UNIVERSIDADE FEDERAL DE PERNAMBUCO
CENTRO DE INFORMÁTICA (CIn)
DOUTORADO EM CIÊNCIAS DA COMPUTAÇÃO

DANIEL ORLANDO MARTÍNEZ RIVILLAS

Towards a Homotopy Domain Theory

Recife

2022

DANIEL ORLANDO MARTÍNEZ RIVILLAS

Towards a Homotopy Domain Theory

Tese apresentada ao Programa de Pós-graduação em Ciência da Computação do Centro de Informática da Universidade Federal de Pernambuco, como requisito parcial para obtenção do grau de Doutor em Ciência da Computação.

Teoria da Computação

Orientador (a): Prof. Dr. Ruy José Guerra Barretto de Queiroz

Recife

2022

Catalogação na fonte
Bibliotecária Nataly Soares Leite Moro, CRB4-1722

M385t Martínez Rivillas, Daniel Orlando
Towards a homotopy domain theory / Daniel Orlando Martínez Rivillas. – 2022.
86 f.: il., fig.

Orientador: Ruy José Guerra Barretto de Queiroz.
Tese (Doutorado) – Universidade Federal de Pernambuco. CIn, Ciência da Computação, Recife, 2022.
Inclui referências.

1. Teoria da computação. 2. Complexo de Kan fracamente ordenado. 3. Ordem parcial de homotopia completo. 4. Equação de domínio de homotopia. 5. Teoria no-tipada de homotopia. I. Queiroz, Ruy José Guerra Barretto de (orientador). II. Título

004 CDD (23. ed.) UFPE - CCEN 2023 – 22

Daniel Orlando Martínez Rivillas

"Towards a Homotopy Domain Theory"

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Ciência da Computação da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutor em Ciência da Computação. Área de Concentração: Teoria da Computação.

Aprovado em: 15/12/2022.

____
Orientador: Prof. Dr. Ruy José Guerra Barretto de Queiroz

BANCA EXAMINADORA

____
Prof. Dr. Edward Hermann Haeusler
Departamento de Informática / PUC/RJ

____
Prof. Dr. Benjamin René Callejas Bedregal
Departamento de Informática e Matemática Aplicada / UFRN

____
Profa. Dra. Alexandra Martins da Silva
Departamento de Ciência da Computação / Cornell Universit

____
Prof. Dr. António Mário da Silva Marcos Florido
Departamento de Ciência de Computadores/Universidade do Porto

____
Prof. Dr. Hugo Luiz Mariano
Instituto de Matemática e Estatística / USP

Dedico o resultado deste esforço à memória dos meus pais: Luz Nelly e José Hilario

AGRADECIMENTOS

Agradeço a toda a minha família: aos meus pais José Hilario e Luz Nelly que continuam me acompanhando na memória, aos meus irmãos Alex, Mao e Juancho que sempre estavam pendentes, apesar da distância, a minha esposa Clarena, que sempre me acompanhava e apoiava mesmo durante toda a estadia neste doutorado. E quero agradecer ao meu orientador, Ruy de Queiroz, por sua colaboração em todo o processo de aprendizado por seu colaboração em tudo o processo de aprendizado e mostrar-me o caminho de pesquisa certo para poder conseguir os objetivos desta tese. Muito obrigado a todos por fazer este trabalho possível.

ABSTRACT

Solving recursive domain equations over a Cartesian closed 0-category is a way to find extensional models of the type-free λ-calculus. In this work we seek to generalize these equations to "homotopy domain equations"; to be able to set about a particular Cartesian closed "$(0,\infty)$-category", which we call the Kleisli ∞-category, and thus find higher λ-models, which we call "λ-homotopic models". To achieve this purpose, we had to previously generalize c.p.o's (complete partial orders) to c.h.p.o's (complete homotopy partial orders); complete ordered sets to complete (weakly) ordered Kan complexes, 0-categories to $(0,\infty)$-categories and the Kleisli bicategory to a Kleisli ∞-category. Continuing with the semantic line of λ-calculus, the syntactical λ-models (e.g., the set D_∞), defined on sets, are generalized to "homotopic syntactical λ-models" (e.g., the Kan complex "K_∞"), which are defined on Kan complexes, and we study the relationship of these models with the homotopic λ-model. Finally, from the syntactic point of view, what the theory of an arbitrary homotopic λ-model would be like is explored, which turns out to contain a theory of higher λ-calculus, which we call Homotopy Type-Free Theory (HoTFT); with higher $\beta\eta$-contractions and thus with higher $\beta\eta$-conversions.

Keywords: ordered weakly Kan complex; complete homotopy partial order; homotopic syntactical λ-model; homotopic λ-model; homotopy domain equation; homotopy type-free theory.

RESUMO

A resolução das equações de domínio recursivas sobre uma 0-categoria Cartesiana fechada é uma maneira de encontrar modelos extensionais do λ-cálculo com Type-free. Neste trabalho buscamos generalizar estas equações para "equações de domínio de homotopia"; definidas sobre uma determinada "$(0, \infty)$-categoria" fechada Cartesiana, que chamamos de Kleisli ∞-category, e assim encontrar λ-modelos superiores, que nós chamar "λ-modelos homotópicos". Para atingir este propósito, tivemos que generalizar previamente c.p.o's (ordens parciais completos) para c.h.p.o's (ordens parciais de homotopia completos); conjuntos ordenados completos para complexos de Kan ordenados (fracamente) completos, 0-categorias para $(0, \infty)$-categorias e a bicategoria Kleisli para uma Kleisli ∞-categoria. Continuando com a linha semântica de λ-cálculo, os λ-modelos sintáticos (e.g., o conjunto D_∞), definidos sobre conjuntos, são generalizados para "λ-modelos sintáticos homotópicos" (e.g., o complexo de Kan "K_∞"), que são definidos em complexos de Kan, e estudamos a relação desses modelos com os λ-modelos homotópicos. Finalmente, do ponto de vista sintático, explora-se como seria a teoria de um λ-modelo arbitrário, que acaba por conter uma teoria de λ-cálculo superior, a qual chamamos Teoria não-tipada de Homotopia; com $\beta\eta$-contrações superiores e daí com $\beta\eta$-conversões superiores.

Palavras-chaves: complexo de Kan fracamente ordenado; ordem parcial de homotopia completo; λ-modelo sintáctico homotópico; λ-modelo homotópico; equação de domínio de homotopia; teoria no-tipada de homotopia.

LISTA DE FIGURAS

SUMÁRIO

1 INTRODUCTION

Dana Scott's Domain Theory (ABRAMSKY; JUNG, 1994), was created in the 60s and 70s with the purpose of finding a semantics for programming languages, specifically for functional programming languages theorized by type-free λ-calculus. Where each computational program, which corresponds to a λ-term, is interpreted as a continuous function $f : D \to D$ defined on an ordered set (poset) D called Domain, which would come a solution of the Recursive Domain Equation

$$X \cong [X \to X],$$

hence D is a model of type-free λ-calculus or also called λ-model, where D_∞ is the first non-trivial λ-model introduced by Dana Scott. Thus, a computable construction can be more easily verified in its interpretation as a continuous function.

In addition to the many recursively defined computational objects, type-free λ-calculus is a classic example of a programming language with recursive definitions of data-types (a classification that specifies the type of value a variable), since the expression $D \cong [D \to D]$, implies a recursive definition of the data-types, in the sense that D is the limit of a non-decreasing sequence of posets $D_i \subseteq D$ recursively defined, where each D_i interprets a type. Thus, a recursively defined computational object (e.g, the factorial function) can be seen as a recursive sequence of partial functions $f_i : D \to D$ which converges to a total function $f : D \to D$ being the interpretation of the λ-term which represents to the computational object.

In the 1970s, λ-calculus was extended by Martin Löf to Intuitionistic Type Theory (ITT) with the purpose of formalizing mathematical proofs to computer programs, i.e., it emerged as a computational alternative to the fundamentals of mathematics. Later, in the 2010s, the Homotopy Type Theory (HoTT) was proposed as an interaction between ITT and Homotopy Theory, where we can affirm that HoTT is the most recent version of the theorization of the notion of algorithm. Finally in (QUEIROZ; OLIVEIRA; RAMOS, 2016) and (RAMOS; QUEIROZ; OLIVEIRA, 2017) one proposes an alternative to HoTT based on computational paths; as a finite sequence of rewriting of computational terms according to the ITT axioms.

Thus, as there is a semantics of type-free λ-calculus with Dana Scott's Domain Theory, we here want to propose a semantics of a untyped version of HoTT based on computational paths starting from a "Domain Theory of Homotopy", that is, we want to start the way for an interpretation of the most recent notion of algorithm given by HoTT, starting with the untyped case.

In (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022a), we introduced a type of topological λ-models called *homotopic λ-models*, in a conceptual way, which is for the present work, a candidate for being a model of this untyped version of HoTT. There λ-terms are interpreted as points and $\beta\eta$-conversions (or $\beta\eta$-equality proofs) between λ-terms are interpreted as continuous paths between two points in the following sense:

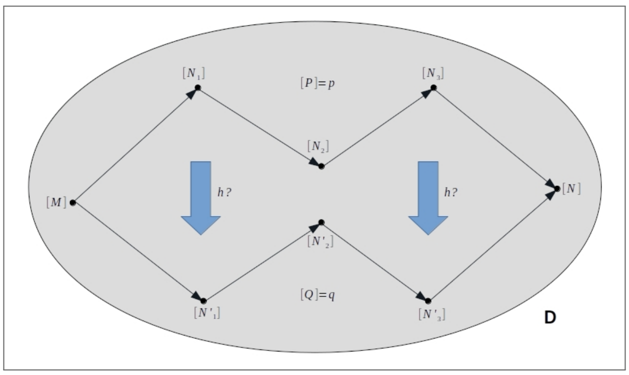
Given the $\beta\eta$-conversion P between the λ-terms M and N, that is,

$$P : M = N_0 =_{1\beta\eta} \cdots =_{1\beta\eta} N_n = N,$$

where the relation $=_{1\beta\eta}$ can be a β-contraction ($\rhd_{1\beta}$), reversed β-contraction ($\lhd_{1\beta}$), η-contraction ($\rhd_{1\eta}$) or reversed η-contraction ($\lhd_{1\eta}$). The proof $P : M =_{\beta\eta} N$ can be interpreted at some homotopic λ-model $\langle D, \bullet, [\![\,]\!]\rangle$ as a continuous path $p : [\![M]\!] \rightsquigarrow [\![N]\!]$ which passes through the intermediate points $[\![N_1]\!], [\![N_2]\!], \ldots, [\![N_{n-1}]\!]$.

The authors wonder, if given two 1-proofs $P, Q : M =_{\beta\eta} N$, in space D, if there is a homotopy h between the paths $p := [\![P]\!]$ and $q := [\![Q]\!]$? (see Figure 1).

Figura 1 – Proofs $P, Q : M =_{\beta\eta} N$ on a homotopic λ-model D



Sink: (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022a)

More so, if one has some intentional definition of equality between the proofs P and Q, such that the interpretation of any 2-proof of this equality, into D, is a homotopy from p to q. One could ask again, for the 2-proofs F and G of the equality $P =_D Q$, if there is a 3-path between homotopies $f := [\![F]\!]$ and $g := [\![G]\!]$?.

And so on, one can continue asking until reaching a getting from a model topology D, a theory of higher equality with ∞-groupoid structure, which could be a typed-free version of the Homotopy Type Theory (HoTT) (PROGRAM, 2013).

It is known that HoTT offers a way of understanding Intuitionistic Type Theory (ITT) by allowing for an interpretation based on the geometric intuition of Homotopy Theory. For example, a type A is interpreted as a topological space, a term $a : A$ as a point $a \in A$, a dependent type $x : A \vdash B(x)$ as the fibration $B \to A$, the identity type Id_A as the path space A^I, a term $p : Id_A(a,b)$ as the path $p : a \to b$, the term $\alpha : Id_{Id_A(a,b)}(p,q)$ as the homotopy $\alpha : p \Rightarrow q$ and so on.

According to Quillen's Theorem, each CW complex topological space is homotopically equivalent to a Kan complex (∞-groupoid), and, conversely, each Kan complex is homotopically equivalent to a CW complex. Then, instead of working directly with topological spaces, we are going to work with Kan complexes, which are ∞-categories whose 1-simplexes or edges are weakly invertible (LURIE, 2009). Thus, to ensure consistency of HoTT (based on topological spaces), Voevodsky (KAPULKIN; LUMSDAINE, 2012) (see (LUMSDAINE; SHULMAN, 2020) for higher inductive types) proved that HoTT has a model in the category of Kan complexes (see (PROGRAM, 2013)).

Consequently, a type A can be interpreted by a Kan complex, a term $a : A$ as the vertex $a \in A$, a dependent type $x : A \vdash B(x)$ as the fibration $B \to A$, the identity type Id_A as the space functors $Fun(\Delta^1, A)$, a term $p : Id_A(a,b)$ as the 1-simplex $p : a \to b$, the term $\alpha : Id_{Id_A(a,b)}(p,q)$ as the 2-simplex $\alpha : p \Rightarrow q$ and so on.

Therefore, this project is a continuation of (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022a), intending to try to answer all these questions in the initial framework of a higher domain theory and see how this relates to HoTT.

1.1 MOTIVATION

The initiative to search for λ-models (HINDLEY; SELDIN, 2008) with a ∞-groupoid structure emerged in (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022a), which studied the geometry of any complete partial order (c.p.o) (e.g., D_∞), and found that the topology inherent in these models generated trivial higher-order groups. From that moment on, the need arose to look for a type of model that presented a rich geometric structure; where their higher-order fundamental groups would not collapse.

It is known in the literature that Dana Scott's Domain-Theory (ABRAMSKY; JUNG, 1994) and (ASPERTI; LONGO, 1991), provides general techniques for obtaining λ-models by solving domain equations over arbitrary Cartesian closed categories. To fulfil the purpose of getting λ-models with non-trivial ∞-groupoid structure, the most natural way would be to adapt Dana Scott's Domain Theory to a "Homotopy Domain Theory". Where the cartesian closed categories (c.c.c) will be replaced by cartesian closed ∞-categories (c.c.i), the c.p.o's will be replaced by "c.h.p.o's (complete homotopy partial orders)", the 0-categories by $(0,\infty)$-categories and the isomorphism between objects in a Cartesian closed 0-categories (at the recursive domain equation) will be replaced by an equivalence between objects in a cartesian closed $(0,\infty)$-category, which we call "homotopy domain equation" such as summarized in Table 1. See (MARTÍNEZ-RIVILLAS; QUEIROZ, 2021a).

Tabela 1 – Generalization of Recursive Domain Equations

Dana Scott's Domain-Theory	**Homotopy Domain Theory**
Cartesian Closed Category (c.c.c)	Cartesian Closed ∞-Category (c.c.i)
Complete Partial Order (c.p.o)	Complete Homotopy Partial Order (c.h.p.o)
0-category	$(0,\infty)$-category
Recursive Domain Equation: $X \cong (X \Rightarrow X)$	Homotopy Domain Equation: $X \simeq (X \Rightarrow X)$

Source: The autor (2022)

Just as the recursive Domain Equation $X \cong [X \to X]$ (in the category of the c.p.o's) has an implicit a the recursive definition of the data-types, the "Homotopy Domain Equation" $X \simeq [X \to X]$ (in the "∞-category of the c.h.p.o's") would also have a recursive definition of the data-types. A recursively defined computational object (e.g., a proof by mathematical induction) would being of a higher order relative to the classical case, whose interpretation would be recursively defined by a sequence of partial functors $F_i : K \to K$, over a Kan complex K weakly ordered, which converges to a total functor $F : K \to K$, whose details are not among the objectives of this thesis, but will be developed in future works, when studying the semantics (case of inductive types) of the version of HoTT based on computational paths.

Thus, intuitively, from the computational point of view, we have that a Kan complex, which satisfies the Homotopy Domain Equation, is not only capable of verifying the computability of constructions typical of classical programming languages, as D_∞ does it, but also, it has the advantage (over D_∞) of verifying the computability of higher constructions, such as a mathematical proof of some proposition, the proof of the equivalence between two proofs of the same proposition etc.

(HYLAND, 2010) lays out several reasons for generalizing the Domain Theory. Particularly, one prove that the Kleisli bicategory (HYLAND, 2014) is cartesian closed and with enough points. This fact, also motives a generalization of the Kleisli bicategory to an ∞-category of Kan complexes which keep the same properties, in order to apply the techniques of the generalized domain theory to this ∞-category Kleisli and thus obtain a reflexive Kan complex (homotopic λ-model) with relevant information.

On other hand, since HoTT has models at the category of the Kan complexes, from the syntactic point of view it would be interesting to see what relationship there is between the theory of any reflexive Kan complex and HoTT. Especially, in exploring this relationship for the case of identity types and additionally for the case of identity types based on computational paths (QUEIROZ; OLIVEIRA; RAMOS, 2016) and (RAMOS; QUEIROZ; OLIVEIRA, 2017). In (MARTÍNEZ-RIVILLAS; QUEIROZ, 2021b) one can see the first sketches still in the process of development.

1.2 OBJECTIVES

To answer each of the questions formulated in the contextualization and motivation of this research project, one proposes the following objectives: To generalize the Kleisli bicategory to a Kleisli ∞-category and to prove that it is Cartesian closed. To initiate a generalization of the Dana Scott's domain theory to a "homotopy domain theory", specifically in the generalization of the recursive equation domains, where the Cartesian closed category (c.c.c) are transformed into Cartesian closed ∞-category (c.c.i), the c.p.o's are replaced by "c.h.p.o's" and the domain equations (isomorphism of objects in a c.c.c) become "equations of homotopy domain" (equivalence of objects in a c.c.i). Having done all this, we can then apply the techniques of solving homotopy domain equations to the particular case of the Kleisli ∞-category to find extensional Kan complexes with relevant information.

2 THEORETICAL FOUNDATION

Since all the objectives will be carried out within the framework of the ∞-categories, some concepts necessary for the development of this project will be presented.

2.1 ∞-CATEGORIES

The ∞-categories or quasi-categories are a type of simplicial sets, introduced by (BOARDMAN; VOGT, 1973) to generalize the classic categories. In the decade of the 2000s, André Joyal made important advances, e.g. showing some results analogous to classical category theory in its ∞-categorical version, as can be seen in (JOYAL, 2002) and (JOYAL; TIERNEY, 2008). In (LURIE, 2009), one can find the most complete treatise written so far on the theory of ∞-categories and in (LURIE, 2017) everything related to the higher operators and algebras on ∞-categories.

2.1.1 Simplicial sets

For a better understanding of the definitions and basic results on ∞-categories, necessary for the development of this work, we present some notions on simplicial sets (GOERSS; JARDINE, 2009).

Definition 2.1.1 (Simplicial indexing category). *Define Δ be the category as follows. The objects are finite ordinals $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$, and morphisms are the (non strictly) order preserving maps. Morphisms in Δ are often called simplicial operators.*

Remark 2.1.1. *There are a coface operator $d^i : [n-1] \to [n]$, which skips the i-th element and a codegeneracy operator $s^i : [n+1] \to [n]$, which maps i and $i+1$ to the same element. All operator f^* in Δ can be obtained as a finite composition of coface and codegeneracy operators.*

Definition 2.1.2 (Simplicial set). *A simplicial set X is a functor $X : \Delta^{op} \to Set$ (or presheaf). A simplicial morphism is just a natural transformation of functors. The category of the simplicial sets $Fun(\Delta^{op}, Set)$ will be denoted by $sSet$ or Set_{Δ}.*

It is typical to write X_n for $X([n])$, and call it the set of n-simplexes in X.

Remark 2.1.2. *Given a simplex $a \in X_n$ and a simplicial operator $f^* : [m] \to [n]$, the function $f : X_n \to X_m$ is given by $f(a) := X(f^*)(a)$. In this explicit language, a simplicial set consists of*

- *a sequence of sets $X_0, X_1, X_2, \ldots,$*
- *functions $f : X_n \to X_m$ for each simplicial operator $f^* : [m] \to [n]$.*
- *$id(a) = a$ and $(gf)(a) = g(f(a))$ for all simplex a and simplicial operators f^* and g^* whenever this makes sense.*

For the coface operator $d^i : [n-1] \to [n]$, the face map is denoted by $d_i : X_n \to X_{n+1}$, $0 \leq i \leq n$. For the codegeneracy operator $s^i : [n+1] \to [n]$, the degeneracy map is written by $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$.

Definition 2.1.3 (Product of simplicial sets (FRIEDMAN, 2012))**.** *Let X and Y be simplicial sets. Their product $X \times Y$ is defined by*

1. *$(X \times Y)_n = X_n \times Y_n = \{(x, y) \,|\, x \in X_n,\, y \in Y_n\}$,*
2. *if $(x, y) \in (X \times Y)_n$, then $d_i(x, y) = (d_i x, d_i y)$,*
3. *if $(x, y) \in (X \times Y)_n$, then $s_i(x, y) = (s_i x, s_i y)$.*

Notice that there are evident projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. These maps are clearly simplicial morphisms.

Definition 2.1.4 (Standard n-simplex)**.** *The standard n-simplex Δ^n is the simplicial set defined by*

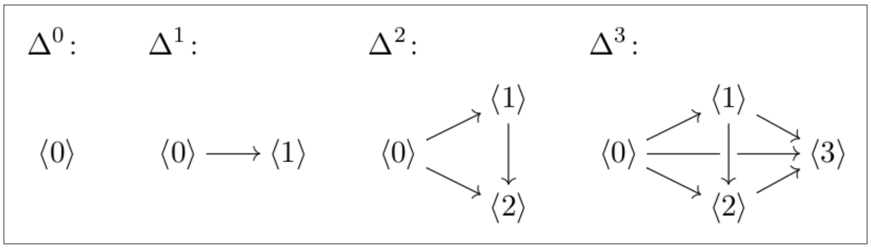
$$\Delta^n := \Delta(-, [n]).$$

That is, the standard n-simplex is exactly the functor represented by the object $[n]$.

The standard 0-simplex Δ^0 is the terminal object in $sSet$; i.e., for every simplicial set X there is a unique map $X \to \Delta^0$. Sometimes we write $*$ instead of Δ^0 for this object. The empty simplicial set $\emptyset$ is the functor $\Delta^{op} \to Set$ sending each $[n]$ to the empty set. It is the initial object in $sSet$, i.e., for every simplicial set X there is a unique map $\emptyset \to X$. Besides, there is a bijection $sSet(\Delta^n, X) \cong X_n$; applying the Yoneda Lemma to category Δ (CISINSKI, 2019).

A graphical representation of the convex hull of Δ^n is made up of the n+1 vertices $\langle 0\rangle, \langle 1\rangle, \ldots, \langle n\rangle$ and the faces are the injective simplicial operators, which are called non-degenerated cells. In the Figure 2, we have some "pictures" of Δ^n for $n = 0, \ldots, 3$, which show their non-degenerate 1-cells as arrows.

Figura 2 – Standard n-simplexes from $n = 0$ to $n = 3$



Source: (REZK, 2017)

Definition 2.1.5 ((GOERSS; JARDINE, 2009))**.** *For two simplicial sets X, Y we have a mapping simplicial set, $Map(X, Y)$ defined as:*

$$Map(X, Y)_n := sSet(X \times \Delta^n, Y).$$

Note that in particular $Map(X, Y)_0 = sSet(X \times \Delta^0, Y) \cong sSet(X, Y)$ (bijection of sets). Sometimes to simplify notation, the simplicial set $Map(X, Y)$ will be written as X^Y or $[X \to Y]$.

Next, a collection of sub-objects of the standard simplexes, called "horns" is defined.

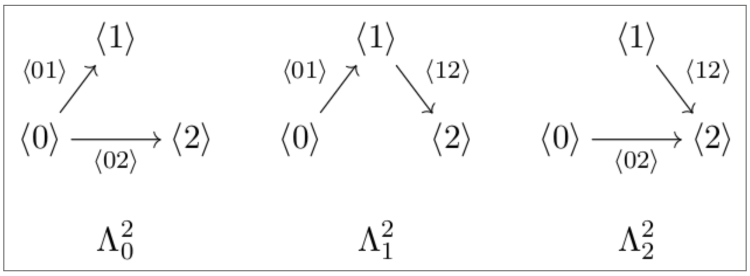
Definition 2.1.6 (Horns)**.** *For each $n \geq 1$, there are subcomplexes $\Lambda^n_i \subset \Delta^n$ for each $0 \leq i \leq n$. The horn Λ^n_i is the subcomplex of Δ^n such that this is the largest sub-object that does not include the face opposing the i-th vertex.*

When $0 < i < n$ one says that $\Lambda^n_i \subset \Delta^n$ is an inner horn. One also says that it is a left horn if $i < n$ and a right horn if $0 < i$.

For example, the horns inside Δ^1 are just the vertices: the left horn, the right horn $\Lambda^1_0 = \{0\} \subset \Delta^1$ and $\Lambda^1_1 = \{1\} \subset \Delta^1$. Neither is an inner horn.

Other example. Δ^2 has three horns: The left horn Λ^2_0, the internal horn Λ^2_1 and the right horn Λ^2_1, see Figure 3.

