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Low dimensional monoidal category theory: A functorial method for constructing monoidal bicategories

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Trabalho apresentado ao Programa de Pós-graduação em Matemática do Centro de Ciências Exatas e da Natureza da Universidade Federal de Pernambuco, como requisito parcial para obtenção do grau de Mestre em Matemática.

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LOW DIMENSIONAL MONOIDAL CATEGORY THEORY: A FUNCTORIAL METHOD FOR CONSTRUCTING MONOIDAL BICATEGORIES

Dissertação apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestrado em Matemática.

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ABSTRACT

In this work, we start studying some basic concepts of classical category theory, such as categories, functors, natural transformations, products and co-products, among other important concepts, understanding its definitions and their main properties. We proceed to the theory of monoidal categories, with the objective of understanding a generalization of the product in categories and of algebraic objects within such categories. We begin this part studying properties of the neutral, the commutativity of certain diagrams and the properties of functors that preserve the monoidal structure, with the aim of being able to prove MacLane's coherence theorem, which gives us the commutativity of a large class of diagrams, and the strictification theorem, which gives us a monoidal category equivalent to the initial one that is algebraically simpler. We finish the study of these categories by looking at additional braiding structures, symmetry and internal algebraic structures (monoids, modules, bimodules and actions in monoidal categories). Finally, we extend the study of monoidal categories to the case of low-dimensional categories to prove a theorem recently proved by Shulman (which says that a certain bicategory associated with an isofibrant monoidal double category is also monoidal through a functorial association) and then we detail the applications of this result to some scenarios.

Keywords: category theory; monoidal categories; monoidal bicategories; monoidal double categories; locally cubical bicategories; low dimensional categories.

RESUMO

Neste trabalho começamos estudando alguns conceitos básicos da teoria de categorias clássica, como as categorias, funtores, transformações naturais, produtos e coprodutos, entre outros conceitos importantes, indo a fundo em suas definições e em suas propriedades gerais. Após este estudo nos é permitido estender o conhecimento para a teoria das categorias monoidais, com o objetivo de entender uma espécie de generalização do produto em categorias e de objetos algébricos dentro de tais categorias. Nesta parte, começamos estudando propriedades do neutro monoidal, a comutatividade de certos diagramas e propriedades de funtores que respeitam esta estrutura monoidal, com o objetivo de conseguirmos provar o teorema de coerência de MacLane, que nos provê a comutatividade de uma grande classe de diagramas, e o teorema de estritificação, que nos dá uma categoria monoidal equivalente à inicial que é mais algebricamente mais simples. Terminamos o estudo destas categorias vendo estruturas adicionais de trançamento, simetria e estruturas algébricas internas (monóides, módulos, bimódulos e ações em categorias monoidais). Por fim, estendemos o estudo de categorias monoidais para o caso de categorias de baixa dimensão para provar um teorema recentemente provado por Shulman (que diz que uma certa bicategoria associada à uma categoria dupla monoidal isofibrante é também monoidal através de uma associação funtorial) e detalhamos aplicações deste resultado em algumas situações.

Palavras-chaves: teoria de categorias; categorias monoidais; bicategorias monoidais; categorias duplas monoidais; bicategorias localmente cúbicas; categorias monoidais de baixa dimensão.

CONTENTS

1 INTRODUCTION AND OVERVIEW OF THE WORK

Category theory is, in the words of Norman Steenrod, an "abstract nonsense". At first glance one can think that this was an offense to the theory of categories, but actually he was just referring to the power of abstraction that category theory brings to other areas of mathematics. Steenrod took category theory as a very serious subject early on, since it allowed him to quickly solve a problem that he was working on: find a proper axiomatic treatment of homology theory ([16]).

Since it was formally founded by Saunders Mac Lane and Samuel Eilenberg in 1945, in the paper [41] category theory has been gaining acknowledgment for its power to see problems from above. It is a theory that allows algebraic geometers and topologists to work with tools that allow to pose problems in an entirely new framework that can abstract the core concepts more easily (see, for example, [10]). It also provides a framework to physicists work with quantum mechanics, conformal quantum field theories, general relativity, quantum computation and even some very well known areas, like thermostatistics and classical mechanics.

The theory of categorical graphical calculi has been developed since the 1980's. More recently, it gained further momentum due to some contributions like the work of Bob Coecke's. He provided an entirely categorical framework for quantum protocols and with quantum mechanics. In the realms of theoretical and mathematical physics, the work of John Baez's has received widespread recognition for posing physical problems in categorical language, as well as using categories to gain more and new perspectives on the topics that are already "well known" (see, for example, the references [27] to [38].) Last, but definitely not least (indeed, the more important contribution to this dissertation), Michael Shulman has studied the monoidal theory of low dimensional categories. His results have been used in research on various topics – E.g.: [18], [19] and [20] are regularly cited in Baez's most recent works. Shulman and Baez are coauthors of a paper on Petri Nets from a categorical viewpoint.

Now, we pose the big question: Why is low dimensional monoidal category theory important? Well, a first reason is that this type of categories appears more and more in applied and theoretical research – e.g. the categorical treatments of quantum theory, Markov chains, monoidal fibrations, profunctors, categorical algebra, bimodules, cobordisms, etc. They are very complicated structures with which to deal, so we need to develop new, clever ways to work with them. A second reason is that the monoidal structure in low dimension is a good way to start thinking on how we are going to generalize monoidal structures to categories of higher dimensions, and gain further insight into the differences between the low- and high-dimensional cases.

In this dissertation, we study a theorem by Shulman ([19]) that gives a functorial construction for the so-called monoidal bicategory, namely, taking the underlying loose structure of an isofibrant monoidal double category. It is a pretty simple and not general way, since we have to require the double category to be fibrant in order to lift the monoidal structure. Nonetheless, it is very useful because a lot of the monoidal bicategories arising in research are built in that manner, and the axioms for a monoidal bicategory are exaustingly larger than those for a monoidal double category. We start this journey acquiring a deep understanding of the most fundamental concepts of category theory in Chapter **2**, such as functors, natural transformations, adjunctions, and some special objects like products and pullbacks. Then, we proceed to carefully study the monoidal structures in 1-categories in Chapter **3**, proving coherence and strictification theorems, as well as other results on the fundamental structure of this type of category. We end our work in chapter **4** introducing some types of low dimensional categories, showing how we can generalize the fundamental concepts of category theory to higher dimensions, as well as the monoidal structures and, finally, discussing the main theorem of this paper: the loose bicategory of a monoidal double category inherit the monoidal structure functorially. Our original contribution to this work consists of more detailed arguments for the propositions and theorems of the paper, checking most of the details left to the reader. We also do a very careful detailing of the construction $\mathbf{Alg}(\mathbf{D})$, where $\mathbf{D}$ is a double category, and also detailing where and how the theorem can be applied. There is no single main reference to chapters **2** and **3**, its content follows from a mix of books (some parts are closer to one book than another), for example [1], [2], [3], [4], [6] and [7].

With all said, now we can proudly start our study on "abstract nonsense"!

2 ELEMENTS OF CATEGORY THEORY

2.1 INITIAL DEFINITIONS AND EXAMPLES

For those who are interested in the rigorous foundations of the subject, there is a detailed discussion in the appendix A. For those who only use categories as a tool for other subjects, a ***class*** can be viewed just as a certain "arbitrary" collection of sets that behaves pretty similarly to sets themselves, and that we have essentially the same concepts involving them (unions, intersections, functions, cartesian products, etc).

Definition 2.1.1. *A* **(1-)Category** ***C*** *is a 5-tuple* $\boldsymbol{C} = (Ob(\boldsymbol{C}), Mor(\boldsymbol{C}), dom, cod, \circ)$*, where* $Ob(\boldsymbol{C})$ *and* $Mor(\boldsymbol{C})$ *are classes (called, respectively, the class of* **objects** *and the class of* **morphisms** *or* **arrows** *of* ***C****), and dom, cod are (class) functions* $dom, cod\colon Mor(\boldsymbol{C}) \to Ob(\boldsymbol{C})$ *(called, respectively,* **domain** *and* **codomain** *of the morphism). A morphism* f *with domain* A *and codomain* B *is called a* **morphism from** A **to** B *and is denoted as* $A \xrightarrow{f} B$.

The symbol $\circ\colon Mor(\boldsymbol{C}) \times Mor(\boldsymbol{C}) \to Mor(\boldsymbol{C})$ *is a partial binary operation called* **composition**, *defined in the pair* (g, f) *(and denoted by* $g \circ f = gf$*) if, and only if,* $cod(f) = dom(g)$*, and, in this case* $dom(gf) = dom(f)$ *and* $cod(gf) = cod(g)$*. When we can compose arrows, we call them* **composable**. *Also, we require the two following conditions concerning morphisms:*

1. *(Associativity). For any three composable arrows* f, g, h*, we have* $h(gf) = (hg)f$.
2. *(Identities). For all* $x \in Ob(\boldsymbol{C})$*, there exists a* $1_x \in Mor(\boldsymbol{C})$ *with* $dom(1_x) = cod(1_x) = x$ *such that for all arrows* g, f *with compatible domain/codomain, respectively, we have* $1_x \circ f = f$ *and* $g \circ 1_x = g$.

We define, for $A, B \in Ob(\boldsymbol{C})$*, the class* $\hom_{\boldsymbol{C}}(A, B)$ *to be the class of all morphisms with domain* A *and codomain* B *(sometimes called "***hom-set***", even though it can be a proper class - see A - and the* ***C*** *is often ommitted, being clear from the context). We say that* C *is* **small** *when* $Ob(\boldsymbol{C})$ *and* $Mor(\boldsymbol{C})$ *are sets, and we say that* ***C*** *is* **locally small** *when, for all objects* $A, B \in Ob(\boldsymbol{C})$*,* $\hom(A, B)$ *is a set. We say that* ***C*** *is* **thin** *when there is a unique morphism between* A *and* B*, when one exists, and we say that* ***C*** *is* **discrete** *when the only morphisms are the identities. An* **endomorphism** *is a morphism with domain equal to its codomain, and the class of endomorphisms of an object* A *is denoted by* $End(A)$*. An*

automorphism *is an invertible endomorphism, and the class of automorphisms of an object* A *is denoted by Aut*(A)*.*

Remark. For the sake of brevity, we simply write $A \in \mathbf{C}$ for $A \in \text{Ob}(\mathbf{C})$.

Remark. It is important to notice that $\text{hom}(A, B)$ does not need to be formed by functions which preserve extra structure. Indeed, it does not need to be formed by functions at all (and can contain elements in common with the class of objects). For example, we can define a category **C** with:

$$\text{Ob}(\mathbf{C}) = \{\mathbb{N}, \mathbb{R}\},\ \text{hom}(\mathbb{N}, \mathbb{N}) = \{\emptyset\},\ \text{hom}(\mathbb{R}, \mathbb{R}) = \{\mathbb{N}\},\ \text{hom}(\mathbb{N}, \mathbb{R}) = \{\mathbb{R}\}.$$

Define a composition $\circ$ in the class of morphisms as:

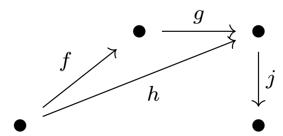
$$\mathbb{N} \circ \mathbb{N} = \mathbb{N},\ \emptyset \circ \emptyset = \emptyset,\ \mathbb{R} \circ \emptyset = \mathbb{R},\ \mathbb{N} \circ \mathbb{R} = \mathbb{R}.$$

It is clear that what we described is a category, but notice that we do not have functions as morphisms. Category theory is a very general branch of mathematics, as we are going to see better with the examples given in this section. Pretty much anything can be a category, and we just have to be creative.

Definition 2.1.2. *A* **directed graph** *will be defined as a pair* $G = (V(G), E(G)) = (V, E)$*, with* V, E *classes such that* $E \subseteq V \times V$*. We say that* V *is the class of* **vertices** *and* E *is the class of* **edges***. Given an edge* $e = (x, y) \in E$*, we say that* e *has domain* x *and codomain* y*, or that* e *is an* **edge from** x **to** y*. For those who are already familiar with notions of graph theory, it is interesting to notice that this definition is very similar to the usual one where* V, E *are sets. The only real difference is that, in our definition,* V, E *can be more general collections of objects. When we are talking about a directed graph with* G, E *sets, we say that this directed graph is* **small***. We also may consider these directed graphs as* **directed multigraphs** *(i.e, directed graphs in which there can be more than only one arrow from one vertex to another). A* **directed graph morphism** $f\colon G \to H$ *from the graph* G *to the graph* H *is a pair of functions* $f_1\colon V(G) \to V(H)$ *and* $f_2\colon E(G) \to E(H)$ *such that* f_2 *preserves incidence relations, i.e,* $f_2((x, y))$ *is an edge from* $f_1(x)$ *to* $f_1(y)$*. Sometimes, by abuse of notation, we denote* f_1 *and* f_2 *by* f*, so the preservation of incidence relation can be easily pictured as below.*

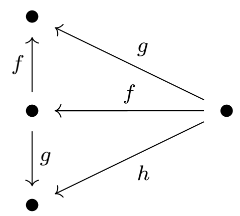
$$x \xrightarrow{\ e\ } y \qquad \stackrel{f}{\Longrightarrow} \qquad f(x) \xrightarrow{f(e)} f(y)$$

With this definition, it is clear that every category can be represented by a directed graph (and this representation of a category that we will call a ***graphic representation***). Nevertheless, not every directed graph can be viewed as a category, as we picture below (identities ommitted):



We can impose $gf = h$ to try turning this graph into a category, but jg and jh cannot be defined with the edges we have, so this graph cannot represent a category.

Another important observation we make is that we cannot label morphisms with different domain or codomain with the same morphism symbol. Graphically, we are saying that we do not admit something like this:



The reason is pretty simple: the domain and codomain functions are not well defined for f and g in this case. So, mathematically, we have that

$$(\forall A, B, A', B' \in \mathbf{C})[(A \neq A' \vee B \neq B') \implies \hom_{\mathbf{C}}(A, B) \cap \hom_{\mathbf{C}}(A', B') = \emptyset].$$

We list here some examples of categories of sets with additional structure that appear extensively throughout mathematics (all identities below are the identity function and the composition of the morphisms is just the usual composition of functions):

C	Ob(**C**)	$\mathrm{Hom}(A, B)$
Set	Sets	Functions
Posets	Partially ordered sets	Order preserving functions
Ord	Ordinal numbers	Order preserving functions
Card	Cardinal numbers	Functions
Grph	(Small) Graphs	Graph morphisms
Grp	Groups	Group homomorphisms
Ab	Abelian groups	Group homomorphisms
Rng	Rings	ring homomorphisms
R-**Mod**	R-Modules	Module homomorphisms
Vect$_k$	k-Vector Spaces	Linear transformations
Top	Topological spaces	Continuous functions
Ban	Banach spaces	Bounded linear transformations
Hilb	Hilbert spaces	Bounded linear transformations
Diff	Smooth manifolds	Differentiable functions

The reader does not need to be familiar with all of these categories for the sake of understanding this work. If any of the more unusual categories appears in the text, it will be detailed. For now, we are satisfied with just listing those categories. We give some detailed examples below.

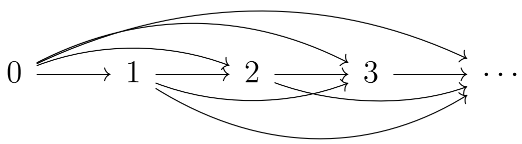
1. A ***partial order*** on a set P is a relation $\leq$ on P which satisfies the following properties:

   a) $(\forall a \in P)[a \leq a]$

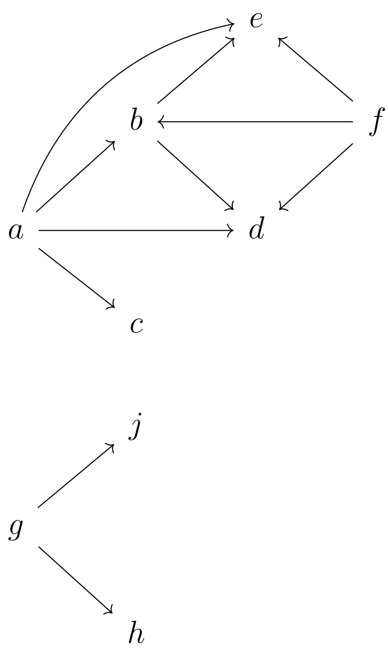
   b) $(\forall a, b \in P)[(a \leq b) \wedge (b \leq a) \implies a = b]$

   c) $(\forall a, b, c \in P)[(a \leq b) \wedge (b \leq c) \implies a \leq c]$

   A pair $(P, \leq)$, where $\leq$ is a partial order on P, is called a ***partially ordered set*** (or just a ***poset***). Every partially ordered set $(P, \leq)$ can be viewed as a category itself. The objects of this category are the elements of P, and there exists a (unique) morphism $a \to b$ if and only if $a \leq b$. The composition of composable arrows always exists by the property 3 above. A good graphical example of this construction is $(\mathbb{N}, \leq)$, where we omitted the identities below:

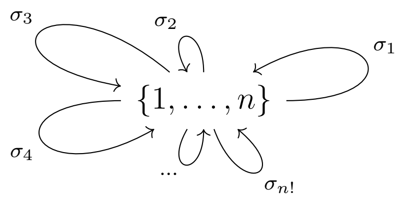


Now, we give a more abstract example. Take the poset $P = \{a, b, c, d, e, f, g, h, j\}$, with the partial order $\leq = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (b, e), (b, d), (c, c), (d, d),$ $(e, e), (f, f), (f, b), (f, d), (f, e), (g, g), (g, h), (g, j), (j, j), (h, h)\}$. Its categorical (or graphical) representation is as below (identities ommitted):



It is easily seen that, in the same way as before, posets can be viewed as directed graphs. Of course, neither every category nor every directed graph can be viewed as a poset.

2. Every group G can be viewed as a category with a single object $*$, and Mor$(G) = G$ (with identity and composition as in G). As a concrete example, take $* = \{1, \ldots, n\}$, and $G = S_n$. Then S_n is pictured as below:



The example above makes it evident that we are seeing the group as its isomorphic group of permutations given by Cayley's Theorem. Indeed, we are going to see this relation much closely when we study the Yoneda Lemma.

3. $\mathbf{Matr}_R$ is the category of matrices over the ring R, where the objects are natural numbers, and morphisms $n \to m$ are $m \times n$ matrices. This mysterious definition (a category of matrices that doesn't have matrices as objects) will make more sense when we discuss the equivalence of categories. For now, we stay with this heuristic argument: Matrices and linear transformations are correlated, so in order to compare those things in some sense, we have to treat matrices as morphisms.

4. $\mathbf{Ch}(R\text{-}\mathbf{Mod})$ is the category of chain complexes of $R-$modules. A ***chain complex*** of R-modules is a sequence of modules and homomorphisms as below:

$$\cdots \overset{\partial_{n+2}}{\to} M_{n+1} \overset{\partial_{n+1}}{\to} M_n \overset{\partial_n}{\to} M_{n-1} \overset{\partial_{n-1}}{\to} \cdots$$

where the sequence $(\partial_j)_{j\in\mathbb{Z}}$ satisfies $\partial_j \circ \partial_{j+1} = 0$ for all $j \in \mathbb{Z}$. Set theoretically we are saying that $\mathsf{Im}(\partial_{j+1}) \subseteq \mathsf{Ker}(\partial_j)$. When this inclusion is an equality for all $j \in \mathbb{Z}$, we say that the sequence is ***exact***. These homomorphisms are called ***boundary*** or ***differentials*** maps, motivated by applications of such operators in algebraic and differential topology. A morphism $f\colon$ M $\to$ N is a sequence of homomorphisms $(f_j\colon M_j \to N_j)_{j\in\mathbb{Z}}$ such that the diagram below commutes:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{n+2}} & M_{n+1} & \xrightarrow{\partial_{n+1}} & M_n & \xrightarrow{\partial_n} & M_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow{\partial'_{n+2}} & N_{n+1} & \xrightarrow{\partial'_{n+1}} & N_n & \xrightarrow{\partial'_n} & N_{n-1} & \xrightarrow{\partial'_{n-1}} & \cdots
\end{array}$$

Composition of these morphisms are defined by pointwise composition, and the identity morphism is the sequence of identities. This class of objects endowed with the above morphism, composition and identities is clearly a category.

5. The category $\mathbf{Rep}_G^k$ is the category of ***group representations*** of G over the field k. A group representation of G is a group homomorphism $\rho\colon G \to \mathsf{GL}(V)$, where V is any k-vector space. The objects of $\mathbf{Rep}_G^k$ are pairs (V, ρ), where V is a k-vector space and ρ is a representation. A morphism $f\colon (V,\rho) \to (W,\sigma)$ in this category is called a ***interwining operator*** (or ***equivariant map***), and it is defined as a linear transformation $f\colon V \to W$ such that the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \rho_g}\downarrow & & \downarrow{\scriptstyle \sigma_g} \\ V & \xrightarrow[f]{} & W \end{array}$$

6. The category **Braid** has natural numbers as objects and hom$(m, n) = \emptyset$ if $m \neq n$, hom$(n, n) = B_n$, the ***braid group on*** n ***strands***. Composition is obtained by just connecting the end of one braid with the beggining of the next, seeing two braids as essentially the same when they are ambient isotopic relative to their endpoints, i.e., when we can deform the space sending one onto the other so that the deformation is a homeomorphism of 3-space onto itself at each instant, keeping their endpoints still. One consequence is that two braid diagrams that can be deformed into each other without changing their crossings clearly represent the same braid. The identity morphism is the identity braid (i.e, the braid free of crossings). For more details, we suggest reading Appendix C.

Remark. We just need a **preorder** to define a category subjacent to it. I.e, we do not need the second condition of a partial order.

It is important to stress that the most familiar examples of categories for the begginer are given by a set with an additional structure, and morphisms between those sets being functions that preserves these extra structures. Categories built in this way are examples of the so-called ***concrete categories*** over Set, in which objects are sets, and the morphisms between any two of them are some of the functions between them. Quoting MacLane, we only define categories in order to define what are functors and natural transformations. The philosophy of category theory tells us that relations between categories are more important than the categories. As usual in mathematics, our definition of functor will be a function that preserves the structure of the category. In the most simple case (i.e., when we do not have additional structures), we only require the preservation of compositionality and units.

Definition 2.1.3. *A (***covariant***)* **functor** $F\colon$ ***C*** $\to$ ***D*** *between categories* ***C*** *and* ***D*** *is defined as a pair of functions, one between objects and one between morphisms, in such a way that a morphism* $A \overset{f}{\to} B$ *in* ***C*** *is mapped to a morphism* $F(A) \overset{F(f)}{\to} F(B)$ *in* ***D****, subject to the relations* $F(gf) = F(g)F(f)$ *(for any pair of composable arrows) and* $F(1_A) = 1_{F(A)}$ *(for* *all objects A). A* **contravariant functor** $F\colon \mathbf{C} \to \mathbf{D}$ *is defined just as above, but with the compositionality condition being* $F(gf) = F(f)F(g)$.

Remark. The (quasi) category of all categories is called **CAT**. It is not an actual category because there is no such thing as the class of all classes in the theory of classes NBG (see A).

For each concrete category **C** over **Set**, we have a functor $U\colon \mathbf{C} \to \mathbf{Set}$, called the ***forgetful*** (or ***underlying***) functor, that sends to each object and each morphism, respectively to its underlying set and function. Another fruitful example is the functor $\pi_0\colon \mathbf{Top}^* \to \mathbf{Grp}$, that sends each pointed topological space to the fundamental group with base point $*$. In a first course on algebraic topology, it is a routine exercise to show that π_0 is, indeed, a functor. A lot of other mathematical constructions can be viewed as functors (or at least a part of a functor), just like the enveloping algebra of a Lie algebra, free groups, free vector spaces, the matrix of a linear transformation (with respect to fixed bases), the Stone-Čech compactification, the quotient field of an integral domain, etc. In most cases, the converse is true, i.e., functors tell us what are the interesting objects in mathematics. Indeed, these functors often can reflect important properties from one category to another, having the power to turn hard problems in much simpler ones. Below, we detail some examples of functors.

1. To each set X, we associate with it the **free** k**-vector space** over X. The idea is to construct a k-vector space $\mathbf{F}(X)$ such that it has X as a basis. First, we consider the set of all functions $f\colon X \to k$ such that $\{x\colon f(x) \neq 0\}$ is a finite set. In this set, we introduce the sum $f + g$ and the scalar action λf as $(f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda f(x)$ respectively. Clearly, both can only have non-zero values in a finite subset of X. It is very straightforward to show that this set with these operations is a vector space. Now, we claim that X can be viewed as a basis for this vector space $\mathbf{F}(X)$.

   Indeed, define for each $x_0 \in X$, the function $f_{x_0}(x_0) = 1$, and $f_{x_0}(x) = 0$ for any $x \neq x_0$. The set of these functions f_{x_0} are in bijection with X, by construction. Moreover, given any $f \in \mathbf{F}(X)$, if $\{x_1, \ldots, x_n\}$ is the set of $x \in X$ such that $f(x) \neq 0$, and if $f(x_i) = k_i$, then we claim that $f = k_1 f_{x_1} + \cdots + k_n f_{x_n}$.

   $f(x) = 0 = f = (k_1 f_{x_1} + \cdots + k_n f_{x_n})(x)$ for each $x \notin \{x_1, \ldots, x_n\}$. Also, by definition, $(k_1 f_{x_1} + \cdots + k_n f_{x_n})(x_i) = 0 + \cdots + k_i f_{x_i}(x_i) + \cdots + 0 = k_i = f(x_i)$, proving that $\{f_{x_1}, \ldots, f_{x_n}\}$ is a basis for $\mathbf{F}(X)$. When we identify this basis with X, we can identify $\mathbf{F}(X)$ with the set of all formal finite linear combinations of elements of X. Therefore, we denote $k_1 f_{x_1} + \cdots + k_n f_{x_n}$ just as $k_1 x_1 + \cdots + k_n x_n$. One could start trying to define $\mathbf{F}(X)$ as this "set of formal finite linear combinations", but it wouldn't be mathematically precise, unless it was already defined mathematically what a formal finite linear combination is. Thus, we need the above construction to capture this intuitive idea.

If we have $f\colon X \to Y$ a function, then there exists a unique linear transformation $\mathbf{F}(f)\colon \mathbf{F}(X) \to \mathbf{F}(Y)$ that, when restricted to X, it is f. This is nothing new. All we are saying here is that, if we define a function in all elements of a basis of a vector space, then, there exists a unique extension to a linear transformation in the whole vector space. By this constructive approach well known in linear algebra, it is clear that $\mathbf{F}(1_X)$ is $1_{\mathbf{F}(X)}$, and that $\mathbf{F}(fg) = \mathbf{F}(f)\mathbf{F}(g)$. Hence, this construction is indeed a functor $\mathbf{F}\colon \mathbf{Set} \to \mathbf{Vect}_k$.

At last, notice that essentially the same construction applies for the construction of a ***free*** R***-module*** over X, which also gives rise to a functor. Indeed, all usual free constructions gives rise to functors.

2. A ***Lie algebra*** is a k-vector space L with a bilinear, antisymmetric product $[,]\colon L \times L \to k$ called ***Lie bracket***, satisfying the ***Jacobi identity***, i.e, for all $u, v, w \in L$, $[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$. E.g.: Given an associative algebra, we have an induced Lie bracket $[u,v] = uv - vu$, known as the commutator. A homomorphism of Lie algebras will be a linear transformation preserving the Lie bracket. Given a Lie algebra L, we wish to construct an algebra that includes L, and such that the induced Lie bracket is the Lie bracket of L when restricted to elements of L. In order to do so, we first define $\mathbf{T}(V) = \bigoplus_{j\in\mathbb{N}}(\bigotimes_{i=0}^{j} V)$, the **tensor algebra** over V. It is the **free algebra** over a given vector space V, with multiplication given by $(v_1 \ldots v_n)\cdot(w_1 \ldots w_m) = v_1 \ldots v_n w_1 \ldots w_m$. Here we identify the elements of the direct sum with the words, i.e., the finite concatenation sequences of elements of the summands of that direct sum. As usual, we grade each vector with the index of the summand. Now, applying this construction to a Lie algebra L, we define the ideal $\mathbf{I}(L)$

of $\mathbf{T}(L)$ generated by all elements of the form $uv - vu - [u, v]$, where $u, v \in L$. The **universal enveloping algebra** $\mathrm{U}(L)$ of the Lie algebra L is defined as the quotient of $\mathbf{T}(V)$ by $\mathbf{I}$.

Given a Lie-algebra homomorphism $f\colon L \to L'$, then we can extend it in a unique way to a morphism of algebras $U(f)\colon U(L) \to U(L')$. We define, for each summand, $U(f)(\overline{v_1 \otimes \cdots \otimes v_j}) = \overline{f(v_1) \otimes \cdots \otimes f(v_j)}$. This can be defined (without the quotient) by the universal property of the tensor product (see ahead the appendix B), and is well defined considering the quotients because, since f preserves the Lie bracket, elements in $\mathbf{I}(L)$ are taken to elements of $\mathbf{I}(L')$.

The assignment of Lie algebras to their respective universal envelopping algebras is functorial by construction. Thus, we have a functor $\mathbf{U}\colon \mathbf{Lie}_k \to \mathbf{Alg}_k$, where the first category is the category of k-Lie algebras and the second category is the category of k-algebras. Both of these categories are defined as a category of sets with additional structure with morphisms being functions that preserves these extra structures.

Another interesting feature to observe in this example is that the function that assigns a given algebra to its induced Lie algebra with commutator as Lie bracket, is itself a functor.

3. Not every important mathematical concept can be regarded as a functor. For example, given a group G, its center $Z(G)$ is defined as the subgroup consisting of each element of G that commutes with all the elements of G.. We claim that there is no way that we can assign a functor $Z\colon \mathbf{Grp} \to \mathbf{Ab}$ to this construction.

   Indeed, from group theory, the dihedral group D_n has trivial center when 2 does not divide n. We can see $\mathbb{Z}_2$ as a subgroup of D_n (generated by the reflection), and $\mathbb{Z}_2$ is the quotient of D_n by the normal subgroup generated by the rotation. Thus, the following composite is an isomorphism:
$$\mathbb{Z}_2 \overset{\subseteq}{\to} D_n \overset{\pi}{\to} \mathbb{Z}_2.$$

   If Z was a functor, then, its application to the above group homomorphism would result in group homomorphisms that compose into a group isomorphism:

$$\mathbb{Z}_2 \overset{Z(\subseteq)}{\to} \{e\} \overset{Z(\pi)}{\to} \mathbb{Z}_2$$

Since $Z(\subseteq)(x) = e$ for all $x \in \mathbb{Z}_2$, and $Z(\pi)(e) = \overline{0}$, their composition is clearly not an isomorphism, contradicting the assumption that Z is a functor.

It is interesting to notice that although the center cannot be viewed as the object function of any functor between categories of groups, it does not mean that it has no categorical counterpart. Indeed, there is a notion of center in the context of 2-categories that we will mention later.

4. We say that a group G has ***presentation*** $\langle S|R\rangle$, where S is a set, called the set of generators, and R is a set of relations among those generators, when G is isomorphic to the quotient of the free group over S by the normal subgroup generated by the relations R. Further in this section, we construct the free monoid over a set, and we give an idea of how to adapt the construction to free groups. Of course, given any set S and relations R in S, we can construct the group $\mathsf{F}(S)/N_R$, where $\mathsf{F}(S)$ is the free group generated by S, and N_R is its normal subgroup generated by the relations R. We define the ***category of presentations*** **Prst**, with objects all pairs $\langle S|R\rangle$, for S any set, and any set of relations R on S, so that a morphism $f\colon \langle S|R\rangle \to \langle S'|R'\rangle$ is any function $f\colon S \to S'$ that sends every pair in R to a pair in R' (i.e, it preserves the relations.) For each $\langle S|R\rangle$, we associate to it the group $\mathsf{F}(S)/N_R$. We can extend each morphism $f\colon \langle S|R\rangle \to \langle S'|R'\rangle$ to a group homomorphism $\overline{f}\colon \mathsf{F}(S)/N_R \to \mathsf{F}(S')/N_{R'}$ setting $\overline{f}(\overline{x}) = \overline{f(x)}$ for each $x \in S$. It is well defined because f takes N_R to $N_{R'}$.

   The construction above is functorial. Indeed, it is a functor $T\colon$ **Prst** $\to$ **Grp**.

5. Given a chain complex M, we define the ***i-th homology module*** of M as $H_i = \mathrm{Ker}(\partial_i)/\mathrm{Im}(\partial_{i-1})$. Given a morphism $f\colon M \to N$ of chain complexes, we can define a sequence of morphisms from the homology modules of M to the homology modules of N as follows: We define $\overline{f}_i\colon H_i \to H_i'$ by $\overline{f}_i(\overline{x}) = \overline{f_i(x)}$. It is a routine exercise to show that each of these functions is, indeed, a module homomorphism. It is important to notice that we can gather all information about the sequence H_i in the graded module $\bigoplus_{i=-\infty}^{\infty} H_i$. Here, a ***graded module*** is just a module with a direct sum decomposition

with $\mathbb{Z}$ as index. Morphisms of graded modules are morphisms of modules that preserve the grading. We can then form the category gr$R$-Mod of graded R-modules. The construction above of homology modules gives us morphisms between the homology graded modules,and is functorial, indeed. It is called the ***homology functor***

$$\mathbf{H}\colon \mathrm{Ch}(R\text{-}\mathbf{Mod}) \to \mathrm{gr}R\text{-}\mathbf{Mod}.$$

6. Given any locally small category $\mathbf{C}$, and any object $c \in \mathbf{C}$, we have two ***hom-functors*** associated to c, one covariant and one contravariant. The (covariant) functor

$$\mathrm{hom}(c, \_)\colon \mathbf{C} \to \mathbf{Set}$$

is defined on objects as $\mathrm{hom}(c, \_)(d) = \mathrm{hom}(c, d)$ and on a morphism $d \xrightarrow{f} d'$ as the function $\mathrm{hom}(c, f)\colon \mathrm{hom}(c, d) \to \mathrm{hom}(c, d')$ given by $\mathrm{hom}(c, f)(g) = f \circ g$. We also have a contravariant functor $\mathrm{hom}(\_, c)$ defined similarly.

Definition 2.1.4. *If* $\mathbf{C}$ *is a category, then we define its* **opposite category** $\mathbf{C}^{op}$ *as the category with same objects as* $\mathbf{C}$*, but with* $\mathrm{hom}_{\mathbf{C}^{op}}(A, B) = \mathrm{hom}_{\mathbf{C}}(B, A)$ *and* $f \circ_{\mathbf{C}^{op}} g = g \circ_{\mathbf{C}} f$ *(identities clearly still the same.)*

Remark. A contravariant functor $F\colon \mathbf{C} \to \mathbf{D}$ is simply a (covariant) functor $F\colon \mathbf{C}^{op} \to \mathbf{D}$.

With this concept in mind, we can talk about a very important metatheorem concerning theorems in the language of category theory. This metatheorem says, that if you prove a theorem about categories, then you actually have proved two theorems about categories. To be more precise, given a predicate P written in the language of category theory and NBG, we can define the dual predicate P^{op} reversing the direction of all arrows that appears in P (only the arrows in the categorical language, not those in set language). Typically, P^{op} is not the same predicate as P. We give some examples below which apply to any category $\mathbf{C}$:

- P: A is an object such that $(\forall B \in \mathbf{C})(\exists! A \xrightarrow{f} B)$. (This means that A is an initial object in $\mathbf{C}$)
  P^{op}: A is an object such that $(\forall B \in \mathbf{C})(\exists! A \xleftarrow{f} B)$. (This meansthat A is a final object in $\mathbf{C}$)
- P: $A \xrightarrow{f} B$ has a left inverse. (That means f is a section)
  P^{op}: $A \xleftarrow{f} B$ has a right inverse. (This means that f is a retraction)

- $P\colon\ A \xrightarrow{f} B$ is such that $(\forall g, h\colon B \to C\colon gf = hf)[g = h]$. (This means that f is an epimorphism)

  $P^{op}\colon\ A \xleftarrow{f} B$ is such that $(\forall g, h\colon C \to B\colon fg = fh)[g = h]$. (This means that f is a monomorphism)

First of all, notice that the above predicates have some free variables, so we cannot say, for example, that they are valid for all categories **C**. We can only say that they are true (or false), for example, for a given object A or a given morphism f, and this truth value depends on the object or morphism being analyzed. In order to say that a predicate P is true for all categories, we need P to have only bounded variables (both object and morphism variables), which now we assume for the sake of the next argument. By construction of the concept of dual statement, it is clear that P holds in a category **C** if and only if P^{op} holds in the category $\mathbf{C}^{op}$. From this observation, we have the following metatheorem, commonly called ***The Duality Principle for Categories***:

$$P \text{ is valid for all categories} \iff P^{op} \text{ is valid for all categories}$$

Proposition 2.1.5. *Initial objects are unique up to isomorphism. Dually, final objects are unique up to isomorphism.*

Proof. Let i_1, i_2 be initial objects. By definition, there exist unique arrows f, g as follows: $i_1 \xrightarrow{f} i_2 \xrightarrow{g} i_1$. The unique arrow $i_1 \to i_1$ (that exists by definition of initial object) has to be the identity, so we conclude $gf = 1$. In the same way we show that $fg = 1$, concluding that those objects are isomorphic. □

As a matter of fact, not every category has initial or final objects (e.g. a poset with no minimum element viewed as a category does not have an initial object.) But, when they exist, they are unique up to isomorphism. Sometimes, the initial object is also terminal, and in this case we call it a zero object (e.g. the trivial k-vector space, the trivial abelian group, the trivial R-module). In Set, $\emptyset$ is an initial object, for the unique function from $\emptyset$ to a set A is the empty function, i.e., the empty set viewed as a left-total, functional relation $\emptyset \subseteq \emptyset \times A$), and $\{\emptyset\}$ is a final object.

Definition 2.1.6. *A functor $F\colon \mathbf{C} \to \mathbf{D}$ is called* **faithful** *if and only if the map*

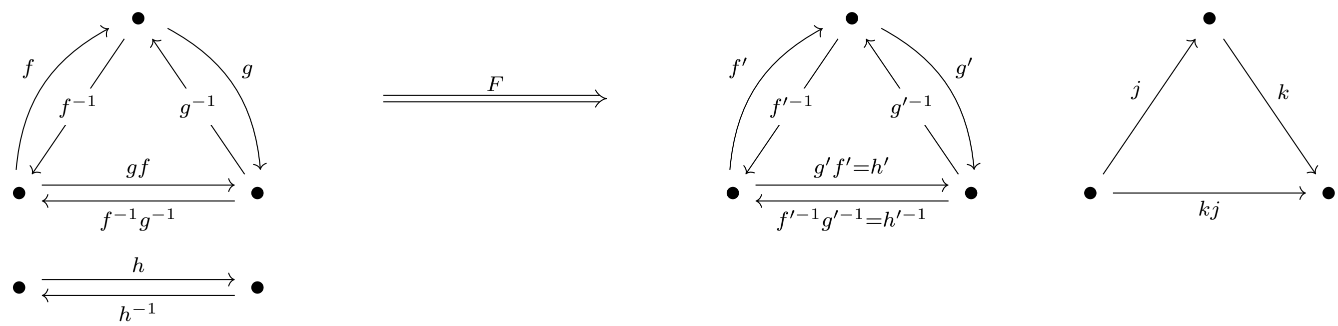
$$F\colon\ \hom_{\mathbf{C}}(A, B) \to \hom_{\mathbf{D}}(FA, FB)$$

assigning f to $F(f)$ is injective. F is called **full** *if and only if this function is surjective. F is called* **essentially surjective on objects** *if and only if, for all $d \in Ob(\boldsymbol{D})$, exists $c \in Ob(\boldsymbol{C})$ such that $F(c) \cong d$. A fully faithful, essentially surjective functor is called an* **equivalence** *of categories.*

Remark. $F\colon \mathbf{FinVect}_k^c \to \mathbf{Matr}_k$ defined as $F(V) = \dim(V)$ and $F(T) = [T]_\beta^\alpha$ is clearly an equivalence, where $\mathbf{FinVect}_k^c$ is the category of the finite k-vectors spaces with a chosen fixed basis.

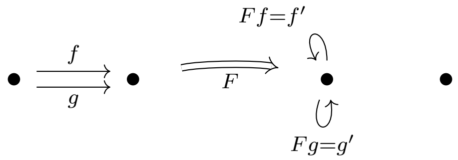
Definition 2.1.7. *A* **subcategory** $\boldsymbol{D}$ *of* $\boldsymbol{C}$ *is a category such that $Ob(\boldsymbol{D}) \subseteq Ob(\boldsymbol{C})$ and, for each $A, B \in \boldsymbol{D}$, $\hom_{\boldsymbol{D}}(A,B) \subseteq \hom_{\boldsymbol{C}}(A,B)$. We also require that the identities of $\boldsymbol{D}$ are the identities of $\boldsymbol{C}$, and that composition in $\boldsymbol{D}$ is the restriction of the composition in $\boldsymbol{C}$. This implies that whenever two morphisms in $Mor(\boldsymbol{D})$ can be composed in $\boldsymbol{C}$, the composition must also belong to $Mor(\boldsymbol{D})$). A subcategory $\boldsymbol{D}$ is called full if and only if, for any $A, B \in \boldsymbol{D}$, $\hom_{\boldsymbol{D}}(A,B) = \hom_{\boldsymbol{C}}(A,B)$. It is clear that a subcategory is full if and only if the inclusion functor is full.*

Remark. A functor F being faithful or full does not say anything about F being itself injective or surjective. Below, we give a graphical example of a fully faithful functor that is neither injective nor surjective (in both objects and morphisms.) Identities are omitted from the diagram:



In this diagram, $F(f) = f', F(g) = g', F(h) = h', F(gf) = g'f'$, where the inverses follow this same pattern. The functor F does not reach the triangle with arrows j, k, kj, so it is not surjective neither on the objects nor on the morphisms. F also is not injective neither on the objects nor on the morphisms, as we are going to argue now. Notice that $\mathrm{dom}(h)$ and $\mathrm{dom}(gf)$ are different, but both are mapped to $\mathrm{dom}(h') = \mathrm{dom}(g'f')$. gf and h are different morphisms, both mapped to the same morphism $h' = g'f'$. Nevertheless, F is locally injective and surjective on the hom-sets, which is what we want. A fully faithful functor F gives us a category formed by the image of F that behaves fairly similar to the category in the domain in terms of morphism properties.

