in both side we obtain:

$$\int \frac{dN}{N(K-N)} = \int \frac{\lambda}{K}dt. \tag{4.4}$$

To solve the integral on the left side, we have to consider the partial fraction on the form:

$$\frac{1}{N(K-N)} = \frac{a}{N} + \frac{b}{K-N} = \frac{a(K-N)+bN}{N(K-N)},$$

which provides $aK + (b-a)N = a(K-N) + bN = 1$ such that has $a = \frac{1}{K}$ and $b = \frac{1}{K}$ as solution. With this we obtain:

$$\begin{aligned}\int \frac{dN}{N(K-N)} &= \int \frac{a(K-N)+bN}{N(K-N)}dN \\ &= \int \frac{a}{N}dN + \int \frac{b}{K-N}dN \\ &= \int \frac{\frac{1}{K}}{N}dN + \int \frac{\frac{1}{K}}{K-N}dN \\ &= \frac{1}{K}\ln|N| - \frac{1}{K}\ln|K-N| \\ &= \frac{1}{K}\ln\left|\frac{N}{K-N}\right|,\end{aligned}$$

and to solve (4.3) just realize that using the above integral in equation (4.4) we get:

$$\frac{1}{K}\ln\left|\frac{N}{K-N}\right| = \frac{\lambda}{K}t + C_1 \Longrightarrow \ln\left|\frac{N}{K-N}\right| = \lambda t + C_2 \Longrightarrow \left|\frac{N}{K-N}\right| = e^{\lambda t}e^{C_2} = Ce^{\lambda t}.$$

Then the solution of (4.3) is given as follow:

$$N(t) = \frac{K}{1 + be^{-\lambda t}} \tag{4.5}$$

where the constant b is determined by the initial condition $N_0 = N(0) = \dfrac{1}{1+b}$.

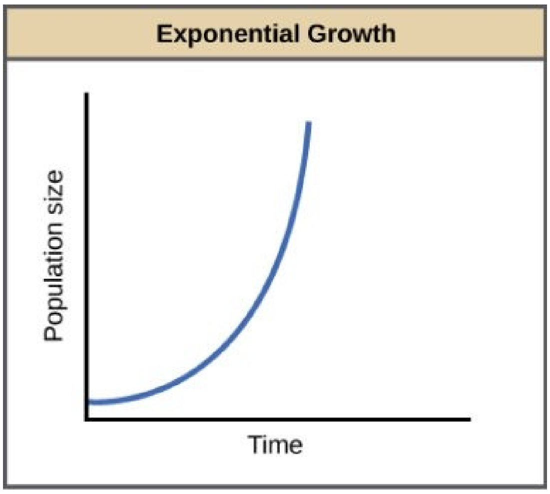
The equation (4.3) is called the *logistic differential equation* and (4.5) is the *logistic curve*. Note that the logistic curve is the simplest equation satisfying assumptions $(1), (2)$ and (3). The upper bound for this curve is the limit:

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{K}{1 + be^{-\lambda t}} = K.$$

The straight line $N = K$ is a horizontal asymptote. For a small interval of time the resources are supposed to be in abundance, then the growth of $N(t)$ is also supposed to be exponentially.

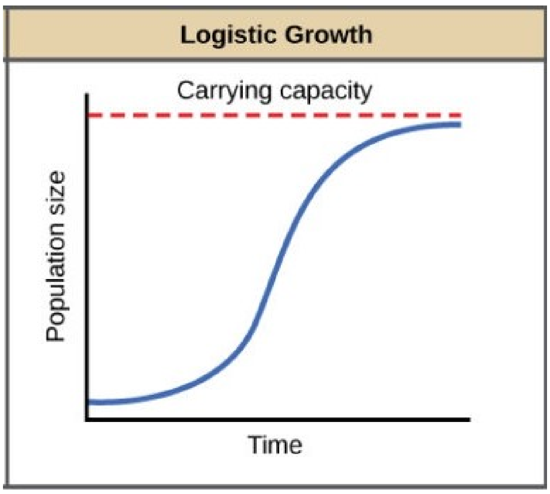
Suppose that for a small time instant t, the carrying capacity K is much bigger than N and for this case we have that $\dfrac{K-N}{K} \approx \dfrac{K}{K} = 1$. Under this hypothesis we obtain $\dfrac{dN}{dt} \approx \lambda N$ for $t \in [0, T]$, for T sufficiently small. Plotting the graph of the Malthusian and Logistic curves one can observe that they are approximately equal in the beginning growth as show the simple sketch:

Figure 1 – Exponential growth



Font: Khan Academy

Figure 2 – Shaped growth



Font: Khan Academy

Note that the red dashed line in figure 2 coincide with the horizontal asymptote $N = K$. This is the reason why K is called Carrying capacity. When $N(t)$ get closer to K this means that the resources are running out from the location in question and the population growth tends to be zero, because there will not have food or other resource for every individual.

For a given real problem, how to determine the constant K? Obviously, this is a question that the answer needs much more information then just the dynamic. Essentially, the constant

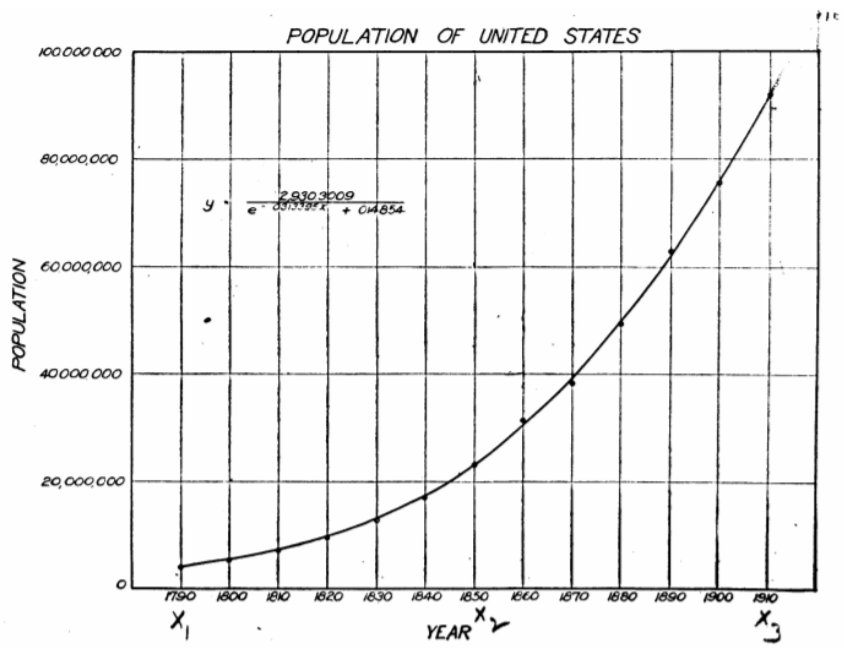
K is closely linked with the specie N needs to survives; there would be no problem in replacing K by $K_{(N)}$ to sign the idea that the Carrying capacity K is a "property" of specie. That is, any important resource for the survival of a given specie is consider to be a upper bound in its population growth. One can realize that to estimate the value of K it is necessary have a look on which resources are available for specie N and how many individuals the location is ready for. For example, we could think about plants and what they need to survival; even though plants are organism that require just water and sunlight, the location can be a problem because this is a finite resource compared to water and sunlight.

Here, we will present three situations involving population growth which can be modelled by the Logistic curve:

- *American Population*

The logistic first arose in 1838 in the work of mathematician P. Verhulst and later in the work of demographers R. Pearl and L. Reed in 1920. They found that it described the human population growth in the U.S.A. from 1790 to 1920.

Figure 3 – USA (1790-1920)

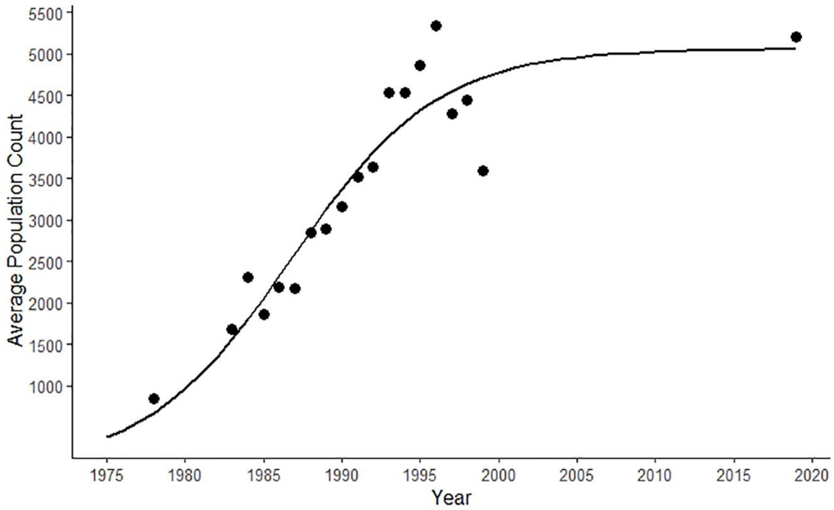


Font: (PARL; REED, 1920)

- *Harbor seal of Washington*

It is very known that the number of Harbor seal were severely reduced by a state-financed population control program. This population control ceased in the second half of the twentieth century and then the population started to grow up. The following graphic shows that the Logistic curve is a great approximation to Harbor seal population growth:

Figure 4 – Harbor seal of Washington

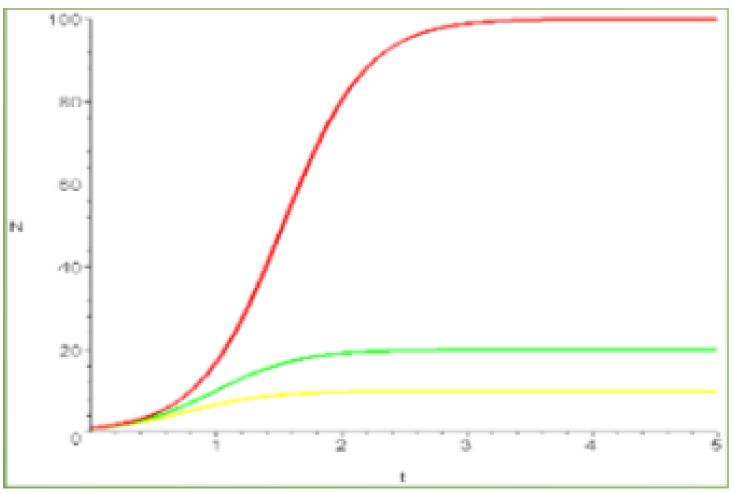


Font: (EA OLSON JK; JK, 2020)

- *Gigogas*

In 2006 the plants *Eichornia crassipes* known as Gigogas had big notoriety in Brazil after beaches, rivers and lagoons had become green with so much plants. Although the gigogas are part of the ecosystem of the city ponds, the abrupt growth was caused by the sewer falling into the rivers and lagoons of Rio de Janeiro in 2006. The sewer was being as large carrying capacity of population of gigogas over the rivers and lagoons.