Figura 3 – Horns inside Δ^2



Source: (REZK, 2017)

2.1.2 Definition of ∞-category and Kan complex

Definition 2.1.7 (∞-category (LURIE, 2009))**.** *An ∞-category is a simplicial set X which has the following property: for any $0 < i < n$, any map $f_0 : \Lambda_i^n \to X$ admits an extension $f : \Delta^n \to X$.*

Definition 2.1.8. *From the definition above, we have the following special cases:*

- *X is a Kan complex if there is an extension for each $0 \leq i \leq n$.*
- *X is a category if the extension exists uniquely (REZK, 2022).*
- *X is a groupoid if the extension exists for all $0 \leq i \leq n$ and is unique (REZK, 2022).*

Next is the definition of Cartesian product of ∞-categories, which generalizes the Cartesian product of categories.

By the bijective correspondence between the set $sSet(K, X \times Y)$ and $sSet(K \to X, K \to Y)$ one has the following proposition (see (REZK, 2017)).

Proposition 2.1.1. *The product of two ∞-categories (as simplicial sets) is an ∞-category.*

2.1.3 Categorical constructions in ∞-categories

Next, one approaches the ∞-categories from the basic notions of the classical categories.

Definition 2.1.9. *A functor of ∞-categories $X \to Y$ is exactly a morphism of simplicial sets. Thus, $Fun(X, Y) = Map(X, Y)$ must be a simplicial set of the functors from X to Y.*

Notation 2.1.1. *The notation $Map(X,Y)$ is usually used for simplicial sets, while $Fun(X,Y)$ is for ∞-categories. One will refer to morphisms in $Fun(X,Y)$ as natural transformations of functors, and equivalences in $Fun(X,Y)$ as natural equivalences.*

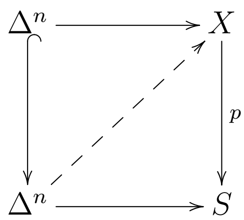
The composition from n-simplex $f:\Delta^n \to X$ (or $f \in X_n$) with a functor $F: X \to Y$, will be denoted as the image $F(f) \in Y_n$, where $n \geq 0$.

A 1-simplex $f:\Delta^1 \to X$, such that $f(0)=x$ and $f(1)=y$ will be denoted as a morphism $f: x \to y$ in the ∞-category X.

An inner horn $\Lambda^2_1 \to X$, which corresponds to composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in the ∞-category X, will be denoted by $(g,-,f)$ or in some cases to simplify notation it will be denoted by $g.f$.

Proposition 2.1.2 ((LURIE, 2009) and (CISINSKI, 2019))**.** *For every ∞-category Y, the simplicial set $Fun(X,Y)$ is an ∞-category.*

Definition 2.1.10 (Trivial Fibration)**.** *A morphism $p: X \to S$ of simplicial sets is a trivial fibration if it has the right lifting property with respect to every inclusion $\partial\Delta^n \subseteq \Delta^n$, i.e., if given any diagram*



there exists a dotted arrow as indicated, rendering diagram commutative.

The proof of the following theorem can be found in (LURIE, 2009) and (CISINSKI, 2019).

Theorem 2.1.1 (Joyal)**.** *A simplicial set X is an ∞-category if and only if the canonical morphism*

$$Fun(\Delta^2, X) \to Fun(\Lambda^2_1, X),$$

is a trivial fibration. Thus, each fibre of this morphism is contractible.

The above theorem guarantees the laws of coherence of the composition of 1-simplexes or morphisms of an ∞-category. In the sense that the composition of morphisms is unique up to homotopy, i.e., the composition is well-defined up to a space of choices is contractible (equivalent to Δ^0).

With respect to the Kan complexes, we have the following equivalence.

Proposition 2.1.3 (Homotopy extension lifting property (LURIE, 2009))**.** *The simplicial set X is a Kan complex if and only if the induced map*

$$Fun(\Delta^1, X) \to Fun(\{0\}, X)$$

is a trivial fibration of simplicial sets.

Definition 2.1.11 (Space of morphisms (REZK, 2017))**.** *For two vertices x, y in an ∞-category X, define the space of morphisms $X(x, y)$ by the following pullback diagram*

$$\begin{array}{ccc} X(x,y) & \longrightarrow & Fun(\Delta^1, X) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \Delta^0 & \xrightarrow[(x,y)]{} & X \times X \end{array}$$

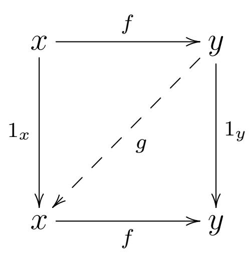
in the category $sSet$.

Proposition 2.1.4 ((LURIE, 2009) and (REZK, 2017))**.** *The morphism spaces $X(x, y)$ are Kan complexes.*

2.1.4 Equivalences in ∞-categories

In category theory, we have the concept of isomorphism of objects. For the case of the ∞-categories, we will have the equivalence of objects (vertices) in the following sense.

Definition 2.1.12 (Equivalent vertices)**.** *A morphism (1-simplex) $f : x \to y$ in an ∞-category X is invertible (an equivalence) if there is morphism $g : y \to x$ in X, a pair of 2-simplexes $\alpha, \beta \in X_2$ such that $(g, -, f) \xrightarrow{\alpha} 1_x$ and $(f, -, g) \xrightarrow{\beta} 1_y$, i.e., we can fill out following diagram*



with the 2-simplexes α and β.

Theorem 2.1.2 ((LURIE, 2009) and (CISINSKI, 2019))**.** *Let X be an ∞-category. The following are equivalent*

1. *Every morphism (1-simplex) in X is an equivalence.*

2. *X is a Kan complex.*

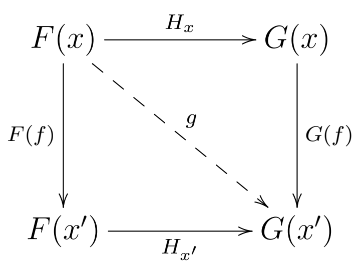
2.1.5 Natural transformations and natural equivalence

Definition 2.1.13 ((CISINSKI, 2019) and (REZK, 2017)). *If X and Y are ∞-categories, and if $F, G : X \to Y$ are two functors, a natural transformation from F to G is a map $H : X \times \Delta^1 \to Y$ such that*

$$H(x, 0) = F(x), \quad H(x, 1) = G(x),$$

for each vertex $x \in X$. Such a natural transformation is invertible or it is a natural equivalence if for any vertex $x \in X$, the induced morphism $F(x) \to G(x)$ (corresponding to the restriction of H to $\Delta^1 \cong \{x\} \times \Delta^1$) is invertible in Y. If there is a natural equivalence from F to G, we write $F \simeq G$.

Remark 2.1.3. *This means that for each vertex $x \in X$, one chooses a morphism $H_x : F(x) \to G(x)$ such that the following diagram*



commutes under some 2-simplexes $\alpha : g \to (G(f), -, H_x)$ and $\beta : (H_{x'}, -, F(f)) \to g$.

2.1.6 Categorical equivalences and homotopy equivalences

Definition 2.1.14 (Categorical equivalence (REZK, 2017) and (LURIE, 2009)). *A functor of ∞-categories $F : X \to Y$ is a categorical equivalence if there is another functor $G : Y \to X$, such that $GF \simeq 1_X$ and $FG \simeq 1_Y$.*

Remark 2.1.4. *From the definition above, if $F : X \to Y$ is a functor of Kan complexes, we say that F is a homotopy equivalence.*

Lemma 2.1.1 ((REZK, 2017) and (CISINSKI, 2019)). *A functor of ∞-categories $F : X \to Y$ is a categorical equivalence if it satisfies the following two conditions:*

- *Fully Faithful (Embedding): For two objects $x, y \in X$ the induced functor of Kan complexes*

$$X(x, y) \to Y(Fx, Fy),$$

is a homotopy equivalence.

- *Essentially Surjective: For every object $y \in Y$ there exists an object $x \in Y$ such that Fx is equivalent to y.*

2.1.7 The join of ∞-categories

Next, the extension from join of categories to ∞-categories. This will make us able to define limit and colimit in an ∞-category.

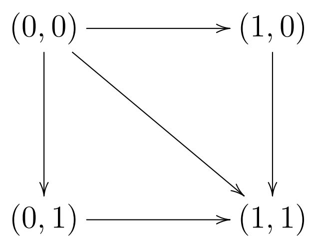
Definition 2.1.15 (Join (LURIE, 2009))**.** *Let K and L be simplicial sets. The join $K \star L$ is the simplicial set defined by*

$$(K \star L)_n := K_n \cup L_n \cup \bigcup_{i+1+j=n} K_i \times L_j, \quad n \geq 0.$$

Example 2.1.1. *(GROTH, 2015)*

1. *If $K \in sSet$ and $L = \Delta^0$, then $K^{\triangleright} = K \star \Delta^0$ is the cocone or the right cone on K. Dually, If $L \in sSet$ then $L^{\triangleleft} = \Delta^0 \star L$ is the cone or the left cone on L.*

2. *Let $K = \Lambda_0^2$. If we see this left horn as a pushout, the cocone $(\Lambda_0^2)^{\triangleright}$ is isomorphic to the square $\square = \Delta^1 \times \Delta^1$, that is, to the filled in diagram*



Proposition 2.1.5. *(LURIE, 2009)*

(i) For the standard simplexes one has an isomorphism $\Delta^i \star \Delta^j \cong \Delta^{i+1+j}$, $i, j \geq 0$, and these isomorphisms are with the obvious inclusions of Δ^i and Δ^j.

(ii) If X and Y are ∞-categories, then the join $X \star Y$ is an ∞-category.

2.1.8 The slice ∞-category

In the case of the classical categories, if A, B are categories and $p : A \to B$ is any functor, one can form the slice category $B_{/p}$ of the object over p or cones on p. The following propositions allow us to define the slice ∞-category.

Proposition 2.1.6 ((JOYAL, 2002))**.** *Let K and S be simplicial sets, and $p : K \to S$ be an arbitrary map. There is a simplicial set $S_{/p}$ such that there exists a natural bijection*

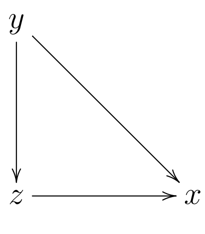
$$sSet(Y, S_{/p}) \cong sSet_p(Y \star K, S),$$

where the subscript on right hand side indicates that we consider only those morphisms $f : Y \star K \to S$ such that $f|K = p$.

Proposition 2.1.7 (Joyal)**.** *Let X be an ∞-category and K be a simplicial set. If $p : K \to X$ be a map of simplicial sets, then $X_{/p}$ is an ∞-category. Moreover, if $q : X \to Y$ is a categorical equivalence, then the induced map $X_{/p} \to Y_{/qp}$ is a categorical equivalence as well.*

Definition 2.1.16 (Slice ∞-Categorical (LURIE, 2009))**.** *Given X be an ∞-category, K be a simplicial set and $p : K \to X$ be a map of simplicial sets. Define the slice ∞-category $X_{/p}$ of the objects over p or cones on p. Dually, $X_{p/}$ is the ∞-category of objects under p or cocones on p.*

Example 2.1.2. *Let X be an ∞-category and $x \in X$ be an object, which correspond to map $x : \Delta^0 \to X$. The objects of the ∞-category $X_{/x}$ of cones on x are morphism $y \to x$ in X, and the morphism from $y \to x$ to $z \to x$ in $X_{/x}$, are the 2-simplexes*



in the ∞-category X.

2.1.9 Limits and colimits

An object t of a category is final if for each object x in this category, there is a unique morphism $x \to t$. Next, one defines the final objects at ∞-categories, under a contractible space of morphisms.

Definition 2.1.17 (Final object (LURIE, 2009)). *An object $\omega \in X$ in an ∞-category X, is a final object if for any object $x \in X$, the Kan complex of morphisms $X(x, \omega)$ is contractible.*

Theorem 2.1.3 (Joyal). *Let an object $\omega \in X$ in a ∞-category X and $\pi : X_{/\omega} \to X$ the canonical projection. The following conditions are equivalent.*

(i) The object $\omega \in X$ is final.

(ii) The map $\pi : X_{/\omega} \to X$ is a trivial fibration.

(iii) The map $\pi : X_{/\omega} \to X$ is a categorical equivalence.

(iv) The map $\pi : X_{/\omega} \to X$ has a section which sends ω to 1_ω.

(v) Any map $f_0 : \partial\Delta^n \to X$, such that $n > 0$ and $f(n) = \omega$, has an extension $f : \Delta^n \to X$.

Corollary 2.1.1 ((CISINSKI, 2019)). *The final objects of an ∞-category X form a Kan complex which is either empty or equivalent to the point.*

Corollary 2.1.2 ((CISINSKI, 2019)). *Let x be a final object in an ∞-category X. For any simplicial set A, the constant map $A \to X$ with value x is a final object in $Map(A, X)$.*

Definition 2.1.18 (Limit and colimit (JOYAL, 2002)). *Let X be an ∞-category and let $p : K \to X$ a map of simplicial sets. A colimit for p is an initial object of $X_{p/}$, and a limit for p is a final object of $X_{/p}$.*

By the dual of Corollary 2.1.1, if the colimit exists, then the Kan complex of initial objects is contractible, i.e., the initial object is unique up to contractible choice.

2.1.10 ∞-categories of presheaves

Definition 2.1.19 (∞-categories of presheaves (LURIE, 2009)). *Let S be a simplicial set. One lets $P(S)$ or PS denote simplicial set $Fun(S^{op}, \mathcal{S})$; where $\mathcal{S}$ denotes the ∞-category of the small Kan complexes or ∞-groupoids, also called the ∞-category of spaces. One will say that $P(S)$ is the ∞-category of the presheaves on S.*

Proposition 2.1.8 ((LURIE, 2009)). *Let S be a simplicial set. The ∞-category $P(S)$ of the presheaves on S admits all small limits and colimits.*

Proposition 2.1.9 (∞-Categorical Yoneda Lemma (LURIE, 2009))**.** *Let S be a simplicial set. Then the Yoneda embedding $j : S \to PS$ is fully faithful.*

Notation 2.1.2. *Let X be an ∞-category and S be a simplicial set. One lets $Fun^L(PS, X)$ denote the full subcategory of $Fun(PS, X)$ spanning those functors $PS \to X$ which preserve small colimits.*

The motivation for this notation stems from Adjoint Functor Theorem (will be seen later), where $Fun^L(PS, X)$ also denotes the full subcategory of $Fun(PS, X)$ spanning those functors which are left adjoints.

Theorem 2.1.4 ((LURIE, 2009))**.** *Let S be a small simplicial set and let X be an ∞-category which admits small colimits. The composition with the Yoneda embedding $j : S \to PS$ induces an equivalence of ∞-categories*

$$Fun^L(PS, X) \to Fun(S, X).$$

Another fundamental result is the following (LURIE, 2009): given a collection of simplicial sets $\mathcal{K}$, $\mathcal{R} \subseteq \mathcal{K}$ and A an ∞-category, there exists an ∞-category $P^{\mathcal{K}}_{\mathcal{R}}A$ and a functor $j : A \to P^{\mathcal{K}}_{\mathcal{R}}A$ with the following properties:

1. $P^{\mathcal{K}}_{\mathcal{R}}A$ admits $\mathcal{K}$-indexed colimits, i.e., admits K-indexed colimits for each $K \in \mathcal{K}$.
2. For every ∞-category B which admits $\mathcal{K}$-indexed colimits, composition with j induces an equivalence of ∞-categories

$$Fun_{\mathcal{K}}(P^{\mathcal{K}}_{\mathcal{R}}A, B) \simeq Fun_{\mathcal{R}}(A, B).$$

   If A admits all the $\mathcal{R}$-indexed colimits, we also have

3. The functor j is fully faithful.

where $Fun_{\mathcal{R}}(A, B)$ is the full subcategory of $Fun(A, B)$ spanned by those functors which preserve $\mathcal{R}$-indexed colimits, i.e., which preserve K-indexed colimits for each $K \in \mathcal{R}$; the same applies to $Fun_{\mathcal{K}}(P^{\mathcal{K}}_{\mathcal{R}}A, B)$.

Example 2.1.3. *Let $\mathcal{R} = \emptyset$ and $\mathcal{K}$ be the class of all small simplicial sets. If A is a small ∞-category, then $P^{\mathcal{K}}_{\mathcal{R}}A \simeq PA$.*

Example 2.1.4. *Let $\mathcal{R} = \emptyset$ and $\mathcal{K}$ be the class of all small κ-filtered simplicial sets for some regular cardinal κ. If A is a small ∞-category, then $P^{\mathcal{K}}_{\mathcal{R}} A \simeq Ind_{\kappa} A$. For the category case, see (GABRIEL; ULMER, 1971).*

Example 2.1.5. *Let $\mathcal{R} = \emptyset$ and $\mathcal{K}$ be the class of all κ-small simplicial sets for some regular cardinal κ. If A is a small ∞-category, then $P^{\mathcal{K}}_{\mathcal{R}} A \simeq P^{\kappa} A$, where $P^{\kappa} A$ is the full subcategory of all κ-compact elements (for more details see Section 2.1.12) of PA.*

Example 2.1.6. *Let $\mathcal{R}$ be the class of all κ-small simplicial sets for some regular cardinal κ and let $\mathcal{K}$ be the collection of all small simplicial sets. Let A be a small ∞-category which admits κ-small colimits, then $P^{\mathcal{K}}_{\mathcal{R}} A \simeq Ind_{\kappa} A$. Also, we have $A \simeq P^{\kappa} C$ for some small ∞-category C which does not necessarily admit κ-small colimits.*

2.1.11 Adjoint Functors

Next is the definition of adjoint functors; motivated from classical definition of functors between categories.

Definition 2.1.20 (Adjunction (CISINSKI, 2019))**.** *Let $F : X \to Y$ and $G : Y \to X$ be functors between ∞-categories. One will say that (F, G) form an adjoint pair, or that F is the left adjoint of G, or that G is the right adjoint of F, if there exists a functorial invertible map of the form*

$$\alpha_{x,y} : X(x, Gy) \to Y(Fx, y)$$

in the ∞-category $\hat{\mathcal{S}}$ of all Kan complexes (not necessarily small), where the word functorial means that this map is the evaluation at (x, y) of a morphism α in the ∞-category of the functors $Fun(X^{op} \times Y, \hat{\mathcal{S}})$.

An adjunction from A to B is a triple (F, G, α), where F and G are the functors as above, while α is an invertible map from $X(-, G(-))$ to $Y(F(-), -)$ which exhibits G as a right adjoint of F.

Proposition 2.1.10 ((CISINSKI, 2019))**.** *Let $F : X \to Y$ be a functor between small ∞-categories. For two adjunctions (F, G, α) and (F, G', α'), the functorial maps $Gy \to G'y$ which are compatibles with α and α' form a contractible Kan complex (i.e., there is unique way to identify G with G' as right adjoint of F).*

Proposition 2.1.11 ((LURIE, 2009)). *Let $F : X \to Y$ and $G : Y \to X$ be functors between ∞-categories. The following conditions are equivalent:*

1. *The functor F is a left adjoint to G.*

2. *There is a unit transformation $u : id_X \to G \circ F$*

Proposition 2.1.12 ((LURIE, 2009)). *Let $F : X \to Y$ be a functor between ∞-categories which has a right adjoint $G : Y \to X$. Then F preserves all the colimits which exist in X, and G preserves all the limits which exist in Y.*

Definition 2.1.21 ((LURIE, 2009)). *Let X and Y be ∞-categories. One lets $Fun^L(X,Y) \subseteq Fun(X,Y)$ denote the full subcategory of $Fun(X,Y)$ spanned by those functors $F : X \to Y$ which are left adjoints. Similarly, one will define $Fun^R(X,Y)$ to be the full subcategory of $Fun(X,Y)$ spanned by those functors which are right adjoints.*

Proposition 2.1.13 ((LURIE, 2009)). *Let X and Y be ∞-categories. Then the ∞-categories $Fun^L(X,Y)$ and $Fun^R(X,Y)^{op}$ are (canonically) equivalent.*

2.1.12 Filtered ∞-Categories and Compact Objects

Recall that the filtered categories are the generalization of the partially ordered set A which are filtered, i.e., those that satisfy the condition: every finite subset of A has an upper bound in A. Next is the definition of filtered ∞-category which generalizes the classical version to categories.

Definition 2.1.22 (κ-Filtered (LURIE, 2009)). *Let κ be a regular cardinal and let X be an ∞-category. We say that X is κ-filtered if, for every κ-small set simplicial S and all map $f : S \to X$ there is a map $\overline{f} : S^{\triangleright} \to X$ (with $S^{\triangleright} = S \star \Delta^0$) which extends to f. (i.e., X is filtered if this the extension property with respect to the inclusion $S \subseteq S^{\triangleright}$ for every κ-small simplicial set S). One will say that X is filtered if this is ω-filtered.*

Definition 2.1.23 (Weakly contractible). *A ∞-category X is weakly contractible if its geometric realization $|X| \in Top$ is contractible.*

Proposition 2.1.14 ((LURIE, 2009)). *Let X be a filtered ∞-category. Then X is weakly contractible.*

Definition 2.1.24 ((LURIE, 2009)). *Let κ be a regular cardinal and X be an ∞-category. One will say that X admits κ-filtered colimits if it admits colimits of all indexed diagrams on any κ-filtered ∞-category.*

Definition 2.1.25 (Continuity (LURIE, 2009)). *Let X be an ∞-category which admits κ-filtered colimits. One will say a functor $f : X \to Y$ is κ-continuous if it preserves κ-filtered colimits.*

Definition 2.1.26 (κ-Compact (LURIE, 2009)). *Let X be an ∞-category, $x \in X$ and let $X(x, -) : X \to \hat{\mathbb{S}}$ be the corepresented by x. If X admits κ-filtered colimits, then one will say that x is κ-compact if the functor corepresented by it is κ-continuous. One will say that x is compact if it is ω-compact (and X admits filtered colimits).*

Remark 2.1.5. *Let X be an accessible ∞-category and κ be a regular cardinal. The full subcategory $X^\kappa \subseteq X$ consisting of all the κ-compact objects of X is essentially small, i.e., there exists a small ∞-category Y equivalent to X^κ.*

Notation 2.1.3. *Let X be an ∞-category and κ be regular cardinal. Denote by $Ind_\kappa(X)$ the closure of X under κ-filtered colimits.*

Proposition 2.1.15 ((LURIE, 2009)). *Let X be a small ∞-category and κ a regular cardinal. The ∞-category $P^\kappa(X)$ of κ-compact objects of $P(X)$ is essentially small: that is, there exists a small ∞-category Y and an equivalence $i : Y \to P^\kappa(X)$. Let $F : Ind_\kappa(Y) \to P(X)$ be a κ-continuous functor such that the composition of F with the Yoneda embedding*

$$Y \to Ind_\kappa(Y) \to P(X)$$

is equivalent to i. Then F is an equivalence of ∞-categories.

2.1.13 Accessible ∞-category

In this section, we define everything related to accessibility, necessary to define presentable ∞-categories.

Definition 2.1.27 (Accessible ∞-Category (LURIE, 2009)). *Let κ be a regular cardinal. An ∞-category X is κ-accessible if there exists a small ∞-category X^0 and an equivalence*

$$Ind_\kappa(X^0) \to X.$$

One will say that X is accessible if it is κ-accessible for some regular cardinal κ.

Definition 2.1.28 (Accessible Functor (LURIE, 2009))**.** *Let X be an accessible ∞-category, then a functor $F : X \to X'$ is accessible if it is κ-continuous for some regular cardinal κ.*

Example 2.1.7. *The ∞-category $\mathcal{S}$ of spaces is accessible. More generally, for any small ∞-category X, the ∞-category $P(X)$ is accessible.*

Notation 2.1.4. *Denote by Cat_∞ the ∞-category of all small ∞-categories, and by CAT_∞ the ∞-category of the all the ∞-categories.*

Definition 2.1.29 ((LURIE, 2009))**.** *Let κ be a regular cardinal. We let $Acc_\kappa \subseteq CAT_\infty$ denote the subcategory defined as follows:*

1. *The objects of Acc_κ are the κ-accessible ∞-categories.*

2. *A functor $F : X \to Y$ between accessible ∞-categories belongs to Acc_κ if and only if F is κ-continuous and preserves κ-compact objects.*

Let $Acc = \bigcup_\kappa Acc_\kappa$. We will refer to Acc as the ∞-category of accessible ∞-categories.

2.1.14 Presentable ∞-Categories

Next, we define presentable ∞-categories and their characterizations.

