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Multi-Objective Control problems for Parabolic and Dispersive Systems

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Tese apresentada ao Programa de Pós-graduação em Matemática, Departamento de Matemática da Universidade Federal de Pernambuco, como parte dos requisitos parciais para obtenção do título de Doutora em Matemática.

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To my loves Maria Inês, Nadjane Alcântara and Maria da Anunciação (in memoriam)

ABSTRACT

This thesis is dedicated to the study of some multi-objective control problems for partial differential equations. Usually, problems containing many objectives are not well-posed, since one objective may completely determine the control, turning the others objectives impossible to reach. For this reason, concepts of equilibrium (or efficiency) are normally applied to find controls which are acceptable, in the sense they make the best decision possible according to some prescribed goals. By applying the so called Stackelberg-Nash strategy, we consider a hierarchy, in the sense that we have one control which we call the leader, and other controls which we call the followers. Once the leader policy is fixed, the followers intend to be in equilibrium according to their targets, this is what we call Stackelberg's Method. Once this hierarchy is established, we determine the followers in such a way they accomplish their objectives in a optimal way, and to do that a concept of equilibrium is applied. In this work, we apply the concept of Nash Equilibrium, which correspond to a non-cooperative strategy. By combining the Stackelberg's Method and the concept of Nash Equilibrium is what we call Stackelberg-Nash strategy. This thesis is divided into two chapters. In each of them, we solve a multi-objective control problems by following the Stackelberg-Nash strategy. In the first chapter, we consider a linear system of parabolic equations and prove that the Stackelberg-Nash strategy can be applied under some suitable conditions for the coupling coefficients. In the second one, we consider the nonlinear Korteweg-de Vries (KdV) equation, which has a very different nature of parabolic equations, and the same method is applied.

Keywords:null controllability; Stackelberg-Nash strategy; Carleman estimates; observability estimates; parabolic systems; KdV equation.

RESUMO

Esta tese é dedicada ao estudo de alguns problemas de controle multiobjetivos para equações diferenciais parciais. Normalmente, problemas desta natureza não são bem colocados, uma vez que um objetivo pode determinar completamente o controle, tornando impossível o alcance dos demais. Por esta razão, conceitos de equilíbrio (ou eficiência) são normalmente aplicados para encontrar controles que são aceitáveis, no sentido em que tomam a melhor decisão possível de acordo com alguns objetivos prescritos. Ao aplicar a chamada estratégia de Stackelberg-Nash, nós consideramos uma hierarquia, no sentido de que temos um controle que chamamos de líder e outros controles que chamamos de seguidores. Um vez que a escolha do líder é fixada, os seguidores pretendem estar em equilíbrio de acordo com seus objetivos, isto é o que chamamos de Método de Stackelberg. Uma vez que essa hierarquia é estabelecida, nós determinamos os seguidores de modo que eles cumpram seus objetivos de maneira ideal, e para isso um conceito de equilíbrio é aplicado. Neste trabalho, nós aplicamos o conceito de Equilíbrio de Nash, que corresponde a uma estratégia não-cooperativa. Combinando o método de Stackelberg e o conceito do equilíbrio de Nash, temos o que chamamos de estratégia de Stackelberg-Nash. Esta tese é dividida em dois capítulos. Em cada um deles, nós resolvemos um problema de controle multiobjetivo seguindo a estratégia Stackelberg-Nash. No primerio capítulo, nós consideramos um sistema linear de equações parabólicas e provamos que a estratégia Stackelberg-Nash pode ser aplicada sob algumas condições adequadas para os coeficientes de acoplamento. No segundo capítulo, nós consideramos a equação de Korteweg-de Vries (KdV) não linear, que tem uma natureza muito diferente de equações parabólicas, e o mesmo método é aplicado.

Palavras-chave: controlabilidade nula; estratégia Stackelberg-Nash; estimativas de Carleman; estimativas de observabilidade; sistemas parabólicos; equação de KdV.

CONTENTS

1 INTRODUCTION

Over the last decades, the society we live in is being shaped by the search for efficiency. Our actions are planned with the aim of reducing time or minimizing costs and waste in the execution of specific objectives. As the objectives grow in number, the necessity to find methods that make them possible in a satisfactory way grows as well. The challenge of this is that efficiency does not always imply lower costs, a low cost does not imply quality, and efficiency cannot demand lesser time. All of these things are associated to the action of controlling, supervising, directing, having something under your domain. We are constantly invited to control situations and this encompasses giving good/optimal destinies to all the variables involved in the situations with which we are faced.

In the history of humanity, many mechanisms have been created with the intention of facilitating people's lives and bringing more efficiency to actions. In ancient Rome (3 BCE), aqueducts began being build with the objective of transporting water from the rivers, supplying cities where water was not as abundant. These structures possessed regulating valves with the objective of maintaining the level of water constant.

In the XVII and XVIII centuries, the measurement of time was relevant to navigation, because determining longitude was directly associated to reliable methods of measuring time intervals. The swaying of vessels altered the oscillation ranges of the used pendula, consequently altering the ships' locations during navigation. That is when the mathematician, physicist and astronomer Christiaan Huygen (1629 - 1695), who dedicated 40 years of his life attempting to develop and improve the marine chronometers, invented what is known as the isochronous pendulum. This pendulum allows for the precise measurement of time and, as a result, of the real location of ships during navigation.

Years later, the works of Huygens were adapted to create a speed regulation mechanism for windmills, whose main objective was to maintain the rotation speed approximately constant. James Watt (1736 - 1819), mathematician and engineer, adapted this model to the steam machine in order to maintain their speed constant, making an important breakthrough in the industrial revolution.

All of these inventions (and many others) are seen today as control mechanisms and are part of the history of their classical theories. Only in the middle of the XX century more modern theories got stronger. In this period, we could highlight as protagonists: the Hungarian mathematician and engineer, naturalized North American Rudolf Emil Kalman, co-inventor of filtering and algebraic approximation techniques for linear systems [34]; the American mathematician Richard Ernest Bellman for the invention of Dynamic Programming [11]; and also the Russian mathematician Lev Semenovich Pontryagin, famous for the Principle of the Maximum for Nonlinear Optimal Control Problems [49].

It is notorious that Mathematics has increasingly shown itself to be a powerful tool to help answer certain problems of efficiency and control. With the advent of Differential Calculus, several phenomena could be described mathematically through the Differential Equations and, naturally, techniques with the intention of controlling such phenomena began being developed. The Ordinary Differential Equations (ODE) and the Partial Differential Equations (PDE) come in as strong tools that make the modelling of different natural phenomena and even phenomena resulted from technological advancements possible.

Let us be more specific about how a control problem can be formulated through the mathematical point of view. A *control system* is an evolution equation (ODE or PDE) which we can describe in the following general way

$$y_t = f(t, y, v). \tag{1.1}$$

In this context, the parameter $T > 0$ is a real fixed number, commonly called final time and $t \in [0, T]$ represents the time instant which the variables are submitted to. The variable $y : [0, T] \longrightarrow H$ is the so-called state function and y_t represents the derivative of y in relation to time. The function $v : [0, T] \longrightarrow U$ is an external force called control and, finally, the sets H, U are adequate function spaces, chosen essentially in a way that guarantees that the problem (1.1) has a solution.

It is clear that different choices for the v control result in alterations in the dynamic of the function y. The control problem therefore aims to find a control v so that the state function y assumes a desired behavior through time. This problematic gives room for a series of controllability concepts that we are going to define next.

Exact controllability: We say that the system (1.1) is exactly controllable if for each $y_0, y_1 \in H$, there is a control $v : [0, T] \longrightarrow U$ such that

$$\begin{cases} y_t = f(t, y, v), & t \in [0, T], \\ y(0) = y_0, & y(T) = y_1. \end{cases}$$

That is, from any initial configuration y_0, we can conduct the solution y to the final state y_1 under the action of the v control.

Approximate controllability: We say that a system (1.1) is approximately controllable when for all $\varepsilon > 0$ given and for each $y_0, y_1 \in H$, we are able to find $v : [0, T] \longrightarrow U$ such that

$$\begin{cases} y_t = f(t, y, v), & t \in [0, T], \\ y(0) = y_0, & ||y(T) - y_1||_H < \epsilon. \end{cases}$$

Observe that the exact controllability implies in the approximate controllability, though, in general, reciprocity is not true. In the first one we ask for the state function to be exactly y_1 in T, while in the second we ask for the state to be arbitrarily approximate to y_1.

Null controllability: We say that a system (1.1) is null controllable if, for each $y_0 \in H$, there exists $v : [0, T] \longrightarrow U$ such that

$$\begin{cases} y_t = f(t, y, v), & t \in [0, T], \\ y(0) = y_0, & y(T) = 0. \end{cases}$$

Exact controllability to trajectories: We say that the system is exactly controllable to trajectories if, for each $\bar{y}$ trajectory (any solution of (1.1) with control $\bar{v} : [0, T] \longrightarrow U$), there exists a control $v : [0, T] \longrightarrow U$ such that

$$\begin{cases} y_t = f(t, y, v), & t \in [0, T] \\ y(0) = y_0, & y(T) = \overline{y}(T). \end{cases}$$

The concepts of null controllability and exact controllability to trajectories are applicable to systems nonreversible in time or in systems with regularizing effects. In these cases, exact controllability is not expected.

The aforementioned control problems are related to a single final objective, that is, we wish to determine a control v so that the solution of (1.1) reaches a single target. The works contained in this thesis are related to problems where more than one objective is required. This problematic, in the context of control problems for differential equations was introduced by J.L. Lions (see [38]) and has many applications. For example, we could be interested in controlling the distribution of temperature in an environment at the same time that we want temperatures of certain regions to be in "reasonable"levels, according to pre-established parameters. Or, we could be interested in controlling the dynamic of a fluid keeping adequate levels of energy in specific regions, which motivates us to define the following category of control problems.

Multi-objective problems: In this class of problems, there are many objectives that must be achieved simultaneously, and controls that strive for this purpose must be built. Here, it is usual that the controls adopt strategies that make the achievement of the objectives possible. These strategies can be cooperative, that is, there is cooperation between controls in a way that all objectives are reached, or noncooperative, where each control is not concerned with whether its action is hindering the achievement of the other controls objectives.

Many theories have been developed for working with multi-objective problems, from which we can highlight three scientists that gave significant contributions in the scope of the problems here described. The first of them was Heinrich Freiherr von Stackelberg who was born in Germany in October 31st, 1946. As an economist, he contributed to the theory of games and oligopolies (from the greek words oligos - few, polens - to sell). The theory of the oligopolies is characterized by the existence of a reduced number of salespeople, where each one must consider the behavior of their competitors in the face of making a decision. In this structure, the goods produced can be made in a homogeneous way and normally competition is had through services such as quality, post-sales, satisfaction or customer retention.

In 1934, Stackelberg published his main work entitled Marktform und Gleichgewicht (Market Structure and Equilibrium [57]) which describes the behavior of a duopoly (a specific case of oligopoly with only 2 producers / companies / salespeople dominating the market). Normally, the duopoly is studied as if it was an oligopoly due to its conceptual simplicity. A current example of the execution of the oligopoly theory are the Solar Energy companies. The services, in general, are offered along with the purchase of equipment included in the billing. Because the equipment has a very high price, the profit for installation depends on the competition's price, because if a company increases its profit too much in relation to the others it can get no service requests. The dispute for clients happens in marketing, environmental conscientization (since it is a clean energy), information about maintenance, duration of equipment, calculations of investment and the time of its return, supply of different consumption units under the same system, among others.

Our second highlighted scientist is the North American mathematician John Forbes Nash, who was born in June 13th 1928 and died in May 23rd 2015 in a car accident. John Nash, as he is popularly known, contributed very much for the game theory, differential geometry and partial differential equations. He was the winner of some awards such as the John von Neumann Theory Prize, in 1978, for developing a theory that, in a game involving one or more players, no player can benefit from carrying out their strategy unilaterally. This situation is known as Nash equilibrium. Another award destined to John Nash was the Prize in Economic Sciences in Memory of Alfred Nobel, in 1994, for his basic analysis of equilibrium in the theory of noncooperative games (see [46]).

The third scientist that we can highlight is the French mathematician Jaques Louis Lions. He was born in May 3rd 1928 and died in May 17th 2001, and was an important codifier for the works of Heinrich Stackelberg and John Nash, establishing the connection with the theory of PDEs (see [20, 42, 43]), having created the entire formalization of multi-objective problems for partial differential equations. More importantly, it is important to stress the great contributions of Lions to the general theory of PDEs and Numeric

Analysis.

Let us be more precise about how the strategies of multi-objective optimization can be applied to control problems in PDEs. Suppose that the control v acting in (1.1) has some of the objectives previously mentioned (exact controllability, approximate, null or to trajectories). Suppose also that, for a fixed $N \in \mathbb{N}$, certain functionals $J_i := J_i(T, y_0, y, v)$ $(i = 1, \dots, N)$ have the function of measuring the state y. In multi-objective problems, we are also interested in knowing if the control v can be determined in such a way that the functionals J_i assume satisfactory values. Notice that the problem above contains $N + 1$ objectives.

It is worth pointing out that multi-objective problems are, in general, not well-posed. The reason for that is that one of the objectives can completely determine the control, making the others impractical. For this reason, the Stackelberg method becomes entirely viable and consists in representing the control v as a sum of $N + 1$ controls, that is,

$$v = v_0 + v_1 + \cdots + v_N,$$

where each control will be responsible for its objective. Furthermore, the objective of the type of controllability is to be understood as the main one, being the control responsible for it called *leader*, while other objectives (related to the functionals J_i) are named *followers*. The objective of the leader is mandatory and the followers must be adapted in order to achieve its objectives given these circumstances. Notice that here we establish a hierarchy between the controls and for this reason we denominate this as a problem of *Hierarchical Control*. The scientist and economist Von Stackelberg was the one who introduced this method in an economy context in a completely different way (see [57]) and the mathematician J. L. Lions adapted it for the case of PDEs control (see [42, 43]). Once the Stackelberg method is implemented, we can choose an optimization strategy that the follower controls must adopt. In this sense, we can talk about cooperative strategies, that is, which the controls cooperate in order to reach their objectives or about noncooperative strategies, where each follower is interested in achieving their objective without worrying about how much this can hinder the objective of the other controls. The cooperative strategies are commonly called *Pareto Strategy* while noncooperative ones are called *Nash Strategies*, these names are due to the fact that the economist Vilfredo Pareto and the mathematician John Nash were the ones who proposed these concepts for efficiency problems in economy. When we unite the Stackelberg method with strategies of the Pareto type, we obtain what is called the Stackelberg-Patero Method and when we unite it with strategies of the Nash type, we obtain the Stackelberg-Nash Method.

Our study is guided by multi-objective control problems, initiated by J. L. Lions in [42] and directed to Parabolic Equation systems, as well as to the Korteweg-de Vries equation

(KdV), which is of the dispersive kind. Next, we are going to talk in detail about the specific objectives to be addressed in this thesis.

Specific Objectives

Now, let us consider a general system in order to specifically understand the mathematical structure of the strategy to which we report. We are going to see how the Stackelberg-Nash strategy can be applied to the control problems of Partial Differential Equations. For simplicity, we will only consider three controls.

Being N a positive natural number, $\Omega \subset \mathbb{R}^N$ an open set and T a real number, define $Q = \Omega \times (0,T)$ e $\Sigma = \partial\Omega \times (0,T)$, where $\partial\Omega$ represents the boundary of Ω. Also consider the evolution equation

$$\begin{cases} y_t + Ay + F(y) = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y(\cdot,0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where A is a linear unbounded operator and the function $F : \mathbb{R} \to \mathbb{R}$ is, possibly, nonlinear. In the equation (1.2), the function f is the leader control while v^1, v^2 are the follower controls. The functions $1_{\mathcal{O}}$ and $1_{\mathcal{O}_i}$ $(i = 1,2)$ are characteristic functions in the regions $\mathcal{O}$, $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Here, the leader chooses the type of controllability of the system (1.2) (exact, approximate, null etc.) and later the followers (v^1, v^2) must meet a certain equilibrium condition in relation to the functionals

$$J_i(f, v_1, v_2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 \, dxdt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v^i|^2 \, dxdt, \quad i = 1,2, \tag{1.3}$$

where $\alpha_i, \mu_i > 0$ are constants and $y_{i,d} = y_{i,d}(x,t)$ are given functions.

In the problems here addressed, we are going to assign as the leader's objective the null controllability, that is, the leader control f must be such that the solution y of (1.3) satisfies $y(\cdot,T) = 0$, while the followers must minimize the functionals (1.3) simultaneously, in the sense that having assigned f and its objective, we are going to look for controls $v_i \in L^2(\mathcal{O}_i \times (0,T))$, $i = 1,2$, that satisfy

$$J_1(f, v^1, v^2) = \min_{\hat{v}^1} J_1(f, \hat{v}^1, v^2), \quad J_2(f, v^1, v^2) = \min_{\hat{v}^2} J_1(f, v^1, \hat{v}^2). \tag{1.4}$$

We can understand this problem in the following manner: each follower v^1 and v^2 is interested in having the y state reach the objectives $y_{1,d}$ and $y_{2,d}$ respectively. However, because of the equation's structure or even because of the quantity of simultaneous objectives of interest, it can be impossible to reach these targets directly. Thus, finding controls that satisfy (1.4) establishes the best condition we can get given the circumstances. It is important to notice that each follower is interested in minimizing their functional without worrying about worsening each other's objective, this characterizes a noncooperative problem (a control that does not have access to the actions of another) and, in this manner, the pair (v^1, v^2) is called Nash Equilibrium. By adopting this hierarchy between the controls and, furthermore, requiring that the followers are in Nash Equilibrium, we are adopting the Stackelberg-Nash method for the system (1.2).

It is important to note that if the functionals $J_i (i = 1, 2)$ are convex, then $(v^1, v^2) \in L^2(\mathcal{O}_1 \times (0,T)) \times L^2(\mathcal{O}_2 \times (0,T))$ is a Nash Equilibrium if, and only if,

$$\begin{aligned} J_1'(f, v^1, v^2)(\hat{v}^1, 0) = 0, \quad \forall \hat{v}^1 \in L^2(\mathcal{O}_1 \times (0,T)); \\ J_2'(f, v^1, v^2)(0, \hat{v}^2) = 0, \quad \forall \hat{v}^2 \in L^2(\mathcal{O}_2 \times (0,T)). \end{aligned} \tag{1.5}$$

In addition, we reinforce that (1.4) and (1.5) are not necessarily equivalent. In fact, if the function F in (1.2) is nonlinear, the functionals J_i cannot be convex and therefore the equivalence is not immediate.

Given what has been exposed, if functionals J_i are convex, we are able to use the condition (1.5) (see [43] for more details) and we will find that the Nash equilibrium can be characterized by the following condition of optimality

$$v^i = -\frac{1}{\mu_i}\varphi^i \mathbb{1}_{\mathcal{O}_i}, \quad i = 1, 2, \tag{1.6}$$

where φ^i is the solution for the system

$$\begin{cases} -\varphi^i_t - A^*\varphi^i + F'(y)\phi^i = \alpha_i (y - y_{i,d}) \mathbb{1}_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \varphi^i(x, T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.7}$$

Once the Nash equilibrium has been determined for a fixed f, the following step is to determine a leader control $\hat{f} \in L^2(\mathcal{O} \times (0,T))$ such that the solution of (1.2) is the value that we desire for the final time. In the cases seen here the objective of the leader is to conduct the system to zero, that is, we want the component y of the coupled system (1.2), (1.6) and (1.7) to satisfy

$$y(\cdot, T) = 0 \quad \text{em} \quad \Omega.$$

This thesis is divided in two chapters, such that the results are presented in the format of scientific articles. In each of them, we will solve a multi-objective problem for a determined evolution equation, using the Stackelberg-Nash method, whose objective of the leader is to conduct the system to the null state. Next, we will make a more specific description of the addressed problems and the obtained results.

Chapter 1 - Null Controllability and Stackelberg-Nash Strategy for a 2×2 Systems of Parabolic Equations

The first presented problem consists of a system of parabolic equations, with zero-order couplings of the structure:

$$\begin{cases} y^1_t - \Delta y^1 = \sum_{p=1}^{2} A_{1\,p} y^p + F^1 & \text{in} \quad Q, \\ y^2_t - \Delta y^2 = \sum_{p=1}^{2} A_{2\,p} y^p + F^2 & \text{in} \quad Q, \\ y^1 = y^2 = 0 & \text{in} \quad \Sigma, \\ y^1(0) = y^1_0,\, y^2(0) = y^2_0 & \text{in} \quad \Omega, \end{cases} \tag{1.8}$$

where (y^1, y^2) represents the state, (F^1, F^2) are controls and $\{A_{i,j}\}_{i,j=1,2}$ are uniformly bounded functions and represent the coupling matrix of the system. Each F^i control has three objectives for the state (y^1, y^2) that are to be described below.

Through the Stackelberg Method, we write

$$F^i = f^i \mathbb{1}_{\mathcal{O}} + v^{i\,1} \mathbb{1}_{\mathcal{O}_{i\,1}} + v^{i\,2} \mathbb{1}_{\mathcal{O}_{i\,2}}, \; i = 1, 2, \tag{1.9}$$

where the sets $\mathcal{O}$, $\mathcal{O}_{i\,1}$ and $\mathcal{O}_{i\,2}$ are open and disjoint. The $\{f^i\}_{i=1}^2$ controls will be the leaders, being responsible for the objectives of the controllability type, while $\{v^{i\,j}\}_{i,j=1}^2$ are called followers and are engaged in the minimization of the cost functionals that will be defined next.

For $k, l = 1, 2$, being $y^{k\,l}_d$ regular functions defined under regions $\mathcal{O}^{k\,l}_d \times (0, T)$ and $\{\mu_{k\,l}\}_{k\,l=1}^2$ real parameters. In these terms, we define the functionals

$$J^{k\,l}(\{v^{i\,j}\}_{i,j=1}^2) = \frac{1}{2} \iint_{\mathcal{O}^{k\,l}_d \times (0,T)} |y^l - y^{k\,l}_d|^2 \, dxdt + \frac{\mu_{k\,l}}{2} \iint_{\mathcal{O}_{k\,l} \times (0,T)} |v^{k\,l}|^2 \, dxdt, \quad k, l = 1, 2. \tag{1.10}$$

Our interest is to assign the objective of the (f^1, f^2) leaders to be the property of null controllability for the variable (y^1, y^2), while the $\{v^{i\,k}\}$ followers must be a Nash equilibrium (see (1.5)) for the functionals (1.10). In this type of problem, we are also interested in the case where less controls are considered, that is, $F^i = 0$ for some $i \in \{1, 2\}$ and we will see that this situation results in some additional technical difficulties.

The situations where the objective of the leader control is only the system controllability, which is characterized as a mono-objective problem, are already found in the literature. Indeed, when we only consider a heat equation instead of the system (1.8) and when the objective of the leader is unique and of the controllability type, we can quote the classical references [23, 31, 32, 37]. In the case of parabolic equation systems where we have only a single objective per control, positive and negative controllability results are known, and for more details see [3, 2, 21, 22]. In the context of multi-objective problems, we can mention the works [5, 6] for the case of a single heat equation and the leader's objective being conducting the system to zero, while in [29] the Stokes system and the objective of the leader were considered as being the approximate controllability. In the case of multi-objective control problem for systems of parabolic equations, in which the leader objective is to conduct the system to the null state, we can cite [30], where a problem similar to the one we propose is formulated, though the news here is that we proved a control result that stays true under the more general conditions not considered in [30].

Considering what has been said, in this chapter we are interested in solving two types of problem:

Problem 1: Consider (y_0^1, y_0^2) *and* $\{A_{j\,p}\}$*, respectively, to be the initial data and the coupling coefficients for* (1.8)*. Utilizing the Stackelberg-Nash method and writing the controls in hierarchy in the structure* (1.9)*, there are* $\{f^i\}$ *leader controls such that the solution of* (1.8) *satisfies*

$$y^i(\cdot, T) = 0, \quad \textit{for}, \quad i = 1, 2,$$

where the followers $\{v^{i\,j}\}$ *are in Nash equilibrium for the functionals* (1.10)*.*

In a second result, we considered a system with one missing control, such as $F^2 = 0$

$$\begin{cases} y_t^1 - \Delta y^1 = \sum_{p=1}^2 A_{1\,p} y^p + F^1 & \text{in} \quad Q, \\ y_t^2 - \Delta y^2 = \sum_{p=1}^2 A_{2\,p} y^p & \text{in} \quad Q, \\ y^1 = y^2 = 0 & \text{in} \quad \Sigma, \\ y^1(0) = y_0^1,\, y^2(0) = y_0^2 & \text{in} \quad \Omega. \end{cases} \tag{1.11}$$

Again, through the Stackelberg method, we put down the controls in a hierarchy

$$F^1 = f^1 1_{\mathcal{O}} + v^{1\,1} 1_{\mathcal{O}_{i\,1}} + v^{1\,2} 1_{\mathcal{O}_{i\,2}}, \; i = 1, 2, \tag{1.12}$$

and considered the functionals

$$J^k(\{v^j\}_{j=1}^2) = \frac{1}{2} \iint_{\mathcal{O}_d^k \times (0,T)} |y^k - y_d^k|^2 \, dxdt + \frac{\mu_k}{2} \iint_{\mathcal{O}_k \times (0,T)} |v^k|^2 \, dxdt, \quad k = 1, 2. \tag{1.13}$$

Next, we wish to solve the problem

Problem 2: Consider (y_0^1, y_0^2) *and* $\{A_{j\,p}\}$*, respectively, to be the initial data and the coupling coefficients for* (1.11)*. Utilizing the Stackelberg-Nash method and writing the controls in hierarchy in the structure* (1.12)*, there is a leader control* f^1 *such that the solution of* (1.11) *satisfies*

$$y^i(\cdot, T) = 0, \quad \textit{for}, \quad i = 1, 2,$$

where the followers $\{v^{1\,j}\}$ *are in Nash equilibrium for the functionals* (1.13)*.*

It is important to stress that, in the reference [30], the authors dedicated themselves to the study of a situation similar to the Problem 2 above. One difference is that they considered functionals of the structure:

$$\tilde{J}^k(\{v^j\}_{j=1}^2) = \frac{1}{2}\iint_{\mathcal{O}_d\times(0,T)} \left(|y^1 - y_d^{1\,k}|^2 + |y^2 - y_d^{2\,k}|^2\right) dxdt + \frac{\mu_k}{2}\iint_{\mathcal{O}_k\times(0,T)} |v^k|^2\, dxdt, \quad k = 1, 2, \tag{1.14}$$

where $\mathcal{O}_d$ is an open of Ω, while here we considered the functionals of the structure of the (1.10). The fact that they have considered in each functional the terms y^1 and y^2 is not a complicating factor, that is, we can solve Problem 2 by replacing the functional (2.9) by

$$J^k(\{v^j\}_{j=1}^2) = \frac{1}{2}\iint_{\mathcal{O}_d^k\times(0,T)} \left(|y^1 - y_d^{1\,k}|^2 + |y^2 - y_d^{2\,k}|^2\right) dxdt + \frac{\mu_k}{2}\iint_{\mathcal{O}_k\times(0,T)} |v^k|^2\, dxdt, \quad k = 1, 2. \tag{1.15}$$

Our reason for choosing functional (1.10) is for merely simplifying some notations adopted throughout the text. On the other hand, note that in (1.14) in both functionals the region $\mathcal{O}_d$ is the same. This hypothesis brings great simplifications from the technical point of view. We emphasize that here we consider the cases in the sets $\mathcal{O}_d^k$, for $k = 1, 2$, as distinct, this being the main contribution we presented in this chapter. Moreover, observe that in Problem 1 we can solve the situation in which two controls are considered. Although intuitively it seems as if Problem 2 is of higher difficulty than Problem 1, since there are less controls to achieve the objective, this is not necessarily true in the problems we have considered here. The reason for this is that, in Problem 1, we have associated six objectives to the controls, while in Problem 2 we have associated three. In other words, we have decreased the control quantity but also the objective quantity, that is to say, the problem with two controls is also relevant for taking into account.

The null controllability in the cases cited above is obtained through the Hilbert Uniqueness Method (HUM) ([39], [40], [41]), which establishes the connection between the controllability property and a quantitative property known as observability inequality.

Chapter 2 - On the Stackelberg-Nash Exact Controllability for KdV Equation

In chapter 2, we have considered the nonlinear Korteweg-de Vries equation (KdV) and have also applied the Stackelberg-Nash method combined with a zero control objective for the leader:

$$\begin{cases} y_t + (1+y)y_x + y_{xxx} = f\mathbb{1}_{\mathcal{O}} + v^1\mathcal{X}_{\mathcal{O}_1} + v^2\mathcal{X}_{\mathcal{O}_2} & \text{in} \quad Q, \\ y\,(0,\cdot) = y\,(L,\cdot) = y_x\,(L,\cdot) = 0 & \text{in} \quad (0,T)\,, \\ y\,(x,\cdot) = y^0 & \text{in} \quad (0,1)\,, \end{cases} \tag{1.16}$$

where $y = y(x,t)$ is the state and y^0 is initial datum. In (1.16), the set $\mathcal{O} \subset (0,1)$ is the *leader control's domain* f while $\mathcal{O}_1, \mathcal{O}_2 \subset (0,1)$ are the *follower control's domains* v^1 and v^2 (all of which are very small and disjoint suppositions). With $\mathbb{1}_{\mathcal{O}}$ we denoted the characteristic function in $\mathcal{O}$ while $\mathcal{X}_{\mathcal{O}_i}$ are positive functions in $C_0^\infty(\mathcal{O}_i)$, $i = 1, 2$.

Let us fix an uncontrolled trajectory, which is a sufficiently regular solution of the system

$$\begin{cases} \overline{y}_t + (1+\overline{y})\overline{y}_x + \overline{y}_{xxx} = 0 & \text{in} \quad Q, \\ \overline{y}\,(0,\cdot) = \overline{y}\,(L,\cdot) = \overline{y}_x\,(L,\cdot) = 0 & \text{in} \quad (0,T)\,, \\ \overline{y}(\cdot,0) = \overline{y}^0 & \text{in} \quad (0,1)\,. \end{cases} \tag{1.17}$$

Here, we establish as objective of the leader control f the one of taking the system exactly to the trajectory in the final time T, that is

$$y(\cdot,T) = \overline{y}(\cdot,T) \quad \text{in} \quad (0,1). \tag{1.18}$$

It is critical to note that, defining a new variable $z = y - \overline{y}$, we can reduce the exact controllability condition above to a null controllability property for z, that is

$$z(x,T) = 0 \quad \text{in} \quad (0,1). \tag{1.19}$$

Let us note that z satisfies the equation

$$\begin{cases} z_t + (1+z)z_x + z_{xxx} + (\overline{y}z)_x = f\,\mathbb{1}_{\mathcal{O}} + v^1\mathcal{X}_{\mathcal{O}_1} + v^2\mathcal{X}_{\mathcal{O}_2} & \text{in} \quad Q, \\ z\,(0,\cdot) = z\,(L,\cdot) = z_x\,(L,\cdot) = 0 & \text{in} \quad (0,T)\,, \\ z(\cdot,0) = z^0 & \text{in} \quad (0,1)\,. \end{cases} \tag{1.20}$$

Once the leader's requirement is assigned, the followers must be in Nash equilibrium (see (1.4)) for the cost functionals

$$J^k(\{v^j\}_{j=1}^2) = \frac{1}{2}\iint_{\mathcal{O}_d^k\times(0,T)} |z - z_d^k|^2\,dxdt + \frac{\mu_k}{2}\iint_{\mathcal{O}_k\times(0,T)} |v^k|^2\,dxdt, \quad k = 1,2. \tag{1.21}$$

We are able to find several results of mono-objective control for the KdV equation in the literature. Let us start citing the pioneering works of Russel and Zhang [56], where periodic boundary conditions are considered and, since then, several controllability and stability results have been considered [15, 16, 18, 27, 26, 52, 54, 60]. In the subject of multi-objective problems, particularly in the application of the Stackelberg-Nash method allied to controllability objectives for the leader, several results are available for parabolic problems (see [5, 6]) and also for fourth-order equations such as the Kuramoto-Sivashinsky equation (see [13]), however, little is currently known for the dispersive equations such as the KdV equation. Because of its completely different structure, compared to the parabolic equations considered in Chapter 1, the arguments must be adapted to this situation. Furthermore, the equation is considered nonlinear, raising the difficulty level of the problem's resolution even more. In fact, we need to assume here that the problem's data are sufficiently small.

Thus, in this chapter we are interested in working with the following problematic: *Problem 3: Being* z^0 *the initial datum for* (3.9). *There is a leader control* f *such that the solution of* (3.9) *satisfies*

$$z(\cdot, T) = 0,$$

where the followers $\{v^j\}$ *are in Nash equilibrium for the functionals* (1.21).