Figure 5 – Gigogas



Font: (RUTZ; SANTOS, 2009)

the red curve is clearly a Logistic curve and represents what happened in 2006.

All what we have done is about the populational dynamics of one specie without considering that this specie could interact with others species. As several species may forage for food ans seek nesting sites in a given habitat, we will consider that various species live and interact with many other species at the same location. For now, we reserve the symbol Π for a *community*. Suppose that n populations species comprise Π, if two species i and j live in the same habitat, there is the possibility of i and j interact with each other or not. If for a given specie in Π it does not interact with any other, the situation of populational growth of this community can be described by a system of logistic equations:

$$\frac{dN^i}{dt} = \lambda_{(i)} N^i \left(1 - \frac{N^i}{K_{(i)}}\right), \quad i = 1, \ldots, n \tag{4.6}$$

where N^i is the density and $\lambda_{(i)}$ is the intrinsic growth rate of specie i and $K_{(i)}$ is its carrying capacity.

Remark 4.1. *Throughout this work, we made use of Einstein summation convention, so, to makes no mistakes we use parentheses to indicate not summed terms. All the products of the form $a_{(i)}b^i$ is supposed to be not in sum because of parentheses.*

4.2 EQUATIONS OF INTERACTION

According to (GAUSE; WITT, 1935), if i and j compete for the same food items and nesting site then the situation is described by

$$\begin{cases} \dfrac{dN^i}{dt} = \lambda_{(i)} N^i \left(1 - \dfrac{N^i}{K_{(i)}} - \delta_{(i)} \dfrac{N^j}{K_{(i)}}\right) \\ \dfrac{dN^j}{dt} = \lambda_{(j)} N^j \left(1 - \dfrac{N^j}{K_{(j)}} - \delta_{(j)} \dfrac{N^i}{K_{(j)}}\right) \end{cases} \tag{4.7}$$

where all coefficients $\lambda'^s, K'^s, \delta'^s$ are positive. These equations were proposed in 1935 and are known as Gause-Witt equations. The constants $\lambda_{(i)}$ and $K_{(i)}$ have the same means as before, however $\delta_{(i)}$ represents how much the specie i is affected by the specie j in the interaction. Equations (4.7) describes the competition between species i and j because $\delta_{(i)}, \delta_{(j)} > 0$, however the dynamics of the species in $\sum$ can be modelled by:

$$\frac{dN^i}{dt} = \lambda_{(i)} N^i \left(1 - \frac{N^i}{K_{(i)}} - \delta_{(i)} \frac{N^j}{K_{(i)}}\right), \quad i, j = 1, 2, \ldots, n \tag{4.8}$$

where $\lambda_{(i)}$ and $K_{(i)}$ are positive constants denoting intrinsic growth rate and carrying capacity for specie i, respectively. The coefficient $\delta_{(i)}$ represents how much the specie i is affected by the specie j in the interaction. The sign of $\delta_{(i)}$ tells what kind of interaction it is. Consider the system of two equations taking any $i, j \in \{1, \ldots, n\}, i \neq j$, in (4.8). There are three possibilities to this system as follow:

- Parasitism: $\delta_{(i)} > 0, \delta_{(j)} < 0$ or $\delta_{(i)} < 0, \delta_{(j)} > 0$;
- Competition: $\delta_{(i)} > 0, \delta_{(j)} > 0$;
- Symbiosis: $\delta_{(i)} < 0, \delta_{(j)} < 0$.

One can check that the positive equilibrium of a Gause-Witt equations system is:

$$N_*^1 = \frac{K_{(1)} - \delta_{(1)} K_{(2)}}{1 - \delta_{(1)} \delta_{(2)}}, \qquad N_*^2 = \frac{K_{(2)} - \delta_{(2)} K_{(1)}}{1 - \delta_{(2)} \delta_{(1)}}. \tag{4.9}$$

According to (GAUSE; WITT, 1935) we see that Competition case in a Gause Witt model. Assuming $\delta_{(i)}, \delta_{(j)} > 0$, the competition case, set $i = 1, j = 2$ for $n = 2$. Then we have the following theorem (ANTONELLI; H.BRADBURY, 1996, page 21):

Theorem 4.1. *For the system* (4.7) *we have the following four cases:*

1. *If* $\delta_{(1)} > \frac{K_{(1)}}{K_{(2)}}$ *and* $\delta_{(2)} > \frac{K_{(2)}}{K_{(1)}}$*, then only one of the two species will persist after the competition and the winner will be determined entirely by the starting proportions. Equilibrium* (4.9) *is unstable;*
2. *If* $\delta_{(1)} > \frac{K_{(1)}}{K_{(2)}}$ *and* $\delta_{(2)} < \frac{K_{(2)}}{K_{(1)}}$*, then the specie* 1 *will be eliminated by the competition;*
3. *If* $\delta_{(1)} < \frac{K_{(1)}}{K_{(2)}}$ *and* $\delta_{(2)} > \frac{K_{(2)}}{K_{(1)}}$*, then the specie* 2 *will be eliminated by the competition;*
4. *If* $\delta_{(1)} < \frac{K_{(1)}}{K_{(2)}}$ *and* $\delta_{(2)} < \frac{K_{(2)}}{K_{(1)}}$*, then both species persist together at equilibrium* (4.9)*. It is stable.*

The following result is a lemma that makes easier the proof of the above theorem. Its prove can be found at (ANTONELLI; H.BRADBURY, 1996).

Lemma 4.1 (Linear Stability Theorem)**.** *Suppose we are given the system:*

$$\begin{cases} \dfrac{dx}{dt} = F(x, y) \\ \dfrac{dy}{dt} = G(x, y) \end{cases} \tag{4.10}$$

for which (x_0, y_0) *is a steady-state, that is,* $F(x_0, y_0) \equiv 0 \equiv G(x_0, y_0)$*. The point* (x_0, y_0) *is stable if and only if* J_0*, the jacobian of* (4.10) *at* (x_0, y_0)*, has both eigenvalues* r_j $(j = 1, 2)$ *with* $\operatorname{Re}[r_j] < 0$*. If* $\operatorname{Re}[r_j] = 0$ (x_0, y_0) *is neutrally stable, otherwise,* (x_0, y_0) *unstable.*

To prove theorem (4.1), let us apply the above lemma to (4.7). Note that

$$F\left(N^1, N^2\right) = \lambda_{(1)} N^1 - \frac{\lambda_{(1)}}{K_{(1)}}\left(N^1\right)^2 - \frac{\lambda_{(1)}\delta_{(1)}}{K_{(1)}} N^1 N^2$$
$$G\left(N^1, N^2\right) = \lambda_{(2)} N^2 - \frac{\lambda_{(2)}}{K_{(2)}}\left(N^2\right)^2 - \frac{\lambda_{(2)}\delta_{(2)}}{K_{(2)}} N^1 N^2,$$

Using $F\left(N_*^1, N_*^2\right) = 0 = G\left(N_*^1, N_*^2\right)$ once $\left(N_*^1, N_*^2\right)$ is a steady state of (4.7) we obtain:

$$J_0 = \begin{pmatrix} \partial F/\partial N^1 & \partial F/\partial N^2 \\ \partial G/\partial N^1 & \partial G/\partial N^2 \end{pmatrix}_{\left(N_*^1, N_*^2\right)} = \begin{pmatrix} -\frac{\lambda_{(1)}}{K_{(1)}} N_*^1 & -\frac{\lambda_{(1)}\delta_{(1)}}{K_{(1)}} N_*^1 \\ -\frac{\lambda_{(2)}\delta_{(2)}}{K_{(2)}} N_*^2 & -\frac{\lambda_{(2)}}{K_{(2)}} N_*^2. \end{pmatrix} \tag{4.11}$$

Therefore, we see that the characteristic polynomial of the matrix J_0, (4.11), is given by $p(r) = r^2 - \mathrm{tr}(J_0) r + \det(J_0)$, where

$$\begin{cases} \det J_0 = \dfrac{\lambda_{(1)}\lambda_{(2)}}{K_{(1)}K_{(2)}} N_*^1 N_*^2 \left(1 - \delta_{(1)}\delta_{(2)}\right) \\ \mathrm{Tr}\left(J_0\right) = -\dfrac{\lambda_{(1)} N_*^1}{K_{(1)}} - \dfrac{\lambda_{(2)} N_*^2}{K_{(2)}} \end{cases} \tag{4.12}$$

Let us now prove item (1) of theorem (4.1). The prove of the others items can be found at the principal reference, (ANTONELLI; H.BRADBURY, 1996), of this chapter.