Definition 2.1.30 (Presentable ∞-category (LURIE, 2009))**.** *An ∞-category X is presentable if X is accessible and admits small colimits.*

Theorem 2.1.5 ((SIMPSON, 1999) and (LURIE, 2009))**.** *Let X be an ∞-category. The following conditions are equivalent:*

1. *The ∞-category X is presentable.*

2. *The ∞-category X is accessible, and for every regular cardinal κ the full subcategory X^κ admits κ-small colimits.*

3. *There exists a regular cardinal κ such that X is κ-accessible and X^κ admits κ-small colimits.*

4. *There exists a regular cardinal κ, a small ∞-category Y which admits κ-small colimits, and an equivalence $Ind_\kappa X \to Y$.*

Proposition 2.1.16 ((LURIE, 2009)). *Let $F : X \to Y$ be a functor between presentable ∞-categories. Suppose that X is κ-accessible. The following conditions are equivalent:*

1. *The functor F preserves small colimits.*
2. *The functor F is κ-continuous, and the restriction $F|X^\kappa$ preserves κ-small colimits.*

Theorem 2.1.6 (Adjoint Functor Theorem (LURIE, 2009)). *Let $F : X \to Y$ be a functor between presentable ∞-categories.*

1. *The functor F has a right adjoint if and only it preserves small colimits.*
2. *The functor F has a left adjoint if and only if it is accessible and preserves small limits.*

Definition 2.1.31. *Define the subcategories $\mathcal{P}r^L, \mathcal{P}r^R \subseteq CAT_\infty$ as follows:*

1. *The objects of both $\mathcal{P}r^L$ and $\mathcal{P}r^R$ are the presentable ∞-categories.*
2. *A functor $F : X \to Y$ between presentable ∞-categories is a morphism in $\mathcal{P}r^L$ if and only if F preserves small colimits. Hence $F \in Fun^L(X, Y)$.*
3. *A functor $G : X \to Y$ between presentable ∞-categories is a morphism in $\mathcal{P}r^R$ if and only if G is accessible and preserves small limits. Thus $G \in Fun^R(X, Y)$.*

Proposition 2.1.17 ((LURIE, 2009)). *The ∞-categories $\mathcal{P}r^L$ and $\mathcal{P}r^R$ admits all small limits, and the inclusion functors $\mathcal{P}r^L, \mathcal{P}r^R \subseteq CAT_\infty$ preserve all small limits.*

Proposition 2.1.18 ((LURIE, 2009)). *Let X be an presentable ∞-category and let S be a small simplicial set. Then $Fun(S, X)$ and $Fun^L(S, X)$ are presentable.*

Remark 2.1.6 ((LURIE, 2009)). *The presentable ∞-category $Fun^L(X, Y)$ can be regarded as an internal mapping object in Pr^L. That is, there exists an operation $\otimes$ that endows Pr^L with the structure of a symmetric monoidal ∞-category. Proposition 2.1.18 can be interpreted as asserting that this monoidal structure is closed.*

2.1.15 Compactly Generated ∞-Categories

Definition 2.1.32 ((LURIE, 2009)). *Let κ be a regular cardinal. We will say that an ∞-category X is κ-compactly generated if it is presentable and κ-accessible. When $\kappa = \omega$, we will simply say that X is compactly generated.*

Proposition 2.1.19 ((LURIE, 2009)). *Let κ be a regular cardinal and let $F : X \rightleftarrows Y : G$ be a pair of adjoint functors, where X and Y admit small κ-filtered colimits and these are κ-compactly generated.*

1. *If G is κ-continuous, then F carries κ-compact objects of X to κ- compact objects of Y.*

2. *Conversely, if X is κ-accessible and F preserves κ-compactness, then G is κ-continuous.*

Definition 2.1.33. *If κ is a regular cardinal, define the subcategories $\mathcal{P}r^L_\kappa, \mathcal{P}r^R_\kappa \subseteq CAT_\infty$ as follows*

1. *The objects of both $\mathcal{P}r^L_\kappa$ and $\mathcal{P}r^R_\kappa$ are κ-compactly generated ∞-categories.*

2. *The morphisms in $\mathcal{P}r^R_\kappa$ are κ-continuous limit-preserving functors.*

3. *The morphisms in $\mathcal{P}r^L_\kappa$ are functors which preserve small colimits and κ- compact objects.*

Proposition 2.1.20. *The ∞-category $\mathcal{P}r^R_\kappa$ admits all the limits and the inclusion $\mathcal{P}r^L_\kappa \subseteq CAT_\infty$ preserves small limits.*

Definition 2.1.34 ((LURIE, 2009)). *Define the subcategory $CAT^{Rex(\kappa)}_\infty$ whose objects are (not necessarily small) ∞-categories which admit κ-small colimits and whose morphisms are functors which preserve κ-small colimits, and let $Cat^{Rex(\kappa)}_\infty = CAT^{Rex(\kappa)}_\infty \cap Cat_\infty$.*

Proposition 2.1.21 ((LURIE, 2009)). *Let κ be a regular cardinal and let*

$$\theta : \mathcal{P}r^L_\kappa \to CAT^{Rex(\kappa)}_\infty$$

be the functor which associates to a κ-compactly generated ∞-category X the full subcategory $X^\kappa \subseteq X$ spanned by the κ-compact objects of X. Then the functor θ is fully faithful.

2.2 THE KLEISLI BICATEGORY

Next, we introduce the Kleisli bicategory according to the works of (HYLAND, 2014) and (HYLAND, 2010).

A Kleisli structure is a 2-dimensional version of a restricted monad, as follows below.

Definition 2.2.1 (Kleisli structure (HYLAND, 2014)). *Let $\mathcal{K}$ be a bicategory and $\mathcal{A}$ be a bicategory contained in $\mathcal{K}$. A Kleisli structure P on $\mathcal{A} \subseteq \mathcal{K}$ is the following.*

- *For each object $a \in \mathcal{A}$ an arrow $y_a : a \to Pa$ in $\mathcal{K}$.*
- *For each $a, b \in \mathcal{A}$ a functor*

$$\mathcal{K}(a, Pb) \to \mathcal{K}(Pa, Pb), \quad f \mapsto f^{\#}.$$

- *Families of invertible 2-cells*

$$\eta_f : f \to f^{\#} y_a, \quad \kappa_a : (y_a)^{\#} \to 1_{Pa}, \quad \kappa_{g,f} : (g^{\#} f)^{\#} \to g^{\#} f^{\#},$$

  where $f : a \to Pb$ and $g : b \to Pc$ be 1-cells at $\mathcal{K}$, and subject to unit and pentagon coherence conditions.

It is clear that P is a pseudo-functor from $\mathcal{A}$ to $\mathcal{K}$ such that for each 1-cell $f : a \to b$ of $\mathcal{A}$, set $Pf = (y_b f)^{\#} : Pa \to Pb$.

Definition 2.2.2 (Kleisli bicategory (HYLAND, 2014)). *Given a Kleisli structure P on $\mathcal{A} \subseteq \mathcal{K}$, define its Kleisli bicategory $Kl(P)$ as follows: The objects of $Kl(P)$ are the objects of $\mathcal{A}$. For the objects a, b one has $Kl(P)(a, b) = \mathcal{K}(a, Pb)$. The identities of $Kl(P)$ are the $y_a : a \to Pa$. The Kleisli composition of $f : a \to Pb$ and $g : b \to Pc$, is $g \cdot f = g^{\#} f : a \to Pc$. This extends to 2-cells so one has composition.*

Theorem 2.2.1 ((HYLAND, 2014)). *Let P be a Kleisli structure on $\mathcal{A} \subseteq \mathcal{K}$. Then $Kl(P)$ is a bicategory.*

Example 2.2.1 ((HYLAND, 2014)). *The pseudo-functor $P : Cat \to CAT$, given by $PA = [A^{op}, Set]$ the category of presheaves over A with the usual Yoneda embedding $y_A : A \to PA$, is a Kleisli structure on $Cat \subseteq CAT$.*

Remark 2.2.1 ((HYLAND, 2014)). *In classical category theory, the Kleisli construction is one universal way to associate an adjunction with a monad. In the 2-dimensional setting of Kleisli structures one gets a restricted (pseudo)adjunction as follows: There is a 'forgetful' pseudo-functor $U : Kl(P) \to K$ taking $f : a \to Pb$ in $Kl(P)$ to $f^{\#} : Pa \to Pb$ in $\mathcal{K}$. And there is a 'free' pseudo-functor $F : A \to Kl(P)$, taking $f : a \to b$ in A to $y_b f : a \to Pb$ considered as a map from a to b in $Kl(P)$. One omits the 2-dimensional structure which is routine, but note that the fact that F is a restricted left pseudo-adjoint is immediate from the identification $Kl(P)(a, b) = \mathcal{K}(a, Pb)$.*

That said, the first objective of our work is to generalize the Kleisli structure P on $Cat \subseteq CAT$ to a Kleisli structure on $Cat_\infty \subseteq CAT_\infty$, where Cat_∞ is the ∞-category of small ∞-categories and CAT_∞ the ∞-category of ∞-categories. Thus getting the Kleisli ∞-category $KL(P)$.

2.2.1 Categories with finite limits

In (HYLAND, 2010)], one gets an example of a Kleisli bicategory on a comonad, which closes small categories to small categories with finite colimit, as we present below.

Let $L : Cat \to Cat$ be the restriction of the 2-monad on CAT which sends categories to categories with finite limits. The Kleisli structure P on $Cat \subseteq CAT$ lifts to the 2-category L-Alg of L-algebras by reason of the following observations.

- Any presheaf category PA has finite limits, and if A has finite limits then the Yoneda embedding $y_A : A \to PA$ preserves finite limits.
- If A and B have finite limits and the functor $f : A \to PB$ preserves finite limits, then the left Kan extension $f^\# : PA \to PB$ preserves finite limits, since $f^\#$ preserves colimits.

This immediately gives a lift of the presheaf Kleisli structure P to L-Alg and hence an extension of $L : Cat \to Cat$ to a pseudo-monad $L_P : Kl(P) \to Kl(P)$.

For L_P^* be the pseudo-comonad (here the term 'pseudo' refers to a functor between 2-categories) of the pseudo-monad L_P. This leads to the following results.

Proposition 2.2.1. *(HYLAND, 2010) The bicategory $Kl(L_P^*)$ is Cartesian closed and does have enough points.*

Taking advantage of the properties of presentable ∞-categories, in our work, instead of addressing an ∞-category $Kl(L_P^*)$, we will work directly with the Kleisli ∞-category $Kl(P)$ and prove that it is cartesian closed.

2.3 RECURSIVE DOMAIN EQUATIONS

2.3.1 Fixed Point Theorem

Definition 2.3.1. *1. An ω-diagram in a category $\mathcal{K}$ is a diagram with the following structure:*

$$k_0 \xrightarrow{f_0} k_1 \xrightarrow{f_1} k_2 \longrightarrow \cdots \longrightarrow k_n \xrightarrow{f_n} k_{n+1} \longrightarrow \cdots$$

(dually, one defines ω^{op}-diagrams by just reversing the arrows).

2. A category $\mathcal{K}$ is ω-complete (ω-cocomplete) if it has limits (colimits) for all ω-diagrams.

3. A functor $F : \mathcal{K} \to \mathcal{K}$ is ω-continuous if it preserves all colimits of ω-diagrams.

Theorem 2.3.1 ((ASPERTI; LONGO, 1991))**.** *Let $\mathcal{K}$ be a category with an initial object 0. Let $F : \mathcal{K} \to \mathcal{K}$ be a ω-continuous (covariant) functor and let the unique morphism $\delta \in \mathcal{K}(0, F0)$. Assume also that $(k, \{\delta_{i,\omega} \in \mathcal{K}(F^i 0, k)\}_{i\in\omega})$ is a colimit for the ω-diagram $(\{F^i 0\}_{i\in\omega}, \{F^i \delta\}_{i\in\omega})$, where $F^0 0 = 0$ and $F^0\delta = \delta$. Then $k \cong Fk$ (isomorphic objects).*

To apply the Theorem 2.3.1 on the domain equation

$$X \cong (X \Rightarrow X) = FX$$

in a c.c.c $\mathcal{K}$, we first need to guarantee that the functor $F : \mathcal{K} \to \mathcal{K}$ is ω-continuous. For that, the contravariant functor problem of the exponential $\Rightarrow: \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ must be solved, so that its composition with the diagonal $\Delta : \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ makes sense.

2.3.2 0-Categories

Definition 2.3.2 (c.p.o.)**.** *A set A is a complete partial order (c.p.o.) if it is a partial order, which has supremum for all ω-chain:*

$$a_0 \leq a_1 \leq a_2 \leq \cdots a_n \leq \cdots$$

Its supremum is denoted by $\bigcup_{i\in\omega} a_i$.

Definition 2.3.3. *A category $\mathcal{K}$ is a 0-category if*

1. every hom-set $\mathcal{K}(a, b)$ is a c.p.o., with a least element $0_{a,b}$,

2. *composition of morphisms is a continuous operation with respect to the order,*

3. *for every f in $\mathcal{K}(a,b)$, $0_{b,c} \circ f = 0_{a,c}$.*

Definition 2.3.4 (Projection)**.** *Let $\mathcal{K}$ be a 0-category, and let $f^+ : a \to b$ and $f^- : b \to a$ be two morphisms in $\mathcal{K}$. Then (f^+, f^-) is a projection pair (from a to b) if $f^- \circ f^+ = I_a$ and $f^+ \circ f^- \leq I_b$. If (f^+, f^-) is a projection pair, f^+ is an embedding and f^- is a projection. The projections composition is defined by $(f^+, f^-) \circ (g^+, g^-) = (f^+ \circ g^+, g^- \circ f^-)$.*

Definition 2.3.5 (Projections pair 0-category.)**.** *Let $\mathcal{K}$ be a 0-category. The 0-category $\mathcal{K}^{Prj}$ has the same object as $\mathcal{K}$, projection pairs as morphisms.*

Remark 2.3.1 ((ASPERTI; LONGO, 1991))**.** *Every embedding i has unique associated projection $j = i^R$ (and, conversely, every projection j has a unique associated embedding $i = j^L$), $\mathcal{K}^{Prj}$ is equivalent to a subcategory $\mathcal{K}^E$ of $\mathcal{K}$ that has embeddings as morphisms (as well to a subcategory $\mathcal{K}^P$ of $\mathcal{K}$ which has projections as morphisms). The uniqueness of the projection j, given the embedding i, one has by the isomorphism $(\mathcal{K}^E)^{op} \cong \mathcal{K}^P$, and reciprocally given a projection j, the uniqueness of the associated embedding one has by $(\mathcal{K}^P)^{op} \cong \mathcal{K}^E$.*

Definition 2.3.6. *Given a 0-category $\mathcal{K}$, and a contravariant functor in the first component $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$, the functor covariant $F^{+-} : \mathcal{K}^{Prj} \times \mathcal{K}^{Prj} \to \mathcal{K}^{Prj}$ is defined by*

$$F^{+-}(A, B) = F(A, B),$$
$$F^{+-}((f^+, f^-), (g^+, g^-)) = (F(f^-, g^+), F(f^+, g^-)),$$

Theorem 2.3.2 ((ASPERTI; LONGO, 1991))**.** *Let $\mathcal{K}$ be a 0-category. Let $(\{k_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ be an ω-diagram in $\mathcal{K}^{Prj}$. If $(k, \{\gamma_i\}_{i\in\omega})$ is a limit for $(\{k_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$ in $\mathcal{K}$, then $(k, \{(\delta_i, \gamma_i)\}_{i\in\omega})$ is a colimit for $(\{k_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ in $\mathcal{K}^{Prj}$ (that is, every γ_i is a right member of a projection pair).*

Definition 2.3.7 (Locally monotonic)**.** *Let $\mathcal{K}$ be a 0-category. A functor $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ is locally monotonic if it is monotonic on the hom-set, i.e., for $f, f' \in \mathcal{K}^{op}(A, B)$ and $g, g' \in \mathcal{K}(C, D)$ one has*

$$f \leq f', g \leq g' \Longrightarrow F(f, g) \leq F(f', g').$$

Proposition 2.3.1 ((ASPERTI; LONGO, 1991))**.** *If $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ is locally monotonic and (f^+, f^-), (g^+, g^-) are projection pairs, then $F^{+-}((f^+, f^-), (g^+, g^-))$ is also a projection pair.*

Definition 2.3.8 (Locally continuous). *Let $\mathcal{K}$ be a 0-category. A $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ is locally continuous if it is ω-continuous on the hom-set. That is, for every directed diagram $\{f_i\}_{i\in\omega}$ in $\mathcal{K}^{op}(A, B)$, and every directed diagram $\{g_i\}_{i\in\omega}$ in $\mathcal{K}(C, D)$, one has*

$$F(\bigcup_{i\in\omega}\{f_i\}, \bigcup_{i\in\omega}\{g_i\}) = \bigcup_{i\in\omega} F(f_i, g_i).$$

Remark 2.3.2. *If F is locally continuous, then it is also locally monotonic.*

Theorem 2.3.3 ((ASPERTI; LONGO, 1991)). *Let $\mathcal{K}$ be a 0-category. Let also $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ be a locally continuous functor. Then the functor $F^{+-} : \mathcal{K}^{Prj} \times \mathcal{K}^{Prj} \to \mathcal{K}^{Prj}$ is ω-continuous.*

Let $\mathcal{K}$ be a Cartesian closed 0-category, ω^{op}-complete and with final object. Since the exponential functor $\Rightarrow: \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ and the diagonal functor $\Delta : \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ are locally continuous, by the Theorem 2.3.3, the associated functors

$$(\Rightarrow)^{+-} : \mathcal{K}^{Prj} \times \mathcal{K}^{Prj} \to \mathcal{K}^{Prj}, \quad (\Delta)^{+-} : \mathcal{K}^{Prj} \to \mathcal{K}^{Prj} \times \mathcal{K}^{Prj}$$

are ω-continuous. But composition of ω-continuous functors is still an ω-continuous functor. Thus, the functor

$$F = (\Rightarrow)^{+-} \circ (\Delta)^{+-} : \mathcal{K}^{Prj} \to \mathcal{K}^{Prj},$$

is ω-continuous. By Theorem 2.3.1 the functor F has a fixed point, that is, there is an object $k \in \mathcal{K}$ such that $k \cong (k \Rightarrow k)$. The category of the fixed points of F is denoted by $Fix(F)$.

Remark 2.3.3. *Set is a c.c.c and cocomplete, but it has no fixed points for the functor $F(X) = X^X$ with cardinality $|X| > 1$. Since if there exists a set X such that $X \cong X^X$ (set isomorphisms), this contradicts Cantor's Theorem $|X| < |X^X|$.*

Example 2.3.1. *(ASPERTI; LONGO, 1991) The category CPO of c.p.o.'s with least (bottom) element and continuous functions for morphisms is a 0-category with respect to the pointwise ordering of morphisms. CPO is a c.c.c and it has limits for every diagram. The functor $\Rightarrow: CPO \times CPO \to CPO$ is defined by:*

$$\Rightarrow (A, B) = (A \Rightarrow B), \qquad \Rightarrow (f, g) = \lambda h.g \circ h \circ f,$$

is locally continuous. The diagonal functor $\Delta : CPO \to CPO \times CPO$, defined by $\Delta(A) = (A, A)$ and $\Delta(f) = (f, f)$ is locally continuous too. Thus, by Theorem 2.3.2 and conclude that the associated functors

1. $(\Rightarrow)^{+-} : (CPO)^{Prj} \times (CPO)^{Prj} \to (CPO)^{Prj}$

   $(\Rightarrow)^{+-}((f^+, f^-), (g^+, g^-)) = (\lambda h.g^+ \circ h \circ f^-, \lambda h.g^- \circ h \circ f^+)$

2. $(\Delta)^{+-} : (CPO)^{Prj} \to (CPO)^{Prj} \times (CPO)^{Prj}$

   $(\Delta)^{+-}(f^+, f^-) = ((f^+, f^-), (f^+, f^-))$

are ω-continuous. But the composition of ω-continuous functors is still an ω-continuous functor, thus, the functor $F = (\Rightarrow)^{+-} \circ (\Delta)^{+-} : (CPO)^{Prj} \to (CPO)^{Prj}$ is ω-continuous. Explicitly, F is defined by

$$F(X) = (X \Rightarrow X)$$
$$F(f^+, f^-) = (\lambda h.f^+ \circ h \circ f^-, \lambda h.f^- \circ h \circ f^+)$$

Thus, there is an X such that $X \cong (X \Rightarrow X)$ in $(CPO)^{Prj}$.

Other examples of 0-categories (Cartesian closed) with fixed points for $F(X) = (X \Rightarrow X)$ are:

Example 2.3.2. *The subcategory $CPOS \subseteq CPO$ with only the function strict functions, i.e., morphisms always take the least element $\perp$ to the least of the target space.*

Example 2.3.3. *The subcategory $D \subseteq CPO$ with Scott Domains (bounded complete algebraic c.p.o's) for objects and continuous functions for morphisms. Where a c.p.o. $(X, \leq)$ is **algebraic (finite)** if for every $x \in X$ the set $x \downarrow = \{x_0 \in X_0 \,|\, x_0 \leq x\}$ is directed and $\bigcup(x \downarrow) = x$, with X_0 being the collection of compact elements of X; a point $x \in X$ is **compact** if for every directed set D such that $x \leq \bigcup D$, there is a $y \in D$ such that $x \leq y$. And, a c.p.o $(X, \leq)$ is **bound complete** if every bound subset of X has a least upper bound (supremum).*

2.4 COMPUTATIONAL PATHS FOR SIMPLE TYPED

Definition 2.4.1 ($\lambda\beta\eta$-theory of equality (HINDLEY; SELDIN, 2008))**.** *The extensional λ-theory consists of the all λ-terms and a relation symbol $=$, where the λ-formulas are just $M = N$ for all λ-terms M and N. The axioms are the cases α, β, η and ρ below, for all λ-terms M, N and all variables x, y. The axiom-schemas for rules μ, ν, ξ, τ and σ are below. Axiom-schemas:*

(α) $\lambda x.M = \lambda y.[y/x]M$ *if $y \notin FV(M)$;*

(β) $(\lambda x.M)N = [N/x]M$;

(η) $\lambda x.Mx = M$ *if* $x \notin FV(M)$;

(ρ) $M = M$,

Rules of inference:

(μ) $\dfrac{M = M'}{NM = NM'}$; (τ) $\dfrac{M = N \quad N = P}{M = P}$;

(ν) $\dfrac{M = M'}{MN = M'N}$; (σ) $\dfrac{M = N}{N = M}$;

(ξ) $\dfrac{M = M'}{\lambda x.M = \lambda x.M'}$.

Definition 2.4.2 (***βη*-equality**)**.** *We say that* P *is* $\beta\eta$*-equal (or* $\beta\eta$*-convertible) to* Q*,* $P =_{\beta\eta} Q$*, if there exists* $P_1, P_2, \ldots, P_n$ *such that*

$$(\forall i \leq n-1)(P_i \rhd_{1\beta\eta} P_{i+1} \ \ or \ \ P_{i+1} \rhd_{1\beta\eta} P_i \ \ or \ \ P_i \equiv_\alpha P_{i+1}),$$

where $P_1 = P$ *and* $P_n = Q$.

Definition 2.4.3 (Theory TA$^{\rightarrow}_{\lambda=\beta\eta}$ of equality (HINDLEY; SELDIN, 2008))**.** *The theory TA*$^{\rightarrow}_{\lambda=\beta\eta}$ *consists of the following rules for the type-assignment system TA*$^{\rightarrow}_{\lambda}$*:*

Axiom-schemas:

(β) *Let* $M : A$ *and* $N : B$*. If* $x : A$*, then* $(\lambda x.M)N = [N/x]M : B$;

(η) *Let* $M : A \rightarrow B$*. If* $x : A$*, then* $\lambda x.Mx = M : A \rightarrow B$*, with* $x \notin FV(M)$;

(ρ) *Let* $M : A$*. Then* $M = M : A$,

Rules of inference:

(μ) $\dfrac{M = M' : A \quad N : A \rightarrow B}{NM = NM' : B}$; ($\tau$) $\dfrac{M = N : A \quad N = P : A}{M = P : A}$;

(ν) $\dfrac{N : A \quad M = M' : A \rightarrow B}{MN = M'N : B}$; ($\sigma$) $\dfrac{M = N : A}{N = M : A}$;

(ξ) $\dfrac{M = M' : B \quad [x : A]}{\lambda x.M = \lambda x.M' : A \rightarrow B}$.

Here, instead of working with Martin Löf's types as in (QUEIROZ; OLIVEIRA; RAMOS, 2016), one will define the computational paths for simple types as follows.

Definition 2.4.4 (Computational path (QUEIROZ; OLIVEIRA; RAMOS, 2016))**.** *Let a and b be elements of a type A. Then, a computational path s from a to b is a composition of rewrites (each rewrite is an application of the inference rules of the equality theory of type theory in the Definition 2.4.3 or is a change of bound variables). One denotes that by $a =_s b$.*

Therefore, all the definitions and theorems referred to in this section should be generalized to the case of ∞-categories, as a prelude to a Homotopy Domains Theory.

2.5 CATEGORICAL λ-MODELS

The definitions, theorems and lemmas should generalize in the framework of the ∞-categories, in the sense that the c.p.o's would be replaced by Kan complexes, the continuous maps by functors and the Cartesian closed category (c.c.c) by Cartesian closed ∞-category (i.c.c).