Now, notice that the following functor is not fully faithful, where we require that $f' \neq g', f'g' = g'f' = 1$ for the category on the right:



Indeed, the functor fails to be full. Let the objects in the left category be, from left to right, A and B. There are no morphisms $B \to A$, while there are from $F(B) \to F(A)$.

Remark. If a functor $F \colon \mathbf{C} \to \mathbf{D}$ is injective on morphisms, it is faithful (obviously), but it is also injective on objects. The reason to such a thing is that $A \neq B \iff 1_A \neq 1_B$, so, assuming $A \neq B$, we have $1_{FA} = F(1_A) \neq F(1_B) = 1_{FB}$ by injectivity of F in the class of morphisms. By the equivalence first stated, we have $F(A) \neq F(B)$, proving that F is injective on objects.

Definition 2.1.8. *We say that a functor F* **reflects** *a property P (about objects or morphisms) if and only if whenever $P(FX)$ holds, $P(X)$ also holds, where X is the suitable object or morphism variable for the predicate P.*

Proposition 2.1.9. *The following hold for a functor $F \colon \mathbf{C} \to \mathbf{D}$:*

1. *F preserves isomorphisms, sections and retractions.*
2. *If F is fully faithful, then, F reflects isomorphisms, sections and retractions.*
3. *If F is fully faithful and surjective on objects, then, F reflects monomorphisms and epimorphisms.*
4. *If F is fully faithful and surjective on objects, then, F preserves and reflects initial and final objects.*

Proof.

1. If f has left inverse g, $1_F = F(1) = F(gf) = F(g)F(f)$, so F preserves sections. Dually, F preserves retractions. An isomorphism is a section that is also a retraction, so F also preserves isomorphisms.

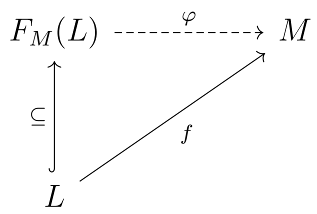
2. Considering the very argument we just gave, we will only prove that F reflects sections. If $F(A) \stackrel{Ff}{\to} F(B)$ is a section, there exists $F(B) \stackrel{h}{\to} F(A)$ such that $hF(f) = 1_F$. F being full implies that there is a morphism $B \stackrel{g}{\to} A$ such that $Fg = h$. Therefore, we have that $1_F = F(g)F(f) = F(gf)$ and, by faithfulness, $gf = 1$, so f is indeed a section.

3. Analogous to the proof of item 2.

4. Analogous to the proof of item 2. □

Using the item 2. of the theorem above, we provide another way to verify that our example given in page 12 of a functor that is not fully faithful is, indeed, not fully faithful ($f' = Ff$ is an isomorphism, but f is not one).

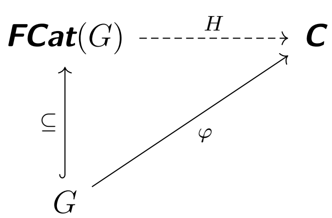
In mathematics, it is very common the occurence of free structures (over something), just like free vector spaces, free monoids, free groups and, as we will see soon, free categories. Usually, these objects are free in the sense that they are free of extra relations (i.e. relations that are not included in the definition of their structure). To exemplify these free structures, let us construct free monoids. If we are given a set $L = \{a, b, c, \dots\}$ (call this set our alphabet), a ***word*** on L is a finite sequence of elements of L. On the set of words on L, we can define the ***concatenation operation***. The ***length*** of a word is tipically defined as the size of this sequence, although this is not always the case (see section 2.3, for the appropriate discussion). Denote by $F_M(L)$ the set L endowed with this operation. It is clear that concatenation has a neutral element, namely, the empty string, and that it is associative, so $F_M(L)$ is indeed a monoid. Moreover, given any monoid N constructed over L, we can associate to each element of $F_M(L)$ an element of N via the map that sends a sequence of symbols to their product. That can make, for example, $ab = c$ in N, while in $F_M(L)$ this cannot be true, because they are different strings. Therefore $F_M(L)$ cannot satisfy any algebraic property other than associativity, existence of neutral and their consequences. Mathematically, we can make this precise saying that, given any function $f\colon L \to M$, where M is a monoid, there is a unique monoid homomorphism $\varphi\colon F_M(L) \to M$ such that the following diagram commutes:



The proof of this statement is a fairly simple exercise in algebra. Another interesting feature is that all of the usual algebraic structures are quotients of free structures. In the case of monoids, given a monoid M we can construct $F_M(M)$ (of course, here we are identifying the monoid M with its underlying set) and, taking $f = 1$ above, we get a unique surjective homomorphism $F_M(M) \to M$ mapping M onto itself. Therefore, it is clear that M is isomorphic to a quotient of $F_M(M)$ (due to the first isomorphism theorem for monoids). This argument is valid in many of the free structures. Also, notice that, categorically, we have $\hom_{\textbf{Set}}(L, U(M)) \cong \hom_{\textbf{Mon}}(F_M(L), M)$, where U is the forgetful functor.

Now, suppose we are given a (class) directed graph $G = (V, E)$. We wish to build a free category over G, which we are going to call **FCat**(G). The obvious way to do this, mimicking our construction of a free monoid, is to take Ob(**FCat**$(G)) = V$, and our morphisms between $A, B \in V$ should be all words of composable arrows starting in A and ending in B, i.e., paths between A and B. The composition is clearly associative, and the identities are the empty paths. We state the following theorem:

Theorem 2.1.10. *If $G = (V, E)$ is a directed graph,* ***C*** *is an arbitrary category and if $\varphi\colon G \to G($**C**$)$ is a graph morphism (where $G($**C**$)$ is the class directed graph underlying* ***C****), then there is a unique functor $H\colon$* ***FCat***$(G) \to$ ***C*** *such that the following diagram commutes:*



*In this diagram, we identified $G($**C**$)$ with* ***C****, for simplicity.*

Proof. Define F on the objects and "edges" (strings of size one, i.e, edges of G) of **FCat**(G) just as φ (in order for the diagram to commute, this is a necessary hypothesis). As every other morphism will be a string of edges $f_1 f_2 \dots f_n$, F be a functor means that we must have that F preserves identities and that $F(f_1 \dots f_n) = F(f_1) \dots F(f_n) = \varphi(f_1) \dots \varphi(f_n)$.

This construction of F as above is clearly functorial and unique, by construction. □

In order to state that every category is isomorphic to the quotient of a free one (just like most algebraic cases), we must define what a quotient of a category is, and what we mean by an isomorphism of categories. It is defined as the following:

Definition 2.1.11. *A* **congruence relation** *on the class of morphisms Mor(****C****) of a category* ***C*** *is a family of equivalence relations $\sim$ (defined locally in hom-classes) such that for all composable f, g, f', g' (composable in respective pairs), it holds that $(f \sim f') \wedge (g \sim g') \implies gf \sim g'f'$. The category quotient of C by $\sim$ is then defined as the category $C/\sim$, where Ob$(C/\sim)$ = Ob(C) and $\hom_{\mathbf{C}/\sim}(A, B) = \hom_{\mathbf{C}}(A, B)/\sim$. The composition is given by $\overline{g}\overline{f} = \overline{gf}$, and domain and codomain are defined just like in* ***C****. This is clearly well defined because $\sim$ is a congruence, so it respects composition in the sense that, whenever we consider other representatives of the congruence classes, the compositions will belong to the same congruence class.*

Definition 2.1.12. *An* **isomorphism** *between two categories is a functor that admits a two-sided inverse functor. We say that two categories* ***C****,* ***D*** *are isomorphic when there is an isomorphism between them and, in this case, we write* ***C*** $\cong$ ***D****.*

Remark. Since isomorphisms are not as important as equivalences in practice, we chose not to discuss any particular example. Nevertheless, a lot of examples of isomorphisms can be constructed quotienting equivalences.

Now, returning to our last theorem, it is clear that if we define a congruence $\sim_H$ in Mor(**FCat**$(G(C))$) as $f \sim_H g \iff Hf = Hg$, **FCat**$(G(C))/\sim$ will be isomorphic to **C**. It is very interesting to notice that this theorem generalizes the case for monoids, because a monoid itself can be viewed as a category. This also generalizes the case for groups (and other algebraic structures), but we would have to impose some other admissible relations in the composition (e.g., $g^{-1}g \sim e$, with g an element in the group).

Another important categorical construction is the product of categories.

Definition 2.1.13. *The* **product category** *of the categories C, D is the category* ***C*** $\times$ ***D****, where Ob(****C*** $\times$ ***D****) = Ob(****C****) $\times$ Ob(****D****), and morphisms from (a, b) to (a', b') are pairs (f, f'), where $a \xrightarrow{f} a', b \xrightarrow{f'} b'$. Composition is taken coordinate-wise, and identities are obviously pairs* *of identities. Furthermore, if we have $F\colon \boldsymbol{A} \to \boldsymbol{C}$ and $G\colon \boldsymbol{B} \to \boldsymbol{D}$ functors, we can define a functor $F \times G\colon \boldsymbol{A} \times \boldsymbol{B} \to \boldsymbol{C} \times \boldsymbol{D}$ as $(F \times G)(x, y) = (F(x), G(y))$ on objects $(x, y) \in \boldsymbol{A} \times \boldsymbol{B}$, and $(F \times G)(f, g) = (F(f), G(g))$ on morphisms $(f, g) \in$ Mor$(\boldsymbol{A} \times \boldsymbol{B})$. $F \times G$ is clearly a functor, because F, G are functors and $F \times G$ is defined componentwise.*

From the definition, it is fairly obvious that $\times\colon$ **Cat** $\times$ **Cat** $\to$ **Cat** is a functor. But this is not a particular feature of this product, all usual products with additional structure are also functors with a product category as domain (e.g., direct product of groups, direct sum of vector spaces, direct sum of modules, topological product, tensor product). Functors with such domain are called ***bifunctors***. There is a pretty straightforward characterization of bifunctors:

Proposition 2.1.14. *A function (on objects and morphisms) $F\colon \boldsymbol{A} \times \boldsymbol{B} \to \boldsymbol{C}$ is a functor if and only if it is functorial in both coordinates and these new functors are coherent with respect to F, i.e.:*

1. *$L_a = F(a, \cdot)\colon \boldsymbol{B} \to \boldsymbol{C}$ is a functor for all $a \in \boldsymbol{A}$ (the morphism held fixed in the left is 1_a).*
2. *$R_b = F(\cdot, b)\colon \boldsymbol{A} \to \boldsymbol{C}$ is a functor for all $b \in \boldsymbol{B}$ (the morphism held fixed in the right is 1_b).*
3. *For $a \xrightarrow{f} a', b \xrightarrow{g} b'$,we have $R_{b'}(g)L_a(f) = L_{a'}(f)R_b(g) = F(f, g)$. By abuse of notation, we can state this condition as $F(f, 1)F(1, g) = F(1, g)F(f, 1) = F(f, g)$.*

Remark. It is interesting to notice that any pair of functors L_a, R_b such that $L_a(b) = R_b(a)$ satisfying $1., 2.$ and $3.$ above determine uniquely a bifunctor $F\colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ by $F(a, b) = L_a(b) = R_b(a)$ on objects and with the obvious formula stated in $3.$ for morphisms.

2.2 NATURAL TRANSFORMATIONS AND UNIVERSALS

Definition 2.2.1. *A powerful and important concept in the theory of categories is that of natural transformation. A* **natural transformation** *from F to G, where $F, G\colon \boldsymbol{C} \to \boldsymbol{D}$ are functors, is a function $\tau\colon$ Ob$(\boldsymbol{C}) \to$ Mor$(\boldsymbol{D})$ such that for each $A \in \boldsymbol{C}$, $\tau(A) = \tau_A$ is a morphism $F(A) \xrightarrow{\tau_A} G(A)$ making, for each $A, B \in \boldsymbol{C}$ and morphism $A \xrightarrow{f} B$, the following* *diagram commute:*

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\tau_B]{} & G(B) \end{array}$$

When all components of a natural transformation are isomorphisms, we call this natural transformation a **natural isomorphism***, and denote this by* $F \cong G$.

Remark. If τ is a natural isomorphism then, given $A, B \in \mathbf{C}$, the following diagram commutes:

$$\begin{array}{ccc} \mathsf{hom}(A,B) & & \\ {\scriptstyle F}\downarrow & \searrow^{G} & \\ \mathsf{hom}(F(A),F(B)) & \xrightarrow[\varphi]{} & \mathsf{hom}(G(A),G(B)) \end{array}$$

Where $\varphi(h) = \tau_B \circ h \circ \tau_A^{-1}$. It is clear, by definition of natural isomorphism, that φ is a bijection between the image of F and the image of G in the diagram above.

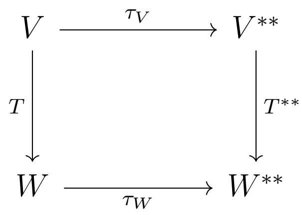
Remark. We denote the class of natural transformations from F to G as $\mathsf{Nat}(F,G)$.

Below, we give some examples of natural transformations:

- The determinant of a matrix with entries over a commutative ring can be seen as a natural transformation. The category of commutative rings is denoted here by CRng. We have a functor $(\_)^\times \colon$ **CRng** $\to$ **Grp** that takes each commutative ring to its group of unities (morphisms are taken to its restrictions), and functors $\mathsf{GL}_n \colon$ **CRng** $\to$ **Grp** that takes each commutative ring to its general linear group of order n (morphisms are taken coordinatewise). To each element in $M \in \mathsf{GL}_n(R)$ we can associate its determinant $\det(M) \in R^\times$. The naturality follows from the commutativity of the following diagram (that is commutative because $f \colon R \to S$ is a ring homomorphism):

$$\begin{array}{ccc} \mathsf{GL}_n(R) & \xrightarrow{\det} & R^\times \\ {\scriptstyle \mathsf{GL}_n(f)}\downarrow & & \downarrow{\scriptstyle f|_R} \\ \mathsf{GL}_n(S) & \xrightarrow[\det]{} & S^\times \end{array}$$

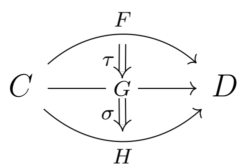
- The bidual can be seen as an endofunctor $(\_)^{**}$ of the category $\mathbf{Vect}_k$, and we have the natural transformation $\tau_V \colon Id_V \to V^{**}$ that sends, to each $v \in V$, the evaluation functional $\tau_V(v) \in V^{**}$ that is defined as $\tau_V(v)(f) = f(v)$. The very definition of T^{**} makes the diagram below commute, sowing that τ is indeed a natural transformation:



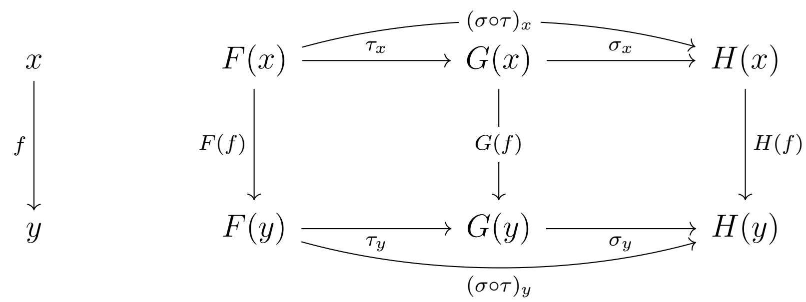
The restriction of τ to the category of finite vector spaces is a natural isomorphism.

- In **Set**, we have a canonical isomorphism $\hom(A \times B, C) \cong \hom(A, \hom(B, C))$, which is easily verified, by construction, to be natural in A, B, C.
- The universal properties of the tensor product of vector spaces, modules or abelian groups gives us natural isomorphisms $\hom(U \otimes V, W) \cong \hom(U, \hom(V, W))$. For more details about this kind of universal property, see the appendix B.

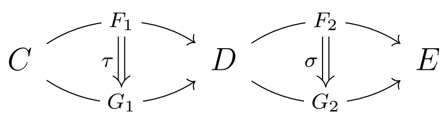
Definition 2.2.2. *Let F, G and H be functors and τ, σ be natural transformations as below*



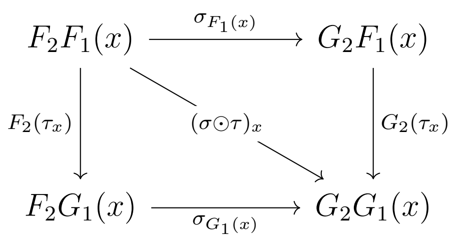
The ***vertical composition*** *of τ and σ, denoted by $\sigma \circ \tau \colon F \to H$. It is defined componentwise as $(\sigma \circ \tau)_x = \sigma_x \circ \tau_x$. Its commutative diagram is pictured below:*



We also have the notion of **horizontal composition** *between natural transformations. To* *define such a composition, suppose that we have F_1, F_2, G_1, G_2 functors and σ, τ natural transformations as below:*



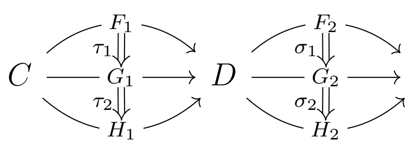
then we define the natural transformation $\sigma \circ \tau \colon F_2F_1 \to G_2G_1$ componentwise as the composite morphism below (the diagram below is, indeed, commutative, by naturality of σ):



This is our ***horizontal composition***.

Remark. We usually call this horizontal composition the ***Godement product***. It is pretty straightforward to verify that these two compositions are associative.

Therefore we have two notions of morphism composition. In one way, we can compose vertically, like the usual composite of morphisms in categories, but we also have an extra notion of horizontal composition. If we have functors $F_1, F_2, G_1, G_2, H_1, H_2$ and natural transformations $\sigma_1, \sigma_2, \tau_1, \tau_2$ as below

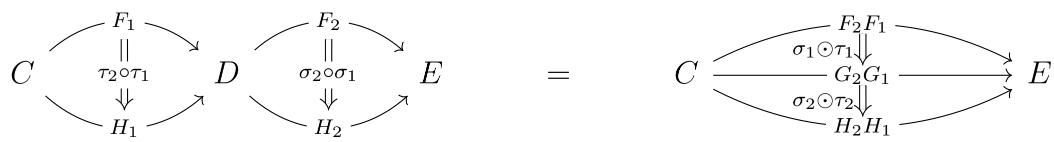


Then we have the **interchange law**:

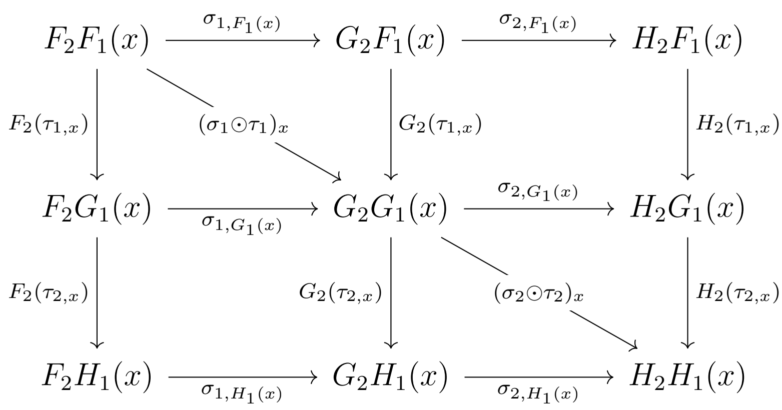
Proposition 2.2.3.

$$(\sigma_2 \circ \sigma_1) \odot (\tau_2 \circ \tau_1) = (\sigma_2 \odot \tau_2) \circ (\sigma_1 \odot \tau_1)$$

Diagramatically, we are saying that, in the diagrams below, the horizontal composition of the first one is equal to the vertical composition of the second.



Proof of the interchange law: In this proof, we denote things like $\tau_1(F(x))$ as $\tau_{1,F(x)}$ (and so on for the remaining natural transformations). Consider the following diagram:



This diagram commutes because the top right and bottom left squares commute by naturality, and the top left and bottom right squares commute by definition. The left hand side of the interchange law is the upper right leg of the diagram, while the right hand side is the diagonal composite. □

Therefore we have a notion of morphism between functors, with two types of composition. We can form the category of functors from one to another, with morphisms being natural transformations between those functors, and the composition being the vertical composition. If we consider the category of functors $F\colon \mathbf{C} \to \mathbf{D}$, we denote it by $\mathbf{D}^{\mathbf{C}}$. The class of natural transformations between two functors $F, G \in \mathbf{D}^{\mathbf{C}}$ is denoted by Nat(F, G) or $[F, G]$. We cannot do horizontal composition in these functor categories $\mathbf{D}^{\mathbf{C}}$. If we consider **Cat** as the category of small categories, with morphisms between categories being functors, we have a notion of morphism between those morphisms (being, in that case, natural transformations), with some nice properties as we saw above. We call this type of category a **two dimensional category**, or simply a 2-**category**. The formal definition of 2-category and some other types of higher dimensional categories are going to arise soon in this work.

Earlier we defined an equivalence of categories as a fully faithful and essentially surjective functor $F\colon \mathbf{C} \to \mathbf{D}$. An isomorphism of categories is a pair of functors such that each functor is the inverse of the other, i.e, F, G such that $FG = \mathrm{Id}_{\mathbf{D}}$ and $GF = \mathrm{Id}_{\mathbf{C}}$. In the categorical point of view, these equalities could have been replaced by natural isomorphisms and this wouldn't alter the categorical properties that are transferred from one category to another. Luckily, there is a nice proposition binding these two concepts:

Proposition 2.2.4. *A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence of categories if and only if there exists a functor $G\colon \mathbf{D} \to \mathbf{C}$ such that $FG \cong \mathrm{Id}_{\mathbf{D}}$ and $GF \cong \mathrm{Id}_{\mathbf{C}}$.*

Proof: First, suppose that the functor G exists. Then each $d \in \mathbf{D}$ is isomorphic to $F(c)$, where $c = G(d') \in \mathbf{C}$ for some $d \in \mathbf{D}$. So F is essentially surjective.

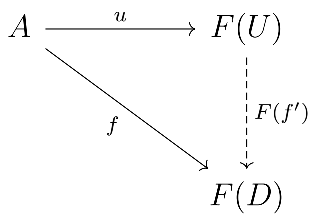
Since $GF \cong \mathrm{Id}_{\mathbf{C}}$, we have that $\hom(C, C') \overset{\varphi}{\cong} \hom(GF(C), GF(C'))$ as in the initial remark of this section. If $F(f) = F(g)$ for $f\colon C \to C'$, then obviously $GF(f) = GF(g)$ and the isomorphism φ gives us that $f = g$. The same argument applies to local surjectivity.

Now, assume that F is an equivalence of categories. For $D, D' \in \mathbf{D}$ we define, on objects, $G(D) = C$ for a chosen $C \in \mathbf{C}$ and a chosen isomorphism $F(C) \cong D$ (there always exists at least one such C, by hypothesis). If $f\colon D \to D'$, we define $G(f)\colon C \to C'$ as the unique arrow $g\colon C \to C'$ making the following diagram commute (C, C' are the choices made for $G(D), G(D')$ on objects, respectively):

$$\begin{array}{ccc} D & \xrightarrow{\quad f \quad} & D' \\ \cong\big\uparrow & & \big\uparrow\cong \\ F(C) & \xrightarrow[F(g)]{\qquad} & F(C') \end{array}$$

The arrow g exists by local surjectivity and is unique by local injectivity. Functoriality of G follows from pasting diagrams and applying uniqueness. By its own construction, it is clear that the composites of F and G are naturally isomorphic to the identities. □

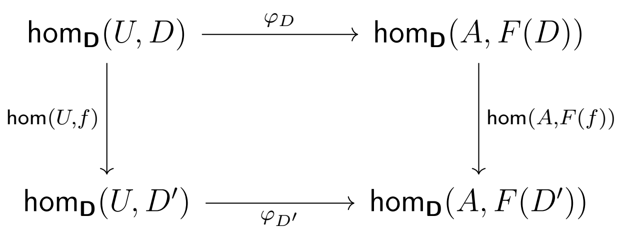
Definition 2.2.5. *A* **universal arrow** *from an object $A \in$ **C** to a functor $F\colon$ **D** $\to$ **C** is a pair (U, u), where $U \in$ **D**, $u\colon A \to F(U)$ and such that for any arrow $f\colon A \to F(D)$, there exists a unique arrow $f'\colon U \to D$ such that the following diagram commutes:*



Remark. It is clear that we also have the notion of a universal arrow from a functor F to an object A. All of the universal properties of mathematics can be fitted in these definitions, with F being the functor in the construction (e.g fields of quotients, free categories, free vector spaces, free groups, tensor products).

Proposition 2.2.6. *Given a functor $F\colon$ **D** $\to$ **C**, a pair $(U, u\colon A \to F(U))$ is universal from A to F if and only if we have a family of functions $\varphi_D\colon$ hom$_{\mathbf{D}}(U, D) \to$ hom$_{\mathbf{D}}(A, F(D))$ that defines a natural isomorphism.*

Proof: Suppose that the pair is a universal arrow. We can choose φ_D to be the function that sends $f'\colon U \to D$ to $\varphi(f') = F(f') \circ u$. It is clear that φ_D as defined is a bijection, by definition of universal arrow, for this function being bijective is just a restatement of the definition. Being natural means that the following diagram commutes for every $f\colon D \to D'$:



The desired naturality actually holds because it is the same as requiring $F(ff') \circ u = F(f) \circ (F(f') \circ u)$.

Now, suppose that we have the natural isomorphism φ as in the statement of the proposition. If we choose $D = U$, we have the following commutative diagram (for any D', $f'\colon U \to D'$):

$$\begin{array}{ccc} \mathrm{hom}_{\mathbf{D}}(U,U) & \xrightarrow{\varphi_U} & \mathrm{hom}_{\mathbf{D}}(A,F(U)) \\ {\scriptstyle \mathrm{hom}(U,f')}\downarrow & & \downarrow{\scriptstyle \mathrm{hom}(A,F(f'))} \\ \mathrm{hom}_{\mathbf{D}}(U,D') & \xrightarrow[\varphi_{D'}]{} & \mathrm{hom}_{\mathbf{D}}(A,F(D')) \end{array}$$

Mapping 1_U in each composite, we have that $\varphi_{D'}(f') = F(f') \circ u$. Since each component of φ is a bijection, the desired result follows. □

The Yoneda lemma and how it establishes the Yoneda embedding may be the most important result in the elementary theory of categories, not only for being a very useful tool, but also for its conceptual role. It describes how any locally small category **C** can be embedded into a category of contravariant functors defined from **C** to **Set**. This generalizes classical results such as the Cayley's theorem from group theory (interpreted categorically).

Theorem 2.2.7. ***(Yoneda lemma)*** *Given a functor $F\colon$ **C** $\to$ **Set**, with **C** locally small, and any object $A \in$ **C**, there exists a bijection $y\colon$ Nat(hom$(A,_), F) \to F(A)$ defined by $y(\tau) = \tau_A(1_A)$.*

Proof: Since τ is natural, we have the commutativity of the following diagram (for any $f\colon A \to B$):

$$\begin{array}{ccc} \mathrm{hom}(A,A) & \xrightarrow{\tau_A} & F(A) \\ {\scriptstyle \mathrm{hom}(A,f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathrm{hom}(A,B) & \xrightarrow[\tau_B]{} & F(B) \end{array}$$

Applying both composites on 1_A, we obtain $\tau_B(f) = F(f)(\tau_A(1_A))$. This implies that every natural transformation τ is uniquely determined by $\tau_A(1_A)$, from where injectivity follows. Surjectivity is equally simple since we have exactly how a natural transformation is related to $\tau_A(1_A)$. More explicitly, if $X \in F(A)$, we can define a natural transformation with $\tau_B(f) = F(f)(X)$ as components. We know that $\tau_B(f) = F(f)(\tau_A(1_A))$. Putting $f = 1_a$, we get $\tau_B(1_A) = \tau_A(1_A) = y(\tau) = X$, proving the desired result. □

Corollary. *If we are given objects A, $B \in$ **C** and a natural transformation $\tau\colon$ hom$(A,_) \to$ hom$(B,_)$, then there exists $h\colon B \to A$ such that $\tau =$ hom$(h,_)$.*

Proof: By the same argument in our last proof, we have $\tau_C(f) = \mathrm{hom}(B,f)(\tau_A(1_A)) = f \circ (\tau_A(1_A))$, where $\tau_A(1_A)\colon B \to A$. We can then take $h = \tau_A(1_A)$. □

Proposition 2.2.8. *The bijection y of theorem **2.22** is a natural isomorphism $y\colon N \to E$, with $N, E\colon$ **Set**$^{\mathbf{C}} \times$ **C** $\to$ **Set** defined as follows:*

1. *E is the evaluation functor, which sends (F, A) to $F(A)$ and $(\tau\colon F \to G, f\colon A \to B)$ to $G(f) \circ \tau_A = \tau_B \circ F(f)$, i.e, the diagonal of the naturality square;*
2. *N is the natural functor, which maps (F, A) to the set Nat(hom$(A, _), F)$ and maps $(\tau\colon F \to G, f\colon A \to B)$ to the function $\Theta\colon$ Nat(hom$(A, _), F) \to$ Nat(hom$(B, _), G)$ defined as $\Theta(\sigma)_C = \tau_C \circ \sigma_C \circ$ hom(f, C) which is indeed a natural transformation, for it is the composite of natural transformations.*

Proof: What we say in this proposition is theorem **2.21** restated in a more categorical viewpoint. □

Theorem 2.2.9. ***(The Yoneda embedding)*** *The assignment $Y\colon$ **C** $\to$ **Set**$^{\mathbf{C}^{op}}$ defined as $Y(A) =$ hom$(_, A)$ on objects and $Y(f) =$ hom$(_, f)$ is a fully faithful functor. Y is called the **Yoneda embedding**.*

Proof: Functoriality is evident. Being fully faithful follows pretty straightforward from our previous results. □

Corollary. *If there exists a natural isomorphism $\varphi\colon$ hom$(A, _) \to$ hom$(B, _)$, then we have that $A \cong B$ via an isomorphism $f\colon B \to A$ such that $\varphi(g) = g \circ f$.*

Proof: Fully faithful functors reflect isomorphisms. □

Corollary. *If hom$(F(_), _) \cong$ hom$(G(_), _)$ naturally, then $F \cong G$ naturally.*

Proof: That $F(A)$ and $G(A)$ are isomorphic follows from the last corollary. We just need to prove that this isomorphism is natural in A. Naturality follows from the fact that hom$(F(A), F(B)) \cong$ hom$(G(A), F(B)) \cong$ hom$(G(A), G(B))$ naturally. □

Remark. The categories in this final discussion are locally small and, usually, when we say that something "follows from Yoneda Lemma" we are actually saying that it follows from any of these results we stated.

Adjunction is a weak form of equivalence between categories that greatly extends the idea of relating categories. There are adjunctions that can be modified to become actual equivalences of categories.

Definition 2.2.10. *An* **adjunction** *from* $\boldsymbol{C}$ *to* $\boldsymbol{D}$ *is a triple* $(F, G, \varphi)\colon \boldsymbol{C} \to \boldsymbol{D}$*, where* $F\colon \boldsymbol{C} \to \boldsymbol{D}$ *and* $G\colon \boldsymbol{D} \to \boldsymbol{C}$ *are functors and* φ *is a family of natural isomorphisms* $\varphi_{a,b}\colon hom_{\boldsymbol{D}}(F(a), b) \to hom_{\boldsymbol{C}}(a, G(b))$ *that are usually denoted just as* φ*. In this case we say that* F *is a* **left adjoint** *for* G*,* G *is a* **right adjoint** *for* F *and we write* $F \dashv G$*.*

If we set $b = F(a)$ above, we can define the arrow $\eta_a = \varphi(1_{F(a)})\colon a \to G(F(a))$. Also, if we set $a = G(b)$, we can define $\epsilon_b = \varphi^{-1}(G(b))$. These arrows are universal, by Yoneda's lemma. The same argument in Yoneda's lemma gives us $\varphi(f) = G(f)\eta_a$, $f\colon F(a) \to b$ and $\varphi^{-1}(g) = \epsilon_b F(g)$, $g\colon a \to G(b)$. By definition, we have $G\epsilon \circ \eta G = 1$ and $\epsilon F \circ F\eta = 1$. We get this following theorem:

Theorem 2.2.11. *Each adjunction* (F, G, φ) *is uniquely determined by natural transformations* $\eta\colon 1 \to GF$ *and* $\epsilon\colon FG \to 1$ *such that* $\epsilon F \circ F\eta = 1$ *and* $G\epsilon \circ \eta G = 1$*.*

Proof: By our previous discussion, we have φ and φ^{-1} defined uniquely by η and ϵ. □

Definition 2.2.12. *An* **adjoint equivalence** *is an adjunction* (F, G, η, ϵ) *such that* η *and* ϵ *are natural isomorphisms, i.e,* F *and* G *form an equivalence.*

Remark. We do not give many examples of this final part since we practically will not use this concept here. One should notice that universal properties usually give rise to an adjunction. For those more familiar with category theory, it is important to say that the concept of an adjunction is so general that most of the core concepts of the theory are a special case of adjunctions.

2.3 WORDS AND ASSOCIATIVITY

In this section we are going to learn the essential tools to develop the theory of associativity in monoidal categories.

Definition 2.3.1. *Given any set* Σ*, we define the set of* **words** *in the* **alphabet** Σ*, denoted by* $W(\Sigma)$*, as the set of all finite sequences of elements of* Σ*. Clearly,* $W(\Sigma)$ *is a monoid with concatenation as the monoid operation, and the empty word (i.e, the empty sequence) as the neutral element.*

Definition 2.3.2. *Given any set* Σ *with a distinguished element* $e_0 \in \Sigma$ *and a distinguished symbol* $\star$*, we define recursively a set* $W(\Sigma, \star, e_0)$ *as below:*

1. *Every element of* Σ *is an element of* $W(\Sigma, \star, e_0)$*;*

2. *If* $x, y \in W(\Sigma, \star, e_0)$*, then the word* $(x \star y) \in W(\Sigma, \star, e_0)$*. Here, parenthesis are also viewed as symbols of the word.*

By abuse of notation, the outter parenthesis of an expression in $W(\Sigma, \star, e_0)$ *will often not be written.*

Clearly, there is an operation in this set induced by the second condition, which is the operation $\star\colon W(\Sigma, \star, e_0) \times W(\Sigma, \star, e_0) \to W(\Sigma, \star, e_0)$, that sends (x, y) to $x \star y$. Clearly this operation is not associative nor does it have a neutral element. The reason why we introduced this new set it is because in category theory we are not interested just in (strict) associativity, but also in associativity up to isomorphism. So, with a little bit of effort we can make this set into a category that will model this idea of associativity up to isomorphism. From now on, we are going to fix $\Sigma = \{(\_), e_0\}$ a binary alphabet.

Definition 2.3.3. *The* **length** *of a word of* $W(\{(\_), e_0\}, \star, e_0)$*, denoted by* L*, is defined recursively as below:*

1. $L((\_)) = 1$*, and* $L(e_0) = 0$*;*

2. $L(x \star y) = L(x) + L(y)$*.*

It is clear from this definition that the set of all words of length n represents all possible parenthesizations of a $\star$ product of n instances of $(\_)$ and an arbitrary number of e's. We define $L(e_0) = 0$ because in a product the neutral element must not alter the size of our formal product (e.g., $xy = xye = xyee$ in a monoid).

Definition 2.3.4. *We define a category* $\mathbf{W}$ *as the category with objects* $W(\{(\_), e_0\}, \star, e_0)$ *and a morphism* $z \to w$ *if and only if* $L(z) = L(w)$*, with identities and composition of arrows defined just like in a poset viewed as a category. If a morphism exists, it is an isomorphism (by symmetry, if there exists a morphism* $z \to w$*, there will exist a morphism* $w \to z$*, and by uniqueness the composition must be the identity). By the comment of our last paragraph, it is a category in which all* e*'s of a word can be deleted up to isomorphism and in which* $\star$ *is associative up to isomorphism. We call this category the* **category of unary words** *(the reason for "unary" and not "binary", althought some authors use binary in this context, is that* *we can delete, up to isomorphism, all instances of e, and e itself has no length, so it can be viewed as a disposable letter).*

Now, suppose that we have a bifunctor $\otimes\colon \mathbf{C}\times\mathbf{C}\to\mathbf{C}$ and a distinguished element $e\in\mathbf{C}$. We can define functors $F\colon \mathbf{C}^n\to\mathbf{C}$ obtained recursively from $\otimes$-iterations using these words defined before, in such way that we can view words as functors associated with this words. Again, we take $\Sigma=\{(\_),e_0\}$ as our alphabet and define recursively, for each element in $W(\Sigma,\star,e)$, functors as below:

1. $e_0\colon \mathbf{C}\to\mathbf{C}$ is the functor that sends every object to e and every morphism to 1_e;

2. $(\_)=Id_{\mathbf{C}}\colon \mathbf{C}\to\mathbf{C}$;

3. For two given words (viewed as functors) $v\colon \mathbf{C}^n\to\mathbf{C}$, $w\colon \mathbf{C}^m\to\mathbf{C}$ given by this process, we define $v\star w\colon \mathbf{C}^{n+m}\to\mathbf{C}$ as the composite $\mathbf{C}^{n+m}\overset{F\times G}{\to}\mathbf{C}\times\mathbf{C}\overset{\otimes}{\to}\mathbf{C}$.

Intuitively, these functors just tell us that, if we have a word of length n with some instances of e_0, then we replace every instance of e_0 with e (or 1_e), each blank space with objects (or morphisms) of the given n-uple in $\mathbf{C}$ (in the order that they are given) and each $\star$ with the bifunctor $\otimes$. Now, it should be clearer that this functor produces to us all instances of $\otimes$-products of elements of $\mathbf{C}$, with all possible parenthesizations. These concepts will be important when we soon start discussing coherence questions in monoidal categories. Below, we give an example of a functor defined recursively as above.

$$\mathbf{C}^5 \xrightarrow{(e_0\star(((\_)\star(\_))\star(\_)))\star e_0} \mathbf{C}$$

$$(a_1,a_2,a_3,a_4,a_5)\longmapsto (e\otimes((a_2\otimes a_3)\otimes a_4))\otimes e$$

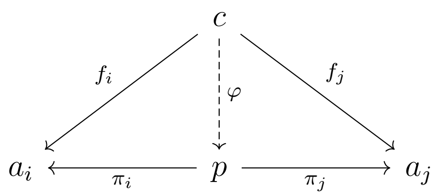
$$(f_1,f_2,f_3,f_4,f_5)\longmapsto (1_e\otimes((f_2\otimes f_3)\otimes f_4))\otimes 1_e$$

This functor gives us all possible $\otimes$-products of objects (and morphisms) with this fixed parenthesization and the two e's fixed in their positions. So, to study associativity issues, we have to compare two functors with the same length, where the length of such a functor is defined as the length of the associated word.

2.4 PRODUCTS, COPRODUCTS, PULLBACKS AND PUSHOUTS

As we saw in the first section, products are an interesting construction, arising with universal diagrams across all mathematics. In general, we can define a product as follows:

Definition 2.4.1. *A* **product** p *of objects* $a_1, \dots, a_n$ *in a category* ***C*** *is an object* $p \in$ ***C*** *together with morphisms (called projections)* $\pi_j \colon p \to a_j$ *such that for any other object* $c \in$ ***C*** *and morphisms* $f_j \colon c \to \pi_j$*, we have a unique arrow* $\varphi \colon c \to p$ *making the following diagram commute (for all* $i, j \in \{1, \dots, n\}$*):*



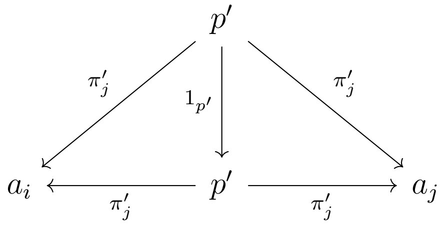
The dual concept of product is called a **coproduct**. *In general, we call the dual of a concept its coconcept. Categorically, the definition of product gives us that we have a natural isomorphism* $\prod_{i=1}^{n}$ *hom*$(\_, a_i) \overset{\cong}{\to}$ *hom*$(\_, p)$. *We give some examples below.*

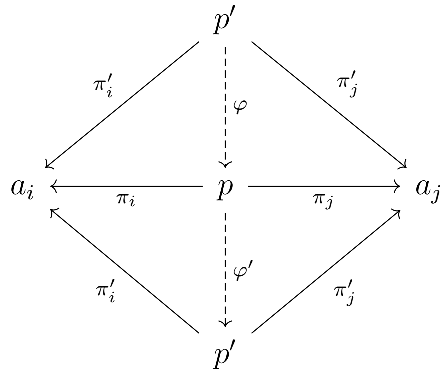
- If a poset P is viewed as a category, then a product is, by definition, a least upper bound.
- In **Vect**$_k$, **Grp**, **Set**, **Top**, the product is the usual direct cartesian product studied in the respective theories.