Proof. Suppose that $\delta_1 > \dfrac{K_{(1)}}{K_{(2)}}$ and $\delta_{(2)} > \dfrac{K_{(2)}}{K_{(1)}}$. Use the hypothesis on the point N_{1_*} to note that

$$0 < N_{1_*} = \frac{K_{(1)} - \delta_{(1)} K_{(2)}}{1 - \delta_{(1)}\delta_{(2)}} = \left[\frac{\frac{K_{(1)}}{K_{(2)}} - \delta_{(1)}}{1 - \delta_{(1)}\delta_{(2)}}\right] K_{(2)}$$

which implies $1 - \delta_{(1)}\delta_{(2)} < 0$ and consequently $\det J_0 < 0$. So, the roots r_1 and r_2 of $p(r)$ satisfies $r_1 r_2 < 0$, which ensures that r_1, r_2 are real with one of them positive and the other is negative. The use of N_{1_*} or N_{2_*} implies the same result. From linear stability theorem (4.1), this shows (4.9) is unstable. □

Now we introduce a natural measure of production x^i of a population N^i, the Volterra's production Variable (VOLTERRA, 1936), by defining:

$$x^i(t) = k_{(i)} \int_0^t N^i(\tau) d\tau + x^i(0) \tag{4.13}$$

where $x(0) > 0$. The per capita production rate, k, is defined by $\dfrac{1}{N} \cdot \dfrac{dx}{dt} = k > 0$ and supposed to be constant.

Let M be an n-dimensional manifold and $\widetilde{TM}$ its slit tangent bundle. If $(U, \phi = (x^i))$ is a local chart on M, then $(\pi^{-1}(U), \Phi = (x^i, N^i))$ will denote the induced local chart on $\widetilde{TM}$, where $\pi : TM \to M$ is the natural projection. By an n-dimensional *Volterra-Hamilton* system we mean a system of ordinary differential equation of the form:

$$\begin{cases} \dfrac{dx^i}{dt} = k_{(i)}N^i & i,j,k = 1,2,\ldots,n \\ \dfrac{dN^i}{dt} = -G^i_{jk}N^jN^k + \lambda_{(i)}N^i + e^i \end{cases} \tag{4.14}$$

where G^i_{jk} are coefficients possible depending on x^i, N^i, t; these n^3 functions are p-homogeneous of degree 0 in N^i. The coordinate x^i are Volterra production variables, whose constant per capita rate is k_i. This system represents not just an populational dynamics but it describes a set of n producer populations whose sizes are denoted by $N^1, \ldots, N^n$.

The Gause-Witt equations and (4.8) are just a particular case of a more general system of equations which describes interactions of species in a simple community Π, which is:

$$\frac{dN^i}{dt} = -G^i_{jk}N^jN^k + \lambda_{(i)}N^i + e^i, \quad i,j,k = 1,\ldots,n \tag{4.15}$$

where e^i means that (4.14) represents a *non-constant environment* and this quantity e^i is how the different species N^i react to external influences and $\lambda_{(i)}$ is already defined as the intrinsic growth rate of specie i.

For a Volterra-Hamilton system as (4.14) we have three possibilities:

- Ecological Interactions: if the coefficients G^i_{jk} are constants;
- Metabolic Interactions: if the coefficients $G^i_{jk} = G^i_{jk}(x^i)$ are explicit functions of coordinates x^i, only;
- Social Interactions: if the coefficients $G^i_{jk} = G^i_{jk}\left(x^i, \dfrac{N^i}{N^j}\right)$ are functions which may depend on x^i and $\dfrac{N^i}{N^j}$.

For our purpose it is convenient assume that (4.14) describes an ecological interaction and take $\lambda_{(i)} = \lambda$ for all $i \in \{1,\ldots,n\}$. This condition is called the *pre-symbiant condition* (ANTONELLI; INGARDEN; MATSUMOTO, 1993). The usual parameter of time t allows us to rewrite (4.14) as the differential equations:

$$\frac{d^2x^i}{dt^2} = -G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + \lambda\frac{dx^i}{dt} + e^i, \quad i,j,k = 1,\ldots,n. \tag{4.16}$$

The pre-symbiant condition allows us to consider other parameter, for exemplo, setting $s = e^{\lambda t}$ $(ds = \lambda e^{\lambda t}dt)$ defines an *intrinsic time sacle, longer than t*, which is related with the individual's own organism (LAIRD, 1965; LAIRD; BARTON; TYLER, 1968). So, at this intrinsic parameter we obtain by chain rule $\frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds}$ and

$$\begin{aligned}\frac{d^2x^i}{ds^2} &= \frac{d}{ds}\left(\frac{dx^i}{dt}\frac{1}{\lambda s}\right)\\ &= \frac{d}{ds}\left(\frac{dx^1}{dt}\right)\frac{1}{\lambda s} + \frac{dx^i}{dt}\left(-\frac{1}{\lambda s^2}\right)\\ &= \frac{d^2x^i}{dt^2}\frac{1}{\lambda^2 s^2} - \frac{dx^i}{dt}\left(\frac{1}{\lambda s^2}\right)\end{aligned}$$

Now, taking $e^i = 0$ representing a constant environment we have:

$$\begin{aligned}\frac{d^2x^i}{ds^2} &= \frac{d^2x^i}{dt^2}\frac{1}{\lambda^2 s^2} - \frac{dx^i}{dt}\frac{1}{\lambda s}\\ &= \left(-G^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + \lambda\frac{dx^i}{dt}\right)\frac{1}{\lambda^2 s^2} - \frac{dx^i}{dt}\frac{1}{\lambda s^2}\\ &= -G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} + \frac{dx^i}{dt}\frac{1}{\lambda s^2} - \frac{dx^i}{dt}\frac{1}{\lambda s^2}\\ &= -G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\end{aligned}$$

where we have take $k_{(i)} = 1$, otherwise, they enter the G^i_{jk} multiplicatively. The Volterra-Hamilton system takes the spray form as defined in Chapter 2:

$$\frac{d^2x^i}{ds^2} + G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{4.17}$$

or could be considered that G^i_{jk} is p-homogeneous of degree 0 to write $G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 2G^i$ by Euler theorem (2.1). Thus, as G^i_{jk} are constant in $(U, \phi = (x^i))$ we often just say (4.17) is a constant spray. For a two-dimensional constant sprays (ANTONELLI, 2003b), equations (4.17) have the following form:

$$\begin{aligned}0 &= \frac{d^2x^1}{ds^2} + \alpha_1\left(\frac{dx^1}{ds}\right)^2 + 2\alpha_2\left(\frac{dx^1}{ds}\right)\left(\frac{dx^2}{ds}\right) + \alpha_3\left(\frac{dx^2}{ds}\right)^2\\ 0 &= \frac{d^2x^2}{ds^2} + \beta_3\left(\frac{dx^1}{ds}\right)^2 + 2\beta_2\left(\frac{dx^1}{ds}\right)\left(\frac{dx^2}{ds}\right) + \beta_1\left(\frac{dx^2}{ds}\right)^2,\end{aligned} \tag{4.18}$$

for which the constants α'^s and β'^s play the role:

1. The case of competition: $\begin{cases}\alpha_1 > 0, \quad \alpha_3 = 0, \quad \alpha_2 > 0\\ \beta_1 > 0, \quad \beta_3 = 0, \quad \beta_2 > 0\end{cases}$

2. The case of parasitism: $\begin{cases}\alpha_1 > 0, \quad \alpha_3 = 0, \quad \alpha_2 < 0\\ \beta_1 > 0, \quad \beta_3 = 0, \quad \beta_2 > 0\end{cases}$

3. The case of mutualism: $\begin{cases} \alpha_1 > 0, & \alpha_3 = 0, \quad \alpha_2 < 0 \\ \beta_1 > 0, & \beta_3 = 0, \quad \beta_2 < 0 \end{cases}$

where the second case means that specie 1 is parasite on specie 2. We choose to evaluate (4.17) for $n = 2$ because that will be useful to us in the next chapter.

Remark 4.2. *Suppose we have a community with n species and each specie produces the amount x^i. Let $(x^1, \ldots, x^n) = (x)$, $\left(\frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}\right) = \left(\frac{dx}{dt}\right)$, be $2n$ coordinates in a open connected subset Ω of the Euclidian $(2n)$-dimensional space $\mathbb{R}^n \times \mathbb{R}^n$. If $F = F(x, dx)$ is a Finslerian function which quantify the cost of total production, then F is called cost functional, so, the pair $\mathbb{F}^n = (\Omega, F)$ is a Finsler space which geodesics are solution of Euler-Lagrange equations associated to $F(x, dx)$, and passing to the canonical parameter s, the geodesics of $\mathbb{F}$ are solution of*

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}\left(x, \frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

where γ^i_{jk} the the Christoffel symbols of the fundamental tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$.

The above remark will be useful to determine when the production of, for example, two species is optimal or not. If the Euler-Lagrange equations of the cost functional F coincide with the Volterra-Hamilton system in the spray form (4.17) we obtain that the interaction of the species minimize the cost of production. Otherwise, the interaction well not be a optimal production.