2.5.1 λ-Models in Concrete Cartesian Closed Categories

Definition 2.5.1 (Extensional $\boldsymbol{\lambda}$-models)**.** *An extensional λ-model is a triple $\langle D, \bullet, [\![\,]\!]\rangle$, where D is a set, $\bullet : D \times D \to D$ is a binary operation and $[\![\,]\!]$ is a mapping which assigns to λ-term M and each valuation $\rho : Var \to D$, a element $[\![M]\!]_\rho$ of D such that*

(*a*) $[\![x]\!] = \rho(x)$;

(*b*) $[\![PQ]\!]_\rho = [\![P]\!]_\rho \bullet [\![Q]\!]_\rho$;

(*c*) $[\![\lambda x.P]\!]_\rho \bullet d = [\![P]\!]_{[d/x]\rho}$ *for all* $d \in D$;

(*d*) $[\![M]\!]_\rho = [\![M]\!]_\sigma$ *if* $\rho(x) = \sigma(x)$ *for* $x \in FV(M)$;

(*e*) $[\![\lambda x.M]\!]_\rho = [\![\lambda y.[y/x]M]\!]_\rho$ *if* $y \notin FV(M)$;

(*f*) *if* $(\forall d \in D)\left([\![P]\!]_{[d/x]\rho} = [\![Q]\!]_{[d/x]\rho}\right)$, *then* $[\![\lambda x.P]\!]_\rho = [\![\lambda x.Q]\!]_\rho$;

(*g*) $[\![\lambda x.Mx]\!]_\rho = [\![M]\!]_\rho$ *if* $x \notin FV(M)$,

where $[d/x]\rho$ means: replace $\rho(x)$ with d in the interpretation of λ-term in question.

Remark 2.5.1. *In the definition above, if $\langle D, \bullet, [\![\,]\!]\rangle$ does not necessarily satisfy (g), one says that it is a λ-model.*

Definition 2.5.2 (Reflexive c.p.o (BARENDREGT, 1984)). *A c.p.o D is called reflexive if $[D \to D]$ is retract of D i.e., there are continuous maps*

$$F : D \to [D \to D], \qquad G : [D \to D] \to D$$

such that $F \circ G = id_{[D \to D]}$.

Remark 2.5.2. *In the definition above, if additionally $G \circ F = id_D$, the c.p.o D is called extensional.*

Definition 2.5.3 ((BARENDREGT, 1984)). *Let D be a reflexive c.p.o via the morphisms F, G*

1. *For $a, b \in D$ define*

$$a \bullet b = F(a)(b),$$

2. *Let ρ be a valuation in D. Define the interpretation $[\![\ ]\!]_\rho : \Lambda \to D$ by induction as follows*

   a) $[\![x]\!]_\rho = \rho(x)$,

   b) $[\![MN]\!]_\rho = [\![M]\!]_\rho \bullet [\![N]\!]_\rho$,

   c) $[\![\lambda x.M]\!]_\rho = G(\boldsymbol{\lambda} d.[\![M]\!]_{[d/x]\rho})$, *where* $\boldsymbol{\lambda} d.[\![M]\!]_{[d/x]\rho} = [\![M]\!]_{[-/x]\rho}$.

Lemma 2.5.1 ((BARENDREGT, 1984)). *$\boldsymbol{\lambda} d.[\![M]\!]_{[d/x]\rho}$ is continuous; hence $[\![\lambda x.M]\!]_\rho$ is well-defined.*

Theorem 2.5.1 ((BARENDREGT, 1984) and (HINDLEY; SELDIN, 2008)). *Let D be a reflexive c.p.o via the morphisms F, G, and let $\mathfrak{M} = \langle D, \bullet, [\![\ ]\!] \rangle$. Then $\mathfrak{M}$ is a λ-model.*

2.5.2 λ-Models in Cartesian Closed Categories

Definition 2.5.4 (Cartesian Closed Category (BARENDREGT, 1984)). *Let $\mathcal{C}$ be a category.*

1. *$\mathcal{C}$ is a Cartesian closed category (c.c.c) iff*

   a) *$\mathcal{C}$ has a terminal object T, i.e., T is such that for every object $A \in T$ there is a unique map $!_A : A \to T$.*

b) For $A_1, A_2 \in \mathcal{C}$ there is an object $A_1 \times A_2$ (Cartesian product) with maps p_i : $A_1 \times A_2 \to A_i$ (projections) such that for all $f_i : c \to A_i$ $(i = 1, 2)$ there is a unique map $(f_1, f_2) : C \to A_1 \times A_2$ with $p_i \circ (f_1, f_2) = f_i$.

Notation. If $g_i : A_i \to B_i$ $(i = 1, 2)$, then $g_1 \times g_2 = (g_1 \circ p_1, g_2 \circ p_2) : A_1 \times A_2 \to B_1 \times B_2$.

c) For $A, B \in \mathcal{C}$ there is an object $B^A \in \mathcal{C}$ (exponent) with the map $ev_{A,B} : B^A \times A \to B$ such that for all $f : C \times A \to B$ there is a unique map $\Lambda f : C \to B^A$ such satisfying $f = ev_{A,B} \circ id_A$.

2. Suppose $\mathcal{C}$ has a terminal object T. A point $A \in \mathcal{C}$ is a map $x : T \to A$. The set of points of i denoted by $|A|$. An object A has enough points if for all $f, g : A \to A$ on has

$$\forall x \in |A|, \;\; f \circ x = g \circ x \Rightarrow f = g.$$

It then follows that the same holds for all $f, g : A \to B$.

Definition 2.5.5 (Reflexive object (BARENDREGT, 1984))**.** *Let $\mathcal{C}$ be a c.c.c. An object $U \in \mathcal{C}$ is reflexive if U^U is a retract of U, i.e, there are maps $F : U \to U^U$ and $G : U^U \to U$ such that*

$$F \circ G = id_{U^U}.$$

If additionally $G \circ F = id_U$, we say that U is extensional.

2.6 IDENTITY TYPES

Homotopy Types Theory (HoTT) (PROGRAM, 2013) corresponds to the axioms and rules of the intensional version of Intuitionistic Type Theory (ITT) plus the univalence axiom and higher inductive types. It was created to give a new foundation of mathematics and facilitate the translation of mathematical proofs into computer programs. In this way, it allows computers to verify mathematical proofs with high deductive complexity.

HoTT facilitates the understanding of ITT by allowing for an interpretation based on the geometric intuition of Homotopy Theory. For example, a type A is interpreted as the topological space, a term $a : A$ as the point $a \in A$, a dependent type $x : A \vdash B(x)$ as the fibration $B \to A$, the identity type I_A as the space path A^I, a term $p : I_A(a, b)$ as the path $p : a \to b$, the term $\alpha : I_{I_A(a,b)}(p, q)$ as the homotopy $\alpha : p \Rightarrow q$ and so on.

Among the dependent types arise the identity types, which were inductively defined by Martin-Löf analogously to the inductive definition of natural numbers, according to the rules:

R1. (I-Formation). If A is a type, and a and b are terms that inhabit it, writing $a, b : A$, there is an identity type denoted by $I_A(a, b)$ (or $a =_A b$).

R2. (I-Introduction). If A is a type and $a : A$, there is a term $ref(a) : a =_A a$ (reflexivity).

R3. (I-Elimination). If A is a type, $a : A$ and $P(b, e)$ is a family of types depending on parameters $b : A$ and $e : I_A(a, b)$. In order to define any term $f(b, e) : P(b, e)$, it suffices to provide a term $p : P(a, ref(a))$. The resulting term f may be regarded as having been completely defined by the single definition $f(a, ref(a)) := p$.

Since the theory of an extensional Kan complex, $K \simeq (K \to K)$, will have higher conversions, these will belong in some iteration from identity type I_K, whose rules must be adapted for λ-terms with free-typed or unique typed K.

The problem is that these higher conversions in HoTT are judgmental equalities ($a \equiv b$), which could lead to a loss of information. To avoid this situation, we would choose to work with identity types based on computational paths, where the higher conversions are intentional equalities ($a =_s b$, with s being an equality proof), which could best preserve the information.

2.6.1 Path-based construction

The identity types based on computational paths are given by the following rules (QUEIROZ; OLIVEIRA; RAMOS, 2016):

- (Id-Formation). Let A a type, $a : A$ and $b : B$. Then $Id_A(a, b)$ is a type.
- (Id-Introduction). Let $a =_s b : A$. Then $s(a, b) : Id_A(a, b)$.
- (Id-Elimination). Let $m : Id_A(a, b)$ and $h(g) : C$. If $a =_s b : A$,

  then $REWR(m, \acute{g}.h(g)) : C$.
- (Reduction β). If $a =_m b : A$ and $[a =_g b : A]\ h(g) : C$.

  Then $REWR(m, \acute{g}.h(g)) \rhd_\beta [a =_m b : A]\ h(m/g)$.

- (Reduction η). $e : Id_A(a,b)$ and $[a =_t b : A]\, t : Id_A(a,b)$. Then $REWR(e, \acute{t}.t(a,b)) \rhd_\beta e : Id_A(a,b)$.

Since the identity types based on computational paths admit higher conversions as intentional equalities, it remains to adapt the rules of these identity types to the case of untyped λ-terms.

3 λ-MODEL HOMOTOPIC AND KLEISLI ∞-CATEGORY

In this chapter we define the necessary concepts for a higher semantic of λ-calculus on framework of the Cartesian closed ∞-category. We adopt the notion of Kleisli structure to the case of the ∞-categories and we define the Kleisli ∞-category of a structure. And, finally, we propose a particular ∞-category and prove that it is Cartesian closed. These first results can also be consulted at (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022b).

3.1 HOMOTOPIC λ-MODELS

Next, we define Cartesian closed ∞-category, points and paths of a ∞-category and homotopic λ-model.

Definition 3.1.1 (Cartesian Closed ∞-Category (LURIE, 2017))**.** *Let $\mathcal{C}$ be an ∞-category whose objects are small ∞-categories. We say that $\mathcal{C}$ is a Cartesian closed ∞-category (c.c.i.) if:*

1. *$\mathcal{C}$ has a terminal object T, i.e., $\mathcal{C}(X,T)$ is contractible for each $X \in \mathcal{C}$.*
2. *For $X, Y \in \mathcal{C}$, the Cartesian product $X \times Y$ belongs to $\mathcal{C}$,*
3. *For $X, Y, Z \in \mathcal{C}$, there exists an internal morphism spaces $Y \Rightarrow Z$ in $\mathcal{C}$ such that it sets the natural equivalence*

$$\mathcal{C}(X \times Y, Z) \simeq \mathcal{C}(X, Y \Rightarrow Z).$$

Definition 3.1.2 (Enough points and n-paths)**.** *Let $\mathcal{C}$ be an ∞-category with a terminal object T. A point of an object X is a morphism $x : T \to X$. The class of points of X is denoted by $|X|_0$.*

1. *We say that $\mathcal{C}$ does have enough points if for each pair of morphisms $f, g : X \to Y$ of $\mathcal{C}$ such that for each point $x : T \to X$ there is an equivalence $\sigma_x : f \circ x \simeq g \circ x$ in $\mathcal{C}(T, Y)$, then there is an equivalence $\sigma : f \simeq g$ in $\mathcal{C}(X, Y)$.*
2. *An object $X \in \mathcal{C}$ does have enough points if one has (1) in the case that $Y = X$.*
3. *Let the points $x, y \in |X|_0 = \mathcal{C}(T, X)$. A 1-path $p : x \to y$ in X is a 1-simplex at $\mathcal{C}(T, X)$. The class of the 1-paths of X is denoted by $|X|_1 = (\mathcal{C}(T, X))_1$. Inductively the class of n-paths of X is correspond to $|X|_n = (\mathcal{C}(T, X))_n$.*

Remark 3.1.1. *Note that in Definition 3.1.2 (3), since $\mathcal{C}$ is an ∞-category then all the 1-paths in X (2-simplexes in X) are invertible. Then we say that $X \in \mathcal{C}$ has a ' homotopic structure'. If $\mathcal{C}$ is an ∞-bicategory with an terminal object, we say that an object $X \in \mathcal{C}$ has '∞-categorical structure'.*

Definition 3.1.3 (Reflexive object)**.** *Let $\mathcal{C}$ be a c.c.i. An object $K \in \mathcal{C}$ is called reflexive if $(K \Rightarrow K)$ is a weak retract of K i.e., there are morphisms*

$$F : K \to (K \Rightarrow K), \qquad G : (K \Rightarrow K) \to K$$

such that there is a natural equivalence $\varepsilon : FG \to id_{(K \Rightarrow K)}$.

If there is a natural equivalence $\eta : id_K \to GF$, then K is an extensional object.

Definition 3.1.4 (Homotopic λ-model)**.** *A homotopic λ-model of a c.c.i. $\mathcal{C}$ is a quadruple $\mathcal{K} = \langle K, F, G, \varepsilon \rangle$ where $K \in \mathcal{C}$ is a reflexive object via F, G and ε of the definition above. The quintuple $\mathcal{K} = \langle K, F, G, \varepsilon, \eta \rangle$ is an extensional homotopic λ-model, with η being the natural equivalence of the same definition.*

Remark 3.1.2. *By virtue of the Remark 3.1.1, if U is a reflexive object in a Cartesian closed ∞-bicategory $\mathcal{C}$, we say that K is an '∞-categorical λ-model'.*

3.2 KLEISLI ∞-CATEGORIES

Next, we define the Kleisli structures on the ∞-categories; a general and direct version of those initially introduced by (HYLAND, 2014) for the case of bicategories.

Definition 3.2.1 (Kleisli structure)**.** *Let $\mathcal{K}$ be an ∞-category and $\mathcal{A}$ be an ∞-subcategory of $\mathcal{K}$. A Kleisli structure P on $\mathcal{A} \subseteq \mathcal{K}$ is the following.*

- *For each vertex $a \in \mathcal{A}$ an arrow $y_a : a \to Pa$ in $\mathcal{K}$.*
- *For each $a, b \in \mathcal{A}$ a functor*

$$\mathcal{K}(a, Pb) \to \mathcal{K}(Pa, Pb), \quad f \mapsto f^{\#}.$$

- *A subcategory $\mathcal{K}^L \subseteq \mathcal{K}$ which sets, for all the vertices $a, b \in \mathcal{A}$, the homotopy equivalence*

$$\mathcal{K}(a, Pb) \underset{(-)y_a}{\overset{(-)^{\#}}{\rightleftarrows}} \mathcal{K}^L(Pa, Pb).$$

such for each horn $(g^{\#}, -, f^{\#}) : \Lambda^2_1 \to \mathcal{K}^L$ one has the equality of fibres

$$F^L_{g^{\#},f^{\#}} = F_{g^{\#},f^{\#}},$$

where $F^L_{g^{\#},f^{\#}}$ and $F_{g^{\#},f^{\#}}$ are the fibres of the canonical maps $Fun(\Delta^2, \mathcal{K}^L) \to Fun(\Lambda^2_1, \mathcal{K}^L)$ and $Fun(\Delta^2, \mathcal{K}) \to Fun(\Lambda^2_1, \mathcal{K})$ respectively, with $f : a \to Pb$ and $g : b \to Pc$ be edges in $\mathcal{K}$.

It is clear that P is a functor from $\mathcal{A}$ to $\mathcal{K}$ such that for each 1-simplex $f : a \to b$ of $\mathcal{A}$, sets $Pf = (y_b f)^{\#} : Pa \to Pb$.

Example 3.2.1. *The functor $P : Cat_\infty \to CAT_\infty$, given by $PA = [A^{op}, \mathcal{S}]$, is a Kleisli structure on $Cat_\infty \subseteq CAT_\infty$, where $\mathcal{K}^L = \mathcal{P}r^L$.*

We have the categorical equivalence

$$Fun(A, PB) \underset{(-)y_A}{\overset{(-)^{\#}}{\rightleftarrows}} Fun^L(PA, PB),$$

where $Fun^L(PA, PB)$ is the ∞-category of functors which preserve small colimits. Restricting to the subcategory $\mathcal{P}r^L \subseteq CAT_\infty$, whose objects are presentable ∞-categories and morphisms are functors which preserve small colimits, then the categorical equivalence is restricted to the homotopy equivalence

$$CAT_\infty(A, PB) \underset{(-)y_A}{\overset{(-)^{\#}}{\rightleftarrows}} \mathcal{P}r^L(PA, PB).$$

Example 3.2.2. *The Kleisli structure P on $Cat_\infty \subseteq CAT_\infty$ of the previous example, satisfies the three conditions of the version bicategory of Kleisli structure of the Chapter 2, That is:*

- *The arrows $y_a : a \to Pa$ are represented by the Yoneda embedding $y_A : A \to PA$.*
- *The functors $\mathcal{K}(a, Pb) \to \mathcal{K}(Pa, Pb)$ are represented by the functors induced by the Yoneda extensions $(-)^{\#} : CAT_\infty(A, PB) \to CAT_\infty(PA, PB)$.*
- *The family of invertible 2-cells $\eta_f : f \to f^{\#} y_a$ and $\kappa_a : (y_a)^{\#} \to 1_{Pa}$ are given by the definition of the categorical equivalences $(-)^{\#} : Fun(A, PB) \to Fun^L(PA, PB)$, and the invertible 2-cells $\kappa_{g,f} : (g^{\#} f)^{\#} \to g^{\#} f^{\#}$ are had, since the functor $g^{\#} f : A \to PC$ can be extended in two ways along of y_A, i.e., $g^{\#} f \to (g^{\#} f)^{\#} y_A$ and $g^{\#} f \to (g^{\#} f^{\#}) y_A$. But the functor $(-)y_A$ is an equivalence, hence $(g^{\#} f)^{\#} \simeq g^{\#} f^{\#}$.*

Definition 3.2.2 (Kleisli ∞-category)**.** *Given a Kleisli structure* P *on* $\mathcal{A} \subseteq \mathcal{K}$*. We define* $Kl(P)$ *as the simplicial set embedded in* $\mathcal{K}^L$*, where the objects of* $Kl(P)$ *are the objects of* $\mathcal{A}$*, the morphism spaces is defined by*

$$Kl(P)(a, b) := \mathcal{K}(a, Pb).$$

for the all objects $a, b \in \mathcal{A}$*. The embedding* $Kl(P) \hookrightarrow \mathcal{K}^L$ *is induced by definition of* $Kl(P)(a, b)$ *and the homotopy equivalence* $\mathcal{K}(a, Pb) \to \mathcal{K}^L(Pa, Pb)$ *of the third item in the Definition 3.2.1. Thus, all the n-simplexes in* $Kl(P)$ *are defined by the inverse image (of the embedding) of the n-simplexes in* $\mathcal{K}^L$*.*

Remark 3.2.1. *By Definition 3.2.1, the homotopy equivalence* $Kl(P)(a, b) \to \mathcal{K}^L(Pa, Pb)$ *gets to establish that* $Kl(P)$ *is the* ∞*-category embedded in* $\mathcal{K}^L$*. Another interesting way would be to define* $Kl(P)$ *as a weighted colimit or the pushout of diagram*

$$(Cat_\infty \subseteq CAT_\infty \stackrel{P}{\leftarrow} Cat_\infty)$$

in the category of simplicial sets $Set_\Delta = Fun(\Delta^{op}, Set)$ *(complete and cocomplete) and so* $Kl(P)$ *would be an* ∞*-category.*

Proposition 3.2.1. $P(A \times B) \simeq PA \otimes PB$ *in the* ∞*-category* $\mathcal{P}r^L$*.*

Proof.

$$\begin{aligned} Fun^L(P(A \times B), C) &\simeq Fun(A \times B, C) \\ &\simeq Fun(A, C^B) \\ &\simeq Fun^L(PA, Fun^L(PB, C)) \end{aligned}$$

Thus, $\mathcal{P}r^L(P(A \times B), C) \simeq \mathcal{P}r^L(PA, PB \multimap C) \simeq \mathcal{P}r^L(PA \otimes PB, C)$. □

3.3 A CARTESIAN CLOSED ∞-CATEGORY (C.C.I)

In this section we prove that the Kleisli ∞-category $Kl(P)$ generated by the structure P on $Cat_\infty \subseteq CAT_\infty$ is Cartesian closed. Thus, $Kl(P)$ is a candidate for a higher λ-model.

Lemma 3.3.1. *The* ∞*-category* $Kl(P)$ *is cartesian closed.*

Proof.

$$\begin{aligned}
Fun(A \times B, PC) &\simeq Fun(A, [B, PC]) \\
&= Fun(A, [B, \mathcal{S}^{C^{op}}]) \\
&\simeq Fun(A, [B \times C^{op}, \mathcal{S}]) \\
&= Fun(A, P(B^{op} \times C)).
\end{aligned}$$

Thus $Kl(P)(A \times B, C) \simeq Kl(P)(A, B^{op} \times C) = Kl(P)(A, B \Rightarrow C)$. □

Proposition 3.3.1. *The ∞-category $Kl(P)$ does have enough points.*

Proof. A morphism $A \to B$ in $Kl(P)$ corresponds to a functor $A \to PB$. Since PA is a closure of A under small colimits, such a functor corresponds to a small colimit preserving functor $PA \to PB$. Since PA is weakly contractible; $(PA)(a, b)$ is contractible for all $a, b \in PA$, the functor $PA \to PB$ is sufficiently determined by all the vertices of PA (points of A). □

4 SOLVING HOMOTOPY DOMAIN EQUATIONS

The purpose of this chapter is to give a follow-up on the project of generalization of Dana Scott's Domain Theory (ABRAMSKY; JUNG, 1994) and (ASPERTI; LONGO, 1991), to a Homotopy Domain Theory, which began in (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022b), in the sense of providing methods to find λ-models that allow raising the interpretation of equality of λ-terms (e.g., β-equality, η-equality etc.) to a semantics of higher equalities.

In theoretical terms, one would be looking for a type of λ-models with ∞-groupoid properties, such as CW complexes, Kan complexes etc. For this task, the strategy is to search in the Cartesian closed ∞-category (cci) $Kl(P)$ (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022b) (generalized version of the Kleisli bicategory of (HYLAND, 2010)), reflexive Kan complexes with relevant information, by means of the solution of domain equations proposed in this cci, which we call homotopy domain equations.

To guarantee the existence of Kan complexes with good information, we define the non-trivial Kan complexes, which in intuitive terms, are those that have holes in all higher dimensions and also have holes in the locality or class of any vertex. We establish conditions to prove the existence of non-trivial Kan complexes which are reflexive, which we call homotopy λ-models -stronger than the homotopic λ-models initially defined in (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022a), and developed in (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022b).

It should be clarified that the literature around Kan complexes is related to some computational theories, such as Homotopy Type Theory (HoTT) (PROGRAM, 2013), so that to ensure the consistency of HoTT, Voevodsky (KAPULKIN; LUMSDAINE, 2012) (see (LUMSDAINE; SHULMAN, 2020) for higher inductive types) proved that HoTT has a model in the category of Kan complexes (see (PROGRAM, 2013)).

To meet the goal of getting homotopy λ-models, in Section 4.1 we define the complete homotopy partial orders (c.h.p.o's), in Section 4.2, we propose a method for solving homotopy domain equations on any c.c.i., namely: first, one solves the contravariant functor problem in a similar way to classical Domain Theory, and later one uses a version from fixed point theorem to find some solutions of this equation. Finally, in Section 4.3, the methods of the previous section are applied in a particular c.c.i., and thus one ends up guaranteeing the existence of homotopy λ-models which, as previously mentioned, present an ∞-groupoid structure with relevant information in higher dimensions.

4.1 COMPLETE HOMOTOPY PARTIAL ORDERS

In this section we introduce the complete homotopy partial orders (c.h.p.o) as a direct generalization of the c.p.o's, where the sets are replaced by Kan complexes and the order relations $\leq$ by weak order relations $\precsim$. The proofs of the propositions, lemmas and theorems are very similar to the classical case of the c.p.o's (BARENDREGT, 1984).

Definition 4.1.1 (h.p.o)**.** *Let $\hat{K}$ be an ∞-category. The largest Kan complex $K \subseteq \hat{K}$ is a homotopy partial order (h.p.o), if for every $x, y \in K$ one has that $\hat{K}(x, y)$ is contractible or empty. Hence, the Kan complex K admits a relation of h.p.o $\precsim$ defined for each $x, y \in K$ as follows: $x \precsim y$ if $\hat{K}(x, y) \neq \emptyset$, hence the pair $(K, \precsim)$ is a h.p.o. (we denote only K). The ∞-category $\hat{K}$ is also called a h.p.o.*

Definition 4.1.2 (c.h.p.o)**.** *Let K be a h.p.o.*

1. *A h.p.o $X \subseteq K$ is directed if $X \neq \emptyset$ and for each $x, y \in X$, there exists $z \in X$ such that $x \precsim z$ and $y \precsim z$.*
2. *K is a complete homotopy partial order (c.h.p.o) if*
   a) *There are initial objects, i.e., $\perp \in K$ is an initial object if for each $x \in K$, $\perp \precsim x$.*
   b) *For each directed $X \subseteq \mathcal{K}$ the supremum (or colimit) $\curlyvee X \in \mathcal{K}$ exists.*

Definition 4.1.3 (Continuity)**.** *Let K and K' be c.h.p.o's. A functor $f : K \to K'$ is continuous if $f(\curlyvee X) \simeq \curlyvee f(X)$, where $f(X)$ is the essential image.*

Proposition 4.1.1. *Continuous functors on c.h.p.o's are always monotonic.*

Proof. Let $f : K \to K'$ be a continuous functor between c.h.p.o's and suppose the non-trivial case $a \precsim b$ in K. Since the h.p.o $\{a, b\} \subseteq K$ is directed, by continuity of f we have

$$f(a) \precsim \bigvee\{f(a), f(b)\} \simeq f(\bigvee\{a, b\}) = f(b).$$

□

The Cartesian product between c.h.p.o's can be considered again as a c.h.p.o.