Let us note that the problem considered above is nonlinear. This makes it so that the currently known methods cannot be applied directly to the system (3.9). Thus, we have considered a linear version of (3.9) and applied the Hilbert Uniqueness Method to arrive to an equivalence between the linear system's controllability and an associated observability inequality. Next, we are going to apply a local inversion theorem in order to retrieve the result for the nonlinear system.

2 NULL CONTROLLABILITY AND STACKELBERG-NASH STRATEGY FOR A 2×2 SYSTEM OF PARABOLIC EQUATIONS

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Abstract

This chapter is dedicated to solve a multi-objective control problem for a 2×2 system of parabolic equations. Here, we have many objectives, possibly conflictive, and a concept of hierarchy must be adopted. We have the *leader* controls, responsible for objectives of controllability type, and the *follower* controls which intend to be in Nash-equilibrium with respect to some given cost functionals. The novelty in here is that we formulate this problem for systems of parabolic equations, meaning that we have much more variables and naturally much more objectives to accomplish.

Keywords: Nash Equilibrium, Stackelberg's Method, Null Controllability, Carleman Estimates.

2.1 Introduction

In a standard controllability problem, we have an evolution equation describing a specific phenomenon, and we look for controls such that the solutions of this system behaves the way we want in a given finite time. In real life, control problems can be much more complex, specially when many objectives are required at the same time. It is not difficult to realize that many objectives may be in conflict, in the sense that solving one of them turns the solvability of the others difficult or even impossible, and that is why concepts of equilibrium are usually considered.

There are several concepts of equilibrium for multi-objective control problems, with origin in game theory and mainly motivated by problems of economy, each of them determines a strategy. Thus, we mention the non-cooperative optimization strategy proposed by J. Nash in [47], the Pareto cooperative strategy in [48], and the Stackelberg hierarchic-cooperative strategy in [57].

Concerning multi-objective control problems for PDEs, a relevant question that can be formulated is whether one is able to exactly (or approximately) control the system to a

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desired state when many other objectives are also being considered at the same time. We refer to the classical works of J. L. Lions in [38, 43] where some results concerning Pareto and Nash strategies are applied for control problems in PDEs, and we also refer the book of I. Diaz and J. Lions in [20] as a complement. We remark that the approaches in [38, 43] are based on the fact that the leader wants to control the state approximately, a more difficult question is whether a similar problem can be solved in the exact controllability level, in this case we can refer the works of Araruna et al. in [5] and [6] for some positive exact controllability results.

Concerning multi-objective control problems for systems of parabolic equations, very few is known. We can cite the work of Hernandez-Santamaria et al. in [30], where a 2×2 system of parabolic equations is considered, and the authors have assumed restrictions over the objectives very similar to the ones in [5]. However, it is proved in [6] that, for the case of one single equation, more general conditions for the objectives can be assumed compared to the ones in [5]. In this way, it natural to ask if the results in [6] can be adapted for the cases of systems of parabolic equations. This will be the main goal of this paper.

2.2 Statement of the Problem

In this section, we will be more precise on the problem under view. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$, $Q = \Omega \times]0,T[$ and $\Sigma = \partial\Omega \times]0,T[$. We consider systems of the form

$$\begin{cases} y^1_t - \Delta y^1 = \displaystyle\sum_{p=1}^{2} A_{1p} y^p + F^1 & \text{in } \quad Q, \\ y^2_t - \Delta y^2 = \displaystyle\sum_{p=1}^{2} A_{2p} y^p + F^2 & \text{in } \quad Q, \\ y^1 = y^2 = 0 & \text{in } \quad \Sigma, \\ y^1(0) = y^1_0,\, y^2(0) = y^2_0 & \text{in } \quad \Omega, \end{cases} \tag{2.1}$$

where (y^1, y^2) represents the state, (F^1, F^2) are the controls and $\{A_{i,j}\}_{i,j=1,2}$ is the coupling matrix, where $A_{i,j} \in L^\infty(Q)$ for every $i, j = 1, 2$. Each control F^i wants the state (y^1, y^2) to accomplish three objectives, one of controllability type and other two of optimal control type.

The problem of finding one control solving more than one objective is in general ill posed, that is why we adopt the Stackelberg's strategy, which consists in divide each control F^i into three controls, each one responsible for its own objective. So we write

$$F^i = f^i \mathbb{1}_{\mathcal{O}} + v^{i\,1} \mathbb{1}_{\mathcal{O}_{i\,1}} + v^{i\,2} \mathbb{1}_{\mathcal{O}_{i\,2}}, \; i = 1, 2, \tag{2.2}$$

where for each $i = 1,2$ the sets $\mathcal{O}$, $\mathcal{O}_{i\,1}$ and $\mathcal{O}_{i\,2}$ are open sets, pairwise disjoint. The controls $\{f^i\}_{i=1}^2$ are called the *leaders*, having the objectives of controllability type, in the sense they want to drive the state to a exact value, while $\{v^{i\,j}\}_{i,j=1}^2$ are called the *followers*, with optimal control type objectives, in the sense they must be optimal with respect some specific functionals.

For $k, l = 1, 2$, let $y_d^{k\,l}$ be regular functions defined on a region $\mathcal{O}_d^{k\,l} \times (0,T)$ and $\{\mu_{k\,l}\}_{k\,l=1}^2$ real positive parameters nonzero. We define the functionals

$$J^{k\,l}\big(\{v^{i\,j}\}_{i,j=1}^2\big) = \frac{1}{2}\iint_{\mathcal{O}_d^{k\,l}\times(0,T)} |y^l - y_d^{k\,l}|^2\,dxdt + \frac{\mu_{k\,l}}{2}\iint_{\mathcal{O}_{k\,l}\times(0,T)} |v^{k\,l}|^2\,dxdt, \quad k,l = 1,2. \tag{2.3}$$

For each $f := \{f^i\}_{i=1}^2$ fixed, the followers intend to be a *Nash equilibrium* for the functionals $\{J^{k\,l}\}_{k,l=1}^2$, meaning that $\{v^{i\,j}\}_{i,j=1}^2$ must satisfy

$$\begin{aligned}
J^{1\,1}(\{v^{i\,j}\}_{i,j=1}^2) &= \min_{\hat{v}^{1\,1}\in L^2(\mathcal{O}_{1\,1}\times(0,T))} J^{1\,1}(\hat{v}^{1\,1}, v^{1\,2}, v^{2\,1}, v^{2\,2}),\\
J^{1\,2}(\{v^{i\,j}\}_{i,j=1}^2) &= \min_{\hat{v}^{1\,2}\in L^2(\mathcal{O}_{1\,2}\times(0,T))} J^{1\,2}(v^{1\,1}, \hat{v}^{1\,2}, v^{2\,1}, v^{2\,2}),\\
J^{2\,1}(\{v^{i\,j}\}_{i,j=1}^2) &= \min_{\hat{v}^{2\,1}\in L^2(\mathcal{O}_{2\,1}\times(0,T))} J^{2\,1}(v^{1\,1}, v^{1\,2}, \hat{v}^{2\,1}, v^{2\,2}),\\
J^{2\,2}(\{v^{i\,j}\}_{i,j=1}^2) &= \min_{\hat{v}^{2\,2}\in L^2(\mathcal{O}_{2\,2}\times(0,T))} J^{2\,2}(v^{1\,1}, v^{1\,2}, v^{2\,1}, \hat{v}^{2\,2}).
\end{aligned} \tag{2.4}$$

This is a non cooperative optimization problem. For each $k, l = 1, 2$, the follower $v^{k\,l}$ wants to minimize the objective $J^{k\,l}$ even if this represents the worst scenario for the other objectives.

Since the equations are linear and the functionals are convex in each direction, it is clear that the conditions in (2.4) are completely equivalent to find $\{v^{i\,j}\}_{i,j=1}^2$ satisfying

$$\frac{\partial J^{k\,l}}{\partial \hat{v}^{k\,l}}\big(\{v^{i\,j}\}_{i,j=1}^2\big) = 0 \ \text{ for every } \ k,l = 1,2. \tag{2.5}$$

Then, it is standard (see Section 2.1.2 of [5]) that the Nash equilibrium is characterized by formula

$$v^{k\,l} = -\frac{1}{\mu_{k\,l}}\varphi^{l,\,k\,l}\mathbb{1}_{\mathcal{O}_{k\,l}}, \quad k,l = 1,2, \tag{2.6}$$

where the function $\varphi^{j,\,k\,l}$ satisfies the following optimality system

$$\begin{cases}
-\varphi_t^{l,\,k\,l} - \Delta\varphi^{l,\,k\,l} = \sum_{p=1}^2 A_{p\,l}\varphi^{p,\,k\,l} + (y^l - y_d^{k\,l})\mathbb{1}_{\mathcal{O}_d^{k\,l}}, & \text{in} \quad Q,\\
-\varphi_t^{j,\,k\,l} - \Delta\varphi^{j,\,k\,l} = \sum_{p=1}^2 A_{p\,j}\varphi^{p,\,k\,l}, \ j \neq l, & \text{in} \quad Q,\\
\varphi^{j,\,k\,l} = 0, & \text{over} \ \ \Sigma,\\
\varphi^{j,\,k\,l}(\cdot, T) = 0, & \text{in} \quad \Omega,
\end{cases} \tag{2.7}$$

for $j, k, l = 1, 2$.

Once the Nash equilibrium is determined, according formula (2.6), we search for leader controls which drives the state (y^1, y^2) to zero. In this way, we are led to prove a null controllability result for the following system,

$$\begin{cases} y_t^j - \Delta y^j = \sum_{p=1}^2 A_{j\,p} y^p + f^j \mathbb{1}_{\mathcal{O}} - \sum_{p=1}^2 \frac{1}{\mu_{jp}} \varphi^{p,\,j\,p} \mathbb{1}_{\mathcal{O}_{jp}}, & \text{in } \ Q, \\ y^j = 0, & \text{in } \ \Sigma, \\ y^j(0) = y_0^j, & \text{in } \ \Omega, \end{cases} \tag{2.8}$$

for $j = 1, 2$. In this way, the first result we are interested to solve is the following:

Theorem 2.1. *Given* $(y_0^1, y_0^2) \in L^2(Q)^2$ *and* $\{v^{i\,j}\}_{i,j=1}^2$ *a Nash equilibrium for the functional* (2.3)*, there exists controls* $(f^1, f^2) \in L^2(\mathcal{O} \times (0,T))^2$*, such that the solution of* (2.1) *with* $\{F^i\}_{i=1}^2$ *given by* (2.2) *satisfies* $(y^1(T), y^2(T)) = (0, 0)$*.*

The main dificulty to prove Theorem 2.1 is that, once the Nash equilibrium is characterized, we are led to prove a partial null controllability result for a system of several equations (ten precisely) acting only over two equations. We remark that, for the case of one single equation (see [5, 6]) the optimality system is composed by three equations only.

Another interesting question which arises is the case where less controls are considered. Indeed, let us consider in (2.1) the control $F^2 = 0$ and let us follow a similar strategy. Now, for $k = 1, 2$ we consider y_d^k regular functions, defined over a region $\mathcal{O}_d^k \times (0,T)$ and $\{\mu_k\}_{k=1}^2$ are real parameters. Here we define the functionals

$$J^k\big(\{v^j\}_{j=1}^2\big) = \frac{1}{2} \iint_{\mathcal{O}_d^k \times (0,T)} |y^k - y_d^k|^2 \, dxdt + \frac{\mu_k}{2} \iint_{\mathcal{O}_k \times (0,T)} |v^k|^2 \, dxdt, \quad k = 1, 2, \tag{2.9}$$

and we search for followers satisfying

$$\begin{aligned} J^1(\{v^j\}) &= \min_{\hat{v}^1 \in L^2(\mathcal{O}_1 \times (0,T))} J^1(\hat{v}^1, v^2), \\ J^2(\{v^j\}) &= \min_{\hat{v}^2 \in L^2(\mathcal{O}_2 \times (0,T))} J^1(v^1, \hat{v}^2). \end{aligned} \tag{2.10}$$

In this case, we find that

$$v^j = -\frac{1}{\mu_j} \varphi^{j,j} \mathbb{1}_{\mathcal{O}_j}, \quad j = 1, 2, \tag{2.11}$$

where, for $i, j = 1, 2$

$$\begin{cases} -\varphi_t^{j,j} - \Delta \varphi^{j,j} = \sum_{p=1}^2 A_{p\,j} \varphi^{p,j} + (y^j - y_d^j) \mathbb{1}_{\mathcal{O}_d^j}, & \text{in } \quad Q, \\ -\varphi_t^{i,j} - \Delta \varphi^{i,j} = \sum_{p=1}^2 A_{p\,i} \varphi^{p,j}, \quad i \neq j & \text{in } \quad Q, \\ \varphi^{i,j} = 0, & \text{over } \ \Sigma, \\ \varphi^{i,j}(\cdot, T) = 0, & \text{in } \quad \Omega. \end{cases} \tag{2.12}$$

In this context, the second problem we want to solve is the following:

Theorem 2.2. *Given* $(y_0^1, y_0^2) \in L^2(Q)^2$ *and* $\{v^j\}_{j=1}^2$ *a Nash equilibrium for the functional* (2.9)*, there exist a control* $f^1 \in L^2(\mathcal{O} \times (0,T))^2$ *such that the solution of* (2.1) *with* F^1 *given by* (2.2) *and* $F^2 = 0$ *satisfies* $(y^1(T), y^2(T)) = (0,0)$.

In Theorem 2.2, we have to partially control a system of several equations (six precisely) acting over one single equation. So, in one hand we have less equations, but in other had we are taking only one leader to exactly control the system. This means that proving Theorem 2.1 can not be seen as a natural step to follow in order to prove Theorem 2.2. Indeed, for problem 1 we have the optimality system with ten equations and two leaders while for problem 2 the optimality system has six equation for only one leader. In this way, it is not clear that Theorem 2.2 is more difficult to prove then Theorem 2.1.

It is well know, by the Hilbert Uniqueness Method, that the controllability of linear systems of PDEs is equivalent to a suitable observability inequality for adjoint states. In order to prove such inequalities we will make use of some suitable Carleman estimates.

2.3 Carleman Estimates

This section is dedicated to prove some Carleman estimates for a suitable adjoint state. We will start this section by introducing some assumptions over the sets $\{\mathcal{O}_d^{k\,l}\}_{k,l=1}^2$ and $\{\mathcal{O}_d^k\}_{k=1}^2$.

To deal with Theorem 2.1, we start assuming that

$$\mathcal{O}_d^{k\,l} \cap \mathcal{O} \neq \emptyset, \quad \text{for every} \quad k,l \in \{1,2\}. \tag{2.13}$$

Under assumption (2.13), there exists open subsets satisfying

$$\omega^{k\,l} \subset\subset \hat{\omega}^{k\,l} \subset\subset \tilde{\omega}^{k\,l} \subset\subset \mathcal{O}_d^{k\,l} \cap \mathcal{O}, \quad \text{for every} \quad k,l \in \{1,2\}. \tag{2.14}$$

We also assume that, for each $l = 1,2$, one of the two conditions

$$\mathcal{O}_d^{1\,l} = \mathcal{O}_d^{2\,l} \tag{2.15}$$

or

$$\mathcal{O}_d^{1\,l} \cap \mathcal{O} \neq \mathcal{O}_d^{2\,l} \cap \mathcal{O} \tag{2.16}$$

hold. Assumptions similar to (2.15) or (2.16) are already used in [5, 6] for the case of one single equation to be controlled.

Notice that, if assumption (2.16) hold for some $l \in \{1,2\}$, two different situations may occur. Or we have that

$$\mathcal{O}_d^{k\,l} \cap \mathcal{O} \subset \mathcal{O}_d^{\tilde{k}\,l} \cap \mathcal{O} \quad \text{and} \quad \mathcal{O}_d^{\tilde{k}\,l} \cap \mathcal{O} \setminus \mathcal{O}_d^{k\,l} \cap \mathcal{O} \neq \emptyset, \quad \text{for some} \quad (k,\tilde{k}) \in \{(1,2),(2,1)\} \tag{2.17}$$

or that

$$\mathcal{O}_d^{k\,l} \cap \mathcal{O} \setminus \mathcal{O}_d^{\tilde{k}\,l} \cap \mathcal{O} \neq \emptyset \quad \text{and} \quad \mathcal{O}_d^{\tilde{k}\,l} \cap \mathcal{O} \setminus \mathcal{O}_d^{k\,l} \cap \mathcal{O} \neq \emptyset, \quad \text{for some} \quad (k, \tilde{k}) \in \{(1,2),(2,1)\}. \tag{2.18}$$

We remark that, for different values of $l \in \{1, 2\}$, we may have combinations of properties (2.15), (2.17) or (2.18). This represents an additional difficulty when dealing with systems of equations. It is not difficult to see that, if (2.17) hold, then the open sets satisfying (2.14) can be assumed in such a way that

$$\tilde{\omega}^{\tilde{k}\,l} \cap \mathcal{O}_d^{k\,l} = \emptyset. \tag{2.19}$$

In a similar way, if (2.18) hold besides (2.19), we can assume that the open sets satisfying (2.14) satisfies (2.19) and

$$\tilde{\omega}^{k\,l} \cap \mathcal{O}_d^{\tilde{k}\,l} = \emptyset. \tag{2.20}$$

These conditions will be important in forthcoming computations.

Concerning the coupling coefficients, we will assume that there exists a constant $C_0 > 0$ such that

$$A_{l\,p} \geq C_0 \quad \text{in} \quad \overline{\tilde{\omega}^{1\,l} \cup \tilde{\omega}^{2\,l}} \times [0, T], \quad \text{for all} \quad (p, l) \in \{(1,2),(2,1)\}. \tag{2.21}$$

To deal with Theorem 2.2, some slightly different conditions are needed. Precisely, in order to prove a controllability result with less controls we have to assume some additional conditions on the coupling coefficients. To start, we assume that

$$\mathcal{O}_d^i \cap \mathcal{O} \neq \emptyset, \quad \text{for every} \quad i = 1, 2. \tag{2.22}$$

As in the previous case, some conditions over the sets $\{\mathcal{O}_d^i\}_i$ are needed. Here, we will assume that

$$(\mathcal{O}_d^i \cap \mathcal{O}) \setminus \overline{supp\, A_{j,\,i} \cap \mathcal{O}} \neq \emptyset, \quad \text{and} \quad Int(supp\, A_{j,\,i} \cap \mathcal{O}) \setminus \overline{(\mathcal{O}_d^i \cap \mathcal{O})} \neq \emptyset, \tag{2.23}$$

for some $i, j = 1, 2,\ i \neq j$. Moreover, under assumption (2.22), we assume the existence of open subsets

$$\omega^i \subset\subset \hat{\omega}^i \subset\subset \tilde{\omega}^i \subset\subset \mathcal{O}_d^i \cap \mathcal{O}, \quad \text{for every} \quad i = 1, 2, \tag{2.24}$$

such that for the pair (i, j) where (2.23) is valid we have that

$$\tilde{\omega}^i \subset (\mathcal{O}_d^i \cap \mathcal{O}) \setminus \overline{supp\, A_{j,\,i} \cap \mathcal{O}}, \quad \text{and} \quad \tilde{\omega}^j \subset supp\, A_{j,\,i} \cap \mathcal{O} \setminus \overline{(\mathcal{O}_d^i \cap \mathcal{O})} \tag{2.25}$$

and there exists $C_0 > 0$ such that

$$A_{i,j} \geq C_0 \quad \text{in} \quad \overline{\tilde{\omega}^i \times (0, T)}, \tag{2.26}$$

for every $i, j = 1, 2$, $i \neq j$. Notice that, for the case of one single leader, more assumptions over the coupling coefficients are needed.

It is well know, by the Hilbert Uniqueness Method, that controllability of linear systems of PDEs is equivalent to a suitable observability inequality for an adjoint variable. In this way, the resolution of Theorems 2.1 or 2.2 presented here are completely equivalent to the following theorem, which is the main result of this paper.

Theorem 2.3.
i) For $j, k, l = 1, 2$ let $(\psi^j, \gamma^{j,kl})$ the solution of the following adjoint system

$$\begin{cases} -\psi^j_t - \Delta\psi^j = \sum_{p=1}^2 A_{pj}\psi^p + \sum_{p=1}^2 \gamma^{j,pj}\mathbb{1}_{\mathcal{O}_d^{pj}} & in \quad Q, \\ \gamma^{j,kl}_t - \Delta\gamma^{j,kl} = \sum_{p=1}^2 A_{jp}\gamma^{p,kl},\ j \neq l & in \quad Q, \\ \gamma^{l,kl}_t - \Delta\gamma^{l,kl} = \sum_{p=1}^2 A_{lp}\gamma^{p,kl} - \frac{1}{\mu_{kl}}\psi^k\mathbb{1}_{\mathcal{O}_{kl}} & in \quad Q, \\ \psi^j = \gamma^{j,kl} = 0, & in \quad \Sigma, \\ \psi^j(T) = \psi^T,\ \gamma^{j,kl}(0) = 0, & in \quad \Omega, \end{cases} \tag{2.27}$$

and assume that, for each $l = 1, 2$ the sets $\mathcal{O}_d^{kl}$ satisfies one of the conditions (2.15) *or* (2.16) *and in case it satisfies* (2.16)*, one of the conditions* (2.17) *or* (2.18) *holds. Then, there exist $C_* > 0$ and a weight function $\rho(t)$ with $\lim_{t\to T}\rho(t) = 0$, such that the following observability inequality holds*

$$\sum_{k=1}^2 \int_\Omega |\psi^k(0)|^2\,dx + \sum_{k,l=1}^2 \iint_{\mathcal{O}_d^{kl}\times(0,T)} \rho^2(t)|\gamma^{l,kl}|^2\,dxdt \leq C_* \sum_{k=1}^2 \iint_{\mathcal{O}\times(0,T)} |\psi^k|^2\,dxdt. \tag{2.28}$$

ii) For $i, j = 1, 2$, let (ψ^j, γ^{ji}) the solution of

$$\begin{cases} -\psi^i_t - \Delta\psi^i = \sum_{p=1}^2 A_{pi}\psi^p + \gamma^{ii}\mathbb{1}_{\mathcal{O}_d^i} & in \quad Q, \\ \gamma^{ii}_t - \Delta\gamma^{ii} = \sum_{p=1}^2 A_{ip}\gamma^{pi} - \frac{1}{\mu_i}\psi^i\mathbb{1}_{\mathcal{O}_i} & in \quad Q, \\ \gamma^{ji}_t - \Delta\gamma^{ji} = \sum_{p=1}^2 A_{jp}\gamma^{pi}, \quad j \neq i & in \quad Q, \\ \psi^i = \gamma^{ii} = \gamma^{ji} = 0 & in \quad \Sigma, \\ \psi^i(T) = \psi^T_i,\ \gamma^{ii}(0) = \gamma^{ji}(0) = 0 & in \quad \Omega. \end{cases} \tag{2.29}$$

Assume that the sets $\{\mathcal{O}_d^k\}_k$ and the coupling coefficients $\{A_{kl}\}_{k,l}$ satisfies (2.22), (2.23) *and* (2.26)*. Then, there exist $C > 0$ and a weight function $\hat\rho(t)$ where $\lim_{t\to T}\hat\rho(t) = 0$, such that the following observability inequalities holds*

$$\sum_{k=1}^2 \int_\Omega |\psi^k(0)|^2\,dx + \iint_{\mathcal{O}_d^k\times(0,T)} \rho^2(t)|\gamma^{kk}|^2\,dxdt \leq C \iint_{\mathcal{O}\times(0,T)} |\psi^1|^2\,dxdt. \tag{2.30}$$

As we have mentioned, if we prove observability inequality (2.28) then we are solving Theorem 2.1, while if we prove (2.30), then we are solving Theorem 2.2.

We remark that, controllability results for systems of parabolic equations is a huge research area in control theory and by now, positive and negative results are known. We refer to [3] for a survey and [2] where, for some specific coupling property, the authors have proved the existence of a minimal time of controllability. More recently, Duprez, M. and Lissy, P. in [21] and [22] have found some sufficient conditions for the controllability of systems with less controls by applying algebraic methods.

We have the following Lemma which is already used in [6] for the case of one single equation. Here we have to adapt it for a more general case.

Lemma 2.4. *Let Λ a finite set and $\{q_m\}_{m\in\Lambda}$ a family of disjoint open subsets of $\mathcal{O}$. Let $\tilde{\mathcal{O}} \subset\subset \mathcal{O}$ be such that $q_m \subset \tilde{\mathcal{O}}$ for every $m \in \Lambda$. There exists a family of functions $\{\eta_m\}_{m\in\Lambda}$ in $C^2(\bar{\Omega})$, such that*

$$\begin{cases} \eta_m > 0 \text{ in } \Omega, & \eta_m = 0 \text{ on } \partial\Omega, \\ \|\nabla \eta_m\| > C \text{ in } \overline{\Omega} \setminus q_m, & \eta_n = \eta_m \text{ in } \Omega \setminus \tilde{\mathcal{O}} \text{ when } n \neq m. \end{cases} \tag{2.31}$$

Demonstração. This result is already proved in [6] for the case $\Lambda = \{1, 2\}$. To generalize it to more sets, we assume that $\Lambda = \{1, \ldots, N_0\}$, for some N_0. The general case can be obtained by a simple identification argument. It is well known that (see [24]) there exists a function η^1 satisfying

$$\begin{cases} \eta_1 > 0 & \text{in} \quad \Omega, \eta_1 = 0 \text{ on } \partial\Omega, \\ \|\nabla \eta_1\| > C & \text{in} \quad \overline{\Omega \setminus q_1}. \end{cases}$$

Using Lemma 5 of [6], we get that, for each $m \neq 1$, we can take $\widetilde{\mathcal{O}}^m \subset \widetilde{\mathcal{O}}$, such that $q^1 \cup q^m \subset \widetilde{\mathcal{O}}^m$, and a weight function n_m such that

$$\begin{cases} \eta_m > 0 & \text{in} \quad \Omega, \ \eta_m = 0 \text{ on } \partial\Omega, \\ \|\nabla \eta_m\| > C & \text{in} \quad \overline{\Omega \setminus q^m}, \ \eta_m = \eta_1 \text{ in } \Omega \setminus \widetilde{\mathcal{O}}^m. \end{cases}$$

It is clear that the family $\{\eta_m\}_{m=1}^{N_0}$ satisfies (2.31) and the lemma is proved. □

Now, for any finite set Λ, let $\{\eta_m\}_{m\in\Lambda}$ be a family given by Lemma 2.4. We define the following weight functions, very common when dealing with Carleman estimates.

$$\sigma_m(x,t) := \frac{e^{4\lambda\|\eta_m\|_\infty} - e^{\lambda(2\|\eta_m\|_\infty + \eta_m(x))}}{t(T-t)}, \ \xi_m(x,t) := \frac{e^{\lambda(2\|\eta_m\|_\infty + \eta_m(x))}}{t(T-t)}, \quad m \in \Lambda, \tag{2.32}$$

and for $n \in \mathbb{N}$ we introduce the notations

$$I_n^m(\psi) := s^{n-4}\lambda^{n-3} \iint_Q e^{-2s\sigma_m}(\xi_m)^{n-4}(|\psi_t|^2 + |\Delta\psi|^2)dxdt + L_n^m(\psi),$$

where

$$L_n^m(\psi) := s^{n-2}\lambda^{n-1}\iint_Q e^{-2s\sigma_m}(\xi_m)^{n-2}|\nabla\psi|^2dxdt + s^n\lambda^{n+1}\iint_Q e^{-2s\sigma_m}(\xi_m)^n|\psi|^2dxdt.$$

For $u^T \in L^2(\Omega)$ and $f, f_1, ..., f_l \in L^2(Q)$, let u be the solution of the following equation

$$\begin{cases} -u_t - \Delta u = f + \sum_{k=1}^{l}\partial_k f_k & \text{on } \sum, \\ u = 0 & \text{on } \sum, \\ u(\cdot, T) = u^T & \text{in } \Omega. \end{cases} \tag{2.33}$$

Then, the following Carleman estimates holds

Proposition 2.5. *Let Λ a finite set, $n \in \mathbb{N}$, and $\{q_m\}_{m\in\Lambda}$ a family of open subsets of Ω. Let $\{\eta_m\}_{m\in\Lambda}$ the functions given by Lemma 2.4. For each $m \in \Lambda$, there exists a constant $C(\Omega, q_m) > 0$ such that, for every $s \geq s^m = C(\Omega, q_m)(T + T^2)$ and every $\lambda \geq C$, the solution u of* (2.33) *for $u^T \in L^2(\Omega)$, $f \in L^2(Q)$ and $f_k = 0$, $k = 1, \ldots, m$, satisfies*

$$I_n^m(u) \leq C\left(s^n\lambda^{n+1}\iint_{q_m\times(0,T)} e^{-2s\sigma_m}(\xi_m)^n|u|^2dxdt + s^{n-3}\lambda^{n-3}\iint_Q e^{-2s\sigma_m}(\xi_m)^{n-3}|f|^2dxdt\right).$$

If the functions f_k are not necessarily zero, the following holds:

Proposition 2.6. *Let Λ a finite set and $\{q_m\}_{m\in\Lambda}$ a family of open subsets of Ω. Let $\{\eta_m\}_{m\in\Lambda}$ the functions given by Lemma 2.4. For each $m \in \Lambda$, there exists a constant $C(\Omega, q_m) > 0$ such that, for every $s \geq s^m = C(\Omega, q_m)(T + T^2)$ and every $\lambda \geq C$, the solution u of* (2.33) *for $u^T \in L^2(\Omega)$, $f \in L^2(Q)$ and $f_k \in L^2(Q)$, $k = 1, \ldots, m$, satisfies*

$$L_n^m(u) \leq C\left(s^n\lambda^{n+1}\iint_{q_m\times(0,T)} e^{-2s\sigma_m}(\xi_m)^n|u|^2dxdt \right.$$
$$\left. + s^{n-3}\lambda^{n-3}\iint_Q e^{-2s\sigma_m}(\xi_m)^{n-3}|f|^2dxdt + s^{n-1}\lambda^{n-1}\sum_{k=1}^{l}\iint_Q e^{-2s\sigma_m}(\xi_m)^{n-1}|f_k|^2dxdt\right).$$

The estimates given in Propositions 2.5 and 2.6 are classical in control theory and are well known nowadays, for a proof see references [24] and [33].

In next section, we prove new Carleman estimates for the solutions of (2.27) and (2.29), these estimates will be the main tool to prove observability inequalities (2.28) and (2.30), respectively.

2.4 New Carleman Estimates

This section is dedicated to prove the following Carleman type estimates

Proposition 2.7.
i) Assume that, for each $l = 1, 2$, the sets $\{\mathcal{O}_d^{k\,l}\}$ satisfies (2.13) *and one of the conditions* (2.15) *or* (2.16), *in case it satisfies* (2.16), *assume that condition* (2.17) *or* (2.18) *holds. Let us also assume that the coupling coefficients $\{A_{i\,j}\}_{i,j}$ satisfies* (2.21). *Then, there exists $C > 0$ such that the following estimate holds*

$$\sum_{j,k,l=1}^{2} I_n^{k\,l}(\gamma^{j,\,k\,l}) + s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2 dxdt \leq$$
$$C s^{n+8}\lambda^{n+9}\sum_{p,j=1}^{2}\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_{p\,j}}\xi_{p\,j}^{n+8}|\psi^p|^2\, dxdt, \quad (2.34)$$

for every $\{\psi^j, \gamma^{j,\,lk}\}_{j,k,l}$ solution of (2.27).
ii) Assume that the sets $\{\mathcal{O}_d^k\}_k$ and the coupling coefficients $\{A_{k\,l}\}_{k,l}$ satisfies (2.22) *and* (2.23) *and there exists $\{\omega^i\}_i$ satisfying* (2.24), (2.25) *and* (2.26). *Then, there exist $C > 0$ such that*

$$s^n\lambda^{n+1}\iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2 dxdt + I_{n-3}^2(\psi^2) + \sum_{i,j=1}^{2} I_{n-3i}^i(\gamma^{j\,i})$$
$$\leq C s^{n+9}\lambda^{n+10}\iint_{\mathcal{O}\times(0,T)} (e^{-2s\sigma_1}(\xi_1)^{n+9} + e^{-2s\sigma_2}(\xi_2)^{n+3})|\psi^1|^2\, dxdt, \quad (2.35)$$

for every $\{\psi^j, \gamma^{ji}\}_{j,i}$ solution of (2.29).

Demonstração. For the proof of i), three distinct situations are possible depending on the geometry of the sets $\{\mathcal{O}_d^{k\,l}\}_{k,l=1}^2$:

A) Condition (2.15) holds for all $l = 1, 2$;

B) Condition (2.15) holds for one value of l, and for the other $\tilde{l}$ different from l we have (2.16);

C) Condition (2.16) holds for all $l = 1, 2$.