4.3 TWO DIMENSIONAL CONSTANT SPRAYS

Let $\mathbb{F}^2 = (M, F(x, y))$ be a two-dimensional Finsler space. We have showed that Volterra-Hamilton system in the canonical parameter has the spray form

$$\frac{d^2x^i}{ds^2} + G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{4.19}$$

and proceeding with the same trajectories variation as in the last chapter to define a new trajectory $\widetilde{\gamma}(t) = (\widetilde{x}^i(t))$, according to:

$$\widetilde{x}^i(t) = x^i(t) + \varepsilon\xi^i(t)$$

where ε denotes a scalar parameter value $\varepsilon > 0$, and $\xi^i(t)$ are components of a contravariant vector field along $\gamma(t)$ with conditions $\xi^i(a) = \xi^i(b) = 0$. Suppose that $\widetilde{\gamma}$ is also solution of the second order differential equations (4.19). Now, using $\widetilde{\gamma}$ with the semispray equations system we obtain:

$$\frac{d^2x^i}{dt^2}+\varepsilon\frac{d^2\xi^i}{dt^2}+2\widetilde{G}^i\left(\widetilde{x},\frac{d\widetilde{x}}{dt}\right)=\frac{d^2\widetilde{x}^i}{dt^2}+2\widetilde{G}^i\left(\widetilde{x},\frac{d\widetilde{x}}{dt}\right)=0.$$

Using the KCC-covariant differential (3.4) we can rewrite the above equation into its invariant form, that is, the Jacobi equation:

$$\frac{\mathbb{D}^2\xi^i}{ds^2}+\mathcal{B}^i_j\xi^j=0, \tag{4.20}$$

where

$$\mathcal{B}^i_j=2\frac{\partial G^i}{\partial x^j}-\frac{\partial G^i_j}{\partial x^r}y^r+2G^i_{jr}G^r-G^i_rG^r_j$$

is the second KCC-invariant. The others four KCC-invariants are given as before in chapter 2, with a remark that $\mathcal{E}^i\equiv 0$ because $2G^i=G^i_{jk}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}$ is p-homogeneous of degree 2 in $y^i=\dfrac{dx^i}{dx}$. But we pay attetion to the third KCC-invariant, defined by:

$$\begin{aligned}\mathcal{R}^i_{jk}&:=\frac{1}{3}\left(\frac{\partial\mathcal{B}^i_j}{\partial y^k}-\frac{\partial\mathcal{B}^i_k}{\partial y^j}\right)\\&=\frac{\partial G^i_j}{\partial x^k}-\frac{\partial G^i_k}{\partial x^j}+G^r_jG^i_{rk}-G^r_kG^i_{rj},\end{aligned}$$

because the geometer L. Berwald has introduced the Berwald's Gaussian curvature $\mathcal{K}$ for two-dimensional Finsler space and defined from his famous formula (BERWALD, 1941):

$$\mathcal{R}^i_{jk}=F\mathcal{K}m^i\left(l_jm_k-l_km_j\right) \tag{4.21}$$

where $l^i=y^i/F$ is the unit vector in the y^i direction, and m^i the unique (up to orientation) unit vector perpendicular to l^i. In such 2-dimensional Finsler space (l^i,m^i) is referred to as the *Berwald frame*. Lowering the index on m^i via the metric tensor gives m_i, which satisfies:

$$\begin{cases}F(x,m)=g_{ij}(x,y)m^im^j\equiv m_im^i=1;\\ g_{ij}(x,y)l^im^j=0.\end{cases} \tag{4.22}$$

The whole discussion about the construction of the vectors $l^i(x,y)$ and $m^i(x,y)$ can be found at (ANTONELLI; INGARDEN; MATSUMOTO, 1993). One can show that a Jacobi field V^i, solution of (4.20), along a geodesic $\gamma(s)$ may be expressed as

$$V^i=v(s)m^i.$$

Use of (4.21) and (4.22) give us

$$\mathcal{R}^i_{jk} y^j = F^2 \mathcal{K} m^i m_k. \tag{4.23}$$

Therefore, the equation (4.23) reduces the Jacobi equation $\frac{\mathbb{D}^2 V^i}{ds^2} + \mathcal{B}^i_j V^j = 0$ to

$$\frac{d^2 V}{ds^2} + \mathcal{K} V = 0.$$

In the chapter 2, the posposition (3.4) discuss about the instability of (4.19) in the sense of Jacobi instability using the eigenvalues of $\mathcal{B}^i_j$. A characterization of Jacobi instability was developed in terms of the Berwald's Gaussian curvature $\mathcal{K}$ as you can see this relation at (ANTONELLI; H.BRADBURY, 1996). According to definition (3.3) we have the following result:

- If $\mathcal{K}(x, y) > 0$ everywhere on $\widetilde{TM}$, then (4.19) is Jacobi-stable;
- If $\mathcal{K}(x, y) \leq 0$ everywhere on $\widetilde{TM}$, then (4.19) is Jacobi-unstable.

This notion of stability is a Lyapunov notion, but it is a whole trajectory concept.

5 APPLICATION

5.1 BLEACHING RECOVERY MODEL

Let $N^1(t)$, $N^2(t)$ and $N^3(t)$ be continuous functions of time which denote coral, symbiotic alga and commensal alga population density, respectively. We split this modelling in three stages: (I) Commensal + Symbiosis; (II) Symbiosis + Competition; (III) Symbiosis. Here, it is initially assumed the pre-symbiant condition $\lambda_{(N^1)} = \lambda_{(N^2)} = \lambda_{(N^3)} = \lambda$, where these constants have the same meaning as in (4.8). Following this pre-symbiont assumptions, we can describe these three stages of interactions between coral and algae.

Remark. The first stage describes how these three species live in the coral reef barrier before bleaching; Second stage is the dynamic produced by bleaching, and we will focus our attention at the competition between the algae while the coral stay in its stable steady state; in the last one, we suppose alga A^2 will develop a symbiotic relation with the coral which was invaded by the outside algae, creating the condition to stop bleaching and start the recovering process.

- Commensal + Symbiosis

At this stage we suppose water temperature is adequate for both species of Alga and to the Coral. First, note that algae A^2 lives outside the *Polyp* (commensal relation), then this interaction is beneficial only one to alga A^2. By the other hand, alga A^1 and coral have a symbiotic interaction. So, we can describe this relation by extended interactive equations (4.15) as follow:

$$\begin{cases} \dfrac{dN^1}{dt} = \lambda N^1 - \dfrac{\lambda (N^1)^2}{K_{(1)}} + \delta_{(1)} \dfrac{\lambda N N^2}{K_{(1)}} \\ \\ \dfrac{dN^2}{dt} = \lambda N^2 - \dfrac{\lambda (N^2)^2}{K_{(2)}} + \delta_{(2)} \dfrac{\lambda N^2 N^1}{K_{(2)}} \\ \\ \dfrac{dN^3}{dt} = \lambda N^3 - \dfrac{\lambda (N^3)^2}{K_{(3)}} + \delta_{(3)} \dfrac{\lambda N^3 N^1}{K_{(3)}} \end{cases} \tag{5.1}$$

where $\delta^{'s} > 0$ to describes the symbiosis. If N^1 was affected by N^3, there would be a $4th$ term in the $1st$ equation, but for now, we are supposing that N^1 and N^3 have no interaction.

- Symbiosis + Competition

We suppose that due to the weakening of the symbiont alga, an external secundary one is allowed in the coral. Here we assume that water warming is less lethal to N^3 than to N^2. The increasing water temperature produce a decreasing population density of algae N^2, since these species are not well adjusted to live in these conditions. This situation provides adequate conditions to algae N^3 penetrate the Polyp to establish the symbiotic relation that coral needs to live. In this case we have a dynamic where each specie interact to each other described as follow:

$$\begin{cases} \dfrac{dN^1}{dt} = \lambda N^1 - \dfrac{\lambda (N^1)^2}{K_{(1)}} + \tilde{\delta}_{(1)} \dfrac{\lambda N^1 (N^2 + N^3)}{K_{(1)}} \\ \\ \dfrac{dN^2}{dt} = \lambda N^2 - \dfrac{\lambda (N^2)^2}{K_{(2)}} + \delta_{(2)} \dfrac{\lambda N^2 N^1}{K_{(2)}} - \mu_{(2)} \dfrac{\lambda N^2 N^3}{K_{(2)}} \\ \\ \dfrac{dN^3}{dt} = \lambda N^3 - \dfrac{\lambda (N^3)^2}{K_{(3)}} + \delta_{(3)} \dfrac{\lambda N^3 N^1}{K_{(3)}} - \mu_{(3)} \dfrac{\lambda N^2 N^3}{K_{(3)}} \end{cases} \tag{5.2}$$

where $\mu_{(i)}$ $(i = 2, 3)$ are positive contants and $\mu_{(i)}$ is the impact that N^i suffers by interection with specie N^j, for $i, j \in \{2, 3\}$. We expect that competition between N^2 and N^3 is so strong that we can assume $\mu_{(2)}, \mu_{(3)} \gg \tilde{\delta}_{(1)}, \delta_{(2)}, \delta_{(3)}$. Therefore (5.2) becomes a classical Gause-Witt competition system:

$$\begin{cases} \dfrac{dN^2}{dt} = \lambda N^2 - \dfrac{\lambda (N^2)^2}{K_{(2)}} - \mu_{(2)} \dfrac{\lambda N^2 N^3}{K_{(2)}} \\ \\ \dfrac{dN^3}{dt} = \lambda N^3 - \dfrac{\lambda (N^3)^2}{K_{(3)}} - \mu_{(3)} \dfrac{\lambda N^2 N^3}{K_{(3)}}. \end{cases} \tag{5.3}$$

As we have supposed that warmer water is more lethal to N^2 than N^3, then $\mu_3 < \mu_2$ because this competition is harder N^2. Thus, by item 2 of theorem 1 we can conclude that N^2 is eliminated by the competition described in (5.3).

- Symbiosis

After elimination of N^1 by competition with N^3, the coral N^1 has a new alga population to establish a symbiotic relation and then stop bleaching. The situation before bleaching and after recovering is quite the same in the sense of system of equations as follow:

$$\begin{cases} \dfrac{dN^1}{dt} = \lambda N^1 - \dfrac{\lambda (N^1)^2}{K_{(1)}} + \delta_{(1)} \dfrac{\lambda N^1 N^2}{K_{(1)}} \\ \\ \dfrac{dN^3}{dt} = \lambda N^3 - \dfrac{\lambda (N^3)^2}{K_{(3)}} + \delta_{(3)} \dfrac{\lambda N^3 N^1}{K_{(3)}} \end{cases} \tag{5.4}$$

Remark. Equations (5.4) have the same form of the equations system to describe interaction of N^1 and N^2 in (5.1). This occurs because N^2 is supplanted by N^3.