Proposition 4.1.2. *Given the c.h.p.o's K, K', let $K \times K'$ the Cartesian product partially ordered by*

$$(x, x') \precsim (y, y') \text{ if } x \precsim y \text{ and } x' \precsim' y'.$$

Then $K \times K'$ is a c.h.p.o with for directed $X \subseteq K \times K'$

$$\curlyvee X = (\curlyvee X_0, \curlyvee X_1)$$

where X_0 is the projection of X on K, and X_1 is the projection of X on K'.

Proof. $(\bot, \bot')$ is an initial object of $K \times K'$. On the other hand, if $X \subseteq K \times K'$ is directed, by definition of order on $K \times K'$ is also directed, so the supremum $\curlyvee X \in K \times K'$ exists. □

Definition 4.1.4. *Let K, K' be c.h.p.o's. Define the full subcategory $[K \to K'] \subseteq Fun(K, K')$ of the continuous functors. Since $\hat{K}$ has enough points (is weakly contractible), we can define the order pointwise on $[K \to K']$ by:*

$$f \precsim g \iff \forall x \in K,\ f(x) \precsim' g(x).$$

Notation 4.1.1. *Let K be a h.p.o and P a predicate. Denote by $\langle x \in K \mid P(x) \rangle$ the h.p.o induced by the order of K, whose objects are the $x \in K$ which satisfy the property P.*

Lemma 4.1.1. *Let $\langle f_i \rangle_i \subseteq [K \to K']$ be a indexed directed of functors. Define*

$$f(x) = \curlyvee_i f_i(x).$$

Then f is well defined and continuous.

Proof. Since $\langle f_i \rangle_i$ is directed, $\langle f_i(x) \rangle_i$ is directed for each $x \in K$, and since $\hat{K}$ has enough points, the functor f exists. One other hand, for directed $X \subseteq K$

$$f(\curlyvee X) = \curlyvee_i f_i(\curlyvee X) \simeq \curlyvee_i \curlyvee f_i(X) \simeq \curlyvee \curlyvee_i f_i(X) = \curlyvee f(X).$$

□

Proposition 4.1.3. *$[K \to K']$ is a c.h.p.o with supremum of a directed $F \subseteq [K \to K']$ defined by*

$$(\curlyvee F)(x) = \curlyvee \langle f(x) \mid f \in F \rangle.$$

Proof. The constant functor $\boldsymbol{\lambda} x.\bot'$ is a initial object of $[K \to K']$. By Lemma 4.1.1 the functor $\boldsymbol{\lambda} x. \curlyvee \langle f(x) \mid f \in F \rangle$ is continuous, which is the supremum of F. □

Lemma 4.1.2. *Let $f : K \times K' \to K''$. Then f is continuous iff f is continuous in its arguments separately, that is, iff $\boldsymbol{\lambda} x.f(x, x'_0)$ and $\boldsymbol{\lambda} x'.f(x_0, x')$ are continuous for all x_0, x'_0.*

Proof. ($\Rightarrow$) Let $g = \boldsymbol{\lambda} x.f(x, x'_0)$. Then for the directed $X \subseteq K$

$$
\begin{aligned}
g(\bigvee X) &= f(\bigvee X, x'_0) \\
&\simeq \bigvee f(X \times \{x_0\}); \quad f \text{ is continuous and } X \times \{x_0\} \text{ is directed} \\
&= \bigvee g(X).
\end{aligned}
$$

□

Similarly $\boldsymbol{\lambda} x'.f(x_0, x')$ is continuous.

($\Leftarrow$) Let $X \subseteq K \times K'$ be directed. So

$$
\begin{aligned}
f(\bigvee X) &= f(\bigvee X_0, \bigvee X_1) \\
&\simeq \bigvee f(X_0, \bigvee X_1); \quad \text{by hypothesis,} \\
&\simeq \bigvee \bigvee f(X_0, X_1); \quad \text{by hypothesis,} \\
&= \bigvee f(X); \quad X \text{ is directed.}
\end{aligned}
$$

Proposition 4.1.4 (Continuity of application)**.** *Define application*

$$Ap : [K \to K'] \times K \to K'$$

by the functor $Ap(f, x) = f(x)$. The Ap is continuous.

Proof. The functor $\boldsymbol{\lambda} x.f(x) = f(x)$ is continuous by continuity of f. Let $h = \boldsymbol{\lambda} f.f(x)$. Then for directed $F \subseteq [K \to K']$

$$
\begin{aligned}
h(\bigvee F) &= (\bigvee F)(x) \\
&= \bigvee \langle f(x) \mid f \in F \rangle \quad \text{by Proposition 4.1.3,} \\
&= \bigvee \langle h(f) \mid f \in F \rangle \\
&= \bigvee h(F).
\end{aligned}
$$

So h is continuous, and by Lemma 4.1.2 the functor Ap is continuous. □

Proposition 4.1.5 (Continuity of abstraction)**.** *Let $f \in [K \times K' \to K'']$. Define the functor $\hat{f}(x) = \boldsymbol{\lambda} y \in K'.f(x, y)$. Then*

1. *$\hat{f}$ is continuous, i.e., $\hat{f} \in [K \to [K' \to K'']]$;*

2. *$\boldsymbol{\lambda} f.\hat{f} : [K \times K' \to K''] \to [K \to [K' \to K'']]$ is continuous.*

Proof. (1) Let $X \subseteq K$ be directed. Then

$$\begin{aligned}\hat{f}(\bigvee X) &= \boldsymbol{\lambda} y.f(\bigvee X, y)\\ &\simeq \boldsymbol{\lambda} y.\bigvee f(X, y)\\ &\simeq \bigvee \boldsymbol{\lambda} y.f(X, y); \quad \text{by Proposition 4.1.3: takes } F = \boldsymbol{\lambda} y.f(X, y),\end{aligned}$$

(2) Let $L = \boldsymbol{\lambda} f.\hat{f}$. Then for $F \subseteq [K \times K' \to K'']$ directed

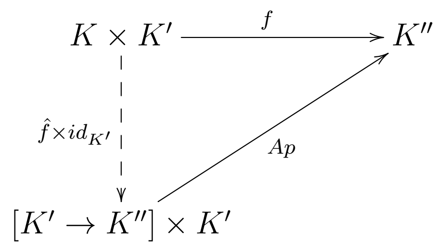
$$\begin{aligned}L(\bigvee F) &= \boldsymbol{\lambda} x \boldsymbol{\lambda} y.(\bigvee F)(x, y)\\ &= \boldsymbol{\lambda} x \boldsymbol{\lambda} y. \bigvee_{f \in F} f(x, y)\\ &\simeq \bigvee_{f \in F} \boldsymbol{\lambda} x \boldsymbol{\lambda} y.f(x, y)\\ &= \bigvee L(F).\end{aligned}$$

□

Definition 4.1.5 ($CHPO$)**.** *Define the subcategory $CHPO \subseteq CAT_\infty$ whose objects are the c.h.p.o's and the morphisms are the continuous functors.*

Proposition 4.1.6. *$CHPO$ is a Cartesian closed ∞-category.*

Proof. One has that the product of c.h.p.o's $K \times K' \in CHPO$. The singleton c.h.p.o Δ^0 is a terminal object. By 4.1.4 and 4.1.5 for each continuous functor $f : K \times K' \to K''$ there is an unique continuous functor $\hat{f} : K \to [K' \to K'']$ such that



commutes in $sSet$ (functors are morphisms of simplicial sets). Thus, the functor $K \times (-)$ has a right adjoint functor $[K \to (-)]$. □

Another alternative:

Remark 4.1.1. *Let $g \in [K \to [K' \to K'']]$. Define the functor $\breve{g}(x,y) = \boldsymbol{\lambda}(x,y) \in K \times K'.g(x)(y)$. Then*

1. *$\breve{g}$ is continuous, i.e., $\breve{g} \in [K \times K' \to K'']$;*
2. *$\boldsymbol{\lambda} g.\breve{g} : [K \to [K' \to K'']] \to [K \times K' \to K'']$ is continuous.*
3. *$\boldsymbol{\lambda} g.\breve{g}$ is an inverse of $\boldsymbol{\lambda} f.\hat{f}$.*

Hence, $CHPO$ is Cartesian closed.

Now, one shows that in $CHPO$, the projective limits exist.

Definition 4.1.6 (The Kan complex K_∞)**.** *Let $\{K_i\}_{i\in\omega}$ be countable sequence of c.h.p.o's and let $f_i \in [K_i \to K_{i+1}]$ for each $i \in \omega$.*

1. *The diagram (K_i, f_i) is called a projective (or inverse) system of c.h.p.o's.*
2. *The projective (or inverse) limit of the system (K_i, f_i) (notation $\underleftarrow{lim}(K_i, f_i)$) is the h.p.o $(K_\infty, \precsim_\infty)$, where K_∞ is the full subcategory of $\Pi_i K_i$ (ω-times Cartesian product) whose objects are the sequences $(x_i)_{i\in\omega}$ (or $x : \omega \to \bigcup_i K_i$) such that $x_i \in K_i$, $f(x_{i+1}) \simeq x_i$ and*
$$(x_i)_i \precsim_\infty (y_i)_i \ \text{ if } \forall i.\ x_i \precsim_i y_i \ \ (\text{in } K_i).$$

Proposition 4.1.7. *Let (K_i, f_i) be a projective system. Then $\underleftarrow{lim}(K_i, f_i) = K_\infty$ is c.h.p.o with*
$$\bigvee X = \boldsymbol{\lambda} i. \bigvee \langle x(i) \,|\, x \in X \rangle,$$
for directed $X \subseteq \underleftarrow{lim}(K_i, f_i)$.

Proof. If X is directed, then $\langle x(i) \,|\, x \in X\rangle$ is directed for each i. Let
$$y_i = \bigvee \langle x(i) \,|\, x \in X \rangle.$$
Then by continuity of f_i
$$\begin{aligned} f_i(y_{i+1}) &\simeq \bigvee f_i(\langle x(i+1) \,|\, x \in X \rangle) \\ &= \bigvee \langle x(i) \,|\, x \in X \rangle \\ &= y_i \end{aligned}$$
Thus, $(y_i)_i \in \underleftarrow{lim}(K_i, f_i)$. Clearly it is the supremum of X. ☐

Therefore, the c.h.p.o K_∞ satisfies the equation $X \simeq [X \to X]$ in the ∞-category $CHPO$.

Definition 4.1.7 (Compact objects and Algebraic c.h.p.o's)**.** *1. $x \in K$ is compact if for every directed $X \subseteq K$ one has*

$$x \precsim \bigvee X \quad \Longrightarrow \quad x \precsim x_0 \text{ for some } x_0 \in X.$$

2. K is an algebraic c.h.p.o if for all $x \in K$ the h.p.o $x\downarrow = \langle y \precsim x \,|\, y \text{ compact}\rangle$ is directed and $x \simeq \bigvee(x\downarrow)$.

Proposition 4.1.8. *Let K be algebraic and $f : K \to K$. Then f is continuous iff $f(x) \simeq \bigvee\langle f(e) \,|\, e \precsim x$ and e compact$\rangle$.*

Proof. ($\Rightarrow$) Let f be continuous. Then

$$\begin{aligned} f(x) &= f(\bigvee\langle e \precsim x \,|\, e \text{ compact}\rangle) \\ &\simeq \bigvee\langle f(e) \,|\, e \precsim x \text{ and } e \text{ compact}\rangle. \end{aligned}$$

($\Leftarrow$) First we check that f is monotonic. If $x \precsim y$, then

$$\langle e \precsim x \,|\, e \text{ compact}\rangle \subseteq \langle e \precsim y \,|\, e \text{ compact}\rangle,$$

hence

$$\begin{aligned} f(x) &\simeq \bigvee\langle f(e) \,|\, e \precsim x \text{ and } e \text{ compact}\rangle \\ &\precsim \bigvee\langle f(e) \,|\, e \precsim y \text{ and } e \text{ compact}\rangle \\ &\simeq f(y). \end{aligned}$$

Now let $X \subseteq K$ directed. Then

$$\begin{aligned} f(\bigvee X) &\simeq \bigvee\langle f(e) \,|\, e \precsim \bigvee X \text{ and } e \text{ compact}\rangle \\ &\precsim \bigvee\langle f(x) \,|\, x \in X\rangle; \quad \text{by compactness and monotonicity of } f \\ &\precsim f(\bigvee X); \quad \text{by monotonicity and definition of supremum.} \end{aligned}$$

Thus, $f(\bigvee X) \simeq \bigvee f(X)$. □

Proposition 4.1.9. *Let K, K' be c.h.p.o's.*

1. $(x, y) \in K \times K'$ is compact iff x and y are compact.

2. K and K' are algebraic, then $K \times K'$ is algebraic.

Proof. 1. ($\Rightarrow$) Let $X \subseteq K$ and $Y \subseteq K'$ be directed such that $x \precsim \curlyvee X$ and $y \precsim \curlyvee Y$. Hence

$$(x, y) \precsim (\bigcurlyvee X, \bigcurlyvee Y) = \bigcurlyvee (X \times Y).$$

By hypothesis, $(x, y) \precsim (x_0, y_0)$ for some $(x_0, y_0) \in X \times Y$. Thus, x and y are compact.

($\Leftarrow$) Let $X \subseteq K \times K'$ be directed such that

$$(x, y) \precsim \bigcurlyvee X = (\bigcurlyvee X_0, \bigcurlyvee X_1) = \bigcurlyvee (X_0 \times X_1).$$

By hypothesis, there are $x_0 \in X_0$ and $y_0 \in X_1$ such that $x \precsim x_0$ and $y \precsim y_0$. Since $(x_0, y_0) \precsim \curlyvee (X_0 \times X_1) = \curlyvee X$, there exists $(x_1, y_1) \in X$ such that $(x, y) \precsim (x_0, y_0) \precsim (x_1, y_1)$. Hence (x, y) is compact.

2. Let $(x, y) \in K \times K'$. Then

$$\begin{aligned} \bigcurlyvee ((x, y) \downarrow) &= \bigcurlyvee \langle (e, d) \precsim (x, y) \,|\, (e, d) \text{ compact} \rangle \\ &= \bigcurlyvee \langle (e, d) \precsim (x, y) \,|\, e \text{ and } d \text{ are compact} \rangle; \quad \text{by 1.} \\ &\simeq (\bigcurlyvee (x \downarrow), \bigcurlyvee (y \downarrow)) \\ &\simeq (x, y); \quad \text{by hypothesis.} \end{aligned}$$

□

Definition 4.1.8 (Alg). *Define the subcategory $Alg \subseteq CHPO$, whose object are the algebraic c.h.p.o's and the morphisms are continuous functors.*

Note that Acc, the ∞-category of accessible ∞-categories, is the generalization of Alg, since all directed is filtered, all supremum is a filtered colimit, and for each $X \in Alg$, X has all supremum and the full subcategory $X_c \subseteq X$ of compacts objects generates X under supremum.

4.2 HOMOTOPY DOMAIN EQUATION ON AN ARBITRARY CARTESIAN CLOSED ∞-CATEGORY

This section is a direct generalization of the traditional methods for solving domain equations in Cartesian closed categories (see (ASPERTI; LONGO, 1991) and (ABRAMSKY; JUNG, 1994)), in the sense of obtaining solutions for certain types of equations, which we call Homotopy Domain Equations in any Cartesian closed ∞-category.

Definition 4.2.1. *(ASPERTI; LONGO, 1991)*

1. *An ω-diagram in an ∞-category $\mathcal{K}$ is a diagram with the following structure:*

$$K_0 \xrightarrow{f_0} K_1 \xrightarrow{f_1} K_2 \longrightarrow \cdots \longrightarrow K_n \xrightarrow{f_n} K_{n+1} \longrightarrow \cdots$$

   (dually, one defines ω^{op}-diagrams by just reversing the arrows).

2. *An ∞-category $\mathcal{K}$ is ω-complete (ω-cocomplete) if it has limits (colimits) for all ω-diagrams.*

3. *A functor $F : \mathcal{K} \to \mathcal{K}$ is ω-continuous if it preserves (under equivalence) all colimits of ω-diagrams.*

Theorem 4.2.1. *Let $\mathcal{K}$ be an ∞-category. Let $F : \mathcal{K} \to \mathcal{K}$ be a ω-continuous (covariant) functor and take a vertex $K_0 \in \mathcal{K}$ such that there is an edge $\delta \in \mathcal{K}(K_0, FK_0)$. Assume also that $(K, \{\delta_{i,\omega} \in \mathcal{K}(F^iK_0, K)\}_{i\in\omega})$ is a colimit for the ω-diagram $(\{F^iK_0\}_{i\in\omega}, \{F^i\delta\}_{i\in\omega})$, where $F^0K_0 = K_0$ and $F^0\delta = \delta$. Then $K \simeq FK$.*

Proof. We have that $(FK, \{F\delta_{i,\omega} \in \mathcal{K}(F^{i+1}K_0, FK)\}_{i\in\omega})$ is a colimit for

$$(\{F^{i+1}K_0\}_{i\in\omega}, \{F^{i+1}\delta\}_{i\in\omega})$$

and $(K, \{\delta_{i+1,\omega} \in \mathcal{K}(F^{i+1}K_0, K)\}_{i\in\omega})$ is a cocone for the same diagram. Then, there is a unique edge (under homotopy; the space of choices is contractible) $h : FK \to K$ such that $h.F\delta_{i,\omega} \simeq \delta_{i+1,\omega}$, for each $i \in \omega$. We add to $(FK, \{F\delta_{i,\omega} \in \mathcal{K}(F^{i+1}K_0, FK)\}_{i\in\omega})$ the edge $F\delta_{0,\omega}.\delta \in \mathcal{K}(K_0, FK)$. This gives a cocone for $(\{F^iK_0\}_{i\in\omega}, \{F^i\delta\}_{i\in\omega})$ and, since $(K, \{\delta_{i,\omega} \in \mathcal{K}(F^iK_0, K)\}_{i\in\omega})$ is its colimit, there is a unique edge (under homotopy) $k : K \to FK$ such that $k.\delta_{i+1,\omega} \simeq F\delta_i$ and $k.\delta_{0,\omega} \simeq F\delta_{0,\omega}.\delta$, for each $i \in \omega$. But, $h.k.\delta_{i+1,\omega} \simeq h.F\delta_{i,\omega} \simeq \delta_{i+1,\omega}$ and $h.k.\delta_{0,\omega} \simeq \delta_{0,\omega}$, for each $i \in \omega$, thus $h.k$ is a mediating edge between the colimit $(K, \{\delta_{i,\omega} \in \mathcal{K}(F^iK_0, K)\}_{i\in\omega})$ and itself (besides I_K). Hence, by unicity (under homotopy) $h.k \simeq I_K$. In the same way, we prove that $k.h \simeq I_{FK}$, and we conclude that F has a fixed point. □

Definition 4.2.2. *An ∞-category $\mathcal{K}$ is a $(0, \infty)$-category if*

1. *every Kan complex $\mathcal{K}(A,B)$ is a c.h.p.o with a least element $0_{A,B}$ under homotopy,*

2. *composition of morphisms is a continuous operation with respect to the homotopy order,*

3. *for every f in $\mathcal{K}(A,B)$, $0_{B,C}.f \simeq 0_{A,C}$.*

Definition 4.2.3 (h-projection par). *Let $\mathcal{K}$ be a $(0,\infty)$-category, and let $f^+ : A \to B$ and $f^- : B \to A$ be two morphisms in $\mathcal{K}$. Then (f^+, f^-) is a homotopy projection (or h-projection) pair (from A to B) if $f^-.f^+ \simeq I_A$ and $f^+.f^- \precsim I_B$. If (f^+, f^-) is an h-projection pair, $f^+ \in \mathcal{K}^{HE}(A,B)$ is an h-embedding (homotopy embedding) and $f^- \in \mathcal{K}^{HP}(A,B)$ is an h-projection (homotopy projection). Where $\mathcal{K}^{HE}$ is the subcategory of $\mathcal{K}$ with the same objects and the h-embeddings as morphisms, and $\mathcal{K}^{HP}$ is the subcategory of $\mathcal{K}$ with the same objects and the h-projections as morphisms.*

Definition 4.2.4 (h-projections pair $(0,\infty)$-category.)**.** *Let $\mathcal{K}$ be a $(0,\infty)$-category. The $(0,\infty)$-category $\mathcal{K}^{HPrj}$ is the ∞-category embedding in $\mathcal{K}^{HE}$ with the same objects of $\mathcal{K}$ and h-projection pairs (f^+, f^-) as morphisms.*

Remark 4.2.1. *Every h-embedding i has unique (under homotopy) associated h-projection $j = i^R$ (and, conversely, every h-projection j has a unique (under homotopy) associated h-embedding $i = j^L$), since if there is j_0 such that $j_0.i \simeq I$ and $i.j_0 \precsim I$ (under homotopy), so $j_0 \succsim j_0.i.j \simeq j$ and $j_0 \precsim j$ (under homotopy). Thus, $j_0 \simeq j$ under homotopy (and, in the same way, we have $i_0 \simeq i$ under homotopy). $\mathcal{K}^{HPrj}$ and is equivalent to a subcategory $\mathcal{K}^{HE}$ of $\mathcal{K}$ that has h-embeddings as morphisms (as well to a subcategory $\mathcal{K}^{HP}$ of $\mathcal{K}$ which has h-projections as morphisms).*

Definition 4.2.5. *Given a $(0,\infty)$-category $\mathcal{K}$, and a contravariant functor in the first component $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$, the functor covariant $F^{+-} : \mathcal{K}^{HPrj} \times \mathcal{K}^{HPrj} \to \mathcal{K}^{HPrj}$ is defined by*

$$F^{+-}(A,B) = F(A,B),$$
$$F^{+-}((f^+, f^-), (g^+, g^-)) = (F(f^-, g^+), F(f^+, g^-)),$$

where A, B are vertices and $(f^+, f^-), (g^+, g^-)$ are n-simplexes pairs in $\mathcal{K}^{HPrj}$.

Given the ω-chain $(\{K_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ in an h-projective $(0,\infty)$-category. Let $(K, \{\gamma_i\}_{i\in\omega})$ be a limit for $(\{K_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$ in $\mathcal{K}$. Note that $\delta_i.\gamma_i \simeq \delta_{i+1}.f_i^+.f_i^-.\gamma_{i+1} \precsim \delta_{i+1}.\gamma_{i+1}$, for each $i \in \omega$. Then, $\{\delta_i.\gamma_i\}$ is an ω-chain and its colimit is $\Theta = \bigvee_{i\in\omega}\{\delta_i.\gamma_i\}$. Now for each $j \in \omega$ one has

$$\begin{aligned}\gamma_j.\Theta_j &= \gamma_j.\bigvee_{i\in\omega}\{\delta_i.\gamma_i\}\\ &\simeq \gamma_j.\bigvee_{i\geq j}\{\delta_i.\gamma_i\}\\ &\simeq \bigvee_{i\geq j}\{(\gamma_j.\delta_i).\gamma_i\}\\ &\simeq \bigvee_{i\geq j}\{f_{i,j}.\gamma_i\}\\ &\simeq \gamma_j.\end{aligned}$$

Thus, Θ is a mediating edge between the limit $(K, \{\gamma_i\}_{i\in\omega})$ for ω^{op}-diagram $(\{K_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$ and itself (besides I_K). So, by unicity (under equivalence) $\Theta \simeq I_K$. This result guarantees the proof of the following theorems of this section. All these proofs are similar to case of the 0-categories (except for uniqueness proofs, which are under homotopy) and, for that, the reader is referred to (ASPERTI; LONGO, 1991).

Theorem 4.2.2. *Let $\mathcal{K}$ be a $(0,\infty)$-category. Let $(\{K_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ be an ω-diagram in $\mathcal{K}^{HPrj}$. If $(K, \{\gamma_i\}_{i\in\omega})$ is a limit for $(\{K_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$ in $\mathcal{K}$, then $(K, \{(\delta_i, \gamma_i)\}_{i\in\omega})$ is a colimit for $(\{K_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ in $\mathcal{K}^{HPrj}$ (that is, every γ_i is a right member of a projection pair).*

Proof. Fix K_j. For each i define $f_{j,i} : K_j \to K_i$ by:

$$f_{j,i} = \begin{cases} f_i^-.f_{i+1}^- \dots f_{j-1}^- & if \quad i < j,\\ I_{K_j} & if \quad i = j,\\ f_{i-1}^+ \dots f_{j+1}^+.f_j^+ & if \quad i > j.\end{cases}$$

$(K_i, \{f_{j,i}\}_{i\in\omega})$ is a cone for $(\{K_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$, since $f_i^-.f_{j,i+1} \simeq f_{j,i}$. Thus there is a unique morphism (under homotopy) $\delta_j : K_j \to K$ such that $\gamma_i.\delta_j \simeq f_{j,i}$ for each $i \in \omega$. If $i = j$, $\gamma_j.\delta_j \simeq I_{K_j}$.