Condition **C)** is the most difficult to handle, that is why we are going to concentrate on it, in Remark 2.8 we make some comments concerning the other possibilities. We remind that, if (2.16) holds for some $l \in \{1, 2\}$, the cases (2.17) or (2.18) are possible and they have to be treated separately, therefore we have divided the proof of i) into two cases, Case 1 where (2.18) hold for ever $l \in \{1, 2\}$ and Case 2 where (2.18) hold for some $l \in \{1, 2\}$ and for the other $\tilde{l} \in \{1, 2\}$ different from l we have (2.17).
Proof of i)

Case 1: Assume that (2.16) and (2.18) hold and consider $\omega^{k\,l} \subset\subset \tilde{\omega}^{k\,l}$ satisfying (2.14), (2.19) and (2.20), for every $l = 1, 2$.

From now on, we are going to make use of the weight functions (2.31) and (2.32) for $\Lambda = \{1,2\} \times \{1,2\}$, $m = (k,l)$ and $\{q_m\} = \{\omega^{k\,l}\}$.

For all $j,k,l \in \{1,2\}$, $j \neq l$, we apply Proposition 2.5 for $u = \gamma^{j,\,k\,l}$ (see (2.27)) and $q_m = \omega^{k\,l}$ and we sum the resulting estimates to obtain:

$$\sum_{j=1}^{2} I_n^{k\,l}(\gamma^{j,\,k\,l}) \leq C \sum_{j=1}^{2} \left(s^n \lambda^{n+1} \iint_{\omega^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}} (\xi_{k\,l})^n |\gamma^{j,\,k\,l}|^2 dxdt + \right.$$
$$\left. s^{n-3}\lambda^{n-3} \iint_Q e^{-2s\sigma_{k\,l}} (\xi_{k\,l})^{n-3} \sum_{p=1}^{2} \|A_{j\,p}\|_\infty^2 |\gamma^{p,\,k\,l}|^2 dxdt \right)$$
$$+ s^{n-3}\lambda^{n-3} \iint_{\mathcal{O}_{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}} (\xi_{k\,l})^{n-3} |\psi^l|^2 dxdt, \quad k,l = 1,2, \quad (2.36)$$

for every $\lambda \geq C$ and $s \geq s^{k\,l} = C(\Omega, \omega^{k\,l})(T + T^2)$. Given a small $\epsilon > 0$, we use the fact that $T^6 \xi_{k\,l} \geq C$ and we get that

$$s^{n-3}\lambda^{n-3} \iint_Q e^{-2s\sigma_{k\,l}} (\xi_{k\,l})^{n-3} \sum_{p=1}^{2} \|A_{j\,p}\|_\infty^2 |\gamma^{p,\,k\,l}|^2 dxdt$$
$$\leq CT^6 \|A_{j\,p}\|_\infty^2 s^{n-3}\lambda^{n-3} \iint_Q e^{-2s\sigma_{k\,l}} (\xi_{k\,l})^n \sum_{p=1}^{m} |\gamma^{p,\,k\,l}|^2 dxdt \leq \epsilon I_n^{k\,l}(\gamma^{j,\,k\,l}), \quad k,l = 1,2, \quad (2.37)$$

for $\lambda \geq C$ and $s \geq \max\{s^{k\,l}, CT^2\|A_{jp}\|^{\frac{2}{3}}\}$. Combining (2.36) and (2.37) we have

$$\sum_{j=1}^{2} I_n^{k\,l}(\gamma^{j,\,k\,l}) \leq C s^n \lambda^{n+1} \sum_{p=1}^{2} \iint_{\omega^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}} (\xi_{k\,l})^n |\gamma^{p,\,k\,l}|^2 dxdt +$$
$$+ s^{n-3}\lambda^{n-3} \iint_{\mathcal{O}_{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}} (\xi_{k\,l})^{n-3} |\psi^k|^2 dxdt, \quad (2.38)$$

for all $k, l = 1, 2$.

Since we are assuming (2.13), (2.14) and (2.21), we take the open subsets $\hat{\omega}^{k\,l}$ and functions $\hat{\theta}^{k\,l} \in C_0^2(\hat{\omega}^{k\,l})$ such that $\hat{\theta}^{k\,l} = 1$ in $\omega^{k\,l}$, an ϵ sufficiently small, and we show that

$$
\begin{aligned}
s^n\lambda^{n+1} & \iint_{\omega^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}}(\xi_{k\,l})^n|\gamma^{p,\,k\,l}|^2\,dxdt \\
&\qquad \leq Cs^n\lambda^{n+1}\iint_{\hat{\omega}^{k\,l}\times(0,T)} \hat{\theta}^{k\,l}e^{-2s\sigma_{k\,l}}(\xi_{k\,l})^n A_{l\,p}|\gamma^{p,\,k\,l}|^2\,dxdt \\
&= s^n\lambda^{n+1}\iint_{\hat{\omega}^{k\,l}\times(0,T)} \hat{\theta}^{k\,l}e^{-2s\sigma_{k\,l}}(\xi_{k\,l})^n(\gamma_t^{l,\,k\,l} - \Delta\gamma^{l,\,k\,l} - A_{l\,l}\gamma^{l,\,k\,l} + \frac{1}{\mu_{k\,l}}\psi^k\mathbb{1}_{\mathcal{O}_{k\,l}})\gamma^{p,\,k\,l}\,dxdt \\
&\qquad \leq \epsilon I_n^{k\,l}(\gamma^{p,\,k\,l}) + Cs^{n+4}\lambda^{n+5}\iint_{\hat{\omega}^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}}\xi_{k\,l}^{n+4}|\gamma^{l,\,k\,l}|^2\,dxdt \\
&\qquad\qquad + Cs^n\lambda^{n+1}\iint_{(\mathcal{O}_{k\,l}\cap\hat{\omega}^{k\,l})\times(0,T)} e^{-2s\sigma_{k\,l}}\xi_{k\,l}^{n}|\psi^k|^2\,dxdt, \quad (2.39)
\end{aligned}
$$

for all $p,k,l = 1,2$ with $p \neq l$.

Then, we combine (2.34) and (2.39), we use the fact that $\mathcal{O}_{k\,l}\cap\hat{\omega}^{k\,l} = \emptyset$, and we obtain

$$
\sum_{j=1}^{2} I_n^{k\,l}(\gamma^{j,\,k\,l}) \leq Cs^{n+2}\lambda^{n+3}\iint_{\hat{\omega}^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}}(\xi_{k\,l})^n|\gamma^{l,\,k\,l}|^2\,dxdt, \quad k,l = 1,2, \qquad (2.40)
$$

for $\lambda \geq C$ and $s \geq \max\{s^{k\,l}, CT^2\|A_{j\,p}\|^{\frac{2}{3}}\}$.

Let $\widetilde{O}$ be given in Lemma 2.4 and let $\theta \in C^2(\bar{\Omega})$ be such that

$$
\begin{cases}
\theta(x) = 0 & \text{for} \quad x \in \tilde{\mathcal{O}}, \\
\theta(x) = 1 & \text{for} \quad x \in \Omega\setminus\mathcal{O}.
\end{cases} \qquad (2.41)
$$

For $i = 1,2$, we have that $\theta\psi^j$ satisfies the equation

$$
\begin{cases}
-(\theta\psi^j)_t - \Delta(\theta\psi^j) = \theta(\sum_{p=1}^2 A_{p\,j}\psi^p + \sum_{p=1}^2 \gamma^{j,\,p\,j}\mathbb{1}_{O_d^{p\,j}}) - 2\nabla\cdot(\nabla\theta\cdot\psi^j) + 2\Delta\theta\psi^j & \text{in} \quad Q, \\
\theta\psi^j = 0 & \text{on} \quad \Sigma, \\
(\theta\psi^j)(\cdot,T) = \theta\psi_T^j & \text{in} \quad Q.
\end{cases}
$$

We apply Proposition 2.6 replacing n for $n+3$, $m = (j,j)$, $u = \theta\psi^j$ and $\{q_m\} = \{\omega^{j\,j}\}$ to obtain

$$
\begin{aligned}
L_{n+3}^{j\,j}(\theta\psi^j) \leq C \Bigg(& s^{n+3}\lambda^{n+4}\iint_{\omega^{j\,j}\times(0,T)} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\theta\psi^j|^2 dxdt \\
& + s^n\lambda^n\sum_{p=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^n|\theta|^2|\gamma^{j,\,p\,j}\mathbb{1}_{O_d^{p\,j}}|^2 dxdt + \\
& s^n\lambda^n\sum_{p=1}^{2}\|A_{p,\,j}\|_\infty^2\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+2}|\theta|^2|\psi^p|^2 dxdt + s^{n+2}\lambda^{n+2}\int_0^T\!\!\int_{\mathcal{O}} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+2}|\psi^j|^2 dxdt \Bigg),
\end{aligned}
$$

for all $j = 1,2$.

Using the definition of θ (see (2.41)), taking $\lambda \geq C$ and $s \geq \max_{p,j}\{s^{jj}, CT^2, CT^2\|A_{pj}\|^{\frac{2}{3}}\}$ we obtain

$$
\begin{aligned}
s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2dxdt \\
\leq C\sum_{j=1}^{2}\Bigg(s^{n+3}\lambda^{n+4}\iint_{O\times(0,T)} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2dxdt \\
+s^n\lambda^n\sum_{p=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^n|\theta|^2|\gamma^{j,pj}1_{O_d^{pj}}|^2dxdt\Bigg), \quad j=1,2.
\end{aligned}
$$

By summing this last estimate with (2.40) and using the fact that all the weights coincide in the support of θ (see (2.31)), we absorb the terms of $\gamma^{j,pj}$ and get

$$
\begin{aligned}
\sum_{j,k,l=1}^{2} I_n^{kl}(\gamma^{j,kl}) + s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2dxdt \\
\leq C\sum_{j=1}^{2}\Bigg(s^{n+3}\lambda^{n+4}\iint_{O\times(0,T)} e^{-2s\sigma_{jj}}(\xi_{jj})^{-3}|\psi^j|^2dxdt \\
+\sum_{p=1}^{2}s^{n+2}\lambda^{n+3}\iint_{\hat{\omega}^{pj}\times(0,T)} e^{-2s\sigma_{pj}}(\xi_{pj})^{n+2}|\gamma^{j,pj}|^2\,dxdt\Bigg), \quad (2.42)
\end{aligned}
$$

for $\lambda \geq C$ and $s \geq \max_{k,l,p,j}\{s^{kl}, CT^2, CT^2\|A_{pj}\|^{\frac{2}{3}}\}$ sufficiently large. The next calculations are dedicated to estimate the local terms of $\gamma^{j,pj}$ in the right hand side of (2.42).

Now, we use the fact that $\tilde{\omega}^{kl}$ can be taken satisfying (2.14), (2.19) and (2.20). Then, it is easy to see that

$$
-\psi_t^j - \Delta\psi^j = \sum_{p=1}^{2}A_{pj}\psi^p + \gamma^{j,pj}1_{O_d^{pj}} \quad \text{in} \quad \tilde{\omega}^{pj}\times(0,T).
$$

Let $\tilde{\theta}^{pj} \in C_0^2(\tilde{\omega}^{pj})$ such that $\tilde{\theta}^{pj} = 1$ in $\hat{\omega}^{pj}$. Then we have

$$
\begin{aligned}
s^{n+2}\lambda^{n+3}\iint_{\hat{\omega}^{pj}\times(0,T)} e^{-2s\sigma_{pj}}(\xi_{pj})^{n+2}|\gamma^{j,pj}|^2\,dxdt = \\
s^{n+2}\lambda^{n+3}\iint_{\tilde{\omega}^{pj}\times(0,T)} \tilde{\theta}^{pj}(\xi_{pj})^{n+2}e^{-2s\sigma_{pj}}\gamma^{j,pj}(-\psi_t^j - \Delta\psi^j - \sum_{p=1}^{2}A_{pj}\psi^p)\,dxdt \\
\leq \epsilon I_n^{pj}(\gamma^{j,pj}) + Cs^{n+8}\lambda^{n+9}\sum_{p=1}^{2}\iint_{\tilde{\omega}^{pj}\times(0,T)} e^{-2s\sigma_{pj}}\xi_{pj}^{n+8}|\psi^p|^2\,dxdt, \quad (2.43)
\end{aligned}
$$

for $\epsilon > 0$ sufficiently small. Finally, we combine (2.42) and (2.43), we obtain (2.34).

In what follows, we prove prove Carleman estimate (2.35), we assume that (2.17) holds for a given value of l and that (2.18) for $\tilde{l} \neq l$.
Case 2: For simplicity, we will assume that (2.17) holds for $l=1$ and $(k,\tilde{k})=(2,1)$ and that (2.18) holds for $l=2$ and $(k,\tilde{k})=(1,2)$, similar arguments can be used for different values l, k and $\tilde{k}$. Then, we are assuming that

$$\mathcal{O}_d^{1\,1} \cap \mathcal{O} \subset \mathcal{O}_d^{2\,1} \cap \mathcal{O} \quad \text{and} \quad \mathcal{O}_d^{2\,1} \cap \mathcal{O} \setminus \overline{\mathcal{O}_d^{1\,1} \cap \mathcal{O}} \neq \emptyset, \tag{2.44}$$

and

$$\mathcal{O}_d^{2\,2} \cap \mathcal{O} \setminus \overline{\mathcal{O}_d^{1\,2} \cap \mathcal{O}} \neq \emptyset, \quad \text{and} \quad \mathcal{O}_d^{1\,2} \cap \mathcal{O} \setminus \overline{\mathcal{O}_d^{2\,2} \cap \mathcal{O}} \neq \emptyset. \tag{2.45}$$

We start considering the open sets satisfying (2.14), and since we have (2.44), we can assume that

$$\tilde{\omega}^{1\,1} \subset \mathcal{O}_d^{2\,1} \cap \mathcal{O} \quad \text{and} \quad \tilde{\omega}^{2\,1} \cap \mathcal{O}_d^{1\,1} = \emptyset. \tag{2.46}$$

Defining $h^j = \gamma^{j,1\,1} + \gamma^{j,2\,1}$ for $j=1,2$, the adjoint system (2.27) becomes

$$\begin{cases} -\psi_t^1 - \Delta\psi^1 = \sum_{p=1}^2 A_{p\,1}\psi^p + h^1 \mathbb{1}_{\mathcal{O}_d^{11}} - \gamma^{1,2\,1}(\mathbb{1}_{\mathcal{O}_d^{1\,1}} - \mathbb{1}_{\mathcal{O}_d^{2\,1}}), & \text{in} \quad Q, \\ -\psi_t^2 - \Delta\psi^2 = \sum_{p=1}^2 A_{p\,2}\psi^p + \sum_{p=1}^2 \gamma^{2,p\,2} \mathbb{1}_{\mathcal{O}_d^{p\,2}}, & \text{in} \quad Q, \\ h_t^1 - \Delta h^1 = \sum_{p=1}^2 A_{1\,p} h^p - \sum_{p=1}^2 \frac{1}{\mu_{p\,1}} \psi^p \mathbb{1}_{\mathcal{O}_{p\,1}}, & \text{in} \quad Q, \\ h_t^2 - \Delta h^2 = \sum_{k=1}^2 A_{2\,p} h^p, & \text{in} \quad Q, \\ \gamma_t^{l,k\,l} - \Delta\gamma^{l,k\,l} = \sum_{p=1}^2 A_{l\,p}\gamma^{p,k\,l} - \frac{1}{\mu_{k\,l}}\psi^k \mathbb{1}_{\mathcal{O}_{k\,l}},\ (k,l) \neq (1,1) & \text{in} \quad Q, \\ \gamma_t^{j,k\,l} - \Delta\gamma^{j,k\,l} = \sum_{p=1}^2 A_{j\,p}\gamma^{p,k\,l},\ (k,l) \neq (1,1) & \text{in} \quad Q, \\ \psi^j = h^j = \gamma^{j,k\,l} = 0, \quad \text{for} \quad j,k,l = 1,2 & \text{in} \quad \Sigma, \\ \psi^j(T) = \psi^T,\ h^j(0) = 0,\ \gamma^{j,k\,l}(0) = 0, \quad \text{for} \quad j,k,l = 1,2 & \text{in} \quad \Omega. \end{cases} \tag{2.47}$$

By using Proposition 2.5 for $\{(\gamma^{1,k\,2}, \gamma^{2,k\,2})\}_{k=1}^2$ for $(\gamma^{1,2\,1}, \gamma^{2,2\,1})$ and for (h^1, h^2), we obtain

$$\begin{aligned} \sum_{j=1}^2 \left(\sum_{\substack{k,\,l=1 \\ (k,l)\neq(1,1)}}^2 I_n^{k\,l}(\gamma^{j,k\,l}) + I_n^{11}(h^j) \right) \leq C \sum_{j=1}^2 \Bigg(s^n \lambda^{n+1} \sum_{\substack{k,\,l=1 \\ (k,l)\neq(1,1)}}^2 \iint_{\omega^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}} (\xi_{k\,l})^n |\gamma^{j,k\,l}|^2 dxdt \\ + s^n \lambda^{n+1} \sum_{j=1}^2 \iint_{\omega^{11}\times(0,T)} e^{-2s\sigma_{11}} (\xi_{11})^n |h^j|^2 dxdt \Bigg) \\ + s^{n-3}\lambda^{n-3} \sum_{p,k=1}^2 \iint_{\mathcal{O}_{p\,k}\times(0,T)} e^{-2s\sigma_{p\,k}} (\xi_{p\,k})^{n-3} |\psi^p|^2 dxdt, \end{aligned} \tag{2.48}$$

for $\lambda \geq C$ and $s \geq \max\{s^{k\,l}, CT^2 \|A_{jp}\|^{\frac{2}{3}}\}_{k,l,j,p}$.

Using the equation of h^1 in (2.47), that is

$$A_{1\,2}h^2 = h^1_t - \Delta h^1 - A_{1\,1}h^1 + \sum_{p=1}^{2} \frac{1}{\mu_{p\,1}}\psi^p \mathbb{1}_{\mathcal{O}_{p\,1}},$$

and from assumption (2.21), we prove that

$$\begin{aligned} s^n\lambda^{n+1} \iint_{\omega^{11}\times(0,T)} e^{-2s\sigma_{11}}(\xi_{11})^n|h^2|^2\,dxdt &\leq \epsilon I_n^{11}(h^2) \\ &+ Cs^{n+4}\lambda^{n+5} \iint_{\hat{\omega}^{11}\times(0,T)} e^{-2s\sigma_{11}}\xi_{11}^{n+4}|h^1|^2\,dxdt \\ &+ Cs^n\lambda^{n+1}\sum_{p=1}^{2}\iint_{(\mathcal{O}_{p\,1}\cap\hat{\omega}^{11})\times(0,T)} e^{-2s\sigma_{11}}\xi_{11}^n|\psi^p|^2\,dxdt, \quad (2.49) \end{aligned}$$

for $\epsilon > 0$ sufficiently small.

In a very similar way, we use (2.21) again, obtaining

$$\begin{aligned} s^n\lambda^{n+1} &\iint_{\omega^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}}(\xi_{k\,l})^n|\gamma^{j,\,k\,l}|^2\,dxdt \\ &\leq \epsilon I_n^{k\,l}(\gamma^{j,\,k\,l}) + Cs^{n+4}\lambda^{n+5}\iint_{\hat{\omega}^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}}\xi_{k\,l}^{n+4}|\gamma^{l,\,k\,l}|^2\,dxdt \\ &\quad + Cs^n\lambda^{n+1}\iint_{(\mathcal{O}_{k\,l}\cap\hat{\omega}^{k\,l})\times(0,T)} e^{-2s\sigma_{k\,l}}\xi_{k\,l}^n|\psi^k|^2\,dxdt, \quad (2.50) \end{aligned}$$

for $\lambda \geq C$ and $s \geq CT^2\|A_{j\,j}\|^{\frac{2}{3}}$, for $j,k,l = 1,2$ where $j \neq l$ and $(k,l) \neq (1,1)$.

Now, applying Carleman inequalities for $\theta\psi^j$ where θ is given in (2.41), summing to (2.48) and using (2.49) and (2.50), we get

$$\begin{aligned} \sum_{j=1}^{2}\left(\sum_{\substack{k,l=1\\(k,l)\neq(1,1)}}^{2} I_n^{k\,l}(\gamma^{j,\,k\,l}) + I_n^{11}(h^j)\right) &+ s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2 dxdt \\ \leq C\Bigg(s^{n+3}\lambda^{n+4}&\sum_{j=1}^{2}\iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2 dxdt \\ +s^{n+2}\lambda^{n+3}&\sum_{\substack{k,l=1\\(k,l)\neq(1,1)}}^{2}\iint_{\hat{\omega}^{k\,l}\times(0,T)} e^{-2s\sigma_{k\,l}}(\xi_{k\,l})^{n+2}|\gamma^{l,\,k\,l}|^2\,dxdt \\ &+ Cs^{n+4}\lambda^{n+5}\iint_{\hat{\omega}^{11}\times(0,T)} e^{-2s\sigma_{11}}\xi_{11}^{n+4}|h^1|^2\,dxdt\Bigg). \quad (2.51) \end{aligned}$$

for $\lambda \geq C$ and $s \geq \max\{s^{k\,l}, CT^2\|A_{j\,p}\|^{\frac{2}{3}}\}_{k,l,j,p}$. To absorb the local terms of $\gamma^{l,k\,l}$ and h^1, we proceed in a similar way as in (2.43). Indeed, that we can use the equation of ψ^1 in (2.27) since we have that

$$-\psi^1_t - \Delta\psi^1 = \sum_{p=1}^{2} A_{p\,1}\psi^p + h^1 \quad \text{in} \quad \omega^{1\,1} \times (0,T), \tag{2.52}$$

and using (2.44), (2.45) and (2.46) we have

$$-\psi^l_t - \Delta\psi^l = \sum_{p=1}^{2} A_{p,l}\psi^p + \gamma^{l,k\,l} \quad \text{in} \quad \omega^{k\,l} \times (0,T), \quad \text{for every} \quad k,l = 1,2,\ (k,l) \neq (1,1). \tag{2.53}$$

Using (2.52) and (2.53), and proceeding in a similar way as in (2.43) we have

$$\sum_{j=1}^{2}\left(\sum_{\substack{k,l=1\\(k,l)\neq(1,1)}}^{2} I_n^{k\,l}(\gamma^{j,k\,l}) + I_n^{11}(h^j)\right) + s^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2dxdt$$
$$\leq Cs^{n+3}\lambda^{n+4}\sum_{j=1}^{2}\iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2dxdt. \tag{2.54}$$

Since we have h^j and $\gamma^{j,2,j}$ on the left hand side of (2.51), if we write $\gamma^{j,1\,j} = h^j - \gamma^{j,2\,j}$, we can add the global terms of $\gamma^{j,1\,j}$ in the left hand side of (2.56), obtaining (2.34).

Remark 2.8. *For the proof of i) of Proposition 2.7, we have assumed only the case* ***C)****, which correspond to the case where* (2.16) *holds for every* $l = 1,2$*. It is not difficult to see that the other possibilities can be threated by adapting the arguments presented here. Indeed, if in the proof of Case 2 we have assumed* (2.44) *and that*

$$\mathcal{O}_d^{2\,2} \cap \mathcal{O} \subset \mathcal{O}_d^{1\,2} \cap \mathcal{O} \quad \textit{and} \quad \mathcal{O}_d^{1\,2} \cap \mathcal{O} \setminus \overline{\mathcal{O}_d^{2\,2} \cap \mathcal{O}} \neq \emptyset, \tag{2.55}$$

then equation (2.53) *does not hold for* $(k,l) = (2,2)$*, then we could not absorb the local term of* $\gamma^{2,2\,2}$ *in the right hand* (2.51)*. To avoid this, we define the function* $g^j = \gamma^{j,1\,2} + \gamma^{j,2\,2}$*, use a similar equation as* (2.52) *and we bound the local terms of* g^j *in a similar way as we have bounded the local terms of* h^j*.*

Now, if (2.15) *holds for some* $l \in \{1,2\}$*, the analysis is much more simpler. Indeed, to fix the ideas, let us assume that*

$$\mathcal{O}_d^{1\,1} = \mathcal{O}_d^{2\,1}.$$

In this case we can define the same functions $h^j = \gamma^{j,1\,1} + \gamma^{j,2\,2}$ *and we see that the equation satisfied by* $\{\psi^j, h^j, \gamma^{j,k\,l}\}_{j,k,l=1}^2$ *is similar to* (2.47) *with the only difference that the equation of* ψ^1 *would be*

$$-\psi_t^1 - \Delta\psi^1 = \sum_{p=1}^{2} A_{p,1}\psi^p + h^1 1\!\!1_{\mathcal{O}_d^{11}}, \quad in \quad Q,$$

which turns the system (2.47) *simpler and gives directly expression* (2.52).

Let us now proceed with the proof of ii).

Proof of ii: Let $\{(\psi^i, \gamma^{i\,i}, \gamma^{j\,i})\}_{i,j=1}^2$ a solution of (2.29), and let us assume that (2.22) is valid. To fix the ideas, we will suppose that (2.23) holds for $(i,j) = (1,2)$. Moreover, we take open sets satisfying (2.24), (2.25) and (2.26).

By using Proposition 2.5 for $(\gamma^{1\,1}, \gamma^{2\,1})$, we get

$$I_{n-3}^1(\gamma^{11}) + I_{n-3}^1(\gamma^{21}) \leq C\Bigg(s^{n-3}\lambda^{n-2}\iint_{\omega^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-3}\big(|\gamma^{11}|^2 + |\gamma^{21}|^2\big)dxdt + s^{n-6}\lambda^{n-6}\iint_{\mathcal{O}_1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-6}|\psi^1|^2 dxdt\Bigg), \quad (2.56)$$

for $\lambda \geq C$ and $s \geq C(T + T^2(1 + \max\{\|A_{ip}\|^{\frac{2}{3}}\}))$. Let $\hat{\theta}^1 \in C_0^2(\hat{\omega}^1)$ such that $\hat{\theta}^1 = 1$ in ω^1, and using that $A_{1,2} \geq C_0 > 0$ in $\omega^1 \times (0,T)$ (see (2.26)) we obtain that

$$\begin{aligned} s^{n-3}\lambda^{n-2} &\iint_{\omega^1\times(0,T)} e^{-2s\sigma_1}|\gamma^{2\,1}|^2 dxdt \\ &\leq Cs^{n-3}\lambda^{n-2}\iint_{\hat{\omega}^1\times(0,T)} |\hat{\theta}^1|^2 e^{-2s\sigma_1}\gamma^{2\,1}\Big(-\gamma_t^{11} - \Delta\gamma^{1\,1} - A_{1\,1}\gamma^{1\,1} + \frac{1}{\mu_1}\psi^1 1\!\!1_{\mathcal{O}_1}\Big)dxdt \\ &\leq \epsilon I_{n-3}^1(\gamma^{2\,1}) + Cs^{n+1}\lambda^{n+2}\iint_{\hat{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+2}|\gamma^{1\,1}|^2 dxdt \\ &\quad + s^{n+1}\lambda^{n+2}\iint_{(\hat{\omega}^1\cap\mathcal{O}_1)\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\psi^1|^2 dxdt, \quad (2.57) \end{aligned}$$

for $\epsilon > 0$ sufficiently small. Combining (2.56) and (2.57) and using that $\mathcal{O} \cap \mathcal{O}_1 = \emptyset$, we get that

$$I_{n-3}^1(\gamma^{11}) + I_{n-3}^1(\gamma^{21}) \leq C\Bigg(s^{n+1}\lambda^{n+2}\iint_{\hat{\omega}^1\times(0,T)} e^{-2s\sigma_i}(\xi_i)^{n+1}|\gamma^{11}|^2\, dxdt + s^{n-6}\lambda^{n-6}\iint_{\mathcal{O}_1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-6}|\psi^1|^2 dxdt\Bigg), \quad (2.58)$$

for $\lambda \geq C$ and $s \geq C(T + T^2(1 + \max\{\|A_{ip}\|^{\frac{2}{3}}\}))$. In a completely analogous way, we can take $\hat{\theta}^2 \in C_0^2(\hat{\omega}^2)$, such that $\hat{\theta}^2 = 1$ in ω^2, and using that $A_{2,1} \geq C_0 > 0$ in $\omega^2 \times (0,T)$ (see

(2.26)), we can prove the following estimate

$$I^2_{n-6}(\gamma^{22}) + I^2_{n-6}(\gamma^{12}) \leq C\left(s^{n-6}\lambda^{n-5}\iint_{\hat{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\gamma^{22}|^2\,dxdt\right.$$
$$\left.+s^{n-6}\lambda^{n-6}\iint_{\mathcal{O}_2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\psi^2|^2 dxdt\right), \quad (2.59)$$

for $\lambda \geq C$ and $s \geq C(T + T^2(1 + \max\{\|A_{ip}\|^{\frac{2}{3}}\}))$. Summing (2.58) and (2.59) we obtain

$$\sum_{i,j=1}^{2} I^i_{n-3i}(\gamma^{ji}) \leq C\left(s^{n+1}\lambda^{n+2}\iint_{\hat{\omega}^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\gamma^{11}|^2\,dxdt\right.$$
$$+s^{n-6}\lambda^{n-5}\iint_{\hat{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\gamma^{22}|^2\,dxdt + s^{n-3}\lambda^{n-3}\iint_{\mathcal{O}_1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-3}|\psi^1|^2 dxdt$$
$$\left.+s^{n-6}\lambda^{n-6}\iint_{\mathcal{O}_2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\psi^2|^2 dxdt\right). \quad (2.60)$$

Let $\theta \in C^2(\bar{\Omega})$ be given such that

$$\begin{cases} \theta(x) = 0 & \text{for} \quad x \in \tilde{\mathcal{O}}, \\ \theta(x) = 1 & \text{for} \quad x \in \Omega \setminus \mathcal{O}. \end{cases} \quad (2.61)$$

From the first equation of system (2.29), we find that

$$\begin{cases} -(\theta\psi^1)_t - \Delta(\theta\psi^1) = \theta(\sum_{p=1}^2 A_{p,1}\psi^p + \gamma^{11}1\!\!1_{O^1_d}) + \psi^1\Delta\theta - 2\nabla(\psi^1\nabla\theta) & \text{in} \quad Q, \\ \theta\psi^1 = 0 & \text{on} \quad \Sigma, \\ (\theta\psi^1)(\cdot, T) = \theta\psi^1_T & \text{in} \quad Q. \end{cases}$$

By applying Proposition 2.6, we obtain that

$$s^n\lambda^{n+1}\iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2 dxdt \leq C\left(s^n\lambda^{n+1}\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2 dxdt\right.$$
$$+s^{n-3}\lambda^{n-3}\sum_{p=1}^{2}\iint_Q e^{-2s\sigma_1}(\xi_1)^{n-3}|\theta|^2|A_{p,1}\psi^p|^2 dxdt$$
$$\left.+s^{n-3}\lambda^{n-3}\iint_{O^1_d\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n-3}|\theta|^2|\gamma^{11}|^2 dxdt\right) \quad (2.62)$$

for $\lambda \geq C$ and $s \geq s^1$. Now, using Proposition 2.5 for ψ^2, we get

$$I^2_{n-3}(\psi^2) \leq C\left(s^{n-3}\lambda^{n-2}\iint_{\omega^2\times(0,T)} (\xi_2)^{n-3}e^{-2s\sigma_2}|\psi^2|^2 dxdt\right.$$
$$+s^{n-6}\lambda^{n-6}\sum_{p=1}^{2}\iint_Q e^{-2s\sigma_2}(\xi_2)^{n-6}|A_{p,2}\psi^p|^2 dxdt$$
$$\left.+\, s^{n-6}\lambda^{n-6}\iint_{O^2_d\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\gamma^{22}|^2 dxdt\right), \quad (2.63)$$

for $\lambda \geq C$ and $s \geq s^2$. Summing (2.62) and (2.63) and absorbing the global terms of ψ^1 and ψ^2 for $\lambda \geq C$ and $s \geq \max_{i,j}\{s^j, CT^2\|A_{i,j}\|_\infty^2\}$ large enough, we get

$$\begin{aligned}
&s^n\lambda^{n+1}\iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2dxdt + I_{n-3}^2(\psi^2)\\
&\leq C\left(s^n\lambda^{n+1}\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt + s^{n-3}\lambda^{n-2}\iint_{\omega^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-3}|\psi^2|^2\,dxdt\right.\\
&\left.+s^{n-3}\lambda^{n-3}\iint_{\mathcal{O}_d^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^n|\gamma^{11}|^2\,dxdt + s^{n-6}\lambda^{n-6}\iint_{\mathcal{O}_d^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\gamma^{22}|^2\,dxdt\right).
\end{aligned} \tag{2.64}$$

Summing (2.60) and (2.64) and absorbing the global terms of ψ^i and γ^{ii} we get

$$\begin{aligned}
&s^n\lambda^{n+1}\iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2dxdt + I_{n-3}^2(\psi^2) + \sum_{i,j=1}^{2} I_{n-3i}^i(\gamma^{ji})\\
&\leq C\left(s^n\lambda^{n+1}\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2\,dxdt + s^{n-3}\lambda^{n-2}\iint_{\omega^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-3}|\psi^2|^2\,dxdt\right.\\
&\left.+s^{n+1}\lambda^{n+2}\iint_{\hat\omega^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\gamma^{11}|^2\,dxdt + s^{n-6}\lambda^{n-5}\iint_{\hat\omega^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\gamma^{22}|^2\,dxdt\right),
\end{aligned} \tag{2.65}$$

$\lambda \geq C$ and $s \geq \max_{i,j}\{s^j, CT^2(1+\|A_{i,j}\|_\infty^2)\}$ large enough.