5.2 PROPOSAL OF THE MODEL

Before bleaching disruption, it is known that coral and symbiotic alga develop a by-product as a result of their interaction. The same occurs after bleaching recovery since we are assuming alga N^3 becomes the symbiotic alga before the coral dies completely. Volterra-Hamilton is well suited to describe this production. For simplicity, we suppose all three populations have the same percapita rate of production (set $k_{(i)} = 1, i = 1, 2, 3$), and hence we present the first situation, before bleaching, with classical ecological interaction equations as follow:

$$\frac{dx^1}{dt} = N^1, \quad \frac{dN^1}{dt} = \lambda N^1 - \frac{\lambda (N^1)^2}{K_{(1)}} + \delta_{(1)} \frac{\lambda N^1 N^2}{K_{(1)}} \tag{5.5}$$

to describe how coral produce and interact with the symbiotic algae. Likewise, the production of algae represented by N^2 is given by equations:

$$\frac{dx^2}{dt} = N^2, \quad \frac{dN^2}{dt} = \lambda N^2 - \frac{\lambda \left(N^2\right)^2}{K_{(2)}} + \delta_{(2)} \frac{\lambda N^2 N^1}{K_{(2)}} \tag{5.6}$$

where the quantities x^1 and x^2 are the Volterra production variable corresponding to each specie.

Let (x^1, x^2), (N^1, N^2) be 4 coordinates in a open connected subset Ω of the Euclidian 4-dimensional space $\mathbb{R}^2 \times \mathbb{R}^2$. Let us define a functional F on $\widetilde{T\Omega}$, the tangent bundle with deleted origin by:

$$F(x, dx) = F\left(x^1, x^2, N^1, N^2\right) = e^{\psi(x^1, x^2)} \frac{\left(N^2\right)^{1+(1/\lambda)}}{(N^1)^{1/\lambda}} \tag{5.7}$$

where ψ is of the form $\psi\left(x^1, x^2\right) = Ax^1 + Bx^2$, with:

$$\begin{aligned} A &= -\left(\frac{\lambda \delta_{(2)}}{K_{(2)}} + \frac{K_{(2)} + \delta_{(2)} K_{(1)}}{K_{(1)} K_{(2)}}\right); \\ B &= \left(\frac{-\lambda \delta_{(1)}}{K_{(1)}} + \frac{(1+\lambda)\left(K_{(1)} + \delta_{(1)} K_{(2)}\right)}{K_{(1)} K_{(2)}}\right); \end{aligned} \tag{5.8}$$

For this functional F let us verify the conditions in definition (2.5).

Claim 2. *The pair* $\mathbb{F}^2 = (\Omega, F)$ *is a Finsler space.*

Proof. Let $(x, N) \in \Omega$, with $x = (x^1, x^2)$ and $N = (N^1, N^2)$. As $\psi(x)$ is clearly smooth for each x, and hence $e^{\psi(x)}$ is also smooth. The smoothness of $F(x, dx)$ on $\widetilde{T\Omega}$ is a simple check that $F(x, dx) = e^{\psi(x)} P(N^2)/Q(N^1)$, with P and Q polynomial functions. As $N^1 > 0$ on $\widetilde{T\Omega}$, we have $F\left(x^1, x^1, N^1, N^1\right) = e^{\psi\left(x^1,x^1\right)} \dfrac{\left(N^1\right)^{1+(1/\lambda)}}{(N^1)^{\frac{1}{\lambda}}} = e^{\psi(x^1,x^1)} N^1 > 0$. This shows that F is positive definite on $\widetilde{T\Omega}$.

Let $r \in (0, +\infty)$. Note that

$$
\begin{aligned}
F\left(x^1, x^2, rN^1, rN^2\right) &= e^{\psi\left(x^1,x^2\right)} \frac{\left(rN^2\right)^{1+1/\lambda}}{\left(rN^1\right)^{1/\lambda}} \\
&= e^{\psi\left(x^1,x^2\right)} \frac{\left(r^{1+1/\lambda}\right)\left(N^2\right)^{1+1/\lambda}}{r^{1/\lambda} \cdot \left(N^1\right)^{1/\lambda}} \\
&= r e^{\psi\left(x^1,x^2\right)} \frac{\left(N^2\right)^{1+(1/\lambda)}}{\left(N^1\right)^{1/\lambda}} \\
&= rF(x^1, x^2, N^1, N^2),
\end{aligned}
$$

which implies F is p-homogeneous of degree 1 in N. $\square$

The equations (5.5) and (5.6) are equivalent to a second order differential equations. Passing $x^1(t)$ and $x^2(t)$ to the natural parameter $ds = \lambda e^{\lambda t} dt$, these equations become to be a constant spray as in (4.18). This can be easily checked by the simple computation:

$$
\begin{aligned}
\frac{d^2x^1}{ds^2} &= \frac{d}{ds}\left(N^1 \frac{1}{\lambda s}\right) \\
&= \frac{d^2x^1}{dt^2}\frac{1}{\lambda^2 s^2} - \frac{dx^1}{dt}\frac{1}{\lambda s^2} \\
&= -\frac{\lambda}{K_{(1)}}\left(\frac{dx^1}{ds}\right)^2 + \frac{\delta_{(1)}\lambda}{K_{(1)}}\left(\frac{dx^1}{ds}\right)\left(\frac{dx^2}{ds}\right),
\end{aligned}
$$

and by symmetry, the constant spray is the following.

$$
\begin{cases}
\dfrac{d^2x^1}{ds^2} + \dfrac{\lambda}{K_{(1)}}\left(\dfrac{dx^1}{ds}\right)^2 - \dfrac{\lambda\delta_{(1)}}{K_{(1)}}\left(\dfrac{dx^1}{ds}\right)\left(\dfrac{dx^2}{ds}\right) = 0 \\
\dfrac{d^2x^2}{ds^2} + \dfrac{\lambda}{K_{(2)}}\left(\dfrac{dx^2}{ds}\right)^2 - \dfrac{\lambda\delta_{(2)}}{K_{(2)}}\left(\dfrac{dx^2}{ds}\right)\left(\dfrac{dx^1}{ds}\right) = 0,
\end{cases}
\tag{5.9}
$$

with $\alpha_1 = \dfrac{\lambda}{K_{(1)}}$, $\beta_1 = \dfrac{\lambda}{K_{(2)}}$, $2\alpha_2 = \dfrac{\lambda\delta_{(1)}}{K_{(1)}}$, $2\beta_2 = \dfrac{\lambda\delta_{(2)}}{K_{(2)}}$ and $\alpha_3 = 0 = \beta_3$.

Theorem 5.1. *Production given by* (5.5) *and* (5.6) *preserves* $F(x, dx)$ *along any solution* γ.

Proof. By the chain rule the derivative of F with respect to t is:

$$\begin{aligned}\frac{dF}{dt} &= \frac{\partial F}{\partial x^1}N^1 + \frac{\partial F}{\partial x^2}N^2 + \frac{\partial F}{\partial N^1}\cdot\frac{dN^1}{dt} + \frac{\partial F}{\partial N^2}\frac{dN^2}{dt}\\ &= F\left(AN^1 + BN^2 - \frac{1}{\lambda N^1}\frac{dN^1}{dt} + \frac{1}{\lambda N^2}\frac{dN^2}{dt} + \frac{1}{N^2}\frac{dN^2}{dt}\right)\\ &= N^1\left(A + \frac{1}{K_{(1)}} + \frac{\delta_{(1)}}{K_{(2)}} + \frac{\lambda\delta_{(2)}}{K_{(2)}}\right) + N^2\left(B - \frac{\delta_{(1)}}{K_{(1)}} - \frac{1}{K_{(2)}} - \frac{\lambda}{K_{(2)}}\right),\end{aligned}$$

where constants (5.8) ensures that $dF/dt = 0$. This is the necessary condition to implies that production given by (5.5) and (5.6) preserves $F(x, dx)$ along any solution γ of these equations systems. □

Under suitable conditions, the Finsler space $\mathbb{F}^2 = (\Omega, F)$ is called to be a Wagner space and there are new results beyond the Finsler case. The paper (ANTONELLI, 2003b) is one of the most complete work about Wagner theory in two-dimension with constant sprays and there one can find a more elegant proof of the above theorem and much more about it. In this theory, the equations (5.9) are called the Wagner autoparallels and these are almost never the equations of the geodesics of the metric function $F(x, dx)$.

We will focus our attention to the constant spray (5.9) which describes the symbiotic relation of production dynamics of algae and coral at barrier coral reef before bleaching. For this one spray we have

$$\begin{aligned}2G^1 &= \frac{\lambda}{K_{(1)}}\left(\frac{dx^1}{ds}\right)^2 - \frac{\lambda\delta_{(1)}}{K_{(1)}}\left(\frac{dx^1}{ds}\right)\left(\frac{dx^2}{ds}\right),\\ 2G^2 &= \frac{\lambda}{K_{(2)}}\left(\frac{dx^2}{ds}\right)^2 - \frac{\lambda\delta_{(2)}}{K_{(2)}}\left(\frac{dx^2}{ds}\right)\left(\frac{dx^1}{ds}\right);\end{aligned} \tag{5.10}$$

then, the coefficients of spray (5.9) are:

$$\begin{aligned}G^1_{11} &= \frac{\lambda}{K_{(1)}}, \quad G^1_{12} = -\frac{\lambda\delta_{(1)}}{K_{(1)}} = G^1_{21}\\ G^2_{11} &= \frac{\lambda}{K_{(2)}}, \quad G^2_{12} = -\frac{\lambda\delta_{(2)}}{K_{(2)}} = G^2_{21}.\end{aligned}$$

Now, we will use these coefficients to compute those five KCC-invariants mentioned in theorem (3.3) to obtain information about the dynamics of the barrier coral reef and how it will be related to dynamics after recovery bleaching. This computation will be evaluate with (MAPLE,) where will make use of (RUTZ, 2001). This is a great tool which has been showed a very important useful in computing KCC-invariants and others Finsler geometrical calculations. Because of this, we will consider (5.10) as in the general semispray (3.2) to obtain the required.