Remark. A product of 0 objects is, by definition, a terminal object.

Proposition 2.4.2. *Products are unique up to natural isomorphism. Dually, coproducts are unique up to natural isomorphism.*

Proof. If p, p' are products of $a_1, \dots, a_n$, then the following diagrams commute:



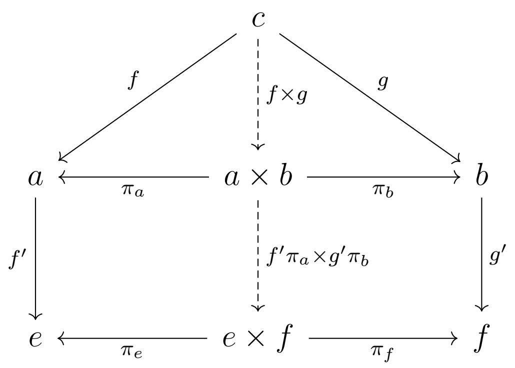


Therefore, $1_{p'}$ and $\varphi'\varphi$ are arrows making the left diagram commute. By uniqueness of this arrow, $\varphi'\varphi = 1_{p'}$. Similarly, we have $\varphi\varphi' = 1_p$, proving that $p \cong p'$. Naturality is pretty straightforward. □

This product object, which is unique up to isomorphism, will be denoted as $a_1 \times \cdots \times a_n$, and the unique arrow φ of the definition will be denoted as $f_1 \times \cdots \times f_n$. Using the same argument of uniqueness as before, it is clear that, when the arrows in discussion are composable, we have

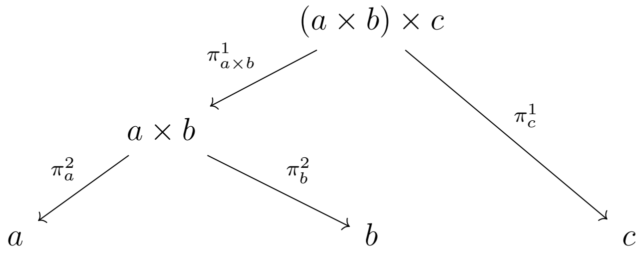
$$(f_1 \dots f_n) \times (g_1 \dots g_n) = (f_1 \times g_1) \dots (f_n \pi^a_{n-1} \times g_n \pi^b_{n-1})$$

. This is illustrated in the diagram below as well as discussed for the case $n = 2$. For arbitrary n, the argument is analogous:

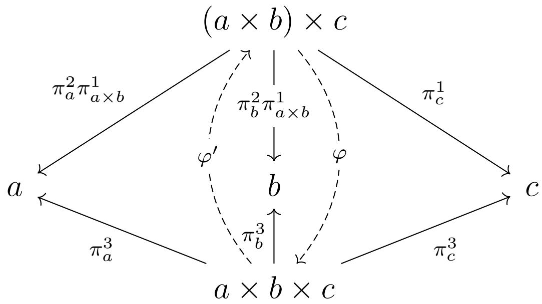


Now, if $a \overset{\varphi}{\cong} a'$ and $b \overset{\varphi'}{\cong} b'$ we have $a \times a' \overset{\psi}{\cong} b \times b'$, where $\psi = (\varphi\pi_a) \times (\varphi'\pi_b)$. The inverse for ψ is $\psi' = (\varphi^{-1}\pi_{a'}) \times (\varphi'^{-1}\pi_{b'})$ due to the last equality proved.

We proved that, when the product exists, it is unique, but we did not prove that all possible parenthesizations for the product of $a_1, \dots, a_n$ are isomorphic. First of all, consider the following morphisms constructed from $(a \times b) \times c$ to a, b and c:



Now, using the universal property of the product, we have unique φ and φ' as below, making the diagram commute. By the same uniqueness argument given before, each φ, φ' are isomorphims, being one the inverse of the other.

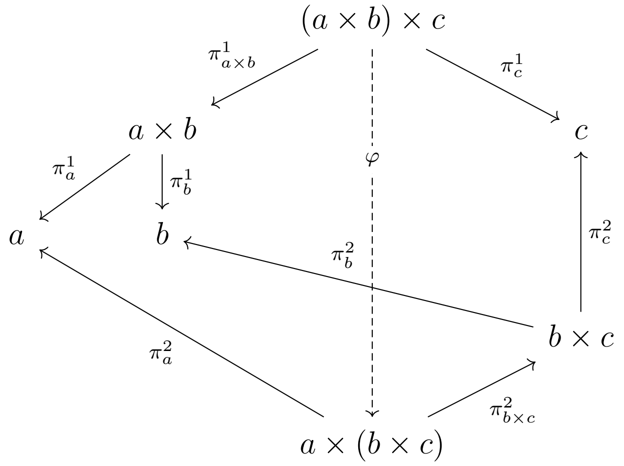


The argument can be mimicked for $a \times (b \times c)$, so in particular we have $(a \times b) \times c \cong a \times (b \times c)$ as the composition of these two isomorphism involving $a \times b \times c$.

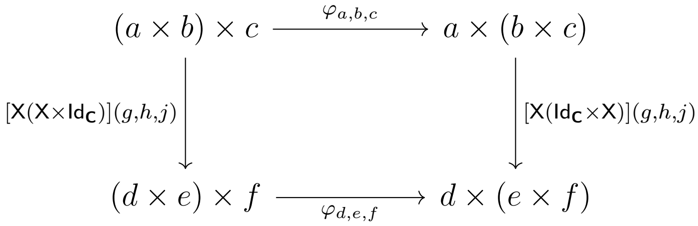
If a category **C** has products for all pair of objects, we can define a bifunctor $\bigtimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ as $\bigtimes(a, b) = a \times b$ on objects and $\bigtimes(f, g) = (f\pi_a) \times (g\pi_b)$ for $a \xrightarrow{f} c$ and $b \xrightarrow{g} d$. $\bigtimes$ is functorial because of what we already proved about products. Notice that $\pi_c \circ \bigtimes(f, g) = f\pi_a$ and $\pi_d \circ \bigtimes(f, g) = g\pi_b$. Our last argument in this section proved that $(a \times b) \times c \cong a \times (b \times c)$ for any $a, b, c \in \mathbf{C}$. In functorial terms, we can rewrite this as $\bigtimes(\mathsf{Id}_{\mathbf{C}} \times \bigtimes) \cong \bigtimes(\bigtimes \times \mathsf{Id}_{\mathbf{C}})$. We claim that this isomorphism (say, α) is, indeed, natural:

Theorem 2.4.3. $\bigtimes(Id_C \times \bigtimes) \cong \bigtimes(\bigtimes \times Id_{\mathbf{C}})$ *constructed as before is a natural isomorphism.*

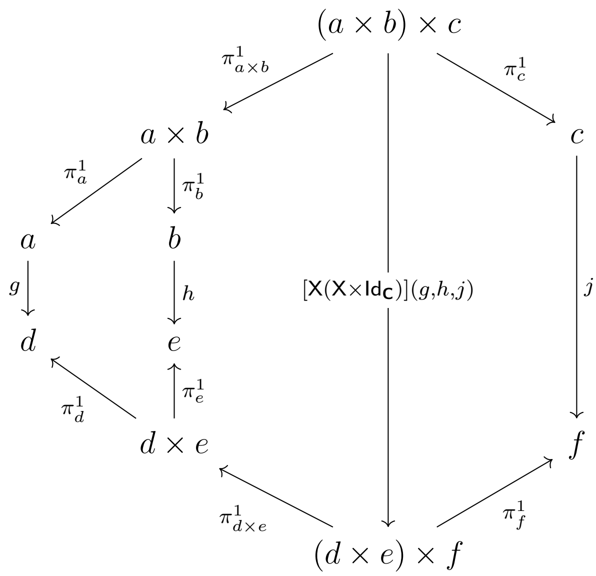
Proof. By construction, our isomorphism φ is the only morphism making the following diagram commute:

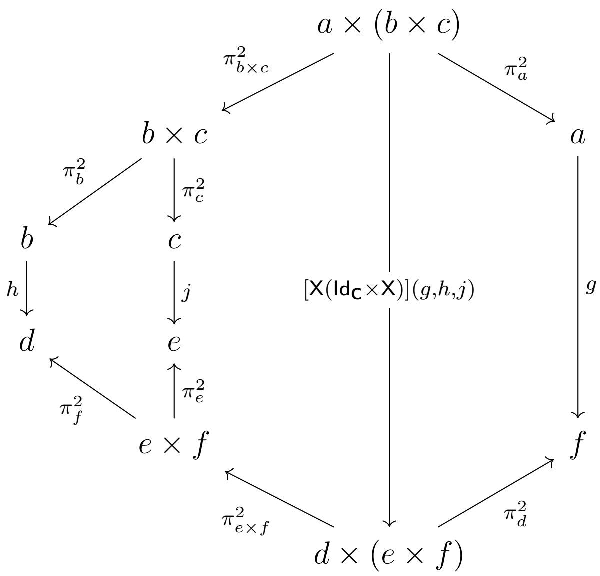


It remains to show that this assignment is natural. We will label the isomorphism φ above as $\varphi_{a,b,c}$. We have to prove that the following diagram commutes for all $a, b, c, d, e, f \in$ **C** and morphisms $a \xrightarrow{g} d, b \xrightarrow{h} e, c \xrightarrow{j} f$:



In order to prove that, we only have to show that the two compositions are the same when we compose them with the projections, due to the universality of the product. We do this with π^2_d, and all the remaining projections are similar. We are going to use the following commutative diagram to do our calculations. Both of them are commutative by definition of the functors involved:





$$\pi^2_d\varphi_{d,e,f}[\bigtimes(Id_C \times \bigtimes)](g,h,j) = \pi^1_d\pi^1_{d\times e}[\bigtimes(\mathsf{Id}_{\mathsf{C}} \times \bigtimes)](g,h,j) = g\pi^1_a\pi^1_{a\times b}.$$

$$\pi^2_d[\bigtimes(\mathsf{Id}_{\mathsf{C}} \times \bigtimes)](g,h,j)\varphi_{a,b,c} = g\pi^2_a\varphi_{a,b,c} = g\pi^1_a\pi^1_{a\times b}.$$

The same argument holds with the same diagrams for obtaining the other equality. Therefore, the isomorphism is natural. □

Theorem 2.4.4. *All parenthesizations of products are naturally isomorphic. More precisely, all words of length n viewed as functors are naturally isomorphic.*

Proof: We prove this by induction on the length n of the word. For $n = 3$ we have the previous proposition. We proceed to adapt the argument that is usually given for the general associativity law for algebraic operations satisfying associativity for any 3 elements.

Assume, by induction, that any word of length $m < n$ is naturally isomorphic to a canonical word, constructed following its action on objects as below (here we are given j objects $a_1, \ldots, a_j$):

1. $\prod_{i=0}^{0} a = a$;
2. $\prod_{i=0}^{j} a_i = a_j \times \left(\prod_{i=0}^{j-1} a_i\right)$.

If z is any word of length n, it is clear by construction that $z = v \times w$, with $L(v), L(w) < L(z)$. By the induction hypothesis, there exist natural isomorphisms $v \cong \prod a_j$, $w \cong \prod b_k$ for some sequence of objects. By our previous results we have the following sequence of natural isomorphisms (here, the c_l, d_p are just relabelings of the sequence of objects involved):

$$v \times w \cong (\prod a_j) \times (\prod b_k) = (a_i \times \prod^{i-1} a_j) \times (\prod b_k) \cong a_i \times (\prod^{i-1} a_j \times \prod b_k) \cong a_i \times \prod^{n-1} c_l = \prod^{n} d_p$$

We applied the induction hypothesis to the last natural isomorphism that is not an equality above, and we could do so because it is a word of length $n-1$.

As any word of length n is naturally isomorphic to this canonical word, we have that all words of length n are naturally isomorphic. □

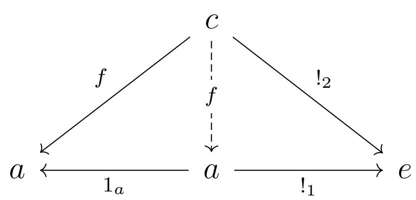
Corollary. *A category has binary products if and only if has finite products.*

Proof: We construct finite products recursively using binary products via the canonical words constructed in the proof above. □

Remark. There exists an isomorphism $a \times b \overset{\tau}{\cong} b \times a$ because we can use π_a, π_b for both products and the diagrams involved obviously will stay commuting.

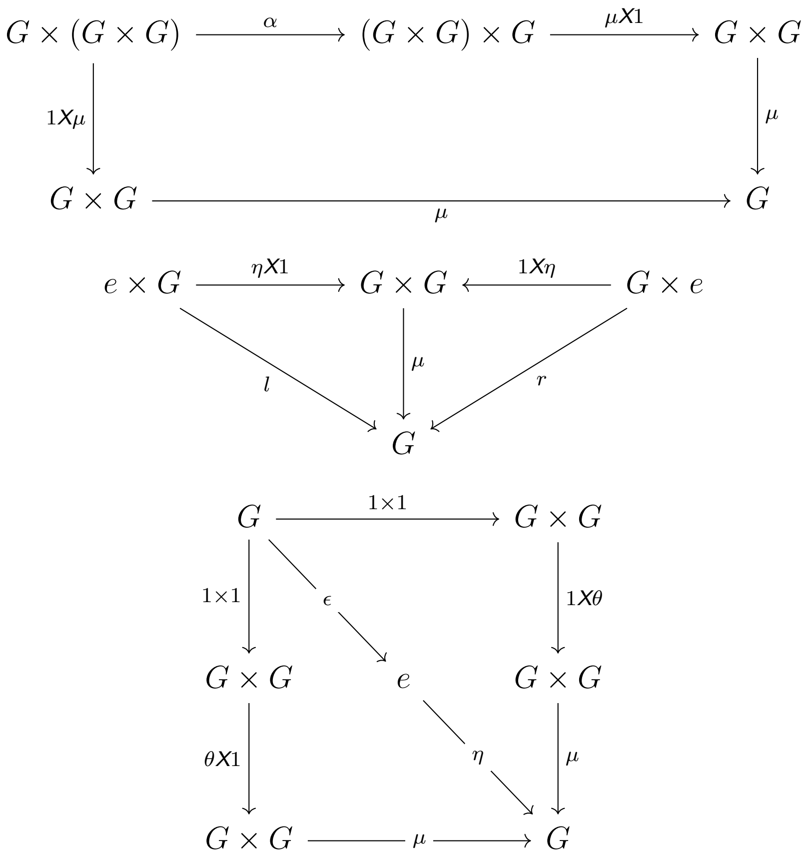
Proposition 2.4.5. *If e is a terminal object in a category **C** with finite products, then $e \times a \cong a \cong a \times e$ naturally for any $a \in$ **C**. We denote this isomorphisms by $r_a \colon a \times e \to a$ and $l_a \colon e \times a \to a$.*

Proof: We can just prove that a is a product of a and e. The unique arrow $a \to e$ will be denoted as $!_1$. The following diagram commutes for any a:



Where f is any morphism and $!_2$ is the only morphism $c \to e$. Uniqueness of the middle arrow $f \colon c \to a$ making the diagram commute follows from the requirement that the left triangle is commutative. It follows that a is a product of a and e. □

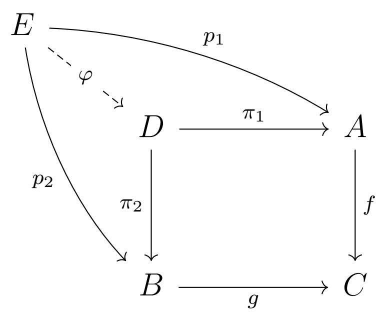
Definition 2.4.6. *A* **group object** *in a category **C** with finite products and a terminal object e is a 4-uple (G, μ, η, θ) such that $G \in$ **C**, $\mu \colon G \times G \to G$, $\eta \colon e \to G$, $\theta \colon G \to G$, and the* *diagrams below commute. ϵ is the unique morphism $G \to e$. Sometimes, μ ,η, θ, and ϵ are called, respectively,* **multiplication**, **unit (morphism)**, **inversion**, *and* **counit**.*:*



Remark. The first two diagrams characterize a ***monoid object*** in a (cartesian) category (see chapter **3**). The morphism θ is introduced, together with its diagram, to introduce inverses in the monoid, making it a group. Nevertheless, the objects in the category do not need to be sets, and we do not require elements to realize the group theoretical ideas. Also, notice that in some morphisms from the diagram we did the product $\bigtimes(f,g) = f \bigtimes g$ and in others the product $f \times g$.

All results in this sections hold dually for coproducts. This theorem can be viewed as a coherence theorem. It is a specific instance of a much more general theorem for the so called monoidal categories.

Definition 2.4.7. *In a category* ***C****, we define a* ***pullback*** *of a pair of arrows $A \xrightarrow{f} C \xleftarrow{g} B$ as an object D together with two arrows $\pi_1 \colon D \to A$, $\pi_2 \colon D \to B$ such that for any other* *object E and any two other arrows $p_1\colon E \to A$, $p_2\colon E \to B$ such that $f \circ p_1 = g \circ p_2$, there exists a unique $\varphi\colon E \to D$ such that the following diagram commutes:*



A **pushout** *in a category is defined as a pullback in the opposite category.*

Categorically, a pullback gives us a natural isomorphism

$$\text{Sub}(\_) \to \hom(\_, D).$$

Where Sub(_) is the functor induced by the subclass of hom$(_, A) \times$ hom$(_, B)$ formed by the pair of morphisms (p_1, p_2) satisfying $f \circ p_1 = g \circ p_2$.

Remark. Just as before, we can show that the pullback is unique up to isomorphism. Sometimes the pullback is called the fiber product, and denoted $D = A \times_C B$. In many situations, $A \times_C B$ can be thought as consisting of pairs $(x, y) \in A \times B$ such that $f(x) = g(y)$, like the pullback in **Set** or **Cat**. Indeed, given functors $\mathbf{B} \overset{S}{\to} \mathbf{C} \overset{T}{\leftarrow} \mathbf{B}$, the morphisms of the pullback $B \times_{\mathbf{C}} B$ are formed by the pairs of morphisms (f, g) such that $S(f) = T(g)$. Later we will see, in a certain context, S, T as source and target functors, so that the pullback gives us the composable morphisms.

Using this interpretation given in the above remark, we can better understand the following definition, which generalizes the concept of small category to be thought as relative to an "ambient" category **C**:

Definition 2.4.8. *Given a category* ***C*** *with pullbacks, an* **internal category** *(or* **category object***) D in* ***C*** *consists of two objects D_0, D_1 of* ***C*** *(called objects and morphisms of D) together with morphisms $s, t\colon D_1 \to D_0$, $e\colon D_0 \to D_1$ and $\circ'\colon D_1 \times_{D_0} D_1 \to D_1$, where the pullback is taken over s, t. We call s the source morphism, t the target morphism, e the identity-assigning morphism and $\circ'$ the composition morphism. They satisfy diagrams* *analogous to the axioms we gave in the definition of a category, abstracting from classes to categories.*

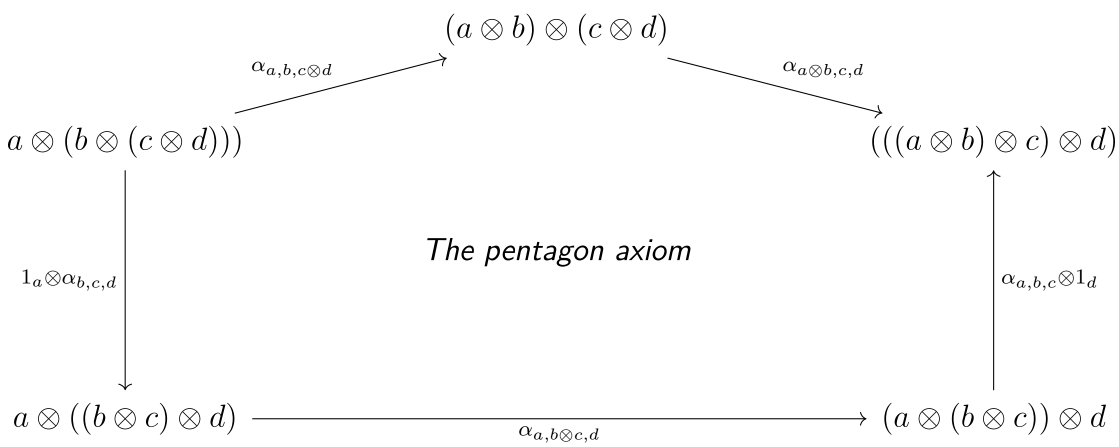
Remark. A small category is just an internal category to **Set**, for example. All the definitions given to categories can be abstracted to internal categories in a reasonably natural way.

3 MONOIDAL CATEGORIES

3.1 MONOIDAL CATEGORIES: THE BEGINNING

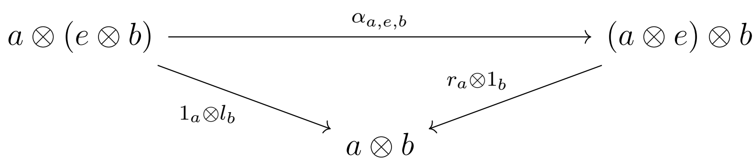
Definition 3.1.1. *A* **(relaxed) monoidal category** *consists of a 6-uple* $(\boldsymbol{C}, \otimes, e, \alpha, r, l)$*, where:*

1. $\otimes \colon \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$ *is a bifunctor (called tensor product);*
2. $\otimes(Id_{\boldsymbol{C}} \times \otimes) \overset{\alpha}{\cong} \otimes(\otimes \times Id_{\boldsymbol{C}})$ *is a natural isomorphism (called* **associator** *or* **associativity constraint***);*
3. $e \in \boldsymbol{C}$ *is a special object (called* **neutral object** *or* **unit object***);*
4. *Viewing* e *as the functor* $e \colon \{*\} \to \boldsymbol{C}$ *that sends the unique object to* e *and the unique morphism to* 1_e*, we have* $\otimes(Id_{\boldsymbol{C}} \times e) \overset{r}{\cong} Id_{\boldsymbol{C}}$ *and* $\otimes(e \times Id_{\boldsymbol{C}}) \overset{l}{\cong} Id_{\boldsymbol{C}}$ *are natural isomorphisms (called* **right** *and* **left unitors** *or* **unit constraints** *respectively);*
5. *The associator is subject to the following condition, for all objects* $a, b, c, d \in \boldsymbol{C}$*:*



6. *The unitors are subject to the following coherence condition, for all objects* $a, b \in C$*:*

The triangle axiom



When these natural isomorphisms are the identities, we call the monoidal category strict. Sometimes the name "Tensor categories" arise in the literature as a monoidal category *with some other additional structures, but here in this text tensor categories and monoidal categories are the same concept.*

Remark. We could have taken the inverse of associators and unitors to define what a monoidal category is. We would just have to adjust the directions of arrows in the coherence axioms. Actually, we are going to use this alternative definition later on in order to stay close to recent research papers.

Remark. Sometimes we will omit the subscripts from the associators and unitors, or else the equations will simply become unreadable.

Remark. One can ask if the other graphs constructed via parenthesizations have a well determined shape (like the parenthesizations for words of 4 letters form a pentagon). One can see this discussion in [42].

There are a lot of diverse examples on monoidal categories, and it is usually pretty easy to prove that a given category is monoidal (the diagrams for the usual examples are extremely simple to be verified). We give some interesting examples below:

1. **Set**, **Top**, **Grp**, **Vect**$_k$, **Rng**, **Grph**, **Cat**, **CAT** and many others are monoidal categories with $\otimes = \times$, their usual categorical products, and e their terminal object (Dually, coproducts with initial objects).

2. The categories **Vect**$_k$, **Ab** and $R-$ **Mod** are monoidal categories with usual tensor product $\otimes$ and $e = k$, $e =$ the trivial group, $e = R$ (respectively). For more details see B. All basic constructions can be generalized to the other two categories.

3. Ch(R-**Mod**) is a monoidal category with tensor product defined on objects by $(M \otimes N)_n = \bigoplus_{i+j=n} (M_i \otimes N_j)$, and on morphisms just as in vector spaces. Here, e is the chain complex $\cdots \to 0 \to 1 \to 0 \to \cdots$, with 1 marked with label $n = 1$ in the chain.

4. In the category **Diff**, we can take $\otimes$ to be the disjoint union $M \sqcup N$. A tensor product of morphisms $f_1 \otimes f_2 \colon M_1 \otimes M_2 \to N_1 \otimes N_2$ is defined as $(f_1 \otimes f_2)(x) = f_1(x)$ if $x \in M_1$ and $(f_1 \otimes f_2)(x) = f_2(x)$ if $x \in M_2$. Here, e is the empty manifold.

5. **Braid** is a strict monoidal category with $m \otimes n = m + n$ and given two braids $B \colon n \to n$, $B' \colon m \to m$, we define $B \otimes B' \colon n + m \to n + m$ as the juxtaposition of the two braids. We regard the only braid $\emptyset \in \hom(0, 0)$ as the neutral object.

6. $\mathbf{Rep}_G^k$ is a strict monoidal category, with tensor product $\rho \otimes \sigma \colon G \to GL(V \otimes W)$ defined as $(\rho \otimes \sigma)_g = \rho_g \otimes \sigma_g$. The tensor product of two equivariant linear transformations is defined as the tensor product of the two underlying linear functions.

7. For any category **C** we can view End(**C**), the category of endofunctors of **C**, as a monoidal category with $\otimes = \circ$.

8. As we showed earlier, any category **C** with finite products can be regarded as a monoidal category (choosing, for each pair of objects, a specified product, and the neutral object being the terminal object). A ***cartesian monoidal category*** is a monoidal category where the tensor product is the categorical product.

Regarding the usual tensor products, like the tensor product of modules, vector spaces, or abelian groups, it is clear that the pentagon axiom and the triangle axiom hold because of the equalities $\alpha_{U,V,W}(u \otimes (v \otimes w)) = (u \otimes v) \otimes w$, $r_U(v \otimes a) = av$ and $l_U(a \otimes v) = av$. One can think at first glance that the pentagon and the triangle axioms could be deduced from the other properties, which is completely false. If we take the natural isomorphism $\alpha' = -\alpha$ in these tensor products, the pentagon axiom simply will not hold for, one way, we get a positive sign and, in the other one, we get a negative sign.

In many contexts within category theory, we identify things that are isomorphic as the same thing, for example in the classification of finite groups, or the classification of compact connected topological surfaces, so one can try to avoid our discussion about coherence and these associators and unitors by just identifying these isomorphic objects and setting all monoidal isomorphisms equal to identities. We now see that this is not the case, as Isbell showed ([1]).

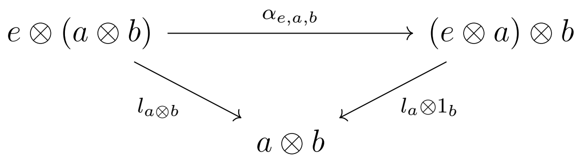
Isbell argument: Let $\mathbf{Card}$ be the usual skeleton of $\mathbf{Set}$ (i.e., it will be the category of cardinal numbers). In $\mathbf{Set}$, our tensor product will be the usual cartesian product, and our neutral element will be the set $\{\emptyset\}$. In the same way we define the tensor product in $\mathbf{Card}$. We have $\aleph_0 \times \aleph_0 = \aleph_0$, and both projections $\pi_1, \pi_2 \colon \aleph_0 \times \aleph_0 \to \aleph_0$ are epimorphisms. Identifying the isomorphic objects, we can identify $\alpha = Id$ to make an attempt of avoiding associators. Now, given any three functions $f, g, h \colon \aleph_0 \to \aleph_0$, we have the following equalities:

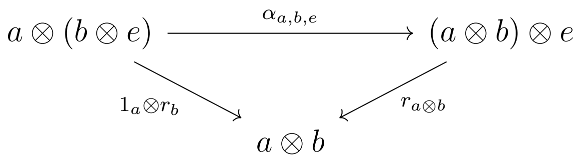
$$f\pi_1 = \pi_1(f \times (g \times h)) = \pi_1((f \times g) \times h) = (f \times g)\pi_1$$

$$g\pi_2 = \pi_2((h \times f) \times g) = \pi_2(h \times (f \times g)) = (f \times g)\pi_2$$

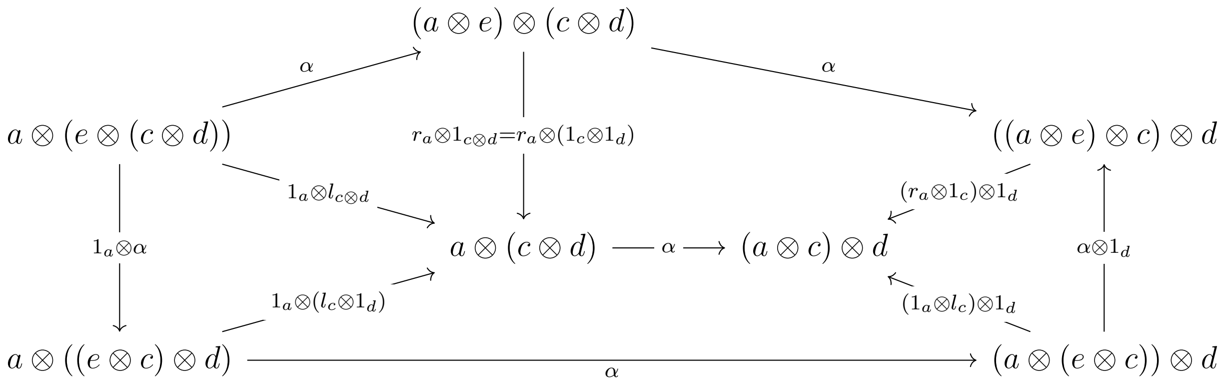
As π_1, π_2 are epimorphisms, we conclude that $f = f \times g = g$ for any functions $f, g\colon \aleph_0 \to \aleph_0$, which is obviously false. We conclude that we cannot just identify isomorphic objects and put $\alpha = Id$ to avoid the theory of monoidal categories. □

Proposition 3.1.2. *The following diagram commutes for all $a, b \in$ **C** in a monoidal category:*





Proof: Consider the following diagram:



The outter pentagon commutes by the pentagon axiom. The two squares (the upper and lower ones) commute by naturality of the associator. The upper left triangle and the right triangle commute by the triangle axiom (we also use bifunctoriality on the right triangle). All terms involved in these equations are isomorphisms, so we have that the left triangle also commutes.

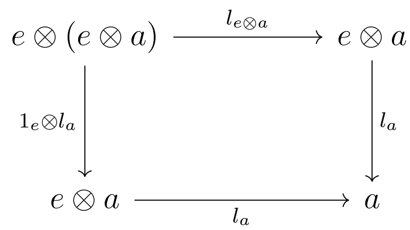
We conclude that $[1_a \otimes (l_c \otimes 1_d)](1_a \otimes \alpha) = 1_a \otimes l_{c\otimes d}$. Taking $a = e$, and using bifunctoriality on the left side of the equation, we get $1_e \otimes [(l_c \otimes 1_d)\alpha] = 1_e \otimes l_{c\otimes d}$. By naturality of the left unitor, we conclude that $l_{c\otimes d} = (l_c \otimes 1_d)\alpha$, and this is exactly what we wanted to prove. The other equality follows from a similar argument, taking $c = d = e$ in the pentagon. □

Proposition 3.1.3. *The following relations hold in a monoidal category **C** (for any $a \in$ **C**):*

$$l_{e\otimes a} = 1_e \otimes l_a,\ r_{a\otimes e} = r_a \otimes 1_e,\ l_e = r_e$$

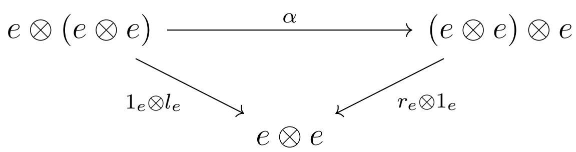
Proof:

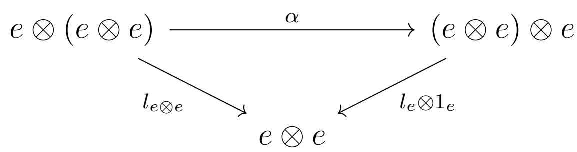
1) The following diagram commutes by naturality of the left unitor:



But as l_a is an isomorphism, we conclude that $l_{e\otimes a} = 1_e \otimes l_a$. A similar argument holds for the identity involving $r_{a\otimes e}$.

2) The two diagrams below commute (the first one by the triangle axiom, and the second one by our previous proposition):





But we already proved that $l_{e\otimes e} = 1_e \otimes l_e$, so, comparing the two diagrams, we obtain $r_e \otimes 1_e = l_e \otimes 1_e$. By naturality of unitors, we get $r_e = l_e$. □

Proposition 3.1.4. *End(e) is a commutative monoid with respect to composition.*

Proof: Let $f, g\colon e \to e$ be endomorphisms of e.

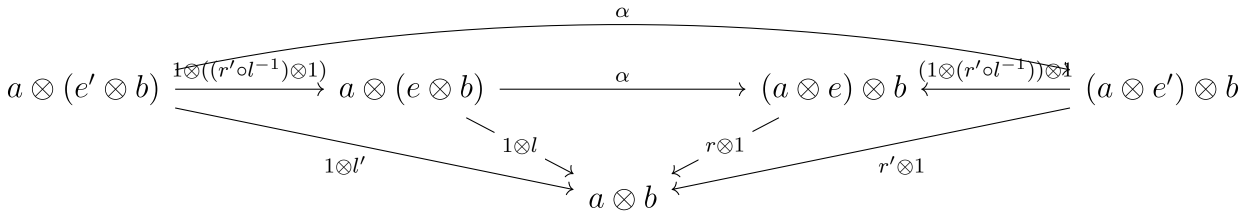
$$f \otimes g = (f \circ 1_e) \otimes (1_e \circ g) = (f \otimes 1_e)(1_e \otimes g) = (r_e^{-1} f r_e)(l_e^{-1} g l_e) = r_e^{-1}(fg) r_e$$

$$f \otimes g = (1_e \circ f) \otimes (g \circ 1_e) = (1_e \otimes g)(f \otimes 1_e) = (l_e^{-1} g l_e)(r_e^{-1} f r_e) = r_e^{-1}(gf) r_e$$

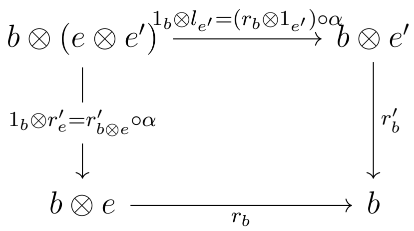
We used bifunctoriality of $\otimes$ and the fact that $r_e = l_e$. As r_e is an isomorphism, we conclude that $fg = gf$. Moreover, we have $f \otimes g = g \otimes f$. □

Proposition 3.1.5. *Given a monoidal category **C**, if there exists e' and natural isomorphisms $\otimes($Id$_{\mathbf{C}} \times e') \overset{r'}{\cong}$ Id$_{\mathbf{C}}$ and $\otimes(e' \times$ Id$_{\mathbf{C}}) \overset{l'}{\cong}$ Id$_{\mathbf{C}}$ satisfying the triangle axiom, then we have a unique coherent isomorphism $e \cong e'$. A **coherent isomorphism** of monoidal units is an isomorphism such that the diagram formed by the two triangle axioms connected by this isomorphism is commutative.*

Proof: $e' \overset{l^{-1}}{\cong} e \otimes e' \overset{r'}{\cong} e$. We want to prove that this isomorphism is coherent, i.e, that the following diagram commutes:



The top square commutes by naturality of α. The inner and outter triangles commute by the triangle axiom. The right triangle commutes because the following diagram commutes:



We can cancel α out of the legs of the diagram, and the commutativity follows from naturality of r'. The left triangle commutes by a similar calculation. Therefore, we found an isomorphism $\varphi\colon e' \to e$ such that the first diagram here commutes. φ being coherent can be summarize by saying that $l'_a = l_a \circ (\varphi \otimes 1_a)$, $r'_a = r_a \circ (1_a \otimes \varphi)$ for each $a \in \mathbf{C}$.

Now we wish to prove that an isomorphism $\varphi\colon e' \to e$ that satisfies the condition above is unique. In order to prove that, it suffices to show that the unique coherent automorphism of e is the identity. Indeed, if we have two coherent isomorphisms $\varphi, \varphi'\colon e' \to e$, then $\varphi' \circ \varphi^{-1}$ is a coherent automorphism of e. Now, if ψ is a coherent automorphism of e, then $l_e = l_e \circ (\psi \otimes 1_e)$, i.e, $1_e \otimes 1_e = 1_{e\otimes e} = \psi \otimes 1_e$, so it follows that $\psi = 1_e$. $\square$

Remark. With this proposition in mind, we can refer about ***the*** monoidal unit of a given monoidal product.

Definition 3.1.6. *A* **(lax) monoidal functor** $F\colon (\boldsymbol{C}, \otimes, e, \alpha, r, l) \to (\boldsymbol{D}, \boxtimes, e', \alpha', r', l')$ *is a functor* $F\colon \boldsymbol{C} \to \boldsymbol{D}$ *with a morphism* F_0 *and a family of natural morphisms* $F_2(\cdot,\cdot)$ *such that:*

$$F_0\colon e' \to F(e)$$

$$F_2(a,b)\colon F(a) \boxtimes F(b) \to F(a \otimes b)$$

making the diagrams below commute for all $a, b, c \in \boldsymbol{C}$*. When these morphisms are isomorphisms, we call the monoidal functor* **strong***. Furthermore, if they are all identities, then we call it*

strict.

$$
\begin{array}{ccc}
e' \boxtimes F(a) & \xrightarrow{l'_{F(a)}} & F(a) \\
{\scriptstyle F_0 \boxtimes 1_{F(a)}} \downarrow & & \uparrow {\scriptstyle F(l_a)} \\
F(e) \boxtimes F(a) & \xrightarrow[F_2(e,a)]{} & F(e \otimes a)
\end{array}
\qquad
\begin{array}{ccc}
F(a) \boxtimes e' & \xrightarrow{r'_{F(a)}} & F(a) \\
{\scriptstyle 1_{F(a)} \boxtimes F_0} \downarrow & & \uparrow {\scriptstyle F(r_a)} \\
F(a) \boxtimes F(e) & \xrightarrow[F_2(a,e)]{} & F(a \otimes e)
\end{array}
$$

$$
\begin{array}{ccc}
F(a) \boxtimes (F(b) \boxtimes F(c)) & \xrightarrow{\alpha} & (F(a) \boxtimes F(b)) \boxtimes F(c) \\
{\scriptstyle 1_{F(a)} \boxtimes F_2(b,c)} \downarrow & & \downarrow {\scriptstyle F_2(a,b) \boxtimes 1_{F(c)}} \\
F(a) \boxtimes F(b \otimes c) & & F(a \otimes b) \boxtimes F(c) \\
{\scriptstyle F_2(a, b \otimes c)} \downarrow & & \downarrow {\scriptstyle F_2(a \otimes b, c)} \\
F(a \otimes (b \otimes c)) & \xrightarrow[F(\alpha)]{} & F((a \otimes b) \otimes c)
\end{array}
$$

Remark. By dualization, that is, reversing the arrows of F_0, F_1, and the diagrams, we obtain the definition of F being a ***colax*** or ***oplax*** monoidal functor.

As usual, a natural transformation has to be from one functor to another. In the case of the family F_2, it is a natural transformation $\boxtimes \circ (F \times F) \xrightarrow{F_2} F \circ \otimes$. Sometimes we will denote $F_2(a, b)$ just as F_2. In these situations, the a, b will be clear from the context. Sometimes we also are going to denote $\boxtimes$ and $\otimes$ with the same symbol by abuse of notation. This can be tolerated because we usually do not work with two different tensor products for the same category, so the tensor product to which we are referring can be easily inferred from the context.

Below, we detail some examples of monoidal functors:

- The underlying functor $U\colon$ **Ab** $\to$ **Set**, with maps $U_e\colon \{*\} \to \mathbb{Z}$ and $U_\otimes\colon U(R) \times U(S) \to U(R \otimes S)$ defined as $U_e(*) = 1$ and $U(x, y) = x \otimes y$.
- The homology functor **H** is a monoidal functor with the mapping $\mathbf{H}_\otimes\colon \mathbf{H}(C) \otimes \mathbf{H}(D) \to \mathbf{H}(C \otimes D)$ given by $\mathbf{H}_\otimes(\overline{x} \otimes \overline{y}) = \overline{x \otimes y}$.
- One can show that the enveloping algebra **U** is a monoidal functor essentially the same way as **H** above.

- Define the monoidal category $\mathbf{Bord}_{(n-1,n)}$ as the category that has $(n-1)$-dimensional closed manifolds as objects, and each morphism is a cobordism, which is an n-dimensional manifold with boundary given by the disjoint union of two disjoint $(n-1)$-dimensional closed manifolds, namely, its domain and codomain. The tensor product is the disjoint union. A ***topological quantum field theory in dimension*** n is a monoidal functor $F\colon \mathbf{Bord}_{(n,n-1)} \to \mathbf{Vect}_k$, often required to satisfy extra conditions with quantum-theoretical motivations.