Since $\Theta = \bigvee_{i\in\omega}\{\delta_i.\gamma_i\} \simeq I_K$, $\delta_i.\gamma_i \precsim I_K$ for each $i \in \omega$. Thus, (δ_i, γ_i) is an h-projection pair for each $i \in \omega$.

One still has to check that $(f_j^+, f_j^-).(\delta_{j+1}, \gamma_{j+1}) \simeq (\delta_j, \gamma_j)$. We have that $f_j^-.\gamma_{j+1} \simeq \gamma_j$ by the definition of a cone in $\mathcal{K}$. In order of to prove that $\delta_{j+1}.f_j^+ \simeq \delta_j$, note that $\gamma_i.(\delta_{j+1}.f_j^+) \simeq$

$f_{j+1,i}.f_j^+ \simeq f_{j,j} \simeq I_{K_j} \simeq \gamma_i.\delta_j$, by unicity (under homotopy) of $\delta_j : K_j \to K$, $\delta_{j+1}.f_j^+ \simeq \delta_j$. Thus, $(K, \{(\delta_i, \gamma_i)\}_{i\in\omega})$ is a cone in $\mathcal{K}^{HPrj}$.

One proves next that $(K, \{(\delta_i, \gamma_i)\}_{i\in\omega})$ is a colimit. Let $(K', \{(\delta'_i, \gamma'_i)\}_{i\in\omega})$ be another cocone for $(\{K_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$. That is, for each $i \in \omega$:

$$\delta'_i.\gamma_i \simeq \delta'_{i+1}.f_i^+.f_i^-\gamma_{i+1} \precsim \delta'_{i+1}.\gamma_{i+1}$$
$$\delta_i.\gamma'_i \simeq \delta_{i+1}.f_i^+.f_i^-\gamma'_{i+1} \precsim \delta_{i+1}.\gamma'_{i+1}.$$

Define thus:

$$h = \bigvee_{i\in\omega}\{\delta'_i.\gamma_i\} : K \to K'$$
$$k = \bigvee_{i\in\omega}\{\delta_i.\gamma'_i\} : K' \to K.$$

Observe that (h, k) is an h-projection pair, since:

$$\begin{aligned} k.h &= \bigvee_{i\in\omega}\{\delta_i.\gamma'_i\}.\bigvee_{i\in\omega}\{\delta'_i.\gamma_i\} \\ &\simeq \bigvee_{i\in\omega}\{\delta_i.(\gamma'_i.\delta'_i).\gamma_i\} \\ &\simeq \bigvee_{i\in\omega}\{\delta_i.\gamma_i\} \\ &\simeq \Theta = I_K \end{aligned}$$

and

$$\begin{aligned} h.k &= \bigvee_{i\in\omega}\{\delta'_i.\gamma_i\}.\bigvee_{i\in\omega}\{\delta_i.\gamma'_i\} \\ &\simeq \bigvee_{i\in\omega}\{\delta'_i.(\gamma_i.\delta_i).\gamma'_i\} \\ &\simeq \bigvee_{i\in\omega}\{\delta'_i.\gamma'_i\} \\ &\precsim I_{K'} \end{aligned}$$

Moreover, (h, k) is a mediating morphism between $(K, \{(\delta_i, \gamma_i)\}_{i\in\omega})$ and $(K', \{(\delta'_i, \gamma'_i)\}_{i\in\omega})$, since for each $i \in \omega$:

$$\begin{aligned}(h,k).(\delta_j,\gamma_j) &= (h.\delta_j\ ,\ \gamma_j.k)\\ &\simeq (\bigvee_{i\in\omega}\{\delta'_i.\gamma_i\}.\delta_j\ ,\ \gamma_j.\bigvee_{i\in\omega}\{\delta_i.\gamma'_i\})\\ &\simeq (\bigvee_{i\geq j}\{\delta'_i.\gamma_i.\delta_j\}\ ,\ \bigvee_{i\geq j}\{\gamma_j.\delta_i.\gamma'_i\})\\ &\simeq (\bigvee_{i\geq j}\{\delta'_i.f_{j,i}\}\ ,\ \bigvee_{i\geq j}\{\gamma_j.f_{i,j}\})\\ &\simeq (\delta'_j,\gamma'_j).\end{aligned}$$

Thus, for each $j \in \omega$, $\mathcal{K}^{HPrj}_{K_j/}((\delta_j,\gamma_j)\ ,\ (\delta'_j,\gamma'_j)) \neq \emptyset$, that is $\mathcal{K}^{HP}_{/K_j}(\gamma'_j,\gamma_j)$ and $\mathcal{K}^{HE}_{K_j/}(\delta_j,\delta'_j)$ spaces that are not empty.

Since $\mathcal{K}^{HP}_{/K_j}(\gamma'_j,\gamma_j) \subseteq \mathcal{K}_{/K_j}(\gamma'_j,\gamma_j)$ and by hypothesis γ_j is an object final in $\mathcal{K}_{/K_j}$ for all $j \in \omega$, then $\mathcal{K}^{HP}_{/K_j}(\gamma'_j,\gamma_j)$ and $\mathcal{K}^{HE}_{K_j/}(\delta_j,\delta'_j)$ are contractible for each $j \in \omega$. Thus, the Kan complex $\mathcal{K}^{HPrj}_{K_j/}((\delta_j,\gamma_j)\ ,\ (\delta'_j,\gamma'_j))$ is contractible for each $j \in \omega$, that is, (h,k) is unique (under homotopy) in the mediating morphism between $(K,\{(\delta_i,\gamma_i)\}_{i\in\omega})$ and $(K',\{(\delta'_i,\gamma'_i)\}_{i\in\omega})$. □

Therefore, the following corollary is an immediate consequence of the previous theorem.

Corollary 4.2.1. *The cocone $(K,\{(\delta_i,\gamma_i)\}_{i\in\omega})$ for the ω-chain $(\{K_i\}_{i\in\omega},\{(f_i^+,f_i^-)\}_{i\in\omega})$ in $\mathcal{K}^{HPrj}$ is universal (a cocone colimit) iff $\Theta = \bigvee_{i\in\omega}\delta_i.\gamma_i \simeq I_K$.*

Definition 4.2.6 (Locally monotonic)**.** *Let $\mathcal{K}$ be a $(0,\infty)$-category. A functor $F:\mathcal{K}^{op}\times\mathcal{K}\to\mathcal{K}$ is locally h-monotonic if it is monotonic on the Kan complexes of 1-simplexes, i.e., for $f,f'\in\mathcal{K}^{op}(A,B)$ and $g,g'\in\mathcal{K}(C,D)$ one has*

$$f \precsim f'\ ,g \precsim g' \Longrightarrow F(f,g) \precsim F(f',g').$$

Proposition 4.2.1. *If $F:\mathcal{K}^{op}\times\mathcal{K}\to\mathcal{K}$ is locally h-monotonic and (f^+,f^-), (g^+,g^-) are h-projection pairs, then $F^{+-}((f^+,f^-),(g^+,g^-))$ is also an h-projection pair.*

Proof. By definition $F^{+-}((f^+,f^-),(g^+,g^-)) = (F(f^-,g^+),F(f^+,g^-))$. Then

$$\begin{aligned}F(f^+,g^-).F(f^-,g^+) &\simeq F((f^+,g^-).(f^-,g^+))\\ &= F(f^-.f^+,g^-.g^+)\\ &\simeq F(id,id)\\ &\simeq id\end{aligned}$$

and

$$
\begin{aligned}
F(f^-, g^+).F(f^+, g^-) &\simeq F((f^-, g^+).(f^+, g^-)) \\
&= F(f^+.f^-, g^+.g^-) \\
&\precsim F(id, id) \\
&\simeq id.
\end{aligned}
$$

□

Definition 4.2.7 (Locally continuous)**.** *Let $\mathcal{K}$ be a $(0,\infty)$-category. A $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ is locally continuous if it is ω-continuous on the Kan complexes of 1-simplexes. That is, for every directed diagram $\{f_i\}_{i\in\omega}$ in $\mathcal{K}^{op}(A, B)$, and every directed diagram $\{g_i\}_{i\in\omega}$ in $\mathcal{K}(C, D)$, one has*

$$F(\curlyvee_{i\in\omega}\{f_i\}, \curlyvee_{i\in\omega}\{g_i\}) \simeq \curlyvee_{i\in\omega} F(f_i, g_i).$$

Remark 4.2.2. *If F is locally continuous, then it is also locally monotonic.*

Theorem 4.2.3. *Let $\mathcal{K}$ be a $(0,\infty)$-category. Let also $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ be a locally continuous functor. Then the functor $F^{+-} : \mathcal{K}^{HPrj} \times \mathcal{K}^{HPrj} \to \mathcal{K}^{HPrj}$ is ω-continuous.*

Proof. Let $(\{A_i\}_{i\in\omega}, \{(f_i^+, f_i^-)\}_{i\in\omega})$ and $(\{B_i\}_{i\in\omega}, \{(f_i^+, f_i^-)\}_{i\in\omega})$ be two ω-chains in $\mathcal{K}^{HPrj}$ and let $(A, \{(\rho_i^+, \rho_i^-)_{i\in\omega}\})$ and $(B, \{(\sigma_i^+, \sigma_i^-)_{i\in\omega}\})$ be the respective limits. We have that

$$
\begin{aligned}
\Theta &= \bigvee_{i\in\omega}\{\rho_i^+.\rho_i^-\} \simeq I_A \\
\Psi &= \bigvee_{i\in\omega}\{\sigma_i^+.\sigma_i^-\} \simeq I_B.
\end{aligned}
$$

One must show that

$$(F^{+-}(A, B), \{F^{+-}(\rho_i^+, \rho_i^-).F^{+-}(\sigma_i^+, \sigma_i^-)\}_{i\in\omega}) = (F(A, B), \{(F(\rho_i^-, \sigma_i^+),\ F(\rho_i^+, \sigma_i^-))\}_{i\in\omega})$$

is a colimit for the ω-chain $(\{F^{+-}(A_i, B_i)\}_{i\in\omega}, \{F^{+-}((f_i^+, f_i^-), (g_i^+, g_i^-))\}_{i\in\omega})$.

It is clearly a cone, by the property of functors. We show that it is universal by proving that $\curlyvee_{i\in\omega}\{F(\rho_i^-, \sigma_i^+).F(\rho_i^+, \sigma_i^-)\} \simeq I_{F(A,B)}$; according to the Corollary 4.2.1. Computing we have

$$\bigvee_{i\in\omega}\{F(\rho_i^-,\sigma_i^+).F(\rho_i^+,\sigma_i^-)\} \simeq \bigvee_{i\in\omega}\{F(\rho_i^+.\rho_i^-,\sigma_i^+.\sigma_i^-)\}$$
$$\simeq F(\bigvee_{i\in\omega}(\rho_i^+.\rho_i^-),\bigvee_{i\in\omega}(\sigma_i^+.\sigma_i^-))$$
$$\simeq F(\Theta,\Psi)$$
$$\simeq F(I_A,I_B)$$
$$= I_{F(A,B)}.$$

□

Remark 4.2.3. *Let $\mathcal{K}$ be a cartesian closed $(0,\infty)$-category, ω^{op}-complete and with final object. Since the exponential functor $\Rightarrow$: $\mathcal{K}^{op}\times\mathcal{K}\to\mathcal{K}$ and the diagonal functor $\Delta:\mathcal{K}\to\mathcal{K}\times\mathcal{K}$ are locally continuous, by the Theorem 4.2.3, the associated functors*

$$(\Rightarrow)^{+-}:\mathcal{K}^{HPrj}\times\mathcal{K}^{HPrj}\to\mathcal{K}^{HPrj},\quad (\Delta)^{+-}:\mathcal{K}^{HPrj}\to\mathcal{K}^{HPrj}\times\mathcal{K}^{HPrj}$$

are ω-continuous. But composition of ω-continuous functors is still an ω-continuous functor. Thus, the functor

$$F=(\Rightarrow)^{+-}.(\Delta)^{+-}:\mathcal{K}^{HPrj}\to\mathcal{K}^{HPrj},$$

is ω-continuous. By Theorem 4.2.1 the functor F has a fixed point, that is, there is a vertex $K\in\mathcal{K}$ such that $K\simeq(K\Rightarrow K)$. The ∞-category of the fixed points of F is denoted by $Fix(F)$.

4.3 HOMOTOPY DOMAIN EQUATION ON $Kl(P)$

In this section we consider $Kl(P)$ of (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022b) to be an ∞-category, in order to apply the homotopy domain theory of the previous section.

For the next proposition, let $\mathcal{P}r_\kappa^L$ be the subcategory of CAT_∞ whose objects are κ-compactly generated ∞-categories and whose morphisms are functors which preserve small colimits and κ-compact objects. Also, let $CAT_\infty^{Rex(\kappa)}$ denote the subcategory of CAT_∞ whose objects are ∞-categories which admit κ-small colimits and whose morphisms are functors which preserve κ-small colimits. Finally, let $L^*:Cat_\infty\to Cat_\infty^{Rex(\kappa)}$ the functor which closes an ∞-category to an ∞-category which admits κ-small colimits, so $L^*A\simeq(PA)^\kappa$.

Proposition 4.3.1. *The ∞-category $Kl(P)$ admits limits for ω^{op}-diagrams in $\mathcal{P}r^L_\kappa$.*

Proof. Given an ω^{op}-diagram of pro-functors in $Kl(P)$ such that this is associated to ω^{op}-diagram p of functors

$$PK_0 \xleftarrow{f_0} PK_1 \xleftarrow{f_1} PK_2 \xleftarrow{f_2} \cdots$$

in the ∞-category $\mathcal{P}r^L_\kappa$, which admits all the small limits. Then, there is a limit $(T, \{\gamma_k\}_{k\in\omega})$ in $\mathcal{P}r^L_\kappa$ for this ω^{op}-diagram p. Since T is presentable, so for any regular cardinal κ, the full subcategory of κ-compacts T^κ is essentially small (LURIE, 2009). Thus $(T^\kappa, \{\gamma_k.i\}_{k\in\omega})$ is a cone for the ω^{op}-diagram in $Kl(P)$, where i is the inclusion functor $T \supseteq T^\kappa$.

Let $(T', \{\tau_k\}_{k\omega})$ be another cone for $(\{PK_k\}_{k\in\omega}, \{f_k\}_{k\in\omega})$ in $Kl(P)$. Then, $(PT', \{\tau_k^\# \in \mathcal{P}r^L_\kappa(PT', PK_k)\}_{i\in\omega})$ is a cone from ω^{op}-diagram p in $\mathcal{P}r^L_\kappa$. Thus, there is a unique edge (under homotopy) $h : PT' \to T$ in $\mathcal{P}r^L_\kappa$ such that $\gamma_k.h \simeq \tau_k^\#$, for each $k \in \omega$. Applying the full faithful functor $(-)^\kappa : \mathcal{P}r^L_\kappa \to CAT_\infty^{Rex(\kappa)}$ (LURIE, 2009), one has that $h^\kappa \in Fun^\kappa(L^*T', T^\kappa) \simeq Fun(T', T^\kappa)$ is the unique edge (under homotopy) such that $(\gamma_k.i).h^\kappa \simeq \tau_k^\#.j$, for each $k \in \omega$ with $j : L^*T' \to PT'$ being a Yoneda embedding. Thus, there is a unique (under homotopy) $h' : T' \to T^\kappa$ such that $(\gamma_k.i).h' \simeq \tau_k$. ☐

Since each $Fun(A, PB)$ has initial object, any $Fun(A, PB)(F, G)$ is contractible or empty. Thus, $K(P)(A, B) \subseteq Fun(A, PB)$ admits a homotopy partial order (h.p.o.). For the following theorem, denote by $0_{PA,PB}$ in $Fun^L(PA, PB)$ as the constant functor in empty Kan complex $\emptyset$, that is

$$0_{PA,PB}f := \boldsymbol{\lambda} x \in B.\emptyset$$

Theorem 4.3.1. *The ∞-category $Kl(P)$ is a $(0, \infty)$-category.*

Proof.

1. Since PB is presentable, $Fun[A, PB]$ is presentable, thus $\langle Kl(P)(A, B), \precsim\rangle$ is complete. On the other hand, let F be an object in $Fun^L(PA, PB)$, then

$$0_{PA,PB}fx = (\boldsymbol{\lambda} x \in B.\emptyset)x = \emptyset \subseteq Ffx,$$

for every object f in PA and x in B. Thus, $0_{PA,PB}$ is the least element (under homotopy) in $Fun^L(PA, PB)$, i.e., $0_{A,B}$ is the least element in $Kl(P)(A, B)$.

2. Let $\{p_i\}_{i\in\omega}$ be a non-decreasing chain of morphisms in $Fun^L(PA, PB)$. By (1), the colimit $\curlyvee_{i\in\omega} p_i$ exists.

i. First let's prove that for each object z in PA, its colimit is given by

$$(\bigvee_{i\in\omega} p_i)z \simeq \bigvee_{i\in\omega} p_i z.$$

Since $p_i \precsim \bigvee_{i\in\omega} p_i$, given a vertex z of PA, by Definition 4.1.1, $p_i z \precsim (\bigvee_{i\in\omega} p_i)z$. On the other hand, $\bigvee_{i\in\omega} p_i z$ is the supremum (under equivalence) of $\{p_i z\}$, hence

$$p_i z \precsim \bigvee_{i\in\omega} p_i z \precsim (\bigvee_{i\in\omega} p_i)z.$$

Let $qz := \bigvee_{i\in\omega} p_i z$, by Definition 4.1.1

$$p_i \precsim q \precsim \bigvee_{i\in\omega} p_i,$$

but $\bigvee_{i\in\omega} p_i$ is the supremum (under equivalence) of $\{p_i\}_{i\in\omega}$, thus $\bigvee_{i\in\omega} p_i \simeq q$, by Definition 4.1.1,

$$(\bigvee_{i\in\omega} p_i)z \simeq \bigvee_{i\in\omega} p_i z.$$

ii. Now let's prove that the composition is continuous on the right. Take a functor F in $Fun_\kappa(PA', PA)$ and a vertex z of PA', then Fz is a vertex of PA, by (i) we have

$$\begin{aligned}((\bigvee_{i\in\omega} p_i).F)z &\simeq (\bigvee_{i\in\omega} p_i)(Fz)\\ &\simeq \bigvee_{i\in\omega} p_i(Fz)\\ &\simeq \bigvee_{i\in\omega} (p_i.F)z\\ &\simeq (\bigvee_{i\in\omega} p_i.F)z,\end{aligned}$$

since $Kl(P)$ does have enough points, it follows $Fun^L(PA', PB)$,

$$(\bigvee_{i\in\omega} p_i).F \simeq \bigvee_{i\in\omega} p_i.F.$$

iii. Finally let's prove that the composition is continuous on the right. Let G be a functor in $Fun^L(PB, PC)$ and z an object in PA. By (i) and the continuity of G, we have

$$\begin{aligned}(G.\bigvee_{i\in\omega} p_i)z &\simeq G((\bigvee_{i\in\omega} p_i)z)\\ &\simeq G(\bigvee_{i\in\omega} p_i z)\\ &\simeq \bigvee_{i\in\omega} G(p_i z)\\ &\simeq \bigvee_{i\in\omega}(G.p_i)z\\ &\simeq (\bigvee_{i\in\omega} G.p_i)z,\end{aligned}$$

since $Kl(P)$ does have enough points, it follows

$$G.\bigvee_{i\in\omega} p_i \simeq \bigvee_{i\in\omega} G.p_i.$$

3. Let F be an object in $Fun^L(PA, PB)$ and f in PA, hence

$$(0_{PB,PC}.F)f = 0_{PB,PC}(Ff) = \boldsymbol{\lambda} x \in C.\emptyset = 0_{PA,PC}f,$$

that is, $0_{PB,PC}.F = 0_{PA,PC}$.

□

Proposition 4.3.2. *For any small ∞-category A, there is an h-projection from A to $A \Rightarrow A$ in $Kl(P)$.*

Proof. We have that there is a diagonal functor

$$\delta : PA \to [PA, PA]^L \simeq P(A^{op} \times A) = P(A \Rightarrow A),$$

where $[PA, PA]^L$ is the ∞-category of the functors which preserve small colimits or left adjoints. Since δ preserves all small colimits, by the Adjoint Functor Theorem, δ has a right adjoint γ (LURIE, 2009). One the other hand, the diagonal functor δ is an h-embedding, then the unit is an equivalence, i.e., $\gamma.\delta \simeq I_{PA}$ and the counit is an h.p.o., that is, $\delta.\gamma \precsim I_{P(A\Rightarrow A)}$ in the c.h.p.o. $Kl(P)(A \Rightarrow A, A \Rightarrow A)$. Thus, (δ, γ) is a projection pair of A to $A \Rightarrow A$ in $Kl(P)$, which we call the diagonal projection. □

Proposition 4.3.3. *There is a reflexive non-contractible object in $Kl(P)$.*

Proof. First the trivial Kan complex Δ^0 is a fixed point from endofunctor $FX = (X \Rightarrow X)$ on $Kl(P)$, since

$$P\Delta^0 \simeq P(\Delta^0 \times \Delta^0) = P(\Delta^0 \Rightarrow \Delta^0).$$

that is, $\Delta^0 \simeq (\Delta^0 \Rightarrow \Delta^0) = F\Delta^0$ in $Kl(P)$. Let's suppose that all the Kan complexes in $Fix(F)$ are equivalent to Δ^0. Since $Kl(P)$ contains all the small ∞-categories, then there is a small non-contractible Kan complex K_0, i.e.,

$$\Delta^0 \prec K_0 \overset{(\delta_0,\gamma_0)}{\longrightarrow} FK_0$$

in $(Kl(P))^{HPrj}$ for all $n \in \omega$, where δ_0 is the diagonal functor, which has its equivalent functor in $\mathcal{P}^R_\kappa = (\mathcal{P}^L_\kappa)^{op}$ (LURIE, 2009). Since $(\{F^i K_0\}_{i\in\omega}, \{F^i(\gamma_0)\}_{i\in\omega})$ is an ω^{op}-diagram in $\mathcal{P}^L_\kappa$, by Proposition 4.3.1 and Theorem 4.2.2, there is a colimit $(K, \{(\delta_{i,\omega}, \gamma_{\omega,i})_{i\in\omega}\})$ in $(Kl(P))^{HPrj}$ for the ω-diagram $(\{F^i K_0\}_{i\in\omega}, \{F^i(\delta_0, \gamma_0)\}_{i\in\omega})$. Thus,

$$\Delta^0 \prec K_0 \overset{\delta_{0,\omega}}{\hookrightarrow} K \in Fix(F)$$

which is a contradiction. □

Definition 4.3.1. *Let X be a Kan complex. Define $\pi_0(X) := \pi_0(|X|)$ and $\pi_n(X, x) := \pi_n(|X|, x)$ for $n > 0$, with $|\ | : Kan \to Top$ be the functor of geometric realization from the category of the Kan complexes to the category of the topological spaces.*

The fact that a Kan complex X is not contractible does not imply that every vertex $x \in X$ contains relevant information, that is, the higher fundamental groups $\pi_n(|X|, x)$ are not trivial, nor that it contains holes in all the higher dimensions. To guarantee the existence of non-trivial Kan complexes as higher λ-models, we present the following definition.

Definition 4.3.2 (Non-trivial Kan complex)**.** *A small Kan complex X is non-trivial if*

1. *$\pi_0(X)$ is infinite.*

2. *for each $n \geq 1$, there is a vertex $x \in X$ such that $\pi_n(X, x) \ncong *$.*

3. *for each vertex x of some k-simplex in X, with $k \geq 2$, there is $n \geq 1$ such $\pi_n(X, x) \ncong *$.*

Example 4.3.1. *For each $n \geq 0$, let the Kan complex $B^n \cong \partial\Delta^n_\bullet$ (isomorphic). Where $\Delta^n_\bullet$ have the same vertices and faces of Δ^n but invertible 1-simplexes. Define the non-trivial Kan complex B_0 as the disjoint union:*

$$B_0 = \coprod_{n<\omega} B^n.$$

Note that B^n is "similar"to sphere S^{n-1}. Furthermore, $\pi_{n-1}(B^n) \ncong *$ for all $n \geq 2$, and there is $k \geq n$ such that $\pi_k(B^n) \ncong *$ for each $n \geq 3$ (HATCHER, 2001).

Example 4.3.2. *For each $n \geq 2$, let D^n be a Kan complex, such that its k-th face set the isomorphism*

$$d_k D^n \cong \begin{cases} \Delta_\bullet^{n-1} & \textit{if} \;\; k = 0, \\ \partial \Delta_\bullet^{n-1} & \textit{if} \;\; 1 \leq k \leq n, \end{cases}$$

Define the non-trivial Kan complex D_0 as the disjoint union:

$$D_0 = \coprod_{n<\omega} D^n.$$

Note that each Kan complex D^n, with $n \geq 2$, is "similar"to the sphere S^{n-1} with n holes. Besides that its higher groups have the same properties from Example 4.3.1, we also have the additional property $\pi_{n-2}(D^n) \ncong *$ for all $n \geq 3$.