$libname := `C:/Finsler`, libname;$

"C:/Finsler", "C:/Finsler", "C:/Finsler", "C:\Program Files\Maple 17\lib", "." **(1)**

$with(Finsler);$

$$[Dcoordinates, Hdiff, K, connection, init, metricfunction, tddiff] \quad \textbf{(2)}$$

$dimension := 2;$

$$2 \quad \textbf{(3)}$$

> $coordinates(x1, x2);$

The coordinates are:

$$X^{1} = x1$$
$$X^{2} = x2 \quad \textbf{(4)}$$

> $Dcoordinates(N1, N2);$

The d-coordinates are:

$$Y^{1} = N1$$
$$Y^{2} = N2 \quad \textbf{(5)}$$

> $G1 := \frac{1}{2} \frac{\text{lambda}}{K1} N1 \cdot N1 - \frac{1}{2} \frac{\text{lambda} \cdot delta1}{K1} N1 \cdot N2;$

$$G1 := \frac{1}{2} \frac{\lambda N1^2}{K1} - \frac{1}{2} \frac{\lambda \delta1 \, N1 \, N2}{K1} \quad \textbf{(6)}$$

> $G2 := \frac{1}{2} \frac{\text{lambda}}{K2} N2 \cdot N2 - \frac{1}{2} \frac{\text{lambda} \cdot delta2}{K2} N1 \cdot N2;$

$$G2 := \frac{1}{2} \frac{\lambda N2^2}{K2} - \frac{1}{2} \frac{\lambda \delta2 \, N1 \, N2}{K2} \quad \textbf{(7)}$$

> $connection(G1, G2);$

$$G^{x1} = \frac{1}{2} \frac{\lambda N1^2}{K1} - \frac{1}{2} \frac{\lambda \delta1 \, N1 \, N2}{K1}$$
$$G^{x2} = \frac{1}{2} \frac{\lambda N2^2}{K2} - \frac{1}{2} \frac{\lambda \delta2 \, N1 \, N2}{K2} \quad \textbf{(8)}$$

> $show(B[i,-j]);$

$$B^{x1}{}_{x1} = \frac{1}{4} \frac{\lambda^2 \delta1 \, N2 \, (K1 \, N1 \, \delta2 - K2 \, N2 \, \delta1 - 2 \, K1 \, N2)}{K1^2 \, K2}$$
$$B^{x1}{}_{x2} = -\frac{1}{4} \frac{\lambda^2 \delta1 \, N1 \, (K1 \, N1 \, \delta2 - K2 \, N2 \, \delta1 - 2 \, K1 \, N2)}{K1^2 \, K2}$$
$$B^{x2}{}_{x1} = \frac{1}{4} \frac{\lambda^2 \delta2 \, N2 \, (K1 \, N1 \, \delta2 - K2 \, N2 \, \delta1 + 2 \, K2 \, N1)}{K2^2 \, K1}$$
$$B^{x2}{}_{x2} = -\frac{1}{4} \frac{\lambda^2 \delta2 \, N1 \, (K1 \, N1 \, \delta2 - K2 \, N2 \, \delta1 + 2 \, K2 \, N1)}{K2^2 \, K1} \quad \textbf{(9)}$$

> $with(linalg);$

> $HH := convert(B[i,-j], matrix);$

$$HH := \left[\left[\frac{1}{4}\frac{\lambda^2\,\delta 1\,N2\,(K1\,N1\,\delta 2 - K2\,N2\,\delta 1 - 2\,K1\,N2)}{K1^2\,K2}, -\frac{1}{4}\frac{\lambda^2\,\delta 1\,N1\,(K1\,N1\,\delta 2 - K2\,N2\,\delta 1 - 2\,K1\,N2)}{K1^2\,K2}\right], \left[\frac{1}{4}\frac{\lambda^2\,\delta 2\,N2\,(K1\,N1\,\delta 2 - K2\,N2\,\delta 1 + 2\,K2\,N1)}{K2^2\,K1}, -\frac{1}{4}\frac{\lambda^2\,\delta 2\,N1\,(K1\,N1\,\delta 2 - K2\,N2\,\delta 1 + 2\,K2\,N1)}{K2^2\,K1}\right]\right] \quad \textbf{(10)}$$

> *eigenvalues*(*HH*);

$$0, -\frac{1}{4}\frac{1}{K1^2\,K2^2}\left(\lambda^2\left(K1^2\,N1^2\,\delta 2^2 - 2\,K1\,K2\,N1\,N2\,\delta 1\,\delta 2 + K2^2\,N2^2\,\delta 1^2 + 2\,K1\,K2\,N1^2\,\delta 2 + 2\,K1\,K2\,N2^2\,\delta 1\right)\right) \quad \textbf{(11)}$$

> *show*(*NLR*[*i*,-*j*,-*k*]);

$$NLR^{x1}{}_{x1\,x2} = \frac{1}{4}\frac{\lambda^2\,\delta 1\,(K1\,N1\,\delta 2 - K2\,N2\,\delta 1 - 2\,K1\,N2)}{K1^2\,K2}$$

$$NLR^{x2}{}_{x1\,x2} = \frac{1}{4}\frac{\lambda^2\,\delta 2\,(K1\,N1\,\delta 2 - K2\,N2\,\delta 1 + 2\,K2\,N1)}{K2^2\,K1} \quad \textbf{(12)}$$

> *show*(*B*[*i*,-*j*,-*k*,-*l*]);

$$B^{x1}{}_{x1\,x1\,x2} = \frac{1}{4}\frac{\lambda^2\,\delta 1\,\delta 2}{K1\,K2}$$

$$B^{x1}{}_{x2\,x1\,x2} = -\frac{1}{4}\frac{\lambda^2\,\delta 1\,(K2\,\delta 1 + 2\,K1)}{K1^2\,K2}$$

$$B^{x2}{}_{x1\,x1\,x2} = \frac{1}{4}\frac{\lambda^2\,\delta 2\,(K1\,\delta 2 + 2\,K2)}{K2^2\,K1}$$

$$B^{x2}{}_{x2\,x1\,x2} = -\frac{1}{4}\frac{\lambda^2\,\delta 1\,\delta 2}{K1\,K2} \quad \textbf{(13)}$$

> *show*(*G*[*i*,-*j*,-*k*,-*l*]);

$$G^{i}{}_{j\,k\,l} = 0 \quad \textbf{(14)}$$

> *definetensor*(epsilon[*i*] = *N*[*i*,-*j*]·*Y*[*j*] − 2 *G*[*i*]);

$$\varepsilon^{i} = N^{i}{}_{j}\,Y^{j} - 2\,G^{i} \quad \textbf{(15)}$$

> *evalt*(epsilon[*i*]);

$$\varepsilon^{i} = 0 \quad \textbf{(16)}$$

>

Note that the matrix associated to the deviation tensor $\mathcal{B}^i_j$ is as follow:

$$(\mathcal{B}^i_j) = \begin{pmatrix} \frac{\lambda^2\delta_{(1)}N^2}{K_{(1)}}\mathcal{B}^1 & -\frac{\lambda^2\delta_{(1)}N^1}{K_{(1)}}\mathcal{B}^1 \\ \frac{\lambda^2\delta_{(2)}N^2}{K_{(2)}}\mathcal{B}^2 & -\frac{\lambda^2\delta_{(2)}N^1}{K_{(2)}}\mathcal{B}^2 \end{pmatrix}$$

where the constants $\mathcal{B}'^s$ are:

$$\mathcal{B}^1 = \frac{1}{4}\frac{K_{(1)}\delta_{(2)}N^1 - K_{(2)}\delta_{(1)}N^2 + 2K_{(1)}N^2}{K_{(1)}K_{(2)}},$$
$$\mathcal{B}^2 = \frac{1}{4}\frac{K_{(1)}\delta_{(2)}N^1 - K_{(2)}\delta_{(1)}N^2 - 2K_{(2)}N^1}{K_{(1)}K_{(2)}}.$$

It is known that the characteristic polynomial of a 2×2 matrix as $(\mathcal{B}^i_j)$ is given by the quadratic polynomial $p(r) = r^2 - \mathrm{tr}[(\mathcal{B}^i_j)]r + \det[(\mathcal{B}^i_j)]$. The above computation by maple shows that the eigenvalues of $\mathcal{B}^i_j$ have non-negative real part. One can check that the system (5.5) and (5.6) has a positive equilibrium stead-point (N^1_*, N^2_*) given by:

$$N^1_* = \frac{K_{(1)} - \delta_{(1)}K_{(2)}}{1 - \delta_{(1)}\delta_{(2)}}, \quad N^2_* = \frac{K_{(2)} - \delta_{(2)}K_{(1)}}{1 - \delta_{(2)}\delta_{(1)}},$$

that is, this symbiotic relation has stability in the sense of population dynamics, however the second KCC-invariant shows that trajectories of (5.9) are Jacobi unstable. This instability means that the production relation between the coral and algae is unstable by proposition (3.4). Besides the trajectories of spray (5.9) are not stable they are not the Euler-Lagrange of functional F. Although one can prove that F is conserved along the Wagner autoparallels, this model proposed to describes the symbiotic relation between the coral and algae is an easy approach of the one of the most important production relation in a barrier coral reef. For this reason we will consider a new Finsler functional that generates equations which are not just ecological interactions.

Consider $\tilde{F}(x, dx) = F$, with $\psi(x) = Ax^1 + Bx^2 + \nu_3 x^1 x^2$. The Euler-Lagrange for $\widetilde{F}$ were once obtained in (MAPLE, ; RUTZ, 2001), and they are:

$$\begin{aligned} \frac{d^2x^1}{ds^2} + \lambda(A - \nu_3 x^2)\left(\frac{dx^1}{ds}\right)^2 &= 0, \\ \frac{d^2x^2}{ds^2} + \lambda\left(B + \frac{\nu_3}{\lambda+1}x^1\right)\left(\frac{dx^2}{ds}\right)^2 &= 0; \end{aligned} \tag{5.11}$$

which is clearly different from (5.9). This tell us that the symbiotic relation between algae and coral before the bleaching is not an optimal production.