Given a monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$, for each unary word w such that $L(w) = n$, we can define families of natural transformations

$$F_w(a_1, \dots, a_n)\colon w(F(a_1), \dots, F(a_n)) \to F(w(a_1, \dots, a_n))$$

recursively as below:

1. $F_{e_0} = F_0$
2. $F_{(\_)}(a) = 1_{F(a)}\colon F(a) \to F(a)$
3. Given $u, v \in \mathbf{W}$ with $L(u) = n, L(v) = m$, we define $F_{u\star v}$ as the composite $F_2 \circ (F_u \otimes F_v)$. Diagramatically, we have the following commutative diagram:

$$\begin{array}{ccccc}
(u \star v)(F(A), F(B)) & & \xrightarrow{\quad F_{u\star v} \quad} & & F((u \star v)(A, B)) \\
\Big\| & & & & \Big\| \\
u(F(A)) \otimes v(F(B)) & \xrightarrow[F_u \otimes F_v]{} & F(u(A)) \otimes F(v(B)) & \xrightarrow[F_2]{} & F(u(A) \otimes v(B))
\end{array}$$

In the diagram above we made the identifications $A = (a_1, \dots, a_n),\ B = (b_1, \dots, b_m)$, $F(A) = (F(a_1), \dots, F(a_n)),\ F(B) = (F(b_1), \dots, F(b_m))$.

If $w \in \mathbf{W}$ with $L(w) = n$, then the natural transformation F_w is defined from the functor $w \circ (F^n)$ to the functor $F \circ w$, where F^n is $F \times \cdots \times F$ (n times).

Definition 3.1.7. *One of the most important concepts in the early discussion of monoidal categories is the concept of* **canonically constructed isomorphism***(or simply* ***canonical isomorphism/morphism/arrow****) in* ***W****, which is defined by structural recursion as follows:*

1. $1_u,\ \alpha_{u,v,w},\ l_u,\ r_u$ *and its inverses are canonically constructed isomorphisms for each words* $u, v, w \in$ ***W****;*

2. *For each pair of composable canonically constructed isomorphisms* f, g*, we have that* $g \circ f$ *is also a canonically constructed isomorphism; and*

3. *For each canonically constructed isomorphisms* f, g*, we have that* $f \otimes g$ *is also a canonically constructed isomorphism.*

Putting this idea into words, the canonically constructed isomorphisms are built by taking all instances of our isomorphisms and their inverses from the monoidal structure of **W**, all identities, and then, recursively, we take tensor products and compositions. With this in mind, given a canonically constructed isomorphism η in **W** and any monoidal category **C**, we can define canonically constructed isomorphisms in **C** just replacing all instances of identities, associators, unitors, tensor products and compositions of η by their interpretations in **C** (i.e, change the symbols of **W** in η to symbols of **C**). Furthermore, we can stretch this idea, and transform any given canonically constructed isomorphism η in **C** into a canonically constructed isomorphism in any other monoidal category **D** just by changing the identities, associators, and so on for their analogues symbols in **D**. In this way, it makes sense to talk about η as we do below, namely, writing η as an arrow of **D** on the left side and $F(\eta)$ on the right side, as if η was an arrow of **C**). Following this idea, we can see canonically constructed arrows as natural isomorphisms $\eta\colon v \to w$.

Definition 3.1.8. *A* ***formal diagram*** *on a monoidal category* ***C*** *is any diagram made of canonically constructed isomorphisms in* ***W*** *interpreted as a diagram of* ***C****.*

Following this definition, we proceed to explain one final important thing: even if, for example, $x = (a \otimes b) \otimes c = a \otimes (b \otimes c)$ in **C**, we cannot use 1_x as a canonical isomorphism when talking about formal diagrams, because $((\_) \star (\_)) \star (\_) \neq (\_) \star ((\_) \star (\_))$. This has a very deep and clear connection with the Isbell argument, since we tried to replace instances of the associator with identities via that argument. Hence, the Isbell argument really shows us that not every diagram commute, but the coherence theorem will show us that any **formal** diagram commute.

Theorem 3.1.9. *Given a monoidal functor $F\colon \boldsymbol{C} \to \boldsymbol{D}$ and a canonnicaly constructed isomorphism η, the following diagram commutes:*

$$\begin{array}{ccc} v(F(a_1),\ldots,F(a_n)) & \xrightarrow{F_v} & F(v(a_1,\ldots,a_n)) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle F(\eta)} \\ w(F(a_1),\ldots,F(a_n)) & \xrightarrow[F_w]{} & F(w(a_1,\ldots,a_n)) \end{array}$$

Proof: We prove this by structural induction on the construction of canonically constructed isomorphisms. For compositions, we just have to vertically paste diagrams and, for tensor products, the result follows from bifunctoriality. Indeed, we can work on each part of the tensor product separately, and apply the induction hypothesis to each of them.

For α, r, l and its inverses, the calculations are pretty similar. For this reason, we are only going to prove the case where $\eta = \alpha_{u,v,w}$. We have to prove that the diagram below commutes, assuming that the corresponding diagrams for u, v, w commute. Below, we identify $A = (a_1, \ldots, a_n)$, and $F(A) = (F(a_1), \ldots, F(a_n))$:

$$\begin{array}{ccc} [u \star (v \star w)](F(A)) & \xrightarrow{F_{u\star(v\star w)}} & F([u \star (v \star w)](A)) \\ \downarrow{\scriptstyle \alpha_{u,v,w}} & & \downarrow{\scriptstyle F(\alpha_{u,v,w})} \\ [(u \star v) \star w](F(A)) & \xrightarrow[F_{(u\star v)\star w}]{} & F([(u \star v) \star w](A)) \end{array}$$

We proceed to calculate the two legs of the diagram. First, the top leg:

$$F(\alpha) \circ F_{u\star(v\star w)} = F(\alpha) \circ F_2 \circ (F_u \otimes F_{v\star w})) = F(\alpha) \circ F_2 \circ (F_u \otimes (F_2 \circ (F_v \otimes F_w))).$$

Now, we are going to do a "smart" trick: Writing $F_u = 1 \circ F_u$, we get $(1 \circ F_u) \otimes (F_2 \circ (F_v \otimes F_w)))$ inside the outter parenthesis, so we can apply bifunctoriality to proceed with our calculations. In the diagrammatic calculus for monoidal categories, this "smart" trick becomes quite justified and, rather, pretty trivial to do in that context (see [2]). In the given context, it may look like a rather magical step. With this trick, we have:

$$F(\alpha) \circ F_{u\star(v\star w)} = F(\alpha) \circ F_2 \circ (1 \otimes F_2) \circ [F_u \otimes (F_v \otimes F_w)].$$

Now, for the down leg (using the same trick as before):

$$\begin{aligned} F_{(u\star v)\star w} \circ \alpha &= F_2 \circ (F_{u\star v} \otimes F_w) \circ \alpha \\ &= F_2 \circ [(F_2 \circ (F_u \otimes F_v)) \otimes F_w] \circ \alpha \\ &= F_2 \circ (F_2 \otimes 1) \circ [(F_u \otimes F_v) \otimes F_w] \circ \alpha \\ &= F_2 \circ (F_2 \otimes 1) \circ \alpha \circ [F_u \otimes (F_v \otimes F_w)] \end{aligned} \tag{3.1}$$

The last equality follows from naturality of α. In order to conclude that both legs of the diagram result in the same composition, we observe that, since F is a monoidal functor, we have that $F(\alpha) \circ F_2 \circ (1 \otimes F_2) = F_2 \circ (F_2 \otimes 1) \circ \alpha$. □

3.2 COHERENCE AND STRICTIFICATION

First, we are going to explain the meaning of both concepts. Then, we formalize them in terms of monoidal categories with no further structure required.

Coherence: When we talk about coherence, we usually mean that going from one situation to another via some canonical "operations" (or arrows) is the same, regardless of the sequence of operations choosed. In the context of arrows, this means that some sort of "canonical diagrams" always commutes. In the case of monoidal categories, we are saying that all canonically constructed isomorphisms from one object to another are equal (i.e, all formal diagrams commute).

Strictification: The idea of strictification is to create, for each category with a loose additional structure of some kind, a category with the same additional structure, equivalent to it via functors that preserve the structure in some sense, such that the additional structure now is strict. In the case of monoidal categories, strictification wants to build a strict monoidal category that is equivalent to the given monoidal category via a strong monoidal functor.

In order to prove coherence theorems, we start by showing that **W** is the free monoidal category with only one non-neutral object. In categorical terms, we wish to prove the following theorem:

Theorem 3.2.1. ***(Mac Lane Coherence Theorem, 1963, version 1)*** *For each monoidal category **C** and $a \in$ **C**, we have a unique strict monoidal functor $F\colon$ **W** $\to$ **C** such that $F((\_)) = a$.*

Proof: If $F((\_)) = a$ and F is a strict monoidal functor, then we have the following:

$$F(e_0) = e,\ F((\_)) = a,\ F(u \star v) = F(u) \otimes F(v).$$

This completely determines F in the objects of **W** by structural recursion. The harder part is determining F in its morphisms. Two words with the same length in **W** always are connected by an isomorphism that can be expressed as a path (regarding the category as its underlying graph – see its definition below) in which each morphism is a canonically constructed isomorphism without composition in its construction. Since F is built to be a monoidal functor, we must have that all paths containing those canonically constructed isomorphisms in the image of F are equal. In other words, we must prove that any two ways to go from one parenthesization of iterated products of a and e to another one (possibly deleting some e's) are equal. All definitions in this proof apply to any monoidal category, so we will choose conveniently, according to the definition, whether we work with the symbols of an arbitrary category or with the symbols of **W**.

First part : We define the ***underlying graph*** of **C**, G_n, with vertices all words of length n over b, without instances of e. The edges of this graph shall be called **basic arrows**. They are defined recursively as below:

1. All instances of α, α^{-1} are basic arrows;
2. If β is a basic arrow, $1 \otimes \beta$ and $\beta \otimes 1$ are also basic arrows.

By construction, a basic arrow possesses an instance of α or an instance of α^{-1} in its construction, but not both. If it possesses α, we call it **directed**, otherwise we call it **antidirected**.

We also define the **canonical word of length** n (denoted as $w^{(n)}$) as the word with all parenthesis in front, i.e., in the leftmost position. Of course, we can always apply a sequence of α's to achieve that, starting from any other word of length n having no instances of e. We define the **rank** (denoted by r) of a word to measure how far from this canonical word an arbitrary word is as below:

$$r(e) = 0,\ r((\_)) = 1,\ r(u \otimes v) = r(u) + r(v) + L(v) - 1.$$

The rank of an arbitrary word is zero if and only if the rank of its components are also zero and $L(v) - 1 = 0$, i.e, $L(v) = 1$. Therefore we conclude, by recursion, that the rank of a word is zero if and only if the word is a canonical word. Now, we verify by induction (on the construction of basic arrows) that, if we have a word and apply to it a basic directed arrow, then the rank will necessarily decrease.

$$r(u \otimes (v \otimes w)) = r(u) + r(v) + r(w) + L(v) + 2L(w) - 2$$

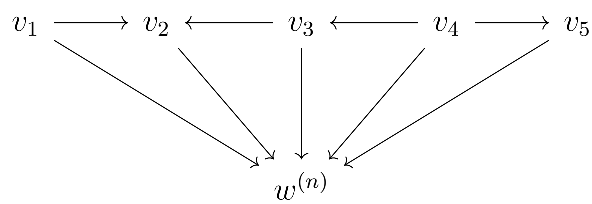
$$r((u \otimes v) \otimes w) = r(u) + r(v) + r(w) + L(v) + L(w) - 2$$

By a direct comparison, we see that the rank after applying α is necessarily lower. Now, if $\beta\colon v \to v'$ satisfies the statement that we wish to prove, we are going to prove that $1 \otimes \beta\colon u \otimes v \to u \otimes v'$ also satisfies it ($\beta \otimes 1$ is analogous):

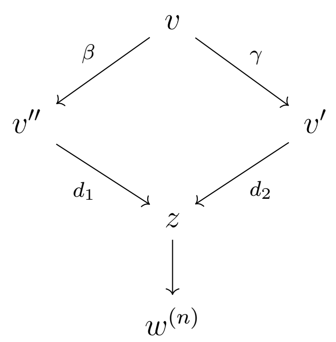
$$r(u \otimes v') = r(u) + r(v') + L(v') - 1 < r(u) + r(v) + L(v) - 1 = r(u \otimes v)$$

as desired.

Second part: Now, we wish to prove that G_n commutes. We start by noting that it is sufficient to prove that any two paths of directed basic arrows from one word of length n to the canonical word have equal composites. This is pretty straightforward: if we have a path of basic arrows from a word v to a word w, we draw the directed paths from v and w to the canonical word, and for each antidirected basic arrow in the path from v to w, we invert its direction and change the instance of α^{-1} for an instance of α. Now, we have a diagram of directed basic arrows connecting words as below:



If we had the uniqueness of directed paths connecting words to the canonical word, this diagram would commute, so the initial path would have a unique composition (by uniqueness of the arrows involved here), showing that G_n commutes. In order to prove that we actually do have it, we apply induction on the rank. Assume, without loss of generality, that the first arrow of the path is the one that is different. We wish to construct a word z in a commutative, diamond-shaped diagram as below:



Above, β, γ are the first directed basic arrows of the paths. In this situation, we can apply the induction hypothesis on v', v'' and prove the statement. Suppose that $v = u \otimes w$. We have, by construction of basic arrows, three possibilites for β (and the same for γ):

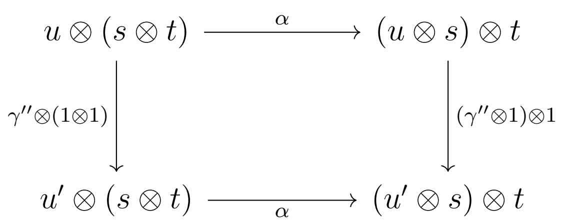
1. $\beta = 1 \otimes \beta'$;

2. $\beta = \beta'' \otimes 1$;

3. $\beta = \alpha_{u,s,t}$, where $w = s \otimes t$.

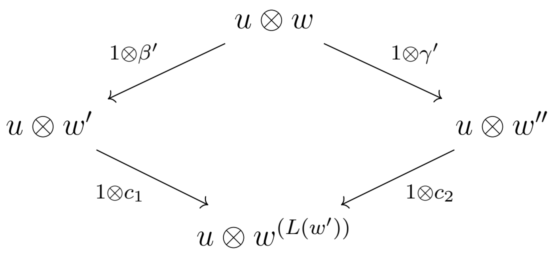
If $\beta = 1 \otimes \beta'$ and $\gamma = \gamma'' \otimes 1$, we choose $d_1 = \gamma'' \otimes 1$ and $d_2 = 1 \otimes \beta'$ to make the diagram commute, by bifunctoriality. If $\beta = \beta'' \otimes 1$ and $\gamma = 1 \otimes \gamma'$, the same argument applies.

If $\beta = \alpha_{u,s,t}$ and $\gamma = \gamma'' \otimes 1$, we can use functoriality to complete the diagram:



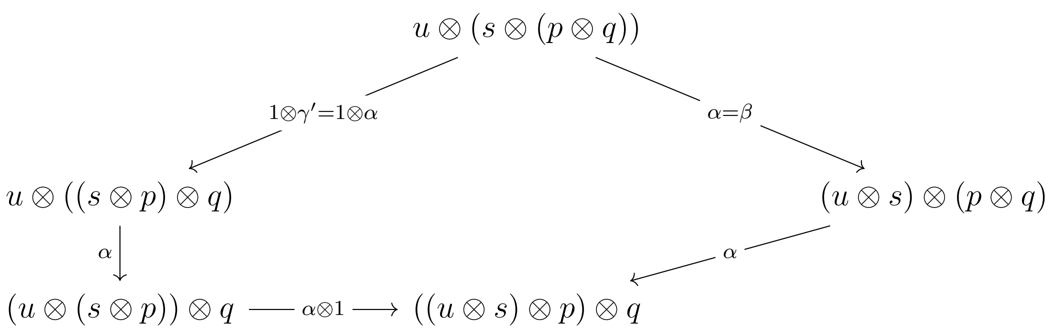
Swapping the roles of β, γ also works this way. If both β, γ are α, we just need to fill $d_1 = d_2 = 1$.

Now we wish to study the case where $\beta = 1 \otimes \beta'$, and $\gamma = 1 \otimes \gamma'$. This case is pretty straightforward, as the diagram below shows:



Here, c_1 and c_2 are the unique composites of paths from w', and w'', respectively, to the canonical word $w^{(L(w'))}$. We have two composites $(c_1 \circ \beta')$ and $(c_2 \circ \gamma')$ from w to the canonical word, which are equal by the induction hypothesis, so this whole diagram commutes.

The most interesting situation is when we have $\beta = \alpha_{u,s,t}$ and $\gamma = 1 \otimes \gamma'$. In this case, we have three possibilities for γ', just like we had for β and γ above. If $\gamma' = 1 \otimes \eta$ or $\gamma' = \eta' \otimes 1$, the reasoning is the same as done before for $\beta = \alpha$ and $\gamma = \gamma'' \otimes 1$, i.e, we are going to see γ' acting through η or η' on s or t while holding the others fixed. It is left the case $\gamma' = \alpha_{s,p,q}$. Here, we can complete the diagram as in the following pentagon, which commutes by the pentagon axiom:



This finishes our proof that G_n commutes.

Third and last part: Now, it only remains the insertion of the unitors in the diagrams. If we had two paths consisting of some kind of "basic arrows" constructed as before, but changing the instances of α for instances of r, l, we would apply a similar method as before and prove that these diagrams commutes (the inductive reasoning would be applied on the number of e's in the word). This is not very difficult, since all the technical details are essentially equal to the reasoning that we did to prove that G_n was indeed commutative. With this in mind, we only have to, for each given path, find a path with same composite such that this path is constructed by a sequence of basic arrows built from associators followed by a sequence of basic arrows built from unitors. Indeed, with the structural recursion, unitors will appear in any desired position as we build new words.

If, in the composite of the path, we have something like $\beta \circ \zeta$, with β a basic arrow built from associators and ζ a basic arrow built from unitors, we would have, inductively, three possibilities for β and six for ζ (because we can built it from l or from r), as we already studied before. With the possibilities in mind, it is pretty straightforward that we can use naturality of the isomorphisms or the commutative diagrams involving α, r, l to find a path with composite $\zeta' \circ \beta' = \beta \circ \zeta$, where β' is a basic arrow built from associators, and ζ' a basic arrow built from unitors. Therefore, we can always find, for an arbitrary path, other path $\zeta_1 \cdots \zeta_n \beta_1 \cdots \beta_n$ with equal composite, where all ζ's are basic unitors arrows and all β's are basic associators arrows, proving the last remaining step. □

Our latest result was the coherence for iterated products of a fixed object (and neutrals), but we can use the universal property of **W** to prove the general coherence theorem. The main idea is to construct a category such that we can take $a = \text{Id}_{\mathbf{C}}$, and the iterated products of b can be regarded as words in the objects of **C**, proving that any (formal) diagram involving the monoidal constraints will commute. Here, it is important to remember the earlier argument that we gave, due to Isbell. It shows that, if we change the monoidal constraints for identities, not all diagrams will commute. Therefore, even if $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, we cannot replace α by $1_{A \otimes (B \otimes C)}$ in a diagram involving this object, or else we cannot assure that the diagram will still commute.

Corollary. ***(Mac Lane Coherence Theorem, 1963, version 2)*** *For each monoidal category*

***C**, we can assign, to each ordered pair of words (v, w), a unique natural isomorphism **can**$(v, w)\colon v \to w$ such that all canonically constructed isomorphisms (viewed as natural transformations between word functors) are canonical.*

Proof: Given a monoidal category $\mathbf{C}$ we construct the monoidal category $\mathrm{It}(\mathbf{C})$ (the category of all $\otimes$-iterations of objects of $\mathbf{C}$), with objects given by the pairs (n, T), where $n \in \mathbb{N}$ and $T\colon \mathbf{C}^n \to \mathbf{C}$ is a functor, and morphisms from $(n, T) \to (m, S)$ only defined when $n = m$, in which case they are all natural transformations $T \to S$. The tensor product $\otimes$ is defined as $(n, T) \otimes (m, S) = (n + m, \otimes \circ [T \times S])$. The neutral element e' is the functor $e'\colon \{*\} = \mathbf{C}^0 \to \mathbf{C}$, and α', r', l' in $\mathrm{It}(\mathbf{C})$ are defined elementwise using α, r, l in $\mathbf{C}$. For example, $r'_T\colon e' \otimes T \to T$ is defined elementwise as the natural isomorphism $r'_T(*, x) = r_{T(x)}\colon e \otimes T(x) \to T(x)$. It is straightforward to show that $\mathrm{It}(\mathbf{C})$ is a monoidal category because the monoidal structure of $\mathrm{It}(\mathbf{C})$ is constructed using the very monoidal structure of $\mathbf{C}$).

Now, we take $a = \mathrm{Id}_{\mathbf{C}}$ in our previous theorem to conclude that there is a unique strict monoidal functor $F\colon \mathbf{W} \to \mathrm{It}(\mathbf{C})$ such that $F((\_)) = \mathrm{Id}_{\mathbf{C}}$. Since F is strict monoidal, the unique morphism $v \to w$ in $\mathbf{C}$ is sent to a natural transformation $v \to w$ (in the latter, the words are viewed as functors). Due to its uniqueness, we denote this natural transformation by **can**(v, w), seeing it as "canonical". Still using the fact that F is strict monoidal, the monoidal constraints are strictly preserved, so the canonically constructed isomorphisms are indeed canonical in the sense of this theorem. □

Remark. By uniqueness of the natural isomorphism in this corollary, we have that all formal diagrams commutes.

Corollary. ***(Coherence theorem for monoidal functors)*** *For any pair of unary words v, w of same length, and any monoidal functor F, there exists a unique canonical arrow η such that the following diagram commutes:*

$$\begin{array}{ccc}
v(F(a_1), \ldots, F(a_n)) & \xrightarrow{F_v} & F(v(a_1, \ldots, a_n)) \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle F(\eta)} \\
w(F(a_1), \ldots, F(a_n)) & \xrightarrow[F_w]{} & F(w(a_1, \ldots, a_n))
\end{array}$$

Proof: It follows immediately from theorem 3.5 and the last coherence theorem. □

There are easier ways to prove coherence, but the proof above is very enlightening, visual, and beautiful, explicitly using only the most basic commutative diagrams of a monoidal category and basic concepts. Now we provide a theorem that implies coherence as a corollary, but the theorem itself gives us something more: it connects the concept of strictification to coherence.

Theorem 3.2.2. *For a given monoidal category* ***C****, if there is a strong monoidal locally injective functor* $F\colon$ ***C*** $\to$ ***D****, where* ***D*** *is a strict monoidal category, then coherence holds for* ***C****. In particular, if* ***C*** *is equivalent to a strict monoidal category Str(****C****) via a strong monoidal functor, then, coherence holds for* ***C****.*

Proof: Let $\eta, \eta'\colon v \to w$ be two canonically constructed isomorphisms. Of course coherence holds for **D**, since **D** is strict, so all canonical isomorphisms are identities. We saw earlier that the following diagram commutes for η and η' separately:

$$\begin{array}{ccc} v(F(A)) & \xrightarrow{F_v} & F(v(A)) \\ \Big\downarrow{\scriptstyle \eta,\eta'} & & \Big\downarrow{\scriptstyle F(\eta),F(\eta')} \\ w(F(A)) & \xrightarrow[F_w]{} & F(w(A)) \end{array}$$

Above, the canonical arrows at the left side are interpreted as canonical arrows in **D**, and those at the right side are interpreted as canonical arrows in **C**. Since coherence holds for **D**, $\eta = \eta'$ in **D**, so we have $F(\eta) = F(\eta')$ because the correspondence $\theta \to F(\theta)$ in the diagram must be bijective, observing that F_v, F_w are isomorphisms and the diagram commutes. Using local injectivity, we conclude that $\eta = \eta'$ in **C**. In this way, coherence also holds for **C**. □

Corollary. *Coherence holds for all monoidal categories.*

Proof: Given any category **C** (monoidal or not), we can view the endofunctor category $\mathbf{C}^{\mathbf{C}} = [\mathbf{C}, \mathbf{C}]$ as a strict monoidal category with composition regarded as its tensor product. When **C** is monoidal, we can construct a strong monoidal functor $F\colon \mathbf{C} \to \mathbf{C}^{\mathbf{C}}$ that acts on objects taking a to the functor $a\otimes\_$ (the functor that acts as a tensor product with a) and, for each morphism $f\colon a \to b$, the natural transformation $f \otimes 1\colon a \otimes \_ \to b \otimes \_$. This is indeed a natural transformation because, due to bifunctoriality, the following diagram is commutative::

$$
\begin{array}{ccccc}
x & \qquad & a \otimes x & \xrightarrow{f\otimes 1_x} & b \otimes x \\
{\scriptstyle g}\downarrow & & {\scriptstyle 1_a\otimes g}\downarrow & & \downarrow{\scriptstyle 1_b\otimes g} \\
y & & a \otimes y & \xrightarrow[f\otimes 1_y]{} & b \otimes y
\end{array}
$$

To show the local injectivity of F is fairly simple: We apply $F(f)$ to e, so we have $f \otimes 1_e$. If $F(f) = F(g)$ (locally), we would have, in particular, $f \otimes 1_e = g \otimes 1_e$, which we already saw that implies $f = g$.

The fact that F is indeed strongly monoidal is a straightforward calculation. $\square$

Theorem 3.2.3. ***(Strictification of monoidal categories)*** *Any monoidal category **C** is equivalent to a strict monoidal category Str(**C**) via a strong monoidal functor F: Str(**C**) $\to$ **C** and a strong monoidal functor G: **C** $\to$ Str(**C**).*

Proof: We define the objects of Str(**C**) as the class of words with alphabet in **C**. The tensor product is chosen to be the concatenation $\cdot$ on objects, so it is strictly associative. Before we define the morphisms of Str(**C**), we define a function $F\colon$ Str(**C**) $\to$ **C** (here Str(**C**) is regarded just as a class of objects) as $F(a_1 \dots a_n) = w^{(n)}(a_1, \dots, a_n)$. The main idea is to set, for $u, v \in$ Str(**C**), $\text{hom}_{\text{Str}(\mathbf{C})}(u, v) = \text{hom}_{\mathbf{C}}(F(u), F(v))$. Composition in Str(**C**) is induced by that of **C** componentwise, so F is extended to morphisms as a functor. The tensor product of two morphisms $f\colon u_1 \to v_1$ and $g\colon u_2 \to v_2$ is defined as following composite:

$$F(u_1 \cdot u_2) \xrightarrow{\textbf{can}} F(u_1) \otimes F(u_2) \xrightarrow{f\otimes g} F(v_1) \otimes F(v_2) \xrightarrow{\textbf{can}} F(v_1 \cdot v_2)$$

Due to the uniqueness of the canonical arrows in the composite above, this tensor product is strictly associative.

In order to see that the functor F defined as above is a strong monoidal functor, we just have to take $F_0 = 1_e$ and $F_2(u, v)$ as the only canonical isomorphism $F(u) \otimes F(v) \to F(u \cdot v)$.

The axioms that F_0 and F_2 must satisfy in order to F be a monoidal functor follow from the coherence theorem for **C**, for F_0 and F_2 are defined as canonical isomorphisms in **C**).

Now we define a functor $G\colon \mathbf{C} \to \mathsf{Str}(\mathbf{C})$ naturally as $G(a) = a$, $G(f) = f$, and G_0, G_2 are just identities. This functor is strong monoidal because all of its instances are identities, making the commutativity of diagrams immediate. It is easy to see that $FG = \mathsf{Id}_{\mathbf{C}}$, while GF is naturally isomorphic to $\mathsf{Id}_{\mathsf{Str}(\mathbf{C})}$ because its only effect is changing the parenthesis order. □

3.3 BRAIDINGS

Definition 3.3.1. *A* **braiding** *for a monoidal category* ***C*** *consists of a natural isomorphism* $\otimes \overset{\sigma}{\cong} \otimes \circ \tau$, *where* $\tau\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ *is the flip functor, i.e,* $\tau(a, b) = (b, a)$ *on objects and morphisms, and the braiding is subject to the following conditions for all* $a, b, c \in \mathbf{C}$*:*

$$\begin{array}{ccc} a \otimes e & \xrightarrow{\ \sigma\ } & e \otimes a \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle l} & \\ & a & \end{array}$$

$$\begin{array}{ccc} (a \otimes b) \otimes c & \xrightarrow{\ \sigma\ } & c \otimes (a \otimes b) \\ \downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \alpha} \\ a \otimes (b \otimes c) & & (c \otimes a) \otimes b \\ \downarrow{\scriptstyle 1\otimes\sigma} & & \uparrow{\scriptstyle \sigma\otimes 1} \\ a \otimes (c \otimes b) & \xrightarrow[\ \alpha\ ]{} & (a \otimes c) \otimes b \end{array} \qquad \begin{array}{ccc} a \otimes (b \otimes c) & \xrightarrow{\ \sigma\ } & (b \otimes c) \otimes a \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha^{-1}} \\ (a \otimes b) \otimes c & & b \otimes (c \otimes a) \\ \downarrow{\scriptstyle \sigma\otimes 1} & & \uparrow{\scriptstyle 1\otimes\sigma} \\ (b \otimes a) \otimes c & \xrightarrow[\ \alpha^{-1}\ ]{} & b \otimes (a \otimes c) \end{array}$$

Remark. When **C** is strict, these conditions just state that

$$\sigma_{a\otimes b,c} = (1_a \otimes \sigma_{b,c})(\sigma_{a,c} \otimes 1_b) \text{ and } \sigma_{a,b\otimes c} = (\sigma_{a,b} \otimes 1_c)(1_b \otimes \sigma_{a,c}).$$

Remark. The two last conditions above are called the ***hexagon axiom.***

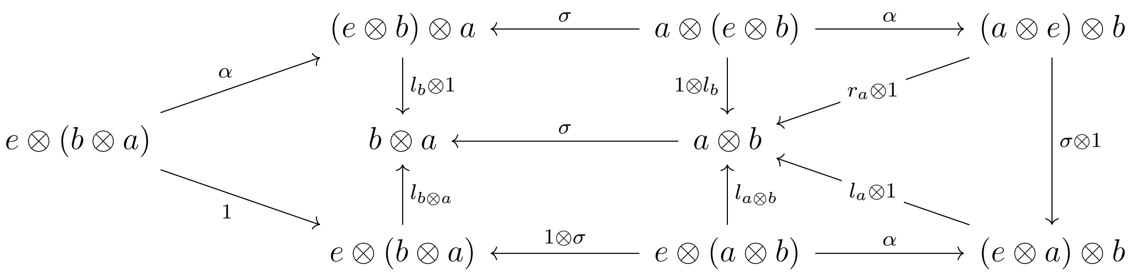
Definition 3.3.2. *A* **braided monoidal category** *is a monoidal category equipped with a braiding. A* **symmetry** *in a monoidal category is a braiding* σ *such that* $\sigma_{a,b}^{-1} = \sigma_{b,a}$*. A* **symmetric monoidal category** *is a monoidal category equipped with a symmetry.*

Proposition 3.3.3. *For any object $a \in$ **C**, we have*

$$l_a \circ \sigma_{a,e} = r_a,\ r_a \circ \sigma_{e,a} = l_a,\ \sigma_{e,a} = \sigma_{a,e}^{-1}.$$

*If **C** is strict, then these relations assert that $\sigma_{e,a} = 1_a = \sigma_{a,e}$.*

Proof: Consider the following diagram:



The outter heptagon commutes by the hexagon axiom, noticing that one of the edges of the heptagon is the identity. We proved earlier the commutativity of the left triangle and of the bottom right one. The upper square commutes by naturality of σ. The bottom square commutes by naturality of l. The upper right triangle commutes by the triangle axiom. We obtain the commutativity of the middle right triangle combining the commutativity of the upper and lower right triangles with that of the heptagon, as well as the fact that all edges are isomorphic (ant, thus, invertible) in a diagram chase. Hence:

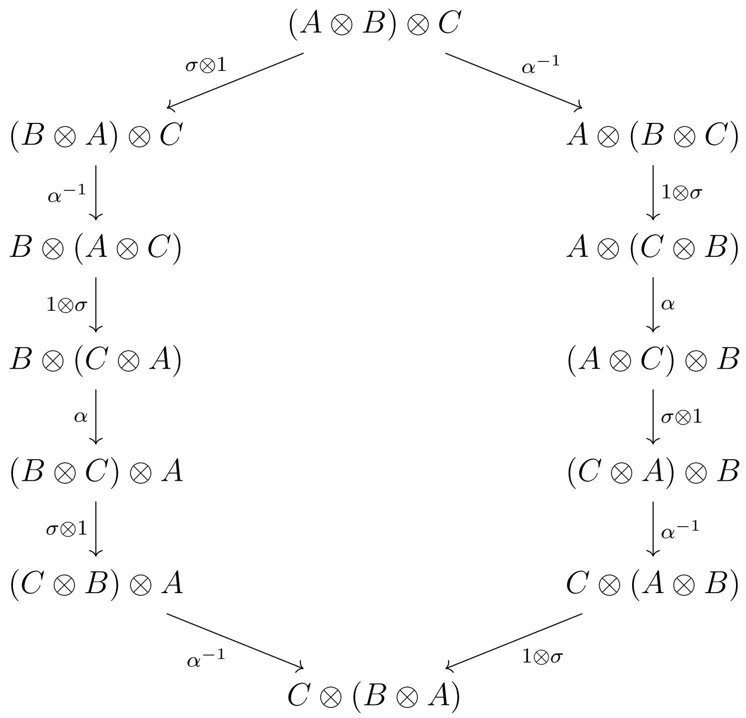
$$r_a \otimes 1_b = (l_a \otimes 1_b) \circ (\sigma_{a,e} \otimes 1_b) = (l_a \circ \sigma_{a,e}) \otimes 1_b.$$

If we take $b = e$, we conclude that $r_a = l_a \circ \sigma_{a,e}$. For the second desired equation, we use the other diagram from the hexagon axiom in a similar way.

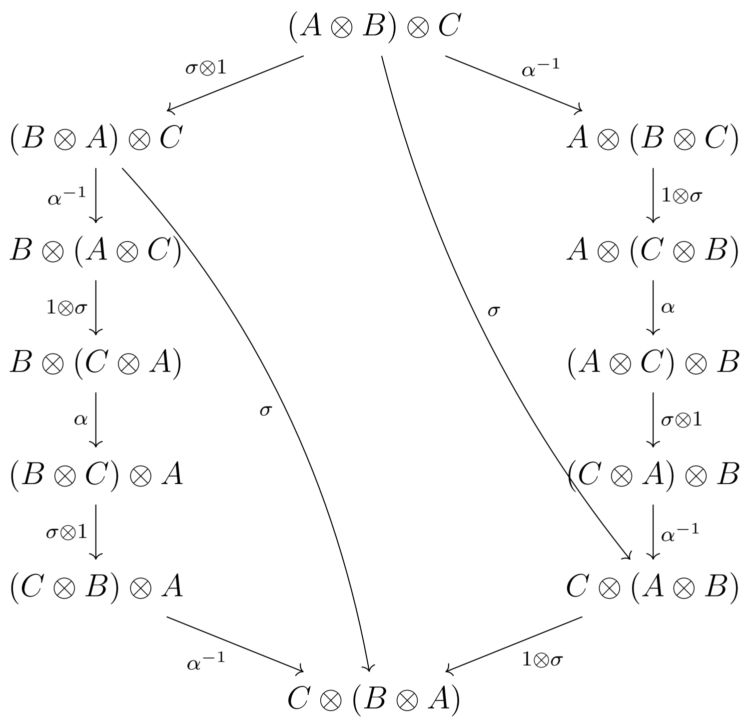
Combining these two relations, we obtain $r_a = l_a \circ \sigma_{a,e} = r_a \circ \sigma_{e,a} \circ \sigma_{a,e}$, so $\sigma_{e,a} = \sigma_{a,e}^{-1}$.

If **C** is strict, then $l_a = r_a = 1_a$, so the final statement of the theorem follows substituting 1_a for r_a, l_a in our relations. □

Proposition 3.3.4. *Braidings follow the loose **Yang-Baxter equation**, i.e, the following* *dodecagon diagram commutes:*



Proof: Consider the following diagram:



The left and right parts of the diagram commute by the hexagon axiom, while the middle square commutes by naturality. Glueing those three commutative diagrams together, we have the commutativity of the dodecagon. □

Remark. Cartesian monoidal categories are symmetric, **Braid** and **Diff** are also symmetric. $\mathbf{Vect}_k$ and R-**Mod** are braided.

3.4 ALGEBRAIC STRUCTURES IN MONOIDAL CATEGORIES

To motivate the generalization of a monoid in a monoidal category, we start categorifying the notion of monoids and k-algebras.

Given a monoid $(M, \cdot, e)$, we know that $\cdot : M \times M \to M$ is a binary operation, and since e is a fixed object, we can understand it as a unary operation $\overline{e} \colon \{*\} \to M$ defined by $\overline{e}(*) = e$. Since this determines $\overline{e}$ completely, we can write $\overline{e} = e$ for simplicity. The associativity axiom becomes

$$
\begin{array}{ccccc}
M \times (M \times M) & \xrightarrow{\ \alpha\ } & (M \times M) \times M & \xrightarrow{\ \cdot \times 1\ } & M \times M \\
\Big\downarrow{\scriptstyle 1 \times \cdot} & & & & \Big\downarrow{\scriptstyle \cdot} \\
M \times M & & \xrightarrow{\qquad\qquad \cdot \qquad\qquad} & & M
\end{array}
$$

and the axiom for the neutral becomes

$$
\begin{array}{ccccc}
\{*\} \times M & \xrightarrow{\ e \times 1\ } & M \times M & \xleftarrow{\ 1 \times e\ } & M \times \{*\} \\
& {\scriptstyle l} \searrow & \Big\downarrow{\scriptstyle \cdot} & \swarrow {\scriptstyle r} & \\
& & M & &
\end{array}
$$

Now, if we are given an k-algebra $(A, \cdot, e)$, we know that $\cdot : A \times A \to A$ is a bilinear map and e is a neutral element for this map. By the universal property of the tensor product, we can view this bilinear map $\cdot$ as a linear map $\mu \colon A \otimes A \to A$. We know that a linear map $T \colon k \to A$ is completely determined by $T(1)$, so we can understand e as the map $\eta \colon k \to A$ that sends 1 to e. Requesting associativity for $\cdot$ is the same as requesting commutativity of the following diagram:

$$\begin{array}{ccccc} A\otimes(A\otimes A) & \xrightarrow{\alpha} & (A\otimes A)\otimes A & \xrightarrow{\mu\otimes 1} & A\otimes A \\ \downarrow{\scriptstyle 1\otimes\mu} & & & & \downarrow{\scriptstyle\mu} \\ A\otimes A & & \xrightarrow{\mu} & & A \end{array}$$

Requesting that e is the neutral for this product is the same as requesting commutativity of this other diagram:

$$\begin{array}{ccccc} k\otimes A & \xrightarrow{\eta\otimes 1} & A\otimes A & \xleftarrow{1\otimes\eta} & A\otimes k \\ & {\scriptstyle l}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle r} & \\ & & A & & \end{array}$$

In both categorifications above, we obtain relations involving the monoidal structure of **Set** and **Vect**$_k$.

Definition 3.4.1. *A* **monoid** M *in a monoidal category* ***C*** *is a triple* (M,μ,η)*, where* $M\in$ ***C****,* $\mu\colon M\otimes M\to M$ *and* $\eta\colon e\to M$ *such that the following diagrams commute:*

$$\begin{array}{ccccc} M\otimes(M\otimes M) & \xrightarrow{\alpha} & (M\otimes M)\otimes M & \xrightarrow{\mu\otimes 1} & M\otimes M \\ \downarrow{\scriptstyle 1\otimes\mu} & & & & \downarrow{\scriptstyle\mu} \\ M\otimes M & & \xrightarrow{\mu} & & M \end{array}$$

$$\begin{array}{ccccc} e\otimes M & \xrightarrow{\eta\otimes 1} & M\otimes M & \xleftarrow{1\otimes\eta} & M\otimes e \\ & {\scriptstyle l}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle r} & \\ & & M & & \end{array}$$

Definition 3.4.2. *A* **monoid (homo)morphism** $f\colon(M,\mu,\eta)\to(M',\mu',\eta')$ *is a morphism* $f\colon M\to M'$ *such that* $f\mu=\mu'(f\otimes f)$ *and* $f\eta=\eta'$*.*

Remark. Comparing to our usual homomorphisms from algebra, the first equality above is the analogous for the preservation of the product and the second equality is the analogous to the preservation of unity.

For a given monoidal category **C**, we define the category $\textbf{Mon}_{\textbf{C}}$ as the category of monoids of **C**. We give some examples of monoids:

1. In **Set** a monoid is the just an ordinary monoid.

2. In $\mathbf{Vect}_k$ a monoid is an k-algebra.

3. A monoid in **Ab** is a ring with the tensor product as the multiplication action.

4. A monoid in Ch($R -$ **Mod**) is what is called a ***differential graded algebra***.

5. In End(**C**) a monoid is called a ***monad***. It is a very important concept in categorical algebra.

6. In **Top** a monoid is a ***topological monoid***, i.e, a topological space with a continuous monoid structure.

Remark. For those who are familiar with the theory of lattices, it is very interesting to notice that a monoid in the category of complete join-semilattices is a unital ***quantale***. This is a useful thing to keep in mind, since quantales can be used as a framework for point-set topology and linear logic.