In order to absorb the local terms of $\gamma^{i\,i}$ in the right hand side of (2.65), we are going to use the equation of ψ^i in (2.29), and the fact that $\hat\omega^i$ satisfy (2.25). Indeed, let $\tilde\theta^i \in C_0^2(\tilde\omega^i)$ a cut-off function such that $\tilde\theta^i = 1$ in $\hat\omega^i$. Then, we have that

$$\begin{aligned}
&s^{n+1}\lambda^{n+2}\iint_{\hat\omega^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\tilde\theta^1|^2|\gamma^{11}|^2\,dxdt\\
&\qquad = s^{n+1}\lambda^{n+2}\sum_{p=1}^{2}\iint_{\tilde\omega^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+1}|\tilde\theta^1|^2\gamma^{11}(-\psi_t^1-\Delta\psi^1-A_{1\,1}\psi^1)\,dxdt\\
&\qquad\qquad \leq \epsilon I_{n-3}^1(\gamma^{11}) + s^{n+9}\lambda^{n+10}\iint_{\tilde\omega^1\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+9}|\psi^1|^2\,dxdt,
\end{aligned} \tag{2.66}$$

and

$$\begin{aligned}
&s^{n-6}\lambda^{n-5}\iint_{\hat\omega^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n+1}|\tilde\theta^2|^2|\gamma^{2\,2}|^2\,dxdt\\
&\qquad \leq s^{n-6}\lambda^{n-5}\sum_{p=1}^{2}\iint_{\hat\omega^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-6}|\tilde\theta^2|^2\gamma^{22}(-\psi_t^2-\Delta\psi^2-A_{2\,2}\psi^2)\,dxdt\\
&\qquad\qquad \leq \epsilon I_{n-6}^2(\gamma^{22}) + s^{n-2}\lambda^{n-1}\iint_{\tilde\omega^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\psi^2|^2\,dxdt,
\end{aligned} \tag{2.67}$$

for $\epsilon > 0$ sufficiently small. Combining (2.65), (2.66) and (2.67), we get that

$$s^n\lambda^{n+1}\iint_Q e^{-2s\sigma_1}(\xi_1)^n|\psi^1|^2dxdt + I^2_{n-3}(\psi^2) + \sum_{i,j=1}^{2} I^i_{n-3i}(\gamma^{j\,i})$$

$$\leq C\left(s^{n+9}\lambda^{n+10}\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_1}(\xi_1)^{n+9}|\psi^1|^2\,dxdt\right.$$

$$\left.+s^{n-2}\lambda^{n-1}\iint_{\tilde{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\psi^2|^2\,dxdt\right). \quad (2.68)$$

Since we are assuming (2.25), we can take an open set ω^2_*, such that

$$\tilde{\omega}^2 \subset \omega^2_* \subset supp\,A_{2,1}\cap\mathcal{O}\setminus\overline{(\mathcal{O}^1_d\cap\mathcal{O})},$$

and a cut-off function $\theta^2_* \in C^2_0(\omega^2_*)$ such that $\theta^2_* = 1$ in $\tilde{\omega}^2$. We use again property (2.26), and we obtain

$$s^{n-2}\lambda^{n-1}\iint_{\tilde{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\theta_*|^2|\psi^2|^2\,dxdt$$

$$= s^{n-2}\lambda^{n-1}\iint_{\tilde{\omega}^2\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n-2}|\theta^2_*|^2\psi^2(-\psi^1_t - \Delta\psi^1 - A_{1,1}\psi^1)\,dxdt$$

$$\leq \epsilon I^2_{n-3}(\psi^2) + s^{n+3}\lambda^{n+4}\iint_{\omega^2_*\times(0,T)} e^{-2s\sigma_2}(\xi_2)^{n+3}|\psi^1|^2\,dxdt \quad (2.69)$$

Combining (2.68) and (2.69) we finally obtain (2.35), for $\lambda \geq C$ and $s \geq \max_{i,j}\{s^j, CT^2\|A_{i,j}\|^2_\infty\}$.
□

In next section, we prove the observability inequalities given in Theorem 3.17.

2.5 Observability Inequalities

In this section we are going to use Proposition 2.7 to prove Theorem 2.3. Here, we will concentrate on the proof of (2.28), and we will omit the proof of (2.30) since it follows in a completely analogous way. In order to do this, we will combine Carleman estimate (2.34) with suitable energy estimates for the solutions of (2.27).

Let $\{\psi^j, \gamma^{j,\,k\,l}\}$ a solution of (2.27). By energy estimates, we have that

$$\sum_{p=1}^{2}\|\gamma^{p,\,k\,l}(\cdot,t)\|^2_2 \leq \frac{C}{\mu_{k\,l}}\int_0^t\|\psi^k(\cdot,s)\|^2_2\,ds, \quad k,l=1,2, \quad t\in[0,T] \qquad (2.70)$$

and

$$\sum_{p=1}^{2}\|\psi^p(\cdot,t)\|_2^2 \le C\sum_{p=1}^{2}\|\psi^p(\cdot,t')\|_2^2 + (1+\max_{i,j}\{\|A_{i,j}\|_\infty^2\})\int_t^{t'}\sum_{p=1}^{2}\|\psi^p(\cdot,s)\|^2\,ds + C\int_t^{t'}\sum_{j,p=1}^{2}\|\gamma^{j,pj}(\cdot,s)\|^2\,ds. \quad (2.71)$$

Combining (2.70) and (2.71) we have

$$\sum_{p=1}^{2}\|\psi^p(\cdot,t)\|_2^2 \le C\sum_{p=1}^{2}\|\psi^p(\cdot,t')\|_2^2 + (1+\max_{i,j}\{\|A_{i,j}\|_\infty^2\})\int_t^{t'}\sum_{p=1}^{2}\|\psi^p(\cdot,s)\|^2\,ds + C\sum_{p,j=1}^{2}\frac{T}{\mu_{pj}}\int_0^{t'}\|\psi^j(\cdot,s)\|_2^2\,ds, \quad k,l=1,2, \quad (2.72)$$

for every $0\le t<t'\le T$. By Gronwall Lemma, we have that

$$\sum_{p=1}^{2}\|\psi^p(\cdot,t)\|_2^2 \le Ce^{(1+\max_{ij}\{\|A_{i,j}\|_\infty^2\})(t'-t)}\sum_{p=1}^{2}\left(\|\psi^p(\cdot,t')\|_2^2+\sum_{j=1}^{2}\frac{T}{\mu_{pj}}\int_0^{t'}\|\psi^j(\cdot,s)\|_2^2\,ds\right). \quad (2.73)$$

Integrating over $[0,t']$ and taking μ_{pj} large enough

$$\sum_{p=1}^{2}\int_0^{t'}\|\psi^p(\cdot,s)\|_2^2\,ds \le Ce^{(1+\max_{ij}\{\|A_{i,j}\|_\infty^2\})(t'-t)}\sum_{p=1}^{2}\|\psi^p(\cdot,t')\|_2^2. \quad (2.74)$$

Combining (2.70), (2.73) and (2.74), we get that

$$\sum_{p=1}^{2}\left(\|\psi^p(\cdot,t)\|_2^2+\sum_{k,l=1}^{2}\|\gamma^{p,kl}(\cdot,t)\|_2^2\right) \le Ce^{(1+\max_{ij}\{\|A_{i,j}\|_\infty^2\})(t'-t)}\sum_{p=1}^{2}\|\psi^p(\cdot,t')\|_2^2, \quad (2.75)$$

for every $0\le t<t'\le T$. Let $\upsilon:[0,T]\to\mathbb{R}$ be such that $\upsilon=1$ in $[0,T/4]$ and $\upsilon=0$ in $[3T/4,T]$. If we apply a similar argument to prove (2.75) for the functions $\{\upsilon\gamma^j,\upsilon\gamma^{j,kl}\}_{j,k,l}$, and taking $t'=T$ we find that

$$\sum_{p=1}^{2}\left(\|\upsilon(t)\psi^p(\cdot,t)\|_2^2+\sum_{k,l=1}^{2}\|\upsilon(t)\gamma^{p,kl}(\cdot,t)\|_2^2\right) \le Ce^{(1+\max_{ij}\{\|A_{i,j}\|_\infty^2\})(T-t)}\int_0^T\sum_{p=1}^{2}\left(\|\upsilon_t(t)\psi^p(\cdot,s)\|_2^2+\sum_{k,l=1}^{2}\|\upsilon_t(s)\gamma^{p,kl}(\cdot,s)\|_2^2\right)ds, \quad (2.76)$$

for $t \in [0,T]$. In particular, we have that

$$\sum_{p=1}^{2}\left(\|\psi^p\|^2_{L^\infty(0,T/2;L^2(\Omega))} + \sum_{k,l=1}^{2}\|\gamma^{p,\,k\,l}\|^2_{L^\infty(0,T/2;L^2(\Omega))}\right)$$
$$\leq Ce^{(1+\max_{i\,j}\{\|A_{i,\,j}\|^2_\infty\})(T-t)}\sum_{p=1}^{2}\left(\|\psi^p\|^2_{L^2(T/2,3T/4;L^2(\Omega))} + \sum_{k,l=1}^{2}\|\gamma^{p,\,k\,l}\|^2_{L^2(T/2,3T/4;L^2(\Omega))}\right). \tag{2.77}$$

Now, we define the following new weight functions

$$l(t) = \begin{cases} T^2/4 & \text{in} \quad [0,T/2], \\ t(T-t) & \text{in} \quad [T/2,T], \end{cases} \tag{2.78}$$

and for a family of sets $\{\eta_m\}_{m\in\Lambda}$ given by Lemma 2.4, we define

$$\tilde{\sigma}_m(x,t) := \frac{e^{4\lambda\|\eta_m\|_\infty} - e^{\lambda(2\|\eta_m\|_\infty+\eta_m(x))}}{l(t)},\ \tilde{\xi}_m(x,t) := \frac{e^{\lambda(2\|\eta_m\|_\infty+\eta_m(x))}}{l(t)}, \quad m\in\Lambda. \tag{2.79}$$

Then, we have that

$$\min\{e^{-2s\sigma_{k\,l}}\xi^q_{k\,l}, e^{-2s\tilde{\sigma}_{k\,l}}\tilde{\xi}^q_{k\,l}\} \geq e^{-C_{max}s/T^2}\frac{1}{T^{2q}} \quad \text{in} \quad \Omega\times(\frac{T}{4},\frac{3T}{4}), \quad k,l=1,2, \tag{2.80}$$

and

$$\max\{e^{-2s\sigma_{k\,l}}\xi^q_{k\,l}, e^{-2s\tilde{\sigma}_{k\,l}}\tilde{\xi}^q_{k\,l}\} \leq e^{-C_{min}s/T^2}\frac{1}{T^{2q}} \quad \text{in} \quad Q, \quad k,l=1,2, \tag{2.81}$$

for

$$C_{max} = e^{4\lambda\|\eta_m\|_\infty} - e^{2\lambda\|\eta_m\|_\infty} \quad \text{and} \quad C_{min} = e^{4\lambda\|\eta_m\|_\infty} - e^{3\lambda\|\eta_m\|_\infty}.$$

In this way, using (2.77), we have that

$$\sum_{p=1}^{2}\left(\int_\Omega|\psi^p(0)|^2\,dx + \sum_{k,l=1}^{2}\int_\Omega\int_0^{T/2}e^{-2s\tilde{\sigma}_{k\,l}}\tilde{\xi}^n_{k\,l}|\gamma^{p,\,k\,l}|^2\,dxdt\right)$$
$$\leq \frac{C}{T^{2n}}e^{-C_{min}s/T^2}e^{(1+\max_{i\,j}\{\|A_{i,\,j}\|^2_\infty\})T}\sum_{p=1}^{2}\left(\|\psi^p\|^2_{L^2(T/2,3T/4;L^2(\Omega))} + \sum_{k,l=1}^{2}\|\gamma^{p,\,k\,l}\|^2_{L^2(T/2,3T/4;L^2(\Omega))}\right)$$
$$\leq Ce^{(C_{max}-C_{min})s/T^2}e^{(1+\max_{i\,j}\{\|A_{i,\,j}\|^2_\infty\})T}\left(\sum_{j,k,l=1}^{2}I^{k\,l}_n(\gamma^{j,\,k\,l}) + \sum_{j=1}^{2}\iint_Q e^{-2s\sigma_{jj}}(\xi_{jj})^{n+3}|\psi^j|^2dxdt\right), \tag{2.82}$$

$\lambda \geq C$ and $s \geq \max\{s^{k\,l}, CT^2\|A_{jp}\|^{\frac{2}{3}}\}_{k,l,j,p}$. Combining (2.82) and (2.34) we get that

$$\sum_{p=1}^{2}\left(\int_{\Omega}|\psi^p(0)|^2\,dx + \sum_{k,l=1}^{2}\int_{\Omega}\int_0^{T/2} e^{-2s\tilde{\sigma}_{k\,l}}\tilde{\xi}_{k\,l}^n|\gamma^{p,\,k\,l}|^2\,dxdt\right)$$
$$\leq Ce^{(1+\max_{i\,j}\{\|A_{i,\,j}\|_{\infty}^2\})T}e^{Cs/T^2}\sum_{p=1}^{2}\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma_{p\,j}}\xi_{p\,j}^{n+8}|\psi^p|^2\,dxdt$$
$$\leq Ce^{C(1+\frac{1}{T}+T+\max\|A_{i,j}\|^{\frac{2}{3}}+T\max_{i\,j}\|A_{i,\,j}\|_{\infty})}\sum_{p=1}^{2}\iint_{\mathcal{O}\times(0,T)}|\psi^p|^2\,dxdt. \quad (2.83)$$

To complete the proof, we just use the fact that the weights coincide in $(T/2, T)$ and we obtain

$$\sum_{k,l=1}^{2}\int_{\Omega}\int_{T/2}^{T} e^{-2s\tilde{\sigma}_{k\,l}}\xi_{k\,l}^n|\gamma^{p,\,k\,l}|^2\,dxdt \leq I_n^{k\,l}(\gamma^{j,\,k\,l}) \leq C\sum_{p=1}^{2}\iint_{\mathcal{O}\times(0,T)}|\psi^p|^2\,dxdt. \quad (2.84)$$

Hence, we have proved (2.28) with $C_* = Ce^{C(1+\frac{1}{T}+T+\max\|A_{i,j}\|^{\frac{2}{3}}+T\max_{i\,j}\|A_{i,\,j}\|_{\infty})}$ and $\rho(t) = \min_{x\in\Omega} e^{-2s\tilde{\sigma}_{k\,l}}\xi_{k\,l}^n$.

2.6 Comments and open questions

On the conditions over $\mathcal{O}_d^{k\,l}$ and $\mathcal{O}_d^k$

In order to solve controllability problems for systems of parabolic equations for every time T, it is usual to assume that the support of the couping coefficients intersects the control domain in a suitable way. In this chapter, we have assumed some similar conditions to the ones in [5, 6], these conditions are given essentially in (2.15), (2.16) or (2.23). Concerning these conditions, some open questions arises naturally. The first one is the case that $\mathcal{O}_d^{k,l}$ coincides only inside $\mathcal{O}$ and differ outside $\mathcal{O}$, we remark that, even for the case of one single equation, similar open problem arises (see [6]). The second one is the case that

$$\mathcal{O}_d^i \cap \mathcal{O} \subset \overline{supp\,A_{j,\,i} \cap \mathcal{O}} \neq \emptyset \quad \text{or} \quad \overline{supp\,A_{j,\,i} \cap \mathcal{O}} \subset \mathcal{O}_d^i \cap \mathcal{O},$$

for some $(i, j) \in \{(1, 2), (2, 1)\}$. The difficulty here is related to the fact that the variables ψ^j are backward in time, while $\gamma^{i\,j}$ are forward in time, turning difficult to follow a similar strategy as in the proof of Case 2 of Proposition 2.7.

On the functionals (2.3) or (2.9)

If we look to the recent results in [30], the authors have solved Theorem 2.2 for the specific case where

$$J^i(\{v^j\}) = \alpha_i\iint_{\mathcal{O}_d^i\times(0,T)}|y - y_d^1|^2 + |y - y_d^2|^2\,dxdt + \mu_i\iint_{\mathcal{O}_i\times(0,T)}|v^i|\,dxdt, \quad (2.85)$$

and

$$\mathcal{O}_d^1 = \mathcal{O}_d^2.$$

The reason why we have defined the functionals (2.9) instead of (2.85) is that it seems more natural to assign to each follower the variable it has to control. In any case, by applying similar arguments as the ones contained here, one can solve the problem of [30], even when $\mathcal{O}_d^1 \neq \mathcal{O}_d^2$. For this, we have to assume that

$$\mathcal{O}_d^1 \cap \mathcal{O} \setminus \mathcal{O}_d^2 \cap \mathcal{O} \neq \emptyset \quad \text{or} \quad \mathcal{O}_d^2 \cap \mathcal{O} \setminus \mathcal{O}_d^1 \cap \mathcal{O} \neq \emptyset,$$

and moreover

$$\mathcal{O}_d^j \cap \mathcal{O} \setminus \overline{suppA_{j,i}} \neq \emptyset \quad \text{and} \quad Int(suppA_{j,i}) \setminus \overline{\mathcal{O}_d^j \cap \mathcal{O}} \neq \emptyset, \quad \text{for every} \quad i \neq j.$$

These are some kind of mixed conditions between the ones used to solve Theorems 2.1 and 2.2.

On the quantity of equations considered

A natural open question which arises is the possibility of proving similar results, when we have m equation, where $m > 2$. The difficulty for this case, is the proof of estimate (2.39), in the case of Theorem 2.1 and (2.66), (2.67) for Theorem 2.2. For each (k, l) fixed, the variables $\{\gamma^{j,k\,l}\}_{j=1}^m$ are the solutions of a system of m equation, and for each $j \neq l$ we must estimate the local terms of $\gamma^{j,k,l}$ by local terms of $\gamma^{l,k\,l}$. This problem can be compared to the one of controlling a general system of m equation with one single control, which is a completely open problem. Even if the coupling coefficients are in cascade we do not know how we can estimate the local terms of $\gamma^{j,k,l}$ by local terms of $\gamma^{l,k\,l}$ for evey $l = 1, \ldots, m$.

3 ON THE STACKELBERG-NASH EXACT CONTROLLABILITY FOR KDV EQUATION

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Abstract

This paper deals with a hierarchical control problem for the *Korteweg-de Vries* (KdV) equation with distributed controls following a Stackelberg-Nash strategy. Here, we have a control problem where many objectives have to be achieved at once and, in order to do that, many controls are needed. We assume that there is a main control called *the leader*, and two secondary controls called *the followers*, each of them has it own objective. Once the leader policy is fixed, the followers must act in order to accomplish their goals. The leader objective is to drive the solutions of the KdV equation to a given trajectory, while the followers must be a Nash equilibrium for their targets. Problems of this kind are already solved for many parabolic and hyperbolic equations, and here is the first time it is considered for dispersive equations such as the KdV equation.

Keywords: Nash equilibrium, Stackelberg's method, null controllability, Carleman estimates, KdV equation.

3.1 Introduction

The Korteweg-de Vries (KdV) equation is a third order partial differential equation which models the propagation of water waves on shallow water surfaces. It was first derived by Boussineq (see [8]) and Korteweg-de Vries (see [36]). In the last few years, this equation was extensively studied in many point of views like well posedness, stability of solitary waves, integrability and long time behavior, see for instance [35, 45]. In the control point of view a lot has been done. For the boundary controllability, we can refer the papers [26, 52, 53, 60] for linear and [15, 16, 18] for nonlinear cases. We can also refer [27] where an uniform null controllability result is adressed in the case of zero-dispersion limit. For the controllability with distributed controls, we can cite [12] where using Carleman estimates, some controllability results for linear KdV equation are proved.

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It is also important to mention reference [4] where an internal controllability result for a coupled system of Schrödinger and KdV system is considered.

All the references cited previously are dedicated to single objective control problems. In this paper, we consider a control problem for the KdV equation in which many objectives have to be accomplished, that is what we call a multi-objective control problem. Here, we want to know if it is possible to control the equation to a desired state when many other objectives are also being considered at the same time. It is not difficult to realize that many objectives may be in conflict in the sense that solving one of them turns the others difficult or even impossible to solve. To overcome this, we apply the concept of Stackelberg's Optimization where a hierarchy for the controls is assumed. We consider one control called the *leader* and other controls called the *followers* (in some sense subordinated to the leader). Once the leader policy is fixed, the followers must accomplish its objectives in a optimal way. To determine the followers, we make use of a equilibrium concept due to John Nash (see [47]) usually applied to noncooperative game problems.

Multiobjective control problems have been considered in many papers in the past few years. The concept of hierarchic controls applied for PDEs was introduced and popularized by Jacques L. Lions in [38, 43], since then, many related papers appeared and nowadays we can find many results for parabolic and hyperbolic equations. For instance, in [19, 20, 29, 50, 51] the authors studied multiobjective control problems in which the leader objective is of approximate controllability type. Results in the exact controllability level are quite recent being [5, 6] the pioneer works for the heat equation. In this context, we can also cite [13] where the arguments of [5, 6] were adapted for the case of a fourth order evolution equation. Up to our knowledge, no results are known for dispersive equations such as the KdV equation.

3.2 Statement of the Problem

Let $(0,1) \subset \mathbb{R}$ be the unitary open interval and $T > 0$ a real number. We consider internal controls supported on a nonempty open subset $\omega \subset (0,1)$ and homogeneous boundary conditions. We define $Q = (0,1) \times (0,T)$ and for any open subset $\omega \subset (0,1)$ we define $Q_\omega = \omega \times (0,T)$.

Consider the nonlinear KdV equation

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = f\mathbb{1}_{\mathcal{O}} + v^1\chi_{\mathcal{O}_1} + v^2\chi_{\mathcal{O}_2} & \text{in} \quad Q, \\ y(0,\cdot) = y(L,\cdot) = y_x(L,\cdot) = 0 & \text{on} \quad (0,T), \\ y(x,\cdot) = y^0 & \text{in} \quad (0,1), \end{cases} \tag{3.1}$$

where $y = y(x,t)$ is the state, y^0 is prescribed. In (3.1), the set $\mathcal{O} \subset (0,1)$ is *the main* *control domain* and (all them are supposed to be small and disjoint). The controls are f, v^1 and v^2, where f is called the *leader* and v^1 and v^2 are the *followers*, each of them will have it own objective. The function $\mathbb{1}_{\mathcal{O}}$ represents the characteristic function in $\mathcal{O}$ while $\chi_{\mathcal{O}_i}$ are non negative functions in $C_0^\infty(\mathcal{O}_i)$, $i = 1, 2$. This way of disposing the controls, in some kind of hierarchy, is called Stackelberg's optimization strategy and is commonly used in multi-objective control problems.

Let us be more precise about the problem under view. Let $\mathcal{O}_{1,d}$, $\mathcal{O}_{2,d} \subset (0, 1)$ be open sets and consider the functionals

$$J_i(f, v^1, v^2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i\,d}|^2 \, dxdt + \frac{\mu_i}{2} \iint_Q \chi_{\mathcal{O}_i} |v^i|^2 \, dxdt, \quad i = 1, 2, \tag{3.2}$$

where the $\alpha_i > 0$, $\mu_i > 0$ are constants and the $y_{i,d} = y_{i,d}(x, t)$ are given functions. The Stackelberg-Nash exact controllability for the KdV equation can be described by the following steps:

1. For each f, the followers v^1 and v^2 intend to be a *Nash equilibrium* for the costs J_i $(i = 1, 2)$. In other words, once f has been fixed, we look for a couple (v^1, v^2), with $v^i \in L^2(\mathcal{O}_i \times (0, T))$, satisfying

$$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \quad J_2\left(f; v^1, v^2\right) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \tag{3.3}$$

   Note that, if the functionals J_i $(i = 1, 2)$ are convex, then (v^1, v^2) is a Nash equilibrium if and only if

$$J_1'(f; v^1, v^2)(\hat{v}^1, 0) = 0, \quad \forall \hat{v}^1 \in L^2\left(\mathcal{O}_1 \times (0, T)\right), \quad v^1 \in L^2(\mathcal{O}_1 \times (0, T)), \tag{3.4}$$

   and

$$J_2'(f; v^1, v^2)(0, \hat{v}^2) = 0, \quad \forall \hat{v}^2 \in L^2\left(\mathcal{O}_2 \times (0, T)\right), \quad v^2 \in L^2(\mathcal{O}_2 \times (0, T)). \tag{3.5}$$

   Since system (3.1) is nonlinear, functionals (3.2) are not necessarily convex. However, we will see further that the convexity of J_i will follow for μ_i sufficiently large.

2. Let us fix an uncontrolled trajectory, that is, a sufficiently regular solution to the system

$$\begin{cases} \overline{y}_t + \overline{y}_x + \overline{y}_{xxx} + \overline{y}\,\overline{y}_x = 0 & \text{in} \quad Q, \\ \overline{y}\,(0, \cdot) = \overline{y}\,(L, \cdot) = \overline{y}_x\,(L, \cdot) = 0 & \text{on} \quad (0, T)\,, \\ \overline{y}(\cdot, 0) = \overline{y}^0 & \text{in} \quad (0, 1)\,. \end{cases} \tag{3.6}$$

   Once the Nash equilibrium has been identified and fixed for each f, we look for a control $\hat{f} \in L^2(\mathcal{O} \times (0, T))$ such that

$$y(\cdot, T) = \overline{y}(\cdot, T) \quad \text{in} \quad (0, 1). \tag{3.7}$$

Defining the new variables $z = y - \overline{y}$ and $z_{i,d} = y_{i,d} - \overline{y}$ it is clear that property (3.7) is equivalent to the null controllability property for z, that is,

$$z(x,T) = 0 \quad \text{in} \quad (0,1), \tag{3.8}$$

where z is the solution of the equation

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + (\overline{y}z)_x = f\,\mathbb{1}_{\mathcal{O}} + v^1\chi_{\mathcal{O}_1} + v^2\chi_{\mathcal{O}_2} & \text{in} \quad Q, \\ z\,(0,\cdot) = z\,(L,\cdot) = z_x\,(L,\cdot) = 0 & \text{on} \quad (0,T)\,, \\ z(\cdot,0) = z^0 & \text{in} \quad (0,1)\,. \end{cases} \tag{3.9}$$

On the other hand, the functionals J_i can be rewritten as

$$J_i(f,v^1,v^2) = \frac{\alpha_i}{2}\iint_{\mathcal{O}_{i,d}\times(0,T)} |z - z_{i\,d}|^2\,dxdt + \frac{\mu_i}{2}\iint_Q \chi_{\mathcal{O}_i}|v^i|^2\,dxdt, \quad i = 1,2, \tag{3.10}$$

where $z_{i\,d} := y_{i\,d} - \bar{y}$. If the functionals J_i of (3.10) are convex, it follows from conditions (3.4) and (3.5), and for μ_i $(i = 1,2)$ sufficiently large, that

$$v_i = -\frac{1}{\mu_i}\phi^i\chi_{\mathcal{O}_i}, \quad i = 1,2, \tag{3.11}$$

where (z,ϕ^1,ϕ^2) is solution of the following optimality system:

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x + (\overline{y}z)_x = f\,\mathbb{1}_{\mathcal{O}} - \sum_{i=1}^2 \frac{1}{\mu_i}\phi^i\chi_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \phi^i_x - \phi^i_{xxx} - (z+\overline{y})\phi^i_x = \alpha_i(z - z_{i,d})\chi_{\mathcal{O}_{i,d}},\ i = 1,2 & \text{in} \quad Q, \\ z\,(0,\cdot) = z\,(L,\cdot) = z_x\,(L,\cdot) = 0 & \text{on} \quad (0,T)\,, \\ \phi^i\,(0,\cdot) = \phi^i\,(L,\cdot) = \phi^i_x\,(0,\cdot) = 0 & \text{on} \quad (0,T)\,, \\ z(\cdot,0) = z^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad (0,1)\,. \end{cases} \tag{3.12}$$

The proof to that is standard, see for instance [5, 38]. It is also standard that for μ_i $(i = 1,2)$ sufficiently large the functionals J_i are indeed convex (the proof is made in the next section) and hence characterization (3.11) may be used.

Finally, once the followers are characterized by (3.11) and (3.12), what remains to do is to find f such that the solutions of (3.12) satisfy (3.8). This motivates the main result of this thesis

Theorem 3.1. *For $i = 1,2$, suppose that*

$$\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset \tag{3.13}$$

and that μ_i are sufficiently large. Also, assume that one of the two conditions holds:

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \tag{3.14}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{3.15}$$

Then, there exist a positive function $\hat{\rho} = \hat{\rho}(t)$ blowing up at $t = T$ and $\delta > 0$ such that if

$$\|z^0\|^2_{H_0^1(0,1)} + \sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} \hat{\rho}^2 |z_{i,d}|^2 \, dx \, dt < \delta, \tag{3.16}$$

there exist controls $f \in L^2(\mathcal{O} \times (0,T))$ and associated Nash equilibria (v^1, v^2) such that the corresponding solutions to (3.1) *satisfies* (3.7).

In order to prove Theorem 3.1, we proceed as follows: we first prove a null controllability result for a linearized version of the optimality system (3.12). This linear system is given by

$$\begin{cases} z_t + z_x + z_{xxx} + (\bar{y}z)_x = f\,\mathbb{1}_{\mathcal{O}} - \sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i \chi_{\mathcal{O}_i} + f^0 & \text{in} \quad Q, \\ -\phi^i_t - \phi^i_x - \phi^i_{xxx} - \bar{y}\phi^i_x = \alpha_i z \chi_{\mathcal{O}_{i,d}} + f^i, \quad i = 1,2 & \text{in} \quad Q, \\ z\,(0,\cdot) = z\,(L,\cdot) = z_x\,(L,\cdot) = 0 & \text{on} \quad (0,T)\,, \\ \phi^i\,(0,\cdot) = \phi^i\,(L,\cdot) = \phi^i_x\,(0,\cdot) = 0 & \text{on} \quad (0,T)\,, \\ z(\cdot,0) = z^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad (0,1)\,, \end{cases} \tag{3.17}$$

where (f^0, f^1, f^2) are given source terms. We prove that there exists a function f such that the solution (z, ϕ^1, ϕ^2) of (3.17) is partially null controllable, in the sense it satisfies $z(T) = 0$. The proof of this linear controllability result is based on dualiy arguments. From the Hilbert Uniqueness Method (HUM), the null controllability for (3.18) is equivalent to a suitable observability estimate for the solutions of an adjoint state. For any M, g^0, g^1 and g^2 in a suitable functional spaces, and for any $(\psi, \gamma^1, \gamma^2)$ solution of

$$\begin{cases} -\psi_t - M\psi_x - \psi_{xxx} = \sum_{i=1}^{2} \alpha_i \gamma^i \chi_{\mathcal{O}_{i,d}} + g^0 & \text{in} \quad Q, \\ \gamma^i_t + (M\gamma^i)_x + \gamma^i_{xxx} = -\frac{1}{\mu_i}\psi\chi_{\mathcal{O}_i} + g^i, \quad 1 = 1,2 & \text{in} \quad Q, \\ \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = 0 & \text{on} \quad (0,T), \\ \gamma^i(0,\cdot) = \gamma^i(L,\cdot) = \gamma^i_x(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \psi(x,T) = \psi^T(x), \quad \gamma^i(\cdot,0) = 0 & \text{in} \quad (0,1), \end{cases} \tag{3.18}$$

we prove the existence of a constant $C > 0$, and a weight function ρ, blowing up at $t = 0$, such that the following observability estimate holds

$$\int_0^1 |\psi(0)|^2 \, dx + \sum_{i=1}^{2} \iint_Q \rho^{-2} |\gamma^i|^2 dxdt \leq C \left(\iint_{Q_\omega} |\psi|^2 \, dxdt + \iint_Q (|g^0|^2 + \sum_{i=1}^{2} |g^i|^2) \, dxdt \right). \tag{3.19}$$

In next sections, we prove that inequality (3.19) holds as a consequence of the so called Carleman estimates.

3.3 Well-posedness results and existence of a Nash equilibrium

The main goal of this section is to prove the existence of a Nash equilibrium. To do this, we will first prove that system (3.12) possesses unique solutions, which from (3.11) will guarantee the existence of a pair satisfying (3.4)-(3.5). After this, we will prove that these controls indeed satisfy (3.3). In this step, the uniqueness of solutions plays a crucial role.

Let us introduce some functional spaces which will be used along the paper:

$$\begin{gathered} X_0 := L^2(0,T;H^{-2}(0,1)), \quad X_1 := L^2(0,T;H_0^2(0,1)), \\ \tilde{X}_0 := L^1(0,T;H^{-1}(0,1)), \quad \tilde{X}_1 := L^1(0,T;H^3(0,1)\cap H_0^1(0,1)), \\ Y_0 := L^2(0,T;L^2(0,1))\cap C([0,T];H^{-1}(0,1)), \\ Y_1 := L^2(0,T;H^4(0,1))\cap C([0,T];H^3(0,1)). \end{gathered}$$

In addition, we define (see e.g. [7]), for each $\theta \in [0,1]$, the interpolation spaces

$$X_\theta := (X_0, X_1)_{[\theta]}, \quad \tilde{X}_\theta := (\tilde{X}_0, \tilde{X}_1)_{[\theta]}, \quad Y_\theta := (Y_0, Y_1)_{[\theta]}.$$

The space

$$Y_{\frac{1}{4}} = L^2((0,T);H^1(0,1))\cap C([0,T];L^2(0,1)).$$

will be needed in forthcoming computations.

For ζ and M to be given, consider the following linear KdV equation:

$$\begin{cases} \zeta_t + M\zeta_x + \zeta_{xxx} = g & \text{in} \quad Q, \\ \zeta(0,t) = \zeta(1,t) = \zeta_x(0,t) = 0 & \text{on} \quad (0,T), \\ \zeta(x,0) = \zeta_0(x) & \text{in} \quad (0,1). \end{cases} \tag{3.20}$$

The first existence result is concerned to the case $M \neq 0$.

Proposition 3.2. *(See [27], Section 2.2.2) If $\zeta_0 \in L^2(0,1)$, $M \in Y_{1/4}$ and $g \in G$ with $G = L^2(0,T;H^{-1}(0,1))$ or $G = L^1(0,T;L^2(0,1))$, then system* (3.20) *has a unique solution $\zeta \in Y_{\frac{1}{4}}$. Moreover, there exists a constant $C > 0$ such that*

$$\|\zeta\|^2_{Y_{\frac{1}{4}}} \le C\Big(\|\zeta_0\|^2_{L^2(0,L)} + \|g\|^2_G\Big)e^{C\int_0^T(1+\|M_x(s)\|_2^2)\,ds}. \tag{3.21}$$

Remark 3.3. *Proposition 3.2 is still valid if we replace equation* (3.20) *for*

$$\begin{cases} \zeta_t + (M\zeta)_x + \zeta_{xxx} = g & \textit{in} \quad Q, \\ \zeta(0,t) = \zeta(1,t) = \zeta_x(0,t) = 0 & \textit{on} \quad (0,T), \\ \zeta(x,0) = \zeta_0(x) & \textit{in} \quad (0,1). \end{cases} \tag{3.22}$$

For the case $M = 0$, we have the following improved regularity results.

Proposition 3.4. *(See [27], Section 2.3.1) Suppose that* $M = 0$. *If* $\zeta_0 \in H^3(0,1)$ *is such that* $\zeta_0(0) = \zeta_0(1) = (\zeta_0)_x(0) = 0$, *and* $g \in G$ *with* $G = L^2(0,T;H_0^2(0,1))$ *or* $G = L^1(0,T;H^3(0,1) \cap H_0^2(0,1))$, *then system* (3.20) *has a unique solution* $\zeta \in Y_1$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|\zeta\|_{Y_1}^2 \leq C\Big(\|g\|_G^2 + \|\zeta_0\|_{H^3(0,1)}^2\Big). \tag{3.23}$$

Proposition 3.5. *(See [27], Section 2.3.2) Let* $\theta \in [\frac{1}{4}, 1]$ *be given and suppose that* $M = 0$ *and* $\zeta_0 = 0$. *If* $g \in G$ *with* $G = X_\theta$ *or* $G = \tilde{X}_\theta$, *then system* (3.20) *has a unique solution* $\zeta \in Y_\theta$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|\zeta\|_{Y_\theta}^2 \leq C\|g\|_G^2. \tag{3.24}$$

Remark 3.6. *In Proposition 3.5, we could assume that* ζ_0 *belongs to the* θ *interpolation space between* $L^2(0,1)$ *and* $\{\zeta_0 \in H^3(0,1); \zeta_0(0) = \zeta_0(1) = \zeta_{0\,x}(0) = 0\}$ *instead of* $\zeta_0 = 0$. *In particular, for* $\theta = 1/4$, *Proposition 3.5 is valid by taking* $\zeta_0 \in H_0^1(0,1)$, *see Theorem 2.6 of [44].*

Remark 3.7. *If* $\zeta_0 \in H_0^1(0,L)$, $M \in Y_{1/2}$ *and* $g \in G$ *where* $G = X_{1/2}$ *or* $G = \tilde{X}_{1/2}$, *then problem* (3.20) *posses a unique solution* $\zeta \in Y_{1/2}$. *Indeed, from Proposition 3.2, the solution* ζ *belongs to* $Y_{1/4}$. *Since* $M \in Y_{1/2}$, *the term* $M\zeta_x$ *is in* $L^2(Q)$. *Thus, the results follows by Proposition 3.5 and Remark 3.6. Moreover,*

$$\|\zeta\|_{Y_{\frac{1}{2}}}^2 \leq C(1 + \|M\|_{Y_{\frac{1}{2}}}^2) e^{C(1+\|M\|_{Y_{\frac{1}{4}}}^2)\,ds} \Big(\|\zeta_0\|_{H_0^1(0,L)}^2 + \|g\|_G^2\Big). \tag{3.25}$$

For the linear system (3.17), we have the following result:

Proposition 3.8. *Let* $z^0 \in L^2(0,L)$, $f \in L^2(Q)$, $(f^0, f^1, f^2) \in G \times G \times G$, *with* $G = L^2(0,T;H^{-1}(0,L))$ *or* $G = L^1(0,T;L^2(0,L))$, *and* $\overline{y} \in Y_{1/4}$. *Then, problem* (3.17) *posses a unique solution* $(z, \phi^1, \phi^2) \in Y_{1/4} \times Y_{1/4} \times Y_{1/4}$.

Demonstração. Let $(\overline{\phi}^1, \overline{\phi}^2) \in G \times G$. From Remark 3.3, there exists a unique solution $\overline{z} \in Y_{1/4}$ for the problem

$$\begin{cases} \overline{z}_t + \overline{z}_x + \overline{z}_{xxx} + (\overline{y}\,\overline{z})_x = f\,\mathbb{1}_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i} \overline{\phi}^i \chi_{\mathcal{O}_i} + f^0 & \text{in} \quad Q, \\ \overline{z}(0,\cdot) = \overline{z}(L,\cdot) = \overline{z}_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \overline{z}(\cdot,0) = z^0 & \text{in} \quad (0,L). \end{cases} \tag{3.26}$$

Moreover, we have that

$$\|\overline{z}\|^2_{Y_{\frac{1}{4}}} \leq C\Big(\sum_{i=1}^{2} \frac{1}{\mu_i^2} \left\|\overline{\phi}^i\right\|^2_{L^2(Q)} + \|z^0\|^2_{L^2(0,L)} + \|f\|^2_{L^2(Q)} + \left\|f^0\right\|^2_G \Big) e^{C\int_0^T (1+\|\overline{y}_x(s)\|_2^2)\,ds}, \quad (3.27)$$

where $C > 0$ is independent on μ_i.

By Proposition 3.2, for each $i = 1, 2$, we can guarantee the existence of $\hat{\phi}^i \in Y_{1/4}$ which solves the system

$$\begin{cases} -\hat{\phi}^i_t - \hat{\phi}^i_x - \hat{\phi}^i_{xxx} - \overline{y}\hat{\phi}^i_x = \alpha_i \overline{z}\chi_{\mathcal{O}_{i,d}} + f^i & \text{in} \quad Q, \\ \hat{\phi}(0,\cdot) = \hat{\phi}(L,\cdot) = \hat{\phi}_x(0,\cdot) = 0 & \text{in} \quad (0,T), \\ \hat{\phi}(\cdot,T) = 0 & \text{in} \quad (0,L), \end{cases} \quad (3.28)$$

and verifies

$$\|\hat{\phi}^i\|^2_{Y_{\frac{1}{4}}} \leq C\Big(\|f^i\|^2_G + \alpha_i\|\overline{z}\|^2\Big) e^{C\int_0^T (1+\|\overline{y}_x(s)\|_2^2)\,ds}. \quad (3.29)$$

Combining (3.27) and (3.29), we obtain

$$\|\hat{\phi}^i\|^2_{Y_{\frac{1}{4}}} \leq C\left(\sum_{i=1}^{2} \frac{1}{\mu_i^2} \left\|\overline{\phi}^i\right\|^2_{L^2(Q)} + \|f\|^2_{L^2(Q)} + \left\|f^0\right\|^2_G + \|f^i\|^2_G\right) e^{C\int_0^T (1+\|\overline{y}_x(s)\|_2^2)\,ds}, \quad (3.30)$$

where $C > 0$ is independent on μ_i $(i = 1, 2)$.

Now, we set $\Lambda : G\times G \to G\times G$ by $\Lambda(\overline{\phi}^1, \overline{\phi}^2) = (\hat{\phi}^1, \hat{\phi}^2)$. Using (3.30), it is not difficult to see that

$$\|\Lambda(\overline{\phi}^1, \overline{\phi}^2) - \Lambda(\phi^1_*, \phi^2_*)\|^2_{G\times G} \leq \frac{C}{\min\{\mu_i^2\}} e^{C\int_0^T (1+\|\overline{y}_x(s)\|_2^2)\,ds} \|(\overline{\phi}^1, \overline{\phi}^2) - (\phi^1_*, \phi^2_*)\|^2_{G\times G},$$

for all $(\overline{\phi}^1, \overline{\phi}^2), (\phi^1_*, \phi^2_*) \in G \times G$. Hence, for μ_i sufficiently large, the functional Λ is a contraction, and from Banach fixed point Theorem it posses a unique fixed point which we call (ϕ^1, ϕ^2). If z is the solution of (3.26) with (ϕ^1, ϕ^2) on the right-hand side instead of $(\overline{\phi}^1, \overline{\phi}^2)$, then $(z, \phi^1, \phi^2) \in Y_{1/4} \times Y_{1/4} \times Y_{1/4}$ and by (3.27) and (3.30), we get, for μ_i large enough, that

$$\|(z, \phi^1, \phi^2)\|^2_{[Y_{1/4}]^3} \leq C\left(\|z^0\|^2_{L^2(0,L)} + \|f\|^2_{L^2(Q)} + \left\|(f^0, f^1, f^2)\right\|^2_{[G]^3}\right) e^{C\int_0^T (1+\|\overline{y}_x(s)\|_2^2)\,ds}, \quad (3.31)$$

where C does not depend on μ_i. This completes the proof. □

We are now ready to prove that (3.12) is well-posed. To do this, the following global result for (3.1) is needed:

Proposition 3.9. *(See [17], Theorem 9) For any $T > 0$, any $f, v^1, v^2 \in L^2(Q)$, and any $y^0 \in L^2(0,L)$, there is a unique solution $y \in Y_{1/4}$ for (3.1) such that*

$$\|y\|^2_{Y_{1/4}} \leq C(\|y^0\|^2_{L^2(0,L)} + \|f\mathbb{1}_{\mathcal{O}}\|^2_{L^2(Q)} + \|v^1\chi_{\mathcal{O}_1}\|^2_{L^2(Q)} + \|v^2\chi_{\mathcal{O}_2}\|^2_{L^2(Q)}).$$

Now, we are able to prove the first main result of this section.

Theorem 3.10. *Let $z^0 \in L^2(0,L)$, $f \in L^2(Q)$ and $(f^0, f^1, f^2) \in G \times G \times G$, with $G = L^2(0,T;H^{-1}(0,L))$ or $G = L^1(0,T;L^2(0,L))$, such that*

$$\|z^0\|_{L^2(0,L)} + \|f\|_{L^2(Q)} + \sum_{i=0}^{2} \|f^i\|_G \leq \epsilon_0, \tag{3.32}$$

for some $\epsilon_0 > 0$ sufficiently small. Then, for $\mu_i > 0$ sufficiently large, system (3.12) possesses a unique solution $(z, \phi^1, \phi^2) \in Y_{1/4} \times Y_{1/4} \times Y_{1/4}$ satisfying

$$\|(z, \phi^1, \phi^2)\|^2_{[Y_{1/4}]^3} \leq C(\|z^0\|^2_{L^2(0,L)} + \|f\|^2_{L^2(Q)} + \sum_{i=0}^{2} \|f^i\|^2_G), \tag{3.33}$$

where $C > 0$ does not depend on μ_i.

Demonstração. By fixing $\hat{z} \in Y_{1/4}$, we have that $\hat{z}\hat{z}_x \in L^1(0,T;L^2(0,L))$ (see [52], Prop. 4.1), and

$$\|\hat{z}\hat{z}_x - \bar{z}\bar{z}_x\|_{L^1(0,T;L^2(0,L))} \leq C(\|\hat{z}\|_{L^2(0,T;H_0^1(0,L))} + \|\bar{z}\|_{L^2(0,T;H_0^1(0,L))})\|\hat{z} - \bar{z}\|_{L^2(0,T;H_0^1(0,L))}, \tag{3.34}$$

for every $\hat{z}, \bar{z} \in Y_{1/4}$.

From Proposition 3.8, system

$$\begin{cases} z_t + z_x + z_{xxx} + (\overline{y}z)_x = -\hat{z}\hat{z}_x + f\,\mathbb{1}_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i \chi_{\mathcal{O}_i} + f^0 & \text{in} \quad Q, \\ -\phi^i_t - \phi^i_x - \phi^i_{xxx} - (\hat{z} + \overline{y})\phi^i_x = \alpha_i(z - z_{i,d})\chi_{\mathcal{O}_{i,d}} + f^i, \ i = 1,2 & \text{in} \quad Q, \\ z(0,\cdot) = z(L,\cdot) = z_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \phi^i(0,\cdot) = \phi^i(L,\cdot) = \phi^i_x(0,\cdot) = 0 & \text{in} \quad (0,T), \\ z(\cdot,0) = z^0, \quad \phi^i(\cdot,T) = 0 & \text{in} \quad (0,L) \end{cases} \tag{3.35}$$

possesses a unique solution $(z, \phi^1, \phi^2) \in Y_{1/4} \times Y_{1/4} \times Y_{1/4}$ satisfying

$$\begin{aligned} \|(z, \phi^1, \phi^2)\|^2_{[Y_{1/4}]^3} & \\ \leq C \Big(& \|\hat{z}\|^4_{L^2(0,T;H_0^1(0,L))} + \|z^0\|^2_{L^2(0,L)} + \|f\|^2_{L^2(Q)} \\ & + \left\|(f^0, f^1, f^2)\right\|^2_{[G]^3} \Big) e^{C\int_0^T (1 + \|\hat{z}_x(s)\|^2_{L^2(0,L)} + \|\overline{y}_x(s)\|^2_{L^2(0,L)})\, ds}, \end{aligned} \tag{3.36}$$

where C does not depend on μ_i.

Now, if $(z_*, \phi_*^1, \phi_*^2)$ is the solution of (3.35) by replacing $\hat{z}$ for $\bar{z}$ in $L^2(0,T;H_0^1(0,L))$, then

$$\begin{aligned}\|(z,\phi^1,\phi^2)-(z_*,\phi_*^1,\phi_*^2)\|^2_{[Y_{1/4}]^3} \\ \leq C\|\hat{z}-\bar{z}\|^2_{Y_{1/4}}\Big((\|\hat{z}\|^2_{L^2(0,T;H_0^1(0,L))}+\|\bar{z}\|^2_{L^2(0,T;H_0^1(0,L))} \\ +\|\phi_x^i\|^2_{L^2(0,L)}\Big)\,e^{C\int_0^T(1+\|\bar{z}_x(s)\|^2_{L^2(0,L)}+\|\bar{y}_x(s)\|^2_{L^2(0,L)})\,ds}. \quad (3.37)\end{aligned}$$

Combining (3.36) and (3.37),

$$\begin{aligned}\|(z,\phi^1,\phi^2)-(z_*,\phi_*^1,\phi_*^2)\|^2_{[Y_{1/4}]^3} \leq C\|\hat{z}-\bar{z}\|^2_{Y_{1/4}}\Big(\|\hat{z}\|^2_{L^2(0,T;H_0^1(0,L))}+\|\hat{z}\|^2_{L^2(0,T;H_0^1(0,L))} \\ +\|\bar{z}\|^2_{L^2(0,T;H_0^1(0,L))}+\|z^0\|^2_{L^2(0,L)}+\|f\|^2_{L^2(Q)} \\ +\left\|(f^0,f^1,f^2)\right\|^2_{[G]^3}\Big)\,e^{C\int_0^T(1+\|\hat{z}_x(s)\|^2_{L^2(0,L)}+\|\bar{z}_x(s)\|^2_{L^2(0,L)}+\|\bar{y}_x(s)\|^2_{L^2(0,L)})\,ds}. \quad (3.38)\end{aligned}$$

Assuming that $\hat{z}$ and $\bar{z}$ belong to a ball $\mathcal{B}=\{w\in Y_{1/4}; \|w\|_{Y_{1/4}}\leq R\}$, and that the data satisfies (3.32), then

$$\|(z,\phi^1,\phi^2)-(z_*,\phi_*^1,\phi_*^2)\|^2_{[Y_{1/4}]^3} \leq C\|\hat{z}-\bar{z}\|^2_{Y_{1/4}}\left(R^4+2R^2+\epsilon_0^2\right)e^{C\int_0^T(1+2R^2+\|\bar{y}_x(s)\|^2_{L^2(0,L)})\,ds}. \quad (3.39)$$

Finally, taking R and ϵ sufficiently small, the map $\hat{z}\mapsto(z,\phi^1,\phi^2)$ is a contraction. Hence, from Banach fixed point Theorem, it possesses a unique fixed point, which is a solution of (3.12). Moreover, from (3.36), estimate (3.33) holds. □

Now, we can still present another way we prove the existence of solutions to the optimality system (3.17). In order to simplify some computations, we will prove the existence of solutions to the reduced system

$$\begin{cases} u_t+u_{xxx}+(Mu)_x=\frac{1}{\mu}\upsilon\chi_{\mathcal{O}_1}+F^0 & \text{in} \quad Q, \\ -\upsilon_t-\upsilon_{xxx}-M\upsilon_x=u\chi_{\mathcal{O}_2}+F^1 & \text{in} \quad Q, \\ u(0,\cdot)=u(L,\cdot)=u_x(L,\cdot)=0 & \text{in} \quad (0,T), \\ \upsilon(0,\cdot)=\upsilon(L,\cdot)=\upsilon_x(0,\cdot)=0 & \text{in} \quad (0,T), \\ u(\cdot,0)=u^0, \quad \upsilon(\cdot,T)=0 & \text{in} \quad (0,1), \end{cases} \quad (3.40)$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are open sets, $\mu>0$, and u^0, F^0 and F^1 are in suitable functional spaces. The proof of existence of solutions to (3.40) can be easily adapted to prove the existence of solutions to (3.17) or (3.18). Before that, we define the sense of solutions by transposition of (3.40).

Definition 3.11. *Given $u^0 \in L^2(0,1)$, $(F^0, F^1) \in [L^2(0,T;H^{-1}(0,1))]^2$ and $M \in Y_{1/4}$. We say that $(u,v) \in [L^2(Q)]^2$ is a solution in the transposition sense for* (3.40) *if*

$$\iint_Q uG^0\,dxdt + \iint_Q vG^1\,dxdt = \int_0^1 u^0(x)p(x,0)\,dx + \int_0^T \langle F^0, p\rangle_{H^{-1}\,H_0^1}\,dt + \iint_Q \langle F^1, q\rangle_{H^{-1}\,H_0^1}\,dt, \quad (3.41)$$

for every $(p,q) \in Y_{1/4} \times Y_{1/4}$, solution of

$$\begin{cases} -p_t - p_{xxx} - Mp_x = q\chi_{\mathcal{O}_2} + G^0 & in \quad Q, \\ q_t + q_{xxx} + (Mq)_x = \frac{1}{\mu} p\chi_{\mathcal{O}_1} + G^1 & in \quad Q, \\ p(0,\cdot) = p(L,\cdot) = p_x(L,\cdot) = 0 & in \quad (0,T), \\ q(0,\cdot) = q(L,\cdot) = q_x(0,\cdot) = 0 & in \quad (0,T), \\ p(\cdot,T) = 0, \quad q(\cdot,0) = 0 & in \quad (0,1), \end{cases} \quad (3.42)$$

where $(G^0, G^1) \in [L^2(Q)]^2$.

In the next result we prove the existence of solution for (3.42). After that, we can apply Riesz Representation Theorem to guarantee the existence of solution for (3.40) in the sense (3.41).

Proposition 3.12. *Let $(G^0, G^1) \in [L^2(Q)]^2$ and $M \in Y_{1/4}$. Then, problem* (3.42) *possess a unique solution $(p,q) \in Y_{1/4} \times Y_{1/4}$.*

Demonstração. Let $\bar{p} \in L^2(Q)$ and consider $\bar{q} \in Y_{1/4}$ solution of

$$\begin{cases} \bar{q}_t + (M\bar{q})_x + \bar{q}_{xxx} = \frac{1}{\mu}\bar{p}\chi_{\mathcal{O}_1} + G^1 & \text{in} \quad Q, \\ \bar{q}(0,\cdot) = \bar{q}(L,\cdot) = \bar{q}_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \bar{q}(\cdot,0) = 0 & \text{in} \quad (0,1). \end{cases} \quad (3.43)$$

The existence of $\bar{q}$ follows from Proposition 3.2 and Remark 3.3. Moreover, we have that

$$\|\bar{q}\|_{Y_{\frac{1}{4}}} \le C \left\| \frac{1}{\mu}\bar{p}\chi_{\mathcal{O}_1} \right\|_{L^2(Q)} + \left\|G^1\right\|_{L^2(Q)}. \quad (3.44)$$

Now, let $\hat{p}$ the solution of

$$\begin{cases} -\hat{p}_t - M\hat{p}_x - \hat{p}_{xxx} = \bar{q}\chi_{\mathcal{O}_0} + G^0 & \text{in} \quad Q, \\ \hat{p}(0,\cdot) = \hat{p}(L,\cdot) = \hat{p}_x(0,\cdot) = 0 & \text{in} \quad (0,T), \\ \hat{p}(\cdot,T) = 0 & \text{in} \quad (0,1). \end{cases} \quad (3.45)$$

From Proposition 3.2 and estimate (3.44), we have that $\hat{p} \in Y_{\frac{1}{4}}$ and the following estimate holds

$$\|\hat{p}\|_{Y_{\frac{1}{4}}} \leq C\left(\|\bar{q}1_{\mathcal{O}_0}\|_{L^2(Q)} + \|G^0\|_{L^2(Q)}\right) \leq C\left(\frac{1}{\mu}\|\bar{p}\|_{L^2(Q)} + \|G^0\|_{L^2(Q)} + \left\|G^1\right\|_{L^2(Q)}\right). \tag{3.46}$$

Now, we set $\Lambda : L^2(Q) \longrightarrow L^2(Q)$ by $\Lambda\bar{p} = \hat{p}$. Using (3.46), it is not difficult to see that

$$\|\Lambda\bar{p} - \Lambda\bar{p}^*\|_{L^2(Q)} \leq \frac{C}{\mu}\|\bar{p} - \bar{p}^*\|, \quad \text{for all} \quad \bar{p}, \bar{p}^* \in L^2(Q).$$

Hence, for μ sufficiently large, the functional Λ is a contraction, and from Banach Fixed point Theorem it possess a unique fixed point, which we will call p^*. If q^* is the solution of (3.43) with p^* on the right-hand side instead of $\bar{p}$, then from (3.44) and (3.46), we get that $(u^*, v^*) \in Y_{1/4} \times Y_{1/4}$ and

$$\|p^*\|_{Y_{1/4}} + \|q^*\|_{Y_{1/4}} \leq C\left(\|G^0\|_{L^2(Q)} + \left\|G^1\right\|_{L^2(Q)}\right), \tag{3.47}$$

for μ sufficiently large. □

Remark 3.13. *If we assume that $M \equiv 0$ and $(G^0, G^1) \in G \times G$ where $G = X_\theta$ or $G \in \tilde{X}_\theta$, for $\theta \in [0,1]$, then using similar arguments we have used to prove Proposition 3.12, we can prove that problem (3.42) possesses a unique solution $(p, q) \in Y_\theta \times Y_\theta$. Similar arguments can be adapted to prove existence of solutions for (3.40) with $M = 0$ in $Y_\theta \times Y_\theta$.*

To prove the existence of solution for (3.40) in the sense (3.41), we proceed as follows. Define the linear functional $R : [L^2((0,1))]^2 \to \mathbb{R}$ by

$$R(G^0, G^1) = \int_0^1 u^0(x)p(x,0)\,dx + \int_0^T \langle F^0, p\rangle_{H^{-1}H_0^1}\,dt + \iint_Q \langle F^1, q\rangle_{H^{-1}H_0^1}\,dt,$$

where (p, q) is the solution of (3.42) with (G^0, G^1) on the right-hand side. It is easy to see that functional R is linear and from estimate (3.47) is continuous. Then, the existence of solutions for (3.40) in the sense (3.41) follows from the Riesz Representation Theorem.

On the existence of a Nash-Equilibrium

Due to the fact that functionals J_i are differentiable in the direction v^i $(i = 1, 2)$, we have proved that if (v^1, v^2) is a Nash-equilibrium, then it satisfies (3.4) and (3.5). Moreover, for μ_i is large enough, it is characterized by (3.11) and, from Theorem 3.10, it is unique at least for small data. Now, we want to know if the converse is true, that is, we want to prove that (v^1, v^2) given by (3.11) is actually a Nash equilibrium. The approach we follow to prove this is new and much simpler compared to the ones in [5, 13]. There, it is applied a second derivative test to show that critical points are minimal, which requires more regularity to the solutions than we have here.

The following result holds.

Theorem 3.14. *For each $v^2 \in L^2(\mathcal{O}_2 \times (0,T))$ ($v^1 \in L^2(\mathcal{O}_1 \times (0,T))$), the infimum of $J_1(\cdot, v^2)$ ($J_2(v^1, \cdot)$) is attained in some point $v^1 = v^1(v^2)$ in $L^2(\mathcal{O}_1 \times (0,T))$ ($v^2 = v^2(v^1)$ in $L^2(\mathcal{O}_2 \times (0,T))$).*

Demonstração. It is sufficient to prove that $J_1(\cdot, v^2)$ is sequentially lower semicontinuous (see [58], Theorem 2.D). Let $v \in L^2(\mathcal{O}_1 \times (0,T))$ and $v_n \in L^2(\mathcal{O}_1 \times (0,T))$ be any sequence converging weakly to v. We have to prove that

$$J_1(v, v^2) \leq \liminf J_1(v_n, v^2). \tag{3.48}$$

Indeed, from Proposition 3.9, the solution y^n of (3.1), with $v^1 = v_n$, is bounded in $Y_{1/4}$. In this way, using that (y^n) solves (3.1), we obtain that (y^n_t) is bounded in $L^1(0,T; H^{-2}(0,L))$, and from Aubin-Lions compactness lemma, there is a subsequence (y^{n_k}) that converges strongly to a function y in $L^2(Q)$. Combined with the fact that (y^{n_k}) converges weakly (up to another subsequence) to y in $L^2(0,T; H^1_0(0,L))$, we get that $y^{n_k} y^{n_k}_x$ converges weakly to yy_x and hence y is a solution of (3.1) with $v^1 = v$. Therefore, we have that $J_1(v, v_2) \leq \liminf J_1(v_{n_k}, v_2)$. To get (3.48), we take a subsequence (v_{n_j}) of (v_n) such that $\lim J_1(v_{n_j}, v_2) = \liminf J_1(v_n, v_2)$ and we repeat the argument for (v_{n_j}) instead of (v_n). □

By using the previous Theorem, we have the following consequence:

Corollary 3.15. *If (z_0, f, f^1, f^2) satisfies (3.32) and μ_i is large enough, then the pair (v^1, v^2) given by (3.11), where (z, ϕ^1, ϕ^2) is the unique solution of (3.12), is a Nash equilibrium for the functional J_i given by (3.2).*

Demonstração. From Proposition 3.9, functionals $J_1(\cdot, -\frac{1}{\mu_2}\phi^2\chi_{\mathcal{O}_2})$ and $J_2(-\frac{1}{\mu_1}\phi^1\chi_{\mathcal{O}_1}, \cdot)$ are well-defined. Now, using Theorem 3.14, the $\inf_{\hat{v}^1} J_1(\hat{v}^1, -\frac{1}{\mu_2}\phi^2\chi_{\mathcal{O}_2})$ and $\inf_{\hat{v}^2} J_2(-\frac{1}{\mu_1}\phi^1\chi_{\mathcal{O}_1}, \hat{v}^2)$ is attained at v^{1*} and v^{2*}, respectively. In particular, we have that

$$J_1'(v^{1*}, -\frac{1}{\mu_2}\phi^2\chi_{\mathcal{O}_2})(\hat{v}^1, 0) = 0 \quad \text{and} \quad J_2'(-\frac{1}{\mu_1}\phi^1\chi_{\mathcal{O}_1}, v^{2*})(0, \hat{v}^2) = 0,$$

for all $(\hat{v}^1, \hat{v}^2) \in \prod_{i=1}^2 L^2(\mathcal{O}_i \times (0,T))$. By following a similiar procedure of proving (3.11), we obtain that

$$v^{i*} = -\frac{1}{\mu_i}\phi^i\chi_{\mathcal{O}_i}, \quad i = 1, 2,$$

and the result follows. □

Remark 3.16. *In the proof Corollary 3.15, it is crucial that system* (3.12) *possesses unique solutions, otherwise, when minimizing* $J_1(\cdot, -\frac{1}{\mu_2}\phi^2\chi_{\mathcal{O}_2})$ *and* $J_2(-\frac{1}{\mu_1}\phi^1\chi_{\mathcal{O}_1}, \cdot)$*, it can happen that* $v^{i*} = -\frac{1}{\mu_i}\hat{\phi}^i\chi_{\mathcal{O}_i}$*, with* $\phi^i \neq \hat{\phi}^i$*.*

3.4 Carleman estimates

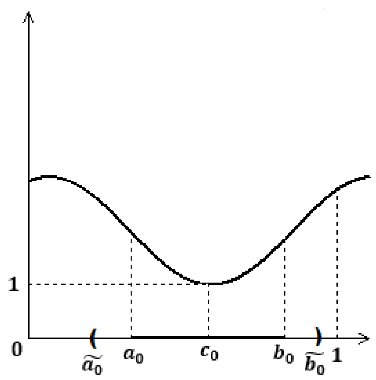
In order to prove the controllability result given in Theorem 3.1, we proceed by duality arguments. We start proving a controllability result for the linear system (3.17) as a consequence of an observability inequality such as (3.19). In order to do that, we are going to make use of Carleman estimates.

When dealing with distributed Carleman estimates for the KdV equation (see [4] or [12]), let us suppose that $\omega = (\tilde{a}_0, \tilde{b}_0) \subset (0,1)$, $[a_0, b_0] \subset \omega$, $c_0 = \frac{(a_0+b_0)}{2}$ and weight functions having the following properties are essential,

$$\begin{cases} \eta \in C^\infty([0,1]) \quad \text{and} \quad \eta > 0, \\ \\ -\eta''(x) \geq C > 0 \quad \text{and} \quad |\eta'(x)|^2 \geq C \quad \text{in} \quad \overline{[0,1]\backslash\omega}, \\ \\ \eta'(0) < 0 \quad \text{and} \quad \eta'(1) > 0, \end{cases} \tag{3.49}$$

for any open subset $\omega \subset\subset (0,1)$. (See figure 1)

Figura 1 – Graphic



Source: Own authorship

For any function η satisfying (3.49), we define the weight functions

$$\xi(t) = \frac{1}{t(T-t)} \quad \text{and} \quad \Phi(x,t) = \eta(x)\xi(t), \tag{3.50}$$

and consider the following notation

$$\hat{\Phi}(t) = \max_{x\in[0,1]} \Phi(t,x) \quad \text{and} \quad \check{\Phi}(t) = \min_{x\in[0,1]} \Phi(t,x).$$

It is not difficult to see that we can choose functions satisfying (3.49), and such that

$$8\check{\Phi}(t) - 7\hat{\Phi}(t) > 0 \quad \text{in} \quad [0,T]. \tag{3.51}$$

To do that, we first take a function η_0 satisfying (3.49), and we define $\eta_\epsilon := \epsilon\eta_0$. Since $\eta_0 \in C^\infty([0,1])$, then $\min \eta_0 \leq \eta_0 \leq \max \eta_0$. In this way, for $\varepsilon > 0$, we have that $\min \varepsilon < \varepsilon\eta_0 < \max \varepsilon$. Then, we define $\eta_0^\varepsilon = \varepsilon\eta_0$ and we have that $\min \eta_0^\varepsilon < \eta_0^\varepsilon < \max \eta_0^\varepsilon$. Note that $\min \eta_0^\varepsilon - \max \eta_0^\varepsilon \to 0$ when $\varepsilon \to 0$. In this way, we just have to fix ε sufficiently small such that $8c_0^\varepsilon - 7c_1^\varepsilon > 0$.

Now, for $m \in \mathbb{Z}$, we introduce the notation

$$\begin{aligned} I^m(v) := s^m \iint_Q e^{-2s\Phi}\xi^m|v|^2 dxdt + s^{m-2}\iint_Q e^{-2s\Phi}\xi^{m-2}|v_x|^2 dxdt \\ + s^{m-4}\iint_Q e^{-2s\Phi}\xi^{m-4}|v_{xx}|^2 dxdt + s^{m-6}\iint_Q e^{-2s\Phi}\xi^{m-6}|v_t + v_{xxx}|^2 dxdt. \end{aligned} \tag{3.52}$$

The following Carleman estimate for the KdV equation holds, and we refer to [4] or [12] for a proof of it.

Proposition 3.17. *Let $\omega_* \subset\subset (0,1)$ and assume that $M \in Y_{1/4} = L^2(0,T;H^1(0,1)) \cap L^\infty(0,T;L^2(0,1))$. There exist $C > 0$ and $s_0 \geqslant 1$ such that*

$$\begin{aligned} I^m(v) \leq C\Big(s^{m-5}\iint_Q e^{-2s\Phi}\xi^{m-5}|Lv|^2 dxdt + s^m \iint_{Q_{\omega_*}} e^{-2s\Phi}\xi^m|v|^2 dxdt \\ + s^{m-4}\iint_{Q_{\omega_*}} e^{-2s\Phi}\xi^{m-4}|v_{xx}|^2 dxdt\Big), \end{aligned} \tag{3.53}$$

for all $s > s_0$, and for every $v \in L^2(0,T;H^2 \cap H_0^1(0,1))$ with $v_x(0,t) = 0$ and $Lv := (v_t + v_{xxx} + Mv_x) \in L^2(Q)$.

The next step is to use (3.53) and prove a new Carleman estimates for the adjoint system (3.18), provided that assumptions of Theorem 3.1 hold. In order to do that, we deal with conditions (3.14) and (3.15) separately.

3.4.1 The Case $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$

Assume that condition (3.14) holds. In this case, it is convenient to denote $\mathcal{O}_d = \mathcal{O}_{1,d} = \mathcal{O}_{2,d}$ and $M = 1 + \bar{y}$. Then, system (3.18) reads as

$$\begin{cases} -\psi_t - M\psi_x - \psi_{xxx} = (\alpha_1\gamma^1 + \alpha_2\gamma^2)\chi_{\mathcal{O}_d} + g^0 & \text{in} \quad Q, \\ \gamma_t^i + (M\gamma^i)_x + \gamma_{xxx}^i = -\frac{1}{\mu_i}\psi\chi_{\mathcal{O}_i} + g^i, \quad i = 1,2 & \text{in} \quad Q, \\ \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = 0 & \text{on} \quad (0,T), \\ \gamma^i(0,\cdot) = \gamma^i(L,\cdot) = \gamma_x^i(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \psi(x,T) = \psi^T(x), \quad \gamma^i(\cdot,0) = 0 & \text{in} \quad (0,1). \end{cases} \tag{3.54}$$

If we take $h := \alpha_1\gamma^1 + \alpha_2\gamma^2$ and $g = \alpha_1 g^1 + \alpha_2 g^2$, we obtain from (3.54) system

$$\begin{cases} -\psi_t - M\psi_x - \psi_{xxx} = h\chi_{\mathcal{O}_d} + g^0 & \text{in} \quad Q, \\ h_t + (Mh)_x + h_{xxx} = -\frac{\alpha_1}{\mu_1}\psi\chi_{\mathcal{O}_1} - \frac{\alpha_2}{\mu_2}\psi\chi_{\mathcal{O}_2} + g & \text{in} \quad Q, \\ \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = 0 & \text{on} \quad (0,T), \\ h(0,\cdot) = h(L,\cdot) = h_x(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \psi(x,T) = \psi^T(x), \quad h(\cdot,0) = 0 & \text{in} \quad (0,1). \end{cases} \tag{3.55}$$

Therefore, we have the following Carleman estimate to the solutions of (3.55). In what follows, we strongly use the theory of fractional Sobolev Spaces. (See [7])

Proposition 3.18. *Assume that $M \in Y_{1/2} = L^2(0,T;H^2(0,1)) \cap L^\infty(0,T;H_0^1(0,1))$ and that*

$$\mathcal{O}_d \cap \mathcal{O} \neq \emptyset.$$

Let $\omega \subset\subset \mathcal{O}_d \cap \mathcal{O}$ be an open subset. Then, there exists a constant $C := C(\mathcal{O}_d, \{\mathcal{O}_i\}_{i=1}^2, \{\alpha_i\}_{i=1}^2, \{\mu_i\}_{i=1}^2$ and $s_0 \geq 1$ such that

$$I^5(\psi) + I^5(h) \leq C\left(s^{31}\iint_{\mathcal{O}\times(0,T)} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{49}|\psi|^2 dxdt \right. \\ \left. + s^{27}\int_0^T e^{12s\hat{\Phi}-14s\Phi}\xi^{45}\left(\|g^0\|^2_{H^{2/3}(0,1)} + \|g\|^2_{H^{2/3}(0,1)}\right)dt\right), \tag{3.56}$$

for all $s > s_0$, and for every solution (ψ, h) of (3.55), with $\psi^T \in L^2(Q)$ and $(g^0, g) \in [L^2(0,T;H^{2/3}(0,1))]^2$.

Demonstração. Let $\tilde{\omega} \subset\subset \omega \subset\subset \mathcal{O}_d \cap \mathcal{O}$. We use Carleman estimate (3.53) with $\omega_* = \tilde{\omega}$ and $m = 5$ to the solutions of (3.55), to get

$$I^5(\psi) + I^5(h) \leq C\left(s^5\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi^5|\psi|^2 dxdt + s\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi|\psi_{xx}|^2 dxdt \right. \\ + s^5\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi^5|h|^2 dxdt + s\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi|h_{xx}|^2 dxdt \\ + \iint_Q e^{-2s\Phi}|g^0|^2 dxdt + \iint_Q e^{-2s\Phi}|g|^2 dxdt \\ \left. + \iint_Q e^{-2s\Phi}|h\chi_{\mathcal{O}_d}|^2 dxdt + \iint_Q e^{-2s\Phi}\left|\sum_{i=1}^2 \frac{\alpha_i}{\mu_i}\psi\chi_{\mathcal{O}_i}\right|^2 dxdt\right). \tag{3.57}$$

Note that the last three terms on the right-hand side of (3.57) can be absorbed by left hand side, by taking s sufficiently large. Therefore, we obtain that

$$\begin{aligned} I^5(\psi) + I^5(h) \leq C \Big(s^5 \iint_{Q_{\tilde\omega}} e^{-2s\Phi}\xi^5|\psi|^2 dxdt + s \iint_{Q_{\tilde\omega}} e^{-2s\Phi}\xi|\psi_{xx}|^2 dxdt \\ + s^5 \iint_{Q_{\tilde\omega}} e^{-2s\Phi}\xi^5|h|^2 dxdt + s \iint_{Q_{\tilde\omega}} e^{-2s\Phi}\xi|h_{xx}|^2 dxdt \\ + \iint_{Q} e^{-2s\Phi}|g^0|^2 dxdt + \iint_{Q} e^{-2s\Phi}|g|^2 dxdt \Big). \quad (3.58) \end{aligned}$$

In what follows, we prove that we can add a weighted $L^2(0,T;H^{8/3}(\Omega))$ norm of ψ and h on the left hand side of (3.58). This will be important to absorb the second and fourth terms on the right-hand side of (3.58).

Let $\sigma_1 = e^{-s\hat\Phi}\xi^{\frac{1}{2}}$ and define $(\psi_1, h_1) := (\sigma_1\psi, \sigma_1 h)$. Then, we have that

$$\begin{cases} -\psi_{1t} - \psi_{1xxx} = k_1 & \text{in} \quad Q, \\ h_{1t} + h_{1xxx} = g_1 & \text{in} \quad Q, \\ \psi_1(0,\cdot) = \psi_1(L,\cdot) = \psi_{1x}(0,\cdot) = 0 & \text{on} \quad (0,T), \\ h_1(0,\cdot) = h_1(L,\cdot) = h_{1x}(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \psi_1(x,T) = 0, \quad h_1(\cdot,0) = 0 & \text{in} \quad (0,1), \end{cases} \quad (3.59)$$

where

$$\begin{aligned} k_1 &= h_1\chi_{\mathcal{O}_d} + M\psi_{1x} + \sigma_1 g^0 - \sigma_1'\psi, \\ g_1 &= -\textstyle\sum_{i=1}^2 \frac{\alpha_i}{\mu_i}\psi_1\chi_{\mathcal{O}_i} - (Mh_1)_x + \sigma_1 g + \sigma_1' h. \end{aligned} \quad (3.60)$$

From the facts that $M \in Y_{1/2}$ and that $Y_{1/2} \hookrightarrow Y_{1/4}$, we get that $(\psi, h) \in Y_{1/4} \times Y_{1/4}$, and using that $|\sigma_1'| \leq Cse^{-s\hat\Phi}\xi^{\frac{5}{2}}$, we get that $(k_1, g_1) \in [L^2(0,T;L^2(0,1))]^2$, and the following estimate holds

$$\|k_1\|^2_{L^2(Q)} + \|g_1\|^2_{L^2(Q)} \leq C\Big(I^5(\psi) + I^5(h) + \|\sigma_1 g^0\|^2_{L^2(Q)} + \|\sigma_1 g\|^2_{L^2(Q)}\Big). \quad (3.61)$$

From Proposition 3.5, we get that $(\psi_1, h_1) \in Y_{1/2} \times Y_{1/2}$, and since $Y_{1/2} \hookrightarrow L^4(0,T;H^{\frac{3}{2}}(0,1))$, we obtain that

$$\|\psi_1\|^2_{L^4(0,T;H^{\frac{3}{2}}(0,1))} + \|h_1\|^2_{L^4(0,T;H^{\frac{3}{2}}(0,1))} \leq C\Big(\|k_1\|^2_{L^2(Q)} + \|g_1\|^2_{L^2(Q)}\Big). \quad (3.62)$$

Hence, combining (3.61) and (3.62)

$$\|\psi_1\|^2_{L^4(0,T;H^{\frac{3}{2}}(0,1))} + \|h_1\|^2_{L^4(0,T;H^{\frac{3}{2}}(0,1))} \leq C\Big(I^5(\psi) + I^5(h) + \|\sigma_1 g^0\|^2_{L^2(Q)} + \|\sigma_1 g\|^2_{L^2(Q)}\Big). \quad (3.63)$$

Consider now $\sigma_2 = e^{-s\hat{\Phi}}\xi^{-\frac{3}{2}}$ and define $(\psi_2, h_2) = (\sigma_2\psi, \sigma_2 h)$. It is clear that

$$\begin{cases} -\psi_{2t} - \psi_{2xxx} = k_2 & \text{in} \quad Q, \\ h_{2t} + h_{2xxx} = g_2 & \text{in} \quad Q, \\ \psi_2(0,\cdot) = \psi_2(L,\cdot) = \psi_{2x}(0,\cdot) = 0, & \text{on} \quad (0,T), \\ h_2(0,\cdot) = h_2(L,\cdot) = h_{2x}(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \psi_2(x,T) = 0, \quad h_2(\cdot,0) = 0 & \text{in} \quad (0,1), \end{cases} \tag{3.64}$$

where

$$\begin{aligned} k_2 &= h_2\chi_{\mathcal{O}_d} + M\psi_{2x} + \sigma_2 g^0 - \sigma_2^{'}\psi \\ &= \sigma_2\sigma_1^{-1}h_1\chi_{\mathcal{O}_d} + M\sigma_2\sigma_1^{-1}\psi_{1x} + \sigma_2 g^0 - \sigma_2^{'}\sigma_1^{-1}\psi_1, \end{aligned} \tag{3.65}$$

and

$$\begin{aligned} g_2 &= -\textstyle\sum_{i=1}^{2}\frac{\alpha_i}{\mu_i}\psi_2\chi_{\mathcal{O}_i} - (Mh_2)_x + \sigma_2 g + \sigma_2^{'}h \\ &= -\textstyle\sum_{i=1}^{2}\frac{\alpha_i}{\mu_i}\sigma_2\sigma_1^{-1}\psi_1\chi_{\mathcal{O}_i} - \sigma_2\sigma_1^{-1}(Mh_1)_x + \sigma_2 g + \sigma_2^{'}\sigma_1^{-1}h_1. \end{aligned} \tag{3.66}$$

Taking into account that the produt of two functions in $H^{\frac{1}{2}}(0,1)$ belongs to $H^{\frac{1}{3}}(0,1)$ and using the fact that M, ψ_1 and h_1 belongs to $L^4(0,T;H^{\frac{3}{2}}(0,1))$, we get that $M\psi_{1x}$ and $(Mh_1)_x$ belongs to $X_{\frac{7}{12}} = L^2(0,T;H^{1/3}(0,1))$. Moreover, since $|\sigma_2\sigma_1^{-1}| \leq C$ and $|\sigma_2^{'}\sigma_1^{-1}| \leq Cs$, it follows that $(k_2, g_2) \in [L^2(0,T;H^{\frac{1}{3}}(0,1))]^2$ and hence, from Proposition 3.5, we get that $(\psi_2, h_2) \in \Big[Y_{7/12}\Big]^2$ and we obtain the following estimate

$$\begin{aligned} \|\psi_2\|^2_{Y_{\frac{7}{12}}} + \|h_2\|^2_{Y_{\frac{7}{12}}} &\leq C\Big(\|k_2\|^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))} + ||g_2||^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))}\Big) \\ &\leq Cs^2\Big(I^5(\psi) + I^5(h) + \|\sigma_1 g^0||^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))} + \|\sigma_1 g||^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))}\Big), \end{aligned} \tag{3.67}$$

where

$$Y_{\frac{7}{12}} = L^2(0,T;H^{\frac{7}{3}}(0,1)) \cap L^\infty(0,T;H^{\frac{4}{3}}(0,1)).$$

Finally, consider $\sigma_3 = e^{-s\hat{\Phi}}\xi^{-\frac{7}{2}}$ and define $(\psi_3, h_3) := (\sigma_3\psi, \sigma_3 h)$. Then, we have the following system

$$\begin{cases} -\psi_{3t} - \psi_{3xxx} = k_3 & \text{in} \quad Q, \\ h_{3t} + h_{3xxx} = g_3 & \text{in} \quad Q, \\ \psi_3(0,\cdot) = \psi_3(L,\cdot) = \psi_{3x}(0,\cdot) = 0, & \text{on} \quad (0,T), \\ h_3(0,\cdot) = h_3(L,\cdot) = h_{3x}(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \psi_3(x,T) = 0, \quad h_3(\cdot,0) = 0 & \text{in} \quad (0,1), \end{cases} \tag{3.68}$$

where

$$\begin{aligned} k_3 &= h_3\chi_{\mathcal{O}_d} + M\psi_{3x} + \sigma_3 g^0 - \sigma_3^{'}\psi \\ &= \sigma_3\sigma_2^{-1}h_2\chi_{\mathcal{O}_d} + M\sigma_3\sigma_2^{-1}\psi_{2x} + \sigma_3 g^0 - \sigma_3^{'}\sigma_2^{-1}\psi_2, \end{aligned} \tag{3.69}$$

and

$$\begin{array}{rcl} g_3 &=& -\sum_{i=1}^{2} \frac{\alpha_i}{\mu_i} \psi_3 \chi_{\mathcal{O}_i} - (Mh_3)_x + \sigma_3 g + \sigma_3^{'} h \\ &=& -\sum_{i=1}^{2} \frac{\alpha_i}{\mu_i} \sigma_3 \sigma_2^{-1} \psi_2 \chi_{\mathcal{O}_i} - \sigma_3 \sigma_2^{-1} (Mh_2)_x + \sigma_3 g + \sigma_3^{'} \sigma_2^{-1} h_2. \end{array} \tag{3.70}$$

Using that the product of functions in $H^{2/3}(0,1)$ belongs to $H^{2/3}(0,1)$, that $M_x \in L^3(0,T;H^{\frac{2}{3}}(0,1))$ and $(\psi_{2x}, h_{2x}) \in [L^6(0,T;H^{\frac{2}{3}}(0,1))]^2$, and using that $|\sigma_3\sigma_2^{-1}| \leq C$ and $|\sigma_3^{'}\sigma_2^{-1}| \leq Cs$, we get that $(k_3, g_3) \in [L^2(0,T;H^{\frac{2}{3}}(0,1))]^2$ and from Proposition 3.5, we obtain

$$\begin{aligned} &\|\psi_3\|^2_{Y_{\frac{2}{3}}} + \|h_3\|^2_{Y_{\frac{2}{3}}} \leq C\big(\|k_3\|^2_{X_{\frac{2}{3}}} + ||g_3||^2_{X_{\frac{2}{3}}}\big) \\ &\qquad \leq Cs^4\Big(I^5(\psi) + I^5(h) + \|\sigma_1 g^0||^2_{L^2(0,T;H^{\frac{2}{3}}(0,1))} + \|\sigma_1 g||^2_{L^2(0,T;H^{\frac{2}{3}}(0,1))}\Big), \end{aligned} \tag{3.71}$$

where $X_{\frac{2}{3}} = L^2(0,T;H^{\frac{2}{3}}(\Omega))$ and $Y_{\frac{2}{3}} = L^2(0,T;H^{\frac{8}{3}}(0,1)) \cap L^\infty(0,T;H^{\frac{5}{3}}(0,1))$.

Now, combining (3.58) and (3.71), we have

$$\begin{aligned} &s^{-4}\int_0^T e^{-2s\hat{\Phi}}\xi^{-7}\|\psi\|^2_{H^{\frac{8}{3}}(0,1)}dt + s^{-4}\int_0^T e^{-2s\hat{\Phi}}\xi^{-7}\|h\|^2_{H^{\frac{8}{3}}(0,1)}dt + I^5(\psi) + I^5(h) \\ &\qquad \leq C\left(s^5\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi^5|\psi|^2dxdt + s\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi|\psi_{xx}|^2dxdt\right. \\ &\qquad\qquad +s^5\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi^5|h|^2dxdt + s\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi|h_{xx}|^2dxdt \\ &\qquad\qquad \left. s^4\int_0^T e^{-2s\check{\Phi}}\xi\|g^0\|^2_{H^{2/3}(0,1)}\,dt + s^4\int_0^T e^{-2s\check{\Phi}}\xi\|g\|^2_{H^{2/3}(0,1)}dt\right). \end{aligned} \tag{3.72}$$

Now, we are going to estimate the terms of second order derivative on the right hand side of (3.72). Using interpolation estimates and Young's inequality, we have

$$\begin{aligned} &s\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi|\psi_{xx}|^2dxdt + s\iint_{Q_{\tilde{\omega}}} e^{-2s\Phi}\xi|h_{xx}|^2dxdt \\ &\quad \leq Cs\left(\int_0^T e^{-2s\check{\Phi}}\xi\|\psi\|^{\frac{1}{2}}_{L^2(\tilde{\omega})}\|\psi\|^{\frac{3}{2}}_{H^{\frac{8}{3}}(\tilde{\omega})}dt + \int_0^T e^{-2s\check{\Phi}}\xi\|h\|^{\frac{1}{2}}_{L^2(\tilde{\omega})}\|h\|^{\frac{3}{2}}_{H^{\frac{8}{3}}(\tilde{\omega})}dt\right) \\ &\qquad \leq C\int_0^T\left(s^4\xi^{25/4}e^{\frac{3}{2}s\hat{\Phi}-2s\check{\Phi}}\|\psi\|^{\frac{1}{2}}_{L^2(\tilde{\omega})}s^{-3}\xi^{-21/4}e^{-\frac{3}{2}s\hat{\Phi}}\|\psi\|^{\frac{3}{2}}_{H^{\frac{8}{3}}(\tilde{\omega})}\right)dt \\ &\qquad + C\int_0^T\left(s^4\xi^{25/4}e^{\frac{3}{2}s\hat{\Phi}-2s\check{\Phi}}\|h\|^{\frac{1}{2}}_{L^2(\tilde{\omega})}s^{-3}\xi^{-21/4}e^{-\frac{3}{2}s\hat{\Phi}}\|h\|^{\frac{3}{2}}_{H^{\frac{8}{3}}(\tilde{\omega})}\right)dt \\ &\qquad\qquad \leq Cs^{16}\int_0^T e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{25}\Big(\|\psi\|^2_{L^2(\tilde{\omega})} + \|h\|^2_{L^2(\tilde{\omega})}\Big)\,dt \\ &\qquad\qquad\qquad + \varepsilon s^{-4}\int_0^T e^{-2s\hat{\Phi}}\xi^{-7}\Big(\|\psi\|^2_{H^{\frac{8}{3}}(\tilde{\omega})} + \|h\|^2_{H^{\frac{8}{3}}(\tilde{\omega})}\Big)\,dt. \end{aligned} \tag{3.73}$$

In this way, we combine (3.72) and (3.73) to get that

$$s^{-4}\int_0^T e^{-2s\hat{\Phi}}\xi^{-7}\|\psi\|^2_{H^{\frac{8}{3}}(0,1)}dt + s^{-4}\int_0^T e^{-2s\hat{\Phi}}\xi^{-7}\|h\|^2_{H^{\frac{8}{3}}(0,1)}dt + I^5(\psi) + I^5(h)$$
$$\leq Cs^{16}\int_0^T e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{25}\left(\|\psi\|^2_{L^2(\tilde{\omega})} + \|h\|^2_{L^2(\tilde{\omega})}\right)dt$$
$$+ Cs^4\int_0^T e^{-2s\check{\Phi}}\xi\left(\|g^0\|^2_{H^{2/3}(0,1)} + \|g\|^2_{H^{2/3}(0,1)}\right)dt. \quad (3.74)$$

Now, we are going to eliminate the local term of h in the right-hand side of (3.74). For this, let ω be an open set satisfying $\tilde{\omega} \subset\subset \omega \subset\subset \mathcal{O}$ and let $\theta_0 \in C_0^3(\omega)$ be such that $\theta_0(x) = 1$ for $x \in \tilde{\omega}$ and $0 \leq \theta_0 \leq 1$. Using the first equation of (3.55), we can see that

$$-\psi_t - M\psi_x - \psi_{xxx} - g^0 = h\chi_{\mathcal{O}_d} \quad \text{in} \quad \omega \cap \mathcal{O}_d \times (0,T).$$

Therefore,

$$s^{16}\iint_{Q_{\tilde{\omega}}} e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{25}|h|^2dxdt \leq s^{16}\iint_{Q_\omega} e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{25}\theta_0 h(-\psi_t - M\psi_x - \psi_{xxx} - g^0)dxdt$$
$$\leq \varepsilon I^5(h) + C_\varepsilon s^{31}\iint_{Q_\omega} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{49}|\psi|^2dxdt$$
$$+ s^{16}\iint_{Q_\omega} e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{25}Mh\psi dxdt + C_\varepsilon s^{27}\iint_{Q} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{45}(|g^0|^2 + |g|^2)\,dxdt. \quad (3.75)$$

In (3.75), we have used the fact that $|\partial_t(e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^n)| \leq Cse^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{n+2}$ for $n > 0$. We also have that

$$s^{16}\iint_{Q_\omega} e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{25}\theta_0 h(M\psi)dxdt \leq \varepsilon\|s^{\frac{5}{2}}e^{-s\hat{\Phi}}\xi^{\frac{5}{2}}h\|^2_{L^2(0,T;L^\infty(\Omega))}\|M\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$+ C_\varepsilon s^{27}\iint_{Q_\omega} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{45}|\psi|^2dxdt$$
$$\leq \varepsilon I^5(h) + C_\varepsilon s^{27}\iint_{Q_\omega} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{45}|\psi|^2dxdt. \quad (3.76)$$

Then, combining (3.74), (3.75) and (3.76) we get Carleman estimate (3.56). □

3.4.2 The Case $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}$

In this case, we have two possible assumptions to be distinguished:

A. $\mathcal{O}_{i,d} \cap \mathcal{O}$ is not contained in $\mathcal{O}_{j,d} \cap \mathcal{O}$ for every $i, j \in \{1,2\}$, $i \neq j$;

B. $\mathcal{O}_{i,d} \cap \mathcal{O}$ is contained in $\mathcal{O}_{j,d} \cap \mathcal{O}$ for some $i, j \in \{1,2\}$, $i \neq j$.

Here, we will concentrate in the case where Assumption A holds. For Assumption B, the proof follows from an argument that mix the proofs of Section 3.4.1 and the proof of case A given below. We refer to [6] where case B is solved for Heat Equation and the adequacy to the KdV equation considered here follows in an analogous way.

The main difficulties that the KdV equation case shows compared to the heat equation case are two. First, the distributed Carleman estimate given in Proposition 3.17 possesses second order derivative localized in a small open set $\omega_* \subset (0,1)$. Second, the weight functions to be considered have to be constructed in a very special way, since the weight functions applied for the heat equation case does not apply for the KdV case. In what follows we construct the new weight functions.

Lemma 3.19. *Let $\omega_1 = (a_1, b_1)$ and $\omega_2 = (a_2, b_2)$ be open subsets of $[0,1]$ such that $b_1 < a_2$ and let $\mathcal{O}_0$ be an open interval such that $\omega_1 \cup \omega_2 \subset \mathcal{O}_0$. There exist two non-negative functions η_1 and η_2 in $C^\infty(\mathbb{R})$, satisfying (3.49) for ω_1 and ω_2 respectively, such that $n_1 = n_2$ in $[0,1] \setminus \mathcal{O}_0$ and that $\|\eta_1\|_\infty = \|\eta_2\|_\infty$.*

Demonstração. Let $\delta > 0$, and define $\omega_{1\,1} = (b_1 - 2\delta, b_1 - \delta)$ and $\omega_{2\,1} = (a_2 + \delta, a_2 + 2\delta)$. It is clear that we can take δ sufficiently small such that $\overline{\omega_{1\,1}} \subset \omega_1$ and $\overline{\omega_{2\,1}} \subset \omega_2$.

Now, let $F_1 \in C^\infty(\mathbb{R})$ such that

$$F_1 > 0, \; -F_1^{''} > 0 \quad \text{and} \quad F_1' < 0,$$

and define $F_2(x) = F_1(-x + 2c_0)$, where $c_0 = (b_1 + a_2)/2$. It is not difficult to see that

$$F_2 > 0, \; -F_2^{''} > 0 \quad \text{and} \quad F_2' > 0.$$

Moreover, the functions F_1 and F_2 are symmetric with respect the line $\{x = c_0\}$, that is, we have that $F_2(c_0 + y) = F_1(c_0 - y)$ for every $y \in \mathbb{R}$.

Consider the functions

$$\lambda_1(x) = \begin{cases} 1 & \text{if} \quad x \leq b_1 - 2\delta, \\ 0 & \text{if} \quad x \geq b_1 - \delta, \end{cases} \quad \text{and} \quad \lambda_2(x) = \begin{cases} 0 & \text{if} \quad x \leq b_1 - 2\delta, \\ 1 & \text{if} \quad x \geq b_1 - \delta, \end{cases} \tag{3.77}$$

and define

$$\eta_1(x) = \lambda_1(x) F_1(x) + \lambda_2(x) F_2(x). \tag{3.78}$$

Since $\eta_1 = F_1$ for $x \leq b_1 - 2\delta$ and $\eta_1 = F_2$ for $x \geq b_1 - \delta$, we have that η_1 and ω_1 satisfies (3.49). Defining

$$\eta_2(x) = \eta_1(-x + 2c_0), \tag{3.79}$$

it is not difficult to see that η_2 also satisfies (3.49), that $\eta_2 = \eta_1$ in $(-\infty, b_1 - 2\delta) \cup (a_2 + 2\delta, +\infty)$, and are symmetric in $(b_1 - 2\delta, a_2 + 2\delta)$ with respect the line $\{x = c_0\}$, which means that

$$\max_{x\in[0,1]}|\eta_1(x)| = \max_{x\in[0,1]}|\eta_2(x)| \quad \text{and} \quad \min_{x\in[0,1]}|\eta_1(x)| = \min_{x\in[0,1]}|\eta_2(x)|.$$

□

Now we turn to assumption (3.15). Since we are assuming case A, we can consider ω_1 and ω_2 to be two disjoint nonempty open subsets of $(0,1)$ such that

$$\overline{\omega}_1 \subset \mathcal{O}_{1,d}\cap\mathcal{O} \quad \text{and} \quad \overline{\omega}_2 \subset \mathcal{O}_{2,d}\cap\mathcal{O}. \tag{3.80}$$

Now, let $\widetilde{\mathcal{O}}_0$ and $\widetilde{\mathcal{O}}$ be two nonempty open subsets of $\mathcal{O}$ such that $\widetilde{\mathcal{O}}_0 \subset \widetilde{\mathcal{O}} \subset \mathcal{O}$ and

$$\overline{\omega_1\cup\omega_2} \subset \widetilde{\mathcal{O}}_0. \tag{3.81}$$

For $i=1,2$, let η_i given by Lemma 3.19, for these open sets ω_1 and ω_2 and $\mathcal{O}_0 = \widetilde{\mathcal{O}_0}$, and define the weight functions

$$\Phi_i(x,t) = \eta_i(x)\xi(t), \quad \xi(t) = \frac{1}{t(T-t)}. \tag{3.82}$$

Also, we denote

$$\hat{\Phi}(t) = \max_{x\in[0,1]}\Phi_1(t,x) = \max_{x\in[0,1]}\Phi_2(t,x) \quad \text{and} \quad \check{\Phi}(t) = \min_{x\in[0,1]}\Phi_1(t,x) = \min_{x\in[0,1]}\Phi_2(t,x). \tag{3.83}$$

In a similar way as in (3.51), we can prove that the functions η_i can be taken in such a way that

$$8\check{\Phi}(t) - 7\hat{\Phi}(t) > 0. \tag{3.84}$$

For each $i=1,2$, we define

$$I_i^m(v) := L_i^m(v) + s^{m-6}\iint_Q e^{-2s\Phi_i}\xi^{m-6}|v_t + v_{xxx}|^2 dxdt, \tag{3.85}$$

where

$$L_i^m(v) := s^m\iint_Q e^{-2s\Phi_i}\xi^m|v|^2 dxdt + s^{m-2}\iint_Q e^{-2s\Phi_i}\xi^{m-2}|v_x|^2 dxdt \\ + s^{m-4}\iint_Q e^{-2s\Phi_i}\xi^{m-4}|v_{xx}|^2 dxdt. \tag{3.86}$$

Then, we will prove the following Carleman estimate.

Proposition 3.20. *Assume that* $M \in Y_{1/2} = L^2(0,T;H^2(0,1)) \cap L^\infty(0,T;H_0^1(0,1))$. *Also, assume that*

$$\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset, \quad \textit{for} \quad i = 1,2,$$

and that $\mathcal{O}_{i,d} \cap \mathcal{O}$ *is not contained in* $\mathcal{O}_{j,d} \cap \mathcal{O}$ *for every* $i,j \in \{1,2\}$, $i \neq j$ *(Assumption A). Assume also that* (3.15) *holds and let* ω_1*,* ω_2 *and* $\widetilde{\mathcal{O}}_0$ *satisfying* (3.80) *and* (3.81)*. If* η_1 *and* η_2 *are weight functions constructed in Lemma 3.19 for* ω_1 *and* ω_2 *respectively, then there exists* $C > 0$ *such that*

$$L_1^5(\psi) + \sum_{i=1}^{2} I_i^1(\gamma^i) \leq C \left(s^5 \int_0^T e^{14s\hat{\Phi}-16s\check{\Phi}} \xi^{41} \Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^{2} \|g^i\|^2_{H^{2/3}(0,1)} \Big)\, dt \right.$$
$$\left. + s^{19} \iint_{\mathcal{O}\times(0,T)} e^{14s\hat{\Phi}-16s\check{\Phi}} \xi^{45} |\psi|^2 \, dxdt \right). \quad (3.87)$$

Demonstração. We start by considering a function $\theta \in C^3([0,1])$ such that

$$\begin{cases} \theta(x) = 0 & \text{for} \quad x \in \widetilde{\mathcal{O}}_0, \\ \theta(x) = 1 & \text{for} \quad x \in [0,1] \setminus \widetilde{\mathcal{O}}. \end{cases} \quad (3.88)$$

It is straightforward to check that $\theta\psi$ satisfies

$$\begin{cases} -(\theta\psi)_t - M(\theta\psi)_x - (\theta\psi)_{xxx} = \sum_{i=1}^2 \theta\alpha_i\gamma^i\chi_{\mathcal{O}_{i,d}} + \theta g^0 + R(\psi) & \text{in} \quad Q, \\ (\theta\psi)(0,\cdot) = (\theta\psi)(L,\cdot) = (\theta\psi)_x(0,\cdot) = 0 & \text{in} \quad (0,T), \\ (\theta\psi)(x,T) = (\theta\psi)^T(x), \quad \gamma^i(\cdot,0) = 0 & \text{in} \quad (0,1), \end{cases} \quad (3.89)$$

where

$$R(\psi) = -M\theta_x\psi - \theta_{xxx}\psi - 3\theta_{xx}\psi_x - 3\theta_x\psi_{xx}.$$

For $i = 1,2$, let $\tilde{\omega}_i \subset \omega_i$. Using Carleman estimate (3.53) for $\theta\psi$, weight function η_1, $m = 5$ and $\omega_* = \tilde{\omega}_1$, we get

$$L_1^5(\theta\psi) \leq C \left(\iint_Q e^{-2s\Phi_1} \left| \sum_{i=1}^{2} \theta\alpha^i\gamma^i\chi_{\mathcal{O}_{i,d}} + \theta g^0 + R(\psi) \right|^2 dxdt \right.$$
$$\left. + s^5 \iint_{Q_{\tilde{\omega}_1}} e^{-2s\Phi_1}\xi^5|\theta\psi|^2 dxdt + s \iint_{Q_{\tilde{\omega}_1}} e^{-2s\Phi_1}\xi|(\theta\psi)_{xx}|^2 dxdt \right). \quad (3.90)$$

Using the fact that $\theta = 1$ in $\overline{[0,1] \setminus \widetilde{\mathcal{O}}}$, we deduce that

$$L_1^5(\theta\psi) \geq L_1^5(\psi) - s \iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1}\xi^1(s^4\xi^4|\psi|^2 + s^2\xi^2|\psi_x|^2 + |\psi_{xx}|^2) dxdt. \quad (3.91)$$

Hence, combining (3.90) and (3.91), we get

$$L_1^5(\psi) \leq C \left(\sum_{i=1}^{2} \iint_Q e^{-2s\Phi_1} |\theta\gamma^i \chi_{\mathcal{O}_{i,d}}|^2 \, dxdt + \iint_Q e^{-2s\Phi_1} (|\theta g^0|^2 + |R(\psi)|^2) \, dxdt \right.$$
$$\left. + s \iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1} \xi (s^4\xi^4 |\psi|^2 + s^2\xi^2 |\psi_x|^2 + |\psi_{xx}|^2) dxdt \right). \quad (3.92)$$

For given $\epsilon > 0$ and small, we can take s sufficiently large such that

$$\iint_Q e^{-2s\Phi_1} |R(\psi)|^2 \, dxdt \leq \epsilon L_1^5(\psi), \quad (3.93)$$

and hence

$$L_1^5(\psi) \leq C \left(\sum_{i=1}^{2} \iint_Q |\theta\gamma^i \chi_{\mathcal{O}_{i,d}}|^2 \, dxdt + \iint_Q e^{-2s\Phi_1} |\theta g^0|^2 \, dxdt \right.$$
$$\left. + s \iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1} \xi (s^4\xi^4 |\psi|^2 + s^2\xi^2 |\psi_x|^2 + |\psi_{xx}|^2) dxdt \right), \quad (3.94)$$

for $s > 0$ sufficiently large.

Now, for $i = 1, 2$, we apply Theorem 3.17 to the functions γ^i, weight functions η_i, assuming $m = 1$ and $\omega_* = \tilde{\omega}_i$, obtaining

$$I_i^1(\gamma^i) \leq C \left(s^{-4} \iint_Q e^{-2s\Phi_i} \xi^{-4} \left| -\frac{1}{\mu_i} \psi \chi_{\mathcal{O}_i} \right|^2 dxdt + s^{-4} \iint_Q e^{-2s\Phi_i} \xi^{-4} |g^i|^2 dxdt \right.$$
$$\left. + s \iint_{Q_{\tilde{\omega}_i}} e^{-2s\Phi_i} \xi |\gamma^i|^2 dxdt + s^{-3} \iint_{Q_{\tilde{\omega}_i}} e^{-2s\Phi_i} \xi^{-3} |\gamma^i_{xx}|^2 dxdt \right), \quad (3.95)$$

for $i = 1, 2$.

Since, for $i = 1, 2$, we have that $\mathcal{O}_i \cap \mathcal{O} = \emptyset$, then $\eta_1 = \eta_2$ in $\mathcal{O}_1 \cup \mathcal{O}_2$. In this way, for given small $\epsilon > 0$, we can take s sufficiently large, such that

$$s^{-4} \iint_Q e^{-2s\Phi_i} \xi^{-4} \left| -\frac{1}{\mu_i} \psi \chi_{\mathcal{O}_i} \right|^2 dxdt \leq \varepsilon L_1^5(\psi), \quad \text{for} \quad i = 1, 2. \quad (3.96)$$

Now, we combine (3.94), (3.95) and (3.96), and we get that

$$L_1^5(\psi) + \sum_{i=1}^{2} I_i^1(\gamma^i) \leq C \left(\iint_Q e^{-2s\Phi_1} |g^0|^2 \, dxdt + s^5 \iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1} \xi^5 |\psi|^2 dxdt \right.$$
$$+ s^3 \iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1} \xi^3 |\psi_x|^2 dxdt + s \iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1} \xi |\psi_{xx}|^2 dxdt$$
$$+ s^{-4} \sum_{i=1}^{2} \iint_Q e^{-2s\Phi_i} \xi^{-4} |g^i|^2 dxdt + s \sum_{i=1}^{2} \iint_{Q_{\tilde{\omega}_i}} e^{-2s\Phi_i} \xi |\gamma^i|^2 dxdt$$
$$\left. + s^{-3} \sum_{i=1}^{2} \iint_{Q_{\tilde{\omega}_i}} e^{-2s\Phi_i} \xi^{-3} |\gamma^i_{xx}|^2 dxdt \right). \quad (3.97)$$

Let $\theta_1 \in C_0^\infty(\mathcal{O})$ a positive function such that $\theta_1 = 1$ in $\widetilde{\mathcal{O}}$. For $\varepsilon > 0$ and small, the following estimate holds

$$\begin{aligned} s^3 \iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1}\xi^3|\psi_x|^2 dxdt &\leq s^3 \iint_{\mathcal{O}\times(0,T)} e^{-2s\Phi_1}\xi^3\theta_1|\psi_x|^2 dxdt \\ &\leq \varepsilon I_1^5(\psi) + Cs^5 \iint_{\mathcal{O}\times(0,T)} e^{-2s\Phi_1}\xi^5|\psi|^2 dxdt. \end{aligned} \tag{3.98}$$

Using (3.98) in (3.97), we obtain that

$$\begin{aligned} L_1^5(\psi) + \sum_{i=1}^2 I_i^1(\gamma^i) \leq C \Bigg(&\iint_Q e^{-2s\Phi_1}|g^0|^2 \, dxdt + s^5 \iint_{\mathcal{O}\times(0,T)} e^{-2s\Phi_1}\xi^5|\psi|^2 dxdt \\ &s \iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1}\xi|\psi_{xx}|^2 dxdt + s^{-4}\sum_{i=1}^2 \iint_Q e^{-2s\Phi_i}\xi^{-4}|g^i|^2 dxdt \\ &+s\sum_{i=1}^2 \iint_{Q_{\tilde{\omega}_i}} e^{-2s\Phi_i}\xi|\gamma^i|^2 dxdt + s^{-3}\sum_{i=1}^2 \iint_{Q_{\tilde{\omega}_i}} e^{-2s\Phi_i}\xi^{-3}|\gamma^i_{xx}|^2 dxdt \Bigg). \end{aligned} \tag{3.99}$$

In a similar way as in the case $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$ (see Section 3.4.1), we can add a weighted $H^{8/3}$ norm of $(\psi, \gamma^1, \gamma^2)$ on the left hand side of (3.99) (see (3.72)). In what follows, we provide a sketch on how we can do this, and in this case, the fact that $\|\eta_1\|_\infty = \|\eta_2\|_\infty$ will be very important.

Let $\beta_1 = e^{-s\hat{\Phi}}\xi^{-\frac{3}{2}}$ and define $(\psi_1, \gamma_1^1, \gamma_1^2) := (\beta_1\psi, \beta_1\gamma^1, \beta_1\gamma^2)$, where $(\psi, \gamma^1, \gamma^2)$ is solution of (3.55). Then, we have that

$$\begin{cases} -\psi_{1t} - (\psi_1)_{xxx} = l_1 & \text{in} \quad Q, \\ (\gamma_1^1)_t + (\gamma_1^1)_{xxx} = g_1^1 & \text{in} \quad Q, \\ (\gamma_1^2)_t + (\gamma_1^2)_{xxx} = g_1^2 & \text{in} \quad Q, \\ \psi_1(0,\cdot) = \psi_1(L,\cdot) = \psi_{1x}(0,\cdot) = 0, & \text{on} \quad (0,T), \\ \gamma_1^i(0,\cdot) = \gamma_1^i(L,\cdot) = (\gamma_1^i)_x(L,\cdot) = 0 & \text{on} \quad (0,T), \quad \text{for} \quad i=1,2, \\ \psi_1(x,T) = 0, \quad \gamma_1^i(\cdot,0) = 0 & \text{in} \quad (0,1), \quad \text{for} \quad i=1,2, \end{cases} \tag{3.100}$$

where

$$\begin{aligned} &l_1 = \textstyle\sum_{i=1}^2 \alpha_i\gamma_1^i\chi_{\mathcal{O}_{id}} + M(\psi_1)_x + \beta_1 g^0 - \beta_1'\psi, \\ &g_1^i = -\textstyle\sum_{i=1}^2 \frac{\alpha_i}{\mu_i}\psi_1\chi_{\mathcal{O}_i} - (M\gamma_1^i)_x + \beta_1 g^i + \beta_1'\gamma_1^i, \quad \text{for} \quad i=1,2. \end{aligned} \tag{3.101}$$

Now, we use the fact that $|\beta_1'| \leq Cse^{-s\hat{\Phi}}\xi^{\frac{1}{2}}$ together with (3.83), to get that l_1 and g_1^i are in $L^2(Q)$, and the following estimate holds

$$\|l_1\|^2_{L^2(Q)} + \sum_{i=1}^2 \|g_1^i\|^2_{L^2(Q)} \leq Cs\Big(I_1^5(\psi) + \sum_{i=1}^2 I_i^1(\gamma^i) + \|\beta_1 g^0\|^2_{L^2(Q)} + \sum_{i=1}^2 \|\beta_1 g^i\|^2_{L^2(Q)}\Big). \tag{3.102}$$

We remark that, in (3.102), we have estimated the weight β_1' for both weights of the form $e^{-s\Phi_i}$, for $i=1,2$. In order to do this, we have used the fact that $\|\eta_1\|_\infty = \|\eta_2\|_\infty$.

From (3.102) and Proposition 3.5, we obtain that

$$\|\psi_1\|^2_{L^4(0,T;H^{\frac{3}{2}}(0,1))} + \sum_{i=1}^{2} \|\gamma_1^i\|^2_{L^4(0,T;H^{\frac{3}{2}}(0,1))} \le C\Big(\|l_1\|^2_{L^2(Q)} + \sum_{i=1}^{2}\|g_1^i\|^2_{L^2(Q)}\Big)$$
$$\le Cs\Big(I_1^5(\psi) + \sum_{i=1}^{2} I_i^1(\gamma^i) + \|\beta_1 g^0\|^2_{L^2(Q)} + \sum_{i=1}^{2}\|\beta_1 g^i\|^2_{L^2(Q)}\Big). \quad (3.103)$$

Consider now $\beta_2 = e^{-s\hat{\Phi}}\xi^{-\frac{7}{2}}$ and define $(\psi_2,\gamma_2^1,\gamma_2^2) := (\beta_2\psi, \beta_2\gamma^1, \beta_2\gamma^2)$. It is clear that

$$\begin{cases} -\psi_{2t} - \psi_{2xxx} = l_2 & \text{in} \quad Q, \\ (\gamma_2^1)_t + (\gamma_2^1)_{xxx} = g_2^1 & \text{in} \quad Q, \\ (\gamma_2^2)_t + (\gamma_2^2)_{xxx} = g_2^2 & \text{in} \quad Q, \\ \psi_2(0,\cdot) = \psi_2(L,\cdot) = \psi_{2x}(0,\cdot) = 0, & \text{on} \quad (0,T), \\ \gamma_2^i(0,\cdot) = \gamma_2^i(L,\cdot) = (\gamma_2^i)_x(L,\cdot) = 0 & \text{on} \quad (0,T), \quad \text{for} \quad i=1,2, \\ \psi_2(x,T) = 0, \quad \gamma_2^i(\cdot,0) = 0 & \text{in} \quad (0,1), \quad \text{for} \quad i=1,2, \end{cases} \quad (3.104)$$

where

$$\begin{aligned} l_2 &= \textstyle\sum_{i=1}^2 \alpha_i \gamma_2^i \chi_{\mathcal{O}_{id}} + M(\psi_2)_x + \beta_2 g^0 - \beta_2'\psi \\ &= \beta_2\beta_1^{-1}\gamma_1^i\chi_{\mathcal{O}_{id}} + M\beta_2\beta_1^{-1}(\psi_1)_x + \beta_2 g^0 - \beta_2'\beta_1^{-1}\psi_1, \end{aligned} \quad (3.105)$$

and

$$\begin{aligned} g_2^i &= -\textstyle\sum_{i=1}^2 \frac{\alpha_i}{\mu_i}\psi_2\chi_{\mathcal{O}_i} - (M\gamma_2^i)_x + \beta_2 g^i + \beta_2'\gamma^i, \\ &= -\textstyle\sum_{i=1}^2 \frac{\alpha_i}{\mu_i}\beta_2'\beta_1^{-1}\psi_1\chi_{\mathcal{O}_i} - \beta_2'\beta_1^{-1}(M\gamma_1^i)_x + \beta_2 g^i + \beta_2'\beta_1^{-1}\gamma_1^i, \quad \text{for} \quad i=1,2. \end{aligned} \quad (3.106)$$

Then, using that $\beta_2' \le Cse^{-s\hat{\Phi}}\xi^{-\frac{3}{2}}$, and using similar arguments to prove (3.67), we obtain that

$$\|\psi_2\|^2_{Y_{\frac{7}{12}}} + \sum_{i=1}^{2}\|\gamma_2^i\|^2_{Y_{\frac{7}{12}}} \le C\Big(\|l_2\|^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))} + \sum_{i=1}^{2}\|g_2^i\|^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))}\Big)$$
$$\le Cs^3\Big(I_1^5(\psi) + \sum_{i=1}^{2} I_i^1(\gamma_i) + \|\beta_1 g^0\|^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))} + \sum_{i=1}^{2}\|\beta_1 g^i\|^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))}\Big). \quad (3.107)$$

Finally, consider $\beta_3 = e^{-s\hat{\Phi}}\xi^{-\frac{11}{2}}$ and define $(\psi_3,\gamma_3^1,\gamma_3^2) := (\beta_3\psi, \beta_3\gamma^1, \beta_3\gamma^2)$. It is not difficult to see that

$$\begin{cases} -\psi_{3t} - \psi_{3xxx} = l_3 & \text{in} \quad Q, \\ (\gamma_3^1)_t + (\gamma_3^1)_{xxx} = g_3^1 & \text{in} \quad Q, \\ (\gamma_3^2)_t + (\gamma_3^2)_{xxx} = g_3^2 & \text{in} \quad Q, \\ \psi_3(0,\cdot) = \psi_3(L,\cdot) = (\psi_3)_x(0,\cdot) = 0, & \text{on} \quad (0,T), \\ \gamma_3^i(0,\cdot) = \gamma_3^i(L,\cdot) = (\gamma_3^i)_x(L,\cdot) = 0 & \text{on} \quad (0,T), \quad \text{for} \quad i=1,2, \\ \psi_2(x,T) = 0, \quad \gamma_3^i(\cdot,0) = 0 & \text{in} \quad (0,1), \quad \text{for} \quad i=1,2, \end{cases} \tag{3.108}$$

where

$$\begin{aligned} l_3 &= \textstyle\sum_{i=1}^2 \alpha_i \gamma_3^i \chi_{\mathcal{O}_{id}} + M(\psi_3)_x + \beta_3 g^0 - \beta_2' \psi \\ &= \beta_3 \beta_2^{-1} \gamma_2^i \chi_{\mathcal{O}_{id}} + M\beta_3\beta_2^{-1}(\psi_2)_x + \beta_3 g^0 - \beta_3' \beta_2^{-1} \psi_2, \end{aligned} \tag{3.109}$$

and

$$\begin{aligned} g_3^i &= -\textstyle\sum_{i=1}^2 \frac{\alpha_i}{\mu_i} \psi_3 \chi_{\mathcal{O}_i} - (M\gamma_3^i)_x + \beta_3 g^i + \beta_3' \gamma^i, \\ &= -\textstyle\sum_{i=1}^2 \frac{\alpha_i}{\mu_i} \beta_3' \beta_2^{-1} \psi_2 \chi_{\mathcal{O}_i} - \beta_3' \beta_2^{-1} (M\gamma_2^i)_x + \beta_3 g^i + \beta_3' \beta_2^{-1} \gamma_2^i, \quad \text{for} \quad i=1,2. \end{aligned} \tag{3.110}$$

Therefore, we use the fact that $\beta_3' \leq Cse^{-s\hat{\Phi}}\xi^{-\frac{7}{2}}$, and proceed similarly as in (3.71), and we find that

$$\begin{aligned} &\|\psi_3\|_{Y_{\frac{2}{3}}}^2 + \sum_{i=1}^2 \|\gamma_3^i\|_{Y_{\frac{2}{3}}}^2 \leq C\Big(\|l_3\|_{X_{\frac{2}{3}}}^2 + \sum_{i=1}^2 \|g_3^i\|_{X_{\frac{2}{3}}}^2\Big) \\ &\leq Cs^5\Big(I_1^5(\psi) + \sum_{i=1}^2 I_i^1(\gamma^i) + \|\beta_1 g^0\|^2_{L^2(0,T;H^{\frac{2}{3}}(0,1))} + \sum_{i=1}^2 \|\beta_1 g^i\|^2_{L^2(0,T;H^{\frac{2}{3}}(0,1))}\Big). \end{aligned} \tag{3.111}$$

Now, combining (3.99) and (3.111), we obtain

$$\begin{aligned} s^{-5}\int_0^T e^{-2s\hat{\Phi}}\xi^{-11}\|\psi\|^2_{H^{\frac{8}{3}}(0,1)}dt + s^{-5}\sum_{i=1}^2\int_0^T e^{-2s\hat{\Phi}}\xi^{-11}\|\gamma^i\|^2_{H^{\frac{8}{3}}(0,1)}dt + L_1^5(\psi) + \sum_{i=1}^2 I_i^1(\gamma^i) \\ \leq C\left(s^5\iint_{\mathcal{O}\times(0,T)} e^{-2s\Phi_1}\xi^5|\psi|^2dxdt + s\iint_{\widetilde{\mathcal{O}}\times(0,T)} e^{-2s\Phi_1}\xi|\psi_{xx}|^2dxdt\right. \\ +s\sum_{i=1}^2\iint_{Q_{\tilde{\omega}_i}} e^{-2s\Phi_i}\xi|\gamma^i|^2dxdt + s^{-3}\sum_{i=1}^2\iint_{Q_{\tilde{\omega}_i}} e^{-2s\Phi_i}\xi^{-3}|\gamma^i_{xx}|^2dxdt \\ \left.+s^5\int_0^T e^{-2s\check{\Phi}}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2\|g^i\|^2_{H^{2/3}(0,1)}\Big)dt\right). \end{aligned} \tag{3.112}$$

The next computations are to eliminate the second order derivatives located on the right-hand side of (3.112). We use again an interpolation argument (see (3.91)) and

Young's inequality, to get that

$$
\begin{aligned}
s\iint_{\widetilde{\mathcal{O}}\times(0,T)} & e^{-2s\Phi_1}\xi|\psi_{xx}|^2dxdt+s^{-3}\sum_{i=1}^{2}\iint_{\tilde{\omega}_i\times(0,T)}e^{-2s\Phi_i}\xi^{-3}|\gamma^i_{xx}|^2dxdt\\
&\le Cs^{19}\int_0^T e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{37}\|\psi\|^2_{L^2(\widetilde{\mathcal{O}})}dt+\varepsilon s^{-5}\int_0^T e^{-2s\hat{\Phi}}\xi^{-11}\|\psi\|^2_{H^{\frac{8}{3}}(\widetilde{\mathcal{O}})}dt\\
&+Cs^3\sum_{i=1}^{2}\int_0^T e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{21}\|\gamma^i\|^2_{L^2(\tilde{\omega}_i)}dt+\varepsilon s^{-5}\sum_{i=1}^{2}\int_0^T e^{-2s\hat{\Phi}}\xi^{-11}\|\gamma^i\|^2_{H^{\frac{8}{3}}(\tilde{\omega}_i)}dt. \quad (3.113)
\end{aligned}
$$

Combining (3.112) and (3.113), we obtain that

$$
\begin{aligned}
& s^{-5}\int_0^T e^{-2s\hat{\Phi}}\xi^{-11}\|\psi\|^2_{H^{\frac{8}{3}}(0,1)}dt+s^{-5}\sum_{i=1}^{2}\int_0^T e^{-2s\hat{\Phi}}\xi^{-11}\|\gamma^i\|^2_{H^{\frac{8}{3}}(0,1)}dt\\
&+L^5_1(\psi)+\sum_{i=1}^{2}I^1_i(\gamma^i)\le C\left(s^{19}\int_0^T e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{37}\|\psi\|^2_{L^2(\widetilde{\mathcal{O}})}dt+s^3\sum_{i=1}^{2}\int_0^T e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{21}\|\gamma^i\|^2_{L^2(\tilde{\omega}_i)}dt\right.\\
&\left.+s^5\int_0^T e^{-2s\check{\Phi}}\|g^0\|^2_{H^{2/3}(0,1)}\,dt+s^5\int_0^T e^{-2s\check{\Phi}}\sum_{i=1}^{2}\|g^i\|^2_{H^{2/3}(0,1)}dt\right). \quad (3.114)
\end{aligned}
$$

Now, for $i=1,2$, we are going to estimate the local term of γ^i on the right hand side of (3.114). For $i=1,2$, let ω_i be open subset such that $\tilde{\omega}_i\subset\subset\omega_i\subset\subset\mathcal{O}_{id}\cap\tilde{\mathcal{O}}$, and $\omega_i\cap\mathcal{O}_{jd}=\emptyset$, $i\neq j$. Let $\theta_2\in C^3_0(\omega_i)$ be such that $\theta_2(x)=1$ for $x\in\tilde{\omega}_i$ and $0\le\theta_2\le 1$. From equation (3.18), we easily see that

$$
-\psi_t-M\psi_x-\psi_{xxx}-g^0=\alpha_i\gamma^i\chi_{\mathcal{O}_{i,d}}\quad\text{in}\quad\omega_i\times(0,T),\quad\text{for}\quad i=1,2.
$$

Proceeding in a similar way as in (3.75) and (3.76), we can prove that

$$
\begin{aligned}
s^3\sum_{i=1}^{2}\int_0^T e^{6s\hat{\Phi}-8s\check{\Phi}}\xi^{21}\|\gamma^i\|^2_{L^2(\tilde{\omega}_i)}dt &\le\varepsilon(I^1_i(\gamma^i)+L^5_1(\psi))+Cs^5\sum_{i=1}^{2}\iint_{Q_{\omega_i}}e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{41}|g^0|^2\,dxdt\\
+Cs\sum_{i=1}^{2}\iint_{Q_{\omega_i}}e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{41}|g^i|^2\,dxdt &+Cs^9\sum_{i=1}^{2}\iint_{Q_{\omega_i}}e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{45}|\psi|^2\,dxdt, \quad (3.115)
\end{aligned}
$$

for $i=1,2$.

Therefore, we combine (3.114) and (3.115) and we get (3.87). □

3.5 The observability inequality

This section is dedicated to prove an observability inequality such as (3.19) to the solutions of System (3.18). In order to do that, we use the following Lemma:

Lemma 3.21. *There exists $C > 0$ such that*

$$\int_0^1 \left(|\psi(t)|^2 + \sum_{i=1}^2 |\gamma^i(t)|^2\right) dx + \int_0^{\frac{T}{4}}\int_0^1 \left(|\psi_x|^2 + \sum_{i=1}^2 |\gamma^i_x|\right) dxdt$$
$$\leq C\left(\int_0^{\frac{T}{2}}\int_0^1 \left(|g^0|^2 + \sum_{i=1}^2 |g^i|^2\right) dxdt + \int_{\frac{T}{4}}^{\frac{T}{2}}\int_0^1 \left(|\psi|^2 + \sum_{i=1}^2 |\gamma^i|^2\right) dxdt\right), \quad (3.116)$$

for all $t \in [0, T/4]$ and for every $(\psi, \gamma^1, \gamma^2)$ solution of (3.18) *with* $(g^0, g^1, g^2) \in [L^2(Q)]^3$.

Demonstração. Define $\chi \in C^2([0, T])$ such that

$$\chi = \begin{cases} 1, & \text{if} \quad t \in [0, T/4], \\ 0, & \text{if} \quad t \in [T/2, T]. \end{cases} \quad (3.117)$$

If $(\psi, \gamma^1, \gamma^2)$ is a solution of (3.54), it is not difficult to see that $(\chi\psi, \chi\gamma^1, \chi\gamma^2)$ is a solution of the system

$$\begin{cases} -(\chi\psi)_t - M(\chi\psi)_x - (\chi\psi)_{xxx} = \sum_{i=1}^2 \alpha_i(\chi\gamma^i)\chi_{\mathcal{O}_{i,d}} + \chi g^0 - \chi_t\psi & \text{in} \quad Q, \\ (\chi\gamma^i)_t + (M\chi\gamma^i)_x + (\chi\gamma^i)_{xxx} = -\frac{1}{\mu_i}(\chi\psi)\chi_{\mathcal{O}_i} + \chi g^i + \chi_t\gamma^i \quad i = 1, 2 & \text{in} \quad Q, \\ (\chi\psi)(0, \cdot) = (\chi\psi)(L, \cdot) = (\chi\psi)_x(0, \cdot) = 0 & \text{on} \quad (0, T), \\ (\chi\gamma^i)(0, \cdot) = (\chi\gamma^i)(L, \cdot) = (\chi\gamma^i)_x(L, \cdot) = 0 \quad i = 1, 2 & \text{on} \quad (0, T), \\ (\chi\psi)(x, T) = 0, \quad (\chi\gamma^i)(\cdot, 0) = 0 \quad i = 1, 2 & \text{in} \quad (0, 1). \end{cases} \quad (3.118)$$

Now, we use Proposition 3.2 (see also Remark 3.3) and following the proof of Proposition 3.12 (see (3.47)), we obtain that

$$\|\chi\psi\|_{Y_{\frac{1}{4}}} + \sum_{i=1}^2 \|\chi\gamma^i\|_{Y_{\frac{1}{4}}} \leq C\left(\|\chi g^0\|_{L^2(Q)} + \|\chi_t\psi\|_{L^2(Q)} + \sum_{i=1}^2 \|\chi g^i\|_{L^2(Q)} + \sum_{i=1}^2 \|\chi_t\gamma^i\|_{L^2(Q)}\right).$$

Then, using that $\chi = 1$ in $(0, T/4)$ and that $\chi_t = 0$ in $(0, T/4) \cup (T/2, T)$, we obtain (3.116). □

Now, we define the new weight functions

$$\Psi_i(x, t) = \eta_i(x)\beta(t), \quad (3.119)$$

where $\beta(t) = \frac{1}{l(t)}$ and

$$l(t) = \begin{cases} T^2/4, & \text{if} \quad t \in [0, T/2], \\ t(T - t), & \text{if} \quad t \in [T/2, T]. \end{cases} \quad (3.120)$$

Also, we denote

$$\hat{\Psi}(t) = \max_{x\in[0,1]} \Psi_1(t, x) = \max_{x\in[0,1]} \Psi_2(t, x) \quad \text{and} \quad \check{\Psi}(t) = \min_{x\in[0,1]} \Psi_1(t, x) = \min_{x\in[0,1]} \Psi_1(t, x). \quad (3.121)$$

Then, we have the following Carleman estimate with new weight functions.

Proposition 3.22. *There exists a constant $C > 0$ (depending on s), such that every solution $(\psi, \gamma^1, \gamma^2)$ of (3.18) with $(g^0, g^1, g^2) \in [L^2(0,T; H^{2/3}(0,1))]^3$, satisfies*

$$\begin{aligned}
&\int_0^1 |\psi(x,0)|^2\,dx + \iint_Q e^{-2s\Psi_1}\beta^5|\psi|^2dxdt + \iint_Q e^{-2s\Psi_1}\beta^3|\psi_x|^2dxdt \\
&\quad + \sum_{i=1}^2 \iint_Q e^{-2s\Psi_i}\beta|\gamma^i|^2dxdt + \sum_{i=1}^2 \iint_Q e^{-2s\Psi_i}\beta^{-1}|\gamma^i_x|^2dxdt \\
&\quad \le C\Bigg(\int_0^T e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2 \|g^i\|^2_{H^{2/3}(0,1)}\Big)\,dt \\
&\qquad + \iint_{\mathcal{O}\times(0,T)} e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{45}|\psi|^2dxdt\Bigg). \qquad (3.122)
\end{aligned}$$

Demonstração. In one hand, since $\beta = \xi$ for every $t \in [T/2, T]$, we get that

$$\begin{aligned}
&\int_{T/2}^T e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{41}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2 \|g^i\|^2_{H^{2/3}(0,1)}\Big)\,dt + \int_{T/2}^T\int_{\mathcal{O}} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{45}|\psi|^2\,dxdt \\
&= \int_{T/2}^T e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2 \|g^i\|^2_{H^{2/3}(0,1)}\Big)dt + \int_{T/2}^T\int_{\mathcal{O}} e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{45}|\psi|^2dxdt. \\
&\qquad (3.123)
\end{aligned}$$

In other hand, since l is constant in $[0, T/2]$, we easily see that

$$\begin{aligned}
&\int_0^{T/2} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{41}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2 \|g^i\|^2_{H^{2/3}(0,1)}\Big)\,dt + \int_0^{T/2}\int_{\mathcal{O}} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{45}|\psi|^2\,dxdt \\
&\quad \le C\Bigg(\int_0^{T/2}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2 \|g^i\|^2_{H^{2/3}(0,1)}\Big)\,dt + \int_0^{T/2}\int_{\mathcal{O}} |\psi|^2\,dxdt\Bigg) \\
&\quad \le C\Bigg(\int_0^{T/2} e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2 \|g^i\|^2_{H^{2/3}(0,1)}\Big)\,dt \\
&\qquad + \int_0^{T/2}\int_{\mathcal{O}} e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{45}|\psi|^2\,dxdt\Bigg). \qquad (3.124)
\end{aligned}$$

Hence, combining (3.123) and (3.124), we get

$$\begin{aligned}
&\int_0^T e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{41}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2 \|g^i\|^2_{H^{2/3}(0,1)}\Big)\,dt + \iint_{\mathcal{O}\times(0,T)} e^{14s\hat{\Phi}-16s\check{\Phi}}\xi^{45}|\psi|^2\,dxdt \\
&\quad \le C\Bigg(\int_0^T e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Big(\|g^0\|^2_{H^{2/3}(0,1)} + \sum_{i=1}^2 \|g^i\|^2_{H^{2/3}(0,1)}\Big)\,dt \\
&\qquad + \iint_{\mathcal{O}\times(0,T)} e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{45}|\psi|^2\,dxdt\Bigg). \qquad (3.125)
\end{aligned}$$

Now, we have that the weight coincides a in $[T/2, T]$ and then

$$\int_{T/2}^{T}\int_{\Omega} e^{-2s\Psi_1}\beta^5|\psi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{-2s\Psi_1}\beta^3|\psi_x|^2dxdt + \sum_{i=1}^{2}\int_{T/2}^{T}\int_{\Omega} e^{-2s\Psi_i}\beta|\gamma^i|^2dxdt$$
$$+\sum_{i=1}^{2}\int_{T/2}^{T}\int_{\Omega} e^{-2s\Psi_i}\beta^{-1}|\gamma^i_x|^2dxdt = \int_{T/2}^{T}\int_{\Omega} e^{-2s\Phi_1}\xi^5|\psi|^2dxdt + \int_{T/2}^{T}\int_{\Omega} e^{-2s\Phi_1}\xi^3|\psi_x|^2dxdt$$
$$+\sum_{i=1}^{2}\int_{T/2}^{T}\int_{\Omega} e^{-2s\Phi_i}\xi|\gamma^i|^2dxdt + \sum_{i=1}^{2}\int_{T/2}^{T}\int_{\Omega} e^{-2s\Phi_i}\xi^{-1}|\gamma^i_x|^2dxdt. \quad (3.126)$$

In the time interval $[0, T/2]$, we use the fact that the weight functions are bounded from above and we get that

$$\int_0^1 |\psi(x,0)|^2\,dx + \int_0^{T/2}\int_0^1 e^{-2s\Psi_1}\beta^5|\psi|^2dxdt + \int_0^{T/2}\int_0^1 e^{-2s\Psi_1}\beta^3|\psi_x|^2dxdt$$
$$+\sum_{i=1}^{2}\int_0^{T/2}\int_0^1 e^{-2s\Psi_i}\beta|\gamma^i|^2dxdt + \sum_{i=1}^{2}\int_0^{T/2}\int_0^1 e^{-2s\Psi_i}\beta^{-1}|\gamma^i_x|^2dxdt$$
$$\leq C\left(\int_0^1 |\psi(x,0)|^2\,dx + \int_0^{T/2}\int_0^1 |\psi|^2dxdt + \int_0^{T/2}\int_0^1 |\psi_x|^2dxdt\right.$$
$$\left.+\sum_{i=1}^{2}\int_0^{T/2}\int_0^1 |\gamma^i|^2dxdt + \sum_{i=1}^{2}\int_0^{T/2}\int_0^1 |\gamma^i_x|^2dxdt\right). \quad (3.127)$$

Now, we use Lemma 3.21 and the fact that the weight functions are bounded from below to obtain

$$\int_0^1 |\psi(x,0)|^2\,dx + \int_0^{T/2}\int_0^1 |\psi|^2dxdt + \int_0^{T/2}\int_0^1 |\psi_x|^2dxdt$$
$$+\sum_{i=1}^{2}\int_0^{T/2}\int_0^1 |\gamma^i|^2dxdt + \sum_{i=1}^{2}\int_0^{T/2}\int_0^1 |\gamma^i_x|^2dxdt$$
$$\leq C\left(\int_0^{\frac{T}{2}}\int_0^1 \left(|g^0|^2 + \sum_{i=1}^{2}|g^i|^2\right)dxdt + \int_{\frac{T}{4}}^{\frac{T}{2}}\int_0^1 \left(|\psi|^2 + \sum_{i=1}^{2}|\gamma^i|^2\right)dxdt\right)$$
$$\leq C\left(\int_0^{\frac{T}{2}}\int_0^1 e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\left(|g^0|^2 + \sum_{i=1}^{2}|g^i|^2\right)dxdt\right.$$
$$\left.+\int_{\frac{T}{4}}^{\frac{T}{2}}\int_0^1 \left(e^{-2s\Phi_1}\xi^5|\psi|^2 + \sum_{i=1}^{2}e^{-2s\Phi_i}\xi|\gamma^i|^2\right)dxdt\right). \quad (3.128)$$

Combining (3.126), (3.127) and (3.128), we have

$$\begin{aligned}
&\int_0^1 |\psi(x,0)|^2\,dx + \iint_Q e^{-2s\Psi_1}\beta^5|\psi|^2 dxdt + \iint_Q e^{-2s\Psi_1}\beta^3|\psi_x|^2 dxdt \\
&\qquad + \sum_{i=1}^2 \iint_Q e^{-2s\Psi_i}\beta|\gamma^i|^2 dxdt + \sum_{i=1}^2 \iint_Q e^{-2s\Psi_i}\beta^{-1}|\gamma^i_x|^2 dxdt \\
&\quad \le C\left(\iint_Q e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\left(|g^0|^2 + \sum_{i=1}^2 |g^i|^2\right) dxdt + \iint_Q e^{-2s\Phi_1}\xi^5|\psi|^2 dxdt\right. \\
&\left. + \iint_Q e^{-2s\Phi_1}\xi^3|\psi_x|^2 dxdt + \sum_{i=1}^2 \iint_Q e^{-2s\Phi_i}\xi|\gamma^i|^2 dxdt + \sum_{i=1}^2 \iint_Q e^{-2s\Phi_i}\xi^{-1}|\gamma^i_x|^2 dxdt\right).
\end{aligned} \tag{3.129}$$

To finish, we combine (3.129) and (3.87), and then we use (3.125) to obtain (3.122). □

3.6 Null Controllability of the Linear System

In this section, we prove that the linear system (3.17) is partially null controllable in the sense that we can find a control $f \in L^2(\mathcal{O} \times (0,T))$ such that $z(\cdot,T) = 0$.

To simplify the notation, let us denote by $\mathcal{L}$ the linear operator

$$\mathcal{L}u = u_t + (Mu)_x + u_{xxx}$$

and by $\mathcal{L}^*$ its formal adjoint

$$\mathcal{L}^*u = -u_t - Mu_x - u_{xxx},$$

in which $M = 1 + \bar{y}$ and $\bar{y}$ is a given trajectory. Also, we consider the functional space

$$\begin{aligned}
B_1 = \Big\{(z,\phi^1,\phi^2,f);\;\; & e^{s\psi_1}\beta^{-5/2}(\mathcal{L}z + \textstyle\sum_{i=1}^2 \frac{1}{\mu_i}\phi^i\chi_{\mathcal{O}_i} - f\mathbb{1}_{\mathcal{O}}) \in L^2(Q), \\
& e^{s\psi_i}\beta^{-1/2}(\mathcal{L}^*\phi^i - \alpha_i z\chi_{\mathcal{O}_{i,d}}) \in L^2(Q) \text{ for } i = 1,2, \\
& e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}z \in Y_{1/2}, e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\phi^i \in Y_{1/2} \quad \text{for} \quad i = 1,2, \\
& e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-41/2}z \in L^2(0,T;H^{-1}(0,1)), \\
& e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-41/2}\phi^i \in L^2(0,T;H^{-1}(0,1)) \quad \text{for} \quad i = 1,2, \\
& e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}f \in L^2(Q), z(\cdot,0) \in H_0^1(0,1)\Big\}.
\end{aligned}$$

We have the following result.

Proposition 3.23. *Assume that $z^0 \in H_0^1(0,1)$, that*

$$e^{s\Psi_1}\beta^{-\frac{5}{2}}f^0 \quad \text{in} \quad L^2(Q) \quad \text{and that} \quad e^{s\Psi_i}\beta^{-\frac{1}{2}}f^i \quad \text{in} \quad L^2(Q) \quad \text{for} \quad i = 1,2. \tag{3.130}$$

There exists $(z,\phi^1,\phi^2,f) \in B_1$ solution of (3.17) with control f, initial data z^0 and (f^0,f^1,f^2) as source terms. In particular $z(\cdot,T) = 0$.

Demonstração. To start, we define

$$\mathcal{P}_0 = \Big\{(\psi,\gamma^1,\gamma^2)\in[C^4([0,1]\times[0,T])]^3;\ \ \begin{aligned}&\psi(0,t)=\psi(1,t)=\psi_x(1,t)=0 \quad \forall t\in(0,T);\\ &\gamma^i(0,t)=\gamma^i(1,t)=\gamma^i_x(0,t)=0 \quad \forall t\in(0,T), \quad i=1,2;\\ &\mathcal{L}^*\psi(0,t)=\mathcal{L}^*\psi(1,t)=0 \quad \forall t\in(0,T);\\ &\mathcal{L}\gamma^i(0,t)=\mathcal{L}\gamma^i(1,t)=0 \quad \forall t\in(0,T), \quad i=1,2;\\ &\gamma^i(x,0)=0 \quad \forall x\in(0,1), \quad i=1,2\Big\}.\end{aligned}$$

Let $b:\mathcal{P}_0\times\mathcal{P}_0\longrightarrow\mathbb{R}$ the bilinear functional given by

$$\begin{aligned}&b((\hat{\psi},\hat{\gamma}^1,\hat{\gamma}^2),(\psi,\gamma^1,\gamma^2)) =\\ &\iint_Q e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\nabla\left(\mathcal{L}^*\hat{\psi}-\sum_{i=1}^2\alpha_i\hat{\gamma}^i\chi_{\mathcal{O}_{i,d}}\right)\nabla\left(\mathcal{L}^*\psi-\sum_{i=1}^2\alpha_i\gamma^i\chi_{\mathcal{O}_{i,d}}\right)dxdt\\ &+\sum_{i=1}^2\iint_Q e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\nabla\left(\mathcal{L}\hat{\gamma}^i+\frac{1}{\mu_i}\hat{\psi}\chi_{\mathcal{O}_i}\right)\nabla\left(\mathcal{L}\gamma^i+\frac{1}{\mu_i}\psi\chi_{\mathcal{O}_i}\right)dxdt\\ &+\iint_{\mathcal{O}\times(0,T)} e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{45}\hat{\psi}\psi dxdt, \quad (3.131)\end{aligned}$$

and the linear form $l:\mathcal{P}_0\longrightarrow\mathbb{R}$ defined by

$$l(\psi,\gamma^1,\gamma^2)=\int_0^T z^0(x)\psi(x,0)dx+\iint_Q f^0\psi dxdt+\sum_{i=1}^2\iint_Q f^i\gamma^i dxdt. \quad (3.132)$$

It is not difficult to see that $\|(\psi,\gamma^1,\gamma^2)\|_{\mathcal{P}}=b((\psi,\gamma^1,\gamma^2),(\psi,\gamma^1,\gamma^2))^{\frac{1}{2}}$ defines a norm on $\mathcal{P}_0$. Indeed, if $b((\psi,\gamma^1,\gamma^2),(\psi,\gamma^1,\gamma^2))=0$, then we obtain from Proposition 3.22 that $\psi=\psi_x=0$ and that $\gamma^i={\gamma^i}_x=0$ in $[0,1]\times[0+\delta,T-\delta]$, for every $\delta>0$. Therefore, we can consider the space $\mathcal{P}$, the completion of $\mathcal{P}_0$ with the norm $\|\cdot\|_{\mathcal{P}}$ and it is straightforward that $b(\cdot,\cdot)$ is continuous and coercive over $\mathcal{P}\times\mathcal{P}$.

By combining Hölder's inequality and Proposition 3.22, we can estimate l as follows

$$\begin{aligned}|l(\psi,\gamma^1,\gamma^2)|&\le C\left(\int_0^1|z^0|^2dx+\iint_Q e^{2s\Psi_1}\beta^{-5}|f^0|^2dxdt+\sum_{i=1}^2\iint_Q e^{2s\Psi_i}\beta^{-1}|f^i|^2dxdt\right.\\ &\left.+\sum_{i=1}^2\iint_Q e^{2s\Psi_i}\beta^{-1}|z_{i,d}|^2dxdt\right)^{\frac{1}{2}}\times\left(\int_0^1|\psi(x,0)|^2dx+\iint_Q e^{-2s\Psi_1}\beta^5|\psi|^2dxdt\right.\\ &\left.+\sum_{i=1}^2\iint_Q e^{-2s\Psi_i}\beta|\gamma^i|^2dxdt\right)^{\frac{1}{2}}\le C\|(\psi,\gamma^1,\gamma^2)\|_{\mathcal{P}}\left(\int_0^1|z^0|^2dx+\iint_Q e^{2s\Psi_1}\beta^{-5}|f^0|^2dxdt\right.\\ &\left.+\sum_{i=1}^2\iint_Q e^{2s\Psi_i}\beta^{-1}|f^i|^2dxdt\right)^{\frac{1}{2}}, \quad (3.133)\end{aligned}$$

for every $(\psi,\gamma^1,\gamma^2)\in\mathcal{P}_0$. By applying the Hahn-Banach extension Theorem, the functional l can be extended to a continuous functional over $\mathcal{P}$. In this way, we apply the Lax-Milgram Theorem and conclude that the variational problem

$$b((\hat{\psi},\hat{\gamma}^1,\hat{\gamma}^2),(\psi,\gamma^1,\gamma^2))=l(\psi,\gamma^1,\gamma^2),\quad \forall(\psi,\gamma^1,\gamma^2)\in\mathcal{P}, \tag{3.134}$$

possesses a unique solution $(\hat{\psi},\hat{\gamma}^1,\hat{\gamma}^2)\in\mathcal{P}$.

Define $(\hat{z},\hat{\phi}^1,\hat{\phi}^2,\hat{f})$ by

$$\begin{cases} \hat{z}=-e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Delta\Big(\mathcal{L}^*\hat{\psi}-\sum_{i=1}^2\alpha_i\hat{\gamma}^i\chi_{\mathcal{O}_{i,d}}\Big) & \text{in} \quad Q,\\ \hat{\phi}^i=-e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Delta\Big(\mathcal{L}\hat{\gamma}^i+\frac{1}{\mu_i}\hat{\psi}\chi_{\mathcal{O}_{i,d}}\Big) & \text{in} \quad Q,\\ \hat{f}=-e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{45}\hat{\psi} & \text{in} \quad Q. \end{cases} \tag{3.135}$$

Combining (3.134) and (3.135), we get that

$$\begin{aligned} -\int_0^T z^0(x)\psi(x,0)dx+\int_0^T\langle\hat{z},\mathcal{L}^*\psi-\sum_{i=1}^2\alpha_i\gamma^i\chi_{\mathcal{O}_{i,d}}\rangle_{H^{-1}H_0^1}dt \\ +\sum_{i=1}^2\int_0^T\langle\hat{\phi}^i,\mathcal{L}\gamma^i+\frac{1}{\mu_i}\psi\chi_{\mathcal{O}_i}\rangle_{H^{-1}H_0^1}dt \\ =\iint_Q f^0\psi dxdt+\sum_{i=1}^2\iint_Q f^i\gamma^i dxdt+\iint_{\mathcal{O}\times(0,T)}\hat{f}\psi dxdt, \end{aligned} \tag{3.136}$$

for all $(\psi,\gamma^1,\gamma^2)\in\mathcal{P}_0\subset\mathcal{P}$.

Let us prove now that $(\hat{z},\hat{\phi}^1,\hat{\phi}^2,\hat{f})$ is solution in the transposition sense for (3.17) (see (3.41)). For this, let $(z^*,\phi^{1*},\phi^{2*},f^*)$ be the transposition solution of (3.17) associated to the control $f^*=\hat{f}$, then it satisfies

$$\begin{aligned} \iint_Q z^*g^0\,dxdt+\sum_{i=1}^2\iint_Q\phi^{i*}g^i\,dxdt=\int_0^1 z^0(x)\psi(x,0)\,dx+ \\ \int_0^T\langle f^0,\psi\rangle_{H^{-1}\,H_0^1}\,dt+\sum_{i=1}^2\iint_Q\langle f^i,\gamma^i\rangle_{H^{-1}\,H_0^1}\,dt+\iint_{\mathcal{O}\times(0,T)}\hat{f}\psi dxdt, \end{aligned} \tag{3.137}$$

for every (g^0,g^1,g^2), where (ψ,γ^1,γ^2) is solution of (3.42) with (g^0,g^1,g^2) on the right-hand side.

By taking a sequence of functions $(\psi^n,\gamma^{1n},\gamma^{2n})\in[C_0^\infty(Q)]^3\subset\mathcal{P}_0$ converging to the solution of (3.18) with $(g^0,g^1,g^2)\in[L^2(0,T;H_0^1(0,1))]^3$ on the right hand side and $\psi^T=$

0, and using (3.136) we obtain that

$$\int_0^T \langle \hat{z}, g^0\rangle_{H^{-1}\,H_0^1}\,dxdt + \sum_{i=1}^{2}\iint_Q \langle \hat{\phi}^i, g^i\rangle_{H^{-1}\,H_0^1}\,dxdt = \int_0^1 z^0(x)\psi(x,0)\,dx + \int_0^T \langle f^0, \psi\rangle_{H^{-1}\,H_0^1}\,dt + \sum_{i=1}^{2}\iint_Q \langle f^i, \gamma^i\rangle_{H^{-1}\,H_0^1}\,dt, \quad (3.138)$$

for all $(g^0, g^1, g^2) \in L^2(0,T;H_0^1(0,1))$, where $(\psi, \gamma^1, \gamma^2)$ is solution of (3.42) with (g^0, g^1, g^2) on the right-hand side. To conclude, we combine (3.137) and (3.138) and we get that $(\hat{z}, \hat{\phi}^1, \hat{\phi}^2) = (z^*, \phi^{1*}, \phi^{2*})$, in particular $(\hat{z}, \hat{\phi}^1, \hat{\phi}^2) \in [L^2(Q)]^3$.

We remark that formula (3.135) implies that $\hat{z}(\cdot, T) = 0$. This is due to the fact that the weight function $e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}$ goes to zero when t tends to T.

Let us prove that $(\hat{z}, \hat{\phi}^1, \hat{\phi}^2)$ is indeed more regular. We start using (3.135) to see that

$$\begin{aligned}
&\|e^{8s\check{\Psi}-7s\hat{\Psi}}\beta^{-\frac{41}{2}}\hat{z}\|_{L^2(0,T;H^{-1}(0,1))} \\
&= \Big(\int_0^T e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Big(\sup_{\|\zeta\|_{H_0^1(0,1)}=1}\big\langle -\Delta\Big(\mathcal{L}^*\hat{\psi} - \sum_{i=1}^{2}\alpha_i\hat{\gamma}^i\chi_{\mathcal{O}_{i,d}}\Big), \zeta\big\rangle_{H^{-1}\,H_0^1}\Big)^2 dt\Big)^{1/2} \\
&= \Big(\int_0^T e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Big(\sup_{\|\zeta\|_{H_0^1(0,1)}=1}\int_0^1 \nabla\Big(\mathcal{L}^*\hat{\psi} - \sum_{i=1}^{2}\alpha_i\hat{\gamma}^i\chi_{\mathcal{O}_{i,d}}\Big)\cdot\nabla\zeta\,dx\Big)^2 dt\Big)^{1/2} \\
&\le \Big(\iint_Q e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Big|\nabla\Big(\mathcal{L}^*\hat{\psi} - \sum_{i=1}^{2}\alpha_i\hat{\gamma}^i\chi_{\mathcal{O}_{i,d}}\Big)\Big|^2 dxdt\Big)^{1/2}. \quad (3.139)
\end{aligned}$$

In a completely analogous way, we can prove that

$$\|e^{8s\check{\Psi}-7s\hat{\Psi}}\beta^{-\frac{41}{2}}\hat{\phi}^i\|_{L^2(0,T;H^{-1}(0,1))} \le \Big(\iint_Q e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{41}\Big|\nabla\Big(\mathcal{L}\hat{\gamma}^i + \frac{1}{\mu_i}\hat{\psi}\chi_{\mathcal{O}_{i,d}}\Big)\Big|^2 dxdt\Big)^{1/2}. \quad (3.140)$$

We also obtain the following estimate to the control

$$\|e^{8s\check{\Psi}-7s\hat{\Psi}}\beta^{-45/2}\hat{f}\|_{L^2(Q)} = \Big(\iint_Q e^{14s\hat{\Psi}-16s\check{\Psi}}\beta^{45}|\hat{\psi}|^2 dxdt\Big)^{1/2}. \quad (3.141)$$

Hence, we have that

$$\int_0^T e^{16s\check{\Psi}-14s\hat{\Psi}}\beta^{-41}\|\hat{z}\|^2_{H^{-1}(0,1)}dt + \int_0^T e^{16s\check{\Psi}-14s\hat{\Psi}}\beta^{-41}\|\hat{\phi}^i\|^2_{H^{-1}(0,1)}dt + \iint_Q e^{16s\check{\Psi}-14s\hat{\Psi}}\beta^{-45}\|\hat{f}\|^2 dxdt \le b((\hat{\psi}, \hat{\gamma}^1, \hat{\gamma}^2), (\hat{\psi}, \hat{\gamma}^1, \hat{\gamma}^2)) < \infty. \quad (3.142)$$

Now, we define

$$\begin{cases} z^* = e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\hat{z}, \\ \phi^{i*} = e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\hat{\phi}^i, \quad i = 1, 2, \\ f^{0*} = e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\Big(\hat{f}\mathbb{1}_{\mathcal{O}} + f^0\Big), \\ f^{i*} = e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}f^i, \end{cases} \quad (3.143)$$

and it is not difficult to see that

$$\begin{cases} \mathcal{L}z^* = f^{0*} - \sum_{i=1}^{2} \frac{1}{\mu_i}(e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2})_t\hat{\phi}^i\chi_{\mathcal{O}_i} & \text{in} \quad Q, \\ \mathcal{L}^*\phi^{i*} = f^{i*} + \alpha_i(e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2})_t\hat{z}\chi_{\mathcal{O}_{i,d}}, \quad i = 1,2 & \text{in} \quad Q, \\ z^*(0,\cdot) = z^*(L,\cdot) = z^*_x(L,\cdot) = 0 & \text{in} \quad (0,T), \\ \gamma^{i*}(0,\cdot) = \gamma^{i*}(L,\cdot) = \gamma^{i*}_x(0,\cdot) = 0 & \text{in} \quad (0,T), \\ z^*(\cdot,0) = e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\Big|_{t=0} z^0, \quad \gamma^{i*}(\cdot,T) = 0 & \text{in} \quad (0,1). \end{cases} \tag{3.144}$$

An easy computation shows that $(e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2})_t \leq Ce^{8s\check{\psi}-7s\hat{\psi}}\beta^{-41/2}$, in this way

$$f^{0*} - \sum_{i=1}^{2} \frac{1}{\mu_i}(e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2})_t\hat{\phi}^i\chi_{\mathcal{O}_i} \in L^2(0,T;H^{-1}(0,1))$$

and

$$f^{i*} + \alpha_i(e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2})_t\hat{z}\chi_{\mathcal{O}_{i,d}} \in L^2(0,T;H^{-1}(0,1)).$$

Together with the fact that $z^0 \in H^1_0(0,1)$, we finally obtain that $(z^*, \phi^{1*}, \phi^{2*}) \in [Y_{1/2}]^3$ (see Remark 3.6). Note that, we have also obtained that $(\hat{z}, \hat{\phi}^1, \hat{\phi}^2, \hat{f}) \in B_1$.

3.7 Null Controllability for the Nonlinear Case

To deal with the nonlinear case, we are going to use a local inversion mapping theorem (see [1]).

Theorem 3.24 (Inverse Mapping Theorem)**.** *Let B_1 and B_2 be Banach spaces and let $\mathcal{A} : B_1 \to B_2$ in $C^1(B_1, B_2)$. Given $b_1 \in B_1$ and $b_2 = \mathcal{A}(b_1)$, suppose that $\mathcal{A}'(b_1) : B_1 \to B_2$ is surjective. Then, there exists $\delta > 0$ such that for all $b' \in B_2$ with $\|b' - b_2\| \leq \delta$, one can find $b \in B_1$ such that $\mathcal{A}(b) = b'$.*

We will also need the following lemma whose the proof is immediate:

Lemma 3.25. *If $v \in Y_{1/4}$ and $w \in Y_{1/2}$, then $vw_x \in L^2(Q)$ and*

$$\|vw_x\|_{L^2(Q)} \leq C\|v\|_{Y_{1/4}}\|w\|_{Y_{1/2}}.$$

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: Let

$$B_2 = L^2(e^{s\psi_1}\beta^{-5/2}(0,T); L^2(0,1)) \times \prod_{i=1}^{2} L^2(e^{s\psi_i}\beta^{-1/2}(0,T); L^2(0,1)) \times H^1_0(0,1).$$

We define $\mathcal{A} : B_1 \to B_2$ by

$$\mathcal{A}(z,\phi^1,\phi^2,f) = \Big(\mathcal{L}z + zz_x + \sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i\chi_{\mathcal{O}_i} - f1\!\!1_{\mathcal{O}}, \mathcal{L}^*\phi^1 - z\phi^1_x - \alpha_1 z\chi_{\mathcal{O}_{1,d}}, \mathcal{L}^*\phi^2 - z\phi^2_x - \alpha_2 z\chi_{\mathcal{O}_{2,d}},$$
$$z(\cdot,0)\Big). \quad (3.145)$$

Note that $A(0,0,0,0) = (0,0,0,0)$ and the surjectivity for $A'(0,0,0,0)$ follows from Proposition 3.23. Let us prove that $\mathcal{A} \in C^1(B_1, B_2)$. Indeed, using that

$$L^2(e^{16s\check{\psi}-14s\hat{\psi}}\beta^{-45}(0,T);L^2(0,1))$$
$$\hookrightarrow L^2(e^{s\psi_1}\beta^{-5/2}(0,T);L^2(0,1)) \cap L^2(e^{s\psi_i}\beta^{-1/2}(0,T);L^2(0,1)), \quad (3.146)$$

for $i = 1, 2$, we get

$$\|\mathcal{A}(z,\phi^1,\phi^2,f) - \mathcal{A}(\tilde{z},\tilde{\phi}^1,\tilde{\phi}^2,\tilde{f})\|^2_{B_2}$$
$$\leq \iint_Q e^{2s\psi_1}\beta^{-5}\Big|\mathcal{L}(z-\tilde{z}) + \sum_{i=1}^{2}\frac{1}{\mu_i}(\phi^i - \tilde{\phi}^i)\chi_{\mathcal{O}_i} - (f-\tilde{f})\Big|^2\,dxdt$$
$$\iint_Q e^{16s\check{\psi}-14s\hat{\psi}}\beta^{-45}|zz_x - \tilde{z}\tilde{z}_x|^2\,dxdt + \sum_{i=1}^{2}\iint_Q e^{2s\psi_i}\beta^{-1}|\mathcal{L}^*(\phi^i - \tilde{\phi}^i) - \alpha_i(z-\tilde{z})\chi_{\mathcal{O}_{i,d}}|^2\,dxdt$$
$$+ \sum_{i=1}^{2}\iint_Q e^{16s\check{\psi}-14s\hat{\psi}}\beta^{-45}|z\phi^i_x - \tilde{z}\tilde{\phi}^i_x|^2\,dxdt + \|z(\cdot,0) - \tilde{z}(\cdot,0)\|^2_{H^1_0(0,1)}. \quad (3.147)$$

Note that the first, third an fifth terms on the right-hand side of (3.147) are exactly a part of the norm of B_1, then it is clear that

$$\iint_Q e^{2s\psi_1}\beta^{-5}\Big|\mathcal{L}(z-\tilde{z}) + \sum_{i=1}^{2}\frac{1}{\mu_i}(\phi^i - \tilde{\phi}^i)\chi_{\mathcal{O}_i} - (f-\tilde{f})\Big|^2\,dxdt$$
$$+ \sum_{i=1}^{2}\iint_Q e^{2s\psi_i}\beta^{-1}|\mathcal{L}^*(\phi^i - \tilde{\phi}^i) - \alpha_i(z-\tilde{z})\chi_{\mathcal{O}_{i,d}}|^2\,dxdt + \|z(\cdot,0) - \tilde{z}(\cdot,0)\|^2_{H^1_0(0,1)}$$
$$\leq \|(z,\phi^1,\phi^2,f) - (\tilde{z},\tilde{\phi}^1,\tilde{\phi}^2,\tilde{f})\|^2_{B_1}. \quad (3.148)$$

To deal with the second and fourth terms, we use Lemma 3.25 and we obtain

$$\iint_Q e^{16s\check{\psi}-14s\hat{\psi}}\beta^{-45}|zz_x - \tilde{z}\tilde{z}_x|^2\,dxdt = \iint_Q e^{16s\check{\psi}-14s\hat{\psi}}\beta^{-45}|(z-\tilde{z})z_x - \tilde{z}(\tilde{z}_x - z_x)|^2\,dxdt$$
$$\leq C\iint_Q e^{16s\check{\psi}-14s\hat{\psi}}\beta^{-45}|(z-\tilde{z})z_x|^2\,dxdt + \iint_Q e^{16s\check{\psi}-14s\hat{\psi}}\beta^{-45}|\tilde{z}(\tilde{z}_x - z_x)|^2\,dxdt$$
$$\leq C\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}(z-\tilde{z})\|^2_{Y_{1/2}}\big(\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}z\|^2_{Y_{1/2}} + \|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\tilde{z}\|^2_{Y_{1/2}}\big). \quad (3.149)$$

Analogously, we can prove that

$$\sum_{i=1}^{2}\iint_Q e^{16s\check{\psi}-14s\hat{\psi}}\beta^{-45}|z\phi_x^i-\tilde{z}\tilde{\phi}_x^i|^2\,dxdt$$
$$\leq C\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}(z-\tilde{z})\|_{Y_{1/2}}\sum_{i=1}^{2}\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\phi^i\|^2_{Y_{1/2}}$$
$$+\sum_{i=1}^{2}\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}(\phi^i-\tilde{\phi}^i)\|_{Y_{1/2}}\|^2 e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\tilde{z}\|_{Y_{1/2}}\big). \quad (3.150)$$

From (3.147)-(3.150) we have that $\mathcal{A}$ is continuous from B_1 to B_2. Now, it just remains to prove that $\mathcal{A}':B_1\mapsto\mathcal{L}(B_1,B_2)$ is also continuous. Notice that

$$A'(z,\phi^1,\phi^2,f)(\tilde{z},\tilde{\phi}^1,\tilde{\phi}^2,\tilde{f})=\Big(\mathcal{L}\tilde{z}+\tilde{z}z_x+z\tilde{z}_x+\sum_{i=1}^{2}\frac{1}{\mu_i}\tilde{\phi}^i-\tilde{f},\mathcal{L}^*\tilde{\phi}^1-z\tilde{\phi}_x^1-\tilde{z}\phi_x^1-\alpha_1\tilde{z}\chi_{\mathcal{O}_{1,d}},$$
$$\mathcal{L}^*\tilde{\phi}^2-z\tilde{\phi}_x^2-\tilde{z}\phi_x^2-\alpha_2\tilde{z}\chi_{\mathcal{O}_{2,d}},\tilde{z}(\cdot,0)\Big). \quad (3.151)$$

In this way,

$$A'(z,\phi^1,\phi^2,f)(\tilde{z},\tilde{\phi}^1,\tilde{\phi}^2,\tilde{f})-A'(\hat{z},\hat{\phi}^1,\hat{\phi}^2,\hat{f})(\tilde{z},\tilde{\phi}^1,\tilde{\phi}^2,\tilde{f})=$$
$$\Big(\tilde{z}(z_x-\hat{z}_x)+(z-\hat{z})\tilde{z}_x,-(z-\hat{z})\tilde{\phi}_x^1-\tilde{z}(\phi_x^1-\hat{\phi}_x^1),-(z-\hat{z})\tilde{\phi}_x^2-\tilde{z}(\phi_x^2-\hat{\phi}_x^2),0\Big). \quad (3.152)$$

Hence, proceeding in a very similar way as in (3.149) and (3.150), we can prove that

$$\|A'(z,\phi^1,\phi^2,f)(\tilde{z},\tilde{\phi}^1,\tilde{\phi}^2,\tilde{f})-A'(\hat{z},\hat{\phi}^1,\hat{\phi}^2,\hat{f})(\tilde{z},\tilde{\phi}^1,\tilde{\phi}^2,\tilde{f})\|^2_{B_2}$$
$$\leq C\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}(z-\hat{z})\|^2_{Y_{1/2}}\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\tilde{z}\|^2_{Y_{1/2}}$$
$$+C\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}(z-\hat{z})\|^2_{Y_{1/2}}\sum_{i=1}^{2}\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\tilde{\phi}^i\|^2_{Y_{1/2}}$$
$$+C\sum_{i=1}^{2}\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}(\phi^i-\hat{\phi}^i)\|^2_{Y_{1/2}}\|e^{8s\check{\psi}-7s\hat{\psi}}\beta^{-45/2}\tilde{z}\|^2_{Y_{1/2}}$$
$$\leq C\|(z,\phi^1,\phi^2,f)-(\hat{z},\hat{\phi}^1,\hat{\phi}^2,\hat{f})\|^2_{B_1}\|(\tilde{z},\tilde{\phi}^1,\tilde{\phi}^2,\tilde{f})\|^2_{B_1} \quad (3.153)$$

Hence, the derivative A' is also continuous and we can apply the Inverse Mapping Theorem (Theorem 3.24) to the operator $\mathcal{A}$. This completes the proof of Theorem 3.1. □

3.8 **Open questions**

In this chapter, we have assumed some similar conditions to the ones in [5, 6], these conditions are given essentially when we make the cases $\mathcal{O}_{1,d}=\mathcal{O}_{2,d}$ and $\mathcal{O}_{1,d}\cap\mathcal{O}\neq$

$\mathcal{O}_{2,d} \cap \mathcal{O}$. Concerning these conditions, some open questions arises naturally. The first one is the case that $\mathcal{O}_d^{k,l}$ coincides only inside $\mathcal{O}$ and differ outside $\mathcal{O}$, we remark that, even for the case of one single equation, similar open problem arises (see [6]).

Other cases where we do not get results the respect is when the controls are on the bondary, leader or followers. For example, in case that f is the leader, v^1, v^2 are followers we don't know what happens when $y(0,t) = f$, $y(L,t) = v^1$ or $y_x(L,t) = v^2$.

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