Note that, if $\nu_3 = 0$, then the original two-dimensional logistic system (4.3) with production are obtained. So, non-vanishes of the parameter ν_3 (called the exchange parameter) is responsible to (5.11) describes a metabolic interaction. These equations have the spray form

$$\frac{d^2x^i}{ds^2} + G^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

with $G^i_{jk} = G^i_{jk}(x)$ being explicit functions of coordinates x^i, only. These coefficients of spray are expressed in (5.11) by:

$$G^1_{11} = -\lambda\nu_3 x^2, \quad G^1_{12} = 0 = G^1_{21},$$
$$G^2_{11} = \frac{\lambda\nu_3}{\lambda+1}x^1, \quad G^2_{12} = 0 = G^2_{21}.$$

Moreover, Lyapunov stability (4.21) of this sytem is completely determined by the sign of the curvature:

$$\mathcal{K} = \frac{\lambda^2}{\lambda+1}\cdot\nu_3\cdot\left(\frac{y^1}{y^2}\right)^{1+2\lambda}\exp\left\{-2\left[-Ax^1 + Bx^2 + \nu_3 x^1 x^2\right]\right\}, \qquad y^i = \frac{dx^i}{ds}. \tag{5.12}$$

The exchange parameter, ν_3, is responsible to intensify the production dynamics between these two species and determine the stability of (5.11) by the sign of ν_3. It is natural take $\nu_3 > 0$ since we are interested in a new modelling with Jacobi stability.

This system is conservative relative to the total production parameter s, along its solution in that $d\widetilde{F}/ds = 0$. In addition, system (5.11) represents an optimal production dynamics and its trajectories are Jacobi stable since the Gaussian curvature (4.21) $\mathcal{K} > 0$ for $\nu_3 > 0$. There are some production models where Volterra-Hamilton system (4.14) are not Euler-Lagrange for any functional F, and hence describes a non optimal production interaction. For example, our first approach was a simple symbiosis which is not the geodesics of any Finsler space, so, this interaction has not a satisfactory production.

This idea of considering a production cost functional $\widetilde{F}$ is commonly used in our applications when we are interested in studying optimally production relation of two species. There is a good reason for that; notice that before we treat the properties of (5.11) we were working on non-optimally productive and an unstable system. The same occur in (ANTONELLI P. L.; RUTZ, 2003). The metabolic characteristic in the interaction has a property about how intense is the production dynamic between the species.

We present the maple calculus to obtain the geodesic spray $5.11)$ generated by the Euler-Lagrange equations of $\widetilde{F}(x, dx)$ as follow:

```
> libname := `C:/Finsler`, libname;
```
libname := "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:/Finsler", "C:\Program Files\Maple 17\lib", "." **(1)**

```
> with(Finsler);
```
[Dcoordinates, Hdiff, K, connection, init, metricfunction, tddiff] **(2)**

```
> dimension := 2;
```
$$dimension := 2 \quad (3)$$

```
> coordinates(x1, x2);
```
The coordinates are:

$$X^{1} = x1$$
$$X^{2} = x2 \quad (4)$$

```
> Dcoordinates(y1, y2);
```
The d-coordinates are:

$$Y^{1} = y1$$
$$Y^{2} = y2 \quad (5)$$

$$> F := \exp((-a\cdot x1 + b\cdot x2 + c\cdot x1\cdot x2))\cdot\left(\frac{(y2)^{\left(1+\frac{1}{\mathrm{lambda}}\right)}}{(y1)^{\frac{1}{\mathrm{lambda}}}}\right);$$

$$F := \frac{e^{c\,x1\,x2 - a\,x1 + b\,x2}\, y2^{1+\frac{1}{\lambda}}}{y1^{\frac{1}{\lambda}}} \quad (6)$$

```
>
metricfunction(F^2);
```
The components of the metric are:

$$g_{x1\,x1} = \frac{\left(e^{c\,x1\,x2 - a\,x1 + b\,x2}\right)^2 \left(y2^{\frac{\lambda+1}{\lambda}}\right)^2 (\lambda+2)}{\left(y1^{\frac{1}{\lambda}}\right)^2 \lambda^2\, y1^2}$$

$$g_{x1\,x2} = -\frac{2\left(e^{c\,x1\,x2 - a\,x1 + b\,x2}\right)^2 \left(y2^{\frac{\lambda+1}{\lambda}}\right)^2 (\lambda+1)}{\left(y1^{\frac{1}{\lambda}}\right)^2 \lambda^2\, y1\, y2}$$

$$g_{x2\,x2} = \frac{\left(e^{c\,x1\,x2 - a\,x1 + b\,x2}\right)^2 \left(y2^{\frac{\lambda+1}{\lambda}}\right)^2 (\lambda+1)(\lambda+2)}{\left(y1^{\frac{1}{\lambda}}\right)^2 \lambda^2\, y2^2} \quad (7)$$

```
show(G[i]);
```

$$G^{x1} = \frac{1}{2}\, y1^2\, \lambda\, (-c\, x2 + a)$$

$$G^{x2} = \frac{1}{2}\, \frac{y2^2\, \lambda\, (c\, x1 + b)}{\lambda + 1} \qquad \textbf{(8)}$$

$K := K(a, b);$

$$\frac{y2\, y1 \left(y1^{\frac{1}{\lambda}}\right)^2 \lambda^2\, c}{\left(e^{c\, x1\, x2 - a\, x1 + b\, x2}\right)^2 \left(y2^{\frac{\lambda+1}{\lambda}}\right)^2 (\lambda + 1)} \qquad \textbf{(9)}$$

6 CONCLUSION

The reader can notice that equations (5.5) and (5.6) have a clear appearance with

$$\begin{aligned} \frac{dx^1}{dt} &= N^1, \quad \frac{dN^1}{dt} = \lambda N^1 - \frac{\lambda\,(N^1)^2}{K_{(1)}} + \delta_{(1)}\frac{\lambda N^1 N^3}{K_{(1)}}; \\ \frac{dx^3}{dt} &= N^3, \quad \frac{dN^3}{dt} = \lambda N^3 - \frac{\lambda\,(N^3)^2}{K_{(3)}} + \delta_{(3)}\frac{\lambda N^3 N^1}{K_{(3)}}, \end{aligned} \tag{6.1}$$

even though they describes different moments in the barrier coral reef. This happens because N^3 occupies N^2 niche. The first one (5.5) and (5.6) accords to the symbiotic relation before coral reef starts bleaching. The second one is the symbiotic relation after bleaching recovery of coral reef. The resemblance between these equations may be justified by the classical models we choose to start the modeling and because both represent a symbiotic relation. Converting t into the natural parameter $s = e^{\lambda t}$, the equations (6.1) becomes:

$$\begin{cases} \dfrac{d^2x^1}{ds^2} + \dfrac{\lambda}{K_{(1)}}\left(\dfrac{dx^1}{ds}\right)^2 - \dfrac{\lambda\delta_{(1)}}{K_{(1)}}\left(\dfrac{dx^1}{ds}\right)\left(\dfrac{dx^3}{ds}\right) = 0, \\ \dfrac{d^2x^3}{ds^2} + \dfrac{\lambda}{K_{(3)}}\left(\dfrac{dx^3}{ds}\right)^2 - \dfrac{\lambda\delta_{(3)}}{K_{(3)}}\left(\dfrac{dx^1}{ds}\right)\left(\dfrac{dx^3}{ds}\right) = 0, \end{cases} \tag{6.2}$$

which is also an ecological interaction.

According to theorem (3.3) we have should compute the five KCC-invariants of $(6.1$ to verify the equivalence between the two systems, however there is no why efforts to compute the KCC-invariants of (6.2) because it has the same form as (5.9) and consequently the KCC-invariants of (6.2) can be obtained from the maple calculus (5.2) with a simple replacement of $K_{(2)}, \delta_{(2)}$ and N^2 by $K_{(3)}, \delta_{(3)}$ and N^3, respectively. Thus, the KCC-invariants of (5.5) and (6.1) are equivalent relative to (3.3). According to the main theorem of KCC-theory (3.3) we have that the production dynamics given before bleaching is "equal" to production dynamics after bleaching recovery. Assuming that the new alga replaces the original one in the same ecological niche to be the new symbiotic alga, we interpret this event as representing an adaptation process, as oppose to an evolutionary one, where the cost of production is supposed to diminish, leading to a more efficient interaction pattern. If we consider the same Finsler space $\mathbb{F}^2 = (\Omega, F)$ of claim (2), with $F = F(x^1, x^3, N^1, N^3)$ as in (5.7) and (5.8) after change x^2 by x^3 and N^2 by N^3. Similarly, one can prove that $dF/ds = 0$ along any solution of (6.2) and furthermore eigenvalues of the second KCC-invariant of the above spray has non-negative real part. So, the new equations have also non stable trajectories in the Jacobi's sense. This was expected from the production relation before coral bleaching and after bleaching recovery because the production of the simple symbiotic approach in (5.4) is not too significant as production of other species. To solve this problem we consider the Finsler functional $\widetilde{F} = F(x^1, x^3, N^1, N^3)$, with $\psi(x^1, x^3) = \widetilde{A}x^1 + \widetilde{B}x^3 + \tilde{\nu_3}x^1x^3$. The geodesics for $\widetilde{F}$ are given by:

$$\begin{aligned} \frac{d^2x^1}{ds^2} + \lambda(\widetilde{A} + \tilde{\nu_3}x^2)\left(\frac{dx^1}{ds}\right)^2 = 0, \\ \frac{d^2x^3}{ds^2} + \lambda\left(\widetilde{B} + \frac{\tilde{\nu_3}}{\lambda+1}x^1\right)\left(\frac{dx^3}{ds}\right)^2 = 0, \end{aligned} \tag{6.3}$$

where $\tilde{\nu_3}$ is the same exchange parameter, responsible for the metabolic interaction between species $\#1$ and $\#2$. This metabolic interaction is completely determined by the Gaussian curvature given by:

$$\mathcal{K} = \frac{\lambda^2}{\lambda+1} \cdot \tilde{\nu_3} \cdot \left(\frac{y^1}{y^3}\right)^{1+2\lambda} \exp\left\{-2\left[-Ax^1 + Bx^3 + \tilde{\nu_3}x^1x^3\right]\right\}, \quad y^i = \frac{dx^i}{ds}$$

which is positive, because $\tilde{\nu_3}$ is supposed to be positive. The correspondent description of these subjects such as sprays and Gaussian curvature is already detailed in the section (4.3). By a similar argument applied to the system describing production dynamics before coral bleaching, one can see that the metabolic interaction (6.3) has the appropriate properties of the production dynamics between the coral and algae. Calculation ensuring this result can be found on appendix A.