Definition 3.4.3. *For each unary word u, we define recursively a product $\mu_u\colon u(M,\ldots,M) \to M$ as:*

1. $\mu_{e_0} = \eta\colon e_0 \to M$;

2. $\mu_{(\_)} = 1_M\colon M \to M$;

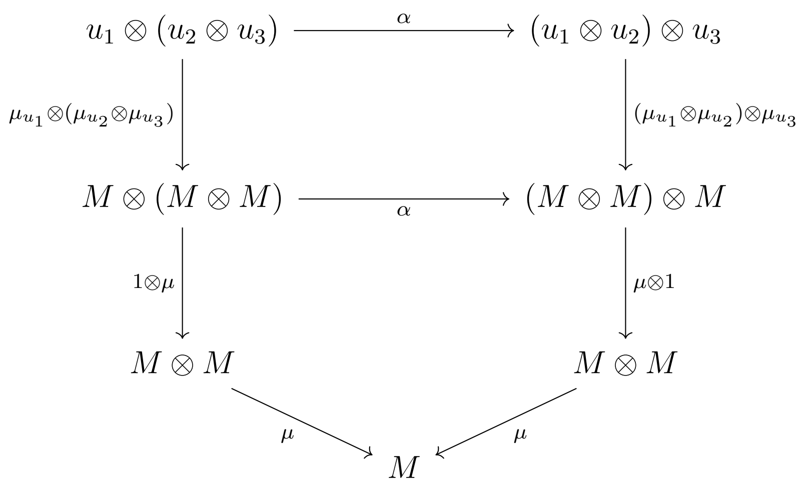
3. $\mu_{(\_)\star(\_)} = \mu\colon M \otimes M \to M$;

4. *If $u = v \star w$, then $\mu_u = \mu \circ (\mu_v \otimes \mu_w)$.*

Remark. Just as we discussed before, this definition of μ_u gives us all the possible ways to do the monoid product for the parenthisizations of the $\otimes$-iterated products of M.

Theorem 3.4.4. ***(General associative law for monoids)*** *If (M, μ, η) is a monoid in* ***C*** *and v, w are unary words with the same length, then $\mu_w \circ$* ***can***$_M(v,w) = \mu_v$*, where we denote* ***can***$_M(v,w) = [$***can***$(v,w)]_{(M,\ldots,M)}$.

Proof: We prove this by structural induction on the construction of canonical arrows. We start by assuming **can**$(v,w) = \alpha$. The following diagram commutes by naturality of α and by the very definition of a monoid in **C**:



But $\mu \circ (\mu \otimes 1) \circ [(\mu_{u_1} \otimes \mu_{u_2}) \otimes \mu_{u_3}] = \mu \circ [(\mu \circ (\mu_{u_1} \otimes \mu_{u_2})) \otimes \mu_{u_3}] = \mu \circ (\mu_{u_1 \otimes u_2} \otimes \mu_{u_3}) = \mu_{(u_1 \otimes u_2) \otimes u_3}$.

In the same way, we prove that $\mu \circ (1 \otimes \mu) \circ [\mu_{u_1} \otimes (\mu_{u_2} \otimes \mu_{u_3})] = \mu_{u_1 \otimes (u_2 \otimes u_3)}$. Since the diagram commutes, we have that $\mu_{u_1 \otimes (u_2 \otimes u_3)} = \mu_{(u_1 \otimes u_2) \otimes u_3} \circ \alpha$. An analogous argument can be applied to the isomorphisms r and l.

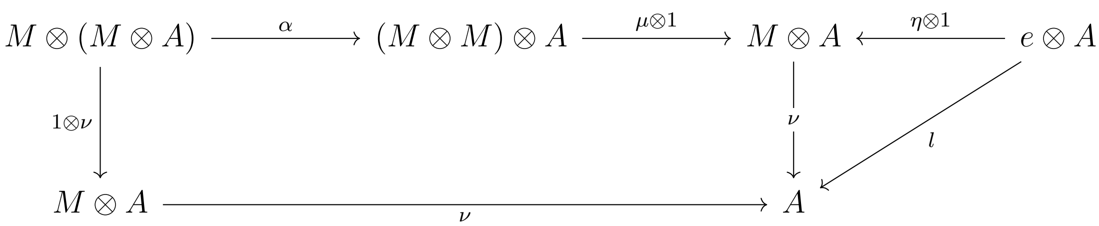
Identities and compositions of canonical arrows follow straightforward from the induction hypothesis. The tensor product of canonical arrows follows from the following calculation, where c_1, c_2 are the canonical arrows $c_1 \colon u \to u'$, $c_2 \colon v \to v'$:

$$\mu_{u' \otimes v'} \circ (c_1 \otimes c_2) = \mu \circ (\mu_{u'} \otimes \mu_{v'}) \circ (c_1 \otimes c_2) = \mu \circ [(\mu_{u'} \circ c_1) \otimes (\mu_{v'} \circ c_2)] = \mu \circ (\mu_u \otimes \mu_v) = \mu_{u \otimes v}$$

This concludes our proof. □

Remark. It is interesting to notice that, just as before, we can state this coherence theorem in terms of commuting paths on a graph.

Definition 3.4.5. *A* **left action** *of a monoid (M, μ, η) on an object A is defined as an arrow $\nu \colon M \otimes A \to A$ such that the following diagram commutes:*



Definition 3.4.6. *A* **left action (homo)morphism** $f\colon \nu \to \nu'$ *is a morphism* $f\colon A \to A'$ *such that* $\nu'(1 \otimes f) = f\nu$*. Here, the monoid* M *is maintained fixed.*

We then define $\mathbf{Lact}_M$ as the category of left actions of M.

Remark. In algebra, when we let an algebraic structure act on itself via its own product, we call this action the ***regular representation***. If we take $A = M$ and $\nu = \mu$ above, we get what we call the ***left regular representation*** of M.

Definition 3.4.7. *A* **(left) module** A *over a monoid* M *is simply an object* A *with a left action* $\nu\colon M \otimes A \to A$*.*

Definition 3.4.8. *Given a left action* $\nu\colon M \otimes A \to A$ *and a unary word* u *with last non-parenthesis character being* (_)*, we define* $\nu_u\colon u(M, \ldots, M, A) \to A$ *recursively by:*

1. $\nu_{(\_)} = 1_A\colon A \to A$;

2. $\nu_{(\_)\star(\_)} = \nu\colon M \otimes A \to A$;

3. *If* $u = v \star w$*, then* $\nu_u = \nu \circ (\mu_v \otimes \nu_w)$*.*

Theorem 3.4.9. ***(Coherence for actions)*** *If* $\nu\colon M \otimes A \to A$ *is a left action and* v, w *are unary words with the same length, then* $\nu_w \circ \textbf{\textit{can}}_{M,A}(v, w) = \nu_v$*, where we denote* $\textbf{\textit{can}}_{M,A}(v, w) = [\textbf{\textit{can}}(v, w)]_{(M,\ldots,M,A)}$*.*

Proof: The proof is completely analogous to the proof of the general associative law for monoids. □

Definition 3.4.10. *A* $M - N$ **bimodule** *is an object* A *together with two actions* $\mu_M\colon M \otimes A \to A$*,* $\mu_N\colon A \otimes N \to A$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} M \otimes (A \otimes N) & \xrightarrow{\alpha} & (M \otimes A) \otimes N & \xrightarrow{\mu_M \otimes 1} & A \otimes N \\ \downarrow{\scriptstyle 1 \otimes \mu_N} & & & & \downarrow{\scriptstyle \mu_N} \\ M \otimes A & & \xrightarrow[\mu_M]{\hspace{6em}} & & A \end{array}$$

Definition 3.4.11. *A* **bimodule homomorphism** *(also called an* **equivariant map***) from the* $M - N$ *bimodule* A *to the* $P - Q$ *bimodule* B *is a triple of morphisms* $\sigma\colon A \to B$*,* $f\colon M \to N$*,* $g\colon P \to Q$ *such that the following diagram commutes for left and right actions (ommiting monoidal constraints):*

$$\begin{array}{ccc}
M \otimes A \otimes N & \xrightarrow{\text{action}} & M \otimes N \\
\Big\downarrow {\scriptstyle f\otimes\sigma\otimes g} & & \Big\downarrow {\scriptstyle f\otimes g} \\
P \otimes B \otimes Q & \xrightarrow[\text{action}]{} & P \otimes Q
\end{array}$$

Remark. Usual monoids, modules, topological monoids, topological modules, algebras, topological algebras and so on are easy examples of the usual algebraic structures equipped with algebraic structures (and are algebraic objects in some category, like $\mathbf{Vect}_k$, **Top**, **Set**, etc). For this reason we have not given a list of examples.

Remark. Algebraic structures of various kinds in categories with additional structure of various kinds are natural extensions of the work on the respectively original algebraic structures. The research on this field has been very active for decades, producing a myriad of interesting structures, and establishing connections between different specialties within mathematics, logic, computer science, and physics.

Remark. We will not develop the general theory of enrichment, so the basic idea of a category enriched over a monoidal category **M** is: rather than having hom-sets, it has hom-objects that are objects of **M**, and there is a notion of composition (associative and unital) between hom-objects inherited from the monoidal structure of **M** ($\circ\colon \text{hom}(B,C) \otimes \text{hom}(A,B) \to \text{hom}(A,C)$). The usual notion of category is that of a category enriched over **Set** endowed with its usual monoidal structure given by the Cartesian product.

4 LOW DIMENSIONAL CATEGORY THEORY

4.1 LOW DIMENSIONAL CATEGORIES

Now we are going to lift the categorical and monoidal structures that we studied to higher dimensions. The nomenclature will remain practically unchanged. For example, adjunctions will remain defined as they are, but reinterpreted with the new concepts of functors, natural transformations, etc.

Definition 4.1.1. *A* ***(strict) 2-category*** *is a category enriched over* ***Cat****. Unravelling this definition, a 2-category* ***C*** *consists of:*

1. *A class of objects, also called 0-**cells**;*
2. *For each ordered pair of objects (A, B), a category hom$_{\boldsymbol{C}}(A, B)$. The objects of this category are called 1-**cells**, and its morphisms are called 2-**cells**. The composition of 2-cells is usually called **vertical composition**, and denoted by $\circ = \circ_1$;*
3. *For each object A, there is a functor id: $\{*\} \to$ hom$_{\boldsymbol{C}}(A, A)$, that picks out the object that we call the **identity 1-cell** and the morphism that we call the **identity 2-cell**. These two are often denoted as id$_A$ and id$_{\text{id}_A}$ respectively;*
4. *For all objects $A,\ B,\ C,$ there is a functor $\circ_0 = \odot$: hom$_{\boldsymbol{C}}(B, C) \times$ hom$_{\boldsymbol{C}}(A, B) \to$ hom$_{\boldsymbol{C}}(A, C)$ called **horizontal composition**, which is associative and which has id$_A$ and id$_{\text{id}_A}$ as identities.*

Remark. From this definition above it is reasonably easy to define what is a **(strict)** n**-category**: We have arrows between arrows and so on, until we reach recursively n-arrows between $(n-1)$-arrows, equipped with strict compositions that behaves in a functorial way.

We give some examples below:

- **Cat** can be regarded as a 2-category, as we saw earlier.
- Every 1-category can be trivially regarded as a 2-category introducing only identities 2-cells.

- We can define **Rel** as the category of sets with relations between them as morphisms. Here, we can define a 2-cell as a function that preserves relations.

Definition 4.1.2. *A **bicategory** is a category "weakly" enriched over **Cat**. The only difference from a bicategory to a 2-category is that the unities and associativity of* $\circ_0$ *only work up to natural isomorphisms that satisfy the triangle and pentagon axiom.*

Remark. The natural isomorphisms above consist of 2-cells. We picture bicategories diagramatically just as **CAT**, with 1-cells drawn horizontally and 2-cells drawn vertically. Diagramatically, we denote 1-cells with $\rightarrow$ and 2-cells with $\Rightarrow$. Sometimes, when it is clear from the context, we denote any type of arrow simply by $\rightarrow$.

We give some examples below:

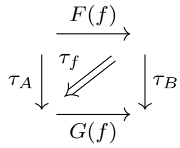
- In **Top**, we can regard homotopies as 2-cells. Actually, we can also consider homotopies between homotopies and so on. Composition of homotopies is a strict operation, but concatenation is associative only up to isomorphism. These morphisms between morphisms between morphisms (and so on) can be generalized in category theory as weak n-categories or even $(\infty, 1)$-categories if we take homotopy classes.
- We define the bicategory $\mathcal{R}ng$ as the bicategory in which 0-cells are rings, 1-cells $f\colon R \rightarrow S$ are $R-S-$bimodules, and 2-cells are bimodule homomorphisms.
- We define the bicategory $n\mathcal{C}ob$ as the bicategory in which 0-cells are closed n-manifolds, 1-cells are cobordisms and 2-cells are diffeomorphisms between these cobordisms.

Definition 4.1.3. *Given the bicategories **D** and **E**, a **(pseudo) functor*** $F\colon$ ***D*** $\rightarrow$ ***E*** *frm **D** to **E** consists of the following:*

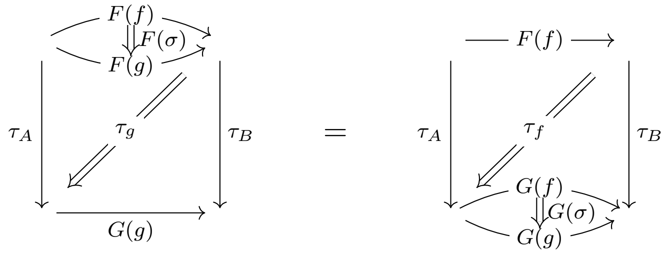
1. *A function* F_{Ob} *sending objects of **D** to objects of **E**, which we denote by* F*;*
2. *For each hom-category* $\mathit{hom}_{\mathbf{D}}(A, B)$, *a functor* $F_{A,B}\colon \mathit{hom}_{\mathbf{D}}(A, B) \rightarrow \mathit{hom}_{\mathbf{E}}(F(A), F(B))$, *which we also denote by* F*;*
3. *For each* $A \in$ ***D****, an isomorphism* $F_{1_A}\colon 1_{F(A)} \Rightarrow F(1_A)$*; and*
4. *A family of natural isomorphisms* $F_{g,f}\colon F(g) \odot F(f) \Rightarrow F(g \odot f)$.

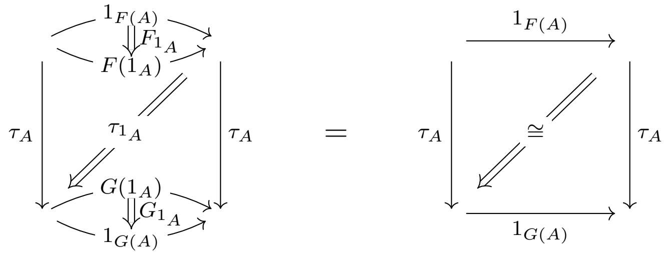
The above isomorphisms are required to satisfy the same coherence diagrams as monoidal functors.

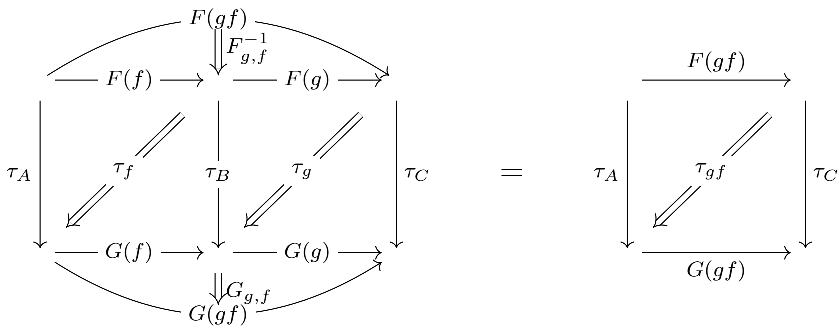
Definition 4.1.4. *An* **oplax (or colax) transformation** $\tau\colon F \to G$ *between two pseudo functors consists of 1-cells* $\tau_A\colon F(A) \to G(A)$ *and 2-cells*



such that, for any object A*, any 1-cells* f *,and* g *and any 2-cell* $\sigma\colon f \Rightarrow g$*, the following conditions hold:*







We call τ *a* **pseudo(natural) transformation** *if the 2-cells* τ_f *are isomorphisms. The concept of* **lax transformation** *is obtained from the concept of oplax transformation reversing the directions of the 2-cells* τ_f*.*

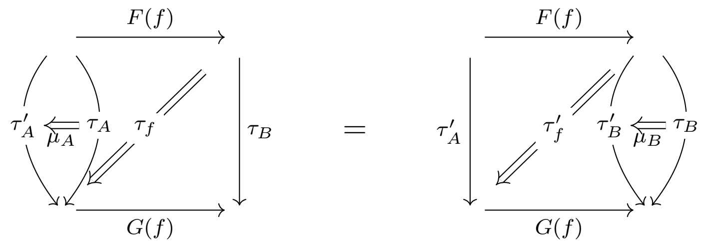
Remark. In order to clarify the diagrams in the above definition, first notice that τ_f is a 2-cell from $\tau_B \odot F(f)$ to $G(f) \odot \tau_A$. In the first equality, the left hand side must be read as the composite $\tau_B \odot F(f) \overset{1_{\tau_B} \odot F(\sigma)}{\Rightarrow} \tau_B \odot F(g) \overset{\tau_g}{\Rightarrow} G(g) \odot \tau_A$. The right hand side of the second equality is the natural isomorphism obtained using the unit constraints for horizontal composition.

Definition 4.1.5. *An* **icon** $\tau \colon F \to G$ *between pseudo functors that agree on objects consists of 2-cells* $\tau_f \colon f \Rightarrow g$ *which are natural in the 1-cells* f*, and such that, for any* $A \xrightarrow{f} B \xrightarrow{g}$*, the following diagrams commutes:*

$$
\begin{CD}
1_{F(A)} @>{F_{1_A}}>> F(1_A) \\
@V{\cong}VV @VV{\tau_{1_A}}V \\
1_{G(A)} @>>{G_{1_A}}> G(1_A)
\end{CD}
\qquad\qquad
\begin{CD}
F(g) \odot F(f) @>{F_{g,f}}>> F(g \odot f) \\
@V{\tau_g \odot \tau_f}VV @VV{\tau_{g \odot f}}V \\
G(g) \odot G(f) @>>{G_{g,f}}> G(g \odot f)
\end{CD}
$$

Remark. The isomorphism above is actually the identity, since $F(A) = G(A)$. Roughly speaking, an icon can be defined as an oplax transformation between functors that agrees on objects such that all 1-cell components are identities.

Definition 4.1.6. *A* **modification** $\mu \colon \tau \to \tau'$ *between oplax transformations consists of a family of 2-cells* $\tau_f \colon \tau_A \Rightarrow \tau'_A$ *such that:*



Remark. All this definitions apply to 2-categories because they can be seen as bicategories.

Remark. With these various concepts in mind, we can define some bicategories:

1. $\mathcal{B}icat_c(\mathbf{C}, \mathbf{D})$;

2. $\mathcal{B}icat_l(\mathbf{C}, \mathbf{D})$;

3. $\mathcal{B}icat_p(\mathbf{C}, \mathbf{D})$.

All of the three have the pseudo functors $F \colon \mathbf{C} \to \mathbf{D}$ as objects (0-cells), and the modifications as 2-cells. Their 1-cells are the colax, lax, and pseudo transformations respectively.

Definition 4.1.7. *Generalizing the definition given for 1-categories, an* **equivalence** *between 2-categories* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *is a pair of pseudo functors* $F \colon \boldsymbol{C} \to \boldsymbol{D}$ *and* $G \colon \boldsymbol{D} \to \boldsymbol{C}$ *with pseudonatural transformations* $F \circ G \cong 1_{\boldsymbol{D}}$, $G \circ F \cong 1_{\boldsymbol{C}}$*. We say that two 2-categories are* equivalent *(or* **biequivalent***) if, and only if, there is an equivalence between them.*

Just as in monoidal 1-categories, we can state a coherence and a strictification theorem for bicategories. We are not going to prove them, because they follows exactly the lines of the theorems we already proved for the monoidal case. A proof can be found in [23], and one can check that the proof is really the same as our proof for monoidal 1-categories.

Theorem 4.1.8. *Every formal diagram in a bicategory* $\boldsymbol{D}$ *commutes (here, a formal diagram is defined just as before).*

Theorem 4.1.9. *Every bicategory is biequivalent to a strict 2-category.*

Definition 4.1.10. *A* **(pseudo) double category** $\boldsymbol{D}$ *consists of:*

1. *Two categories,* $\boldsymbol{D}_0$*, and* $\boldsymbol{D}_1$*;*

2. *The following functors, called* ***structure functors****:*

$$U\colon \boldsymbol{D}_0 \to \boldsymbol{D}_1$$

$$S, T\colon \boldsymbol{D}_1 \to \boldsymbol{D}_0$$

$$\odot\colon \boldsymbol{D}_1 \times_{\boldsymbol{D}_0} \boldsymbol{D}_1 \to \boldsymbol{D}_1,$$

where the pullback is over S, T*. These structure functors are required to satisfy the following conditions, naturally:*

$$S(U(A)) = A \qquad S(M \odot N) = S(N)$$

$$T(U(A)) = A \qquad T(M \odot N) = T(M)$$

3. *The following natural isomorphisms:*

$$\mathfrak{a}\colon \odot(\odot \times Id_{\boldsymbol{D}}) \cong \odot(Id_{\boldsymbol{D}} \times \odot)$$

$$\mathfrak{l}\colon \odot(U \circ T \times Id_{\boldsymbol{D}}) \cong Id_{\boldsymbol{D}}$$

$$\mathfrak{r}\colon \odot(Id_{\boldsymbol{D}} \times U \circ S) \cong Id_{\boldsymbol{D}}$$

such that $T(\mathfrak{a})$*,* $S(\mathfrak{a})$*,* $T(\mathfrak{l})$*,* $S(\mathfrak{l})$*,* $T(\mathfrak{r})$*,* $S(\mathfrak{r})$ *are all identities, and such that the triangle and pentagon axiom hold for these natural isomorphisms.*

Remark. In the same way that we thought of a bicategory as a category weakly enriched over **CAT**, a pseudo double category can be seen as a category weakly internal to **CAT**. With this in mind, we say that S and T are the ***source*** and ***target functors***, respectively. U is the ***loose identity (or unit) functor***, and $\odot$ is the ***loose composition functor***. The natural transformation $\mathfrak{a}$ makes the composition functor associative up to isomorphism, on both objects and morphisms of $\mathbf{D}_1$. In their turn, the natural transformations $\mathfrak{l}$, $\mathfrak{r}$ make U give units up to isomorphisms, on both objects and morphisms of $\mathbf{D}_1$. Also notice that what appears in the unit constraints above is $U \circ T$ and $U \circ S$, for the identities are taken on different sides. For example, in $\mathfrak{l}$, we compose an arrow with the identity on the left, so the identity must be the identity of the target of that arrow.

Remark. As done for bicategories, we may strictify the loose structure by the argument given in the coherence/strictification theorem. A proof of these results can be found in [24], but, just as in the case of bicategories, the proof is essentially the one we presented for the coherence theorem for monoidal 1-categories.

Definition 4.1.11. *The compositions on $\boldsymbol{D}_0$ and $\boldsymbol{D}_1$ are said to be* **tight compositions** *because they are strictly associative and have a strict unit. The composition $\odot$ is called a* **loose composition***, since associativity and unitality hold only up to natural isomorphism. The objects of $\boldsymbol{D}_0$ are called* **objects** *or* **0-cells***, and the morphisms of $\boldsymbol{D}_0$ are called* **tight 1-cells***. We denote the 1-cells as $f \colon A \to B$. The objects of $\boldsymbol{D}_1$ are called* **loose 1-cells***, and the morphisms of $\boldsymbol{D}_1$ are called* **2-cells***. If M is a 1-cell such that $S(M) = A$, and $T(M) = B$, we denote M as $M \colon A \nrightarrow B$. If $\alpha \colon M \to N$ is a 2-cell such that $S(\alpha) = f$, and $T(\alpha) = g$, we denote α as below:*

$$\begin{array}{ccc} A & \overset{M}{\nrightarrow} & B \\ {\scriptstyle f}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle g} \\ C & \underset{N}{\nrightarrow} & D \end{array}$$

Remark. Sometimes k-cells are called k-***morphisms***. Also notice that, due to the functoriality of $\odot$, we have the following **interchange law**:

$$(M_1 \odot M_2) \circ (N_1 \odot N_2) = (M_1 \circ N_1) \odot (M_2 \circ N_2).$$

This property allows us to perform calculations using diagrams without caring about what type of composition we do first.

Definition 4.1.12. *A 2-cell with identities as source and target is called* **globular***.*

Remark. With this definition, $\mathfrak{a}$, $\mathfrak{l}$, $\mathfrak{r}$ are globular. It is also important to notice that a globular 2-cell is pictured just like a natural transformation: it is a 2-morphism between parallel loose 1-morphisms.

Definition 4.1.13. *Given a double category* $\boldsymbol{D}$*, we define* $\mathcal{L}(\boldsymbol{D})$ *as the* **loose bicategory***, consisting of the objects of* $\boldsymbol{D}$*, loose 1-cells, and globular 2-cells.*

Remark. Sometimes this category is called "horizontal bicategory" or "vertical bicategory" in the literature, but this nomenclature depends on how the author draws the diagrams, or what he or she calls a horizontal or vertical arrow.

Remark. To gain a better understanding on why $\mathcal{L}(\mathbf{D})$ is indeed a bicategory, see theorem **4.39**.

Definition 4.1.14. *Given two double categories* $\boldsymbol{C}$*, and* $\boldsymbol{D}$*, we can define their* ***product*** *as a double category* $\boldsymbol{C} \times \boldsymbol{D}$ *in a way similar to the product of 1-categories: We define objects and all types of morphisms as pairs, and define compositions componentwise.*

Here are some good examples of double categories, pointing out their respective loose bicategories:

- We define n**Cob** as the double category which has closed n-manifolds as objects, diffeomorphisms as tight 1-cells, cobordisms as loose 1-cells, and diffeomorphisms between cobordisms as 2-cells. The bicategory $\mathcal{L}(n\mathbf{Cob})$ is the bicategory of cobordisms that we discussed earlier.
- We define **Mod** as the double category which has rings as objects, ring homomorphisms as tight 1-cells, bimodules as loose 1-cells and equivariant bimodule maps as 2-cells. We have $\mathcal{L}(\mathbf{Mod}) = \mathcal{M}od$, the bicategory that we already discussed.
- A ***profunctor*** $A \to B$ is defined as a functor $B^{\mathsf{op}} \times A \to \mathbf{Set}$. We define **Prof** as the double category that has categories as 0-cells, functors as tight 1-cells, profunctors as loose 1-cells, and natural transformations as 2-cells. The bicategory $\mathcal{L}(\mathbf{Prof})$ is commonly encountered in category theory ([18], [19], [20]).

Definition 4.1.15. *A* **(pseudo double) functor** $F\colon \boldsymbol{D} \to \boldsymbol{E}$ *between double categories consists of the following data:*

1. *Functors $F_0\colon \boldsymbol{D}_0 \to \boldsymbol{E}_0$ and $F_1\colon \boldsymbol{D}_1 \to \boldsymbol{E}_1$ such that $S \circ F_1 = F_0 \circ S$ and $T \circ F_1 = F_0 \circ T$;*

2. *Natural transformations $F_\odot\colon F_1(M) \odot F_1(N) \to F_1(M \odot N)$ and $F_U\colon U_{F_0(A)} \to F_1(U_A)$ whose components are globular isomorphisms and which satisfy the same coherence diagrams as monoidal functors.*

Remark. Both F_0, F_1 are usually denoted as F.

Definition 4.1.16. *A* **(tight) natural transformation** *$\tau\colon F \to G$ between two pseudo double functors consists of two natural transformations $\tau_0\colon F_0 \to G_0$ and $\tau_1\colon F_1 \to G_1$ (both usually denoted by τ) with $S(\tau_M) = \tau_{S(M)}$ and $T(\tau_M) = \tau_{T(M)}$ such that the following equalities holds for all objects and 1-cells:*

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(M)} & F(B) & \xrightarrow{F(N)} & F(C) \\ 1 \| & & \Downarrow F_\odot & & \| 1 \\ F(A) & & \xrightarrow{F(N\odot M)} & & F(C) \\ \tau_A \downarrow & & \Downarrow \tau_{N\odot M} & & \downarrow \tau_C \\ G(A) & & \xrightarrow[G(N\odot M)]{} & & G(C) \end{array} \quad = \quad \begin{array}{ccccc} F(A) & \xrightarrow{F(M)} & F(B) & \xrightarrow{F(N)} & F(C) \\ \tau_A \downarrow & \Downarrow \tau_M & \tau_B \downarrow & \Downarrow \tau_N & \downarrow \tau_C \\ G(A) & \xrightarrow[G(M)]{} & G(B) & \xrightarrow[G(N)]{} & G(C) \\ 1 \| & & \Downarrow G_\odot & & \| 1 \\ G(A) & & \xrightarrow[G(N\odot M)]{} & & G(C) \end{array}$$

$$\begin{array}{ccc} F(A) & \xrightarrow{U_{F(A)}} & F(A) \\ 1 \| & \Downarrow F_U & \| 1 \\ F(A) & \xrightarrow{F(U_A)} & F(A) \\ \tau_A \downarrow & \Downarrow \alpha_{U_A} & \downarrow \tau_A \\ G(A) & \xrightarrow[G(U_A)]{} & G(A) \end{array} \quad = \quad \begin{array}{ccc} F(A) & \xrightarrow{U_{F(A)}} & F(A) \\ \tau_A \downarrow & \Downarrow U_{\alpha_A} & \downarrow \tau_A \\ G(A) & \xrightarrow{U_{G(A)}} & G(A) \\ 1 \| & \Downarrow G_U & \| 1 \\ G(A) & \xrightarrow[G(U_A)]{} & G(A) \end{array}$$

Remark. We have three different opposites for a double category **D**:

1. $\mathbf{D}^{l\text{-op}}$ is the ***loose opposite***, in which we and maintaining the tight 1-cells, reverse the loose 1-cells, and reverse the 2-cells only in the loose direction, i.e, we just interchange the roles of S and T;

2. $\mathbf{D}^{t\text{-op}}$ is the ***tight opposite***, obtained reversing the tight 1-cells, and maintaining loose 1-cells and 2-cells;

3. $\mathbf{D}^{tl\text{-op}}$ is the ***double opposite***, where we reverse both types of morphisms.

Definition 4.1.17. *We define $\mathcal{D}bl$ as the strict 2-category of double categories, pseudo double functors, and tight natural transformations. We define **Dbl** as its underlying 1-category.*

Remark. A 2-cell τ in $\mathcal{D}bl$ is invertible if, and only if, each τ_A and each τ_M are invertible.

Definition 4.1.18. *A* **monoidal double category** $\boldsymbol{D}$ *is a double category equipped with pseudo double functors* $\otimes\colon \boldsymbol{D} \times \boldsymbol{D} \to \boldsymbol{D}$ *and* $e\colon \{*\} \to \boldsymbol{D}$ *and tight natural isomorphisms*

$$\otimes(Id_{\boldsymbol{D}} \times \otimes) \overset{\alpha}{\cong} \otimes(\otimes \times Id_{\boldsymbol{D}})$$

$$\otimes(e \times Id_{\boldsymbol{D}}) \overset{l}{\cong} Id_{\boldsymbol{D}}$$

$$\otimes(Id_{\boldsymbol{D}} \times e) \overset{r}{\cong} Id_{\boldsymbol{D}}$$

satisfying the pentagon and triangle axioms. Unraveling this definition, we have the following data:

1. *Monoidal categories* $\boldsymbol{D}_0$ *and* $\boldsymbol{D}_1$*;*
2. *For the monoidal neutral* e *of* $\boldsymbol{D}_0$*,* U_e *is the monoidal neutral of* $\boldsymbol{D}_1$*;*
3. *Strict monoidal functors* S *and* T*, meaning that* $S(M \otimes N) = S(M) \otimes S(N)$*,* $T(M \otimes N) = T(M) \otimes T(N)$*, and that* S *and* T *preserve associators and unitors;*
4. *Globular isomorphisms* $\mathfrak{x} = \otimes_{\odot}$ *and* $\mathfrak{u} = \otimes_U$ *that satisfy the axioms of a pseudofunctor for* $\otimes$*; and*
5. α, r, l *that satisfy the diagrams of a tight natural transformation.*

Remark. Note that we did not introduce anything more than what is stated in the definition. For example, 3 follows from the first condition of the definition of a double functor combined with the conditions required for $T(\alpha)$ and $S(\alpha)$ for tight natural transformations. We unpacked this definition just to show that it can be done, althought it may not help much.

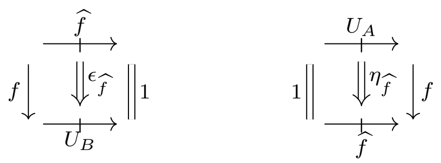
Definition 4.1.19. *A* **braided monoidal double category** $\boldsymbol{D}$ *is a monoidal double category equipped with a natural isomorphism* $\otimes \overset{\sigma}{\cong} \otimes \circ \tau$ *that satisfies the usual axioms for a braiding, where* τ *is the flip functor. If* σ *is involutory, the category is called* **symmetric***.*

- n**Cob** is a symmetric monoidal double category if we take $\otimes$ to be the disjoint union of manifolds;
- **Mod** is a symmetric monoidal category with respect to the usual tensor product of rings and bimodules;

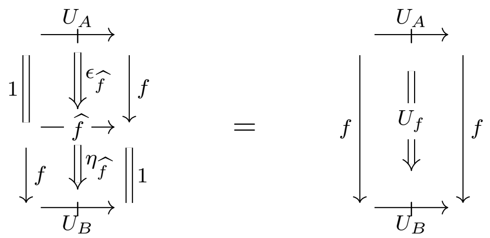
- **Prof** is a symmetric monoidal with respect to the cartesian product of categories.

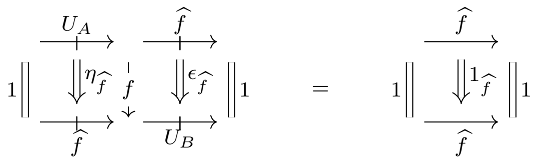
Definition 4.1.20. *A **lax monoidal double functor** $F\colon \boldsymbol{D} \to \boldsymbol{E}$ is a pseudo double functor F together with tight natural transformations $F_0\colon e_{\boldsymbol{E}} \to F \circ e_{\boldsymbol{D}}$ and $F_2\colon \otimes \circ (F \times F) \to F \circ \otimes$ satisfying the usual diagrams for a monoidal functor. In the same way we define colax monoidal double functors, braided functors and symmetric functors.*

Definition 4.1.21. *Given a tight 1-cell $f\colon A \to B$ in a double category, a* **companion** *of f is an arrow $\widehat{f}\colon A \nrightarrow B$ together with two 2-morphisms*



such that the following equalities hold:

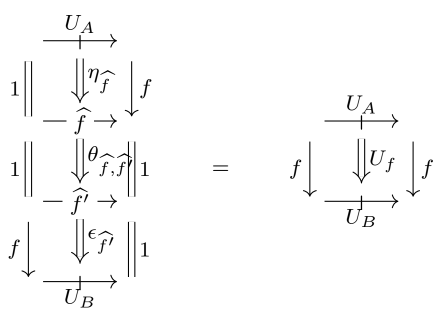




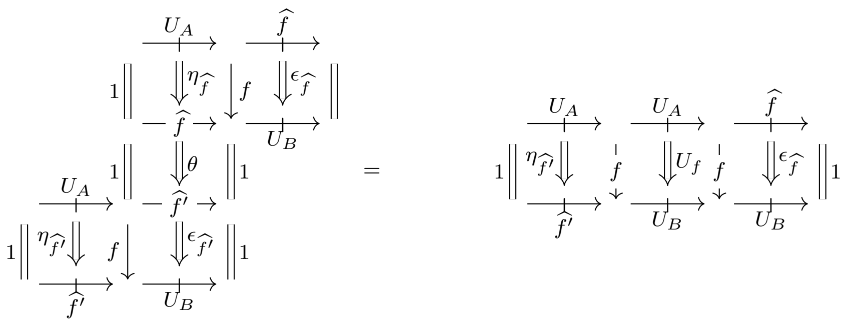
A **conjoint** *for f is defined as a companion of f in $\boldsymbol{D}^{l\text{-op}}$.*

Remark. When doing diagrammatic calculus as below, even when we have associativity and unit isomorphisms, we write "=" and ommit them in the calculation. This can be done because, when we reach a conclusion, we just have to put the canonical arrows that are missing and, by coherence, it does not matter what the order or what isomorphisms we choose.

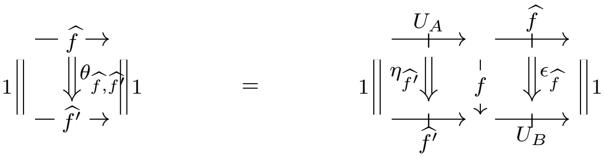
Proposition 4.1.22. *Let $\widehat{f}$, $\widehat{f}'$ be companions for $f\colon A \to B$. Then there is a unique globular isomorphism $\theta_{\widehat{f},\widehat{f}'}\colon \widehat{f} \to \widehat{f}'$ such that*



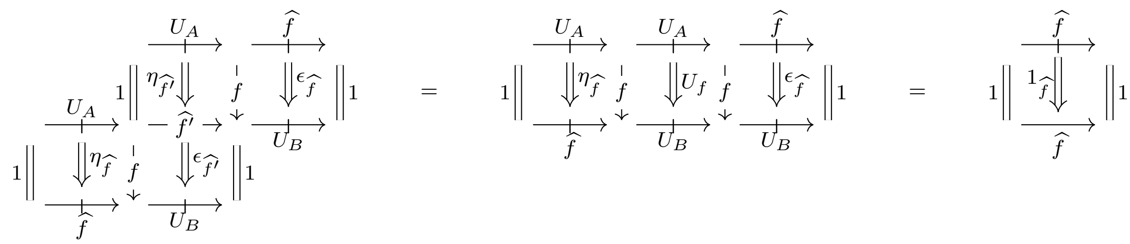
Proof: Notice that, if the above equality holds, then so does the following one:



Applying the definition of companion on the left hand side of the equation, we obtain that the top and bottom rectangles are identities, so when we compose these 2-cells we get θ. The right hand side is similar, since U_f is an identity. Therefore, we have



This proves that, if θ exists, then it must be the above loose composite. Reversing the steps, it is clear that θ defined as above satisfies the condition that we stated in the proposition. Now, it remains to show that the 2-cell θ is an isomorphism. If we interchange the roles of $\widehat{f}$ and $\widehat{f}'$ in the equation above, we get $\theta_{\widehat{f}',\widehat{f}}$. Now we prove that this 2-cell is equal to $\theta^{-1}_{\widehat{f},\widehat{f}'}$. Composing them, we get:



The equalities follow from the definition of companions. For the other composite of the θ's, the calculation is essentially the same. □

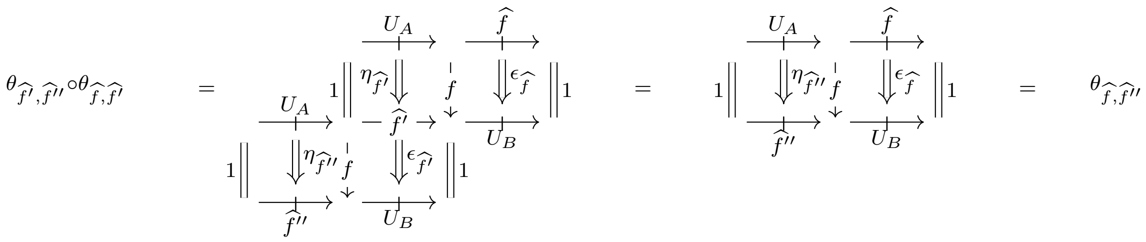
Remark. Note that θ as defined in the proposition is actually an isomorphism $\widehat{f} \odot U_A \cong U_B \odot \widehat{f}'$ but, as we already commented, we are ommitting the constraints so that we can say that θ is an isomorphism $\widehat{f} \cong \widehat{f}'$, as long as we keep this in mind!

Corollary. $\theta_{\widehat{f},\widehat{f}} = 1_{\widehat{f}}$.