Example 4.3.3. *For each $n \geq 2$, let E^n be a Kan complex, such that its k-th face set the isomorphism*

$$d_k E^n \cong \begin{cases} \partial^{n-2} \Delta_\bullet^{n-1} & \textit{if} \;\; 0 \leq k \leq 2, \\ \partial^{n-k} \Delta_\bullet^{n-1} & \textit{if} \;\; 3 \leq k \leq n. \end{cases}$$

Define the non-trivial Kan complex E_0 as the disjoint union

$$E_0 = \coprod_{n<\omega} E^n.$$

Note that E_0 has more information than the non-trivial Kan complexes B_0 and D_0 from the previous examples, in the sense that for all $n \geq 2$, it satisfies the property $\pi_k(E^n) \ncong *$ for each $1 \leq k \leq n-1$.

Proposition 4.3.4. *For every non-trivial Kan complex K_0, there exists a non-trivial Kan complex $K \in Fix(F)$ above K_0.*

Proof. Let K_0 be a non-trivial Kan complex and $(K, \{(\delta_{i,\omega}, \delta_{\omega,i})\}_{i \in \omega})$ the colimit from ω-diagram $(\{F^i K_0\}_{i \in \omega}, \{F^i(\delta_0, \gamma_0)\}_{i \in \omega})$ in $(Kl(P))^{HPrj}$, with (δ_0, γ_0) the first projection from K_0 to $K_1 := FK_0$. Let $z = (x, y)$ be a vertex of some k-simplex in K_{i+1}, with $k \geq 2$. By induction on i, there is $n_1, n_2 \geq 1$, such that $\pi_{n_1}(K_i, x), \pi_{n_2}(K_i, x) \ncong *$. For any $n \in \{n_1, n_2\}$, one has

$$\begin{aligned}\pi_n(K_{i+1}, z) &= \pi_n(K_i \Rightarrow K_i, z)\\ &= \pi_n(K_i \times K_i, (x, y))\\ &\cong \pi_n(K_i, x) \times \pi_n(K_i, y)\\ &\ncong *\end{aligned}$$

Given any vertex y of some k-simplex in K, with $k \geq 2$. There is $i \geq 0$ and $x \in K_i$ such that $\delta_{i,\omega} x \simeq y$. Since there is $\pi_n(K_i, x) \ncong *$, then

$$\pi_n(K, y) \cong \pi_n(K, \delta_{i,\omega} x) \ncong *.$$

□

Definition 4.3.3 (Non-Trivial Homotopy λ-Model)**.** *Let $\mathcal{K}$ be a Cartesian closed ∞-category with enough points. A Kan complex $K \in \mathcal{K}$ is a non-trivial homotopy λ-model if K is a reflexive non-trivial Kan complex.*

Note that every non-trivial homotopy λ-model is a homotopic λ-model as defined in (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022a) and (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022b), which only captures information up to dimension 2 (for equivalences (2-paths) of 1-paths between points). While the non-trivial homotopy λ-models have no dimensional limit to capture relevant information, and therefore, those can generate a richer higher λ-calculus theory.

Example 4.3.4. *Given the non-trivial Kan complexes B_0, D_0 and E_0 of the Examples 4.3.1, 4.3.2 and 4.3.3 respectively. Starting from the diagonal projection as the initial projection pair, these initial objects will generate the respective non-trivial Kan complexes B, D and E in $Fix(F)$. One can see that the non-trivial homotopy λ-model E has more information than the homotopic λ-models B and D.*

5 ARBITRARY SYNTACTICAL HOMOTOPIC λ-MODELS

In this chapter we discuss some consequences of the arbitrary syntactic homotopic lambda models introduced in (MARTÍNEZ-RIVILLAS; QUEIROZ, 2022a), which correspond to a direct generalization (2-dimensional) of the traditional structured set models of a Cartesian closed category (1-dimensional) as can be seen in (BARENDREGT, 1984) and (HINDLEY; SELDIN, 2008).

Notation 5.0.1. *For K be a Kan simplex and $n \geq 0$, let $K_n = Fun(\Delta^n, K)$ be the Kan complex of the n-simplexes.*

Denote by $\Omega_n(K, a)$ the class of all the spheres $\partial\Delta^n \to K$ with initial vertex $a \in K$.

Let Var be the set of all variables of λ-calculus, for all $n \geq 0$, each assignment ρ : $Var \to K_n$ ($\rho(t)$ is a n-simplex of K, for each $t \in Var$), $x \in Var$ and $f \in K_m$. Denote by $[f/x]\rho : Var \to K$ the assignment

$$([f/x]\rho)(t) = \begin{cases} f & \text{if } \; t = x \\ \rho(t) & \text{if } \; t \neq x. \end{cases}$$

Definition 5.0.1 (Syntactical Homotopic λ-model)**.** *A homotopic λ-model is a triple $\langle K, \bullet, [\![\,]\!]\rangle$, where K is a Kan complex, $\bullet : K \times K \to K$ is a functor, and $[\![\,]\!]$ is a mapping which assigns to λ-term M and each assignment $\rho : Var \to K_n$, an n-simplex $[\![P]\!]_\rho$ in K for each $n \geq 0$ such that*

1. $[\![x]\!]_\rho = \rho(x)$;
2. $[\![MN]\!]_\rho = [\![M]\!]_\rho \bullet [\![N]\!]_\rho$;
3. *For each $f \in K_n$, there is a limit $\beta_f : [\![\lambda x.M]\!]_\rho \bullet f \to [\![M]\!]_{[f/x]\rho}$ from $[\![M]\!]_{[f/x]\rho} \in K_n$;*
4. $[\![M]\!]_\rho = [\![M]\!]_\sigma$ *if $\rho(x) = \sigma(x)$ for $x \in FV(M)$;*
5. $[\![\lambda x.M(x)]\!]_\rho = [\![\lambda y.M(y)]\!]_\rho$ *if $y \notin FV(M)$;*
6. *if $(\forall a \in K_0)(\forall n \geq 1)(\forall \omega \in \Omega_n(K, a))\left([\![M]\!]_{[\omega/x]\rho} = [\![N]\!]_{[\omega/x]\rho}\right)$, then*

   $[\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho$.

The homotopic model $\langle K, \bullet, [\![\,]\!]\rangle$ is an extensional syntactical homotopic model if it satisfies the additional property: there is a colimit $\eta : [\![M]\!]_\rho \to [\![\lambda x.Mx]\!]_\rho$ from $[\![M]\!]_\rho \in K_n$ with $x \notin FV(M)$.

Remark 5.0.1. *Note that the condition (3) of the Definition 5.0.1, by the Homotopy Extension Lifting Property (LURIE, 2009), if $x \in FV(P)$ any cone $\beta_f : [\![\lambda x.P]\!]_\rho \bullet f \to [\![P]\!]_{[f/x]\rho}$ in $K_{/[\![P]\!]_{[f/x]\rho}}$ is a limit from n-simplex $[\![P]\!]_{[f/x]\rho}$. Since K_n is a Kan complex, by the theorem mentioned above, the induced functor $Fun(\Delta^1, K_n) \to Fun(\Delta^0, K_n)$ is a trivial fibration, hence the fibre $(K_n)_{/[\![P]\!]_{[f/x]\rho}}$ is contractible, that is $K_{/[\![P]\!]_{[f/x]\rho}}$ is contractible. Thus the condition (3) is reduced to the existence of a cone $\beta_f : [\![\lambda x.P]\!]_\rho \bullet f \to [\![P]\!]_{[f/x]\rho}$ in $(K_n)_{/[\![P]\!]_{[f/x]\rho}}$.*

Definition 5.0.2. *Let $\mathfrak{M} = \langle K, \bullet, [\![\,]\!]\rangle$ be a syntactic homotopic λ-model. The notion of satisfaction in $\mathfrak{M}$ is defined as*

$$\mathfrak{M}, \rho \models M = N \iff [\![M]\!]_\rho \simeq [\![N]\!]_\rho$$

$$\mathfrak{M} \models M = N \iff \forall \rho\, (\mathfrak{M}, \rho \models M = N)$$

Lemma 5.0.1. *Let $\mathfrak{M} = \langle K, \bullet, [\![\,]\!]\rangle$ be a syntactical homotopic λ-model. Then, for all M, N, x, $n \geq 0$ and $\rho : Var \to K_n$,*

(i) $[\![[z/x]M]\!]_\rho = [\![M]\!]_{[\rho(z)/x]\rho}$,

(ii) *if* $[\![[N/x]M]\!]_\rho = [\![M]\!]_{[[\![N]\!]_\rho/x]\rho}$, *then* $[\![\lambda y.[N/x]M]\!]_\rho = [\![\lambda y.M]\!]_{[[\![N]\!]_\rho/x]\rho}$,

(iii) $[\![[N/x]M]\!]_\rho = [\![M]\!]_{[[\![N]\!]_\rho/x]\rho}$.

Proof. (i) One has that,

$$[\![[z/x]M]\!]_\rho = [\![[z/x]M]\!]_{[\rho(z)/z]\rho} \xleftarrow{\beta_{\rho(z)}} [\![\lambda z.[z/x]M]\!]_\rho \bullet \rho(z) = [\![\lambda x.M]\!]_\rho \bullet \rho(z);$$
$$[\![M]\!]_{[\rho(z)/x]\rho} \xleftarrow{\beta_{\rho(z)}} [\![\lambda x.M]\!]_\rho \bullet \rho(z)$$

That is, $[\![[z/x]M]\!]_\rho = [\![M]\!]_{[\rho(z)/x]\rho}$.

(ii) First suppose $x \notin FV(N)$. Let $y \neq x$ and $y \notin FV(N)$. For $\rho' = [[\![N]\!]_\rho/x]\rho$ and any $\omega \in \Omega_n(K, a)$, with an arbitrary vertex $a \in K$ and any $n \geq 1$, one has

$$\begin{aligned}[\![[N/x]M]\!]_{[\omega/y]\rho'} &= [\![[N/x]M]\!]_{[\omega/y]\rho}\\ &= [\![M]\!]_{[\omega/y][[\![N]\!]_\rho/x]\rho}; \quad \text{by hypothesis}\\ &= [\![M]\!]_{\omega/y]\rho'}.\end{aligned}$$

By Definition 5.0.1 (6), $[\![\lambda y.[N/x]M]\!]_{\rho'} = [\![\lambda y.M]\!]_{\rho'}$, hence

$$[\![\lambda y.[N/x]M]\!]_\rho = [\![\lambda y.[N/x]M]\!]_{\rho'} = [\![\lambda y.M]\!]_{[[\![N]\!]_\rho/x]\rho}.$$

If $x \in FV(N)$, the proof is identical to (BARENDREGT, 1984, p.103).

(iii) Follows easily by induction on the λ-term M. □

Notation 5.0.2. *Let $\lambda\beta$ be the theory of β-equality. The notation $\lambda\beta \vdash M = N$ means that the equality $M = N$ is proved from the theory $\lambda\beta$.*

Theorem 5.0.1. *Let $\mathfrak{M} = \langle K, \bullet, [\![\ ]\!] \rangle$ be a syntactical homotopic λ-model. Then*

$$\lambda\beta \vdash M = N \implies \mathfrak{M} \models M = N.$$

Proof. By induction on the length of proof. For the axiom $(\lambda x.M)N = [N/x]M$ we proceed

$$\begin{aligned} [\![(\lambda x.M)N]\!]_\rho &= [\![\lambda x.M]\!]_\rho \bullet [\![N]\!]_\rho \\ &\xrightarrow{\beta_{[\![N]\!]_\rho}} [\![M]\!]_{[[\![N]\!]_\rho/x]\rho} \\ &= [\![[N/x]M]\!]_\rho \end{aligned}$$

The rule $M = N \Longrightarrow \lambda x.M = \lambda x.N$ follows from Definition 5.0.1 (6). The other rules are trivial. □

Definition 5.0.3 (Reflexive and Extensional Kan complex)**.** *Let K be a c.h.p.o. The Kan complex K is called reflexive if the full subcategory $[K \to K] \subseteq Fun(K, K)$ of the continuous functors is a retract of K, i.e., there are continuous functors*

$$F : K \to [K \to K], \qquad G : [K \to K] \to K$$

such that there is a natural equivalence $\varepsilon : FG \to id_{[K \to K]}$.

If there is a natural equivalence $\eta : id_K \to GF$, we call to K an extensional Kan complex.

Example 5.0.1. *The c.h.p.o K_∞, of Definition 4.1.6, is an extensional Kan complex. Since $CHPO$ is a $(0, \infty)$-category, by Remark 4.2.3, K_∞ is a solution for the Homotopy Domain Equation $X \simeq [X \to X]$ in $CHPO$.*

Remark 5.0.2. *In the Definition 5.0.3, note that the quadruple $\langle K, F, G, \varepsilon \rangle$ is a homotopic λ-model in the c.c.i $CHPO$, and the quintuple $\langle K, F, G, \varepsilon, \eta \rangle$ is an extensional homotopic λ-model in the same ∞-category.*

Definition 5.0.4. *Let K be a reflexive Kan complex (via F, G and Δ).*

1. *For $f, g : \Delta^n \to K$ (or also $f, g \in K_n$) define the n-simplex*

$$f \bullet_{\Delta^n} g = F(f)(g).$$

In particular for vertices $a, b \in K$,

$$a \bullet b = a \bullet_{\Delta^0} b = F(a)(b),$$

besides, $F(a)(-) = a \bullet (-)$ and $F(-)(b) = (-) \bullet b$ are functors on K, then for $f \in K_n$ one defines the n-simplexes

$$a \bullet f = F(a)(f), \qquad f \bullet b = F(f)(b).$$

2. *For each $n \geq 0$, let ρ be a valuation at K_n. Define the interpretation $[\![\ ]\!]_\rho : \Lambda \to K_n$ by induction as follows*

   a) $[\![x]\!]_\rho = \rho(x)$,

   b) $[\![MN]\!]_\rho = [\![M]\!]_\rho \bullet [\![N]\!]_\rho$,

   c) $[\![\lambda x.M]\!]_\rho = G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})$, *where* $\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho} = [\![M]\!]_{[-/x]\rho}$.

Lemma 5.0.2. *If $n \geq 0$ and $\rho : Var \to K_n$, then $\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}$ defines a continuous functor $\Delta^n \to [K \to K]$; hence $[\![\lambda x.M]\!]_\rho$ is well-defined in Definition 5.0.4 (2.c).*

Proof. By induction on P we show that $\boldsymbol{\lambda} f.[\![P]\!]_{[f/(x)(i)]\rho}$ defines a continuous functor $K \times \{i\} \to K$ for each vertex $i \in \Delta^n$ and all ρ in K_n, where the map $[\![P]\!]_{[-/(x)(-)]\rho}$: $K \times \Delta^n \to K$ (with $\rho(x)(-) : \Delta^n \to K$) depicts to $\boldsymbol{\lambda} f.[\![P]\!]_{[f/x]\rho} : K \to K_n$.

For each $f : \Delta^m \to K$ one has:

(a) $[\![x]\!]_{[f/(x)(i)]\rho} = f \in K_m$. So $\boldsymbol{\lambda} f.[\![x]\!]_{[f/(x)(i)]\rho} = I_K$ (Identity functor), which is continuous.

(b) $[\![x]\!]_{[f/(y)(i)]\rho} = s^m(\rho(x)(i)) \in K_m$, with s^m the degeneration operator applied m-times to vertex $\rho(x)(i)$. Then $\boldsymbol{\lambda} f.[\![x]\!]_{[f/(y)(i)]\rho}$ is the constant functor in the vertex $\rho(x)(i)$, which is continuous.

(c) $[\![MN]\!]_{[f/(x)(i)]\rho} = [\![M]\!]_{[f/(x)(i)]\rho} \bullet_{\Delta^m} [\![N]\!]_{[f/(x)(i)]\rho} \in K_m$; since by I.H (Induction Hypothesis) $[\![M]\!]_{[f/(x(i))]\rho}$, $[\![N]\!]_{[f/(x)(i)]\rho}$ are m-simplexes (can be degenerates), hence $[\![MN]\!]_{[f/(x)(i)]\rho}$ is a m-simplex. Besides, the functor $[\![MN]\!]_{[-/(x)(i)]\rho} = F([\![M]\!]_{[-/(x)(i)]\rho})([\![N]\!]_{[-/(x)(i)]\rho})$ is continuous by I.H and continuity of F.

(d) $[\![\lambda y.M]\!]_{[f/(x)(i)]\rho} = G(\boldsymbol{\lambda} g.[\![M]\!]_{[g/y][f/(x)(i)]\rho}) \in K_m$; by I.H $\boldsymbol{\lambda} f.\boldsymbol{\lambda} g.[\![M]\!]_{[g/y][f/x(i)]\rho} : K \to [K \to K]$ is a continuous functor in f and g separately, so by Lemma 4.1.2 is continuous. Thus, $\boldsymbol{\lambda} g.[\![M]\!]_{[g/y][f/(x)(i)]\rho}$ is an m-simplex at $[K \to K]$, applying the continuous functor $G : [K \to K] \to K$ on it, one has an m-simplex in K, and hence the functor $[\![\lambda y.M]\!]_{[-/(x)(i)]\rho} = G \circ [\![M]\!]_{[-/y][-/x(i)]\rho}$ is continuous.

For the proof of Theorem 5.0.2, we make the following remark.

Remark 5.0.3. *Just as the category Set has enough points, the ∞-category $\mathcal{S}$ has enough points in the sense: Let $f, g : X \to Y$ be functors between Kan complexes.If for each $x \in X$, $n \geq 0$ one has $f^n_x = g^n_x$, with $f^n_x : \pi_n(X, x) \to \pi_n(Y, f(x))$ and $g^n_x : \pi_n(X, x) \to \pi_n(Y, g(x))$ as maps induced by f and g respectively, then one has functorial equivalence $f \simeq g$. The property '$\mathcal{S}$ has enough points' can also be interpreted as: given a morphism $f : X \to Y$ in $\mathcal{S}$, if induced map $f^n_x : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism of groups for each $n \geq 0$ and $x \in X$, then f is a homotopy equivalence.*

Theorem 5.0.2. *Let K be a reflexive Kan complex via the morphism F, G, and let $\mathfrak{M} = \langle K, \bullet, [\![\ ]\!] \rangle$. Then*

1. *$\mathfrak{M}$ is a syntactic homotopic λ-model.*

2. *$\mathfrak{M}$ is extensional iff there is a natural equivalence $\eta : id_K \to GF$.*

Proof. 1. The conditions in Definition 5.0.1 (1), (2) are trivial. As to (3), given $g \in K_n$,

$$\begin{aligned} [\![\lambda x.M]\!]_\rho \bullet g &= G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}) \bullet g \\ &= F(G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}))(g) \\ &\xrightarrow{(\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}})_g} (\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})(g) \\ &= [\![M]\!]_{[g/x]\rho}, \end{aligned}$$

where $\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}}$ is the natural equivalence, induced by ε, between the functors $F(G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})$, $\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho} : K \to K_n$. Hence $(\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}})_g$ is the equivalence induced by the n-simplex g in K.

The condition (4) is trivial, since if $[\![M]\!]_\rho \neq [\![M]\!]_\sigma$ so there is $x \in FV(M)$ such that $\rho(x) \neq \sigma(x)$. The condition (5), given any vertex $a \in K$ and $y \notin FV(M)$

$$\boldsymbol{\lambda} f.[\![M(y)]\!]_{[f/y]\rho} = \boldsymbol{\lambda} f.[\![M(x)]\!]_{[f/x]\rho}.$$

Applying G and by Definition 5.0.4 (c), it follows that

$$\begin{aligned}[\![\lambda y.M(y)]\!]_\rho &= G(\boldsymbol{\lambda} f.[\![M(y)]\!]_{[f/y]\rho}) \\ &= G(\boldsymbol{\lambda} f.[\![M(x)]\!]_{[f/x]\rho}) \\ &= [\![\lambda x.M(x)]\!]_\rho\end{aligned}$$

Condition (6). By hypothesis, for every vertex $a \in K$, $n \geq 1$ and $\omega \in \Omega_n(K, a)$

$$\begin{aligned}(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})(\omega) &= [\![M]\!]_{[\omega/x]\rho} \\ &= [\![N]\!]_{[\omega/x]\rho} \\ &= (\boldsymbol{\lambda} f.[\![N]\!]_{[f/x]\rho})(\omega),\end{aligned}$$

since K does have enough points, then

$$\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho} = \boldsymbol{\lambda} f.[\![N]\!]_{[f/x]\rho}$$

applying G and by Definition 5.0.4 (c),

$$\begin{aligned}[\![\lambda x.M]\!]_\rho &= G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}) \\ &= G(\boldsymbol{\lambda} f.[\![N]\!]_{[f/x]\rho}) \\ &= [\![\lambda x.N]\!]_\rho\end{aligned}$$

2. Suppose that $\mathfrak{M}$ is extensional. Let $\omega \in \Omega_n(K, a)$. Then for all $a \in K$

$$(GF(\omega)) \bullet a = F(GF(\omega))(a) = ((FG)F(\omega))(a) \xrightarrow{(\varepsilon_\omega F)_a} F(\omega)(a) = \omega \bullet a,$$

by extensionality

$$GF(\omega) \xrightarrow{(\varepsilon_\omega F)} \omega = id_K(\omega),$$

since K does have enough points, hence

$$GF \xrightarrow{\varepsilon F} Id_K.$$

If $GF \xrightarrow{\eta} Id_K$. For all $\omega \in \Omega_n(K, a)$ by hypothesis and Definition 5.0.4

$$F(a)(\omega) = a \bullet \omega \xrightarrow{\sigma_\omega} a' \bullet \omega = F(a')(\omega),$$

since K does have enough points, $Fa \xrightarrow{\sigma} Fa'$. Applying G, it follows that

$$a \xrightarrow{\eta_a} GF(a) \xrightarrow{G\sigma} GF(a') \xrightarrow{\tilde{\eta}_{a'}} a',$$

where $\tilde{\eta}_{a'}$ is an inverse of $\eta_{a'}$. □

5.1 THE THEORY OF AN ARBITRARY HIGHER λ-MODEL

One takes advantage of some basic properties of every homotopic λ-model (e.g. extensional Kan complex) to explore the higher $\beta\eta$-conversions, which would correspond to proofs of equality between terms of a theory of equality of any extensional Kan complex.

If we understand an arbitrary higher λ-model as an extensional Kan complex, the following question arises: What would be the syntactic structure of the equality theory of any higher λ-model?. In this chapter we will try to answer this question exploring the theory of any extensional Kan complex in order to generalize the $\beta\eta$-conversions to $(n)\beta\eta$-conversions in a set $\Lambda_{n-1}(a,b)$ by $(n)\beta\eta$-contractions induced by the extensionality from a Kan complex.

We shall see some consequences of the equality theory $Th(\mathcal{K})$ of an extensional Kan complex $\mathcal{K}$ with some examples of equality and nonequality of terms. This paves the way for a definition of the $(n)\beta\eta$-conversions, which will belong to the set of n-conversions Λ_n induced by the least theory of equality on all the extensional Kan complexes, denoted by HoTFT.

Let $\mathcal{K} = \langle K, F, G, \varepsilon\rangle$ be a reflexive Kan complex, we have the following remark.

Remark 5.1.1. *Given $g \in K_n$ and $\rho : Var \to K$, the higher β-contraction is interpreted by*

$$\begin{aligned}
[\![\lambda x.M]\!]_\rho \bullet g &= G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}) \bullet g \\
&= F(G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}))(g) \\
&\xrightarrow{(\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}})_g} (\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})(g) \\
&= [\![M]\!]_{[g/x]\rho},
\end{aligned}$$

where $\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}}$ is the natural equivalence, induced by ε, between the functors

$$F(G(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}),\ \boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho} : K \to K.$$

Hence $(\varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}})_g$ is the equivalence induced by the n-simplex g in K.

If $\langle K, F, G, \varepsilon, \eta\rangle$ is extensional, so that the β-contraction is modelled by $\varepsilon : FG \to 1$; the η-contraction (reverse) is modelled by $\eta : 1 \to GF$.

Proposition 5.1.1. *Let x, y, M, N, P be λ-terms. The interpretations of β-reductions*

$$\begin{array}{ccc} (\lambda x.M)((\lambda y.N)P) & \xrightarrow{1\beta} & [(\lambda y.N)P/x]M \\ {\scriptstyle 1\beta}\downarrow & & \downarrow{\scriptstyle [1\beta]} \\ (\lambda x.M)([P/y]N) & \xrightarrow[1\beta]{} & [([P/y]N)/x]M \end{array}$$

are equivalent in every reflexive Kan complex $\langle K, F, G, \varepsilon\rangle$.