A perspective of new researches from this work arise by the insertion of an external force in the environment. If we consider again (5.5) and (5.6), but now with $e^i = -(\delta^i_j\sigma_k(x))N^jN^k$, and $\sigma_k(x)$ being a smooth covariant vector field on production space there is a new dynamic to investigate. That is, the new equations are Volterra-Hamilton system (4.14) representing a *non-constant environments*.

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APPENDIX A – MAPLE CALCULUS

```
libname := `C:/Finsler`, libname;
```

"C:/Finsler", "C:/Finsler", "C:\Program Files\Maple 17\lib", "." **(1)**

```
with(Finsler);
```

[*Dcoordinates, Hdiff, K, connection, init, metricfunction, tddiff*] **(2)**

```
dimension := 2;
```

$$2 \quad (3)$$

```
> coordinates(x1, x3);
```

The coordinates are:

$$X^{1} = x1$$

$$X^{2} = x3 \quad (4)$$

```
> Dcoordinates(N1, N3);
```

The d-coordinates are:

$$Y^{1} = N1$$

$$Y^{2} = N3 \quad (5)$$

$$> G1 := \frac{1}{2}\frac{\text{lambda}}{K1} N1 \cdot N1 - \frac{1}{2}\frac{\text{lambda} \cdot delta1}{K1} N1 \cdot N3;$$

$$G1 := \frac{1}{2}\frac{\lambda N1^2}{K1} - \frac{1}{2}\frac{\lambda\,\delta 1\, N1\, N3}{K1} \quad (6)$$

$$> G3 := \frac{1}{2}\frac{\text{lambda}}{K3} N3 \cdot N3 - \frac{1}{2}\frac{\text{lambda} \cdot delta3}{K3} N1 \cdot N3;$$

$$G3 := \frac{1}{2}\frac{\lambda N3^2}{K3} - \frac{1}{2}\frac{\lambda\,\delta 3\, N1\, N3}{K3} \quad (7)$$

```
> connection(G1, G3);
```

$$G^{x1} = \frac{1}{2}\frac{\lambda N1^2}{K1} - \frac{1}{2}\frac{\lambda\,\delta 1\, N1\, N3}{K1}$$

$$G^{x3} = \frac{1}{2}\frac{\lambda N3^2}{K3} - \frac{1}{2}\frac{\lambda\,\delta 3\, N1\, N3}{K3} \quad (8)$$

```
> show(B[i,-j]);
```

$$B^{x1}{}_{x1} = \frac{1}{4}\frac{\lambda^2\,\delta 1\, N3\,(K1\, N1\,\delta 3 - K3\, N3\,\delta 1 - 2\, K1\, N3)}{K1^2\, K3}$$

$$B^{x1}{}_{x3} = -\frac{1}{4}\frac{\lambda^2\,\delta 1\, N1\,(K1\, N1\,\delta 3 - K3\, N3\,\delta 1 - 2\, K1\, N3)}{K1^2\, K3}$$

$$B^{x3}{}_{x1} = \frac{1}{4}\frac{\lambda^2\,\delta 3\, N3\,(K1\, N1\,\delta 3 - K3\, N3\,\delta 1 + 2\, K3\, N1)}{K3^2\, K1}$$

$$B^{x3}{}_{x3} = -\frac{1}{4}\frac{\lambda^2\,\delta 3\, N1\,(K1\, N1\,\delta 3 - K3\, N3\,\delta 1 + 2\, K3\, N1)}{K3^2\, K1} \quad (9)$$

```
> with(linalg);
```

[*BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat,* **(10)** *charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian*]

```
> HH := convert(B[i,-j], matrix);
```

$$HH := \left[\left[\frac{1}{4}\,\frac{\lambda^2\,\delta 1\,N3\,(K1\,N1\,\delta 3 - K3\,N3\,\delta 1 - 2\,K1\,N3)}{K1^2\,K3}, -\frac{1}{4}\,\frac{\lambda^2\,\delta 1\,N1\,(K1\,N1\,\delta 3 - K3\,N3\,\delta 1 - 2\,K1\,N3)}{K1^2\,K3}\right], \left[\frac{1}{4}\,\frac{\lambda^2\,\delta 3\,N3\,(K1\,N1\,\delta 3 - K3\,N3\,\delta 1 + 2\,K3\,N1)}{K3^2\,K1}, -\frac{1}{4}\,\frac{\lambda^2\,\delta 3\,N1\,(K1\,N1\,\delta 3 - K3\,N3\,\delta 1 + 2\,K3\,N1)}{K3^2\,K1}\right]\right] \quad \textbf{(11)}$$

```
> eigenvalues(HH);
```

$$0, -\frac{1}{4}\,\frac{1}{K1^2\,K3^2}\left(\lambda^2\left(K1^2\,N1^2\,\delta 3^2 - 2\,K1\,K3\,N1\,N3\,\delta 1\,\delta 3 + K3^2\,N3^2\,\delta 1^2 + 2\,K1\,K3\,N1^2\,\delta 3 + 2\,K1\,K3\,N3^2\,\delta 1\right)\right) \quad \textbf{(12)}$$

```
> show(NLR[i,-j,-k]);
```

$$NLR^{x1}{}_{x1\,x3} = \frac{1}{4}\,\frac{\lambda^2\,\delta 1\,(K1\,N1\,\delta 3 - K3\,N3\,\delta 1 - 2\,K1\,N3)}{K1^2\,K3}$$

$$NLR^{x3}{}_{x1\,x3} = \frac{1}{4}\,\frac{\lambda^2\,\delta 3\,(K1\,N1\,\delta 3 - K3\,N3\,\delta 1 + 2\,K3\,N1)}{K3^2\,K1} \quad \textbf{(13)}$$

```
> show(B[i,-j,-k,-l]);
```

$$B^{x1}{}_{x1\,x1\,x3} = \frac{1}{4}\,\frac{\lambda^2\,\delta 1\,\delta 3}{K1\,K3}$$

$$B^{x1}{}_{x3\,x1\,x3} = -\frac{1}{4}\,\frac{\lambda^2\,\delta 1\,(K3\,\delta 1 + 2\,K1)}{K1^2\,K3}$$

$$B^{x3}{}_{x1\,x1\,x3} = \frac{1}{4}\,\frac{\lambda^2\,\delta 3\,(K1\,\delta 3 + 2\,K3)}{K3^2\,K1}$$

(14)

$$B^{x3}{}_{x3\,x1\,x3} = -\frac{1}{4}\,\frac{\lambda^2\,\delta 1\,\delta 3}{K1\,K3} \quad \textbf{(14)}$$

> show(G[i,-j,-k,-l]);

$$G^{i}{}_{j\,k\,l} = 0 \quad \textbf{(15)}$$

> definetensor(epsilon[i] = N[i,-j]·Y[j] − 2 G[i]);

$$\varepsilon^{i} = N^{i}{}_{j}\,Y^{j} - 2\,G^{i} \quad \textbf{(16)}$$

> evalt(epsilon[i]);

$$\varepsilon^{i} = 0 \quad \textbf{(17)}$$

> restart;

libname := \`C:/Finsler\`, libname;

"C:/Finsler", "C:\Program Files\Maple 17\lib", "." **(18)**

with(Finsler);

[Dcoordinates, Hdiff, K, connection, init, metricfunction, tddiff] **(19)**

dimension := 2;

2 **(20)**

> coordinates(x1, x3);

The coordinates are:

$$X^{1} = x1$$

$$X^{2} = x3 \quad \textbf{(21)}$$

> Dcoordinates(y1, y3);

```
`Y assigned to DCoordinateName`
```

The d-coordinates are:

$$Y^{1} = y1$$

$$Y^{2} = y3 \quad \textbf{(22)}$$

$$> F := \exp(-ax1 + bx3 + c\cdot x1\cdot x3)\cdot\left(\frac{(y3)^{\left(1+\frac{1}{\mathrm{lambda}}\right)}}{(y1)^{\frac{1}{\mathrm{lambda}}}}\right);$$

$$F := \frac{e^{c\,x1\,x3 - ax1 + bx3}\,y3^{1+\frac{1}{\lambda}}}{y1^{\frac{1}{\lambda}}} \quad \textbf{(23)}$$

> metricfunction(F^2);

The components of the metric are:

$$g_{x1\,x1} = \frac{\left(e^{c\,x1\,x3 - ax1 + bx3}\right)^2 \left(y3^{\frac{\lambda+1}{\lambda}}\right)^2 (\lambda+2)}{\left(y1^{\frac{1}{\lambda}}\right)^2 \lambda^2\, y1^2}$$

$$g_{x1\,x3} = -\frac{2\left(e^{c\,x1\,x3 - a\,x1 + b\,x3}\right)^2 \left(y3^{\frac{\lambda+1}{\lambda}}\right)^2 (\lambda+1)}{\left(y1^{\frac{1}{\lambda}}\right)^2 \lambda^2\, y1\, y3}$$

$$g_{x3\,x3} = \frac{\left(e^{c\,x1\,x3 - a\,x1 + b\,x3}\right)^2 \left(y3^{\frac{\lambda+1}{\lambda}}\right)^2 (\lambda+1)\,(\lambda+2)}{\left(y1^{\frac{1}{\lambda}}\right)^2 \lambda^2\, y3^2} \tag{24}$$

> $K := K(a, b);$

$$K := \frac{y3\, y1 \left(y1^{\frac{1}{\lambda}}\right)^2 \lambda^2\, c}{\left(e^{c\,x1\,x3 - a\,x1 + b\,x3}\right)^2 \left(y3^{\frac{\lambda+1}{\lambda}}\right)^2 (\lambda+1)} \tag{25}$$

>