Proof: The identity clearly satisfies the equality stated in the last proposition, so the equality above follows from the uniqueness of θ. □

Corollary. *If f has three companions $\widehat{f}$, $\widehat{f}'$, and $\widehat{f}''$, then $\theta_{\widehat{f},\widehat{f}''} = \theta_{\widehat{f}',\widehat{f}''} \circ \theta_{\widehat{f},\widehat{f}'}$.*

Proof:



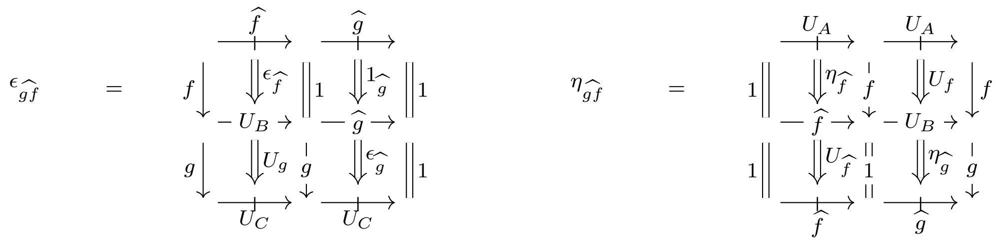
Where the second equality follows from the definition of companion. □

Proposition 4.1.23. *U_A is, canonically, a companion for 1_A.*

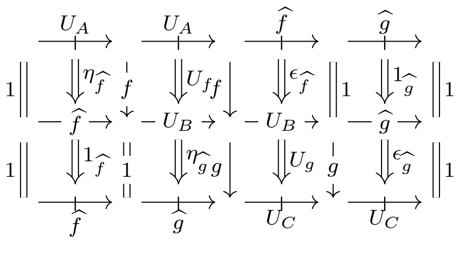
Proof: We define $\epsilon_{\widehat{1_A}} = 1_{U_A} = \eta_{\widehat{1_A}}$. The equalities in the definition of companion are automatically satisfied. □

Proposition 4.1.24. *If $\widehat{f}$ is a companion for $f\colon A \to B$, and $\widehat{g}$ is a companion of $g\colon B \to C$, then $\widehat{g} \odot \widehat{f}$ is a companion of gf.*

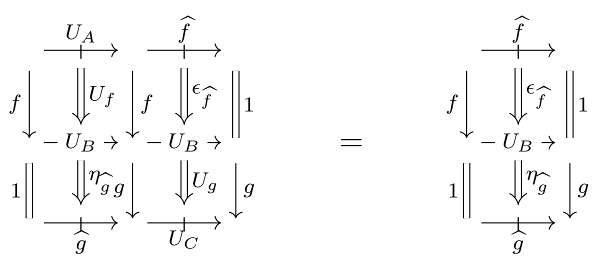
Proof: We define



We will check only the second equality in the definition of companion, since the calculations for the first equality are essentially the same. The left hand side of the equality is:



In the big central square:

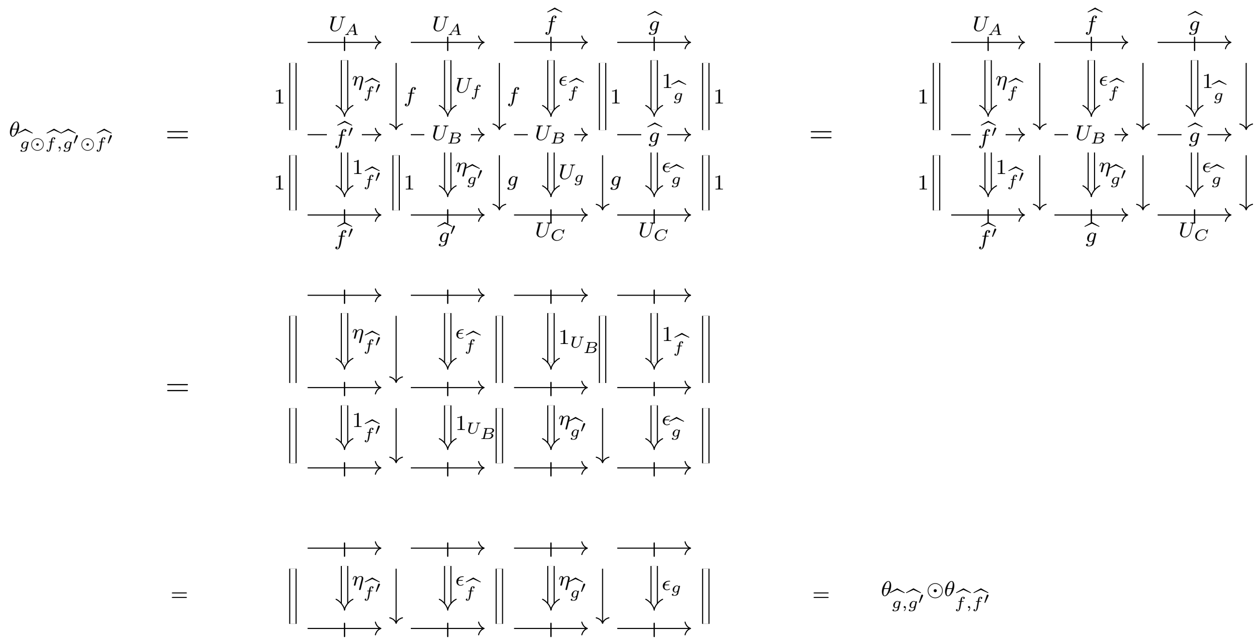


When we replace this in our diagram, we can apply the definition of companion to conclude that the composite of the diagram must be $1_{\widehat{gf}}$ because $\eta \odot \epsilon$ appears at the top and at the bottom. □

Remark. Sometimes, when we are only interested in 2-cells, and the respecive 1-cells can be inferred in the context, we ommit those 1-cells for brevity. The next proposition is one such example.

Proposition 4.1.25. *If $f\colon A \to B$ has $\widehat{f}$ as a companion and $g\colon B \to C$ has $\widehat{g}$ as a companion, then $\theta_{\widehat{g},\widehat{g'}} \odot \theta_{\widehat{f},\widehat{f'}} = \theta_{\widehat{g}\odot\widehat{f},\widehat{g'}\odot\widehat{f'}}$.*

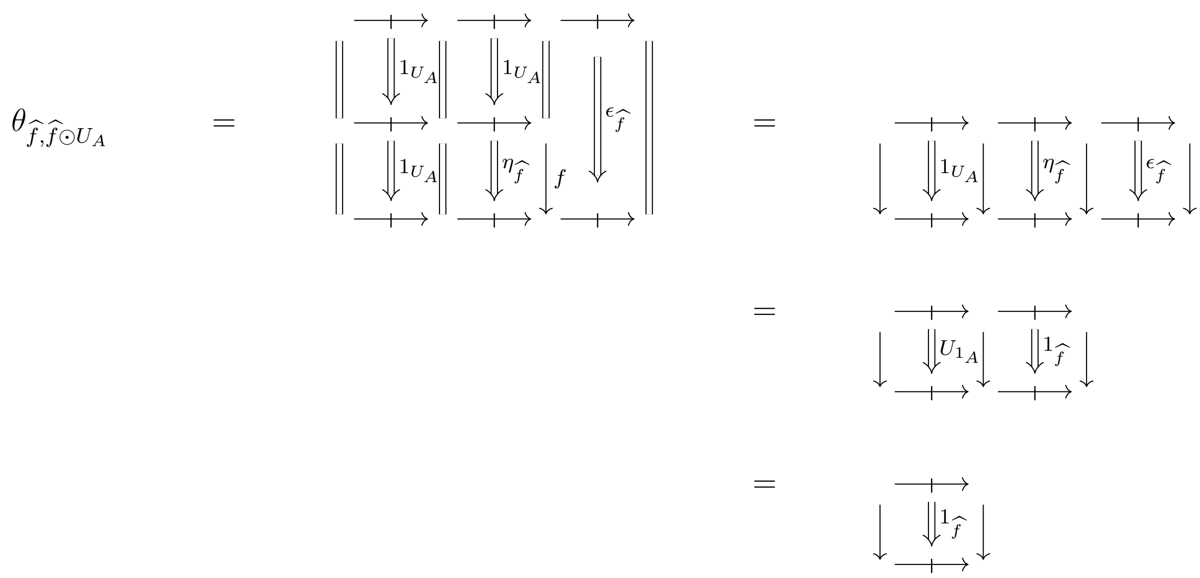
Proof: Due to our last proposition and our last corollary, the first equality of the following sequence of equalities holds:



The other equalities follow easily from the properties of identities. □

Proposition 4.1.26. *If $\widehat{f}$ is a companion for $f\colon A \to B$, then $\theta_{\widehat{f},\widehat{f}\odot U_A}$ and $\theta_{\widehat{f},U_B\odot\widehat{f}}$ are equal to the unit constraints.*

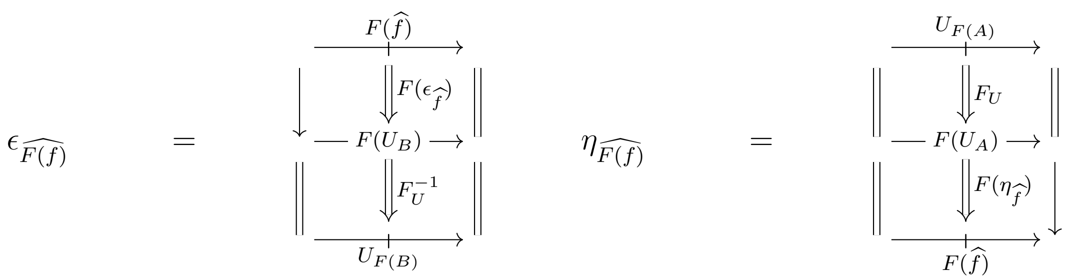
Proof: Using the definition of θ, our previous propositions, the definition of companion, and the functoriality of U, we obtain the following equalities:



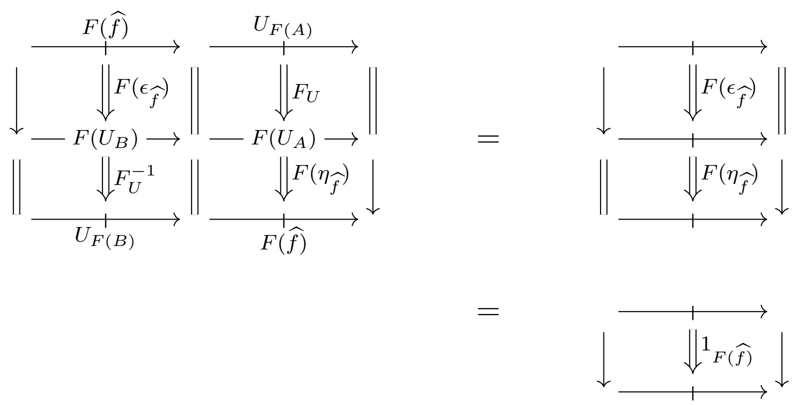
Ommitting the isomorphism constraints as we usually do, we have the desired isomorphism. The other statement is equally easy. □

Proposition 4.1.27. *If $F\colon \boldsymbol{D} \to \boldsymbol{E}$ is a pseudo double functor between the double categories $\boldsymbol{D}$, $\boldsymbol{E}$ and $\widehat{f}$ is a companion of $f\colon A \to B$ in $\boldsymbol{D}$, then $F(\widehat{f})$ is a companion of $F(f)$ in $\boldsymbol{E}$. Symbolically: $F(\widehat{f}) = \widehat{F(f)}$ up to isomorphism.*

Proof: We define ϵ and η as:



Checking the second diagram of the definition:



Where, in the first equality, we used the same trick used in proposition **4.6** (adding loose unities) and, in the second equality, we just used the functoriality of F and the definition of a companion. The first diagram follows similarly. □

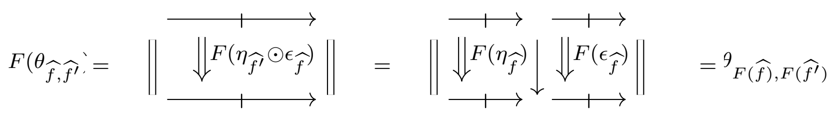
Remark. We ommitted the isomorphisms F_U and $F_\odot$ in our diagrammatic calculations for the same reason as before. We will continue to ommit these arrows.

Corollary. *If* $\mathbf{D}$ *is a monoidal double category and* $f\colon A \to B$, $g\colon C \to D$ *have companions* $\widehat{f}$ *and* $\widehat{g}$ *respectively, then* $\widehat{f} \otimes \widehat{g}$ *is a companion of* $f \otimes g$.

Proof: Since $\otimes$ is a pseudo double functor $\otimes\colon \mathbf{D} \times \mathbf{D} \to \mathbf{D}$, and since a companion in $\mathbf{D} \times \mathbf{D}$ is just, by definition, a pair of companions in $\mathbf{D}$, the result follows directly from our last proposition. □

Proposition 4.1.28. *If* $F\colon \mathbf{D} \to \mathbf{E}$ *is a pseudo double functor and* $\widehat{f}$, $\widehat{f}'$ *are companions for* $f\colon A \to B$, *then* $\theta_{F(\widehat{f}),F(\widehat{f}')} = F(\theta_{\widehat{f},\widehat{f}'})$.

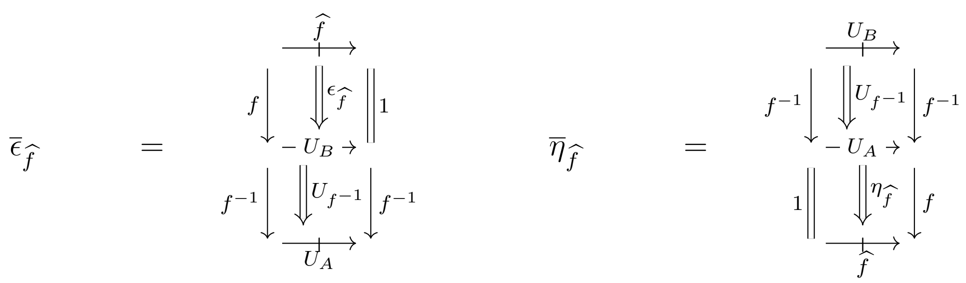
Proof:



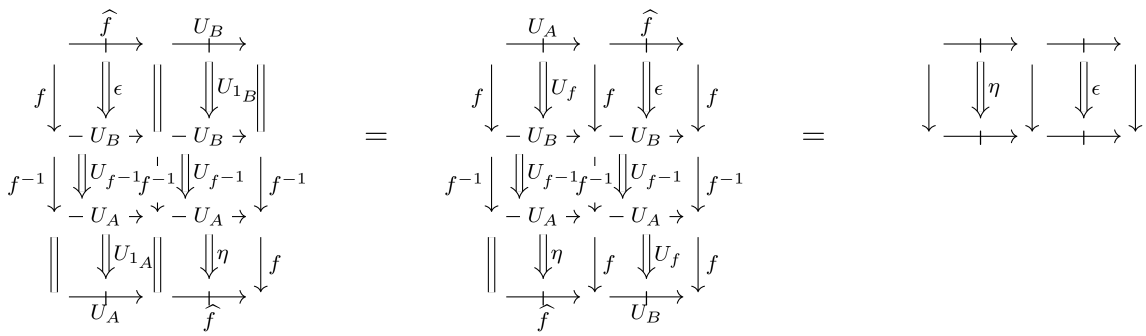
As we wanted. □

Proposition 4.1.29. *If* $\widehat{f}$ *is a companion for a tight isomorphism* $f\colon A \to B$, *then* $\widehat{f}$ *is a conjoint for* $f^{-1}\colon B \to A$.

Proof: Recall that a conjoint is a companion obtained reversing the loose arrows in the definition of conjoint. We want to proof that $\widehat{f}$, together with $\overline{\epsilon}$ and $\overline{\eta}$ as defined below, is a conjoint for f:



In order to prove this, let us paste the diagrams together, reducing the problem to the diagrams of a companion, obtaining the first equality below. For the second one, we add identities on the bottom of the first diagram and on the top of the second diagram, for this will not alter the tight 2-cell composition, and then we perform the loose composition, obtaining:



The last diagram is equal to $1_{\widehat{f}}$ by definition of companion. In the second equality, we used the functoriality of U to conclude $U_{f^{-1}} \circ U_f = U_1 = 1_U$, so this is indeed a tight identity 2-cell.

The verification of the first equality that defines a conjoint is straightforward . and we do not need any trick of adding identities, so we just look at the diagrams pasted vertically in order to conclude the proof. □

Definition 4.1.30. *We call a double category **D** **(iso)fibrant** if every tight isomorphism has a companion.*

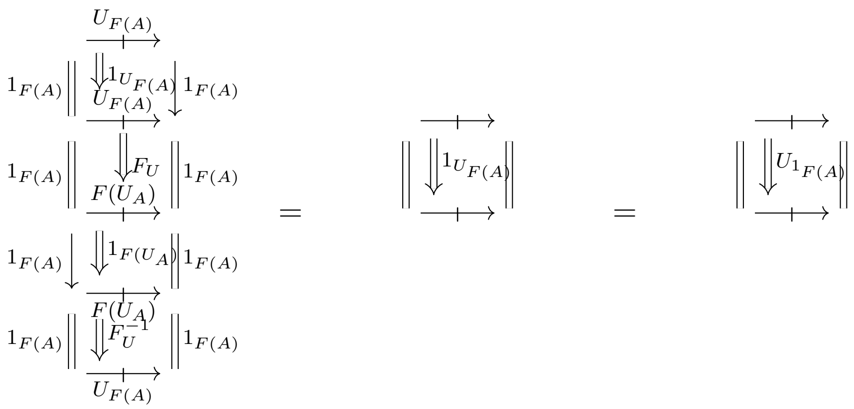
Remark. We denote as $\mathcal{D}bl_{\mathbf{cf}}$ the sub-2-category of $\mathcal{D}bl$ that contains all double categories and all pseudo double functors between them, but only the tight transformations such that each of the components α_A has a companion.

Remark. We denote as $\mathcal{D}bl_{\mathbf{cf}}$ the sub-2-category of $\mathcal{D}bl$ that contains all double categories and all pseudo double functors between them, but only the tight transformations such that each component α_A has a companion.

Proposition 4.1.31. *If $F\colon$ **D** $\to$ **E** is a pseudo double functor, then $\theta_{U_{F(A)}, F(U_A)} = F_U$.*

Proof: Due to propositions **4.1.23** and **4.1.27**, we have that $U_{F(A)}$ and $F(U_A)$ are companions for $1_{F(A)}\colon F(A) \to F(A)$, so it makes sense to talk about the isomorphism $\theta_{U_{F(A)}, F(U_A)}$.

We are going to check that the equality of proposition **4.1.22** holds for the globular isomorphism F_U. We substitute the η and ϵ as in propositions **4.1.23** and **4.1.27**. A direct substitution gives us



The equality stated in this proposition follows from the uniqueness of the globular isomorphism θ. $\square$

Although we did not defined yet what a monoidal 2-category is, we observe what happens in the general case of enriched bicategories. However, we can appreciate their definition restricting it to $\mathbf{V} = \mathcal{D}bl$ and $\otimes = \times$ (See definition **4.1.33**).

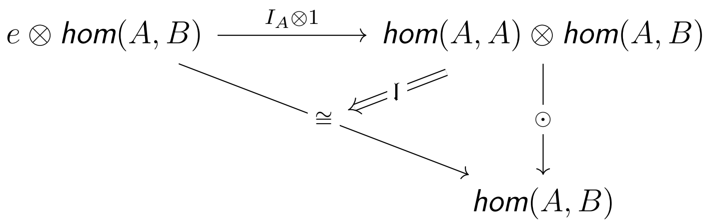
Definition 4.1.32. *Suppose we are given a monoidal 2-category* ***V****. A bicategory* ***C*** **enriched** *over* ***V*** *(or simply* ***V-bicategory****) consists of the following data:*

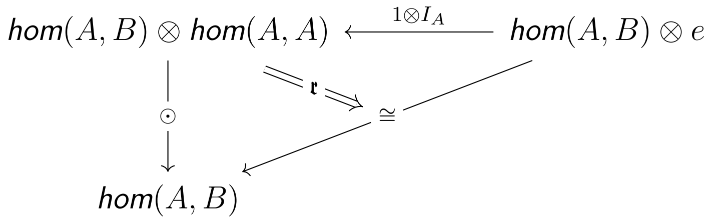
1. *A collection of objects;*
2. *For each ordered pair of objects $(A,\ B)$, an object hom$_{\mathbf{C}}(A,B) \in$* ***V****;*
3. *For each ordered triple of objects $(A,\ B,\ C)$, a 1-cell $\odot\colon$ hom$(B,C) \otimes$ hom$(A,B) \to$ hom(A,C) of* ***V****;*
4. *For each object A, a 1-cell $I_A\colon e \to$ hom(A,A);*
5. *For each ordered quadruple of objects $(A,\ B,\ C,\ D)$, an invertible 2-cell of* ***V***

$$\begin{array}{ccc} hom(C,D)\otimes hom(B,C)\otimes hom(A,B) & \xrightarrow{\odot\otimes 1} & hom(B,D)\otimes hom(A,B) \\ {\scriptstyle 1\otimes\odot}\downarrow & \swarrow_{\mathfrak{a}} & \downarrow{\scriptstyle \odot} \\ hom(C,D)\otimes hom(A,C) & \xrightarrow[\odot]{} & hom(A,D) \end{array}$$

Where the associativity constraint of $\otimes$ is supressed from the diagram; and

6. *For each ordered pair of objects $(A,\ B)$, invertible 2-cells of* ***V***





We impose that the usual axioms for a bicategory hold, albeit interpreted in a new language.

Remark. Our idea of enrichment over a monoidal 1-category is pretty similar to this one, and can easily be abstracted.

Definition 4.1.33. *A* **locally cubical bicategory** ***C*** *is a bicategory enriched over the monoidal 2-category $\mathcal{D}bl$, where the tensor product is $\times$, and $e = \{*\}$ (So it is a cartesian monoidal category.) Furthermore, it is called 1-**strict** if, and only if, $\mathfrak{a}$, $\mathfrak{r}$, $\mathfrak{l}$ are globular.*

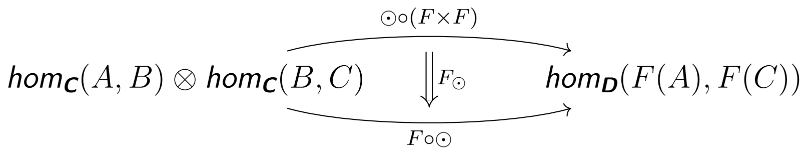
Remark. The objects of a locally cubical bicategory are called 0-**cells**. The objects of hom(A,B) are called 1-**cells**. The tight/loose 2-**cells** of a locally cubical bicategory are the tight/loose 1-cells of hom(A,B). In the same way, we define the 3-**cells** of a locally cubical bicategory as the 2-cells of hom(A,B). The notation is fixed as:

1. $\longrightarrow$ for 1-cells;
2. $\Longrightarrow$ for tight 2-cells;
3. $=\!\!\!|\!\!\!\Rightarrow$ for loose 2-cells;
4. $\Rrightarrow$ for 3-cells.

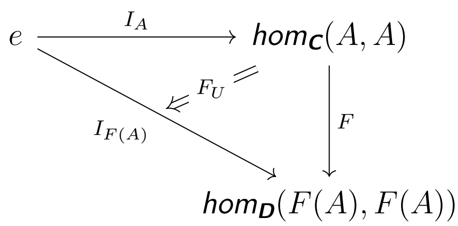
We denote the constraints for the loose composition as $a^\bullet$, $r^\bullet$ and $l^\bullet$. However, we may ommit them, strictifying the hom-double categories. Due to coherence, one can adjust the domain/codomains of the diagrams with any composition of canonical arrows that fits. Also, the loose composition in the hom-double categories is denoted as $\bullet$ and $\alpha \bullet \beta$, and it is read as β after α. Tight composition in the hom-double categories is denoted as $\cdot$. Finally, the composition along objects is written just as $\circ$. The compositions $\cdot$ and $\circ$ follow the usual order of composition, in contrast with $\bullet$.

Definition 4.1.34. *If **C**, **D** are **V**-bicategories, a **V**-enriched functor $F\colon$ **C** $\to$ **D** consists of the following data:*

1. *A function from the objects of **C** to the objects of **D**;*
2. *For each ordered pair of objects $(A,\ B)$, a 1-cell hom$_{\boldsymbol{C}}(A,B) \to$ hom$_{\boldsymbol{D}}(F(A),F(B))$ of **V**;*
3. *For each ordered triple of objects $(A,\ B,\ C)$, an invertible 2-cell*



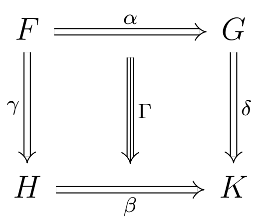
4. *For each object A, an invertible 2-cell*



We require these cells to satisfy the usual coherence diagrams.

Definition 4.1.35. *A* **functor of locally cubical bicategories** *is defined as an enriched functor over the monoidal 2-category $\mathcal{D}bl$. Furthermore, it is called* 1-***strict*** *if, and only if, F_U and $F_\odot$ are globular.*

Definition 4.1.36. *If $F,G,H,K\colon$ **C** $\to$ **D** are pseudo functors between bicategories, $\alpha\colon F \Rightarrow G$, $\beta\colon H \Rightarrow K$ are pseudo transformations, $\gamma\colon F \Rightarrow H$, and $\delta\colon G \Rightarrow K$ are icons, then a* **cubical modification**



is defined as a family of 2-cells $\Gamma_A\colon \alpha_A \Rrightarrow \beta_A$ such that the following equality holds:

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_B} & G(B) \\ {}_1\Vert & & \Downarrow \alpha_f & & \Vert_1 \\ F(A) & \xrightarrow{\alpha_A} & G(A) & \xrightarrow{G(f)} & G(B) \\ {}_1\Vert & \Downarrow \Gamma_A & \Vert_1 & \Downarrow \delta_f & \Vert_1 \\ H(A) & \xrightarrow[\beta_A]{} & K(A) & \xrightarrow[K(f)]{} & K(B) \end{array} \quad = \quad \begin{array}{ccccc} F(A) & \xrightarrow{F(f)} & F(B) & \xrightarrow{\alpha_B} & G(B) \\ {}_1\Vert & \Downarrow \gamma_f & \Vert_1 & \Downarrow \Gamma_B & \Vert_1 \\ H(A) & \xrightarrow[H(f)]{} & H(B) & \xrightarrow[\beta_B]{} & K(B) \\ {}_1\Vert & & \Downarrow \beta_f & & \Vert_1 \\ H(A) & \xrightarrow[\beta_A]{} & K(A) & \xrightarrow[K(f)]{} & K(B) \end{array}$$

Definition 4.1.37. *The following locally cubical bicategories are going to be important for our work:*

1. *Any category can be regarded as a strict double category in the loose direction, as we already saw. This operation clearly preserves products so, in this way, any strict 2-category can be viewed as a locally cubical bicategory. We regard $\mathcal{D}bl_{\boldsymbol{f}}$ as the locally cubical bicategory $\mathfrak{Dbl}_{\boldsymbol{f}}$.*

2. *The locally cubical bicategory $\mathfrak{Bicat}$ is defined having bicategories as 0-cells, pseudo functors as 1-cells, pseudonatural transformations as loose 2-cells, icons as tight 2-cells, and cubical modifications as 3-cells.*

Remark. Loose composition of pseudo transformations and icons is given by the usual Godement product. The loose composition of cubical modifications is defined as

$$\begin{array}{ccccc} FF'(A) & \xrightarrow{\alpha_{F'(A)}} & GF'(A) & \xrightarrow{G(\alpha'_A)} & GG'(A) \\ {}_1\Vert & \Downarrow \Gamma_{F'(A)} & \Vert_1 & \Downarrow \delta_{\alpha'_A} & \Vert_1 \\ HF'(A) & \xrightarrow[\beta_{F'(A)}]{} & KF'(A) & \xrightarrow[K(\alpha'_A)]{} & KG'(A) \\ {}_1\Vert & = & \Vert_1 & \Downarrow K(\Gamma'_A) & \Vert_1 \\ HH'(A) & \xrightarrow[\beta_{H'(A)}]{} & KH'(A) & \xrightarrow[K(\beta'_A)]{} & KK'(A) \end{array}$$

Remark. All conditions for a locally cubical bicategory are satisfied for 0,1 and 2-cells as to we studied so far. We will not verify the conditions for 3-cells because they follow the exact same lines as before, performing simple graphical calculations with our diagrams.

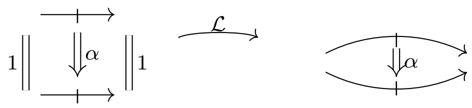
A final important point regarding higher monoidal category theory is the notion of an ***equivalence*. For an n-category, this** is a morphism that is invertible in a maximally weak sense, i.e, up to all higher equivalences. For example, in a 0-category (i.e, a class), an equivalence is equality. In a 1-category, an equivalence is an isomorphism. In a 2-category, an equivalence between f and g consists of 2-cell isomorphisms $fg \cong 1$, $gf \cong 1$. In a 3-category, an equivalence between f and g consists of two 2-cells $fg \to 1$, $gf \to 1$ that are equivalences in their hom-2-category. Recursively, we can define what is an equivalence in a (strong or weak) n-category, in a bicategory, or in any other kind of higher dimensional category that we treated.

4.2 LIFTING THE ASSIGNMENT $\mathcal{L}$

Here, we denote the underlying 1-categories of $\mathcal{Dbl}$ and $\mathcal{Bicat}$ by **Dbl** and **Bicat**, respectively.

Theorem 4.2.1. *If **D** is a double category, then $\mathcal{L}(\boldsymbol{D})$ is a bicategory, and if we are given a pseudo double functor $F\colon \boldsymbol{D} \to \boldsymbol{E}$, we have an induced pseudofunctor $\mathcal{L}(F)\colon \mathcal{L}(\boldsymbol{D}) \to \mathcal{L}(\boldsymbol{E})$. In this way, the assignment $\mathcal{L}$ defines a functor of 1-categories $\mathcal{L}\colon$ **Dbl** $\to$ **Bicat**.*

Proof: Since the constraints for **D** are globular, they collapse like below

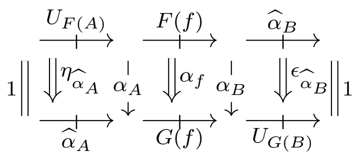


Therefore this constraints are transported to $\mathcal{L}$ as the constraints of a bicategory. Also, when we have a pseudo double functor F, the constraints $F_\odot$ and F_U become the constraints of the underlying pseudo functor, by definition. Since we are taking just an underlying structure, it is clear that $\mathcal{L}$ also preserves unities and compositions. □

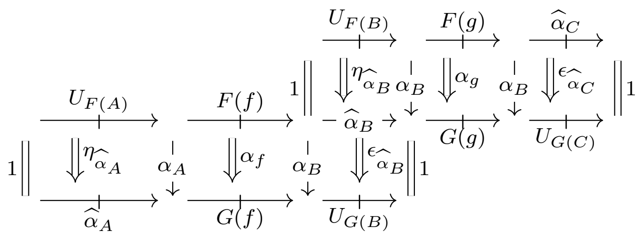
Remark. The functor $R\colon$ **Bicat** $\to$ **Dbl** that regards a bicategory as a double category (in the way we said earlier) is, by definition, a left adjoint to $\mathcal{L}$.

Theorem 4.2.2. *We have a functor of bicategories $\mathcal{L}\colon \mathcal{Dbl}_{cf}(\boldsymbol{C}, \boldsymbol{D}) \to \mathcal{Bicat}_c(\mathcal{L}(\boldsymbol{C}), \mathcal{L}(\boldsymbol{D}))$ defined as $\mathcal{L}(F)$ on functors and $\mathcal{L}(\alpha) = \widehat{\alpha}$ on colax transformations. Here we are regardind the domain 1-category as a bicategory with identity 2-cells.*

Proof: Define the 1-cells of $\widehat{\alpha}$ choosing fixed companions $\widehat{\alpha}_A = \widehat{\alpha_A}$. The 2-cell $\widehat{\alpha}_f$ is defined as



We can see that this composition is indeed a colax transformation. To check the diagrams involved, we can just paste simple diagrams together and manipulate like we have been manipulating everything. For example (suppresing canonical isomorphisms):



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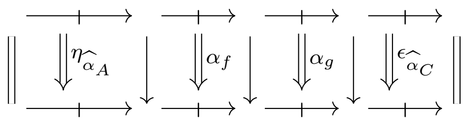

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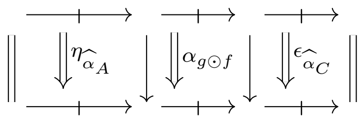


= $\widehat{\alpha}_{g\odot f}$

The other axioms for a colax transformations are easier than this one, and one should be able to see that they hold just examining their diagrams pasted.

Now we only have to construct and check the axioms for the functor constraints. Using propositions **4.23** and **4.24**, we have isomorphisms

$$\theta_{\widehat{\beta\alpha}_A,\widehat{\beta}_A\odot\widehat{\alpha}_A} : \widehat{\beta\alpha}_A \to \widehat{\beta}_A \odot \widehat{\alpha}_A$$

$$\theta_{\widehat{1}_A,U_A} : \widehat{1}_A \to U_A$$

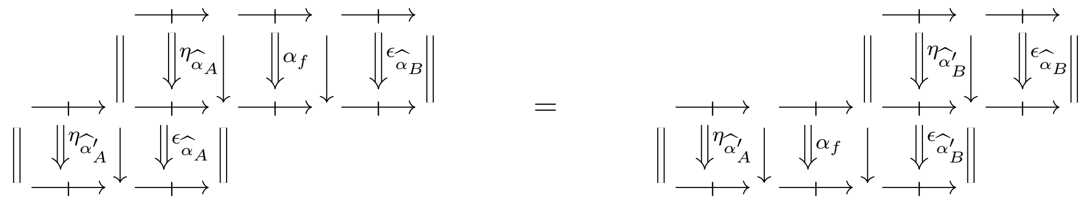
The first family of constraints is natural since the domain category of the functor has no non-identity 2-cells (so any 2-cell $f \to f'$ must be identities, and the diagram for naturality of $F_{g,f}$ commutes trivially).

The associativity and unit constraints follow from the uniqueness of the θ's, recalling that those θ's preserve compositions in a suitable way. □

Remark. Note that $\mathcal{L}$ above actually depends on the choice of the companions of α_A. Nevertheless, we will now prove a proposition connecting functors constructed with different choices, providing a sort of covariance for them via icons.

Proposition 4.2.3. *If we construct functors $\mathcal{L}$ and $\mathcal{L}'$ as in the last theorem, perhaps with different choices for companions of tight transformations, we have an invertible icon $\mathcal{L} \cong \mathcal{L}'$ constructed from the isomorphisms θ.*

Proof: Given a transformation $\alpha\colon F \to G$, we will show that the θ's corresponds to 2-cells $\widehat{\alpha} \cong \widehat{\alpha}'$ in $\mathcal{B}icat_c(\mathcal{L}(\mathbf{C}), \mathcal{L}(\mathbf{D}))$ (i.e, invertible modifications). We need to show this because an icon consists of, among other things, 2-cells in the codomain category. Using the definitions of $\widehat{\alpha}_f$ and θ, the statement that the θ's constitute a modification is, by definition, equivalent to the validity of the following equality:



By definition of companions, it actually holds.

Now we can see that these θ's form, indeed, an invertible icon. We do not need to do any calculation with 2-cells, for all of the 2-cells in our domain category are identities. Functoriality follows easily from our propositions about how θ behaves. □

Theorem 4.2.4. *The assignment $\mathcal{L}$ defines a 1-strict functor of locally cubical bicategories*

$$\mathcal{L}\colon \mathfrak{Dbl}_{\mathfrak{f}} \to \mathfrak{Bicat}.$$

Proof: The first two conditions of definition **4.34** follow from theorems **4.38** and **4.39**. The third condition requires a tight transformation $F_\odot\colon \widehat{\beta} \odot \widehat{\alpha} \cong \widehat{\beta \odot \alpha}$, for any $\alpha\colon F \Rightarrow G$ and $\beta\colon H \Rightarrow K$ horizontally composable transformations with loosely strong companions. Moreover, this isomorphism must make the following diagrams commutes:

$$\begin{array}{ccc} 1_{\mathcal{L}(H)\odot\mathcal{L}(F)} & \xrightarrow{\theta} & \widehat{1}_H \odot \widehat{1}_F \\ \Big\| {=} & & \Big\downarrow F_\odot \\ 1_{\mathcal{L}(H\odot F)} & \xrightarrow[\theta]{} & \widehat{1_H \odot 1_F} \end{array} \qquad \begin{array}{ccc} \widehat{\gamma\alpha} \odot \widehat{\delta\beta} & \xrightarrow{\theta} & (\widehat{\gamma} \odot \widehat{\delta}) \circ (\widehat{\alpha} \odot \widehat{\beta}) \\ F_\odot \Big\downarrow & & \Big\downarrow F_\odot \odot F_\odot \\ \widehat{\gamma\alpha \odot \delta\beta} & \xrightarrow[\theta]{} & (\widehat{\gamma \odot \delta}) \circ (\widehat{\alpha \odot \beta}) \end{array}$$

Since $\odot$ is a functor, we have that $\widehat{\beta} \odot \widehat{\alpha} = \widehat{\beta}_{G(A)} \circ H(\widehat{\alpha}_A)$ is a companion of $(\beta \odot \alpha)_A$ because of propositions **4.24** and **4.27**. Therefore, we can take $F_\odot = \theta_{\widehat{\beta}_{G(A)} \circ H(\widehat{\alpha}_A), \widehat{\beta\odot\alpha}_A}$, and the commutativity of the coherence diagrams above follows from the coherence of θ's that our results from this section provide. An analogous argument can be applied to F_U. □

Proposition 4.2.5. *The functor* $\mathcal{L}\colon \mathfrak{Dbl}_{\mathbf{f}} \to \mathfrak{Bicat}$ *preserves products.*

Proof: Since $\mathcal{L}$ acts like a forgetful functor, we have that $\mathcal{L}(\mathbf{C} \times \mathbf{D}) = \mathcal{L}(\mathbf{C}) \times \mathcal{L}(\mathbf{D})$. Projections also follows with $\pi_{\mathcal{L}(\mathbf{C})} = \mathcal{L}(\pi_{\mathbf{C}})$. The terminal object is also preserved because: In $\mathfrak{Dbl}_{\mathbf{f}}$, the initial object is the category with only one object and only identities cells. The underlying bicategory is the terminal object of $\mathfrak{Bicat}$ as well. □

4.3 MONOIDAL STRUCTURE IN LOCALLY CUBICAL BICATEGORIES

We start this section noticing that a bicategory **C** with just one object can be regarded as a monoidal 1-category, and vice-versa. Indeed, the objects of this monoidal 1-category are the 1-cells of **C**, its morphisms are the 2-cells of **C**, irs composition is the tight composition of the bicategory, and its tensor product $\otimes$ is $\odot$. This is a rather compact, convenient description of monoidal 1-categories. We can produce a notion of a ***monoidal bicategory*** in the same exact way, first giving a suitable concept of ***tricategory*** and, then, defining a monoidal bicategory to be a tricategory with only one object.