Proof. Let $a = [\![P]\!]_\rho$, $[\![\lambda y.N]\!]_\rho \bullet a \xrightarrow{f} [\![N]\!]_{[a/y]\rho}$, $R = FG(\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho})$, $L = \boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}$ and $\varepsilon' = \varepsilon_{\boldsymbol{\lambda} f.[\![M]\!]_{[f/x]\rho}}$. One has that the natural equivalence $\varepsilon' : R \to L$ makes the following diagram (weakly) commute:

$$\begin{array}{ccc} R([\![\lambda y.N]\!]_\rho \bullet a) & \xrightarrow{\varepsilon'_{[\![\lambda y.N]\!]_\rho \bullet a}} & L([\![\lambda y.N]\!]_\rho \bullet a) \\ {\scriptstyle R(f)}\downarrow & & \downarrow{\scriptstyle L(f)} \\ R([\![N]\!]_{[a/y]\rho}) & \xrightarrow[\varepsilon'_{[\![N]\!]_{[a/y]\rho}}]{} & L([\![N]\!]_{[a/y]\rho}) \end{array}$$

which, by Remark 5.1.1, corresponds to the (weakly) commutative diagram

$$\begin{array}{ccc} [\![\lambda x.M]\!]_\rho \bullet ([\![\lambda y.N]\!]_\rho \bullet a) & \xrightarrow{\varepsilon'_{[\![\lambda y.N]\!]_\rho \bullet a}} & [\![M]\!]_{[[\![\lambda y.N]\!]_\rho \bullet a/x]} \\ {\scriptstyle R(f)}\downarrow & & \downarrow{\scriptstyle L(f)} \\ [\![\lambda x.M]\!]_\rho \bullet [\![N]\!]_{[a/y]\rho} & \xrightarrow[\varepsilon'_{[\![N]\!]_{[a/y]\rho}}]{} & [\![M]\!]_{[[\![N]\!]_{[a/y]\rho}/x]} \end{array}$$

□

Example 5.1.1. *The λ-term $(\lambda x.u)((\lambda y.v)z)$ has two β-reductions:*

$$\begin{array}{ccc} (\lambda x.u)((\lambda y.v)z) & \xrightarrow{1\beta} & [(\lambda y.v)z/x]u \\ {\scriptstyle 1\beta}\downarrow & & \downarrow{\scriptstyle [1\beta]} \\ (\lambda x.u)([z/y]v) & \xrightarrow[1\beta]{} & [v/x]u \end{array}$$

making $u = M$, $v = N$ and $z = P$, by Proposition 5.1.1, the interpretations of these β-reductions are equivalent in all reflexive Kan complexes $\langle K, F, G, \varepsilon\rangle$.

Next, we shall give examples where the reductions of λ-terms are not equivalent.

Example 5.1.2. *The λ-term $(\lambda x.(\lambda y.yx)z)v$ has the β-reductions*

$$\begin{array}{ccc} (\lambda x.(\lambda y.yx)z)v & \xrightarrow{1\beta} & (\lambda y.yv)z \\ \downarrow{\scriptstyle 1\beta} & & \downarrow{\scriptstyle 1\beta} \\ (\lambda x.zx)v & \xrightarrow{1\beta} & zv \end{array}$$

Given a reflexive Kan complex $\langle K, F, G, \varepsilon \rangle$. Let $\rho(v) = c$, $\rho(z) = d$ vertices at K and $R = FG$. The interpretation of the β-reductions of $(\lambda x.(\lambda y.yx)z)v$ depends on solving the diagram equation

$$\begin{array}{ccc} R(\boldsymbol{\lambda} a.R(\boldsymbol{\lambda} b.b \bullet a)(d))(c) & \xrightarrow{(\varepsilon_f)_c} & R(\boldsymbol{\lambda} b.b \bullet c)(d) \\ {\scriptstyle (R(?))_c}\downarrow & & \downarrow{\scriptstyle (\varepsilon_{\boldsymbol{\lambda} b.b \bullet c})_d} \\ R(\boldsymbol{\lambda} a.d \bullet a)(c) & \xrightarrow[(\varepsilon_g)_c]{} & d \bullet c \end{array}$$

where $f = \boldsymbol{\lambda} a.R(\boldsymbol{\lambda} b.b \bullet a)(d)$ and $g = \boldsymbol{\lambda} a.d \bullet a$ are functors at $[K \to K]$. One has $h_a = (\varepsilon_{\boldsymbol{\lambda} b.b \bullet a})_d : f(a) \to g(a)$ for each vertex $a \in K$, but h_a is not necessarily a functorial equivalence in any reflexive Kan complex $\langle K, F, G, \varepsilon \rangle$ to get the diagram to commute:

$$\begin{array}{ccc} R(f)(c) & \xrightarrow{(\varepsilon_f)_c} & f(c) \\ {\scriptstyle (R(h?))_c}\downarrow & & \downarrow{\scriptstyle h_c} \\ R(g)(c) & \xrightarrow[(\varepsilon_g)_c]{} & g(c) \end{array}$$

Example 5.1.3. *The λ-term $(\lambda z.xz)y$ has the $\beta\eta$-contractions*

$$(\lambda z.xz)y \underset{1\eta}{\overset{1\beta}{\rightrightarrows}} xy$$

Take an extensional Kan complex $\langle K, F, G, \varepsilon, \eta \rangle$. Let $\rho(x) = a$ and $\rho(y) = b$ be vertices of K. The interpretation of a λ-term is given by: $[\![(\lambda z.xz)y]\!]_\rho = [\![\lambda z.xz]\!]_\rho \bullet b = G(\boldsymbol{\lambda} c.F(a)(c)) \bullet b = G(F(a)) \bullet b = (FGF)(a)(b)$. The interpretation of the $\beta\eta$-contractions corresponds to the degenerated diagrams

$$\begin{array}{ccc} (FGF)(a)(b) & & \\ {\scriptstyle (\varepsilon_{F(a)})_b}\downarrow & \searrow^{1} & \\ F(a)(b) & \xrightarrow[(F(\eta_a))_b]{} & (FGF)(a)(b) \end{array} \qquad \begin{array}{ccc} F(a)(b) & & \\ {\scriptstyle (F(\eta_a))_b}\downarrow & \searrow^{1} & \\ (FGF)(a)(b) & \xrightarrow[(\varepsilon_{F(a)})_b]{} & F(a)(b) \end{array}$$

But the diagrams do not necessarily commute in any extensional Kan complex $\langle K, F, G, \varepsilon, \eta \rangle$.

It is known that the types of HoTT correspond to ∞-groupoids. Taking advantage of this situation, for a reflexive Kan complex, we define the theory of equality on that Kan complex (∞-groupoid) as follows:

Definition 5.1.1 (Theory of an extensional Kan complex)**.** *Let $\mathcal{K} = \langle K, F, G, \varepsilon, \eta \rangle$ an extensional Kan complex. Define the theory of 1-equality of $\mathcal{K}$ as the class*

$$Th_1(\mathcal{K}) = \{M = N \mid [\![M]\!]_\rho \simeq [\![N]\!]_\rho \text{ for all } \rho : Var \to K\}$$

where $[\![M]\!]_\rho \simeq [\![N]\!]_\rho$ be the equivalence between vertices of K for some equivalence $[\![s]\!]_\rho$: $[\![M]\!]_\rho \to [\![N]\!]_\rho$, and "$s$" denote the conversion between λ-terms M and N induced by $[\![s]\!]_\rho$ for all evaluation ρ.

In the Definition 5.1.1, notice that the equivalence $[\![M]\!]_\rho \simeq [\![N]\!]_\rho$ for all ρ, induces the intentional equality $M = N$, which can be seen as an identity type based on computational paths (QUEIROZ; OLIVEIRA; RAMOS, 2016); the conversion s may also be seen as a computational proof (a finite sequence of basic rewrites (QUEIROZ; OLIVEIRA; RAMOS, 2016) induced by K) of the proposition $M = N$ in the theory $Th_1(\mathcal{K})$.

Remark 5.1.2. *If s is a β-contraction or η-contraction and the functor F is not surjective for objects, the equality $M =_{1\beta} N : K$ or $M =_{1\eta} N : K$ is not necessarily a judgmental equality (as it happens in HoTT); $[\![M]\!]_\rho$ and $[\![N]\!]_\rho$ may be different vertices in K. Thus, the theory $Th_1(\mathcal{K})$ may be seen as the family of all the identity types which are inhabited by paths which are not necessarily equal to the reflexive path $\texttt{refl}_M$.*

Notation 5.1.1. *Let M and N be λ-terms ($M, N \in \Lambda_0$) and $\mathcal{K}$ be an extensional Kan complex. Denote by $\Lambda_0(K)(M, N)$ is the set of all the 1-conversions from M to N induced by $\mathcal{K}$. We write by $\Lambda_1(\mathcal{K}) := \bigcup_{M,N\in\Lambda_0} \Lambda_0(\mathcal{K})(M, N)$ be the family of all the 1-conversions induced by $\mathcal{K}$.*

Let $s, t \in \Lambda_0(\mathcal{K})(M, N)$. Denote by $\Lambda_0(\mathcal{K})(M, N)(s, t)$ is the set of all the 2-conversions from s to t. And let $\Lambda_2(\mathcal{K}) := \bigcup_{s,t\in\Lambda_1} \bigcup_{M,N\in\Lambda_0} \Lambda_0(\mathcal{K})(M, N)(s, t)$ is the family of all the 2-conversions induced by $\mathcal{K}$, and so on we keep iterating for the families $\Lambda_3(\mathcal{K})$, $\Lambda_4(\mathcal{K}), \ldots$.

Since $\mathcal{K}$ is a reflexive Kan complex, $Th_1(\mathcal{K})$ is an intentional λ-theory of 1-equality which contains the theory $\lambda\beta\eta$. Iterate again, we have the λ-theory of 2-equality

$$Th_2(\mathcal{K}) = \{r = s \mid \forall\rho\,([\![r]\!]_\rho \simeq [\![s]\!]_\rho) \text{ and } r, s \in \Lambda_0(\mathcal{K})(M, N)\}.$$

If we keep iterating, we can see that the reflexive Kan complex $\mathcal{K}$ will certainly induce a λ-theory of higher equality

$$Th(\mathcal{K}) = \bigcup_{n\geq 1} Th_n(\mathcal{K}).$$

Just as $Th_1(\mathcal{K})$ contains $\lambda\beta\eta$, $Th(\mathcal{K})$ will contain a (simple version of) 'Homotopy Type-Free Theory', defined as follows.

Definition 5.1.2 (Homotopy Type-Free Theory)**.** *A Homotopy Type-Free Theory (HoTFT) consists of the least theory of equality, that is*

$$HoTFT := \bigcap\{Th(\mathcal{K}) \,|\, \mathcal{K} \textit{ is an extensional Kan complex}\}.$$

And for each $n \geq 0$ let

$$\Lambda_n := \bigcap\{\Lambda_n(\mathcal{K}) \,|\, \mathcal{K} \textit{ is an extensional Kan complex}\}$$

be the set of $n\beta\eta$-conversions.

For example, let $\mathcal{K} = \langle K, F, G, \varepsilon, \eta\rangle$ be an extensional Kan Complex and x, M and N λ-terms. By Definition 5.1.2, the β-contraction $(\lambda x.M)N \xrightarrow{1\beta} [N/x]M$ inhabit the set $\Lambda_0((\lambda x.M)N, [N/x]M)$;

$$[\![1\beta]\!]_\rho = (\varepsilon_{[\![M]\!]_{[-/x]\rho}})_{[\![N]\!]_\rho} \in K([\![(\lambda x.M)N]\!]_\rho, [\![[N/x]M]\!]_\rho),$$

and the η-contraction $\lambda x.Mx \xrightarrow{1\eta} M$, $x \notin FV(M)$, belong to $\Lambda_0(K)(\lambda x.Mx, M)$;

$$[\![1\eta]\!]_\rho = \eta_{[\![M]\!]_\rho} \in K([\![\lambda x.Mx]\!]_\rho, [\![M]\!]_\rho).$$

If t is a $\beta\eta$-conversion from λ-term M to N, by Definition 5.1.2, $t \in \Lambda_0(M, N)$. For x, P λ-terms, we have the vertices $[\![\lambda x.P]\!]_\rho \in K$ and $[\![t]\!]_\rho \in K([\![M]\!]_\rho, [\![N]\!]_\rho)$. Thus, $[\![(\lambda x.P)t]\!]_\rho = [\![\lambda x.P]\!]_\rho \bullet [\![t]\!]_\rho \in K([\![(\lambda x.P)M]\!]_\rho, [\![(\lambda x.P)N]\!]_\rho)$ and $[\![P]\!]_{[[\![t]\!]_\rho/x]\rho} \in K([\![P]\!]_{[[\![M]\!]_\rho/x]\rho}, [\![P]\!]_{[[\![N]\!]_\rho/x]\rho})$, where $[[\![t]\!]_\rho/x]\rho : Var \to K_1$ is an evaluation $\rho'(x) = [\![t]\!]_\rho$ and (n-times degeneration of vertex $\rho(r)$) $\rho'(r) = s^n(\rho(r))$ if $r \neq x$. By Definition 5.1.2, $(\lambda x.P)t \in \Lambda_0((\lambda x.P)M, (\lambda x.P)N)$ and $[\![P]\!]_{[[\![t]\!]_\rho/x]\rho} \in K([\![P]\!]_{[[\![M]\!]_\rho/x]\rho}, [\![P]\!]_{[[\![N]\!]_\rho/x]\rho})$. But

$$[\![(\lambda x.P)t]\!]_\rho \xrightarrow{(\varepsilon_{[\![P]\!]_{[-/x]\rho}})_{[\![t]\!]_\rho}} [\![P]\!]_{[[\![t]\!]_\rho/x]\rho},$$

So $(\lambda x.P)t = [t/x]P$ and induces the 2β-contraction

$$(\lambda x.P)t \xrightarrow{2\beta_{P,t}} [t/x]P,$$

corresponding to a similar diagram to that of Proposition 5.1.1, i.e.,

$$\begin{array}{ccc} (\lambda x.P)M & \xrightarrow{1\beta_M} & [M/x]P \\ {\scriptstyle (\lambda x.P)t}\downarrow & \Longrightarrow_{2\beta_t} & \downarrow{\scriptstyle [t/x]M} \\ (\lambda x.P)N & \xrightarrow[1\beta_N]{} & [N/x]P \end{array}$$

Hence $2\beta_t \in \Lambda_0((\lambda x.P)M, [N/x]P)(\tau(1\beta_M, [t/x]M), \tau((\lambda x.P)t, 1\beta_N))$, where $\tau(r, s)$ is the concatenation of the conversions $r \in \Lambda(a, b)$ and $s \in \Lambda(b, c)$. On the other hand, for $y \notin FV(t)$ one has the equivalence

$$[\![t]\!]_\rho \xrightarrow{\eta_{[\![t]\!]_\rho}} [\![\lambda y.ty]\!]_\rho,$$

that is, $(\lambda y.ty) = t$ and induces the 2η-contraction

$$(\lambda y.ty) \xrightarrow{2\eta_t} t,$$

which corresponds to diagram

$$\begin{array}{ccc} \lambda y.My & \xrightarrow{n\eta_r} & M \\ {\scriptstyle \lambda y.ty}\Big\downarrow & \Rightarrow_{2\eta_t} & \Big\downarrow{\scriptstyle t} \\ \lambda y.Ny & \xrightarrow[n\eta_s]{} & N \end{array}$$

In general, if $t \in \Lambda_{n-1}$, the equivalences

$$[\![(\lambda x.P)t]\!]_\rho \xrightarrow{(\varepsilon_{[\![P]\!]_{[-/x]\rho}})_{[\![t]\!]_\rho}} [\![P]\!]_{[[\![t]\!]_\rho/x]\rho}, \quad [\![t]\!]_\rho \xrightarrow{\eta_{[\![t]\!]_\rho}} [\![\lambda y.ty]\!]_\rho$$

in every extensional Kan complex K, induce the $(n)\beta\eta$-contractions

$$(\lambda x.P)t \xrightarrow{n\beta_t} [t/x]P, \quad (\lambda y.ty) \xrightarrow{n\eta_t} t.$$

which explains the following corollary.

Corollary 5.1.1. *If x, y, P be λ-terms, $n \geq 1$ and $t \in \Lambda_n(r, s)$ with $y \notin FV(t)$, then the interpretation from diagrams*

$$\begin{array}{ccc} (\lambda x.P)r & \xrightarrow{n\beta_r} & [r/x]P \\ {\scriptstyle (\lambda x.P)t}\Big\downarrow & & \Big\downarrow{\scriptstyle [t/x]M} \\ (\lambda x.P)s & \xrightarrow[n\beta_s]{} & [s/x]P \end{array} \qquad \begin{array}{ccc} \lambda y.ry & \xrightarrow{n\eta_r} & r \\ {\scriptstyle \lambda y.ty}\Big\downarrow & & \Big\downarrow{\scriptstyle t} \\ \lambda y.sy & \xrightarrow[n\eta_s]{} & s \end{array}$$

commutes in every extensional Kan complex K.

Thus, any reflexive Kan complex inductively induces, for each $n \geq 1$, from an $(n)\beta\eta$-conversion t to the $(n+1)\beta\eta$-contractions

$$\begin{array}{ccc} (\lambda x.P)r & \xrightarrow{n\beta_r} & [r/x]P \\ {\scriptstyle (\lambda x.P)t}\Big\downarrow & \Rightarrow_{(n+1)\beta_t} & \Big\downarrow{\scriptstyle [t/x]M} \\ (\lambda x.P)s & \xrightarrow[n\beta_s]{} & [s/x]P \end{array} \qquad \begin{array}{ccc} \lambda y.ry & \xrightarrow{n\eta_r} & r \\ {\scriptstyle \lambda y.ty}\Big\downarrow & \Rightarrow_{(n+1)\eta_t} & \Big\downarrow{\scriptstyle t} \\ \lambda y.sy & \xrightarrow[n\eta_s]{} & s \end{array}$$

and these, in their turn, define the $(n+1)\beta\eta$-conversions, of $(n)\beta\eta$-conversion, which would inhabit the set Λ_{n+1}.

We have previously realized that the homotopic λ-models (e.g., extensional Kan complexes) of Homotopy Domain Theory (HoDT) are able to provide techniques for the generalization of Church-like conversion relations (such as e.g., β-equality, η-equality) to higher term-contraction induced equivalences. For a future work, it would be interesting to see the relationship between the higher $\beta\eta$-conversions and the computation paths, specifically, to establish a relation between the least theory of all the homotopic λ-models with a type-free version of computational paths.

6 CONCLUSIONS

Some methods were established for solving homotopy domain equations, which further contributes to the project of a generalization of the Domain Theory to a Homotopy Domain Theory (HoDT). Besides, using those methods of solving equations, it was possible to obtain some specific homotopy models in a Cartesian closed ∞-category, which could help to define a general higher λ-calculus theory.

Specially, we generalize the Kleisli bicategory to a Kleisli ∞-category $Kl(P)$ and to prove that it is Cartesian closed with enough points, in order to apply the techniques of HoDT to this Kleisli ∞-category and thus obtain homotopic λ-models (e.g., reflexive Kan complexes in $Kl(P)$) with relevant information. Besides, we prove the existence of an extension of the set D_∞ to a Kan complex K_∞, which models a type-free version of HoTT, which we call HoTFT (Homotopy Type-Free Theory), which could have the advantage of rescuing the $\beta\eta$-conversions as relations of intentional equality and not as relations of judgmental equality as occurs in HoTT.

On other hand, we define the interpretation of the $\beta\eta$-contractions in a reflexive Kan complex, whose ∞-groupoid structure induces higher $\beta\eta$-contractions, which if we type would inhabit a type of identity (based on computational paths). Therefore, in this work, we would be facing the beginning of the semantics of another version of HoTT, but, based on computational paths, its type-free version would be similar to a theory of higher λ-calculus, which would be pending development.

The above would be for the typed case for future work. For the case type-free, HoDT should continue to be developed, parallel to Dana Scott's Domain Theory, and all the semantics of higher λ-calculus and its relation with HoTFT.

BIBLIOGRAPHY

ABRAMSKY, S.; JUNG, A. *Domain theory*. [S.l.]: In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, Handbook of Logic in Computer Science, volume 3, pages 1–168, Clarendon Press, 1994.

ASPERTI, A.; LONGO, G. *Categories, Types and Structures: An Introduction to Category Theory for the working computer scientist*. [S.l.]: Foundations of Computing Series, M.I.T Press, 1991.

BARENDREGT, H. *The Lambda Calculus, its Syntax and Semantics*. [S.l.]: North-Holland Co., Amsterdam, 1984.

BOARDMAN, J.; VOGT, R. *Homotopy invariant algebraic structures on topological spaces*. [S.l.]: Lecture Notes in Mathematics, 347, Berlin, New York: Springer-Verlag, 1973.

CISINSKI, D.-C. *Higher Categories and Homotopical Algebra*. [S.l.]: Cambridge University Press, 2019.

FRIEDMAN, G. An elementary illustrated introduction to simplicial sets. *Rocky Mountain J. Math.*, v. 42, n. 2, p. 353–423, 2012.

GABRIEL, P.; ULMER, F. *Lokal präsentierbare Kategorien*. [S.l.]: Lecture Notes in Mathematics (LNM, volume 221), 1971.

GOERSS, P.; JARDINE, J. *Simplicial Homotopy Theory*. [S.l.]: Birkhäuser Basel, Springer Nature Switzerl and AG, 2009.

GROTH, M. *A short course on ∞-categories*. [s.n.], 2015. Available at: <https://arxiv.org/abs/1007.2925>.

HATCHER, A. *Algebraic Topology*. [S.l.]: Cambridge University Press, New York, NY, 2001.

HINDLEY, J.; SELDIN, J. *Lambda-Calculus and Combinators, an Introduction*. [S.l.]: Cambridge University Press, New York, 2008.

HYLAND, M. Some reasons for generalizing domain theory. *Mathematical Structures in Computer Science*, v. 20, p. 239–265, 2010.

HYLAND, M. Elements of a theory of algebraic theories. *Theoretical Computer Science*, v. 546, p. 132–144, 2014.

JOYAL, A. Quasi-categories and kan complexes. *Journal of Pure and Applied Algebra*, v. 175, n. 1, p. 207–222, 2002.

JOYAL, A.; TIERNEY, M. Quasi-categories vs segal spaces. *Categories in algebra, geometry and mathematical physics, Contemp. Math.*, v. 431, p. 277–326, 2008.

KAPULKIN, C.; LUMSDAINE, P. The simplicial model of univalent foundations (after voevodsky). *arXiv:1211.2851*, 2012. Available at: <https://arxiv.org/abs/1211.2851>.

LUMSDAINE, P.; SHULMAN, M. Semantics of higher inductive types. *Mathematical Proceedings of the Cambridge Philosophical Society*, v. 169, p. 159–208, 2020.

LURIE, J. *Higher Topos Theory*. [S.l.]: Princeton University Press, Princeton and Oxford, 2009.

LURIE, J. *Higher Algebra*. [S.l.]: Harvard University, 2017.

MARTÍNEZ-RIVILLAS, D.; QUEIROZ, R. de. Solving homotopy domain equations. *arXiv:2104.01195*, 2021. Available at: <https://arxiv.org/abs/2104.01195>.

MARTÍNEZ-RIVILLAS, D.; QUEIROZ, R. de. The theory of an arbitrary higher λ-model. *arXiv:2111.07092*, 2021. Available at: <https://arxiv.org/abs/2111.07092>.

MARTÍNEZ-RIVILLAS, D.; QUEIROZ, R. de. The ∞-groupoid generated by an arbitrary topological λ-model. *Logic Journal of the IGPL*, v. 30, n. 3, p. 465–488 https://doi.org/10.1093/jigpal/jzab015, (also arXiv:1906.05729), June 2022.

MARTÍNEZ-RIVILLAS, D.; QUEIROZ, R. de. Towards a homotopy domain theory. *Archive for Mathematical Logic*, 2022. Available at: <https://doi.org/10.1007/s00153-022-00856-0>.

PROGRAM, T. U. F. *Homotopy Type Theory: Univalent Foundations of Mathematics*. [S.l.]: Princeton, NJ: Institute for Advanced Study, 2013.

QUEIROZ, R. de; OLIVEIRA, A. de; RAMOS, A. Propositional equality, identity types, and direct computational paths. *South American Journal of Logic*, v. 2, n. 2, p. 245–296, 2016.

RAMOS, A.; QUEIROZ, R. de; OLIVEIRA, A. de. On the identity type as the type of computational paths. *Logic Journal of the IGPL*, v. 25, n. 4, p. 562–584, 2017.

REZK, C. *Stuff about quasicategories*. [S.l.]: Lecture Notes for course at University of Illinois at Urbana-Champaign, 2017.

REZK, C. *Introduction to Quasicategories*. Lecture Notes for course at University of Illinois at Urbana-Champaign, 2022. Available at: <https://faculty.math.illinois.edu/~rezk/quasicats.pdf>.

SIMPSON, C. Giraud-type characterization of the simplicial categories associated to closed model categories as ∞-pretopoi. *arxiv:9903167*, 1999. Available at: <https://arxiv.org/abs/math/9903167>.