Definition 4.3.1. *A* **tricategory** ***T*** *is a category weakly enriched over the cartesian monoidal 2-category* $\mathcal{B}icat$*. Unfolding this definition, we get the following data:*

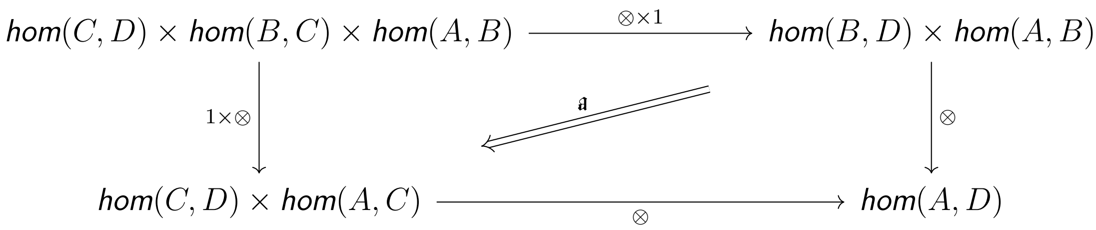
1. *A class of objects, called the* 0-***cells*** *of* ***T****;*

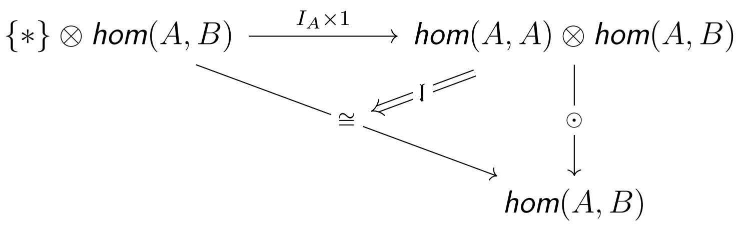
2. *For each ordered pair of objects* $(A,\ B)$*, a bicategory* $\textbf{\textit{T}}(A, B) = hom(A, B)$*. Its* n*-cells are called the* $(n+1)$-***cells*** *of* ***T****, where* $n \in \{0,\ 1,\ 2\}$*. Vertical composition is denoted as usual, and horizontal composition is denoted as* $\odot$*. We will not write explicitly the constraints, since we can supress them by coherence;*

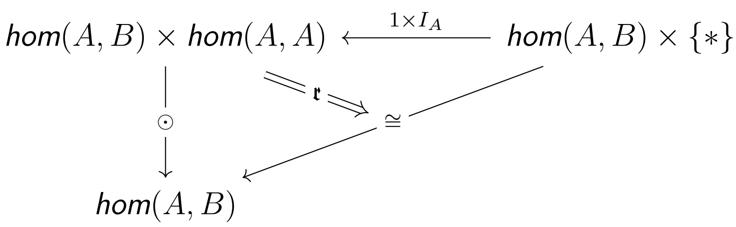
3. *For each ordered triple ofobjects* $(A,\ \ B,\ \ C)$*, a pseudo functor* $\otimes\colon hom(B, C) \times hom(A, B) \to hom(A, C)$*, simply called* ***composition****. Again, due to coherence, we will not write explicitly the constraints of the functor;*

4. *For each object* A*, a pseudo functor* $I_A\colon \{*\} \to hom(A, A)$*;*

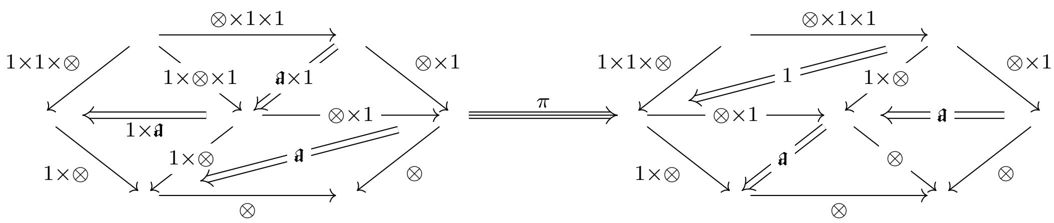
5. *For each ordered quadruple of objects* $(A,\ B,\ C,\ D)$, *adjoint equivalences*

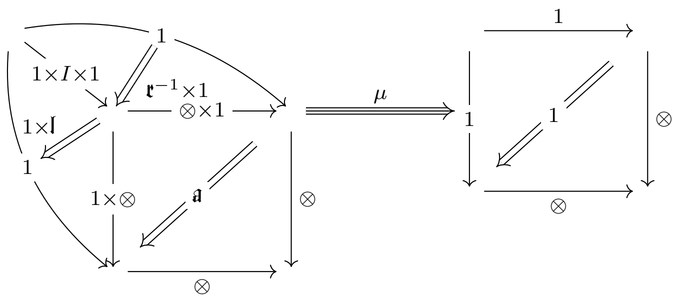


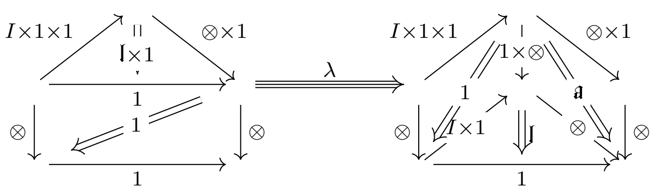


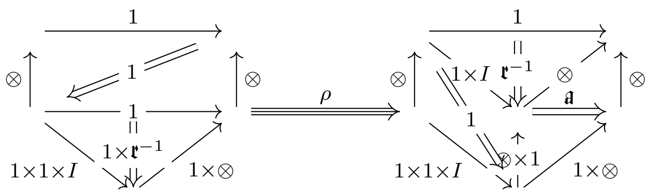


6. *For objects* A, B, C, D, E, *invertible modifications* π, μ, λ, ρ *as below:*

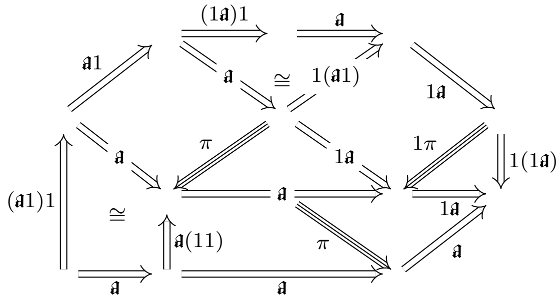




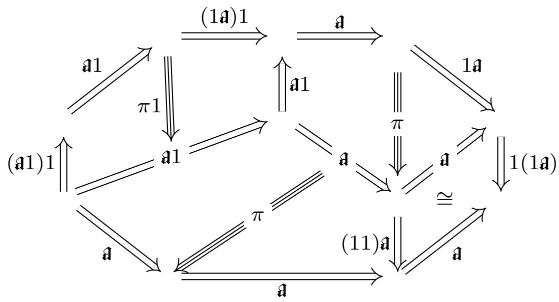


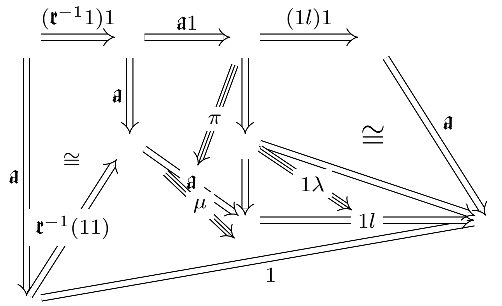


The modifications above are required to satisfy the axioms below, in which we ommit $\otimes$ from the diagram, writing ab to denote $a \otimes b$. We also ommit the 1-cells:



$$=$$





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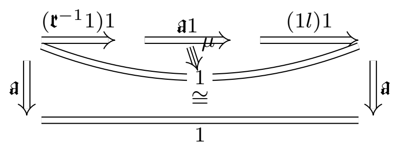
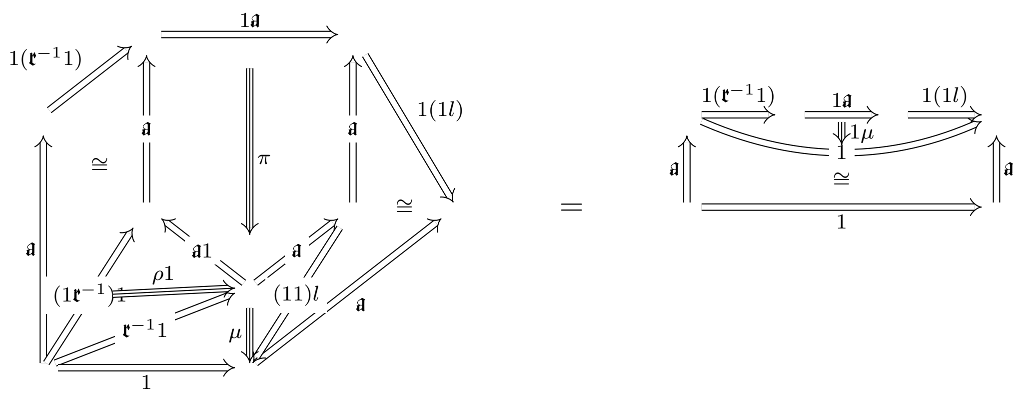




Remark. For the same reasons that we did in the previous section, for companions, we ommit the vertices of the diagrams.

Remark. Note that π generalizes the pentagon axiom, and μ generalizes the triangle axiom. Since we want a theory based on equivalences, and we are in a category of dimension 3, we have to take in account that the equality is not the last step when we have two diagrams of 2-cells, so π is not necessarily an equality, and so is not the pentagon axiom.

Remark. From now one, assume that we have a 1-strict locally cubical bicategory **C**.

Definition 4.3.2. *Let $\{*\}$ be the terminal object of the locally cubical bicategory **C**, and $i_1\colon A \to A \times \{*\}$ and $i_2\colon A \to \{*\} \times A$ be the canonical isomorphisms. A* **monoidal object** *in **C** is an object A equipped with 1-cells $\otimes\colon A \times A \to A$ and $I_A\colon \{*\} \to A$, loose 2-cell equivalences $\alpha\colon \otimes(\otimes \times 1) \Longrightarrow \otimes(1 \times \otimes)$, $l\colon \otimes(I \times 1)i_2 \Longrightarrow 1$ and $r\colon \otimes(1 \times I)i_1 \Longrightarrow 1$ ($i_1\colon A \to A \times \{*\}$ and $i_2\colon A \to \{*\} \times A$ are the canonical isomorphisms), and invertible globular 3-cells $1(\alpha \times 1) \bullet \alpha 1 \bullet 1(1 \times \alpha) \overset{\pi}{\Longrightarrow} \alpha 1 \bullet 1 \bullet \alpha 1$, $1(r^{-1} \times 1) \bullet \alpha \bullet 1(1 \times l) \overset{\mu}{\Longrightarrow} 1$, $1(l \times 1) \overset{\lambda}{\Longrightarrow} \alpha 1 \bullet l1$ and $1(1 \times r^{-1}) \overset{\rho}{\Longrightarrow} r^{-1}1 \bullet \alpha 1$ generalizing the 3-cells from definition **4.43**. They are required to satisfy axioms very similar to those in that definition, the only difference being that they are translated into the language of locally cubical bicategories. The axioms may be found explictly in appendix D, but since they are so huge, we rather keep in mind the small diagrams of definition **4.43**, since they are essentially the same.*

Remark. We choose to call the composition $\otimes$ because it plays the role of the tensor product of a monoidal bicategory.

Remark. We can define additional structures (e.g braidings and symmetries) in an entirely similar way. For more details, see [19] and [25].

Remark. When we take **C** to be $\mathfrak{Bicat}$ and $\mathfrak{Dbl}_{\mathfrak{f}}$, respectively, we have the usual notions of monoidal bicategory and monoidal double category.

Definition 4.3.3. *A 1-cell $f\colon A \to B$ between monoidal objects is called* **lax monoidal** *if, and only if, there exist 2-cells* $\chi\colon \otimes(f \times f) \longrightarrow f\otimes$ *and* $\iota\colon I_B \longrightarrow fI_A$ *, and invertible globular 3-cells* $\omega\colon 1(\chi \times 1) \bullet \chi 1 \bullet 1\alpha \Longrightarrow \alpha 1 \bullet 1(1 \times \chi) \bullet \chi 1$ *,*

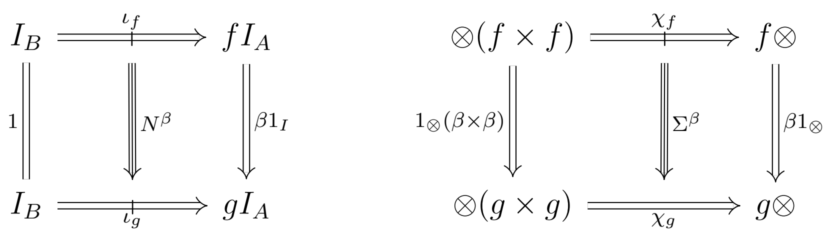
$$\gamma\colon 1(\iota \times 1)1 \bullet \chi 1 \bullet 1l \Longrightarrow l1$$

and $\delta\colon 1r^{-1} \Longrightarrow r^{-1}1 \bullet 1(1 \times \iota)1 \bullet \chi 1$ *that satisfy axioms analogous to the axioms of definition for a* **trihomomorphism***,i.e, a 1-morphism between tricategories, interpreted in the language of locally cubical bicategories. All of the coherence axioms are in appendix D.*

Definition 4.3.4. *A* **lax monoidal 2-cell** *is a loose 2-cell β in* ***C*** *equipped with globular 3-cells* $\Pi\colon 1_{\otimes}(\beta \times \beta) \bullet \chi_g \Longrightarrow \chi_f \bullet \beta 1_{\otimes}$ *and* $M\colon 1_I \bullet \iota_g \Longrightarrow \iota_f \bullet \beta 1_{I_A}$ *satisfying the axioms given in appendix D.*

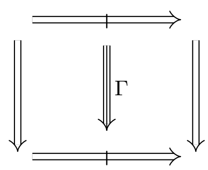
Definition 4.3.5. *A* **lax monoidal 2-cell** *is a loose 2-cell β in* ***C*** *equipped with globular 3-cells* $\Pi\colon 1_{\otimes}(\beta \times \beta) \bullet \chi_g \Longrightarrow \chi_f \bullet \beta 1_{\otimes}$ *and* $M\colon 1_I \bullet \iota_g \Longrightarrow \iota_f \bullet \beta 1_{I_A}$ *satisfying the axioms stated in the appendix.*

Definition 4.3.6. *A* ***monoidal icon*** *is a tight 2-cell in* ***C*** *equipped with non-globular 3-cells*



satisfying the axioms given in appendix D.

Definition 4.3.7. *A* **monoidal 3-cell** Γ *is a 3-cell in* ***C*** *with the following shape:*



so that the vertical arrows are icons, and the horizontal arrows are monoidal 2-cells. We also require them to satisfy the axioms given in appendix D.

Remark. The obvious examples of these concepts are precisely those in which we are interested, namely, the locally cubical bicategories $\mathfrak{Bicat}$ and $\mathfrak{Dbl}_{\mathfrak{f}}$. By their very definition, they give us exactly the monoidal bicategories/double categories, and their respective 1, 2, and 3-cells (and icons).

Definition 4.3.8. *We are now going to define mathematical objects denoted with two indices, in the form X_{ij}. The subscripts i and j refer to the laxity of the 1-cells and the laxity of the 2-cells, respectively. The subscript l means lax, c means colax, and p means strong/pseudo. Sometimes, s is used, meaning strict. In order to obtain locally cubical bicategories, we define $\mathcal{M}on_{ij}(\boldsymbol{C})$ with monoidal objects of $\boldsymbol{C}$ as 0-cells, monoidal 1-cells of correct laxity as 1-cells, and so on, respecting the laxity of the subscripts. Similarly, we can define $\mathcal{B}r\mathcal{M}on_{ij}(\boldsymbol{C})$ and $\mathcal{S}ym\mathcal{M}on_{ij}(\boldsymbol{C})$ for the braided and symmetric cases, respectively.*

Remark. Our verifications given in this work are mostly for the monoidal case with no further structure. The cases with braidings, symmetries, etc., are similar, so they will not be verified explicitly.

Lemma 4.3.9. *The hom-spaces of $\mathcal{M}on_{ij}$, i.e., hom$_{\mathcal{M}on_{ij}(\boldsymbol{C})}(A, B)$ are, indeed, double categories.*

Proof: Observe that colax cases (indices c) lead to lax cases (indices l) by duality, and vice-versa. Also, once we have proved the case $j = c$, the strong (pseudo) case $j = p$ will basically follow from our calculations because the composite of invertible cells is invertible.

First, let us show that 1-cells and icons (in the respective hom's) form a category. Given a lax 1-cell $f\colon A \to B$, the identity icon 1_f is a lax monoidal icon equipped with the 3-cells $N^{1_f} = 1_{\iota_f}$ and $\Sigma^{1_f} = 1_{\chi_f}$. Coherence equations are trivially satisfied because we are dealing with identities. For any 1-cells f and g, and monoidal icons $\alpha, \beta\colon f \Rightarrow g$, we can equip the composite $\beta \cdot \alpha$ with $N^{\beta\cdot\alpha} = N^{\beta} \cdot N^{\alpha}$ and $\Sigma^{\beta\cdot\alpha} = \Sigma^{\beta} \cdot \Sigma^{\alpha}$. We have that $\beta \cdot \alpha$ is a monoidal icon because its coherence diagram can be obtained pasting the coherence diagrams for α and β.

For pseudo 1-cells and icons, we need to verify, additionally, that the above cells N^{1_f}, $N^{\beta\cdot\alpha}$, Σ^{1_f} and $\Sigma^{\beta\cdot\alpha}$ are inverses to their colax counterparts in the loose direction, i.e, with respect to $\bullet$. This holds because of the interchange law for $\bullet$ and $\cdot$. Therefore, we have that $N^{\alpha\cdot\beta} \bullet N^{\alpha'\cdot\beta'} = (N^{\alpha} \cdot N^{\beta}) \bullet (N^{\alpha'} \cdot N^{\beta'}) = (N^{\alpha} \bullet N^{\alpha'}) \cdot (N^{\beta'} \bullet N^{\beta}) = 1$ for $x \in \{\alpha, \beta\}$, and for x' an inverse in the loose direction for x, which is the case because we are studying the strong (pseudo) case.

Now, we will show that lax monoidal 2-cells together with monoidal 3-cells form a category. If we are given a lax monoidal 2-cell α, then the identity 3-cell 1_α is trivially lax monoidal because all parts of the coherence diagrams are identities. If we are given monoidal 3-cells $\alpha \overset{L}{\Longrightarrow} \beta$ and $\beta \overset{K}{\Longrightarrow} \gamma$, then the composition $K \cdot L$ is a monoidal 3-cell because, as before, the diagram can be decomposed as the pasting of the diagrams of K and L. The colax and pseudo cases follows from the same reasoning.

Now, we are ready to describe the loose composition structure: $\bullet$ and 1 corresponds to $\odot$ and U from the definition of double category, so we need to see that these operations are indeed well defined. For instance, we need to show that the loose composition of lax monoidal 2-cells is indeed a lax monoidal 2-cell in a way that respect compositions, at least, up to canonical isomorphisms. We start by noticing that, if f is a lax monoidal 1-cell, then 1_f is a strong monoidal 2-cell with structure 3-cells being M^{1_f} and Π^{1_f} as below:

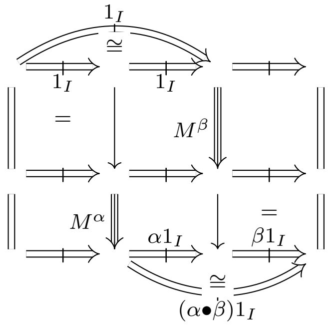
$$M^{1_f} \quad = \quad \begin{array}{ccccc} I_B & \overset{1_I}{\Longrightarrow} & I_A & \overset{\iota_f}{\Longrightarrow} & fI_A \\ {\scriptstyle 1}\Big\| & & \Big\Downarrow \cong & & \Big\|{\scriptstyle 1} \\ I_B & \underset{\iota_f}{\Longrightarrow} & fI_A & \underset{1_f 1_I}{\Longrightarrow} & fI_B \end{array}$$

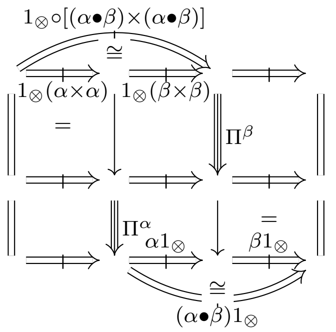
$$\Pi^{1_f} \quad = \quad \begin{array}{ccccc} \otimes(f \times f) & \overset{1_\otimes(1_f \times 1_f)}{\Longrightarrow} & \otimes(f \times f) & \overset{\chi_f}{\Longrightarrow} & f\otimes \\ {\scriptstyle 1}\Big\| & & \Big\Downarrow \cong & & \Big\|{\scriptstyle 1} \\ \otimes(f \times f) & \underset{\chi_f}{\Longrightarrow} & f\otimes & \underset{1_f 1_\otimes}{\Longrightarrow} & fI_B \end{array}$$

The coherence diagrams for these cells follows from the coherence of $\mathfrak{r}^\bullet$ and $\mathfrak{l}^\bullet$.

If γ is a monoidal icon, then 1_γ is a monoidal 3-cell by a similar reasoning.

Now, if we are given composable monoidal 2-cells α, β, then $\alpha \bullet \beta$ is a monoidal 2-cell with structure 3-cells being their respective 3-cells pasted vertically as below:

$M^{\alpha\bullet\beta} \quad = $ 

$\Pi^{\alpha\bullet\beta} \quad = $ 

Again, the huge coherence diagrams are decomposed simply as diagrams in the given 2-cells.

If Γ and Δ are composable monoidal 3-cells, then $\Gamma \bullet \Delta$ is a monoidal 3-cell using arguments analogous to the ones we already gave.

Functoriality of $\bullet$ and 1 follows from their functoriality in $hom_{\mathbf{C}}(A, B)$. Coherence of these functors also follows from the properties of the associators and unitors $\mathfrak{a}^\bullet$ and so on. □

Lemma 4.3.10. *The data from definition **4.3.9** define locally cubical bicategories.*

Proof: The detailed proof in [19] is neither considerably different from our proof of proposition **4.3.9** nor more enlightening than it, so we chose to just provide a brief outline. We already proved that the hom-sets are double categories, so what remains to prove in order to show that these categories are locally cubical is that the external composition $\odot$ between hom-sets works like a external composition of a bicategory. Indeed, the result follows the same ideas in the proof of proposition **4.3.9**, namely, pasting diagrams together to define the structure cells of the composites, and pasting diagrams to prove the coherence equations in appendix D. □

Theorem 4.3.11. *Given 1-strict locally cubical bicategories **C** and **D** with products, if $F\colon \mathbf{C} \to \mathbf{D}$ is a 1-strict functor that preserves those products, then F preserves monoidal objects, 1-cells, 2-cells, icons and 3-cells. Any braided or symmetric structure is also preserved.*

Proof: Observe that, since F preserves products, if we have a monoidal object A, then $F(A \times A)$ can be regarded as a product $F(A) \times F(A)$. Since we have a 1-cell $\otimes\colon A \times A \to A$, then, applying the functor F, we obtain induced 1-cells $\otimes_F\colon F(A) \times F(A) \to F(A)$ and $I_F = F(I_A)$ in view of our first observation. Since F_1 and $F_\bullet$ are globular, we have $F(f \circ g) = F(f) \circ F(g)$ and $F(1_A) = 1_{F(A)}$. Applying the functor F to the diagrams of monoidal objects, we have that the only thing that differs from the respective coherence diagrams for the cells on the image of F is that we get values with the form $F(\beta\gamma)$ instead of $F(\beta)F(\gamma)$. Nevertheless, these expressions are canonically isomorphic due to coherence of the enriched pseudo functors, so this is not really a problem. The exact same argument applies to the monoidal cells (braided and symmetric structures included.) Therefore, F preserves all these structures. □

Theorem 4.3.12. *Given 1-strict locally cubical bicategories **C** and **D** with products, if $F\colon$ **C** $\to$ **D** is a 1-strict functor that preserves those products, then, for $i, j \in \{l, c, p\}$, F lifts to a functor*

$$\mathcal{M}on_{ij}F\colon \mathcal{M}on_{ij}(\boldsymbol{C}) \to \mathcal{M}on_{ij}(\boldsymbol{D}).$$

We have similar liftings in the presence of braided or symmetric additional structure.

Proof: By theorem **4.3.11**, the assignment F gives a well-defined function $\mathcal{M}on_{ij}F$ from $\mathcal{M}on_{ij}(\mathbf{C})$ to $\mathcal{M}on_{ij}(\mathbf{D})$. Now we have to show that F gives rise to a pseudo double functor

$$\mathcal{M}on_l F\colon \mathrm{hom}_{\mathcal{M}on_l(\mathbf{C})}(A, B) \to \mathrm{hom}_{\mathcal{M}on_l(\mathbf{D})}(\mathcal{M}on_l F(A), \mathcal{M}on_l F(B)).$$

This actually holds because $F(N^{\alpha\cdot\beta}) = F(N^{\alpha} \cdot N^{\beta}) = F(N^{\alpha}) \cdot F(N^{\beta})$, $F(N^{1_f}) = F(1_{\iota_f}) = 1_{\iota_{F(f)}} = N^{1_{F(f)}}$. The natural transformations $F_\odot$ and F_U are 3-cells in $\mathcal{M}on_l(\mathbf{D})$, and the coherence diagrams follow from the coherence of pseudo double functors. Now, we just need to prove that $F_\bullet$ and F_1 are tight transformations in $\mathrm{hom}_{\mathcal{M}on(\mathbf{D})}(F(A), F(B))$. Since their components are globular 3-cells, it only remains to show that they are monoidal, but this holds due to the coherence of enriched pseudo functors. □

Corollary. *The functor $\mathcal{L}\colon \mathfrak{Dbl}_f \to \mathfrak{Bicat}$ lifts to a functor*

$$\mathcal{M}on_{ij}\mathcal{L}\colon \mathcal{M}on_{ij}(\mathfrak{Dbl}_f) \to \mathcal{M}on_{ij}(\mathfrak{Bicat})$$

with $i, j = l, c, p$. We have similar liftings for the braided and symmetric extra structures.

Proof: This is merely the application of theorem **4.3.12** to the locally cubical bicategories of theorem **4.2.4**. □

4.4 APPLICATIONS

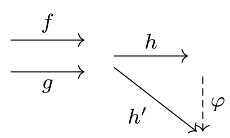
Definition 4.4.1. *Suppose that we have a double category **D**. We can define the double category Alg(**D**) consisting of the following data:*

1. *The monoids in the monoidal category $\boldsymbol{D}_1$ are the 0-cells of Alg(**D**);*
2. *Monoid homomorphisms in $\boldsymbol{D}_1$ are the tight 1-cells of Alg(**D**);*
3. *Bimodules in $\boldsymbol{D}_1$ are the loose 1-cells of Alg(**D**); and*
4. *Equivariant maps in $\boldsymbol{D}_1$ are the 2-cells of Alg(**D**).*

Remark. To be precise, we need to impose that **D** has **local coequalizers** to define the loose composition of bimodules and equivariant maps. See the definition below.

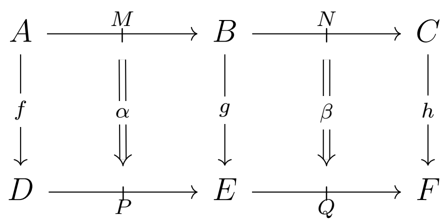
Remark. We consider the structures above as we defined them in monoidal categories, replacing $\otimes$ by $\odot$ in their definitions.

Definition 4.4.2. *A* **coequalizer** *of a pair of arrows $f, g\colon A \to B$ in a 1-category is an arrow $h\colon B \to C$ such that $h \circ f = h \circ g$ and, for any other $h'\colon B \to C'$ satisfying these equations, we have a unique morphism $\varphi\colon C \to C'$ such that:*

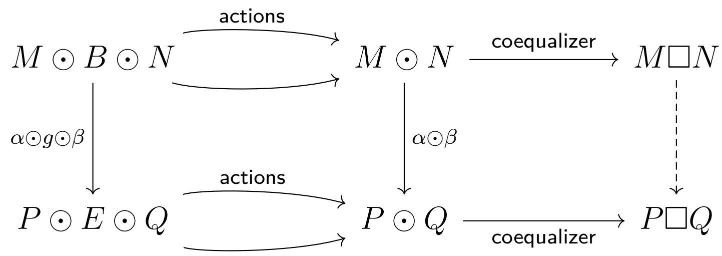


Definition 4.4.3. *We say that a double category **D** has* **local coequalizers** *if, and only if, for each objects A, B, hom(A, B) has coequalizers.*

If **D** has local coequalizers, then, given a $A-B$ bimodule M and a $B-C$ bimodule N, we define their horizontal composition $M\Box N = K$ as the coequalizer of the maps $M \odot B \odot N \to M \odot N$ obtained by left and right actions of the bimodules. The $A - C$ bimodule structure in K is the structure induced by those maps. Loose composition of 2-cells is as follows. First, suppose that we have



in **Alg**(**D**). We define the horizontal composition $\alpha\Box\beta$ as below:



The first square commutes for left and right actions because α and β are equivariant. The existence of the dotted arrow follows from the definition of a coequalizer. The dotted arrow is the definition of $\alpha\Box\beta$.

The other compositions in the definition of **Alg**(**D**) are easier to deal with: tight composition of monoid homomorphisms and equivariant maps are just the usual composition.

Theorem 4.4.4. *If **D** is a fibrant monoidal double category with local equalizers such that $\otimes$ preserves local coequalizers, then the loose bicategory $\mathcal{A}lg(\textbf{D}) = \mathcal{L}(Alg(\textbf{D}))$ is a monoidal bicategory, where its monoidal structure is lifted from $\otimes$. Furthermore, if **D** is braided or symmetric, then so is $\mathcal{A}lg(\textbf{D})$.*

Proof: First, we impose that $\otimes$ preserves local coequalizers so it can work like a interchange law for our horizontal composition, i.e, $(R\Box A) \otimes (T\Box B) = (R \otimes T)\Box(A \otimes B)$. If A is a $R - S$ bimodule, and B is a $T - U$ bimodule, then $A \otimes B$ is a $R \otimes T - S \otimes U$ bimodule. The neutral object for this operation is the same neutral of $\otimes$ in $\textbf{D}_1$. Note that the domain of the left action is $(R \otimes T)\Box(A \otimes B) = (R\Box A) \otimes (T\Box B)$, so we can apply each left action of the bimodules separately, and take their tensor product. The same argument applies to the right actions. The required axioms follow from the functoriality of the tensor product added to the fact that we are defining the action on the tensor product as the tensor product of the actions. Due to this argument, this $\otimes$ gives Alg(**D**) a (double) monoidal structure with the constraints being essentially the constraints of the tensor product in **D**. For instance, the associativity constraint is given below:

$$((M \otimes N) \otimes P)\Box((A \otimes B) \otimes C) \xrightarrow{\alpha\Box\alpha} (M \otimes (N \otimes P))\Box(A \otimes (B \otimes C))$$

Also, Alg($\mathbf{D}$) is fibrant because any monoid homomorphism $f\colon A \to B$ induces an $A - B$ bimodule structure on teal B with f as the left action, and the identity as the right action - The axioms are easily verified. Note that this argument indeed shows that not only Alg($\mathbf{D}$) is fibrant, but also that it has companion and conjoints for all tight 1-morphisms.

The result follows immediately from their statements of theorems **4.3.10** and **4.3.11**. □

Remark. Using our last proof we can easily see that Alg is a functor Alg: $\mathcal{M}on_{pp}(\mathcal{D}bl_{\mathbf{f}}) \to \mathcal{M}on_{pp}(\mathcal{D}bl_{\mathbf{f}})$.

Proposition 4.4.5. *The construction Alg gives rise to a functor $\mathcal{M}on_{pp}(\mathcal{D}bl_{\mathbf{f}}) \to \mathcal{M}on(\mathcal{B}icat)$.*

Proof: Compose the functor Alg: $\mathcal{M}on_{pp}(\mathcal{D}bl_{\mathbf{f}}) \to \mathcal{M}on_{p}p(\mathcal{D}bl_{\mathbf{f}})$ with the functor obtained from theorem **4.53**. □

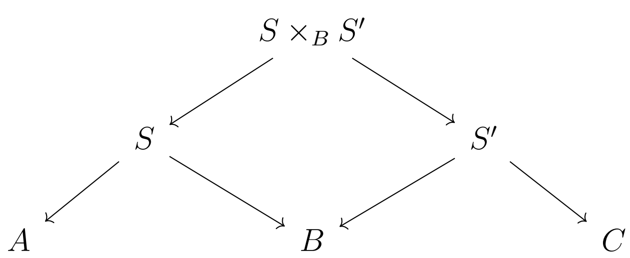
Definition 4.4.6. *We define $2\mathcal{V}ect$ as $\mathcal{A}lg$(**FVect**), where **FVect** is the braided monoidal category of finite dimensional vector spaces.*

Remark. Recalling that **Vect** is the category of all vector spaces, we can also define $2\mathcal{V}ect$ as a bicategory consisting of the internal categories to **Vect**. These objects are also called **Baez-Crans vector spaces** (see [39] for more details).

Proposition 4.4.7. *The bicategory $2\mathcal{V}ect$ is symmetric monoidal.*

Proof: A direct application of theorem **4.4.4** gives us this result. □

Definition 4.4.8. *A* **span** *in a 1-category **C** from A to B is a diagram of the form $A \longleftarrow S \to B$. If **C** has pullbacks, then we can compose spans using the following diagram:*



Remark. Using the same arguments we gave earlier in this work, we can show that the above composition is associative up to isomorphism. Hence, we can define a bicategory $\mathcal{S}pan$($\mathbf{C}$) that has objects of $\mathbf{C}$ as 0-cells, spans as 1-cells, and morphisms between spans as 2-cells. We can also define a double category **Span**($\mathbf{C}$) that has the objects of $\mathbf{C}$ as 0-cells, morphisms of

C as tight 1-cells, spans as loose 1-cells, and morphisms between spans as 2-cells. We usually denote $\mathbf{Span}(\mathbf{Set}) = \mathbf{Span}$. By construction, $\mathcal{L}(\mathbf{Span}(\mathbf{C})) = \mathcal{S}pan(\mathbf{C})$. A monoidal structure in a category of spans can be easily obtained if **C** has all finite products making it a cartesian monoidal category.

From now on, we assume that when we talk about the category of spans of **C**, **C** has pullbacks and finite products in order to compose spans.

Remark. By definition, a monoid in $\mathbf{Span}(\mathbf{C})$ is an internal category to **C**, by definition. From the involved definitions, it is easy to show that $\mathbf{Prof}(\mathbf{C}) \cong \mathrm{Alg}(\mathbf{Span}(\mathbf{C}))$.

Remark. A companion of a functor $F: A \to B$ in **Prof** is obtained regarding F as a representable profunctor $\mathrm{hom}(F(\_), \_)$.

Proposition 4.4.9. *The bicategory $\mathcal{P}rof(\mathbf{C})$ of internal categories and profunctors in a category **C** with pullbacks and coequalizers preserved by pullback is symmetric monoidal.*

Proof: Since $\mathbf{Prof}(\mathbf{C})$ can be constructed as $\mathrm{Alg}(\mathbf{Span}(\mathbf{C}))$, we have that $\mathcal{P}rof(\mathbf{C}) \cong \mathcal{A}lg(\mathbf{Span}(\mathbf{C}))$. $\mathbf{Span}(\mathbf{C})$ has local coequalizers and companions. Indeed, a companion of $f: A \to B$ is this morphism regarded as a span $B \xleftarrow{f} A \xrightarrow{1} A$, while a coequalizer for a span is just an equalizer in **C** for the pair of arrows in the composite diagram. With all this in hand, the result follows from a direct application of theorem **4.4.4**. □

Remark. We can easily generalize this construction to categories and profunctors enriched over a so-called complete closed symmetric monoidal category ([19], [20]).

Remark. If **C** and **D** are categories with pullbacks and coequalizers, and $F: \mathbf{C} \to \mathbf{D}$ is a functor that preserves pullbacks and coequalizers, then it is easy to see that F induces a strong symmetric monoidal double functor $\mathbf{Span}(\mathbf{C}) \to \mathbf{Span}(\mathbf{D})$. From this, F induces a monoidal functor $\mathrm{Alg}(\mathbf{Span}(\mathbf{C})) \to \mathrm{Alg}(\mathbf{Span}(\mathbf{D}))$ and, finally, a symmetric monoidal functor of bicategories $\mathcal{P}rof(\mathbf{C}) \to \mathcal{P}rof(\mathbf{D})$.

It is also interesting to notice that these theorems solve a conjecture due to Baez and Courser ([40]) as a simple application of theorem **4.3.11** to the strong symmetric double functor they construct in the paper. The conjecture was solved in [19] in the theorem **6.17**.

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APÊNDICE A – FOUNDATIONS AND LOGIC

To define what a category is, in a way such that all the collections of usual mathematical objects of a kind (e.g. groups, vector spaces, sets, topological spaces, manifolds, graphs, propositions, modules) form a category, we are tempted to go beyond Zermelo-Frankel set theory with choice (ZFC), and immerse ourselves in alternative set theories, such as set theories with universe or extensions of ZFC (i.e, theories that includes a notion of sets in such a way that all axioms of ZFC still holds for these so called sets). Indeed, ZFC is a well known and widely spread theory, doing a great job supporting the mathematical theories. Nevertheless, in ZFC, we cannot talk about a well defined mathematical object that contains all sets (or virtually any other type of mathematical structure) so, in order to make these objects mathematically precise, we use a theory that contains as objects not only sets, but also these other collections of objects, like the collection of all sets, or all groups, or all vector spaces (called proper classes). These proper classes arise typically from unrestricted comprehension, which is basically associating to a predicate P a mathematical object that contains all sets that satisfies P. For example, the predicate $P(x)\colon x = x$ gives us the collection of all sets, which we know that is not a set, allowing the well known Russell's paradox. Our theory of categories will be developed under the assumption that we are working within the Neumann-Bernays-Gödel (NBG) set theory, which we detail in this section. We choose NBG over any other theory because it treats sets exactly like ZFC. Shoenfield proved that, indeed, NBG is a conservative extension of ZFC, meaning that everything about sets proved in NBG can also be proved in ZFC. Furthermore, NBG treats classes in a very similar way to how ZFC treats sets. If we wish to talk about a "class of classes", we cannot do that in NBG, we would have to appeal, for example, to a theory of conglomerates.

Of course, we are talking about the objects of the category. In a lot of examples, the morphisms between mathematical objects are functions that preserves some structure, so we have only a set of such morphisms, not a proper class. This will be the case of locally small categories.

The NBG set theory uses the primitive notions of **sets**, **classes** and the relation of ***membership*** (***belonging***) denoted by $\in$. Again, we reiterate that classes are nothing more than just a collection of objects that captures not only the notion of sets but also of other bigger collections of objects (like the collection of all sets). We will denote the sentence "X is a set" by $Set(X)$, and the sentence "X is a class, but not a set" (i.e, X is a proper class) by $Proper(X)$. Throughout this section latin letters, as well as latin letters with subscript, will denote arbitrary classes. The axioms for NBG are the following:

1. $\forall x(x \in X \iff x \in Y) \implies X = Y$ (the converse is a clear logical truth, so it does not need to be stated as an axiom)

2. $\forall X(\exists Y\colon X \in Y \implies Set(X))$

3. $(\forall x, y\colon Set(x), Set(y))[(\exists Z\colon Set(Z))\colon (a \in Z \iff a = x \vee a = y)]$

4. If φ is a first order formula in which the only quantified variables are set variables, then we have the following:

$$(\forall X_1, \ldots, X_n)[\exists Y\colon x \in Y \iff \varphi(x, X_1, \ldots, X_n)]$$

5. The infinity, union, power set, replacement and regularity axioms of ZF hold for sets.

6. There exists a (class) function F between the class of all sets and itself such that

$$(\forall X\colon Set(X) \wedge X \neq \emptyset)[F(X) \in X]$$

The first axiom is the axiom of extensionality, meaning that equality for classes occurs if and only if they have the same elements. The second axiom means that proper classes cannot be elements of other classes, i.e, all elements of a class are sets. The third axiom is the axiom of pairing, that states that for all x, y sets, we have a set $\{x, y\}$. The fourth axiom is the extension of the restricted comprehension axiom of ZFC, meaning that for any first order formula P (i.e, a logic statement which does not quantifies over predicates) that only quantifies over set variables, there is a class C such that all elements of C are fully characterized by P. In ZFC, this cannot occur, as we explained in the first paragraph of this section. What we have in ZFC is that, given a predicate P and a set a, there exists a $b \subseteq a$ such that elements of b are characterized by being those elements of a satisfying P. The fifth axiom is just the remaining axioms of ZF. The sixth axiom interesting because it is a global version of the axiom of choice. As we are not working with sets anymore, we can define a more general notion of function, such that the domain or the codomain can be proper classes. In the usual axiom of choice, for every set of non-empty sets, we would have (at least) one choice function associated with it, but this axiom states that there is a global function such that, for any non-empty set, it chooses an element of it. This axiom implies the axiom of choice because we can define a choice function on a set of non-empty sets using this global F that we postulated in the axiom. Also, it is interesting to know that global choice is strictly stronger than the axiom of choice ([13]). Here, the notions of ***small sets*** and ***large sets*** that arise in other set theories (just as set theories with a universe set) corresponds, respectively, to ***sets*** and ***proper classes***. When we speak about something that is "small" in this dissertations, it means that it is a set.

Of course, the set $\emptyset$, as written above, is well defined as a consequence of the axiom 4 because we can determine the empty set as set formed by the sets x that satisfies the property $x \neq x$. Just as that, we can define the class of all sets to be the class of those sets x such that $x = x$. We will denote this class by Set). Functions between classes are also well defined just as we define them for sets. The union of classes indexed by sets can also be done. Some classic examples of ***proper classes*** (i.e, classes that are not sets) are:

1. The class of sets;
2. The classes of groups, rings and fields;
3. The classes of topological spaces;
4. The classes of ordinals and cardinals. For a proof, see the Burali-Forti paradox in [13].

Now, for those with the proper background in logic, we provide a brief sketch of the proof that NBG with local choice (i.e., with the axiom 6 - the axiom of global choice - replaced by the usual axiom of choice of ZFC) is a conservative extension of ZFC, assuming soundness and completeness for countable vocabularies. Denote by NBG* the theory that assumes the axioms 1 to 5 (i.e., NBG without choice). First of all, notice that it is obvious that anything proved in ZF can be proved in NBG, for NBG has all axioms of ZF for sets.

Theorem A.0.1. *NBG** $\vdash \varphi \implies$ *ZF* $\vdash \varphi$ *(φ any sentence in the language of set theory)*

Idea for a proof: For each model of ZF we construct a model of NBG* as the set of all subsets of our initial ZF model. It is intuitive that this will be indeed a model of NBG, taking our subsets that are sets of our model of ZF to be the sets of our model of NBG, and those subsets that are not sets of our initial model will be the proper classes of our model of

NBG. A sentence about sets is true in our model of ZF if and only if it is in our model of NBG, by construction. Then, by hypothesis and by soundness, our model of NBG will model φ. From this, we can conclude that our initial model will also model φ, but as we started with an arbitrary model of ZF, we infer that $ZF \vDash \varphi$ and, by completeness, $ZF \vdash \varphi$. So, it is clear thar NBG* $+$ (local) choice is a conservative extension of ZFC, using the fact that the axiom of choice is independent of ZF. □

One easy consequence of conservative extension is that ZFC and NBG are equiconsistent: If either ZFC or NBG is inconsistent, we have that the proposition $(\emptyset = \emptyset) \wedge (\emptyset \neq \emptyset)$ is true due to the principle of explosion. But notice that this is a statement about sets, so the other theory would also be inconsistent.

Another interesting feature of NBG that we can discuss (pretty informally) is that it is a finitely axiomatizable theory, while ZF is not. Moreover, if ZF is consistent, then it is not finitely axiomatizable, as Montague once proved in [15]. In ZF, the axiom schema of restricted comprehension consists of infinitely many axioms, namely, one axiom for each predicate in the language of ZF that are used to form subsets. Predicates and sets aren not in direct bijection, so we cannot try to change something in this axiom to replace this "quantification over predicates" by a real quantification over sets. But classes are in direct bijection with these predicates, by our axiom 4. Therefore, intuitively, in axiom 4 we are really quantifying over classes, which is an object of our theory, making it possible to build a finite axiomatization. This is more a heuristic argument than a mathematical argument, but these formal arguments run out of the scope of this work. A similar problem occurs with the axiom schema of replacement.

APÊNDICE B – MULTILINEAR ALGEBRA

In this appendix we try to give a more categorical viewpoint of some of the points of multilinear algebra. We start remembering a fairly simple proposition from linear algebra. We denote the vector space of bilinear functions $f\colon V \times W \to U$ by Bil$(V, W; U)$.

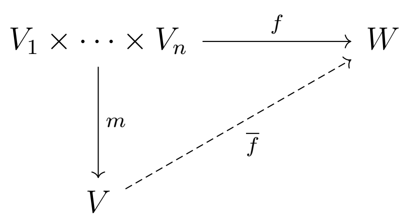
Remark. The basic theory done here can be carried to the tensor products of modules and those of abelian groups.

Proposition B.0.1. *The function $\varphi\colon$ Bil$(V, W; U) \to$ hom$(V,$ hom$(W, U))$ given by $\varphi(f)(v)(w) = f(v, w)$ defines a natural isomorphism.*

Definition B.0.2. *A function $f\colon V_1 \times \cdots \times V_n \to W$ defined from a product of n vector spaces to a vector space, all over the same ground field, is said to be an n-linear function (or just* **multilinear function***) if and only if $f(v_1, \ldots, v_i + \lambda v_i', \ldots, v_n) = f(v_1, \ldots, v_i, \ldots, v_n) + \lambda f(v_1, \ldots, v_i', \ldots, v_n)$ for any $\lambda \in k$, $v_j \in V_j$, $v_i' \in V_i$, $i = 1, \ldots, n$. We define Mult$(V_1, \ldots, V_n; W)$ to be the vector space of multilinear functions.*

Proposition B.0.3. *The function $\varphi\colon$ Mult$(V_1, \ldots, V_n; W) \to$ hom$(V_1,$ hom$(\ldots$ hom$(V_n, W)) \ldots)$ given by $\varphi(f)(v_1) \ldots (v_n) = f(v_1, \ldots, v_n)$ defines a natural isomorphism.*

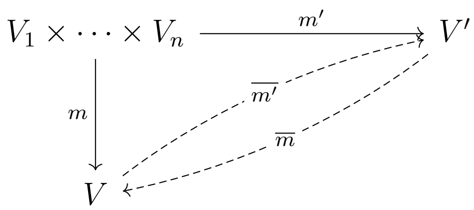
Definition B.0.4. *A* **tensor product** *of the vector spaces $V_1, \ldots, V_n$ is a vector space V together with a n-linear map $m\colon V_1 \times \cdots \times V_n \to V$ such that for any vector space W and any n-linear map $f\colon V_1 \times \cdots V_n \to W$, there exists a unique linear transformation $\overline{f}\colon V \to W$ such that the following diagram commutes:*



Remark. This property of a tensor product is called the ***universal property of the tensor product***.

Proposition B.0.5. *Any two tensor products are naturally isomorphic.*

Proof: By the universal property, we have the commutativity of the following diagram:



By uniqueness of the arrows involved, it is clear that $\overline{m}$ and $\overline{m'}$ are inverses for each other. Naturality of these isomorphisms is proved by pasting those diagrams together. □

Corollary. *hom*$(U \otimes V, W)$ *is naturally isomorphic to hom*$(U, \text{hom}(V, W))$.

Proof: This follows from our first proposition and by the universal property of the tensor product. □

Remark. In general, we have a natural isomorphism

$$\text{hom}(V_1 \otimes \cdots \otimes V_n, W) \cong \text{hom}(V_1, \text{hom}(\dots \text{hom}(V_n, W)) \dots).$$

Theorem B.0.6. *For any $V_1, \dots V_n$ there exists a tensor product $V_1 \otimes \cdots \otimes V_n$. We denote the tensor product by $V_1 \otimes \cdots \otimes V_n$, and refer to it as* **the** *tensor product because we know that tensor products are unique up to natural isomorphisms from Proposition B.0.5.*

Proof: First we consider the free vector space $\mathbf{F} = \mathbf{F}(V_1 \times \cdots \times V_n)$. Now, we consider the subspace $U_0 = \langle (v_1, \dots, v_i + \lambda v_i', \dots, v_n) - (v_1, \dots, v_i, \dots, v_n) - \lambda(v_1, \dots, v_i', \dots, v_n) \colon \lambda \in k, v_j \in V_j, v_i' \in V_i, i = 1, \dots, n \rangle$.

We define $V_1 \otimes \cdots \otimes V_n = \mathbf{F}(V_1 \times \cdots \times V_n)/U_0$. It is clear, by definition of U_0, that the composite map

$$V_1 \times \cdots \times V_n \xrightarrow{\quad \subseteq \quad} \mathbf{F}(V_1 \times \cdots \times V_n) \xrightarrow{\quad \pi \quad} \mathbf{F}(V_1 \times \cdots \times V_n)/U_0$$

is multilinear. We denote this composite as $\otimes(v_1, \dots, v_n) = v_1 \otimes \cdots \otimes v_n = \overline{(v_1, \dots, v_n)}$.

Now, suppose that we are given a multilinear map $f \colon V_1 \times \cdots \times V_n \to W$. Due to the freeness of $\mathbf{F}$, we have a unique linear transformation $T \colon \mathbf{F} \to W$ that is equal to f when restricted to elements of the basis. This unique linear transformation gives us a linear transformation $\overline{f} \colon V_1 \otimes \cdots \otimes V_n \to W$, since U_0 is annihilated by T due to the multilinearity of f. □

Remark. The set of vectors of the form $v_1 \otimes \cdots \otimes v_n$ generate the tensor product space, but is not a basis in general.

Proposition B.0.7. *There are natural isomorphisms*

$$U \otimes (V \otimes W) \overset{\alpha}{\cong} (U \otimes V) \otimes W$$

$$k \otimes V \overset{l}{\cong} V \overset{r}{\cong} V \otimes k$$

$$V \otimes U \overset{\tau}{\cong} U \otimes V$$

Defined by $\alpha(u \otimes (v \otimes w)) = (u \otimes v) \otimes w$, $l(\lambda \otimes v) = \lambda v$, $r(v \otimes \lambda) = \lambda v$, $\tau(v \otimes u) = u \otimes v$. *Moreover, choosing* $V = k$*, we obtain* $k \otimes k = k$ *via the multiplication operation.*

Proof: We are going to give a proof of the existence of α. The existence of r, l, τ is proved using similar ideas. We have the following sequence of natural isomorphisms:

$$\mathsf{hom}(U \otimes (V \otimes W), O) \cong \mathsf{hom}(U, \mathsf{hom}(V \otimes W, O)) \cong \mathsf{hom}(U, \mathsf{hom}(V, \mathsf{hom}(W, O))).$$

Applying the first corollary in this appendix and the Yoneda lemma, we conclude that $U \otimes (V \otimes W) \cong U \otimes V \otimes W$ naturally. The other parenthesization is shown to be naturally isomorphic to these two in the same way. Due to the formulae in Yoneda lemma, this particular isomorphism is given by the equation stated in the proposition. □

Proposition B.0.8. *We have* $(\bigoplus_{i\in I} U_i) \otimes V \cong \bigoplus_{i \in I}(U_i \otimes V)$ *via the transformation* $T((u_i)_{i\in I} \otimes v) = (u_i \otimes v)_{i \in I}$.

Proof: Just like the last proposition, we are going to prove this categorically.
First, notice that $\mathsf{hom}((\bigoplus_{i\in I} U_i) \otimes V, W) \cong \mathsf{hom}(\bigoplus_{i\in I} U_i, \mathsf{hom}(V, W)) \cong \prod_{i\in I} \mathsf{hom}(U_i, \mathsf{hom}(V, W))$ $\cong \prod_{i \in I} \mathsf{hom}(U_i \otimes V, W) \cong \mathsf{hom}(\bigoplus_{i\in I}(U_i \otimes V), W)$. □

Corollary. *If* $\{v_i\}_{i\in I}$ *is a basis for* V *and* $\{w_j\}_{j \in J}$ *is a basis for* W*, then* $\{v_i \otimes w_j\}_{(i,j)\in I \times J}$ *is a basis for* $V \otimes W$*. Moreover, we have dim*$(V \otimes W) =$ *dim*(V)*dim*(W).

Proof:

$$U \otimes V \cong (\bigoplus_{i\in I} \langle v_i \rangle) \otimes (\bigoplus_{j \in G} \langle w_j \rangle) = \bigoplus_{(i,j)\in I\times J} (\langle v_i \rangle \otimes \langle w_j \rangle) = \bigoplus_{(i,j)\in I \times J} \langle v_i \otimes w_j \rangle$$

As we wanted. □

Definition B.0.9. *Given two linear maps* $f\colon V \to W$ *and* $g\colon V' \to W'$*, we define a linear map* $f \otimes g\colon V \otimes V' \to W \otimes W'$ *as* $(f \otimes g)(v \otimes v') = f(v) \otimes g(v')$.

This definition gives rise to a linear mapping λ: $\text{hom}(U', V') \otimes \text{hom}(U, V) \to \text{hom}(U \otimes U', V \otimes V')$ defined as $\lambda(f \otimes g)(u \otimes u') = f(u) \otimes g(u')$.

We can also study the notions of duality, trace and dimension in a categorical point of view, but we will not do this here. The reason to this is that the categorical structure of this study are categories with duality, such as the (right/left) rigid ones, ribbon categories, and traced categories, which we do not explore here. In some of these type of categories with additional structure, we have sufficient structural resources to introduce the notions of (quantum) trace and dimension to abstract what happens in the category $\mathbf{Vect}_k$. Another interesting feature of these categories is how rich their graphical calculi is. Unfortunately, we do not deal with this type of situation in this dissertation. If one is interested, one should look up, for instance, [2].

APÊNDICE C – ARTIN BRAID GROUPS

There are two main approaches to the theory of Artin braid groups, being one algebraic and the other topological. Here, we shall give both and mention their connection. For the formal details of this connection one can see [17].

One well known fact from group theory is that S_n, the permutation group of n objects, can be presented by the ***Coxeter presentation***

$$\langle \tau_i,\ 1 \leq i \leq n-1 :\ \tau_j\tau_{j+1}\tau_j = \tau_{j+1}\tau_j\tau_{j+1},\ \tau_j\tau_k = \tau_k\tau_j,\ \tau_j^2 = 1,\ 1 \leq j \leq n,\ |i-j| \geq 2\rangle.$$

One can define the ***Artin braid group on*** n ***strands*** B_n just by forgetting the idempotence relations above:

$$B_n = \langle \sigma_i,\ 1 \leq i \leq n-1 :\ \sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1},\ \sigma_j\sigma_k = \sigma_k\sigma_j,\ 1 \leq j \leq n,\ |i-j| \geq 2\rangle$$

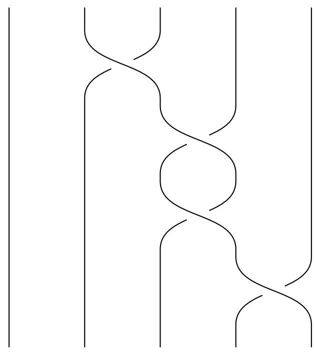
The first set of relations of the presentation is often called ***cubic relations*** or ***braid relations***. It follows from this definition and standard results in combinatorial group theory that there is an epimorphism $\rho_n \colon B_n \to S_n$ defined on generators by

$$\rho_n(\tau_j) = \sigma_j,\ 1 \leq j \leq n-1.$$

Thus, we conclude that $B_n = S_n/\mathrm{Ker}(\rho_n)$. It is also clear that $B_n \leq B_{n+1}$ for all $n \in \mathbb{N}$, so we can define the ***Artin braid group on*** ∞ ***strands*** as $B_\infty = \bigcup_{n=0}^{\infty} B_n$.

A lot can be said about the braid groups using only these algebraic properties and working with them in a similar way as we do for basic properties of S_n, since both look algebraically pretty similar. There is a very deep mathematical interpretation to braid groups: one can define them in an entirely topological viewpoint as the fundamental group of certain configuration spaces.

The name of the subject is very suggestive. A braid, intuitively, is something like this:



With this picture in mind, we inform that our formal topological definition of a braid on n strands must have the following features:

1. They consists of a set of n paths that starts on n different labels and end in a permutation of those labels;
2. These paths are strictly monotonic functions of their projections onto a preferred direction (in our picture, the vertical one), so that each path advances without turning back along it;
3. These paths cannot intersect each other.

If M is a manifold with dimension at least 2, we can define the ***braid group of*** M ***on*** n ***strands*** as follows:

- We define the permutation action $\gamma\colon S_n \times M^n \to M^n$ as $\gamma(\sigma, (x_1, \ldots, x_n)) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$;
- We take the quotient $M^n/\sim$ where $\sim$ is defined as the equivalence relation induced by the orbit partition of γ. This quotient is called the n-***fold symmetric product*** of M. A path in the n-fold symmetric product is a set of n strings traced independently;
- Since we do not want paths crossing each other, we have to remove the subspaces of M^n defined by conditions $x_i = x_j$, with $1 \leq i < j \leq n$. Call this new set $\text{Conf}_n(M)$, the n-***th (ordered) cofiguration space of*** M;
- $\text{Conf}_n(M)$ is invariant under γ, since if we exchange the order of the points, they remain distinct from each other;
- With this perspective, we have that $\text{Conf}_n(M)/\sim$ is a subspace of $M^n/\sim$. We call $\text{Conf}_n(M)/\sim = \text{UConf}_n(M)$ the n-***th unordered cofiguration space of*** M;

- We define the ***braid group of*** M ***on*** n ***strands*** as $\pi_1(\text{UConf}_n(M))$.

Graphically, an element of $\pi_1(\text{UConf}_n(M))$ can be understood as a path that starts in some fixed set of distinct points $(x_1, \ldots, x_n) \in M^n$ and ends in some permutation $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ of that initial point, each set contained in one of two parallel planes, with the component-paths never crossing each other, and consider up to ambient homotopy relative to the planes containing the fixed endpoint. Hence, the whole ambient space (say $\mathbb{R}^2 \times [0, 1]$) is deformed continuously so that we have a homemorphism at each instant and, via the restriction of the deformation to the strands, these are moved along, always mantaining that pair of parallel planes fixed. But this is exactly the naive idea of what a braid on n strands is! To understand this better, we will focus on the case where $M = \mathbb{R}^2$, which was the original case due to Artin.

Let us see how we can visualize elements of $\pi_1(\text{UConf}_n(\mathbb{R}^2))$ in $\mathbb{R}^2 \times [0, 1]$.

Definition C.0.1. *A* **braid on** n **strands** *is a set of* n *pairwise disjoint curves starting at the points* $(k, 0, 1)$, $0 \leq k \leq n$ *and ending at the points* $(k, 0, 0)$, $0 \leq n$ *(each of these curves do not need to end in the same* x *coordinate that it started). We also need to require that these curves cannot contain closed arcs and the intersection of these* n *curves with the plane* $\mathbb{R}^2 \times z$ *consists of exactly* n *points.*

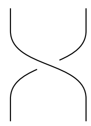
Definition C.0.2. *A* **braid projection** *is the union of a finite number of arcs in* $\mathbb{R}^2$ *starting at the points* $(k, 1)$, $0 \leq k \leq n$ *and ending at the points* $(k, 0)$, $0 \leq n$ *(as before, each of these curves do not need to end in the same* x *coordinate that it started) such that the intersection of each of these curves with* $\mathbb{R} \times y$ *consists of exactly one point. A* **crossing** *in this projection is a point that belongs to the interior of at least two of these curves. The* **order** *of a crossing point is the number of distinct curves passing through this point.*

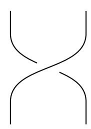
Definition C.0.3. *A* **braid diagram** *is a braid projection such that the order of all crossing points is 2 (this is called a* **regular** *projection) and, for each crossing point* P, *the set* E_P *of the two edges passing through* P *is ordered. The smallest is called the* **overcrossing** *edge and the other is called* **undercrossing** *edge.*

In this better formalized ground, our initial picture of a braid is actually a braid diagram. It is pretty intuitive that braids have braid diagrams associated to it (projecting them into the plane xz and the order of each E_P is given by the order induced by their y coordinate), and that for each braid diagram D we can construct a braid B such that the braid diagram of B can be deformed continuously to D (maintaining the properties of braid diagrams in each instant of the deformation). The complete formal details of this paragraph can be seen in any book containing the basics of knot theory (see, for example, [2]). It is also important to say that Reidemester gave a way to characterize the ambient isotopies of knots (embeddings of the circle into 3-space considered up to ambient isotopy) via three types of elementary movements on their associated knot diagrams, called the ***Reidemester moves***, and that these movements can be adapted to the connection between any two braid diagrams of the same braid. Actually, knots, braids, and their generalizations all the way to certain topological graphs form the ground setting for the visualization of various categories with additional structures and the resulting graphical calculi.

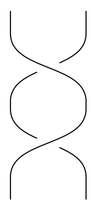
Remark. We are only considering ***progressive*** braids and braid projections/diagrams: They only "follow one direction" (descending the last coordinate).

We can give a structure of group to the set containing the class of equivalence of these braids (or braid diagrams), glueing them vertically (here our distinguished direction is the vertical one.) For example, the following two braids are inverses:





since their composition is



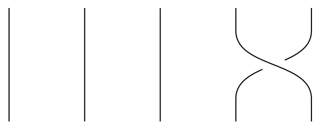
which clearly can be deformed onto two vertical lines via an adequate isotopy.

Elements of $\pi_1(\mathsf{UConf}_n(\mathbb{R}^2))$ and braids on n strands as we defined are "essentially the same" things, for: given a $\gamma = (\gamma_1, \dots, \gamma_n) \in \pi_1(\mathsf{UConf}_n(\mathbb{R}^2))$, we can associate it to a braid via the union of the images of the γ_i, $1 \leq i \leq n$, and, for each braid B on n strands, we can assign a path γ componentwise choosing $\gamma_i((x,y))$ to be the projection onto $\mathbb{R}^2$ of the intersection of the plane $\mathbb{R}^2$ with the curve of B that ends at $(i,0,0)$. It is not hard to convince oneself that these assignments are inverse to each other, and preserve products and neutrals, since we are not changing pretty much anything in their structures. Indeed, $\pi_1(\mathsf{UConf}_n(\mathbb{R}^2))$ is isomorphic to the group structure described for the equivalence classes of braids/braid diagrams on n strands.

Theorem C.0.4. ***(Artin, 1925)***

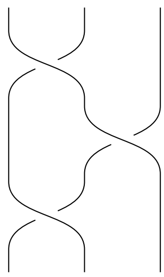
$$B_n \cong \pi_1(\mathit{UConf}_n(\mathbb{R}^2)).$$

Partial proof: We shall give a part of the proof that is very tangible, and the rest can be found at [17]. For a given braid diagram, we can move the crossing points up or down in a way that for each fixed y_0 coordinate there is at most one crossing point with y_0 as y coordinate. In this new representation, we can partition $[0,1]$ as intervals $[t_i,t_{i+1}]$, $0 \leq i \leq N$ such that for each fixed interval $[t_j,t_{j+1}]$ there is exactly one crossing point with y coordinate lying on it. If it has only one crossing point, then, it is equivalent to something like this:

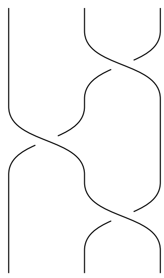


We conclude that any braid diagram is the composition of braid diagrams similar to the diagram above, with just one crossing between two immediately neighboring strands so that the one with leftmost endpoints may pass either over or under the other strand. Braids like the above are one called c_j, $1 \leq j \leq n-1$, where c_j maintain all strands straight except for the strands starting at j, $j+1$, that we require to end at $j+1$, j respectively, with the curve starting at j being the overcrossing. Its inverse braid, c_j^{-1} changes it into the undercrossing.

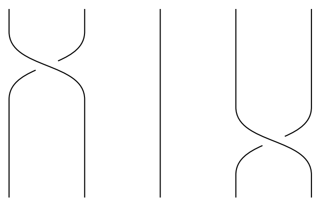
In [17] Artin proved that the assignment $\sigma_j \to c_j$ is an isomorphism. It is not hard to see that exists an homomorphism $\varphi\colon B_n \to \pi_1(\mathsf{UConf}_n(\mathbb{R}^2))$ given by $\sigma_j \to c_j$, since the generators c_j satisfies the presentation relations of B_n. The braid relation is just the third Reidemester move, that says that one can freely slide a strand that is above two other strands:



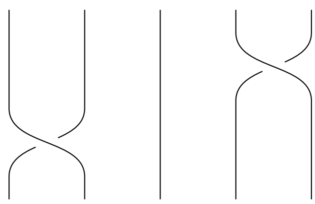
is equivalent to



The remaining relation just says that we can move crossing points involving not immediately neighboring strands up and down, i.e:



is equivalent to



This ends the idea of the partial proof that we wanted to describe. □

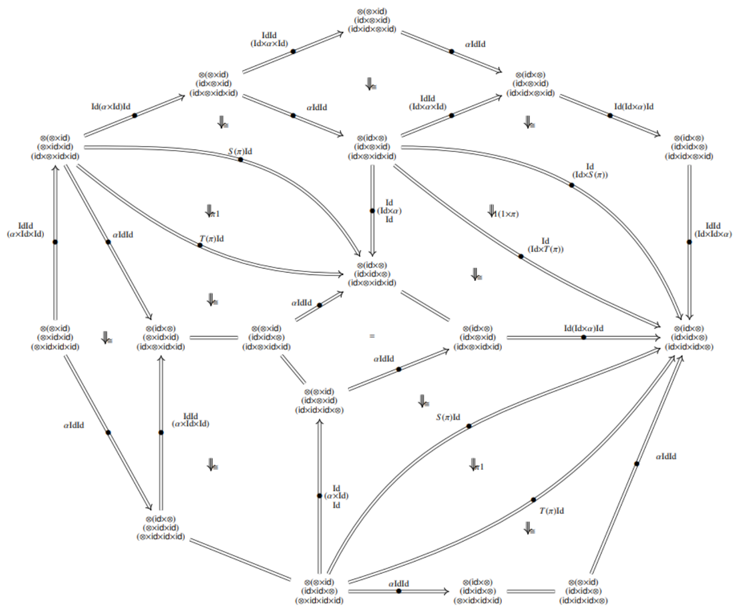
As one can expect from their names, braids and braidings on monoidal categories have a very deep connection. Unfortunately this connection is out of the scope of this dissertation, so we strongly suggest the reading of the respective chapter on [2].

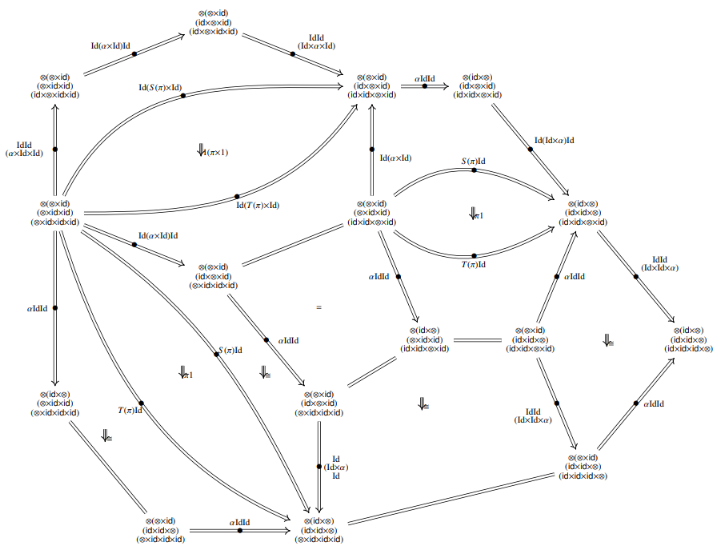
APÊNDICE D – COHERENCE AXIOMS FOR SOME CONCEPTS

The coherence diagrams here were extracted directly exactly as they were in the article [19]. The author of this present dissertation did not write the diagrams of this section, just extracted them from this article as images (since the software used for the diagrams in this present work did not support well the size of these diagrams - the diagrams appeared bigger than the pages in the main latex style of CCEN, hidding most of its content).

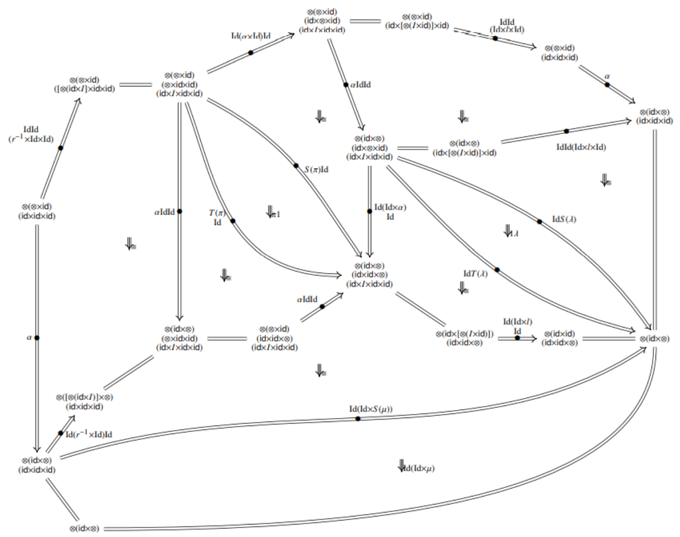
Axioms for monoidal objects:

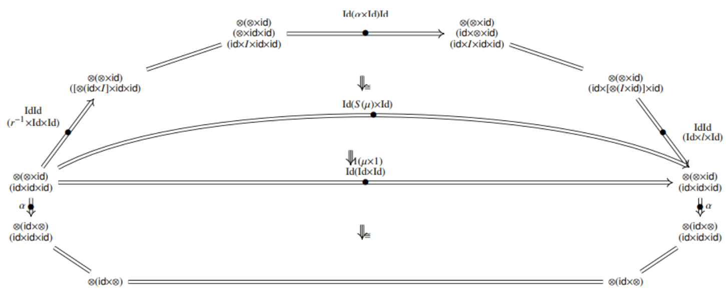
First we require that these two following diagrams are equal:



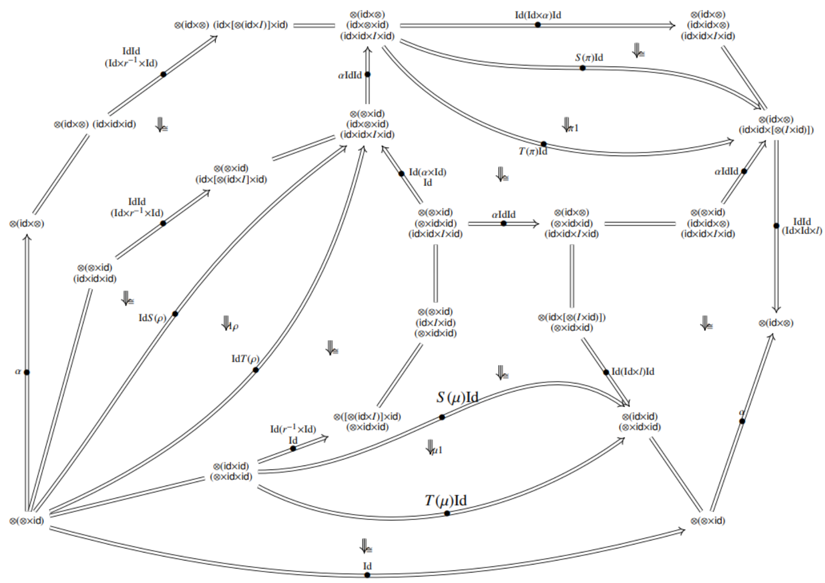


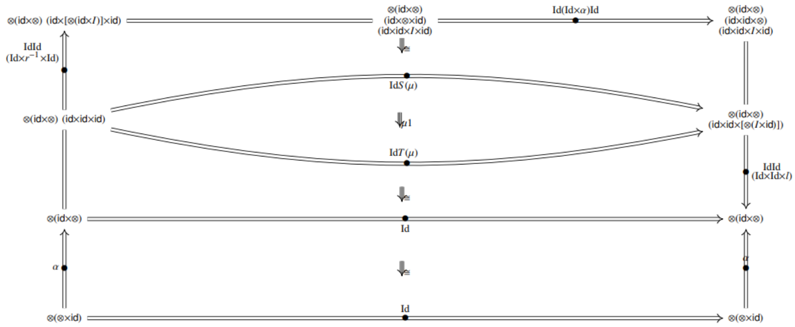
We also require the equality of the following two diagrams:





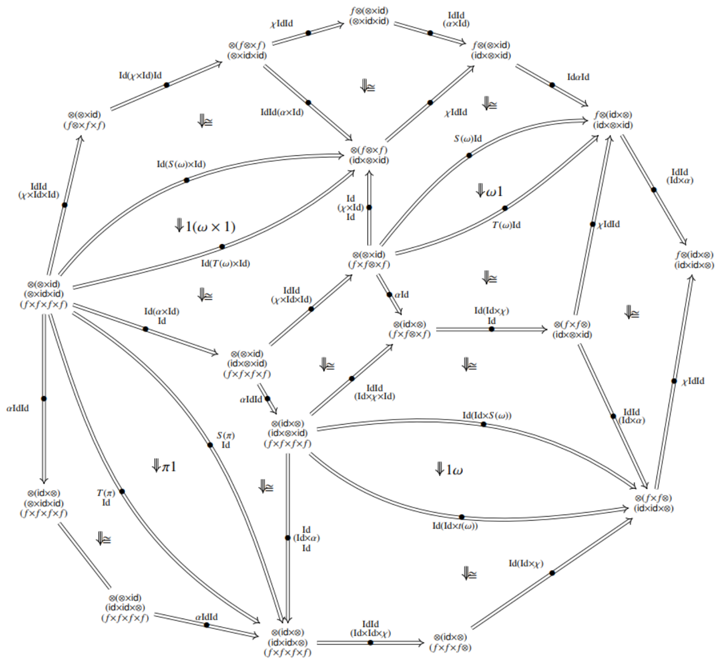
Finally, we require the equality of the following two diagrams:

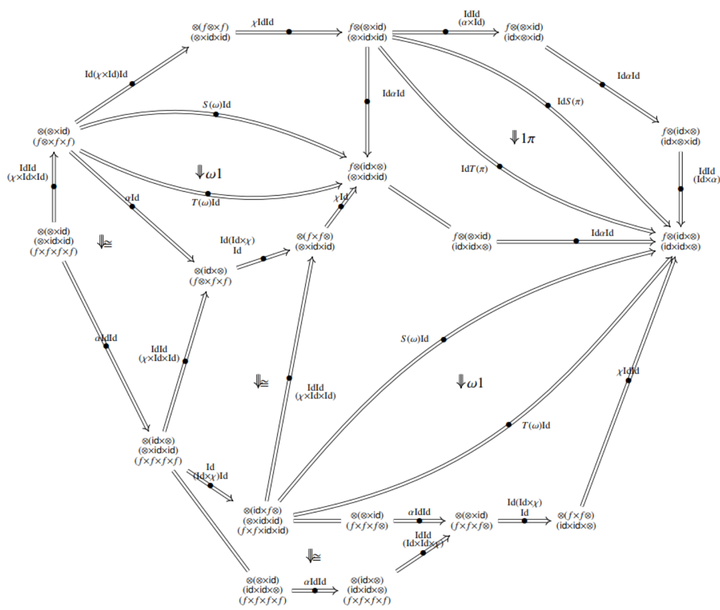




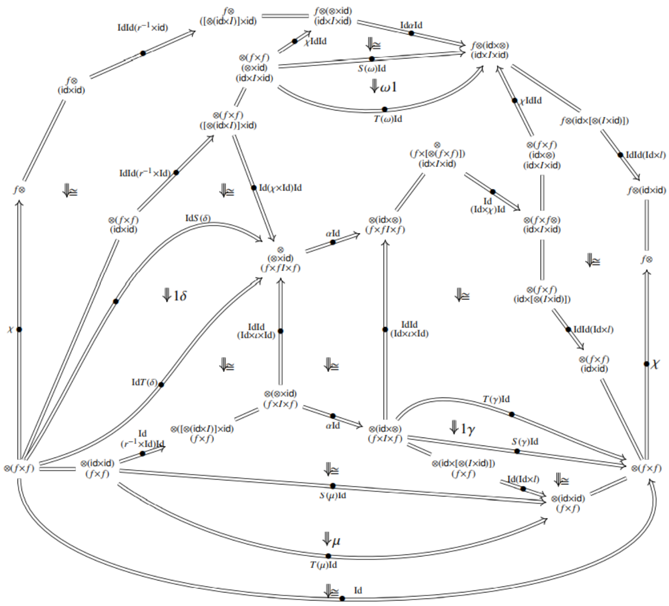
Axioms for monoidal 1-cells:

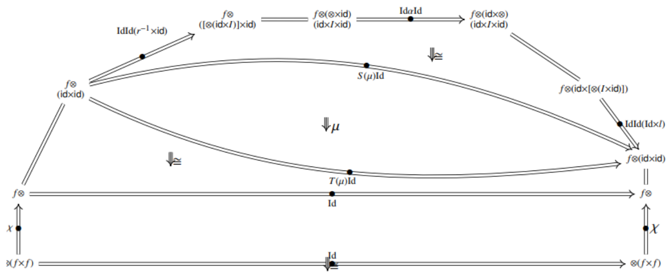
The following two diagrams are equal:





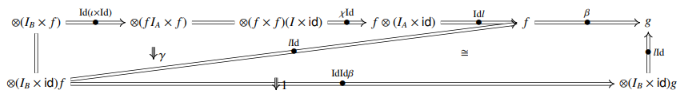
We also have the equality of the following two diagrams:





Axioms for monoidal 2-cells:

We require the following equalities:



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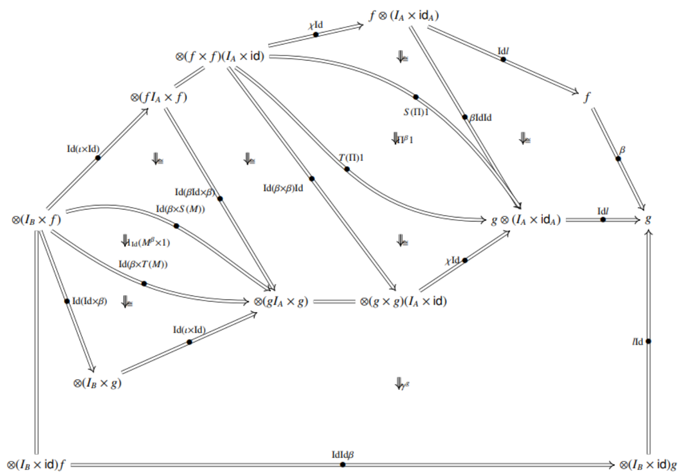
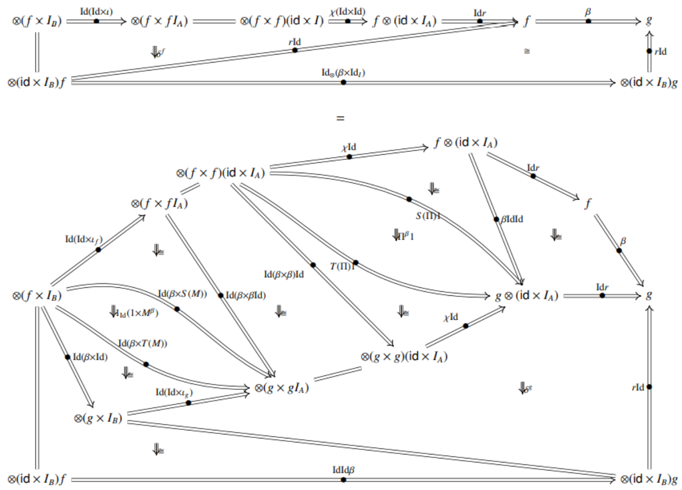
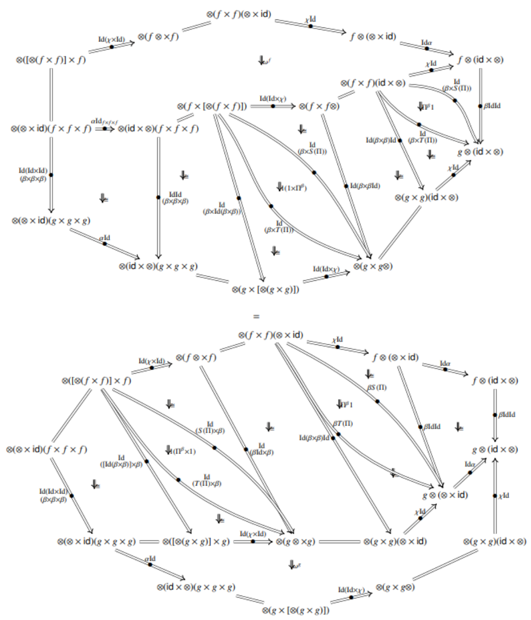
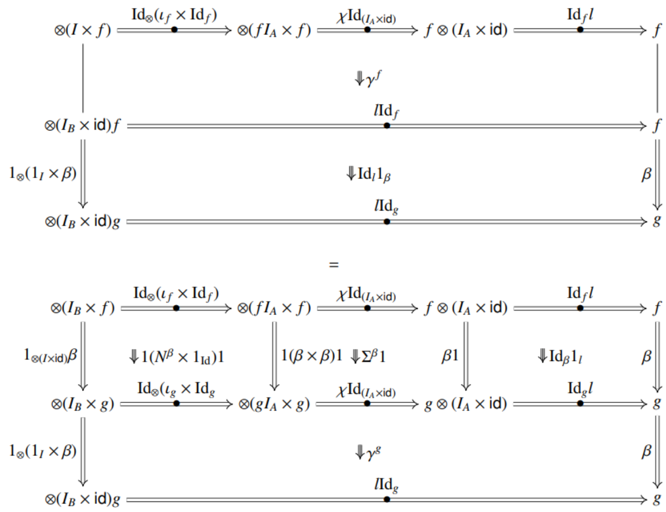


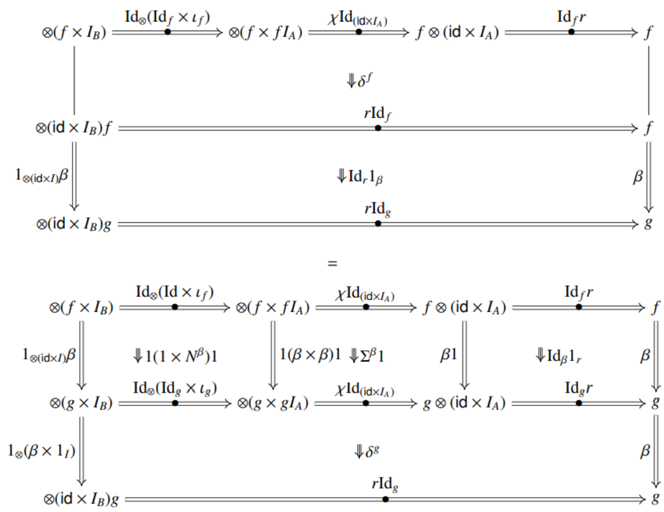


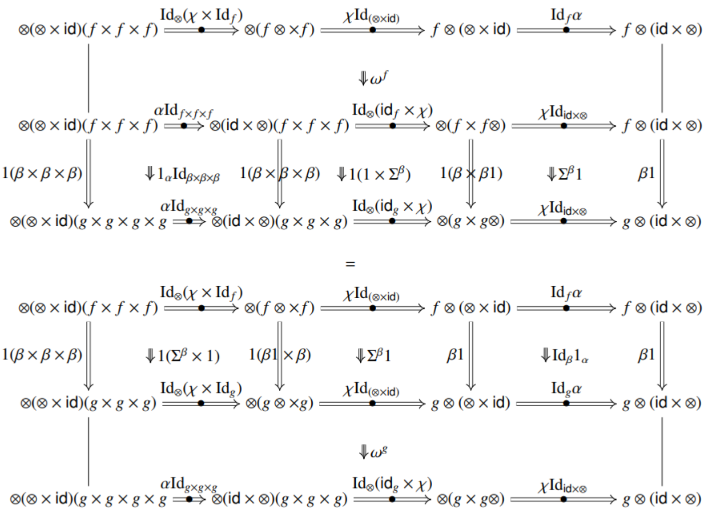


Axioms for monoidal icons:

We require the following equalities:


⊗(I × f)
Id⊗(ι_f × Id_f)
⊗(f I_A × f)
χId_(I_A×id)
f ⊗ (I_A × id)
Id_f l
f
⇓γ^f
⊗(I_B × id)f
lId_f
f
1⊗(1_l × β)
⇓Id_l 1_β
β
⊗(I_B × id)g
lId_g
g
=
⊗(I_B × f)
Id⊗(ι_f × Id_f)
⊗(f I_A × f)
χId_(I_A×id)
f ⊗ (I_A × id)
Id_f l
f
1_⊗(I×id)β
⇓1(N^β × 1_Id)1
1(β × β)1
⇓Σ^β 1
β1
⇓Id_β 1_l
β
⊗(I_B × g)
Id⊗(ι_g × Id_g)
⊗(g I_A × g)
χId_(I_A×id)
g ⊗ (I_A × id)
Id_g l
g
1⊗(1_l × β)
⇓γ^g
β
⊗(I_B × id)g
lId_g
g


⊗(f × I_B)
Id⊗(Id_f × ι_f)
⊗(f × f I_A)
χId_(id×I_A)
f ⊗ (id × I_A)
Id_f r
f
⇓δ^f
⊗(id × I_B)f
rId_f
f
1_⊗(id×I)β
⇓Id_r 1_β
β
⊗(id × I_B)g
rId_g
g
=
⊗(f × I_B)
Id⊗(Id × ι_f)
⊗(f × f I_A)
χId_(id×I_A)
f ⊗ (id × I_A)
Id_f r
f
1_⊗(id×I)β
⇓1(1 × N^β)1
1(β × β)1
⇓Σ^β 1
β1
⇓Id_β 1_r
β
⊗(g × I_B)
Id⊗(Id_g × ι_g)
⊗(g × g I_A)
χId_(id×I_A)
g ⊗ (id × I_A)
Id_g r
g
1⊗(β × 1_l)
⇓δ^g
β
⊗(id × I_B)g
rId_g
g



Axioms for monoidal 3-cells:

We require the following equalities: