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Discrete Calculus: Applications in Stochastic Processes

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LUCAS GABRIEL BEZERRA DE SOUZA

DISCRETE CALCULUS:
APPLICATIONS IN STOCHASTIC PROCESSES

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This dissertation is dedicated to all the victims of the COVID-19 pandemic around the world, especially in Brazil.

Esta dissertação é dedicada a todas as vítimas da pandemia de COVID-19 no mundo, em especial no Brasil.

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[1] Vale ressaltar que este nome tem caráter lúdico. A "profissão" de coaches quânticos é um desserviço para a ciência e para a sociedade por não apresentar nenhum respaldo científico para as suas ações.

"*[...] o cosmos é uma gigantesca massa de informação em potencial à espera de um observador inteligente o suficiente para extrair desse universo conhecimento e, em um mesmo sopro de intuição, conferir algum significado a toda essa vastidão cósmica.*" (NICOLELIS, 2020, p. 16).

ABSTRACT

In this dissertation we explore the relationship between the infinitesimal and discrete descriptions of Nature and how these descriptions are connected in a systematic way via an integral transform called mimetic map, which we propose. We start presenting a brief review of sequences and difference equations, detailing some solving methods. In particular, we see that solving techniques of differential equations of infinitesimal calculus can be transferred to a calculus used to describe and solve finite difference equations, known as discrete calculus, been such techniques widely applied in the research field of difference equations. Then we show how the whole structure of infinitesimal calculus can be transferred to the discrete calculus via the mimetic map, generalizing and systematizing the already known discrete calculus of sequences, using the discrete functions, and interpreting the difference equations as discrete versions of differential equations. Also via the mimetic map we extend the notion of generating functions of sequences to discrete functions, where such extensions depend on a parameter h, returning the sequence case when $h = 1$. With the mimetic map as well we obtain discrete versions of integral transforms, such as the discrete Laplace and Mellin transforms, relating the former with the Z transform. We also present a complex mimetic map used to construct a complex discrete calculus starting from the calculus on the complex plane. As applications in physics, we present a review of discrete and continuous stochastic processes and show how the mimetic transform and the corresponding discrete calculus are capable to map the descriptions of these processes into one another continuous processes one onto the other. In particular, we obtain a discrete version of the H theory for the background variables using the mimetic map and for the observable variable using the tools of stochastic processes. And lastly we present how the formulations of epidemic models, given as continuous and discrete stochastic processes, which is connected by construction in the literature, now could be connected via the discrete calculus and the mimetic map.

Keywords: discrete calculus; finite difference equations; stochastic processes; H theory; epidemic models.

RESUMO

Nesta dissertação exploramos a relação entre as descrições infinitesimal e discreta da natureza e como essas descrições estão conectadas de modo sistemático por uma transformação integral denominada mapa mimético, a qual propomos. Iniciamos apresentando uma breve revisão de sequências e equações de diferença, detalhando alguns métodos de resolução. Em particular nós vemos que técnicas de resolução de equações diferenciais do cálculo infinitesimal podem ser transferidas para um cálculo utilizado para descrever e solucionar equações de diferença finita, conhecido como cálculo discreto, sendo estas técnicas amplamente empregadas na área de equações de diferença. Em seguida mostramos como toda a estrutura do cálculo infinitesimal pode ser transferida para o cálculo discreto através do mapa mimético, generalizando e sistematizando o já conhecido cálculo discreto de sequências, utilizando as funções discretas, e interpretando as equações de diferença como versões discretas das equações diferenciais. Também através do mapa mimético estendemos a noção de funções geradoras de sequências para as funções discretas, onde tais extensões dependem de um parâmetro h, retornando o caso de sequências quando $h = 1$. Com o mapa também obtemos versões discretas de transformadas integrais, como as transformadas de Laplace e Mellin discretas, relacionando a primeira com a transformada Z. Nós também apresentamos um mapa mimético complexo usado para construir um cálculo discreto complexo partindo do cálculo no plano complexo. Como aplicações na física, nós apresentamos uma revisão de processos estocásticos discretos e contínuos e mostramos como o mapa mimético e seu respectivo cálculo discreto são capazes de mapear as descrições destes processos uma na outra. Em particular, nós obtemos uma versão discreta da teoria H para as variáveis de background utilizando o mapa mimético e para a variável observável utilizando as ferramentas de processos estocásticos. E por último mostramos como as abordagens de processos epidêmicos, dadas como processos estocásticos contínuos e discretos, que eram conectadas por construção na literatura, agora poderiam ser conectadas através do cálculo discreto e do mapa mimético.

Palavras-chave: cálculo discreto; equações de diferença finita; processos estocásticos; teoria H; modelos epidêmicos.

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1 INTRODUCTION

To comprehend a theory it is necessary to have an interconnection between a mathematical model, its predictions and experimental observations, made to accept or refuse the predictions, been such predictions analytical or numerical. However, in most cases are observed distinctions between the experiments and numerical simulations, and the mathematical model with respect to its mathematical domains.

The mathematical models in most of the theories assume that all physical quantities (time, space and so on) are infinitesimal, been described in a continuum space-time. In fluid mechanics it is known as the Continuum Assumption (KESSLER; GREENKORN, 1999). As a counterpoint, the numerical methods are always dependent on discrete step parameters, discretizing all quantities involved in the mathematical model, e.g., finite difference methods. Also, experiments are always performed using equipments that work using finite units of increment, e.g., radiation intensity measurements of a laser obtained in time intervals of $100ms$ (GONZáLEZ I. R. R., 2018), presenting the values of the observable as a discrete time series. Thus, the mathematical substratum in terms of which the observables are described distinguish from the model and experimental/numerical approach, been a continuum and discrete respectively.

Furthermore, some mathematical models already involve in its constructions a discrete substratum. For example, lattice gauge theories (KOGUT, 1983), like lattice chromodynamics, and other lattice models, like Ising and Hubbard models, are described in terms of a finite lattice parameter. The regularization procedure, found in quantum field theories (KLAUBER, 2013), depends on a finite cutoff. And also, from a more fundamental level, some quantum gravity theories suggest that, at Planck scale, space-time itself could be discretized (HOOFT, 1979).

Those and other arguments are mentioned to justify the necessity of a construction of a discrete substratum to describe nature. However the attempts made to construct such a discrete substratum did not resemble a continuous substratum in terms of its properties (BALACHANDRAN et al., 1995), which provided a diminishiment of precision of numerical calculations and of predictions of lattice models (BALACHANDRAN et al., 1994). All that changed with the work of Sorkin (SORKIN, 1991), which provided a discrete topological space which converges to a continuous one as it becomes described by a larger number of sets.

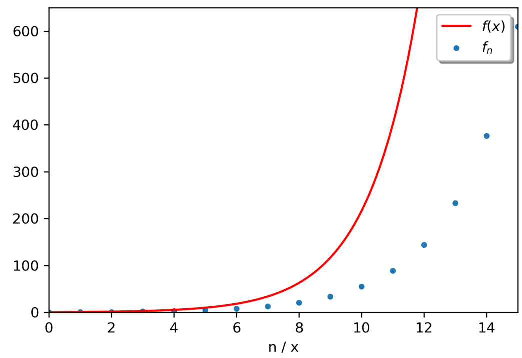
Thus, to reconcile the experimental/numerical and mathematical model approach it is reasonable to expect that a discrete substratum can be used to construct mathematical models of nature, providing discrete objects like discrete vectors and discrete derivative operators or, in general, a discrete differential calculus can be used to rewrite the laws of physics. Indeed, such approach is already used to, starting from a discrete calculus, obtain physical models, such as in (MERCAT, 2001), where a description of the Ising model is obtained from a complex discrete calculus, and as in (MARSDEN; WEST, 2001) and (DESBRUN et al., 2005), where a discrete variational mechanics is formulated.

In particular, such a discrete construction can be made via the **Discrete Exterior Calculus** (DEC) (GILLETTE, 2009), which is a discrete version of the exterior calculus (BASSALO; CATTANI, 2009) that preserves all the properties associated with the latter. The DEC consists of a reformulation of differential geometry where instead of a smooth manifold, we have a discrete manifold described by a graph, where the fundamental units of such a graph are sets of ordered vertices, called simplexes. In such a construction the derivative operator is defined as a matrix representation of the connection between vertices of such graph. With that, it is possible, for instance, to obtain a discrete version of the Stokes theorem and find many applications in physics, computational science and mathematics, such as circuit theory and image processing (GRADY; POLIMENI, 2010). Here we will limit ourselves to the one-dimensional case, relating the usual infinitesimal calculus with the discrete calculus.

Due to this mimicking of the properties of exterior calculus by DEC, we should have a discrete calculus which mimics the properties of the infinitesimal calculus. As a matter of fact, this is done in the literature by trial and error (IZADI, 2018), setting equations called difference equations, satisfied by sequences, which resemble the differential equations. In these cases, educated guesses of sequences that can be seen as discrete versions of functions are made, aiming to satisfy such difference equations. For instance, observe the figure 1 bellow. We see that the behavior of the Fibonacci sequence (blue) and the solution of some differential equation (red) have a similar evolution. Indeed, as will become clear along chapter 2, the difference equation satisfied by the Fibonacci sequence can be expressed in a similar manner, using operations defined in the discrete calculus that resembles such differential equation.

Thus, what we propose in this dissertation is a systematic way to obtain a discrete calculus starting from the infinitesimal calculus via an integral transform approach, called mimetic map, which essentially maps functions onto a generalization of sequences, called discrete functions (KHAN; NAJMI, 1999). Such a map is capable to provide not only all the structure of the known one-dimensional discrete calculus presented in literature (IZADI, 2018), but also generalize

Figure 1 – Fibonacci sequence (dotted blue curve) and the solution (red curve) of a differential equation related via the Fibonacci sequence via discrete calculus.



Source: the author (2021).

and systematizes it to until now unknown discrete versions of some special functions and its respective differential equations. In some cases such discrete functions are given in terms of transcendental functions.

Beyond that, as physical applications, we tried to provide via the mimetic map and the structure of the discrete calculus, a way to connect discrete and continuous stochastic processes in the context of H Theory and Stochastic Epidemic Models, in a similar manner as the random walk can be seen as the discrete version of the Brownian motion.

The H theory consists of a stochastic approach used to model multiscale hierarchical complex systems associating a system of coupled stochastic differential equations to it, which unveil the intermittency and energy cascade features observed in turbulent systems. On the other hand, stochastic epidemic models consist of compartmental models used to describe epidemics which take into account the inherent fluctuating feature of the number of infected individuals in a population contaminated by some transmissive disease.

The dissertation is divided as follows:

In chapter 2 we start presenting the definition of sequences and difference equations, giving some methods to solve these, including the generating function and the Z transform, presenting the existing parallel between discrete and infinitesimal calculus. The chapter ends with a construction of the discrete h-calculus, which will serves as base for the discrete mimetic calculus obtained by the mimetic map.

In chapter 3 we define the mimetic map, presenting its consequences, such as the connection between differential and difference equations and functions and discrete functions. As well, we show how is possible to translate some mathematical objects like the integral transforms of the infinitesimal calculus to the space of discrete functions and how to generalize the generating functions, used for sequences, to discrete functions. In the end of this chapter, we present a two-dimensional version of the mimetic map, which is capable to map the infinitesimal calculus on a complex plane to a complex discrete calculus, discretizing real and imaginary axes.

In chapter 4 we make a review of probability theory, essential for the foundation of stochastic processes which we present right after. After that, we define the Master and the Fokker-Planck equations, which are used to describe the dynamics of the probability distributions of discrete and continuous stochastic processes. We finish the chapter presenting the stochastic differential equations, the tools of Ito's calculus and how the Fokker-Planck equations can be obtained with those.

Along chapter 5 we explain the basic concepts of H theory, presenting some applications in the description of turbulent fluids and in the description of turbulence features in random fibre lasers. As an application of the mimetic map, we show how a discrete H theory can be constructed for the background variables of the theory, providing new relations between continuous and discrete stochastic processes. Then we show how a discrete H theory, in the central limit theorem sense, can be constructed for the observable variable using the tools of stochastic processes only.

We conclude with chapter 6, where we start discussing the theory of a deterministic compartmental epidemic model, called SIR model, in terms of a system of coupled differential equations called Kernack-McKendrick equations. Here we present how the solutions behave in the stationary condition, leading to endemic or disease-free equilibrium, and how analytic parametric solutions of the system of equations can be obtained. We end the chapter presenting the stochastic epidemic SIR model from three approaches which distinguishes from each other according to the discrete or continuous description of time and random variables, which we try to motivate the connection between those via the discrete calculus and the mimetic map.

2 DIFFERENCE EQUATIONS AND DISCRETE CALCULUS

In this chapter we explore the relation between the discrete and infinitesimal substratum of a mathematical description of nature. It is done by a review of difference equations, studying how sequences and difference equations are related with functions and differential equations. In section 2.1 we define the notion of sequence and difference equations, given the former as solution of the latter. Then we discuss some methods to solve difference equations, giving proper mention to the generating function and Z transform methods. In section 2.2 we show how the difference equations can be described from the point of view of discrete calculus, constructing it from polytopic numbers and extending it for the discrete h-calculus.

2.1 DIFFERENCE EQUATIONS AND SOLVING METHODS

Consider the ordered set of real numbers $\{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$. Such set presents a pattern: the third number is the sum of the first and the second, the fourth is the sum of the second and the third, and so on. So, if we associate a number n, for each element of this set, which will be denoted by a variable f_n, say

$$\begin{array}{lcccccccccc} n = & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & \\ f_n \in & \{ & 0 & 1, & 1, & 2, & 3, & 5, & 8, & 13, & \ldots & \} \end{array}$$

we have a map $f : \mathbb{N}_0 \to \mathbb{R}$, which takes natural numbers into real numbers. We might express such pattern on the elements of the set by the relation

$$f_{n+2} = f_{n+1} + f_n, \text{ for } n \geqslant 0.$$

This relation, which is a recurrence relation, defines the term f_n of the mathematical object called sequence, which we denote by $(f_n)_{n\in\mathbb{N}_0}$. Such set is widely found in nature, and is known as the Fibonacci sequence.

In general, given the ordered set $F = \{f_0, f_1, \ldots, f_n, \ldots\}$, we define a **Sequence** $(f_n)_{n\in\mathbb{N}_0}$ as the application of the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ to F. The number f_n is called the n**th term of the sequence**. Sometimes we denote a sequence by just (f_n). A sequence can be classified as (IZADI, 2018)

(i) **Finite sequence**. A sequence (f_n) is said finite if the independent index n is defined in a finite subset of natural numbers:

$$(f_n) := \{f_n \in \mathbb{R} \mid n \in \{0, 1, \ldots, m\},\, m < \infty\} \tag{2.1}$$

(ii) **Infinite sequence**. A sequence (f_n) is said infinite if the independent index n can assume any natural value:

$$(f_n) := \{f_n \in \mathbb{R} \mid n \in \mathbb{N}_0\} \tag{2.2}$$

(iii) **n-periodic sequence**. A sequence (f_j) is said n-periodic if its elements satisfy

$$f_{j+n} = f_j. \tag{2.3}$$

(iv) **Non-periodic sequence**. A sequence (f_n) is said non-periodic if it is not periodic.

In many cases the terms of a sequence does not satisfy a recurrence relation, such as the prime numbers $\{2, 3, 5, 7, 13, 17, 19, \ldots\}$. However when we do have a recurrence relation, we have what is called as a difference equation. If someone is capable to find out what is the expression of the nth term $f_n = f(n)$ which satisfies the difference equation, we can define the image set completely. Wherefore, we will present some features of the difference equations and how to solve it in certain cases.

Most of the classification of the difference equations are quite similar to the classification of the differential equations of infinitesimal calculus, and each one have particular properties, which implies particular methods to solve it. In general, we have finite or infinite, linear or non-linear, periodic or non-periodic, homogeneous or non-homogeneous, difference equations. Thus, along this section, we will give some clues of how the theory of difference equations are directly related with the differential equations.

In general, given a sequence (f_n) with a recurrence relation, we define a **Difference Equation** by such a recurrence relation associated with it. Namely,

$$F(s, f_s, \ldots, f_{s+n}) = 0; \quad s \in \mathbb{N}_0. \tag{2.4}$$

The particular case of linear difference equations can be simply written as

$$\sum_{j=0}^{n} \alpha_j(s) f_{j+s} = g_s; \quad 0 \leqslant s < m, \tag{2.5}$$

for a given sequence (g_s). We have a **Finite Linear Non-homogeneous Difference Equation** if n is finite and a **Infinite Linear Non-homogeneous Difference Equation** if n is infinite. As an example, in the Fibonacci sequence discussed above, it is obtained from (2.5) if we set $n = 2$, $g_s = 0\ \forall s$, and $\alpha_0 = \alpha_1 = -\alpha_2$.

What we get from definition (2.5) is a linear system of equations with $n+m$ variables and m equations, where the variables to be found are the $n+m$ elements of the sequence (f_n):

$$\begin{array}{ccccccccc} \alpha_0 f_0 & + & \alpha_1 f_1 & + & \cdots & + & \alpha_n f_n & = & g_0, \\ \alpha_0 f_1 & + & \alpha_1 f_2 & + & \cdots & + & \alpha_n f_{n+1} & = & g_1, \\ & & & & \vdots & & & & \\ \alpha_0 f_{m-1} & + & \alpha_1 f_m & + & \cdots & + & \alpha_n f_{n+m-1} & = & g_{m-1}. \end{array} \tag{2.6}$$

By this representation we can use matrix methods to solve it.

As we know, a linear system of equations can be uniquely solved if and only if the number of variables is the same as the number of equations. To ensure this for the system (2.6) we need to constraint the values of n terms of (f_n), say $\{f_0, \ldots, f_{n-1}\}$, in the first equation of (2.6), specifying the term $f_n = g_0 - \sum_{j=0}^{n-1} \alpha_j f_j$, which left a system of $m-1$ variables and $m-1$ equations. Those constraints are the **Initial Conditions** of the difference equation, just like an ordinary differential equation of nth order needs $n-1$ initial conditions to be completely specified. Thus the Fibonacci sequence can be completely specified if expressed as

$$\begin{cases} f_{s+2} = f_{s+1} + f_s, \text{ for } \quad s \in \mathbb{N}_0; \\ f_0 = 0,\ f_1 = 1. \end{cases} \tag{2.7}$$

Such similarity between difference equations and differential equations will become more and more evident along this chapter, culminating in the direct connection via the mimetic map 3.1, in chapter 3. For a fixed value of s, (2.5) is also called as an ordinary linear difference equation of nth order. In what follows, we call as difference equations all linear difference equations.

2.1.1 Solutions of Difference Equations

Bellow we describe some methods to solve difference equations. Here we emphasize the similarity between difference and differential equations solving methods, which usually is not mentioned in the literature (IZADI, 2018).

(i) *Periodic finite sequence and non-homogeneous difference equations*

Consider a difference equation which have a n-periodic sequence as solution given by

$$\sum_{j=0}^{n} \alpha_j f_{j+s} = g_s;\ s = 0, \ldots, m-1. \tag{2.8}$$

Ordinary Differential Equations (ODE) satisfied by functions presenting some periodic feature, can be solved using the Fourier series

$$f(x) = \sum_{j=-\infty}^{\infty} \tilde{f}_j e^{-i\frac{2\pi j}{L}x}, \tag{2.9}$$

with Fourier coefficients given by

$$\tilde{f}_j = \frac{1}{L}\int_a^b e^{i\frac{2\pi j}{L}x} f(x)dx, \tag{2.10}$$

for $L = b - a$. Following the same reason, we use a discrete analogy of the Fourier series to solve the difference equation above. The finite periodic sequence $(F_k)_{k\in\mathbb{N}_0}$ defined by (IZADI, 2018)

$$F_k := \sum_{j=0}^{n-1} f_j e^{-i\frac{2\pi j}{n}k}, \tag{2.11}$$

where $k = 0, 1, \ldots, n-1$. (2.11) is called the **Discrete Transform** of $(f_j)_{j\in\mathbb{N}_0}$, with f_j given by

$$f_j = \frac{1}{n}\sum_{k=0}^{n-1} F_k e^{i\frac{2\pi j}{n}k}. \tag{2.12}$$

Clearly there is a resemblance to the Fourier series and the discrete transform. From the Fourier series, consider $x = a + k\Delta x$ and rewrite (2.10) as a Riemann sum:

$$\frac{1}{L}\int_a^b e^{i\frac{2\pi n}{L}x} f(x)dx = \lim_{\Delta x\to 0}\sum_{k=0}^{n-1}\frac{\Delta x}{L} f(a + k\Delta x)e^{i\frac{2\pi aj}{L}} e^{i2\pi j\frac{\Delta x}{L}k}. \tag{2.13}$$

If $\lim\limits_{\Delta x\to 1}\frac{\Delta x}{L} = \frac{1}{n}$, we have

$$\tilde{f}_j \approx e^{i\frac{2\pi aj}{L}}\frac{1}{n}\sum_{k=0}^{n-1} f(a+k)e^{i\frac{2\pi j}{n}k} = e^{i\frac{2\pi aj}{L}} f_j. \tag{2.14}$$

From (2.9), restricting $j \in \mathbb{Z}$ to $j \in \{0, 1, \ldots, n-1\}$, we see that

$$f(a+k) = \sum_{j=0}^{n-1}\tilde{f}_j e^{-i\frac{2\pi aj}{L}} e^{-i\frac{2\pi j}{n}k} \equiv F_k, \tag{2.15}$$

if $f_j = \tilde{f}_j e^{-i\frac{2\pi aj}{L}}$. That connection is a glimpse of a construction of a discrete calculus, which will be systematically constructed in section 2.2.

With that well established, we can use definition (2.11) to solve the difference equation (2.8). Multiplying (2.8) by $e^{-i\frac{2\pi s}{m}k}$ and summing in s, we have

$$\sum_{s=0}^{m-1} e^{-i\frac{2\pi s}{m}k} \sum_{j=0}^{n} \alpha_j f_{j+s} = \sum_{s=0}^{m-1} e^{-i\frac{2\pi s}{m}k} g_s \equiv G_k. \tag{2.16}$$

Rewriting the sum and using that (f_j) is m-periodic, $e^{-i\frac{2\pi s}{m}k} f_s = e^{-i\frac{2\pi(s+m)}{m}k} f_{s+m}$, we have

$$\sum_{s=0}^{m-1} e^{-i\frac{2\pi s}{m}k} f_{j+s} = e^{i\frac{2\pi j}{m}k} \left(\sum_{s=j}^{m-1} e^{-i\frac{2\pi s}{m}k} f_s + \sum_{s=m}^{m+j-1} e^{-i\frac{2\pi(s+m)}{m}k} f_{s+m} \right) = e^{i\frac{2\pi j}{m}k} F_k, \tag{2.17}$$

Finding the relation

$$F_k = \frac{G_k}{A_k}, \tag{2.18}$$

where $A_k = \sum_{j=0}^{n} \alpha_j e^{i\frac{2\pi j}{n}k} \neq 0$.

Substituting F_k into (2.12) we obtain the solution of (2.8) as

$$f_j = \sum_{s=0}^{m-1} g_s \sum_{k=0}^{m-1} \frac{1}{m} \frac{e^{i\frac{2\pi(j-s)}{m}k}}{A_k} \equiv \sum_{s=0}^{m-1} g_s \Phi(j-s), \tag{2.19}$$

for $0 \leqslant j < m$ and

$$\Phi(j-s) \equiv \sum_{k=0}^{m-1} \frac{1}{m} \frac{e^{i\frac{2\pi(j-s)}{m}k}}{A_k}. \tag{2.20}$$

Again, we see here a great similarity with the infinitesimal calculus, where (2.19) seems as the discrete version of a solution of a ODE expressed in term of a Green function $\Phi(j-s)$. As a matter of fact (DENNERY; KRZYWICKI, 1996), an ODE

$$L_x f(x) = g(x),\ a \leqslant x \leqslant b;$$

where L_x is an Hermitian differential operator, have a particular solution

$$f(x) = \int_a^b G(x,x') g(x') dx'.$$

for the Green function $G(x,x')$ given by

$$G(x,x') = \sum_{n=0}^{\infty} \frac{\psi_n^*(x')\psi_n(x)}{\lambda_n},$$

with $\psi_n(x)$ and λ_n been the eigenfunctions and eigenvalues of the differential operator L_x, i.e., $L_x\psi_n(x) = \lambda_n\psi_n(x)$. Comparing with (2.19), we see that what makes the role of the eigenvalue λ_n is the term A_k and of the eigenfunction $\psi_n(x)$ is $\frac{e^{i\frac{2\pi j}{m}k}}{\sqrt{m}}$.

(ii) *Non-periodic finite sequences and non-homogeneous difference equations*

Consider again the difference equation

$$\sum_{j=0}^{n} \alpha_j f_{j+s} = g_s; \quad s = 0, \ldots, m-1; \tag{2.21}$$

but now with the solution (f_n) been a non-periodic finite sequence. We can reuse the solution obtained in the previous method imposing the periodic property for (f_n).

Assuming the terms $\{f_0, \ldots, f_{n-1}\}$ are known, only remains to find $\{f_n, \ldots, f_{n+m-1}\}$. And extending the values $s = 0, \ldots, m-1$ to $s = 0, \ldots, m-1, m, \ldots, n+m-1$, the terms of the sequence satisfy a square linear system. Then, (f_n) became a (n+m)-periodic sequence, i.e.,

$$f_{j+n+m} = f_j, \text{ for } 0 \leqslant j < n. \tag{2.22}$$

Under such construction we have the difference equation

$$\sum_{j=0}^{n} \alpha_j f_{j+s} = g_s; \; s = 0, \ldots, n+m-1; \tag{2.23}$$

with the periodic condition (2.22) imposed, which is a system of $2n+m$ variables and $n+m$ equations, with n variables already known. Hence, we have a $n+m$ square linear system. Therefore, we can solve the equation (2.21) using

$$f_j = \sum_{s=0}^{n+m-1} g_s \Phi(j-s), \tag{2.24}$$

for $0 \leqslant j < n+m$,

$$\Phi(j-s) \equiv \sum_{k=0}^{n+m-1} \frac{1}{(n+m)} \frac{e^{i\frac{2\pi(j-s)}{n+m}k}}{A_k}. \tag{2.25}$$

However, another problem appears: the terms $\{g_m, \ldots, g_{n+m-1}\}$ are unknown. Those can be found noting that (2.24) can be rewritten as

$$\sum_{s=m}^{n+m-1} \Phi(j-s) g_s = f_j - \sum_{s=0}^{m-1} \Phi(j-s) g_s, \text{ for } 0 \leqslant j < n, \tag{2.26}$$

where the right-hand side is completely known from our initial problem. The equation (2.26) can be expressed in matrix form as

$$\mathit{\Phi}\, \vec{g} = \vec{f_0} - \mathit{\Phi_0}\, \vec{g_0}, \tag{2.27}$$

with the matrices

$$(\mathit{\Phi})_{j,s} = \Phi(j-s), \text{ with } j = 0, \ldots, n-1, \; s = m, \ldots, n+m-1;$$

and

$$(\Phi_0)_{j,s} = \Phi_0(j-s);\text{ with } j = 0,\dots,n-1,\ s = 0,\dots,m-1.$$

The other symbols are the vectors

$$\vec{g} = (g_m,\dots,g_{n+m-1})^T,\ \vec{g_0} = (g_0,\dots,g_{m-1})^T \text{ and } \vec{f_0} = (f_0,\dots,f_{n-1})^T.$$

Supposing that the determinant of Φ is non-null, we find the terms $\{g_m,\dots,g_{n+m-1}\}$ by

$$\vec{g} = \Phi^{-1}\ \vec{f_0} - \vec{g_0}, \tag{2.28}$$

where Φ^{-1} is the inverse matrix of Φ. With that, the solution of (2.21), given by (2.24), is completely determined.

(iii) *Infinite sequence and homogeneous difference equations*

Consider the finite difference equation given by

$$\sum_{j=0}^{n} \alpha_j f_{j+s} = 0;\ s \in \mathbb{N}_0; \tag{2.29}$$

with the n initial conditions $f_0,\dots,f_{n-1}$ known. Just like linear ODEs with constant coefficients can be solved by a solution proportional to the exponential function, we make a similar assumption for (2.29).

Assume that the particular solution of the difference equation (2.29) is of the form

$$f_s = C\lambda^s,\ \lambda \in \mathbb{C}. \tag{2.30}$$

Substituting this in (2.29), we get the **Characteristic Equation of a difference equation**, given by

$$\sum_{j=0}^{n} \alpha_j \lambda^j = 0, \tag{2.31}$$

with n roots denoted by $\{\lambda_k\}_{k=1}^n$.

If we assume that all roots of such equation are non-null and distinct, we obtain the general solution of the difference equation (2.29) given by

$$f_s = \sum_{k=1}^{n} C_k \lambda_k^s, \tag{2.32}$$

where each root gives a particular solution $C_k\lambda_k^s$.

However, if there is at least two roots equal to each other, we need to assure that the particular solutions are linearly independent. Consider for example that all roots are the same, i.e., $\{\lambda_k = \lambda_0\}_{k=1}^{n}$. To assure the linear independence of the particular solutions $C_k\lambda_0^s$, $k = 1, \ldots, n$, we multiply each solution by a factor s^k, giving the general solution

$$f_s = \lambda_0^s \sum_{k=0}^{n-1} C_k s^k. \tag{2.33}$$

As a matter of fact, substituting (2.33) in the difference equation (2.29) and manipulating the sums, we have

$$\sum_{j=0}^{n} \alpha_j f_{j+s} = \lambda_0^s \sum_{k=0}^{n-1} C_k \sum_{l=0}^{k} \binom{k}{l} s^l \left(\lambda_0 \frac{d}{d\lambda_0}\right)^{k-l} \sum_{j=0}^{n} \alpha_j \lambda_0^j = 0, \tag{2.34}$$

as it should be.

The solutions are completely determined in any of those cases imposing the initial conditions of the problem.

2.1.2 Solving Difference Equations Using the Generating Functions and the Z Transform

2.1.2.1 Generating Functions

A practical way to solve a difference equation is transform it in an ODE via a discrete transform of the difference equation, called generating function. We define the **Generating Function** of a sequence (f_n) through the quantity

$$G(z) := \sum_{n=0}^{\infty} f_n z^n. \tag{2.35a}$$

From which the sequence (f_n) is obtained using the formula

$$f_n = \frac{1}{n!} \frac{\partial^n}{\partial z^n} G(z)\bigg|_{z=0}. \tag{2.35b}$$

Equivalently, we can define the **Exponential Generating Function** given by

$$G(z) := \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}, \tag{2.36a}$$

with

$$f_n = \frac{\partial^n}{\partial z^n} G(z)\bigg|_{z=0} . \tag{2.36b}$$

Later we will see that the same quantity come out in the context of probability theory (subsection 4.2.1). In addition, see the following examples.

Example 2.1 (Fibonacci sequence). Consider the difference equation

$$\begin{cases} f_{n+2} = f_{n+1} + f_n; \\ f_0 = 0,\ f_1 = 1. \end{cases} \tag{2.37}$$

Multiplying both sides by $\frac{z^n}{n!}$, summing from 0 to ∞, and noting that

$$\sum_{n=0}^{\infty} f_{n+m} \frac{z^n}{n!} = \frac{d^m G(z)}{dz^m},$$

we get

$$G''(z) - G'(z) - G(z) = 0.$$

The conditions $f_0 = 0$ and $f_1 = 1$ gives the initial conditions $G(0) = 0$ and $G'(0) = 1$. Therefore the generating function associated to the Fibonacci sequence satisfies

$$\begin{cases} G''(z) - G'(z) - G(z) = 0; \\ G(0) = 0,\ G'(0) = 1. \end{cases} \tag{2.38}$$

Solving (2.38), we get

$$G(z) = \frac{e^{\frac{1+\sqrt{5}}{2} z} - e^{\frac{1-\sqrt{5}}{2} z}}{\sqrt{5}} \tag{2.39}$$

and so

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right], \tag{2.40}$$

which reproduces completely the image of the sequence.

Example 2.2 (Hanoi Tower). Consider three stakes where the first is occupied with a pile of n disks ordered by size from the bigger to the smaller (bottom to top) and the others two are unoccupied. The Hanoi Tower problem consists in find the smallest number f_n of moves to transfer the n disks from the first to the third stake, using the following rules:

- A move consists in take one disk from one stake to another;
- Only the uppermost disk can be moved;

- The moved disk should be placed on a unoccupied stack or over a disk bigger than it.

Turns out that the number of moves satisfies the difference equation

$$\begin{cases} f_{n+1} = 2f_n + 1; \\ f_0 = 0. \end{cases} \tag{2.41}$$

Using the generating function. in a similar manner as in the previous example, we find the differential equation for the generating function as

$$\begin{cases} G'(z) - 2G(z) = e^z; \\ G(0) = 0; \end{cases} \tag{2.42}$$

with solution

$$G(z) = e^{2z} - e^z, \tag{2.43}$$

giving

$$f_n = 2^n - 1. \tag{2.44}$$

It is easy to see that the number above gives the number of moves.

2.1.2.2 The Z Transform

Equivalently to the generating functions, the Z transform can be used to solve linear difference equations.

Consider a sequence (f_n). The **Z Transform** $F(z)$ of such sequence is defined by

$$F(z) = \mathcal{Z}\{f_n; z\} := \sum_{n=0}^{\infty} \frac{f_n}{z^n}, \tag{2.45}$$

for $z \in \mathbb{C}$, which converges only if exists a positive number R such that $|z| > R$ (KELLEY; PETERSON, 2001). The inverse Z transform is given by

$$f_n = \mathcal{Z}^{-1}\{F(z); n\} = \oint_C \frac{ds}{2\pi i} F(s) s^{n-1}, \tag{2.46}$$

where C is a closed curve which encloses the origin and is contained in the region of convergence of $F(z)$. In the following example we can see how the Z transform can be applied.

Example 2.3. Consider the difference equation

$$\begin{cases} f_n = af_{n-1};\ a \in \mathbb{R} \\ f_0 = 1. \end{cases}$$

Clearly the solution is the sequence (a^n), but let us solve it using the Z transform. Multiplying by $\frac{1}{z^n}$ and summing from 0 to ∞, we get

$$F(z) = \frac{z}{z-a} = \sum_{n=0}^{\infty} \frac{a^n}{z^n},$$

which converges if $|z| > |a|$. Also

$$f_n = \oint_C \frac{dz}{2\pi i} \frac{z^n}{z-a} = a^n,$$

confirming the validity of the inverse Z transform.

In the subsection 3.1.5 will be shown that such transform is in fact a particular case of the discrete version of the Laplace transform.

2.2 DISCRETE CALCULUS

Once we have characterized the difference equations which resembles as discrete versions of differential equations, one should expect that exists a formalism similar to the infinitesimal calculus to manage difference equations. In fact, here we construct such a formalism, called **Discrete Calculus**

2.2.1 Constructing the Discrete Calculus

The usual approach to present discrete calculus (KAC; CHEUNG, 2001)(IZADI, 2018) consists in simply take the infinitesimal calculus definition of derivative,

$$\frac{df(x)}{dx} := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and ignore the limit usually taken, which is a very intuitive approach although brings the discrete calculus to the ground of a consequence of infinitesimal calculus.

However, as proposed by Oliver Knill (KNILL, 2014) the discrete calculus could be developed independently, starting from the concept of function and using polytopic numbers (UFUOMA;

IKHILE, 2019), which are numbers used to describe patterns of objects arranged in a polyhedron. In this construction the infinitesimal calculus arise just as a idealized limit of the discrete calculus. As an example, the polytopic number $\frac{\widehat{x^n}}{n!}$ can be set as a sequence given by

$$\frac{\widehat{x^n}}{n!} \equiv \frac{x(x-1)\dots(x-n+1)}{n!} = \frac{x!}{(x-n)!n!}, \tag{2.47}$$

for $n \in \mathbb{N}_0$ and $x \geqslant n$. The number $\widehat{x^n}$ have a remarkable property:

$$D(\widehat{x^n}) \equiv (\widehat{x+1})^n - \widehat{x^n} = n\widehat{x^{n-1}}. \tag{2.48}$$

$$D(\widehat{x^n}) = n\widehat{x^{n-1}}$$

That is, such polytopic number is a sequence $(f_x^{[n]})$, depending on a parameter n, that satisfies the difference equation

$$f_{x+1}^{[n]} - f_x^{[n]} = nf_x^{[n-1]}, \tag{2.49}$$

which mimics the differential equation

$$\frac{dx^n}{dx} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}, \tag{2.50}$$

leading us to interpret $\widehat{x^n}$ as the discrete analogue of the power function x^n.

Other sequences satisfying difference equations who mimic differential equations can be presented. For example,

$$\widehat{x^{-n}} \equiv \frac{1}{(x+1)(x+2)\dots(x+k)}, \tag{2.51}$$

satisfies

$$D(\widehat{x^{-n}}) = (\widehat{x+1})^{-n} - \widehat{x^{-n}} = -n\widehat{x^{-n-1}}. \tag{2.52}$$

The sequence

$$\widehat{e^{ax}} \equiv (1+a)^x;\ x \in \mathbb{N}_0, \tag{2.53}$$

can be seen as the discrete analogue of the exponential function since it satisfies

$$D(\widehat{e^{ax}}) = a\widehat{e^{ax}}. \tag{2.54}$$

With such expression is possible to define discrete versions of the trigonometric functions

$$\widehat{\sin}(x) = \frac{\widehat{e^{ix}} - \widehat{e^{-ix}}}{2i} \tag{2.55}$$

and

$$\widehat{\cos}(x) = \frac{\widehat{e^{ix}} + \widehat{e^{-ix}}}{2}, \tag{2.56}$$

such that

$$D(\widehat{\sin}(x)) = \widehat{\cos}(x) \tag{2.57}$$

and

$$D(\widehat{\cos}(x)) = -\widehat{\sin}(x). \tag{2.58}$$

Also there is a discrete logarithm function

$$\widehat{\ln}(x) = H_x = \sum_{n=1}^{x} \frac{1}{n}, \tag{2.59}$$

where H_x is a harmonic number. It satisfies

$$D(\widehat{\ln}(x)) = \frac{1}{x+1} = \widehat{x^{-1}}. \tag{2.60}$$

More general, for a sequence $f_x \equiv \widehat{f}(x)$, $x \in \mathbb{N}_0$, define

$$D\widehat{f}(x) \equiv \widehat{f}(x+1) - \widehat{f}(x) \tag{2.61}$$

as the **Difference Operator**. With it, we can obtain discrete analogues of the properties of the derivative operator $\frac{d}{dx}$ of infinitesimal calculus:

(i) Additivity: Let $\widehat{f}(x) = \widehat{g}(x) + \widehat{h}(x)$, then

$$D(\widehat{f}(x)) = D(\widehat{g}(x)) + D(\widehat{h}(x));$$

(ii) Linearity: Let $\widehat{f}(x) = c\widehat{g}(x)$, then

$$D(\widehat{f}(x)) = cD(\widehat{g}(x));$$

(iii) Difference of the product: Let $\widehat{f}(x) = \widehat{g}(x)\widehat{h}(x)$, then

$$D(\widehat{f}(x)) = D(\widehat{g}(x))\widehat{h}(x) + \widehat{g}(x)D(\widehat{h}(x)) + D(\widehat{g}(x))D(\widehat{h}(x))$$

(iv) Difference of the division: Let $\widehat{f}(x) = \frac{\widehat{g}(x)}{\widehat{h}(x)}$, then

$$D(\widehat{f}(x)) = \frac{D(\widehat{g}(x))\widehat{h}(x) - \widehat{g}(x)D(\widehat{h}(x))}{\widehat{h}(x)\widehat{h}(x+1)}.$$

Above, we observe the lack of the chain rule. It occurs because the proof of the chain rule uses the concept of limit and continuous function as well, which are not taken into account here once we are working with sequences.

Equivalently we can define a **Sum Operator** S, given by

$$S(\widehat{f}(x)) = \sum_{y=0}^{x-1} \widehat{f}(y);\ x \in \mathbb{N}. \tag{2.62}$$

Such operator also reproduces some properties of the integral operation of infinitesimal calculus:

(i) Additivity: Let $\widehat{f}(x) = \widehat{g}(x) + \widehat{h}(x)$, then

$$S(\widehat{f}(x)) = S(\widehat{g}(x) + S(\widehat{h}(x));$$

(ii) Linearity: Let $\widehat{f}(x) = c\widehat{g}(x)$, then

$$S(\widehat{f}(x)) = cS(\widehat{g}(x));$$

(iii) Sum by parts: Let $\widehat{f}(x) = D(\widehat{g}(x))D(\widehat{h}(x))$, then

$$S(\widehat{f}(x)) = \widehat{g}(y-1)\widehat{h}(y-1) - \widehat{g}(0)\widehat{h}(0) - S(D(\widehat{g}(x))\widehat{h}(x)) - S(\widehat{g}(x)D(\widehat{h}(x))$$

We can see examples of applications of the sum operator bellow.

Example 2.4. Consider the sum

$$\sum_{y=0}^{x-1} \widehat{y^n} = \sum_{y=0}^{x-1} y(y-1)\dots(y-n+1).$$

With the fundamental theorem of discrete calculus above and the result (2.48), we can easily calculate this sum obtaining

$$S(\widehat{x^n}) = \frac{x(x-1)\dots(x-n)}{n+1} = \frac{\widehat{x^{n+1}}}{n+1}.$$

Example 2.5. Consider the sum

$$\sum_{n=0}^{\infty} 2^{-n} n(n-1)\dots(n-k+1),\ k \geqslant 2.$$

Using that $\widehat{e^{-n}} = 2^{-n}$, $\widehat{n^k} = n(n-1)\dots(n-k+1)$, and the sum by parts property, we get

$$\sum_{n=0}^{\infty} 2^{-n} n(n-1)\dots(n-k+1) = \sum_{n=0}^{\infty} \widehat{e^{-n}}\widehat{n^k} = k!.$$

The result of example 2.5 is expected if we compare this sum with the definition of the gamma function, $\Gamma(z+1) = \int_0^\infty e^{-x}x^z dx$. We can see the result above as the discrete version of the gamma function. It points out that the discrete version of the gamma function is itself if $z = k$ is a positive integer, which in fact will be shown in subsection 3.1.5 using the discrete version of the Mellin transform.

With such operations is possible to construct a discrete analogue of the fundamental theorem of calculus. Let $\widehat{f}(x)$ be a sequence, D the difference operator and S the sum operator. Then,

$$S(D(\widehat{f}(x))) = \widehat{f}(x) - \widehat{f}(0) \tag{2.63a}$$

and

$$D(S(\widehat{f}(x))) = \widehat{f}(x) \tag{2.63b}$$

follow directly from the definitions of D and S above.

Here we see a remarkable connection between the infinitesimal and discrete calculus. Such formalism gives us another method to solve difference equations, as can be seen in the example bellow.

Example 2.6. Consider the difference equation of the arithmetic progression given in terms of the difference operator

$$\begin{cases} D(\widehat{f}(x)) = d;\ x \in \mathbb{N}; \\ \widehat{f}(1) = f_1 = const. \end{cases}$$

Applying the sum operator in both sides, we get

$$\widehat{f}(x) = f_1 + d(x-1).$$

The operators above could have been defined in a similar manner by

$$D(\widehat{f}(x)) = \widehat{f}(x) - \widehat{f}(x-1) \tag{2.64}$$

and

$$S(\widehat{f}(x) = \sum_{y=1}^{x} \widehat{f}(y), \tag{2.65}$$

which would have lead to similar results. But since the definition $\widehat{f}(x+1) - \widehat{f}(x)$ is more similar to the infinitesimal derivative operator, we used that.

Continuing the discrete analogues, we can also define a discrete Taylor series in terms of the sequences $\widehat{x^n}$. We can define the **Discrete Taylor Series** of the sequence $\widehat{f}(x)$ near the point $x_0 \in \mathbb{N}_0$, as (IZADI, 2018)

$$\widehat{f}(x) = \sum_{n=0}^{\infty} D^n(\widehat{f}(x))\Big|_{x=x_0} \frac{\widehat{(x-x_0)^n}}{n!}, \tag{2.66}$$

or just

$$\widehat{f}(x) = \sum_{n=0}^{x-x_0} D^n(\widehat{f}(x))\Big|_{x=x_0} \frac{\widehat{(x-x_0)^n}}{n!}, \tag{2.67}$$

due to the definition of $\widehat{x^n}$.

Apart of this parallel traced above and the usefulness of solve difference equations using the operators D and S, the discrete calculus is useful to solve summation analytically using the discrete analogue of the fundamental theorem of calculus and the discrete Taylor series, as can be seen in the example bellow.

Example 2.7. In principle, it is not trivial to see what is the closed form of

$$\sum_{n=0}^{x} \frac{\widehat{x^n}}{n!} = \sum_{n=0}^{\infty} \frac{x(x-1)\dots(x-n+1)}{n!}.$$

However, it is possible with the assistance of the discrete Taylor series to get

$$\sum_{n=0}^{x} \frac{x(x-1)\dots(x-n+1)}{n!} = \widehat{e^x} = 2^x.$$

2.2.2 Discrete h-Calculus

The discussion above can be extended for a general increment $h \in \mathbb{R}^+$ instead of a unit. Therefore, instead of the definition in terms of polytopic numbers, we can define the discrete analogue of the power function x^n as (KNILL, 2014)

$$\widehat{x^n}_h \equiv x(x-h)(x-2h)\dots[x-(n-1)h], \tag{2.68}$$

for $x \in$ h$\mathbb{Z}$ such that $\frac{x}{h} \geqslant n$. We recover the case $\widehat{x^n}$ when $h = 1$, which we denote by $\widehat{x^n}_{h=1} = \widehat{x^n}$.

In this sense, we have a generalization of a sequence called **Discrete Function** (KHAN; NAJMI, 1999), which is a function with discrete domain defined in terms of steps of size h, such that $\widehat{f}_h$: h$\mathbb{Z} \to \mathbb{R}$. Here we use the notation $\widehat{f}_h$ to emphasize that the discrete function now depends on the parameter h. Thus, we can generalize the operators difference D and sum S defined previously by the **Forward Difference Operator** D^Δ defined as

$$D^\Delta \hat{f}_h(x) = \frac{\hat{f}_h(x+h) - \hat{f}_h(x)}{h} \tag{2.69}$$

and the **Indefinite Forward Sum** of $\widehat{f_h}$ defined as

$$S^{\Delta}\hat{f}_h(x) = h \sum_{k=0}^{[x/h]-1} \hat{f}_h(kh), \tag{2.70}$$

where $[x/h]$ denotes the integer part of $\frac{x}{h}$.

Equivalently, we define the **Backward Difference Operator** D^{∇} as

$$D^{\nabla}\check{f}_h(\nu) = \frac{\check{f}_h(\nu) - \check{f}_h(\nu - h)}{h} \tag{2.71}$$

and the **Indefinite Backward Sum** of $\widehat{f_h}$ as

$$S^{\nabla}\check{f}_h(\nu) = h \sum_{k=1}^{[\nu/h]} \check{f}_h(kh). \tag{2.72}$$

The definitions above satisfy the fundamental theorem of discrete calculus (GOODRICH, 2016):

Theorem 2.2.1 (Fundamental Theorem of Discrete Calculus)**.** Let $\widehat{f_h}$: h$\mathbb{Z} \to \mathbb{R}$ be a discrete function. Then,

$$D^{\Delta}(S^{\Delta}\hat{f}_h(x)) = \hat{f}_h(x), \quad S^{\Delta}(D^{\Delta}\hat{f}_h(x)) = \hat{f}_h(x) - \hat{f}_h(0), \tag{2.73}$$

$$D^{\nabla}(S^{\nabla}\check{f}_h(\nu)) = \check{f}_h(\nu), \quad S^{\nabla}(D^{\nabla}\check{f}_h(\nu)) = \check{f}_h(\nu) - \check{f}_h(0). \tag{2.74}$$

Also we can define the **Forward Definite Sum** of $\hat{f}_h$ by

$$\int_{x_0}^{x_1} \hat{f}_h(x)\Delta x \equiv h \sum_{k=[x_0/h]}^{[x_1/h]-1} \hat{f}_h(hk). \tag{2.75}$$

From it we can get

$$\int_{x_0}^{x_1} D^{\Delta}\hat{f}_h(x)\Delta x = \hat{f}_h(x_1) - \hat{f}_h(x_0), \tag{2.76}$$

confirming the theorem above. Analogously we could have defined the **Backward Definite Sum**. We observe that if we take the limit $h \to 0$ we recover the definitions of derivative and integral from infinitesimal calculus, making the infinitesimal calculus as a particular case of a more possibly general calculus. With that we are capable to present the main object of this dissertation, which a map connecting the infinitesimal calculus and the discrete h-calculus which we described above.

3 DISCRETE MIMETIC CALCULUS FROM AN INTEGRAL TRANSFORM APPROACH

In this chapter we systematically relate the infinitesimal and discrete description of nature connecting the infinitesimal and the discrete h-calculus via the integral transform mimetic map. With that, a novel approach to the construction of a discrete calculus is presented in a systematic way, mimicking all the properties present in the infinitesimal one. In section 3.1 we present our approach, the mimetic map, involving an integral transform which maps continuous functions into discrete functions as well as differential equations into difference equations. We also present a generalization of the generating function in terms of discrete functions and show how the discrete versions of the Laplace and Mellin transforms can be obtained via the mimetic map, relating the former with the Z transform. In section 3.2 we briefly explore a extension of the mimetic map, the complex mimetic map which can be used to construct the complex mimetic calculus from complex infinitesimal calculus.

3.1 DISCRETE MIMETIC CALCULUS

Among such parallels between discrete and infinitesimal calculus presented up to this point, a problem remains. There is no systematic method to obtain discrete functions, e.g., $\widehat{x^n}_h$, apart from educated guesses aiming to satisfy the difference equations which mimics the differential equations. To circumvent it, in this section we present an integral transform, called mimetic map, which maps functions onto discrete functions. Further, with it we can reproduce the entire discrete calculus mimicking the infinitesimal calculus properties, allowing for instance associate differential equations to difference equations by a general procedure.

3.1.1 The Mimetic Map

Consider the function $f : \mathbb{R} \to \mathbb{R}$ and its discrete counterpart $\widehat{f}_h$: h$\mathbb{Z} \to \mathbb{R}$. Then we define the **Mimetic Map** of $f(x)$ by the integral transform

$$\check{f}_h(\nu) := \int_0^\infty d\lambda_{\nu/h}(t) f(-ht), \tag{3.1}$$

where

$$d\lambda_\alpha(t) \equiv \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)} dt \tag{3.2}$$

is the Euler measure and $\Gamma(\alpha)$ is the gamma function, which can be seen as a deformed Mellin-Laplace transform. It provides the properties presented in section 2.2 using the backward operators (2.71) and (2.72). Taking the limit $\nu \to -x$ we get the discrete function $\widehat{f}_h(x)$,

$$\widehat{f}_h(x) = \lim_{\nu \to -x} \check{f}_h(\nu), \tag{3.3}$$

providing the properties of the forward operators (2.69) and (2.70), as will be shown bellow.

Furthermore, this map is one-to-one since we can recover the function $f(x)$ by the **Continuum Limit**

$$f(x) = \lim_{h \to 0} \widehat{f}_h(x). \tag{3.4}$$

So, not only we have a simple way to obtain discrete functions related to functions, but also we can recover the initial function from the discrete function. As we will see along the current chapter, such features will provide us with new methods to extend the set of discrete functions presented in the literature and the parallel between discrete and continuous mathematics.

3.1.1.1 The Mimetic Difference and Mimetic Sum

Let $\check{f}_h(\nu)$ be the mimetic image of $f(x)$ defined by (3.1), then we define the **Mimetic Difference** as

$$\hat{D}\check{f}_h(\nu) := -\int_0^\infty d\lambda_{\nu/h}(t) f'(-ht) \tag{3.5}$$

and the **Mimetic Indefinite Sum**

$$\hat{S}\check{f}_h(\nu) := \frac{h}{\Gamma(\nu/h)} \int_0^\infty dt\, \Gamma(\nu/h, t) f(-ht), \tag{3.6}$$

with

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$$

been the upper incomplete gamma function.

Those operators can be represented in terms of the already defined, and more familiar, backward difference and sum operators, D^∇ and S^∇. Integrating by parts the definition (3.5), we have

$$\hat{D}\check{f}_h(\nu) = -\frac{1}{h}\int_0^\infty \frac{dt}{\Gamma(\nu/h)} f(-ht) \frac{d}{dt}\left(t^{\nu/h-1} e^{-t}\right) = \frac{\check{f}_h(\nu) - \check{f}_h(\nu - h)}{h} \tag{3.7}$$

or just

$$\hat{D}\check{f}_h(\nu) = D^\nabla \check{f}_h(\nu). \tag{3.8}$$

From definition (3.6), we use the identity (ABRAMOWITZ, 1974)

$$\frac{\Gamma(n,x)}{\Gamma(n)} = e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}, \tag{3.9}$$

to rewrite it as

$$\hat{S}\check{f}_h(\nu) = h\int_0^\infty \left(\sum_{k=1}^{[\nu/h]}\frac{t^{k-1}}{(k-1)!}\right)e^{-t}f(-ht)dt = h\sum_{k=1}^{[\nu/h]}\check{f}_h(kh), \tag{3.10}$$

giving

$$\hat{S}\check{f}_h(\nu) = S^{\nabla}\check{f}_h(\nu). \tag{3.11}$$

Even more, from the limit (3.3) we can obtain

$$D^{\Delta}\hat{f}_h(x) = -\lim_{\nu\to -x}\hat{D}\check{f}_h(\nu) \tag{3.12}$$

and

$$S^{\Delta}\hat{f}_h(x) = \lim_{\nu\to -x}\hat{S}\check{f}_h(\nu). \tag{3.13}$$

3.1.1.2 Formulation Using Meijer G-Function Transform

Since the goal of the mimetic map is to generalize the procedure of discretization of functions, it is reasonable that we try to represent such transform in a general way to include a larger class of discrete functions. One way to do this is describing the mimetic map in terms of the Meijer G-function (see appendix A), which is a generalization of the generalized hypergeometric function.

First of all, the Euler measure can be rewritten as

$$d\lambda_\alpha(t) \equiv \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}dt = \frac{dt}{\Gamma(\alpha)}G^{1,0}_{0,1}\left[{--- \atop \alpha-1} \mid t\right], \tag{3.14}$$

giving the mimetic map

$$\check{f}_h(\nu) = \int_0^\infty \frac{dt}{\Gamma(\nu/h)}G^{1,0}_{0,1}\left[{--- \atop \frac{\nu}{h}-1} \mid t\right]f(-ht). \tag{3.15}$$

Thus, by definition, the mimetic derivative will be

$$\hat{D}\check{f}_h(\nu) = -\frac{1}{h}\int_0^\infty \frac{dt}{\Gamma(\nu/h)}f(-ht)\frac{d}{dt}G^{1,0}_{0,1}\left[{--- \atop \frac{\nu}{h}-1} \mid t\right]. \tag{3.16}$$

Consider the Mellin transform defined as

$$\tilde{f}(s) = \mathcal{M}\{f(t);s\} := \int_0^\infty dt t^{s-1}f(t), \tag{3.17}$$

with inverse

$$f(t) = \mathcal{M}^{-1}\{\tilde{f}(s);t\} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} t^{-s}\tilde{f}(s);\; c \in \mathbb{R}. \tag{3.18}$$

With that we find

$$\mathcal{M}\{G^{1,0}_{0,1}\left[{---\atop \alpha-1} \mid t\right];s\} = \Gamma(s+\alpha-1) \tag{3.19}$$

and using the property

$$\mathcal{M}\{f'(t);s\} = -(s-1)\tilde{f}(s-1) \tag{3.20}$$

we have

$$\mathcal{M}\left\{\frac{d}{dt}G^{1,0}_{0,1}\left[{---\atop \frac{\nu}{h}-1} \mid t\right];s\right\} = -(s-1)\Gamma(s+\alpha-2). \tag{3.21}$$

Now let

$$g(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}G^{1,0}_{0,1}\left[{---\atop \frac{\nu}{h}-1} \mid t\right], \tag{3.22}$$

giving

$$\tilde{g}(s) = \frac{\Gamma(2-s)\Gamma(s+\alpha-1)}{\Gamma(1-s)\Gamma(\alpha)}. \tag{3.23}$$

From the inverse Mellin transform we find

$$g(t) = \frac{1}{\Gamma(\alpha)}G^{1,1}_{1,2}\left[{1\atop 0,\alpha-1} \mid t\right], \tag{3.24}$$

resulting in

$$\hat{D}\check{f}_h(\nu) = -\frac{1}{h}\int_0^\infty \frac{dt}{\Gamma(\nu/h)}G^{1,1}_{1,2}\left[{1\atop 0,\frac{\nu}{h}-1} \mid t\right]f(-ht). \tag{3.25}$$

For the mimetic integral, we use the representation

$$\Gamma(\alpha,t) = G^{2,0}_{1,2}\left[{1\atop 0,\alpha} \mid t\right] \tag{3.26}$$

to rewrite (3.6) as

$$\hat{S}\check{f}_h(\nu) = h\int_0^\infty \frac{dt}{\Gamma(\nu/h)}G^{2,0}_{1,2}\left[{1\atop 0,\frac{\nu}{h}} \mid t\right]f(-ht). \tag{3.27}$$

Therefore, from (3.15), (3.25) and (3.27) we have a general mimetic map, allowing us to obtain discrete functions of a large set of functions represented in terms of the Meijer G-function, just using the identities found in literature (ERDéLYI, 1953).

3.1.2 Discrete Versions of Functions

In this section we present the first and direct application of our mimetic map (3.1) in some functions commonly used. We show that the discrete functions presented in 2.2 are recovered by the map, and also some new discrete counterparts of special functions can be obtained.

(i) *Power function:*

Let $f(x) = x^n$, then from (3.1) we have

$$\check{f}_h(\nu) = \int_0^\infty \frac{dt}{\Gamma(\frac{\nu}{h})} e^{-t} t^{\frac{\nu}{h}-1}(-ht)^n = (-h)^n \frac{\Gamma(\nu/h + n)}{\Gamma(\nu/h)}. \tag{3.28}$$

Writing the above expression in a convenient way and taking the limit $\nu \to -x$ we get

$$\widehat{x^n}_h \equiv h^n \frac{\Gamma(\frac{x}{h}+1)}{\Gamma(\frac{x}{h}+1-n)} = x(x-h)\cdots[x-(n-1)h], \tag{3.29}$$

which is the discrete power function.

Using the forward difference operator (3.5) we have

$$D^{\Delta}\widehat{x^n}_h = h^{n-1} n \frac{\Gamma(\frac{x}{h}+1)}{\Gamma(\frac{x}{h}+1-(n-1))}, \tag{3.30}$$

or

$$D^{\Delta}\widehat{x^n}_h = n\widehat{x^{n-1}}_h. \tag{3.31}$$

Therefore, the mimetic map reproduces the property observed in 2.2.1, mimicking the infinitesimal calculus. However, the backward difference operator D^{∇} does not plays the same role as D^{Δ} here. If we try to obtain a difference equation like the one above, we get

$$D^{\nabla}\widehat{x^n}_h = \left(\frac{1+\frac{x}{h}-n}{\frac{x}{h}}\right) n\widehat{x^{n-1}}_h \neq n\widehat{x^{n-1}}_h. \tag{3.32}$$

In fact, applying the mimetic map in $\frac{dx^n}{dx} = nx^{n-1}$, we obtain (3.31).

(ii) *Exponential function:*

Consider the function $f(x) = e^{ax}$, then from (3.1) we get

$$\check{f}_h(\nu) = \int_0^\infty \frac{dt}{\Gamma(\frac{\nu}{h})} e^{-t} t^{\frac{\nu}{h}-1} e^{-hat} = (1+ah)^{-\nu/h}, \tag{3.33}$$

resulting in

$$\widehat{e^{ax}}_h \equiv (1+ah)^{x/h}, \tag{3.34}$$

which is the discrete exponential function.

Again, we can reproduce the properties of infinitesimal calculus:

$$D^{\Delta}\widehat{e^{ax}}_h = \frac{1}{h}[(1+ah)^{x/h+1} - (1+ah)^{x/h}] = a(1+ah)^{x/h} \tag{3.35}$$

or

$$D^{\Delta}\widehat{e^{ax}}_h = a\widehat{e^{ax}}_h, \tag{3.36}$$

as expected. As before, we observe that the difference operator D^{∇} does not mimics the properties of infinitesimal calculus:

$$D^{\nabla}\widehat{e^{ax}}_h = \frac{1}{h}[(1+ah)^{x/h} - (1+ah)^{x/h-1}] = a\widehat{e^{a(x-h)}}_h, \tag{3.37}$$

that is

$$D^{\nabla}\widehat{e^{ax}}_h \neq a\widehat{e^{ax}}_h. \tag{3.38}$$

(iii) *Logarithm function:*

Let $f(x) = \ln x$, then we have

$$\check{f}_h(\nu) = \ln(-h) + \psi\left(\frac{\nu}{h}\right), \tag{3.39}$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. From (ERDéLYI, 1953), we have

$$\psi(n) = -\gamma + H_{n-1}, \tag{3.40}$$

$$\psi(z) - \psi(-z) = -\frac{1}{z} + \gamma + \pi\cot(\pi z) \quad \text{and} \tag{3.41}$$

$$\psi(z) = \ln(z) - \sum_{k=0}^{\infty}\left[\frac{1}{k+z} - ln\left(1 + \frac{1}{k+z}\right)\right], \tag{3.42}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is a harmonic number and $\gamma = 0.5772$ is the Euler-Mascheroni constant. These identities gives

$$\widehat{\ln}_h(x) \equiv \ln(-h) - \gamma + \pi\cot\left(\pi\frac{x}{h}\right) + H_{\frac{x}{h}} \tag{3.43}$$

or

$$\widehat{\ln}_h(x) = \ln x - \sum_{k=0}^{\infty}\left[\frac{1}{k-\frac{x}{h}} - \ln\left(1 + \frac{1}{k-\frac{x}{h}}\right)\right]. \tag{3.44}$$

From (3.43) and (3.44), respectively we find

$$D^{\Delta}\hat{f}_h(x) = \frac{1}{x+h} = \widehat{x^{-1}}_h, \tag{3.45}$$

and

$$\lim_{h\to 0}\widehat{\ln}_h(x) = \ln x, \tag{3.46}$$

as expected.

(iv) *Trigonometric functions:*

Consider $f(x) = \sin x$. From (3.1), we have

$$\check{f}_h(\nu) = \int_0^\infty \frac{dt}{\Gamma(\frac{\nu}{h})} e^{-t} t^{\frac{\nu}{h}-1} \sin(-ht). \tag{3.47}$$

Although such integral seems to be complicated, it becomes easy to solve using not the Meijer G-function but a generalization of it, the Fox H-function (appendix A). Using the identity

$$\sin(z) = \sqrt{\pi} H_{0,2}^{1,0}\left[{------ \atop (\frac{1}{2},1),(0,1)} \mid \frac{z^2}{4}\right], \tag{3.48}$$

we have

$$\check{f}_h(\nu) = \int_0^\infty t^{\nu/h-1} e^{-t} H_{0,2}^{1,0}\left[{------ \atop (\frac{1}{2},1),(0,1)} \mid \frac{h^2t^2}{4}\right] dz. \tag{3.49}$$

And with

$$\int_0^\infty z^{\rho-1} e^{-\lambda z} H_{p,q}^{m,n}\left[{\{(a_p,A_p)\} \atop \{(b_q,B_q)\}} \mid \alpha z^\sigma\right] dz = \lambda^{-\rho} H_{p+1,q}^{m,n+1}\left[{(1-\rho,\sigma),\{(a_p,A_p)\} \atop \{(b_q,B_q)\}} \mid \frac{\alpha}{\lambda^\sigma}\right], \tag{3.50}$$

we get

$$\check{f}_h(\nu) = \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{h})} H_{1,2}^{1,1}\left[{(1-\frac{\nu}{h},2) \atop (\frac{1}{2},1),(0,1)} \mid \frac{h^2}{4}\right], \tag{3.51}$$

or

$$\widehat{\sin}_h(x) \equiv \sqrt{\pi}\Gamma\left(1+\frac{x}{h}\right) H_{1,2}^{1,0}\left[{(1+\frac{x}{h},2) \atop (\frac{1}{2},1),(0,1)} \mid \frac{h^2}{4}\right]. \tag{3.52}$$

By the same procedure above, but using

$$\cos(z) = \sqrt{\pi} H_{0,2}^{1,0}\left[{------ \atop (0,1),(\frac{1}{2},1)} \mid \frac{z^2}{4}\right], \tag{3.53}$$

we obtain

$$\widehat{\cos}_h(x) \equiv \sqrt{\pi}\Gamma\left(1+\frac{x}{h}\right) H_{1,2}^{1,0}\left[{(1+\frac{x}{h},2) \atop (0,1),(\frac{1}{2},1)} \mid \frac{h^2}{4}\right]. \tag{3.54}$$

Of course, is trivial to see that using (3.34), $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, we obtain

$$\widehat{\sin}_h(x) = \frac{\widehat{e^{ix}}_h - \widehat{e^{-ix}}_h}{2i} \tag{3.55}$$

and

$$\widehat{\cos}_h(x) = \frac{\widehat{e^{ix}}_h + \widehat{e^{-ix}}_h}{2}. \tag{3.56}$$

Again the properties of infinitesimal calculus should be satisfied, so we expect a similar relation to the differential relation $(\sin x)' = -\cos x$ holds. Using (3.55) and (3.56) we get

$$D^\Delta \widehat{\sin_h(x)} = -\widehat{\cos_h(x)}. \tag{3.57}$$

(v) *Incomplete gamma function:*

Let $f(x) = x^{z-1}e^{-x}$, then

$$\check{f}_h(\nu) = \int_0^\infty \frac{dt}{\Gamma(\frac{\nu}{h})} e^{-t} t^{\frac{\nu}{h}-1}(-ht)^{z-1}e^{ht} = (-h)^{z-1}\frac{\Gamma(\nu/h + z - 1)}{\Gamma(\nu/h)(1-h)^{\nu/h+z-1}}, \tag{3.58}$$

which gives

$$\hat{f}_h(x) = \frac{\widehat{x^{z-1}}_h \widehat{e^{-x}}_h}{(1-h)^{z-1}}. \tag{3.59}$$

The discrete incomplete gamma function can be defined as

$$\hat{\gamma}_h(z,x) = S^{\Delta}\hat{f}_h(x). \tag{3.60}$$

Using the limit (3.13) we obtain

$$\hat{\gamma}_h(z,x) = \frac{\widehat{x^z}_h \widehat{e^{-x}}_h}{z(1-h)^z}\,{}_2F_1\left(1, z - \frac{x}{h}; 1 + z, -\frac{h}{1-h}\right), \tag{3.61}$$

where ${}_2F_1(\alpha, \beta, \gamma; t)$ is the Gauss hypergeometric function.

Summarizing, observe the table bellow.

Table 1 – Some functions and their respective discrete counterparts.

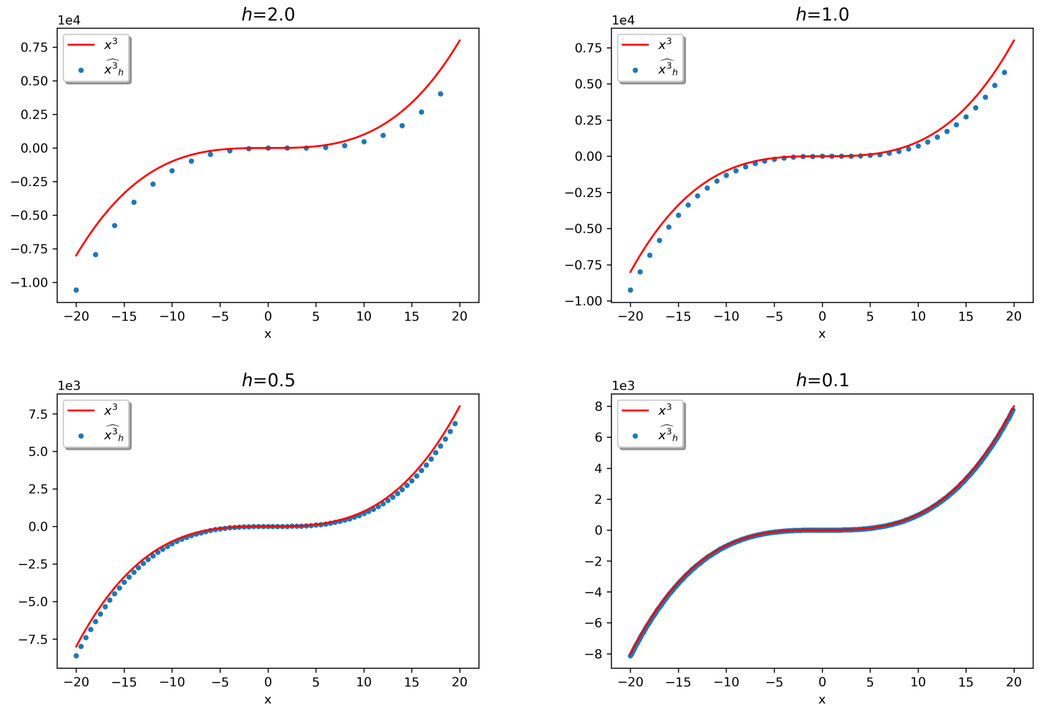
$f(x)$	$\hat{f}_h(x)$
x^n	$\widehat{x^n}_h = h^n \frac{\Gamma(\frac{x}{h}+1)}{\Gamma(\frac{x}{h}+1-n)}$
e^{ax}	$\widehat{e^{ax}}_h = (1+ah)^{x/h}$
$\ln x$	$\widehat{\ln}_h(x) = \ln(-h) - \gamma + \pi\cot\left(\pi\frac{x}{h}\right) + H_{\frac{x}{h}}$
$\sin x$	$\widehat{\sin}_h(x) = \frac{\widehat{e^{ix}}_h - \widehat{e^{-ix}}_h}{2i} = \sqrt{\pi}\Gamma\left(1+\frac{x}{h}\right) H^{1,0}_{1,2}\left[\begin{matrix}(1+\frac{x}{h},2) \\ (\frac{1}{2},1),(0,1)\end{matrix} \middle\vert \frac{h^2}{4}\right]$
$\cos x$	$\widehat{\cos}_h(x) = \frac{\widehat{e^{ix}}_h + \widehat{e^{-ix}}_h}{2} = \sqrt{\pi}\Gamma\left(1+\frac{x}{h}\right) H^{1,0}_{1,2}\left[\begin{matrix}(1+\frac{x}{h},2) \\ (0,1),(\frac{1}{2},1)\end{matrix} \middle\vert \frac{h^2}{4}\right]$
$\gamma(z,x)$	$\hat{\gamma}_h(z,x) = \frac{\widehat{x^z}_h \widehat{e^{-x}}_h}{z(1-h)^z}\,{}_2F_1\left(1, z-\frac{x}{h}; 1+z, -\frac{h}{1-h}\right)$

Source: the author (2021).

We observe that the mimicking feature obtained via the mimetic map appears in a natural way, but for a given specific combination of our discrete operators. It will become more clear when we obtain the expressions for more general discrete operators while we aim to solve differential equations using their respective difference equations, obtained by the mimetic map.

Note that the validity of the continuum limit (3.4) can also be confirmed graphically. See the figure bellow as a particular example.

Figure 2 – The discrete power function $\widehat{x^3}_h$ (blue) approaches to the power function x^3 (red) as h tends to zero. The plot of x^3 uses an increment of size 0.01.



Source: the author (2021).

3.1.3 Discrete Versions of Differential Equations: Difference Equations

As has been observed, there is a clear connection between differential equations and difference equations, and now we see that the mimetic map is able to map those onto each other. That gives a new approach to solve differential equations as well as difference equations.

Let us illustrate the procedure with some classical differential equations. Here, let us refer to the forward difference operator D^{Δ} simply as D.

(i) *Hermite polynomials:*

The Hermite differential equation is given by

$$y'' - 2xy' + 2ny = 0, \tag{3.62}$$

with solution given by the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{3.63}$$

From the map (3.1), definition (3.5) and (3.12), we trivially sees that $\frac{d^2}{dx^2} \to D^2$. Thus we need only to find the map $x\frac{d}{dx} \to E$. Consider the transformation

$$\hat{E}\check{f}_h(\nu) = \int_0^\infty \frac{dt}{\Gamma(\nu/h)} t^{\nu/h-1} e^{-t}(-ht)f'(-ht). \tag{3.64}$$

Integrating by parts

$$\hat{E}\check{f}_h(\nu) = -\int_0^\infty \frac{dt}{\Gamma(\nu/h)} f(-ht)\frac{d}{dt} t^{\nu/h} e^{-t} \tag{3.65}$$

gives

$$\hat{E}\check{f}_h(\nu) = \nu \frac{\check{f}_h(\nu+h) - \check{f}_h(\nu)}{h} = \nu D^{\Delta}(\check{f}_h(\nu)) \tag{3.66}$$

And for the discrete function $\hat{f}_h(x)$,

$$E(\hat{f}_h(x)) = x\frac{\hat{f}_h(x) - \hat{f}_h(x-h)}{h} = xD^{\nabla}(\hat{f}_h(x)), \tag{3.67}$$

from (3.12). Therefore, the difference equation associated with (3.62) is the discrete Hermite equation given by

$$D^2(\hat{f}_h(x)) - 2E(\hat{f}_h(x)) + 2n\hat{f}_h(x) = 0, \tag{3.68}$$

where

$$D^2(\hat{f}_h(x)) = \frac{\hat{f}_h(x+2h) + \hat{f}_h(x) - 2\hat{f}_h(x+h)}{h^2}. \tag{3.69}$$

The solutions are the discrete Hermite polynomials

$$\widehat{H}_n(x)_h = \lim_{\nu\to -x} \int_0^\infty d\lambda_{\nu/h}(t) H_n(-ht). \tag{3.70}$$

Given the definition of the Hermite polynomial, we obviously find $\widehat{H}_0(x)_h = 1$, $\widehat{H}_1(x)_h = x$, $\widehat{H}_2(x)_h = 4\,\widehat{x^2}_h - 2$, and so on.

(ii) *Laguerre polynomials:*

The Laguerre differential equation is

$$xy'' + (1-x)y' + ny = 0, \tag{3.71}$$

with solution given by the Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) = \frac{1}{n!}\left(\frac{d}{dx} - 1\right)^n x^n. \tag{3.72}$$

The corresponding difference equation is

$$H(\hat{f}_h(x)) + (D-E)(\hat{f}_h(x)) + n\hat{f}_h(x) = 0, \tag{3.73}$$

where

$$H(\hat{f}_h(x)) = x\frac{\hat{f}_h(x+h) - 2\hat{f}_h(x) + \hat{f}_h(x-h)}{h^2} = xD^{\Delta}(D^{\nabla}(\hat{f}_h(x))). \tag{3.74}$$

The solution is the discrete Laguerre polynomials, defined by

$$\widehat{L}_n(x)_h = \lim_{\nu \to -x} \int_0^{\infty} d\lambda_{\nu/h}(t) L_n(-ht). \tag{3.75}$$

By the definition of the Laguerre polynomial, we find $\widehat{L}_0(x)_h = 1$, $\widehat{L}_1(x)_h = 1 - x$, $\widehat{L}_2(x)_h = 2 - 4x + \widehat{x^2}_h$ and so on.

(iii) *Confluent hypergeometric function:*

The confluent hypergeometric differential equation is given by

$$xy'' + (c - x)y' - ay = 0. \tag{3.76}$$

The corresponding difference equation is

$$H(\hat{f}_h(x)) + (cD - E)(\hat{f}_h(x)) - a\hat{f}_h(x) = 0. \tag{3.77}$$

The solution is the discrete confluent hypergeometric function defined by

$$\widehat{F}(a; c; x)_h = \lim_{\nu \to -x} \int_0^{\infty} d\lambda_{\nu/h}(t) F(a; c; -ht). \tag{3.78}$$

With the power series solution given by

$$\widehat{F}_h(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{\widehat{x^n}_h}{n!}, \tag{3.79}$$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ is the Pochhammer symbol. A closed form expression can be obtained if we substitute the expression of $\widehat{x^n}_h$, in (3.29). Then we find

$$\widehat{F}_h(a; c; x) = {}_2F_1\left(a, -\frac{x}{h}; c; -h\right). \tag{3.80}$$

Using the definition of the confluent hypergeometric function,

$$F(a; c; z) = \lim_{b \to \infty} {}_2F_1\left(a, b; c; \frac{z}{b}\right), \tag{3.81}$$

we can see that

$$F(a, c; x) = \lim_{h \to 0} \widehat{F}(a, c; x)_h, \tag{3.82}$$

as expected.

Besides, the expression (3.79) also validate the definition of the discrete Taylor series (2.67), which can be obtained from the linearity of the mimetic map.

(iv) *Fibonacci sequence:*

Observe that we can transform the difference equation of the Fibonacci sequence (2.37) in a differential equation if we use $D(\widehat{f}(x)) = \widehat{f}(x+1) - \widehat{f}(x)$, and visualize D in the context of the limit $h \to 0$. Such differential equation is

$$\begin{cases} f''(x) + f'(x) - f(x) = 0; \\ f(0) = 0, \ f'(0) = 1; \end{cases} \tag{3.83a}$$

which have solution

$$f(x) = \frac{e^{\frac{-1+\sqrt{5}}{2}x} - e^{\frac{-1-\sqrt{5}}{2}x}}{\sqrt{5}}. \tag{3.83b}$$

Now, if we use the mimetic map in (3.83), we obtain

$$\hat{f}_h(x+2h) = (2-h)\hat{f}_h(x+h) + (h^2+h-1)\hat{f}_h(x), \tag{3.84a}$$

with solution given by the discrete function

$$\hat{f}_h(x) = \frac{1}{\sqrt{5}}\left[\left(1+\left(\frac{-1+\sqrt{5}}{2}\right)h\right)^{x/h} - \left(1+\left(\frac{-1-\sqrt{5}}{2}\right)h\right)^{x/h}\right]. \tag{3.84b}$$

We recover the Fibonacci sequence (2.40) when $h = 1$ and $x = nh$.

(v) *Hanoi tower:*

Consider the difference equation of the Hanoi sequence (2.41). Using $D(\widehat{f}(x)) = \widehat{f}(x+1) - \widehat{f}(x)$ we associate to it the ODE,

$$\begin{cases} f'(x) - f(x) = 1; \\ f(0) = 0; \end{cases} \tag{3.85a}$$

which have as solution

$$f(x) = e^x - 1. \tag{3.85b}$$

Via mimetic map, we get

$$\hat{f}_h(nh+h) = (1+h)\hat{f}_h(nh) + h. \tag{3.86a}$$

and

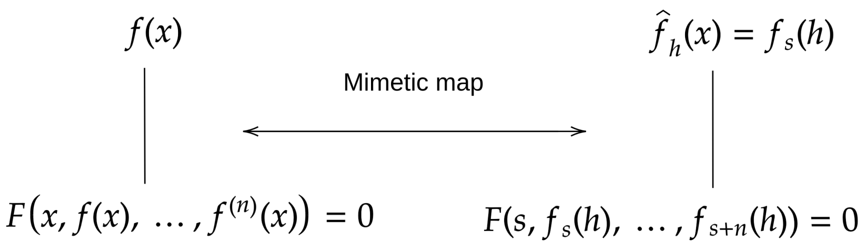
$$\hat{f}_h(x) = (1+h)^{x/h} - 1. \tag{3.86b}$$

Setting $h = 1$ and $x = nh$, we recover the Hanoi sequence

$$\hat{f}_h(x) = \hat{f}(n) = f_n = 2^n - 1. \tag{3.87}$$

So, every time there is a differential equation which have unknown solution, we can map it in a difference equation and solve it using the methods in 2.1.1 or 2.1.2 (or 3.1.4, in general, as will be shown soon). Equally well, we can try to rewrite a difference equation as a differential equation which could be more easily solved (see figure 3 bellow). In this way, the mimetic map can be seen as any other integral transform, e.g., Laplace transform, which can be used to solve differential or difference equations.

Figure 3 – The mimetic map maps differential equations and its solutions onto difference equations and its solutions, in a one-to-one way.



Source: the author (2021).

Moreover, the mimetic map provides the construction of a calculus which will depend of a parameter h that is adaptable to fit the data obtained from an experiment, which usually assumes the continuum assumption. If an experiment gives some data related to some quantities, we can fit these in terms of some value of h and connect those via a difference equation.

For instance, consider the simple example of the movement of a particle with constant velocity. One with a chronometer and a length measure tool can set instants of time separated by a finite increment of time and measure the position of the particle with constant velocity, finding similarly spaced values for it. Such data comes out to be represented by the kinematic relation $x_{i+1} = x_i + v\Delta t$, which is known to describe the observed behavior if x_i is the position of the particle in the instant $t_i = i\Delta t$, v is its velocity and Δt is the time increment. We clearly see it is a difference equation with Δt representing the parameter h, which is directly connected with the differential equation $\frac{dx}{dt} = v$. More elaborate situations, in many areas of knowledge, can be explored where the discrete calculus formulation can be used to set an analytical expression for a data,, but with the advantage of having a parameter to adjust the data.

Further, once the mimetic map provides an exact discrete version of infinitesimal models no information, such as the symmetries, will be lost when the system is discretized. But rather, more information could be obtained once we can look for all cases obtained from values of h instead of $h \to 0$ only.

3.1.4 The h-Generating Function

As discussed in subsection 2.1.2, we can solve difference equations using generating functions. What was discussed was the case of difference equations with step size $h = 1$, satisfied by sequences. Here we generalize the definition of generating function for a h-generating function which applies for difference equations satisfied by discrete functions.

Let $\widehat{f}_h(nh)$ be a discrete function. We define the **h-Generating Function** associated to it by

$$G_h(z) := \sum_{n=0}^{\infty} \widehat{f}_h(nh) \frac{z^n}{n!}, \tag{3.88a}$$

where

$$\widehat{f}_h(nh) = \left. \frac{\partial^n}{\partial z^n} G_h(z) \right|_{z=0}. \tag{3.88b}$$

For $h = 1$ we recover the discussion of subsection 2.1.2. See the examples bellow.

(i) *Discrete power function:*

Consider the discrete power function (3.29) given by

$$P_{n,m}(h) = \widehat{(nh)^m}_h = h^m \frac{\Gamma(n+1)}{\Gamma(n+1-m)} = h^m \frac{n!}{(n-m)!}, \tag{3.89}$$

for $n \geqslant m$.

From the generating function (3.88a), we have

$$G_h(m, z) = \sum_{n=0}^{\infty} \left[h^m \frac{n!}{(n-m)!} \right] \frac{z^n}{n!} = (hz)^m e^z. \tag{3.90}$$

For particular values of m $(= 1, 2, 3, \ldots < n)$ we can conclude that

$$\left. \frac{d^n}{dz^n} \left(z^m e^z \right) \right|_{z=0} = \frac{n!}{(n-m)!}, \tag{3.91}$$

resulting in

$$P_{n,m}(h) = h^m \frac{n!}{(n-m)!}, \tag{3.92}$$

as should be.

(ii) *Discrete exponential function:*

Consider the discrete exponential function 3.34) given by

$$P_n(h) = \widehat{e_h^{a(nh)}} = (1+ah)^n. \tag{3.93}$$

Via (3.88a), we get

$$G_h(z) = e^{(1+ah)z}. \tag{3.94}$$

And, from (3.88b) we recover the expression above.

(iii) *Discrete trigonometric function:*

Consider the discrete sine function (3.52)

$$P_n(h) = \sqrt{\pi} n! H_{1,2}^{1,0} \left[{}^{(n+1,2)}_{\left(\frac{1}{2},1\right),(0,1)} \mid \frac{h^2}{4} \right]. \tag{3.95}$$

Then, the generating function is

$$G_h(z) = \sqrt{\pi} \sum_{n=0}^{\infty} H_{1,2}^{1,0} \left[{}^{(n+1,2)}_{\left(\frac{1}{2},1\right),(0,1)} \mid \frac{h^2}{4} \right] z^n. \tag{3.96}$$

Or, using (3.55) and (3.94), we have

$$G_h(z) = \frac{e^{(1+ih)z} - e^{(1-ih)z}}{2i}. \tag{3.97}$$

From both is trivial to see that (3.52) or (3.55) are recovered.

(iv) *Fibonacci discrete function:*

Consider the Fibonacci discrete function (3.84b) with $x = nh$. From (3.88a) and (3.94), we get

$$G_h(z) = \frac{\exp\left\{\left(1 + \left(\frac{-1+\sqrt{5}}{2}\right) h\right) z\right\} - \exp\left\{\left[1 + \left(\frac{-1-\sqrt{5}}{2}\right) h\right] z\right\}}{\sqrt{5}}, \tag{3.98}$$

which returns (3.84b) via (3.88b). The same can be obtained applying (3.88a) directly in the difference equation (3.84a).

(v) *Hanoi tower:*

Consider the Hanoi discrete function (3.86b) with $x = nh$. Then, the h-generating function will be

$$G_h(z) = e^z(e^{hz} - 1), \tag{3.99}$$

which returns (3.86b). Again, it can be obtained also from (3.86a).

(vi) *Discrete inverse gamma function:*

Consider the function $f(x) = \frac{1}{\Gamma(x+m)}$, where (DENNERY; KRZYWICKI, 1996)

$$\frac{1}{\Gamma(x+m)} = \int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega^{x+m}}. \tag{3.100}$$

Using the mimetic map we have

$$\check{f}_h(\nu) = \int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega^m}\int_0^\infty \frac{dt}{\Gamma(\frac{\nu}{h})} t^{\frac{\nu}{h}-1} e^{-(1-h\ln\omega)t} \tag{3.101}$$

which using the discrete exponential function becomes the discrete inverse gamma function

$$\widehat{\Gamma^{-1}}_h\left(\frac{x}{h}+m\right) \equiv \int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega^m}(1-h\ln\omega)^{x/h}. \tag{3.102}$$

Setting $x = nh$, we have the h-generating function is given by

$$G_h(m,z) = \sum_{n=0}^{\infty}\int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega^m}\frac{[(1-h\ln\omega)z]^n}{n!} = e^z\int_C \frac{d\omega}{2\pi r}\frac{e^\omega}{w^{hz+m}}. \tag{3.103}$$

or

$$G_h(m,z) = \frac{e^z}{\Gamma(hz+m)},\ m\in\mathbb{N}. \tag{3.104}$$

From it we recover the discrete function via

$$P_{n,m}(h) = \frac{d^n}{dz^n}\left(\frac{e^z}{\Gamma(hz+m)}\right)\Bigg|_{z=0} = \widehat{\Gamma^{-1}}_h\left(n+m\right). \tag{3.105}$$

Due the complexity of the expression above, the validation of such approach can be done case by case for some particular values of $x = nh$ and of m.

- $x = 0,\ m\in\mathbb{N}$:

$$P_{0,m}(h) = \frac{1}{\Gamma(m)} = \int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega^m} = \widehat{\Gamma^{-1}}_h(m). \tag{3.106}$$

- $x = h,\ m = 1$: Expanding (3.105), we get

$$P_{1,1}(h) = \left[\frac{e^z(1-h\psi(hz+1))}{\Gamma(hz+1)}\right]\Bigg|_{z=0} = 1-h\psi(1) = 1+h\gamma. \tag{3.107}$$

From (3.102) we observe that

$$\widehat{\Gamma^{-1}}_h(1+1) = \int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega}(1-h\ln\omega) = 1+h\gamma. \tag{3.108}$$

So $P_{1,1}(h) = \widehat{\Gamma^{-1}}_h(1+1)$.

- $x = h$, $m = 2$:
$$P_{1,2}(h) = \left[\frac{e^z(1 - h\psi(hz+2))}{\Gamma(hz+2)}\right]\Bigg|_{z=0} = 1 - h\psi(2) = 1 - h(1-\gamma). \quad (3.109)$$
Again, from (3.102),
$$\widehat{\Gamma^{-1}}_h(1+2) = \int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega^2}(1 - h\ln\omega) = 1 - h(1-\gamma). \quad (3.110)$$
Then $P_{1,2}(h) = \widehat{\Gamma^{-1}}_h(1+2)$.

- $x = 2h$, $m = 1$:
$$\begin{aligned} P_{2,1}(h) = &\left[\frac{e^z[1 - 2h\psi(hz+1) + h^2(\psi(hz+1)^2 - \psi^{(1)}(hz+1))]}{\Gamma(hz+1)}\right]\Bigg|_{z=0} \\ =&1 + 2h\gamma + h^2(\gamma^2 - \frac{\pi^2}{6}). \end{aligned} \quad (3.111)$$
And (3.102) gives
$$\begin{aligned} \widehat{\Gamma^{-1}}_h(2+1) =&1 - h\int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega}\ln(\omega^2) + h^2\int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega}(\ln\omega)^2 \\ =&1 + 2h\gamma + h^2(\gamma^2 - \frac{\pi^2}{6}) \end{aligned} \quad (3.112)$$
Therefore, $P_{2,1}(h) = \widehat{\Gamma^{-1}}_h(2+1)$.

At these calculations, $\psi^{(n)}(z)$ is the Polygamma function defined by (ERDéLYI, 1953)
$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\ln\Gamma(z). \quad (3.113)$$
In particular,
$$\psi(z) = \Gamma(z)\int_C \frac{d\omega}{2\pi i}\frac{e^\omega}{\omega^z}\ln\omega, \quad \psi^{(1)}(z) = \psi(z)^2 - \Gamma(z)\int_C \frac{d\omega}{2\pi i}, \quad (3.114a)$$
$$\psi(1) = -\gamma \quad \text{and} \quad \psi^{(1)}(1) = \frac{\pi^2}{6}, \quad (3.114b)$$
where γ is the Euler-Mascheroni constant. Also, in the last equality of (3.109) was used
$$\psi(z+1) = \psi(z) + \frac{1}{z}. \quad (3.115)$$
Henceforth, we conclude that $P_{n,m}(h) = \widehat{\Gamma^{-1}}_h(n+m)$ is true for $n, m \in \mathbb{N}_0$.

We saw with the examples above that all the structure of sequences can be extended for the discrete functions, with the definition agreeing reasonably well with the approach developed using the mimetic map. In particular we will see in the next subsection that the result presented for the discrete inverse gamma in the case $x = 0$ holds in general, the discrete gamma function is the gamma function itself.

3.1.5 Discrete Integrals Transforms

Once the mimetic map provides a direct connection of the whole structure of the infinitesimal calculus with the one of the discrete calculus, one should expect that we can construct discrete integral transforms involving only discrete functions, allowing the use of discrete versions of usual techniques used to solve differential equations to solve difference equations. In fact, this is what occurs in the following examples, where we obtain discrete versions of the Laplace and Mellin transforms.

(i) *h-Laplace transform and the h-Z transform*

We define the Laplace transform by

$$g(s) = \mathcal{L}\{f(x); s\} := \int_0^\infty dx e^{-sx} f(x), \tag{3.116a}$$

with the inverse Laplace transform

$$f(x) = \mathcal{L}^{-1}\{g(s); x\} = \int_\gamma \frac{ds}{2\pi i} e^{sx} g(s),\ \gamma = (s_0 - i\infty, s_0 + i\infty). \tag{3.116b}$$

Using the mimetic map in (3.116b) we have

$$\hat{f}_h(x) = \int_\gamma \frac{ds}{2\pi i} \widehat{e^{sx}}_h g(s) = \int_\gamma \frac{ds}{2\pi i} (1 + hs)^{x/h} g(s), \tag{3.117}$$

using the discrete exponential function (3.34). With that, we can construct a discrete Laplace transform (BOHNER; GUSEINOV, 2010).

Consider the discrete version of the function $g(s)$, denoted by $\hat{g}_h(s)$, and deform the curve γ to $\Gamma = \{z \in \mathbb{C} \mid |z - h_*| = A\}$, where $A > \frac{R}{h}$, $h_* = -\frac{1}{h}$, and

$$R = \limsup_{k\to\infty} \sqrt[k]{|\hat{f}_h(kh)|}. \tag{3.118}$$

Then, let $\hat{f}_h(kh)$ be the **Inverse h-Laplace Transform** defined by

$$\hat{f}_h(x) := \int_\Gamma \frac{ds}{2\pi i} h^k \left(s - h_*\right)^k \hat{g}_h(s) \tag{3.119}$$

and the **h-Laplace Transform** will be

$$\hat{g}_h(s) := \frac{1}{s - h_*} \sum_{k=0}^\infty \frac{\hat{f}_h(kh)}{h^k (s - h_*)^k}, \tag{3.120}$$

for $s \in \mathbb{C}$, $s \neq h_*$.

We can confirm it substituting (3.120) in the right-hand side of (3.119),

$$\int_\Gamma \frac{ds}{2\pi i} h^k (s-h_*)^k \hat{g}_h(s) = \sum_{j=0}^{\infty} h^{k-j} \hat{f}_h(jh) \int_\Gamma \frac{ds}{2\pi i} \frac{1}{(s-h_*)^{j+1-k}}, \tag{3.121}$$

and using the residue theorem,

$$\int_\Gamma \frac{ds}{2\pi i} \frac{1}{(s-h_*)^{j+1-k}} = \delta_{j,k}, \tag{3.122}$$

to obtain

$$\int_\Gamma \frac{ds}{2\pi i} h^k (s-h_*)^k \hat{g}_h(s) = \hat{f}_h(kh). \tag{3.123}$$

An interesting characteristic of such construction is that the h-Laplace transform is independent of h. That is

$$\widehat{g}_h(z) = g(z),\ \forall h \in \mathbb{R}^+. \tag{3.124}$$

This result follows directly from the construction of the Laplace transform in (BOHNER; GUSEINOV, 2010), where the Laplace transform is the same, only changing the domain of the inverse Laplace transform using $x \in \mathbb{R}$ or $x \in h\mathbb{N}_0$, thus changing the definition from an integral to a sum. We verify it at the example bellow. The same result holds for the construction of the discrete Mellin transform, as we will see.

Example 3.1. Consider $f(x) = x^n$. From (3.116a), its Laplace transform is

$$g(z) = \int_0^\infty e^{-zx} x^n dx = \frac{n!}{z^{n+1}}.$$

And using (3.120) we obtain the same result, i.e.,

$$\hat{g}_h(z) = \sum_{k=n}^{\infty} \frac{h^{n+1}}{(hz+1)^{k+1}} \frac{k!}{(k-n)!} = \sum_{l=0}^{\infty} \frac{h^{n+1}}{(hz+1)^{l+n+1}} \frac{(l+n)!}{l!} = \frac{n!}{z^{n+1}}.$$

Now, consider that following the same reason that a sequence f_n was generalized for a discrete function $\widehat{f}_h(nh)$ and the generating function $G(z)$ was generalized for the h-generating function $G_h(z)$, we can generalize the Z transform $F(z)$, subsection 2.1.2, for a **h-Z transform** defined by

$$F_h(z) = \mathcal{Z}\{\widehat{f}_h(nh); z\} := \sum_{n=0}^{\infty} \frac{\widehat{f}_h(nh)}{z^n} \tag{3.125}$$

with inverse h-Z transform

$$\widehat{f}_h(nh) = \mathcal{Z}^{-1}\{F_h(z); n\} := \oint_C \frac{ds}{2\pi i} F_h(s) s^{n-1}. \tag{3.126}$$

Here the usual Z transform is recovered for the case $h = 1$.

It implies that the h-Z transform is intimately related with the h-Laplace transform by the relation

$$F_h(z) = z\widehat{g}_h(z + h_*). \tag{3.127}$$

Such relation is an extension of the case $h = 1$ presented in (BOHNER; GUSEINOV, 2010).

Thus, not only is possible to obtain a discrete version of the Laplace transform via the mimetic map, but also it can be see as a generalization of the Z transform, the h-Z transform, in such a manner that the h-Laplace becomes the standard Laplace transform in the limit $h \to 0$ and the Z transform when $h = 1$.

(ii) *Mellin Transform*

We define the Mellin transform of the function $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$\tilde{f}(s) = \mathcal{M}\{f(x); s\} = \int_0^\infty x^{s-1} f(x) dx, \tag{3.128}$$

with the inverse Mellin Transform given by

$$f(x) = \mathcal{M}^{-1}\{\tilde{f}(s); x\} = \int_\gamma \frac{ds}{2\pi i} x^{-s} \tilde{f}(s), \tag{3.129}$$

where $\gamma \in (s_0 - i\infty, s_0 + i\infty)$.

Using the mimetic map in the inverse Mellin transform (3.129), we have

$$\hat{f}_h(kh) = \int_\gamma \frac{ds}{2\pi i} \frac{h^{-s} k!}{\Gamma(k+1+s)} \tilde{f}(s) = \int_\gamma \frac{ds}{2\pi i} \widehat{x^{-s}}_h \tilde{f}(s), \tag{3.130}$$

for $x = kh$.

Just as in the previous case, we deform $\tilde{f}(s)$ to its discrete version $\hat{\tilde{f}}_h(s)$. With that, we can state the discrete version of the Mellin transform. Let $\hat{f}_h : h\mathbb{N}_0 \to \mathbb{R}$ be a discrete function. Then the **h-Mellin Transform** is defined by

$$\hat{\tilde{f}}_h(s) := \sum_{k=0}^{\infty} \frac{h}{\widehat{(kh)^{1-s}}_h} \hat{f}_h(kh) = \sum_{k=0}^{\infty} \frac{h^s \Gamma(s+k)}{k!} \hat{f}_h(kh), \tag{3.131}$$

with the **Inverse h-Mellin Transform**

$$\hat{f}_h(kh) := \int_\gamma \frac{ds}{2\pi i} \widehat{(kh)^{-s}}_h \hat{\tilde{f}}_h(s) = k! \int_\gamma \frac{ds}{2\pi i} \cdot \frac{h^{-s} k!}{\Gamma(s+k+1)} \hat{\tilde{f}}_h(s), \tag{3.132}$$

where $\gamma \in (s_0 - i\infty, s_0 + i\infty)$.

To confirm the validity of it, substituting (3.131) in (3.132), we have

$$\hat{f}_h(kh) = k! \sum_{j=0}^{\infty} \frac{\hat{f}_h(jh)}{j!} \int_\gamma \frac{ds}{2\pi i} \frac{\Gamma(s+j)}{\Gamma(s+k+1)} = k! \sum_{j=0}^{\infty} \frac{\hat{f}_h(jh)}{j!} I_{k,j}. \tag{3.133}$$

Here

$$\int_\gamma \frac{ds}{2\pi i} \frac{\Gamma(s+j)}{\Gamma(s+k+1)} = \sum_{n=0}^{k-j} \frac{(-1)^n}{n!} \frac{1}{\Gamma(k+1-j-n)}. \tag{3.134}$$

Therefore,

$$\hat{f}_h(kj) = k! \sum_{j=0}^{\infty} \frac{\hat{f}_h(jh)}{j!} \delta_{j,k}, \tag{3.135}$$

In the same way as the h-Laplace transform, the h-Mellin transform is h independent:

$$\hat{\tilde{f}}_h(z) = \tilde{f}(z),\ \forall h \in \mathbb{R}^+. \tag{3.136}$$

It can be seen from (3.131) and (3.130) giving

$$\hat{\tilde{f}}_h(s) = \sum_{k=0}^{\infty} \frac{h^s \Gamma(s+k)}{k!} \hat{f}_h(kh) = \int_\gamma \frac{dz}{2\pi i} \tilde{f}(z) h^{s-z} \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{\Gamma(k+1+z)}. \tag{3.137}$$

Which can be rewritten as

$$\hat{\tilde{f}}_h(s) = \int_\gamma \frac{dz}{2\pi i} \frac{h^{s-z} \tilde{f}(z)}{(z-s)} \frac{\Gamma(s)}{\Gamma(z)} = \tilde{f}(s). \tag{3.138}$$

We can confirm it in the examples bellow.

Example 3.2. Let $f(x) = e^{-ax}$, for $a > 0$. Its Mellin transform is

$$\tilde{f}(s) = \int_0^\infty x^{s-1} e^{-ax} dx = a^{-s} \Gamma(s).$$

Thus

$$\hat{f}_h(kh) = k! \int_\gamma \frac{ds}{2\pi i} \cdot \frac{\Gamma(s)}{(ah)^s \, \Gamma(s+k+1)}.$$

And since

$$\mathrm{Res}\{\Gamma(s), s = -n\} = \frac{(-1)^n}{n!},$$

we have

$$\hat{f}_h(kh) = \sum_0^k \binom{k}{n} (-ah)^n = (1-ah)^k = \widehat{e^{-ax}}_h.$$

Example 3.3. For the particular case of $f(x) = e^{-x}$, with $a = 1$, we have $\tilde{f}(s) = \Gamma(s)$. Thus if we take $\hat{f}_h(hk) = (1-h)^k$, we have

$$\hat{\Gamma}_h(z) = \sum_{k=0}^{\infty} \frac{h^z \Gamma(z+k)}{k!} (1-h)^k = h^z \Gamma(z) {}_1F_0(z;;1-h).$$

But

$$ {}_1F_0(a;;x) = \frac{1}{(1-x)^a}, $$

which results in

$$ \hat{\Gamma}_h(z) = \Gamma(z). $$

Besides exhibit the characteristic that the discrete version of the gamma function is the gamma function itself, this example validates the result (3.136).

Thus, the systematic procedure conceived by the mimetic map is not only capable to map functions onto discrete functions, differential equations onto difference equations and extend the definition of sequences and the tools applied to it, but also allows us to extend the notion of integral transforms to discrete integral transforms. In such a situation, via the (Laplace or Mellin) transform ((3.116a) or (3.128)) of the function $f(x)$, or via the h-(Laplace or Mellin) transform ((3.120) or (3.131)) of the discrete function $\widehat{f}_h(x)$, we get the same complex function ($g(z)$ or $\tilde{f}(z)$). This is achieved only by discretizing the function $f(x)$. That is, the discrete transforms provide a new way to obtain the (Laplace or Mellin) transform of a function $f(x)$ obtaining its discrete version $\widehat{f}_h(x)$.

3.2 COMPLEX DISCRETE MIMETIC CALCULUS

A possible generalization of our procedure is to present a complex mimetic map, which maps complex functions onto discrete complex functions.

3.2.1 Discrete Calculus in the Discrete Complex Plane

Before we define the complex mimetic map, let us establish some necessary objects to construct the discrete calculus over a complex discrete plane (IZADI, 2018).

For $z = x + iy$, where $x, y \in h\mathbb{Z}$, we define the **Complex Discrete Functions** $\widehat{f}_h : (h\mathbb{Z})^2 \to \mathbb{C}$ by

$$ \widehat{f}_h(z) = \widehat{f}_h(x,y) = \widehat{u}_h(x,y) + i\widehat{v}_h(x,y), \tag{3.139} $$

where

$$ \widehat{u}_h(x,y) = \mathrm{Re}(\widehat{f}_h(z)) \;\; and \;\; \widehat{v}_h(x,y) = \mathrm{Im}(\widehat{f}_h(z)), \tag{3.140} $$

and $(h\mathbb{Z})^2 \equiv h\mathbb{Z} \times h\mathbb{Z}$ denotes the discrete complex plane. The discrete complex function can be seen as a generalization of a double sequence $(f_{n,m})_{(n,m)\in\mathbb{N}_0^2}$ which takes pairs of natural numbers into complex numbers.

Then, in analogy with partial derivatives we define the **Forward Partial Difference Operators** D_x^{Δ} and D_y^{Δ} by

$$D_x^{\Delta}\widehat{f}_h(x,y) = \frac{\widehat{f}_h(x+h,y) - \widehat{f}_h(x,y)}{h} \tag{3.141}$$

and

$$D_y^{\Delta}\widehat{f}_h(x,y) = \frac{\widehat{f}_h(x,y+h) - \widehat{f}_h(x,y)}{h}. \tag{3.142}$$

And the **Backward Difference Operators** D_x^{∇} and D_y^{∇} by

$$D_\nu^{\nabla}\check{f}_h(\nu,\mu) = \frac{\check{f}_h(\nu,\mu) - \check{f}_h(\nu-h,\mu)}{h} \tag{3.143}$$

and

$$D_\mu^{\nabla}\check{f}_h(\nu,\mu) = \frac{\check{f}_h(\nu,\mu) - \check{f}_h(\nu,\mu-h)}{h}. \tag{3.144}$$

If a function $\widehat{f}_h : (h\mathbb{Z})^2 \to \mathbb{C}$ satisfies the **Discrete Cauchy-Riemann Equations**

$$\begin{cases} D_x^{\Delta}\widehat{u}_h(x,y) = D_y^{\Delta}\widehat{v}_h(x,y), \\ D_y^{\Delta}\widehat{u}_h(x,y) = -D_x^{\Delta}\widehat{v}_h(x,y), \end{cases} \tag{3.145}$$

it is called an **Analytic Complex Discrete Function** in the discrete variables x and y.

The **Distance** d between two points $A_k = (x_k, y_k)$, $A_l = (x_l, y_l) \in (h\mathbb{Z})^2$ is defined by

$$d^2 := |A_k - A_l| = (x_k - x_l)^2 + (y_k - y_l)^2.$$

From the above definition and the construction of the discrete complex plane, we can define the **Directed Length** $\overrightarrow{A_{k-1}A_k} \equiv \Delta l_k$ as the quantity connecting two subsequent points which assumes the values $+h$, $-h$, $+ih$ or $-ih$ only. That is, $\sqrt{|A_k - A_{k-1}|} = h$ is the smallest unit of length in the discrete complex plane.

With that, a curve on $(h\mathbb{Z})^2$ can be defined. Suppose that $n+1$ points $\{A_k\}_{k=0}^n$ in $(h\mathbb{Z})^2$ are given such that two subsequent points gives a directed length $\overrightarrow{A_{k-1}A_k}$, $1 \leqslant k \leqslant n$. This set of points is called a **Broken Curve in** $(h\mathbb{Z})^2$ and is denoted by $\vec{\gamma}$. In particular, for some point $(x,y) \in (h\mathbb{Z})^2$, the five consecutive points $[(x,y),(x+h,y),(x+h,y+h),(x,y+h),(x,y)]$ is called a **Simple Closed Curve** in $(h\mathbb{Z})^2$, denoted by $\gamma(x,y)$.

Just as in vectorial calculus, we can define integrals of functions over such curves. Let $\widehat{f}_h : (h\mathbb{Z})^2 \to \mathbb{C}$ be a discrete complex function and $\vec{\gamma} = \{A_k = (x_k, y_k)\}_{k=0}^n$ a broken curve in $(h\mathbb{Z})^2$. Then the **Integral of $\widehat{f}_h$ over $\vec{\gamma}$** is defined by

$$S^\Delta \widehat{f}_h(z) := \sum_{k=0}^{n-1} \widehat{f}_h(x_k + iy_k)\Delta l_k \tag{3.146}$$

and the **Integral of $\widehat{f}_h$ over a simple closed curve** $\gamma(x,y)$ is defined by

$$S^\Delta \widehat{f}_h(\xi,\eta) := h[\widehat{f}_h(x,y) - \widehat{f}_h(x,y+h)] + ih[\widehat{f}_h(x+h,y) - \widehat{f}_h(x,y)]. \tag{3.147}$$

Particularly for $\widehat{f}_h(x,y) = \widehat{u}_h(x,y) + i\widehat{v}_h(x,y)$ we have

$$S^\Delta \widehat{f}_h(\xi,\eta) = h^2\left[-D_y^\Delta \widehat{u}_h(x,y) - D_x^\Delta \widehat{v}_h(x,y)\right] + ih^2\left[-D_y^\Delta \widehat{v}_h(x,y) + D_x^\Delta \widehat{u}_h(x,y)\right]. \tag{3.148}$$

Theorem 3.2.1. Let $\widehat{f}_h(z) = \widehat{u}_h(x,y) + i\widehat{v}_h(x,y)$ be a complex discrete function and $\gamma(x,y)$ be a simple closed curve. $\widehat{f}_h(z)$ is an analytic function if and only if its integral over $\gamma(x,y)$ vanishes.

Proof. It follows from equations (3.145) and (3.148). □

Theorem 3.2.2 (Discrete Cauchy theorem)**.** Let $\widehat{f}_h : (h\mathbb{Z})^2 \to \mathbb{C}$ be an analytic complex discrete function at some region $R \subset (h\mathbb{Z})^2$ bounded by a closed broken curve $\vec{\gamma}$. Then the integral of $\widehat{f}_h$ over $\vec{\gamma}$ is

$$S^\Delta \widehat{f}_h(\xi,\eta) = 0. \tag{3.149}$$

Proof. Denote by $S^\Delta_{\vec{\gamma}} \widehat{f}_h(\xi,\eta)$ the integral of $\widehat{f}_h$ over $\vec{\gamma}$. Suppose we can decompose $\vec{\gamma}$ in a simple closed curve $\gamma(x,y)$, where $(x,y) \in R$, and $\vec{\gamma}' = \vec{\gamma}/\gamma(x,y)$ such that

$$S^\Delta_{\vec{\gamma}} \widehat{f}_h(\xi,\eta) = S^\Delta_{\gamma(x,y)} \widehat{f}_h(\xi,\eta) + S^\Delta_{\vec{\gamma}'} \widehat{f}_h(\xi,\eta). \tag{3.150}$$

By induction, we have

$$S^\Delta_{\vec{\gamma}} \widehat{f}_h(\xi,\eta) = \sum_{(x,y)\in R} S^\Delta_{\gamma(x,y)} \widehat{f}_h(\xi,\eta),$$

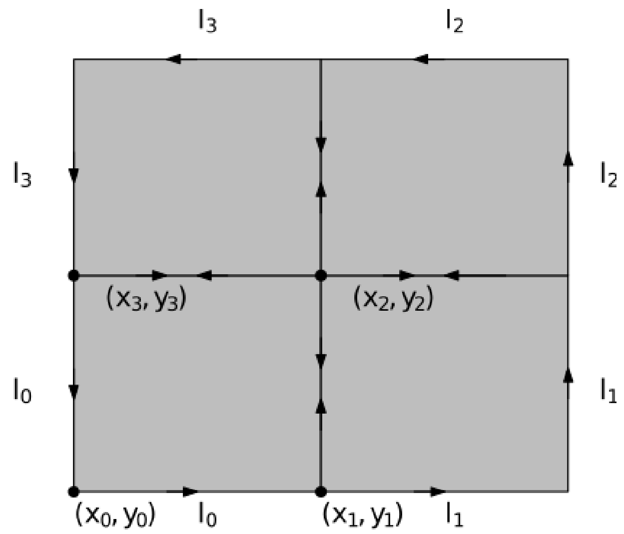
because the edge terms on the integrals over the simple closed curves will cancel each other (see Figure 4 for a particular case).

And from Theorem 3.2.1, $S^\Delta_{\gamma(x,y)} \widehat{f}_h(\xi,\eta) = 0$, $\forall (x,y) \in R$, resulting in

$$S^\Delta_{\vec{\gamma}} \widehat{f}_h(\xi,\eta) = 0. \tag{3.151}$$

□

Figure 4 – In grey, the region R subdivided by the simple closed curves.



Source: the author (2021).

In what follows we present the complex mimetic map defined over the plane $(h\mathbb{Z})^2$. As a consequence, the results above can be recovered while presenting a new way to look towards complex discrete analysis (MERCAT, 2001)(SMIRNOV, 2010).

3.2.2 Complex Mimetic Map

Similarly to what was done for functions defined over $\mathbb{R}$, we can define a complex mimetic map for functions with domain in $\mathbb{C}$. Consider an analytic complex function $f : \mathbb{C} \to \mathbb{C}$ and its respective complex discrete function $\widehat{f_h} : (h\mathbb{Z})^2 \to \mathbb{C}$. The **Complex Mimetic Map** connecting f to $\widehat{f_h}$ is defined by the integral transform

$$\check{f}_h(\nu + i\mu) = \int_0^\infty d\lambda_{\nu/h}(t) \int_0^\infty d\lambda_{\mu/h}(\tau) f(-h(t + i\tau)). \tag{3.152}$$

The complex discrete function is obtained from the limit

$$\hat{f}_h(x + iy) = \lim_{(\nu \to -x,\, \mu \to -y)} \check{f}_h(\nu + i\mu). \tag{3.153}$$

And, as before, the analytic function is recovered from the continuum limit

$$f(x + iy) = \lim_{h \to 0} \hat{f}_h(x + iy). \tag{3.154}$$

3.2.3 Discrete Version of Complex Functions

Here we have some examples validating the complex mimetic map.

(i) *Power function*

Let $f(z) = z^n = (x+iy)^n$. Then from

$$(x+iy)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(iy)^k \tag{3.155}$$

and (3.152) we get

$$\widehat{z^n}_h \equiv \sum_{k=0}^{n} \binom{n}{k} \widehat{x^{n-k}}_h i^k \widehat{y^k}_h. \tag{3.156}$$

(ii) *Exponential function:*

Let $f(z) = e^{az}$, then from the identity $f(x+iy) = e^{ax}e^{iay}$ and (3.152) we get

$$\widehat{e^{az}}_h \equiv (1+ah)^{x/h}(1+iah)^{y/h}. \tag{3.157}$$

(iii) *Power times exponential function:*

Let $f(z) = e^{az}z^n$, then from (3.155) we get

$$\widehat{(e^{az}z^n)}_h \equiv (1+ha)^{x/h}(1+iha)^{y/h} \sum_{k=0}^{n} \binom{n}{k} \frac{\widehat{x^{n-k}}_h i^k \widehat{y^k}_h}{(1+ha)^{n-k}(1+iha)^k}. \tag{3.158}$$

(iv) *Confluent hypergeometric function:*

As an example of application in difference equations, let us find the discrete version of the confluent hypergeometric differential equation and its solution on complex plane. The differential equations is given by

$$zf''(z) + (c-z)f'(z) - af(z) = 0. \tag{3.159}$$

To transform in a difference equation we search for the maps $\frac{d}{dz} \to \tilde{D}$, $z\frac{d}{dz} \to \tilde{E}$ and $z\frac{d^2}{dz^2} \to \tilde{H}$, such that

$$\tilde{H}\hat{f}_h(z) + (c\tilde{D} - \tilde{E})\hat{f}_h(z) - a\hat{f}_h(z) = 0. \tag{3.160}$$

Define

$$\hat{\tilde{D}}\check{f}_h(\nu+i\mu) = \int_0^\infty d\lambda_{\nu/h}(t) \int_0^\infty d\lambda_{\mu/h}(\tau) \frac{1}{2}\left(\frac{\partial}{\partial(-ht)} - i\frac{\partial}{\partial(-h\tau)}\right) f(-h(t+i\tau)), \tag{3.161}$$

$$\hat{\check{E}}\check{f}_h(\nu{+}i\mu) = \int_0^\infty d\lambda_{\nu/h}(t) \int_0^\infty d\lambda_{\mu/h}(\tau)(-h(t{+}i\tau))\frac{1}{2}\left(\frac{\partial}{\partial(-ht)} - \frac{\partial}{\partial(-h\tau)}\right) f(-h(t{+}i\tau)) \tag{3.162}$$

and

$$\hat{\check{H}}\check{f}_h(\nu{+}i\mu) = \int_0^\infty d\lambda_{\nu/h}(t) \int_0^\infty d\lambda_{\mu/h}(\tau)[-h(t{+}i\tau)] \left[\frac{1}{2}\left(\frac{\partial}{\partial(-ht)} - \frac{\partial}{\partial(-h\tau)}\right)\right]^2 f(-h(t{+}i\tau)). \tag{3.163}$$

Integrating by parts and taking the limit (3.153) we obtain

$$\tilde{D}\hat{f}_h(x+iy) = \frac{1}{2}\left(D_x^\Delta - iD_y^\Delta\right)\hat{f}_h(x+iy), \tag{3.164}$$

$$\begin{aligned}\tilde{E}\hat{f}_h(x+iy) = &\frac{1}{2}\left\{xD_x^\nabla + yD_y^\nabla + i\left[yD_x^\Delta - xD_y^\Delta\right]\right\}\hat{f}_h(x+iy) \\ &- \frac{h}{2}i\left[yD_x^\Delta D_y^\nabla - xD_y^\Delta D_x^\nabla\right]\hat{f}_h(x+iy)\end{aligned} \tag{3.165}$$

and

$$\begin{aligned}\tilde{H}\hat{f}_h(x+iy) = &\frac{1}{4}\left[x\left(D_x^2 - D_y^2\right) + 2yD_y^\nabla D_x^\Delta\right]\hat{f}_h(x+iy) \\ &+ \frac{i}{4}\left[y\left(D_x^2 - D_y^2\right) - 2xD_x^\nabla D_y^\Delta\right]\hat{f}_h(x+iy) \\ &- \frac{h}{4}\left[x\left(D_x^2 - D_y^2\right)D_x^\nabla + iy\left(D_x^2 - D_y^2\right)D_y^\nabla\right]\hat{f}_h(x+iy),\end{aligned} \tag{3.166}$$

where $D_x^2 = \left(D_x^\Delta\right)^2$. As before, the power series solution is given by

$$\hat{F}_h(a,c;z) = \sum_{n=0}^\infty \frac{(a)_n}{(c)_n}\frac{\widehat{z^n}_h}{n!}, \tag{3.167}$$

which is easily verified to satisfy (3.160) by numerical methods.

Although not showed explicitly, all complex discrete functions presented here satisfy the discrete Cauchy-Riemann equations in (3.145) as should be, once their infinitesimal counterparts satisfy the standard Cauchy-Riemann equations. Also, one can see from the construction of the complex mimetic map that it can be easily extended for a multivariable function $f(x_1,\ldots,x_n)$. Further, one can also discretize just some of the variables.

4 STOCHASTIC PROCESSES

Randomness is ubiquitous in Nature. The draw of a card from a deck well shuffled, the result of throw an unaddicted die, some genetic feature observed in a individual of a population, the stock price at a stock market, all of those results are seen as random. Such randomness is well described via the Theory of Stochastic Processes, which have Probability Theory as its cornerstone.

We start this chapter discussing the probability theory in section 4.1, where we construct the concepts of probability space, probability distribution and discrete random variables, probability density function and continuous random variables, and the consequences of these. Then we present how the probability theory is used to construct the stochastic processes theory 4.2, presenting the central limit theorem and the Markovian stochastic processes.

After that, we present the formalism to study stochastic processes. The first formalism, used to describe the evolution of discrete stochastic processes, is the one in terms of the Master equation 4.3. The second one is given in terms of the Fokker-Planck equation 4.5, used to describe the evolution of continuous stochastic processes.

The bridge between sections 4.3 and 4.5 is made by section 4.4 presenting how the Master equation and the Fokker-Planck equation formalism are intimately connected by a continuum limit, exemplified with the relation between the random walk and the Brownian motion and the relation, provided by the discrete calculus, relating the Pauli process master equation with the gamma distribution Fokker-Planck equation.

Later, in subsection 4.6 we show the Fokker-Planck equation formalism can be established in a more formal basis using the Stochastic Calculus, also known as Ito Calculus.

4.1 PROBABILITY THEORY

The main objects of Probability Theory are the Random Variables and the Probability Distribution those random variables follows. These and other objects are defined along this section.

4.1.1 Foundations

In the usual sense we may define the probability of some event by the frequency of such event occurs. For example, if we throw a fair coin many times, we expect that roughly half of the total times we get head and the other half we get tails, in a way that we can define the probability of get head or tail equals $\frac{1}{2}$, which is an excellent intuitive approximation. Performing an experiment, such approach agrees very well. However to make it mathematically precise, Andrey N. Kolmogorov axiomatized the concept of probability in his work "Foundation of the Theory of Probability" (KOLMOGOROV, 1950). We define probability here in a different way, although equally precise (APPLEBAUM, 2009).

Let Ω be the **Set of All Possible Outcomes** of an experiment and $\mathcal{F}$ be a collection of subsets of Ω, called **Set of Events**. We say $\mathcal{F}$ is a σ- **algebra** if

(i) $\Omega \in \mathcal{F}$;

(ii) $A \in \mathcal{F} \Rightarrow \bar{A} = \mathcal{F}/A \in \mathcal{F}$;

(iii) If exists a sequence $(A_n)_{n\in\mathbb{N}}$, $A_n \in \mathcal{F}$, such that $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Associated to such a σ-algebra we can define a quantity called **Probability** as a map P : $\mathcal{F} \to [0,1]$ which satisfies

(i) $P(A) \geqslant P(\varnothing) = 0$, $\forall A \in \mathcal{F}$;

(ii) $P\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P\left(A_n\right)$ $(i \neq j)$, where $A_i \cap A_j = \varnothing$, $\forall i, j \in \mathbb{N}$;

(iii) $P(\Omega) = 1$.

Here the space denoted by $(\Omega, \mathcal{F}, P)$ is called the **Probability Space**.

As discussed in (GARDINER, 2004) and (KOLMOGOROV, 1950), what we have here is the generalization of what was pointed out above for a coin toss. If we get n outcomes $\omega \in \Omega$ at random, and happen that $\omega \in A$ m times, we establish a number $\frac{m}{n}$ which is very close to the probability $P(A)$ of the event A occurs. In fact, for a large number n, $P(A) = \frac{m}{n}$. Also, if the probability $P(A)$ is a number arbitrarily small, is almost certain that $\omega \in A$ will not occur if ω is got only once. This point of view is called **Frequency Interpretation**.

The assertions above implicitly assume the existence of quantities $P(A)$ for the probability space constructed. Such probabilities are known as *a priori* probabilities, which are assumed when we define the probability $P : \mathcal{F} \to [0, 1]$. Such point becomes clearer with the example bellow.

Example 4.1. (HASSLER, 2016) Consider the probability space generated by the outcome of a fair die. Then $\Omega = \{1, 2, 3, 4, 5, 6\}$, where the probability of each outcome $\omega \in \Omega$ is $P(\omega) = \frac{1}{6}$ and the probability of an arbitrary event $A \in \mathcal{F}$ is

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{6},$$

where $|A|$ is the number of elements of a given set A. If one want to know the probability to get at least one odd number as outcome, we define the set of events as

$$\mathcal{F} = \{\varnothing, A, \bar{A}, \Omega\},$$

for $A = \{1, 3, 5\}$ and $\bar{A} = \{2, 4, 6\}$. Therefore, the *a priori* probability of the event A is

$$P(A) = \frac{1}{2} = P(\bar{A}).$$

Otherwise, if we are interested in any outcome,

$$\mathcal{F} = \{\varnothing, \{1\}, \ldots, \{6\}, \{1, 2\}, \{1, 3\}, \ldots, \{5, 6\}, \ldots, \Omega\}.$$

Then the *a priori* probabilities are

$$P(\{1, 2\}) = P(\{1, 3\}) = \cdots = P(\{5, 6\}) = \frac{1}{3}, \text{ etc.}$$

Thus, for each set of events $\mathcal{F}$ of interest, we have a specific probability space with specific *a priori* probabilities.

The term "random" here seems to be arbitrary, since as humans we have biases which compels us to make a apparently random choice not random as it should be. We can circumvent this problem, at least in theoretical level, assuming that all outcomes of our experiment are equally possible (see the example bellow). In particular, this notion is useful when we construct the notion of random variables (REICHL; LUSCOMBE, 1998).

Example 4.2. (DURRETT, 2019) Consider a countable (finite or infinite) set of outcomes Ω, with σ-algebra $\mathcal{F}$ constructed with all subsets of Ω. Then, the *a priori* probabilities $P(A)$, $A \in \mathcal{F}$, are

$$P(A) = \sum_{\omega \in A} P(\omega).$$

which gives

$$P(\Omega) = \sum_{\omega \in \Omega} P(\omega) = 1.$$

An educated guess can be $P(\omega) = \frac{1}{|\Omega|}$, if Ω is a finite set.

Correspondingly of what was done for the set of events $\mathcal{F}$, we can define a σ-algebra for the set of real numbers. The **Borel σ-algebra** of the set $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra constructed with all the open intervals of $\mathbb{R}$. An element of a Borel σ-algebra is called a Borel set.

Sometimes we are not interested in events, but rather in values associated with these events. For example, the number associated with the side face-up of a die, the position or velocity of a Brownian particle (subsection 4.4.1.2), the quantity of persons which access some particular website or the number of individuals of a population infected by a disease (chapter 6). To describe those values is necessary to define a new statistical quantity called random variable. Thus, let B be a Borel set. The map $X : \Omega \to \mathbb{R}$ is a **Random Variable** if

$$X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{F}. \tag{4.1}$$

Such property is the $\mathcal{F}$**-measurability** of a probability space and X is said a $\mathcal{F}$**-measurable map**. Even more, any function $f(X)$ of a random variable X is also a random variable (KOLMOGOROV, 1950). It means that the map X takes elements $\omega \in \Omega$ into numbers contained in subintervals of $\mathbb{R}$, such that the image of the inverse map X^{-1} is contained in one of the possible events of $\mathcal{F}$.

With the definition of a σ-algebra over $\mathbb{R}$, we can say that the random variable $X : \Omega \to \mathbb{R}$ induces the probability $P_X : \mathcal{B}(\mathbb{R}) \to [0, 1]$ of the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$, which is known as the **Probability Distribution** of the random variable X, defined by

$$F_X(x) := P_X(X \leqslant x) = P(X^{-1}((-\infty, x])), \text{ for } x \in \mathbb{R}. \tag{4.2}$$

It means that we take a set $B \in \mathcal{B}(\mathbb{R})$ and maps it into $X^{-1}(B) \in \mathcal{F}$, obtaining a value in the interval $[0, 1]$ with the probability P.

We see from the definitions above that the random variable is actually a map between probability spaces (HASSLER, 2016), i.e.,

$$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X).$$

Every time we have a set of n random variables, we can extend the definition above to a random vector $\vec{X} = (X_1, \ldots, X_n)$ in such a way that

$$F_{\vec{X}}(x_1, \ldots, x_n) = P_{\vec{X}}(X_1 \leqslant x_1; \ldots; X_n \leqslant x_n), \tag{4.3}$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. In the following example, we see how a discrete random variable can be used.

Example 4.3. Consider again the set of outcomes of a die, $\Omega = \{1, 2, 3, 4, 5, 6\}$. We define a discrete random variable as $X(\omega) = \omega$, for $\omega = 1, 2, 3, 4, 5, 6$, which follows the multinomial distribution given by

$$P(n_1, n_2, n_3, n_4, n_5, n_6) = P(\{n_i\}) = \frac{n!}{\prod_{i=1}^{6} n_i!} \prod_{i=1}^{6} p_i^{n_i},$$

for $n = \sum_{i=1}^{6} n_i$ and $p_i = 1/6$.

Such distribution gives the probability of obtain n_1 times $X = 1$, n_2 times $X = 2$, n_3 times $X = 3$, n_4 times $X = 4$, n_5 times $X = 5$, n_6 times $X = 6$ if we observe the value of X n times.

In particular, if we have $n_3 = 1$ and $n_1 = n_2 = n_4 = n_5 = n_6 = 0$, we get a probability of $\frac{1}{6}$, as expected. The same occurs for the other cases $n_i = 1$.

The definition constructed above have a particular case where there is no need to define a set of outcomes Ω and probability P. If we assume that $F_X(x)$ is a differentiable function, we can define it in terms of another function $\mathcal{P}_X(x)$, or simply $\mathcal{P}(x)$, known as the **Probability Density Function** (PDF) of X, defined by

$$F_X(x) := \int_{-\infty}^{x} \mathcal{P}(x')dx', \tag{4.4a}$$

such that

$$\mathcal{P}(x) \geqslant 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \mathcal{P}(x)dx = 1. \tag{4.4b}$$

The random variable which follows such probability distribution is a **Continuous Random Variable**.

An example of continuous random variable are the stock price on the stock exchange or the position or velocity of a Brownian particle. And as an example of PDF, the PDF used to describe the Brownian motion is the Gaussian distribution defined by

$$\mathcal{P}(x) = \frac{e^{-\frac{(x-\langle X\rangle)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}. \tag{4.5}$$

A particular case is the Normal distribution, which occurs when $\langle x \rangle = 0$ and $\sigma = 1$ (subsection 4.1.1.2). Such a probability distribution is of great relevance once it is used to describe a wide range of stochastic processes as will be shown in subsection 4.2.2.

Also, if we have a discrete random variable assuming values $n \in \mathbb{Z}$, with probability $P(X \leqslant n) \equiv p_n$, we also can express this in terms of a PDF by

$$\mathcal{P}(x) = \sum_{n=-\infty}^{\infty} p_n \delta(x - n), \tag{4.6}$$

where $\delta(x - x_0)$ is the Dirac delta function.

4.1.1.1 Joint and Conditional Probabilities

An important notion in probability theory, and particularly in stochastic processes, are the joint and conditional probabilities. Consider two possible events A and B and assume that there is some outcomes ω on both events, i.e., $\omega \in A \cap B$. We say that exists a **Joint Probability** given by

$$P(A \cap B) = Prob\{\omega \in A \text{ and } \omega \in B\}. \tag{4.7}$$

In particular, if $\omega \in B$ does not affect the probability that event A occurs, $P(A)$, and if $\omega \in A$ does not affect the probability that event B occurs, $P(B)$, we say that those events are independent from each other which is denoted by

$$P(A \cap B) = P(A)P(B). \tag{4.8}$$

Let us consider that given $\omega \in B$ exist a non-null probability that $\omega \in A$. In other words, we have a non-null probability that $\omega \in A$ will occur given that $\omega \in B$ is known to had occurred. It describe the **Conditional Probability**, defined by

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}. \tag{4.9}$$

That will be of great relevance in the formulation of stochastic processes formulation.

Another important probability relation is the law of total probability. Consider a collection of disjoint events A_i, i.e., $A_i \cap A_j = \varnothing$, $\forall i \neq j$ such that $\cup_i A_i = \Omega$. Then we define the **Law of Total Probability** by

$$\sum_i P(A \mid A_i)P(A_i) = P(A). \tag{4.10}$$

For a random vector $\vec{X} = (X_1, \ldots, X_n)$, $X_i : \Omega \to \mathbb{R}$, the joint probability can be rewritten as

$$P(x_1; \ldots; x_n) \equiv Prob(X_1(\omega_1) \leqslant x_1, \ldots, X_n(\omega_n) \leqslant x_n), \tag{4.11}$$

with independence between random variables defined by

$$P(x_1; \ldots; x_n) = Prob(X_1(\omega_1) \leqslant x_1) \ldots Prob(X_n(\omega_n) \leqslant x_n). \tag{4.12}$$

And the conditional probability will be

$$P(x_{s+1}; \ldots; x_n \mid x_1; \ldots; x_s) = \frac{P(x_1; \ldots; x_n)}{P(x_1; \ldots; x_s)}, \tag{4.13}$$

for some $s \leqslant n$, which now gives the probability of $X_{s+1}(\omega_{s+1}) \leqslant x_{s+1}, \ldots, X_n(\omega_n) \leqslant x_n$ occur given that $X_1(\omega_1) \leqslant x_1, \ldots, X_s(\omega_s) \leqslant x_s$ had occurred.

4.1.1.2 Mean Value and Moments of a Random Variable

Let $f(X)$ be a function of a random variable $X : \Omega \to \mathbb{R}$ defined over a countable space of outcomes. Then we define the **Mean Value** $\langle f(X) \rangle$ by

$$\langle f(X) \rangle = \sum_{\omega \in \Omega} P(\omega) f(X(\omega)). \tag{4.14}$$

In particular, we obtain the **kth Moment of X** given by

$$\left\langle X^k \right\rangle = \sum_{\omega \in \Omega} P(\omega) X(\omega)^k. \tag{4.15}$$

With the first two moments, $\langle X \rangle$ and $\langle X^2 \rangle$, we can define a important quantity called **Variance** given by

$$\sigma_X^2 := \left\langle (X - \langle X \rangle)^2 \right\rangle = \left\langle X^2 \right\rangle - \langle X \rangle^2 , \tag{4.16}$$

which measures the mean-square of the deviation of a random variable from its mean. σ_X is known as the **Mean-Square Root** (or **Standard Deviation**) of the random variable X.

Of course, if we have a random vector $\vec{X} = (X_1, \ldots, X_n)$, $X_i : \Omega \to \mathbb{R}$, we can define the kth moments as

$$\left\langle \prod_{i=1}^{n} X_i^{k_i} \right\rangle = \sum_{\omega_1 \in \Omega} \ldots \sum_{\omega_n \in \Omega} P(\omega_1; \ldots; \omega_n) \prod_{i=1}^{n} X_i(\omega_i)^{k_i}. \tag{4.17}$$

As well, the variance can be generalized to a quantity called **Covariance Matrix** given by

$$\langle X_i, X_j \rangle := \langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \rangle = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle , \tag{4.18}$$

where the diagonal elements $\langle X_i, X_i \rangle = \sigma^2_{X_i} \equiv \sigma^2_i$ are the variance of each of the random variable X_i and the off diagonal elements are called the **Covariance** between X_i and X_j. If the random variables are independent the covariance vanishes.

More precisely, the covariance measures if X_i and X_j are linearly dependent to each other. To make it more clear we define a quantity called **Correlation coefficient** as

$$C_{i,j} := \frac{\langle X_i, X_j \rangle}{\sigma_i \sigma_j}, \tag{4.19}$$

which is independent of the units of the random variables. When two random variables are independent $C_{i,j} = 0$ they are said uncorrelated to each other. When the random variables follow the same probability distribution or are linear to each other, the variables are completely correlated and $C_{i,j} = C_{i,i} = 1$.

4.2 STOCHASTIC PROCESSES

We can precisely define a **Stochastic Process** as a family of random variables $\{X_t(\omega)\}_{t \in T}$, where $T \subset \mathbb{R}$ and $\omega \in \Omega$. If $T = \mathbb{Z}$, we have a stochastic process with discrete parameter. If $T = \mathbb{R}$, we have a stochastic process with continuous parameter. In almost all cases of interest, the parameter t represents time, thus we presume $T = \{0, 1, 2, \ldots\}$ for **Discrete Time Stochastic Processes** and $T = [0, \infty)$ for **Continuous Time Stochastic Processes**.

A stochastic process is classified according to the type of random variable and time, as above. We say a stochastic process is discrete or continuous if it is defined with discrete or continuous random variables.

As expressed above, a stochastic process is a map of two parameters, time and an outcome. For a fixed outcome ω, the quantity $X_t \equiv X_t(\omega)$ denotes a **Realization** (or **Trajectory**) of a stochastic process.

Considering a stochastic process defined in an interval $[t_i, t_f]$ such that

$$t_i \equiv t_1 < t_2 < \cdots < t_n \equiv t_f, \tag{4.20}$$

a given trajectory $\{X_t\}_{t \in [t_i, t_f]}$ with $X_{t_j} \equiv X_j = x_j$ means that the random variable X gives the value x_j when measured at instant t_j. With that we can represent the joint probability of this trajectory as

$$P(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n) = \int_{-\infty}^{x_1} dx'_1 \cdots \int_{-\infty}^{x_n} dx'_n \mathcal{P}(x'_1, t_1 \cdots ; x'_n, t_n), \tag{4.21}$$

where $\mathcal{P}(x_1,t_1;\cdots;x_n,t_n)$ is the joint probability density function. In particular, if the values $X_j = X_{t_j}(\omega) = x_j$ are independent

$$\mathcal{P}(x_1,t_1;\cdots;x_n,t_n) = \prod_{j=1}^{n} \mathcal{P}(x_j,t_j). \tag{4.22}$$

For some $s < n$, we can define the **Marginal Probability** of the trajectory as

$$\mathcal{P}(x_1,t_1;\cdots;x_s,t_s) = \int_{-\infty}^{\infty} dx_{s+1}\cdots\int_{-\infty}^{\infty} dx_n \mathcal{P}(x_1,t_1;x_2,t_2;\cdots;x_s,t_s;x_{s+1},t_{s+1};\ldots;x_n,t_n). \tag{4.23}$$

In the same way, we represent the conditional probability density of the last $n-s$ values of the trajectory X_t occur given that the first s occurred by

$$\mathcal{P}(x_{s+1},t_{s+1};\cdots x_n,t_n \mid x_1,t_1;\cdots;x_s,t_s) = \frac{\mathcal{P}(x_1,t_1;\cdots x_n,t_n)}{\mathcal{P}(x_1,t_1;\cdots;x_s,t_s)}. \tag{4.24}$$

With the definition above and the marginal probability, the law of total probability becomes

$$\mathcal{P}(x_2,t_2) = \int \mathcal{P}(x_2,t_2 \mid x_1,t_1)\mathcal{P}(x_1,t_1)dx_1. \tag{4.25}$$

In the case of a continuous random variable, we define the mean value in terms of the probability density function and the kth moment as

$$\langle f(X)\rangle = \int_{-\infty}^{\infty} \mathcal{P}(x)f(x)dx \tag{4.26}$$

and

$$\left\langle X^k \right\rangle = \int_{-\infty}^{\infty} \mathcal{P}(x)x^k dx. \tag{4.27}$$

Observe that if we treat the random variables $X_1,\ldots,X_n$ not for fixed ω but for fixed instants of time t_j $(j=1,\ldots,n)$, we have now n trajectories, one for each t_j. Therefore we can present the kth moment by

$$\left\langle \prod_{i=1}^{n} X_i^{k_i} \right\rangle = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \mathcal{P}(x_1,\ldots,x_n)x_i^{k_i} d^n x. \tag{4.28}$$

4.2.1 Characteristic and Generating functions

A mathematical quantity of great usefulness to calculate the moments is the **Characteristic Function** of a random variable X, defined by the Fourier transform of the probability distribution function $\mathcal{P}(x)$ by

$$\Phi_X(k) := \int_{-\infty}^{\infty} \mathcal{P}(x)e^{ikx}dx = \left\langle e^{ikx} \right\rangle, \tag{4.29}$$

from where we recover the probability distribution via

$$\mathcal{P}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(k)e^{-ikx}dk. \tag{4.30}$$

The characteristic function satisfies

$$\Phi_X(0) = 1,\ |\Phi_X(k)| \leqslant 1 \text{ and } \Phi_X(-k) = \Phi_X^*(k). \tag{4.31}$$

Observe that expanding the exponential in (4.29), we get

$$\Phi_X(k) = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\langle X^n\rangle . \tag{4.32}$$

Thus, the characteristic function can be actually seen as a **Moment Generating Function** for the nth moment of the random variable X,

$$\langle X^n\rangle = \frac{1}{i^n}\frac{d^n}{dk^n}\Phi_X(k)\bigg|_{k=0}. \tag{4.33}$$

As done previously, we can generalized for a random vector of n components. Then we define the characteristic function as

$$\Phi_{\vec{X}}(\vec{k},n) = \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{x}}\mathcal{P}(\vec{x})d^n x \tag{4.34}$$

with

$$\mathcal{P}(\vec{x}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} \Phi_X(k)e^{-i\vec{k}\cdot\vec{x}}d^n k, \tag{4.35}$$

for $\vec{x} = (x_1,\dots,x_n)$, $\vec{k} = (k_1,\dots,k_n)$, $d^n x = dx_1\dots dx_n$ and $d^n k = dk_1\dots dk_n$. If the components of the random vector are independent to each other, using (4.22), we have

$$\Phi_{\vec{X}}(\vec{k},n) = \prod_{i=1}^{n}\Phi(k_i). \tag{4.36}$$

Particularly, if all random variables follows the same distribution,

$$\Phi(\vec{k}) = (\Phi(k))^n. \tag{4.37}$$

In addition, consider a sum of independent random variables given by $Y = X_1 + \cdots + X_n$. Since the characteristic function is a Fourier transform, from the convolution theorem, the characteristic function becomes

$$\Phi_Y(k,n) = \prod_{i=1}^{n}\Phi_i(k), \tag{4.38}$$

for $\Phi_i(k) \equiv \Phi_{X_i}(k)$. If the independent random variables follows the same probability distribution, we have

$$\Phi_Y(k,n) = (\Phi(k))^n. \tag{4.39}$$

Another quantity that will be relevant later is the **Averaged Characteristic Function** defined by

$$\chi_Y(k,t) := \sum_{n=-\infty}^{\infty} P_n(t)\Phi_Y(k,n;t) \equiv \langle \Phi_Y(k,n;t)\rangle . \tag{4.40}$$

Another way to express this quantity is (MACêDO, 2020)

$$\chi_Y(k) = 2\int_0^{\infty} \cos(ky)\mathcal{P}(y)dy \tag{4.41}$$

returning the PDF by

$$\mathcal{P}_Y(y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky}\chi_Y(k). \tag{4.42}$$

Such a quantity is used to generalize the central limit theorem to include other correlations (APPLEBAUM, 2009), apart of the one defined in (4.19). It will be useful when we explore the distributions which satisfies the central limit theorem.

Just as appears in section 2.1.2, we can define the **Probability Generating Function** by

$$F(z,t) := \sum_{n=0}^{\infty} P_n(t) z^n;\ z \in \mathbb{C} \tag{4.43}$$

with the probability distribution given by

$$P_n(t) = \oint \frac{dz}{2\pi i}\frac{F(z)}{z^{n+1}} = \frac{1}{n!}\frac{d^n}{dz^n}F(z,t)\bigg|_{z=0}. \tag{4.44}$$

By that, we can see that the probability distribution of a discrete random variable can be seen as a sequence. Such quantity allow us to obtain the mean $\langle n \rangle$ of the discrete stochastic variable n by

$$\langle n \rangle_t = -\frac{d}{dz}F(1-z,t)\bigg|_{z=0}. \tag{4.45}$$

We observe here that the form of (4.43) is very similar to the (4.40) just changing z by $\Phi(k)$, if (4.39) is considered. This analogy will be the main point in section 5.3 to construct a discrete H theory.

4.2.2 Central Limit Theorem

In many aspects of statistics, the Gaussian distribution (4.5) is present, which can be explained by the following theorem:

Theorem 4.2.1 (Central Limit Theorem (CLT))**.** Let $X_1, \ldots, X_n$ be n independent random variables with the same probability distribution and finite variance $\sigma^2_{X_i} = \sigma^2_X < \infty$. Define the random variable

$$Y = \frac{\sum_{i=1}^{n} X_i - \langle X_i \rangle}{\sqrt{n}}. \tag{4.46}$$

Then, the probability distribution of Y is a Gaussian distribution with mean $\langle Y \rangle = 0$ and variance σ_X, as n goes to infinity.

Proof. Consider the random variable $Z_i = \frac{X_i - \langle X_i \rangle}{\sqrt{n}}$. Its characteristic function is

$$\Phi_Z(k, n) = \int e^{i\frac{k}{\sqrt{n}}(x - \langle x \rangle)} \mathcal{P}_X(x), \tag{4.47}$$

which up to second order is given by

$$\Phi_Z(k, n) \approx 1 - \frac{k^2}{2n}\sigma^2_X. \tag{4.48}$$

Then from (4.39), taking the limit $n \to \infty$, we get

$$\Phi_Y(k, n) \approx \left(1 - \frac{k^2}{2n}\sigma^2_X\right)^n, \tag{4.49}$$

and so

$$\Phi_Y(k) = \lim_{n \to \infty} \Phi_Y(k, n) = e^{-\frac{k^2}{2}\sigma^2_X}. \tag{4.50}$$

Thus,

$$\mathcal{P}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_Y(k) e^{-iky} dk = \frac{e^{-\frac{y^2}{2\sigma^2_X}}}{\sqrt{2\pi\sigma^2_X}}. \tag{4.51}$$

□

The theorem above explains, for example, why the distribution of the errors in an experiment is a Gaussian distribution centered in the mean value of the measured quantity. Although a variety of situations in Nature confirm the validity of the CLT, also many situations exist where it is not valid. See the following examples.

Example 4.4. Poisson distribution:

Consider the probability distribution

$$P_N = \frac{\mu^N}{N!} e^{-\mu},$$

with mean and variance $\langle N \rangle = \sigma^2 \equiv \mu$. For n random variables $\{N_i\}_{i=1}^{n}$ following this probability distribution, we have $Z_i = \frac{N_i - \mu}{\sqrt{n}}$ with characteristic function

$$\Phi_Z(k, n) = \exp\left[\mu\left(e^{\frac{ik}{\sqrt{n}}} - 1 - \frac{ik}{\sqrt{n}}\right)\right] \approx 1 - \frac{k^2}{2n}\sigma^2,$$

assuming the limit $n \to \infty$, resulting in the Gaussian distribution for the random variable $Y = \sum_{i=1}^{n} Z_i$.

In the same way we can confirm the result via the averaged characteristic function (4.40)

$$\chi_Y(k) = e^{-\mu} \sum_{n=0}^{\infty} \frac{(\mu \Phi_Z(k))^n}{n!} = e^{\mu(\Phi_Z(k)-1)}.$$

Rewriting $Z = \frac{N_i - \mu}{\sqrt{\mu}}$, we have $\Phi_Z(k) = 1 - \frac{k^2}{2\mu}\sigma^2$ and

$$\chi_Y(k) = e^{-\frac{k^2}{2}\sigma^2}.$$

So, if $\chi_Y(k)$ is a Gaussian, the CLT is valid.

Example 4.5. Pauli process:

Consider the probability distribution

$$P_N = (1-u)u^N;\ 0 < u < 1$$

whith mean $\langle N \rangle = \frac{u}{1-u}$ and variance $\sigma^2 = \frac{u}{1-u}\left(1 + \frac{u}{1-u}\right)$. As u gets closer to 1, the variance diverges, violating the CLT. To see that, the averaged characteristic function for Y will be

$$\chi_Y(k) = \frac{1-u}{1-u\Phi_Z(k)} = \frac{1}{1 - \langle N \rangle (\Phi_Z(k) - 1)}.$$

Substituting $\Phi_Z(k) = 1 - \frac{k^2}{2\langle N \rangle}\sigma_X^2$, we get

$$\chi_Y(k) = \frac{1}{1 + \frac{k^2}{2}\sigma_X^2}.$$

From the second example above, we conclude there is distributions which violates the CLT. We will explore it in chapter 5, when we construct the discrete H theory.

4.2.3 Markovian Stochastic Processes

If a stochastic process have a conditional probability for the value x_{s+1} occurs at time t_{s+1} only depending on the value x_s at time t_s, it is called a **Markov Process**. It means that the evolution of the system does not retain any information of the instants t_j, $j < s$. Such property, known as **Markov assumption**, can be mathematically translated by the conditional probability density expression

$$\mathcal{P}(x_{s+1}, t_{s+1} \mid x_1, t_1; \ldots; x_s, t_s) = \mathcal{P}(x_{s+1}, t_{s+1} \mid x_s, t_s). \tag{4.52}$$

In this context, the conditional probability above receives the name of **Transition Probability**, since it gives the probability of a transition from one state s to a state $s+1$ in a time interval $\tau = t_{s+1} - t_s$.

An important property that such stochastic process gives is, from (4.24),

$$\mathcal{P}(x_1, t_1; \ldots; x_n; t_n) = \prod_{i=1}^{n-1} \mathcal{P}(x_{i+1}, t_{i+1} \mid x_i, t_i)\mathcal{P}(x_1, t_1). \tag{4.53}$$

Integrating it in $x_2, \ldots, x_{n-1}$, we have the **Chapman-Kolmogorov equation**

$$\mathcal{P}(x_n, t_n \mid x_1, t_1) = \mathcal{P}(x_1, t_1) \int_{-\infty}^{\infty} \prod_{i=1}^{n-1} \mathcal{P}(x_{i+1}, t_{i+1} \mid x_i, t_i) dx_2 \ldots dx_{n-1}. \tag{4.54}$$

A special class of Markov process is the **Markov Chain**, which is characterized by a discrete stochastic process in which Markov assumption holds. Here we present all definitions in terms of continuous random variables, but all previous equations can be translated to discrete random variables just using the relation in (4.6). As will be discussed in the section about stochastic epidemic models in chapter 6, the Markov chain can be defined with discrete or continuous time.

4.3 MASTER EQUATION

Consider a discrete Markov process described by the law of total probability

$$P(n, t+\tau) = \sum_{m=0}^{\infty} P(n, t+\tau \mid m, t)P(m, t) \equiv \sum_{m=0}^{\infty} P(m, t)Q_{mn}(t; \tau), \tag{4.55}$$

where the $Q_{mn}(t;\tau)$ is the transition probability from state m to state n. Here we assume that the transition occurs in a interval τ such that $\tau \ll t$.

Expanding $Q_{mn}(t;\tau)$ up to first order in τ, we have the general expression

$$Q_{mn}(t;\tau) \approx \delta_{nm} \left(1 - \sum_{k=0}^{L} w_{mn}(t)\right) + \tau w_{mn}(t), \tag{4.56}$$

where

$$w_{mn}(t) \equiv \left. \frac{\partial P(n, \bar{t} \mid m, t)}{\partial \bar{t}} \right|_{\bar{t}=t} \geqslant 0$$

is the **Transition Rate** from state m to state n. Here was assumed the initial condition of the transition as

$$P(n, t \mid m, t) = \delta_{n,m}.$$

The term $\tau w_{mn}(t)$ is the probability to occur the transition m to n and $\left(1-\sum_{k=1}^{L} w_{mn}(t)\right)$ is the probability of such transition not occur.

Substituting Q_{mn} in (4.55) and taking $\tau \to 0$, we have

$$\lim_{\tau\to 0}\frac{P(n,t+\tau)-P(n,t)}{\tau} = \dot{P}(n,t) = \sum_{m}\left(w_{mn}(t)P(m,t) - w_{nm}(t)P(n,t)\right). \tag{4.57}$$

Thus, setting $P(n,t) \equiv P_n(t)$ we obtain the **Master equation** given by

$$\dot{P}_n(t) = \sum_{m}^{L}\left(w_{mn}(t)P_m(t) - w_{nm}(t)P_n(t)\right), \tag{4.58}$$

where L is an integer number in the domain of n.

The master equation have the name of **Differential Chapman-Kolmogorov equation** when written in terms of conditional probabilities, which is given by

$$\frac{\partial P(n,t \mid m,t')}{\partial t} = \sum_{l=0}^{L}\left(w_{ln}P(m,t' \mid l,t'') - w_{nl}P(n,t \mid m,t')\right). \tag{4.59}$$

In reality, the master equation in the form of (4.55) can be seen as a differential-difference equation with variable coefficients, just comparing the expression above with (2.5). Particularly, after a long time the process considered reach an **equilibrium (stationary) state**, where the time derivative on the left-hand side of (4.58) vanishes, giving the equilibrium probability distribution

$$\lim_{t\to\infty} P_n(t) = P_n^{eq}. \tag{4.60}$$

For the conditional probability of a Markov chain we say it is time independent if $P(x_{s+1}, t_{s+1} \mid x_s, t_s) = P(x_{s+1} \mid x_s)$, $\forall s$. In such case becomes clear that what we have is a difference equation, with solution given by a sequence $(P_n^{eq})_{n\in\mathbb{Z}}$.

That will be the motivation to apply the mimetic map in Fokker-Planck equations to map it into master equations in section 5.1. Because of this, the methods used to find the solution of difference equations can be applied to find the equilibrium solution of a master equation. The out of equilibrium solution (time-dependent) can be obtained by many methods, some of them found in (HAAG, 2017). One of those methods is exemplified in section 4.4, used to solve the random walk master equation.

4.3.1 Detailed Balance

From the equilibrium condition above, we see that the master equation (4.58) gives the identity

$$\sum_m w_{mn} P_m^{eq} = \sum_m w_{nm} P_n^{eq}, \tag{4.61}$$

or simply

$$w_{mn} P_m^{eq} = w_{nm} P_n^{eq}, \tag{4.62}$$

which is known as **Detailed Balance Relation**.

The detailed balance can be justified as follows. When the process reaches the equilibrium, one should expect that there is no preferred state, which is just the ergodic hypothesis of the Boltzmann-Gibbs statistical mechanics (SALINAS, 1997). So the joint probability of two events should be the same, whatever was the event that occurred first. Thus consider a process which in one case it is in state m at instant t and in state n at instant $\bar{t}$, and in another it happens to be in state n at instant t and in state m at instant $\bar{t}$, for $\bar{t} \geqslant t$. In the equilibrium regime, we have

$$P^{eq}(m, \bar{t}; n, t) = P^{eq}(n, \bar{t}; m, t). \tag{4.63}$$

From the definition of conditional probability, deriving the expression above in $\bar{t}$ and setting $\bar{t} = t$, we got (4.62). A more physically verification of detailed balance can be found in (KAMPEN, 2007), using the parity invariance property of the Hamilton equations.

4.3.2 Birth and Death Processes

The master equation (4.58) can be rewritten in terms of new transition probabilities $W_{nm}(t)$ defined by

$$W_{nm}(t) = w_{mn} - \delta_{nm} \sum_{k=1}^{L} w_{nk}, \tag{4.64}$$

giving the more compact expression

$$\dot{P}_n(t) = \sum_{m=1}^{L} W_{nm}(t) P_m(t), \tag{4.65}$$

or in matrix notation,

$$\dot{\vec{P}}(t) = W \vec{P}(t). \tag{4.66}$$

Here $W(t)$ is a $L \times L$ matrix called **Transition Matrix** which satisfies

$$W_{nm} \geqslant 0, \text{ if } n \neq m, \tag{4.67a}$$

and

$$\sum_{n=1}^{L} W_{nm} = 0, \text{ for each } m. \tag{4.67b}$$

Here $\vec{P}(t) = (P_1(t), \ldots, P_L(t))^T$ is a column vector. The usual method to solve (4.66) consists in find the eigenvectors and eigenvalues by left and by right of W, which is extensively discussed in (REICHL; LUSCOMBE, 1998).

Let us consider stochastic processes with only transitions between subsequent states. That is, only transitions

$$n \to n-1,\ n \to n,\ n \to n+1,$$

are allowed. It can be imposed in (4.65) defining the transition rates as

$$W_{nm}(t) = \zeta_m(t)\delta_{n,m-1} + \gamma_m(t)\delta_{n,m+1} - (\zeta_m(t) + \gamma_m(t))\delta_{n,m}, \tag{4.68}$$

where $\zeta_m, \gamma_m \geqslant 0$. Which results in the master equation of **Birth-Death Processes**

$$\dot{P}_n(t) = \zeta_{n+1}(t)P_{n+1}(t) + \gamma_{n-1}(t)P_{n-1}(t) - (\zeta_n(t) + \gamma_n(t))P_n(t). \tag{4.69}$$

The birth-death master equations can be classified according to the shape of the transition factors ζ_n and γ_n, which can be constant, linear or nonlinear in n. In most cases of interest, those are time-independent, and linear, namely,

$$\zeta_n = a(n+r) \tag{4.70a}$$

and

$$\gamma_n = b(n+g). \tag{4.70b}$$

Examples of birth-death processes can be found in the rest of the dissertation.

4.4 FROM DISCRETE TO CONTINUOUS STOCHASTIC PROCESSES

In this section we present how a discrete stochastic process can be used to obtain a continuous stochastic process. Specifically, we show how the random walk can be transformed into the Brownian motion using the tools of stochastic processes and how the Pauli process can be transformed into a particular case of the gamma process using the tools of the mimetic map. The latter will be the first application of the discrete calculus to connect discrete and continuous stochastic processes, which will be the main goal of the remaining chapters of this dissertation.

4.4.1 From Random Walk to Brownian Motion

4.4.1.1 Random Walk

Consider the birth-death process described by the master equation

$$\begin{cases} \dot{P}_n(t) = \gamma(P_{n+1}(t) + P_{n-1}(t) - 2P_n(t)); \\ P_n(0) = \delta_{n,0}; \end{cases} \tag{4.71}$$

for $\zeta_n = \gamma_n = \gamma = const.$. It describes a one dimensional stochastic process with equal and constant transition rate of make a transition from one of the adjacent states, imposing that the system starts in $n = 0$ at $t = 0$. Such process is known as a **Random Walk** of continuous time. This can be interpreted as a particle allowed to move in steps of size a in a one-dimensional lattice, where the state n is a site of such lattice.

In general, we can solve any linear master equation obtaining an ODE for the characteristic function of the stochastic process (KAMPEN, 2007). For (4.71), let $x = na$ be the position of the random walker after a time $t = N\tau$. The characteristic function for a discrete process is easily obtained from (4.29) and (4.6) as

$$\Phi(k,t) = \sum_{n=-\infty}^{\infty} e^{ikna} P_n(t). \tag{4.72}$$

So, from (4.71), we get the differential equation given by

$$\begin{cases} \partial_t \Phi(k,t) = \gamma(e^{ika} + e^{-ika} - 2)\Phi(k,t); \\ \Phi(k,0) = 1, \end{cases} \tag{4.73}$$

which have solution given by

$$\Phi(k,t) = e^{-2\gamma t} e^{2\gamma t \cos ka}. \tag{4.74}$$

Using the identity $e^{\frac{z}{2}(u-\frac{1}{u})} = \sum_{n=-\infty}^{\infty} J_n(z) u^n$ (DENNERY; KRZYWICKI, 1996), with $z = i2\gamma t$ and $u = \frac{e^{ika}}{i}$, we obtain

$$P_n(t) = e^{-2\gamma t} I_n(t), \tag{4.75}$$

where $J_\alpha(z)$ is the Bessel function and $I_\alpha(z)$ is the modified Bessel function of first kind.

4.4.1.2 Brownian Motion

The Brownian motion is usually dated from 1828 when Robert Brown observed strange movements of pollen particles suspended in an aqueous solution (BROWN, 1828). However, in 1784 the same was observed by J. Ingen-Housz [1] (INGEN-HOUSZ, 1784) when identical movements were observed for particles of charcoal floating in alcohol, which was explained in 1819 by J. Bywater (BYWATER, 1824) associating such behavior with inorganic particles and microscopic fluctuations, a bit earlier than Brown..

The theoretical description came only in 1905, when A. Einstein explained the behavior of such particle from a probabilistic point of view resulting in an equation of motion for the probability density of find a particle at position x at time t (SILVA; LIMA, 2007), which was later verified experimentally by the 1926 Nobel Prize laureate J. B. Perrin (PERRIN, 2013), in 1908. Not only him, but many others contributed to the theoretical description of the Brownian motion, such as M. V. Smoluchowski (SMOLUCHOWSKI, 1906), P. Langevin (LEMONS; GYTHIEL, 1997) and M. Kac (KAC, 1947). In (ABBOTT et al., 1996) a summarized timeline of the Brownian motion history can be found.

The Brownian motion can emerge as a continuum limit of the random walk. In the random walk, since γ is the transition rate per unit time and τ is the time of each transition, the product $\gamma\tau = \frac{\gamma t}{N}$ gives the frequency of transitions, i.e., probability of the transition for N large. And since the transition rate is the same for the two accessible states, $n-1$ and $n+1$, we have $\gamma = \frac{1}{2\tau}$.

Assuming that the quantity

$$D \equiv \lim_{a\to 0} a^2\gamma = \frac{1}{2}\lim_{a\to 0}\frac{a^2}{\tau} \tag{4.76}$$

is constant and imposing the continuum limit $a \to 0$ on the characteristic function of the random walk, up to second order in ika, we obtain the characteristic function of the Brownian motion given by

$$\Phi(k,t) = e^{-Dtk^2}. \tag{4.77}$$

It results in the probability density

$$\mathcal{P}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\Phi(k,t)dk = \frac{e^{\frac{-x^2}{4Dt}}}{\sqrt{4\pi Dt}}, \tag{4.78}$$

which satisfies the Partial Differential Equation (PDE)

$$\begin{cases} \partial_t\mathcal{P}(x,t) = D\partial_x^2\mathcal{P}(x,t); \\ \mathcal{P}(x,0) = \delta(x); \end{cases} \tag{4.79}$$

[1] Jan Ingen-Housz is better known for the discovery of photosynthesis (GEERDT, 2007).

which is the first example of a Fokker-Planck equation. Such equation describes the evolution of the probability density of a Brownian particle moving in a one-dimensional space.

The parameter D is usually called diffusion coefficient. Taking the system studied by R. Brown in 1820s as an example, with pollen grains of size b, suspended over a fluid with viscosity coefficient η at temperature T, D is given by

$$D = \frac{k_B T}{6\pi\eta b}, \tag{4.80}$$

where k_b is the Boltzmann constant. That was found by A. Einstein on his work (SILVA; LIMA, 2007).

We can see a great similarity of the characteristic function (4.72) and the h-generating function (3.88a) of the previous chapter, making $h = a$ and $z = e^{ika}$, if we had define it not using a exponential generating function (although, now we have a differential-difference equation been solved). Thus, the limit $a \to 0$ can be seen as the continuum limit of the mimetic map, with the constraint (4.76). Further, the same can be seen by the use of difference operators. Observe that (4.71) can be rewritten as

$$\dot{P}_n(t) = \gamma(P_{n+1}(t) + P_{n-1}(t) - 2P_n(t)) = \gamma a^2 {(D^{\Delta})}^2 (P_n(t)), \tag{4.81}$$

assuming $n = \frac{x}{a}$, which converges for (4.79) as a goes to zero with the constraint (4.76) imposed.

4.4.2 From Pauli to Gamma Distribution

The connection between discrete and continuous stochastic processes can also be done through the mimetic map. Consider the Pauli process, which is described by the birth-death master equation

$$\dot{P}_n(t) = \zeta(n+1)P_{n+1}(t) + \gamma n P_{n-1}(t) - [\zeta n + \gamma(n+1)]P_n(t), \tag{4.82}$$

where $\zeta_n = \zeta n$ and $\gamma_n = \gamma(n+1)$. The stationary solution is given by the Pauli distribution

$$P_n^{eq} = (1-u)u^n, \tag{4.83}$$

giving the mean $\langle n \rangle = \frac{u}{1-u}$, where $u = \frac{\gamma}{\zeta}$.

We can rewrite this master equation in terms of the difference operators defined in chapter 2 such that

$$\begin{aligned} \dot{P}_n(\tau) &= (n+1)P_{n+1}(\tau) + unP_{n-1}(\tau) - [n + u(n+1)]P_n(\tau) \\ &= HP_n(\tau) + DP_n(\tau) + (1-u)EP_n(\tau) + (1-u)P_n(\tau) \end{aligned} \tag{4.84}$$

for $h = 1$ and $\tau = \zeta t$. Taking the continuum limit we obtain the PDE

$$\partial_\tau \mathcal{P}(x,\tau) = x\partial_x^2 \mathcal{P}(x,\tau) + [1 + (1-u)x]\partial_x \mathcal{P}(x,\tau) + (1-u)\mathcal{P}(x,\tau) \tag{4.85}$$

or

$$\partial_\tau \mathcal{P}(x,\tau) = \partial_x[((1-u)x - 1)\mathcal{P}(x,\tau)] + \partial_x^2[x\mathcal{P}(x,\tau)]. \tag{4.86}$$

The equation above is another example of a Fokker-Planck equation, been a particular case of the Fokker-Planck of the gamma distribution, which will be found in chapter 5. Observe that the solution of this Fokker-Planck is proportional to $e^{-(1-u)x}$, which gives the solution of the master equation (4.82) as $\widehat{e^{(1-u)x}}_{h=1}$, or in our case $(1-u)\widehat{e^{(1-u)x}}_{h=1}$, validating the use of the mimetic map.

As presented above, we can find ways to connect discrete and continuous stochastic processes. Although the relation between the random walk and the Brownian motion was already known, a new relation now can be seen between the Pauli distribution and the particular case of the gamma distribution. Once the Pauli distribution (4.83) is a particular case of the negative binomial distribution, in chapter 5 we will show via mimetic map that the gamma distribution indeed is directly related with the negative binomial distribution as its discrete version.

4.5 FOKKER-PLANCK EQUATION

We can generalize (4.79) and (4.86) as the **Fokker-Planck equation** (F-P equation), also known as Smoluchowski equation, by

$$\begin{cases} \partial_t \mathcal{P}(x,t) = -\partial_x[D^{(1)}(x,t)\mathcal{P}(x,t)] + \partial_x^2[D^{(2)}\mathcal{P}(x,t)]; \\ \mathcal{P}(x,t_0) = \delta(x - x_0); \end{cases} \tag{4.87}$$

Here $D^{(2)}$ makes the role of the **Diffusion Coefficient** and $D^{(1)}$ is the **Drift Coefficient**. $D^{(1)}$ represents the presence of a external force acting on the particle under a stochastic behavior (KAMPEN, 2007). Such equation describes the dynamics of the PDF $\mathcal{P}(x,t)$, assured that the Brownian particle is at $x = x_0$ at time $t = t_0$.

If we have a multivariate stochastic process, described by a random vector $\vec{X}$ of n components, the definition of the F-P equation can be extended to

$$\begin{cases} \partial_t \mathcal{P}(\vec{x},t) = -\sum_{i=1}^n \partial_{x_i}[D_i^{(1)}(x,t)\mathcal{P}(\vec{x},t)] + \sum_{i=1}^n \sum_{j=1}^n \partial_{x_i}\partial_{x_j}[D_{ij}^{(2)}\mathcal{P}(\vec{x},t)]; \\ \mathcal{P}(\vec{x},t_0) = \prod_{i=1}^n \delta(x_i - x_{i0}); \end{cases} \tag{4.88}$$

where now we have $D^{(2)}$ as the **Diffusion Coefficient Matrix** and $\vec{D^{(1)}}$ as the **Drift Coefficient Vector**.

The stationary (equilibrium) solution of (4.87) can be obtained assuming that the left-hand side (time derivative) is null. So,

$$\frac{D^{(1)}(x)}{D^{(2)}(x)}[D^{(2)}(x)\mathcal{P}_{eq}(x)] = \partial_x[D^{(2)(x)}\mathcal{P}_{eq}(x)]. \tag{4.89}$$

Then, the equilibrium solution is given by

$$\mathcal{P}_{eq}(x) = \frac{\mathcal{N}}{D^{(2)}(x)}e^{\left(\int_{x_0}^{x}\frac{D^{(1)}(\xi)}{D^{(2)}(\xi)}d\xi\right)}. \tag{4.90}$$

Exists quite many methods to solve the F-P equation (4.87), which part of then can be found on (RISKEN; HAKEN, 1989). Also, as will be see in section 5.1, we can solve it as any PDE.

We observe here that the F-P equation is just a particular case of the Master equations, assuming the size that establish the distance between state is infinitesimal and the transition matrix W is a differential operator (KAMPEN, 2007). However, we were treating them separately: the F-P equation describes continuous processes and the Master equation describes discrete processes.

4.5.1 Mapping Into a Brownian Motion Influenced by an External Force

A continuous stochastic behavior can be seen as a Brownian motion affected by an external force. Under a change of space variable we can transform the general F-P equation (4.87) into the F-P equation of the Brownian motion (4.79) with an extra drift term (RISKEN; HAKEN, 1989).

Consider the change of space variable $y = y(x)$ such that

$$\bar{D}^{(2)}(y) \equiv D = y'(x)^2 D^{(2)}(x) \quad \Rightarrow \quad y(x) = \int_{x_0}^{x}\sqrt{\frac{D}{D^{(2)}(\xi)}}d\xi. \tag{4.91}$$

It turns $D^{(1)}(x)$ into

$$\bar{D}^{(1)}(y) = \sqrt{\frac{D}{D^{(2)}(x)}}\left(D^{(1)}(x) - \frac{1}{2}\frac{dD^{(2)}(x)}{dx}\right) \tag{4.92}$$

and (4.87) becomes

$$\partial_t\bar{\mathcal{P}}(y,t) = D\partial_y^2\bar{\mathcal{P}}(y,t) - \partial_y(\bar{D}^{(1)}(y)\bar{\mathcal{P}}(y,t)). \tag{4.93}$$

The first term is the same of the Brownian motion F-P equation, and the second can be interpreted as a external force acting in the Brownian particle.

4.6 LANGEVIN EQUATIONS AND STOCHASTIC DIFFERENTIAL EQUATIONS

4.6.1 Wiener Process and Stochastic Differential Equations

The discussion presented in the subsection 4.4.1.2 about the Brownian motion can also be discussed in terms of Newton's second law, considering the influence of a stochastic force. Consider the system studied by R. Brown again, a particle of mass m and size b suspended in a solution of viscosity coefficient η in thermal equilibrium with a reservoir at temperature T. The viscosity of the solution will cause a drag force depending on the velocity v of the particle, given by $F_d = 6\pi\eta b v$.

The random behavior of the particle will be described by the influence of a fluctuating force $F_r(t) = F_0\xi(t)$, for which ξ is known as **White Noise** if satisfy

$$\langle \xi(t) \rangle = 0 \tag{4.94a}$$

and

$$\langle \xi(t)\xi(t') \rangle = \frac{2\alpha k_B T}{m}\delta(t-t'). \tag{4.94b}$$

Thus, from Newton's second law the equation of motion for the velocity of the particle is

$$\dot{v}(t) = -\alpha v(t) + \sigma\xi(t), \tag{4.95}$$

where $\alpha \equiv \frac{6\pi\eta b}{m}$ and $\sigma \equiv \frac{F_0}{m}$.

(4.95) is known as the **Langevin Equation** and is the first example of an stochastic differential equation, which was firstly obtained by Paul Langevin, in 1908, as an attempt to generalize the results obtained by A. Einstein, in 1905. $\xi(t)$ characterizes it as a stochastic differential equation by rewriting (4.95) as

$$dv = -\alpha v dt + \sigma dW(t), \tag{4.96}$$

for

$$dW(t) \equiv \xi(t)dt, \tag{4.97}$$

which is the usual way that the stochastic differential equations are presented. The quantity $\sigma dW(t)$ is itself a Brownian motion, and the quantity $W(t)$ is the Wiener process. Since in most of the literature those terms are interchangeable (HASSLER, 2016), we do the same here. In general, an **Stochastic Differential Equation** (SDE) can be defined as

$$dX(t) = A(X,t)dt + B(X,t)dW(t). \tag{4.98}$$

Formally, consider the stochastic process $W(t)$ defined in the interval $[0,t]$, with partition $\{t_0 = 0, t_1, \ldots, t_n = t\}$, with $t_{i-1} \leqslant t_i$ and $n \in \mathbb{N}_0$. $W(t)$ is a **Wiener Process** (or a Brownian motion) if satisfies:

(i) Prob(W(0)=0) $= 1$;

(ii) The increments $\Delta W_i = W(t_i) - W(t_{i-1})$, $\forall i$, are stationary and independent of each other;

(iii) For $s < t$, the difference $W(t) - W(s)$ follows a normal distribution with null mean and variance $t - s$;

(iv) The trajectories are continuous.

From the definition above it is easy to see that a Wiener process $W(t)$ is a Gaussian process with $\langle W(t) \rangle = 0$ and $\langle W(t)W(s) \rangle = min\{t, s\}$. Also, some relevant properties are

- $W(t)$ is differentiable nowhere;
- For $a > 0$, the Wiener processes $W(at)$ and $\sqrt{a}W(t)$ have the same probability distribution.

In particular, the first means that the relation $dW(t) = \xi(t)dt$ is not well defined in terms of the usual Riemann integrals once

$$\xi(t) = \frac{dW}{dt}$$

does not formally exist. To remediate that we use the stochastic calculus, which basically ensure that integrals like

$$\int_0^t g(t')dW(t') \tag{4.99}$$

converges in some defined sense.

4.6.2 Stochastic Calculus

The Stochastic Calculus appears as an alternative to solve SDEs. In general, we have the Riemann-Stieltjes integrals, which can be reduced to Riemann, Stratonovich or Ito integrals (GARDINER, 2004)(HASSLER, 2016). The latter is the one of our interest here due to it preserves the statistical properties expected for the Brownian motion, such as the null mean.

Consider the functions $f, g : [a, b] \to \mathbb{R}$ and let $P = \{t_0 = a, \ldots, t_n = b\}$, with $t_{i-1} < t_i$, $i = 1, \ldots, n$, be a partition of the interval $[a, b]$, with partition norm defined by $|P| = \max\{t_1 - t_0, \ldots, t_n - t_{n-1}\}$ and $t_i^* \in [t_{i-1}, t_i]$. The **Riemann-Stieltjes integral** of f with respect to g is defined by

$$\int_a^b f(t)d\,g(t) := \lim_{|P|\to 0} S_n(P), \tag{4.100}$$

where

$$S_n(P) \equiv \sum_{i=1}^{n} f(t_i^*)\Delta g_i \tag{4.101}$$

and $\Delta g_i = g(t_i) - g(t_{i-1})$. The Riemann integral is recovered if we set $g(t) = t$. If $g(t) = W(t)$ is a Wiener process, we have the Riemann-Stieltjes integral as a **Stochastic Integral** of a function $f(t) = F(t)$ which can be either a function or a stochastic process (hence the capital letter).

For such stochastic integral be completely characterized, the number t_i^* need to be set, otherwise the stochastic properties of the system will depend on it, causing statistical inconsistencies. For instance suppose $F(t) = W(t)$ and let

$$t_i^* = (1 - \lambda)t_{i-1} + \lambda t_i$$

with $0 \leqslant \lambda \leqslant 1$. Then,

$$\langle S_n(P)\rangle = \sum_{i=1}^{n} \langle W(t_i^*)\rangle\langle \Delta W_i\rangle = (b - a)\lambda \tag{4.102}$$

using $\langle W(t)W(s)\rangle = min\{t, s\}$. From that we see that the definition of t_i^* is influencing the statistics.

Thus Ito chose $t_i^* = t_{i-1}$ ($\lambda = 0$), which gives null mean of any stochastic integral, defining the **Ito integral** of a function $F(t)$ as

$$\int_{t_0}^{t} F(t)\, dW(t') := \lim_{n\to\infty} {}^{(ms)}S_n, \tag{4.103}$$

for

$$S_n = \sum_{i=1}^{n} F(t_{i-1})\Delta W_i.$$

In that case is assumed that $t_i - t_{i-1} = \frac{t-t_0}{n}$, making the partition norm dependent of n. Here $\lim^{(ms)}$ denotes what is known as the **Mean Square Limit** defined by

$$\lim_{n\to\infty}{}^{(ms)} S_n = I < \infty \Longleftrightarrow \lim_{n\to\infty} \left\langle (S_n - I)^2 \right\rangle = 0. \tag{4.104}$$

The statistical inconsistency is now avoided since (HASSLER, 2016)

$$\left\langle \int_{t_0}^{t} F(t')\, dW(t') \right\rangle = 0 \tag{4.105}$$

and

$$\left\langle \left(\int_{t_0}^{t} F(t')\, dW(t') \right)^2 \right\rangle = \int_{t_0}^{t} \left\langle F^2(t') \right\rangle dt'. \tag{4.106}$$

And with Ito's integrals at hand is possible to solve SDEs using a useful toolkit. In fact, the SDE

$$dX(t) = A(X(t), t)dt + B(X(t), t)dW(t) \tag{4.107}$$

is the differential form of a stochastic process

$$X(t) = X_0 + \int_{t_0}^{t} A(X(t'), t')dt' + \int_{t_0}^{t} B(X(t'), t')dW(t'), \tag{4.108}$$

where X_0 is another stochastic process, the second term is an Riemann integral and the third is an Ito integral. Processes defined by this are called **Ito's or Diffusion Processes**. Thus, some useful consequences and properties are worthy to be mention.

4.6.2.1 Fundamental Theorem of Ito's Calculus

A function $F(t)$ is called a **Non Anticipating Function** if and only if it is statically independent of increments $W(s) - W(t)$, for $t < s$. It basically establish that a process cannot depend on its future history. Using this we can state the **Fundamental Theorem of Ito's Calculus**.

Theorem 4.6.1. Let $F(t)$ be a non anticipating function and define the stochastic integral

$$\int_{t_0}^{t} F(t')dW(t')^{N+2} := lim_{n\to\infty}{}^{(ms)} \sum_{i=1}^{n} F(t_{i-1}) \Delta W_i^{N+2}. \tag{4.109}$$

Then

$$\int_{t_0}^{t} F(t')dW(t')^{N+2} = \delta_{N,0} \int_{t_0}^{t} F(t')dt'. \tag{4.110}$$

The proof of the theorem above can be found in (GARDINER, 2004), using Wick Theorem for the case $N > 0$. In differential form notation, the theorem gives

$$dW(t)^{N+2} = \delta_{N,0}dt, \tag{4.111}$$

which can be intuitively expected from the third item of the definition of a Wiener process, i.e., $\langle (W(t) - W(s))^2 \rangle = t - s$.

4.6.2.2 Differentiation Rule

Consider again a non anticipating function $F(W(t), t)$, where $W(t)$ is an Ito process. Then

$$\begin{aligned} dF(W(t), t) &:= F(W(t) + dW(t), t + dt) - F(W(t), t) \\ &= \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} \right) dt + \frac{\partial F}{\partial W} dW(t). \end{aligned} \tag{4.112}$$

The proof consists in expand $F(W(t)+dW(t), t+dt)$ in powers of $dW(t)$ and $dt = dW(t)^2$ and retain terms up to second order in $dW(t)$.

4.6.2.3 Change of Variables: Ito's Formula

Consider the Ito process given by

$$dX(t) = A(X(t), t)dt + B(X(t), t)dW(t). \tag{4.113}$$

We can obtain an SDE for a new stochastic variable $Y(t) = F(X(t))$. Using $dY(t) := F(X(t) + dX(t)) - F(X(t))$ and $dX(t)$ as defined above, we obtain the **Ito's formula** as

$$dY(t) \approx \left(A(X(t), t)F'(X(t)) + \tfrac{B^2(X(t),t)}{2} F''(X(t)) \right) dt + B(X(t), t)F'(X(t))dW(t). \tag{4.114}$$

As a consequence of this result, we can obtain the F-P equation of the PDF $\mathcal{P}(x, t)$ of $X(t)$. Consider the mean of $dF(X(t))$. Since the mean of $dW(t)$ vanishes,

$$\frac{d\langle F(X(t))\rangle}{dt} = \left\langle A(X(t), t)F'(X(t)) + \frac{B^2(X(t), t)}{2} F''(X(t)) \right\rangle, \tag{4.115}$$

or

$$\int F(x)\partial_t \mathcal{P}(x, t)dx = \int \left(A(x, t)F'(x) + \frac{B^2(x, t)}{2} F''(x) \right) \mathcal{P}(x, t)dx. \tag{4.116}$$

Integrating by parts the right-hand side of the equality above, eliminating the boundary terms and considering that the interval of integration and $F(x)$ are arbitrary, we get

$$\partial_t \mathcal{P}(x,t) = -\partial_x[A(x,t)\mathcal{P}(x,t)] + \partial_x^2 \left[\frac{B^2(x,t)}{2}\mathcal{P}(x,t)\right], \tag{4.117}$$

which is just the F-P equation defined in (4.87).

Thus, the functions $A(x,t)$ and $B^2(x,t)$ are just the drift coefficient and the diffusion coefficient, given by

$$D^{(1)}(x,t) \equiv A(x,t) \text{ and } D^{(2)}(x,t) \equiv \frac{B^2(x,t)}{2}. \tag{4.118}$$

Such a result can be extended for a multivariate stochastic process represented by the random vector $\vec{X}$ and described by the system of SDEs

$$d\vec{X}(t) = \vec{A}(\vec{X}(t),t)dt + B(\vec{X}(t),t)d\vec{W}(t), \tag{4.119}$$

where each component of the random vector $d\vec{W}(t)$ itself is a Wiener process independent of each other. For such a system of SDEs, the multivariate PDF satisfies the F-P equation

$$\partial_t \mathcal{P}(\vec{x},t) = -\sum_{i=1}^{n} \partial_{x_i}[A_i(\vec{x},t)\mathcal{P}(\vec{x},t)] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{x_i}\partial_{x_j}\left[B(\vec{x},t)B^T(\vec{x},t)\mathcal{P}(\vec{x},t)\right], \tag{4.120}$$

resulting in

$$\vec{D^{(1)}}(\vec{x},t) \equiv \vec{A}(\vec{x},t) \text{ and } D^{(2)}(\vec{x},t) \equiv \frac{1}{2}B(\vec{x},t)B^T(\vec{x},t). \tag{4.121}$$

Such results are of the most importance once states the connection between SDEs and F-P equations, which will be used to find the probability distributions appearing in the H theory and in the stochastic epidemic models, which is the usual way that stochastic systems are presented.

5 H THEORY

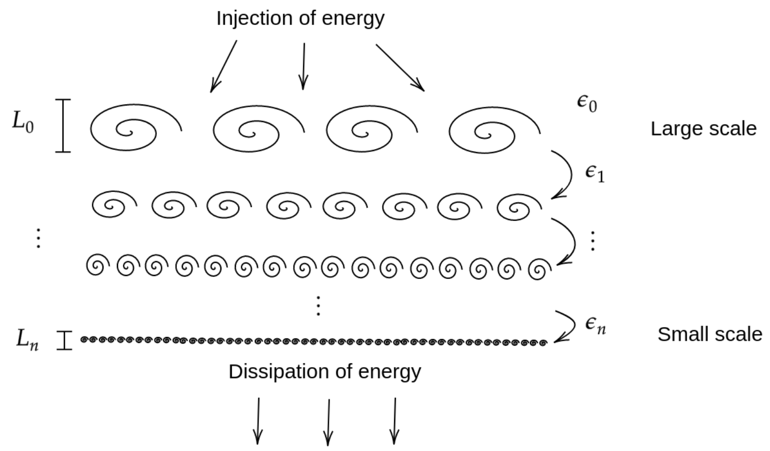
In this chapter we present a physical application of the mimetic map, connecting continuous and discrete stochastic processes, in a similar manner as was briefly discussed in section 4.4. The model in which the mimetic map is applied is the H theory, which consists of a stochastic model used to describe multiscale complex systems, using the fundamental concepts taken from the statistical description of turbulent fluids, the intermittency and the energy cascade. The application of the mimetic map reveals the underneath discrete stochastic process from which emerges the continuous stochastic processes observed in the H theory. Such discrete processes are called as discrete H theory here. Here we will focus our analysis in the case $s = 1/2$ of the theory. In section 5.1 we briefly discuss the H theory formalism and the physical context it is applied. We also present the out of equilibrium solution of the case $s = 1/2$. In section 5.2 we apply the mimetic map to obtain the discrete H theory for the background variables in terms of already known discrete stochastic processes. And in section 5.3 we show how we can construct discrete stochastic processes for the observable variable using the decomposition of variables used in the central limit theorem context, described in the previous chapter. In the central limit theorem sense the result of the continuous H theory for the observable is recovered as well as the one for the background variables, which is recovered in the continuum limit sense.

5.1 THE H THEORY

To model turbulent flow (FRISCH; KOLMOGOROV, 1995) from a statistical point of view is based in two main phenomena, the intermittency and the energy cascade. The energy cascade consists in an energy transfer from a size scale to a smaller one (see figure 5), while the intermittency phenomenon consists of the fluctuating feature of energy transfer rates associated to each scale.

In 1941 A. N. Kolmogorov proposed a statistical model which, although provided great contributions, presented disagreement with experimental results giving the probability distribution of the velocity increments as Gaussian distributions, while experiments give non-Gaussian ones. That was caused by the assumption that the energy rate transfer of the small size scale to be a constant, presenting only the energy cascade feature. In 1962 Kolmogorov modified his model (KOLMOGOROV, 1962) allowing the energy transfer rate float around an average value

Figure 5 – Scheme of an energy cascade in a turbulent fluid.



Source: the author (2021).

according to a log-normal probability distribution, i.e., $\mathcal{F}(\epsilon) = \frac{e^{-\frac{(\ln\epsilon - \langle \ln\epsilon\rangle)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2\epsilon^2}}$, which made the model more accurate with the experimental results, presenting the energy cascade and intermittency features. However such choice of distribution gave some unphysical results (FRISCH; KOLMOGOROV, 1995).

As an attempt to match theory and experiment, a model describing the energy rate as continuous stochastic variables was developed, that is known as H Theory. In such a context, the energy rates are described by a system of coupled SDEs. Such a formalism is not only capable to describe turbulence flow but also other multiscale complex systems, capturing the non-Gaussianity of the experiments, which causes violation of the central limit theorem defined in the subsection 4.2.2.

For instance (SALAZAR; VASCONCELOS, 2010), consider a turbulent fluid with a cascade of $n+1$ eddies, with sizes

$$L_j = \frac{L}{b^j};\ j = 0, 1, \ldots, n;$$

with $b > 1$. For each eddy we associate an energy transfer rate ϵ_j which gives the energy transferred from the scale L_{j-1} to the scale L_j. An amount of energy is injected in the large scale $L_0 = L$ of the cascade and is transferred from the larger to the smaller scale until be dissipated at the small scale L_n. Then, the SDEs of the ϵ_j are given by

$$d\epsilon_j = -\gamma_j(\epsilon_j - \epsilon_{j-1})dt + k_j\epsilon_j dW_j, \tag{5.1}$$

where dW_j are independent Wiener processes. Such SDEs will be justified bellow. The quantity of interest here are the velocity increments δv of the fluid, which will have its probability distribution affected by the intermittency.

As said above, such a formalism can be used to describe a large set of multiscale complex systems described by many scales of length and/or time. Then, to present the formalism consider a multiscale complex system with n time scales τ_j $(j = 0, 1, \ldots, n)$, such that $\tau_0 \gg \tau_1 \gg \cdots \gg \tau_n$. For each time scale τ_j we have a quantity ϵ_j, called **Background Variable**, which fluctuates according to a PDF $\mathcal{F}_j(\epsilon_j)$, with exception in the large scale τ_0 where ϵ_0 is a constant.

The **Observable Variable** X, is the relevant quantity of the study and have the role of the velocity increment δv in the example above. It is measured in the smaller scale τ_n and is assumed to vary faster than the background variable ϵ_n, which gives the interpretation of a quantity evolving in time in a background almost static, as a local equilibrium (MACêDO et al., 2017). Due to this local equilibrium, we can define the conditional PDF $\mathcal{P}_n(x|\epsilon_n)$ for a measurement $X = x$ given the value ϵ_n of the background variable of the nth scale as

$$\mathcal{P}_n(x|\epsilon_n) = \frac{e^{-\frac{x^2}{2\epsilon_n}}}{\sqrt{2\pi\epsilon_n}}. \tag{5.2}$$

In that case the distribution of X is affected by the intermittency phenomenon, due the fluctuation of ϵ_n, which is represented by the marginal distribution

$$\mathcal{P}_n(x) = \int_0^\infty \mathcal{P}(x|\epsilon_n)\mathcal{F}_n(\epsilon_n)d\epsilon_n. \tag{5.3}$$

Such process of compose two or more statistics is called superstatistics (BECK, 2001).

Because of the energy cascade, the probability distribution of ϵ_n is directly affected by the larger scales. Via Chapman-Kolmogorov equation (4.54) we have

$$\mathcal{F}_n(\epsilon_n) = \int \prod_{i=1}^{n} \mathcal{F}(\epsilon_i \mid \epsilon_{i-1})d\epsilon_1 \ldots d\epsilon_{n-1}, \tag{5.4}$$

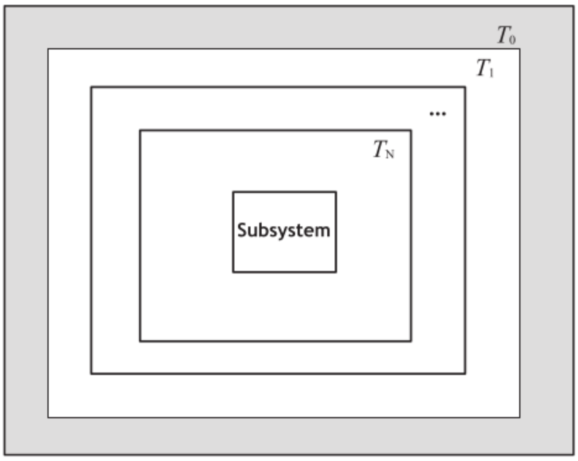
for $\mathcal{F}(\epsilon_0) = 1$.

The definition of (5.2) is only made by assumption taking into account the turbulence flow as the reference model, once with the constant energy transfer rate was obtained a Gaussian probability distribution (FRISCH; KOLMOGOROV, 1995). Another proposal for the conditional PDF can be found in the context of canonical formalism where the system can be seen as nested heat baths with fluctuating temperatures $\beta_j = \frac{1}{k_B T_j}$ (VASCONCELOS; SALAZAR; MACÊDO, 2018), where k_B is the Boltzmann constant (see figure 6).

Here the conditional PDF is given by

$$\mathcal{P}_n(q|\beta_n) = \frac{e^{-\beta_n E(q)}}{Z(\beta_n)}, \tag{5.5}$$

Figure 6 – Nested reservoirs representing a multiscale complex system in the canonical formalism. The outermost reservoir is in thermal equilibrium.



Source: (VASCONCELOS; SALAZAR; MACÊDO, 2018).

where q denotes the degrees of freedom of the subsystem contained in the innermost reservoir and $Z(\beta_n) = \int e^{-\beta_n E(q)} dq$ is the canonical partition function of the nth scale (SALINAS, 1997). The temperature T_0 of the larger scale, outermost reservoir, is held fixed, i.e., the system is in thermal equilibrium with the environment. In this context, the PDF found for the observable variable are given in terms of the H-function (appendix A), not the G-function as is shown bellow.

Aiming to describe the system completely is necessary to find the conditional PDFs $\mathcal{F}(\epsilon_j \mid \epsilon_{j-1})$. To do so, we can start by constructing the stochastic differential equations obeyed by the background variables ϵ_j, which in general is given by

$$d\epsilon_j = F(\epsilon_j, \epsilon_{j-1})dt + G(\epsilon_j, \epsilon_{j-1})dW_j, \tag{5.6}$$

where dW_j are Wiener processes independent of each other. Here the dependence in ϵ_j and ϵ_{j-1} exhibit the local hierarchy of the system, showing that the larger scale affects the shorter scale indirectly, under the hierarchical chain. To obtain the functional form of $F(\epsilon_j, \epsilon_{j-1})$ and $G(\epsilon_j, \epsilon_{j-1})$ we impose the following physics requirements:

(i) Equilibrium condition:

$$\lim_{t\to\infty} \langle \epsilon_j(t) \rangle = \epsilon_0;$$

(ii) Invariance under change of scale:

$$F(\lambda\epsilon_j, \lambda\epsilon_{j-1}) = F(\epsilon_j, \epsilon_{j-1}) \quad \text{and} \quad G(\lambda\epsilon_j, \lambda\epsilon_{j-1}) = G(\epsilon_j, \epsilon_{j-1});$$

(iii) Positivity:

$$Prob(\epsilon_j(t) \geqslant 0) = 1 \text{ if } \epsilon_j(0) \geqslant 0, \forall t \text{ and } \forall j.$$

That will gives (MACêDO et al., 2017)

$$d\epsilon_j = -\gamma_j(\epsilon_j - \epsilon_{j-1})dt + k_j \epsilon_{j-1}^{1-s} \epsilon_j^s dW_j, \tag{5.7}$$

where $\gamma_j \propto \tau_j^{-1}$, k_j are positive constants and $s \in [0, 1]$. Observe that the first term of (5.7) is the deterministic one: if we take the mean of this equation, once $\langle dW_j \rangle = 0$, we obtain ODEs for the $\langle \epsilon_j(t) \rangle$ which solutions tend to ϵ_0 when $t \to \infty$. The second is the one which encodes the fluctuation of the variable.

Observe that ϵ_{i-1} is still a variable, but due to the time scale separation, we can treat it as approximately constant in comparison with ϵ_i, allowing the obtainment of the F-P equation. Thus, from Ito's formula we obtain the F-P equation associated with (5.7) as

$$\begin{aligned} \partial_t \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1}) = & \left[\gamma_i + \frac{k_i^2}{2} 2s(2s-1) \epsilon_{i-1}^{2(1-s)} \epsilon_i^{2(s-1)} \right] \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1}) \\ & + \left[\gamma_i(\epsilon_i - \epsilon_{i-1}) + k_i^2 (2s) \epsilon_{i-1}^{2(1-s)} \epsilon_i^{2s-1} \right] \partial_i \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1}) \\ & + \frac{k_i^2}{2} \epsilon_{i-1}^{2(1-s)} \epsilon_i^{2s} \partial_i^2 \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1}). \end{aligned} \tag{5.8}$$

Here we denoted $\partial_i = \frac{\partial}{\partial \epsilon_i}$ and $\partial_t = \frac{\partial}{\partial t}$. The time scale separation could be dropped, but then the SDEs can only be solved numerically.

Imposing that the F-P equation have an analytic solution, the only acceptable values for s are $s = \frac{1}{2}$ and $s = 1$. In those cases, the stationary solutions are the gamma distribution

$$\mathcal{F}_{s=\frac{1}{2}}(\epsilon_i \mid \epsilon_{i-1}) = \frac{(\beta_i/\epsilon_{i-1})^{\beta_i}}{\Gamma(\beta_i)} \epsilon_i^{\beta_i - 1} e^{-\beta_i \frac{\epsilon_i}{\epsilon_{i-1}}}, \tag{5.9}$$

for $s = \frac{1}{2}$, and the inverse-gamma distribution

$$\mathcal{F}_{s=1}(\epsilon_i \mid \epsilon_{i-1}) = \frac{(\beta_i \epsilon_{i-1})^{\beta_i + 1}}{\Gamma(\beta_i + 1)} \epsilon_i^{-\beta_i - 2} e^{-\beta_i \frac{\epsilon_{i-1}}{\epsilon_i}}, \tag{5.10}$$

for $s = 1$, where $\beta_i = \frac{2\gamma_i}{k_i^2}$. Observe that $s = 1$ is the case of (5.1), thus making (5.10) the stationary conditional distribution used to describe the turbulent fluid statistics. Such analytic cases form universality classes (MACêDO et al., 2017) in which multiscale systems fit in, with the conditional probability of their respective background variables been described by either the gamma or the inverse-gamma distribution.

Substituting (5.9) or (5.10) into (5.4), we obtain the probability distribution for the background variable ϵ_n as

$$\mathcal{F}_n(\epsilon_n) = \frac{\omega}{\epsilon_0 \Gamma(\vec{\beta})} G_{0,n}^{n,0} \left[\begin{matrix} --- \\ \vec{\beta} - \vec{1} \end{matrix} \mid \omega \frac{\epsilon_n}{\epsilon_0} \right], \tag{5.11}$$

or

$$\mathcal{F}_n(\epsilon_n) = \frac{1}{\epsilon_0 \omega \Gamma(\vec{\beta}+\vec{1})} G^{0,n}_{n,0}\left[\begin{smallmatrix} -\vec{\beta}-\vec{1} \\ --- \end{smallmatrix} \mid \frac{\epsilon_n}{\omega \epsilon_0} \right] \tag{5.12}$$

where $\omega = \prod_{j=1}^{n} \beta_j$, $\vec{\beta} = (\beta_1, \ldots, \beta_n)$ and $\Gamma(\vec{\alpha}) = \prod_{j=1}^{n} \Gamma(\alpha_j)$.

And with that we can then obtain the probability distribution of the observable X as

$$\mathcal{P}_n(x) = \sqrt{\frac{\omega}{2\pi\epsilon_0}} \frac{1}{\Gamma(\vec{\beta})} G^{n+1,0}_{0,n+1}\left[\begin{smallmatrix} --- \\ \vec{\beta}-\frac{1}{2}\vec{1} \end{smallmatrix} \mid \frac{\omega}{2\epsilon_0} x^2 \right], \tag{5.13}$$

for $s = \frac{1}{2}$, and

$$\mathcal{P}_n(x) = \sqrt{\frac{1}{2\pi\omega\epsilon_0}} \frac{1}{\Gamma(\vec{\beta}+\vec{1})} G^{1,n}_{n,1}\left[\begin{smallmatrix} -\vec{\beta}-\frac{1}{2}\vec{1} \\ --- \end{smallmatrix} \mid \frac{x^2}{2\omega\epsilon_0} \right], \tag{5.14}$$

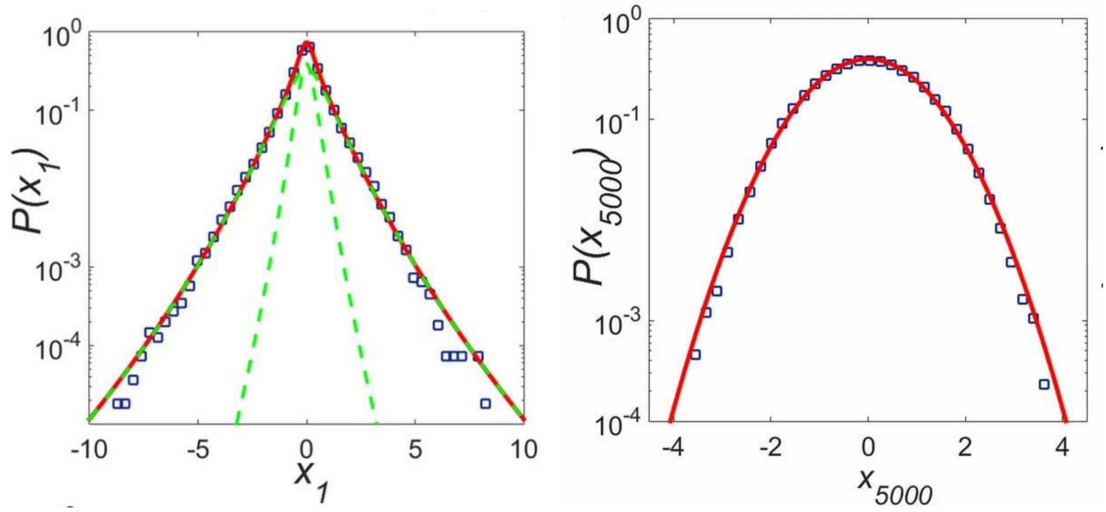
for $s = 1$.

To confirm the agreement of the H theory with experiments, an important application can be found in photonics. The intensity increments δI_τ of the radiation emitted by a random fibre laser (LETOKHOV, 1968) present the turbulent features (intermittency and energy cascade) for a given excitation power threshold (GONZáLEZ I. R. R., 2018).

In such a construction the multiscale is represented by the time separation between points of the time series of the intensity increments, $\tau = \beta - \alpha$, for β and α been given points of the time series. The multiscale is defined in such a way that as smaller τ is, the bigger is the label n of the scale. The background variables are the variances ϵ_τ of the intensity increments. For a normalized intensity increment x_τ, which is the observable variable, the distribution for the small scale $\tau = 1$ is given in the left hand side of figure 7 and the distribution for the large scale $\tau = 5000$ in the right hand side of the figure. The class of hierarchical theory describing the system is $s = \frac{1}{2}$ and the analytic solution, which fits the distributions found in the experiment, is given by a statistical mixture of (5.13) for $n = 6$ given by $\mathcal{P}(x_\tau) = p\mathcal{P}_{n,a}(x_\tau) + (1-p)\mathcal{P}_{n,b}(x_\tau)$, where p is adjusted to fit the curves adequately.

We observe the Gaussian behavior in the semi-log graphs for great values of τ (large scale, $n = 0$), which is revealed by the parabolic curve observed. The non Gaussian behavior is observed for small values of τ (small scale, $n = 6$), which is noted by the deviation from a parabola. In both cases the system is observed with the excitation power set above the threshold.

Figure 7 – Left: probability distribution of the normalized intensity increments x_1 for $\tau = 1$. The solid red line is the statistical mixture solution with parameters $p = 0.3$, $\beta_a = 8.53$, $\epsilon_{0,a} = 0.16$, $\beta_b = 6.47$, $\epsilon_{0,b} = 1.36$ and scale $n = 6$. The dashed green line are the probability distributions of the statistical mixture. Right: probability distribution of the normalized intensity increments x_{5000} for $\tau = 5000$. The solid red line is the Gaussian distribution provided for the large scale $n = 0$. The analytic solution fits well the experimental data given by the blue squares.



Source: (GONZáLEZ I. R. R., 2018).

5.1.1 Out of Equilibrium Solution for $s = \frac{1}{2}$

Let us solve (5.8). However, the out of equilibrium solutions, i.e., time dependent solutions, could be complicated to be solved, so we focus in the case $s = \frac{1}{2}$ only. In that case, we have

$$\partial_t \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1}) = \gamma_i \partial_i \left[(\epsilon_i - \epsilon_{i-1})\mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1})\right] + \frac{k_i^2 \epsilon_{i-1}}{2} \partial_i^2 \left[\epsilon_i \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1})\right]. \quad (5.15)$$

The PDE (5.15) is known as a parabolic equation (FELLER, 1951) and appears in many contexts of stochastic processes like econophysics (COX; INGERSOLL; ROSS, 1985) and in systems of particles confined under a harmonic potential (SALAZAR; LIRA, 2016).

Consider the SDE

$$dr = -\kappa(r - \theta)dt + \sigma\sqrt{r}dW. \quad (5.16)$$

The solution of the F-P equation associated with it is given by (COX; INGERSOLL; ROSS, 1985)

$$f(r(s), s; r(t), t) = Ce^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q\left(2(uv)^{1/2}\right), \quad (5.17)$$

where

$$
\begin{aligned}
C(s,t) &\equiv \tfrac{2\kappa}{\sigma^2}\left(1-e^{-\kappa(s-t)}\right)^{-1}, \\
u(s,t) &\equiv C(s,t)r(t)e^{-\kappa(s-t)}, \\
v(s,t) &\equiv C(s,t)r(s), \\
q &\equiv \tfrac{2\kappa\theta}{\sigma^2}-1
\end{aligned}
\tag{5.18}
$$

and $I_q(z)$ is the modified Bessel function of the first kind.

Using this, we find

$$
\begin{aligned}
\mathcal{F}(\epsilon_i(t),t;\epsilon_i(t_0),t_0|\epsilon_{i-1}) = C(t,t_0)e^{-C(t,t_0)\left(\epsilon_i(t)+\epsilon_i(t_0)e^{-\gamma_i(t-t_0)}\right)}\cdot \\
\cdot\left(\tfrac{\epsilon_i(t)}{\epsilon_i(t_0)e^{-\gamma_i(t-t_0)}}\right)^{\frac{\beta_i-1}{2}} I_{\beta_i-1}\left(2C(t,t_0)\sqrt{\epsilon_i(t_0)e^{-\gamma_i(t-t_0)}\epsilon_i(t)}\right),
\end{aligned}
\tag{5.19}
$$

where

$$
C(t,t_0)=\frac{\beta_i/\epsilon_{i-1}}{1-e^{-\gamma(t-t_0)}}=\frac{1-u_i}{1-e^{-\gamma(t-t_0)}}. \tag{5.20}
$$

Indeed, if we consider the stationary limit $t\to\infty$, we recover the result (5.9), using (ABRAMOWITZ, 1974)

$$
I_\alpha(z)\sim\frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^{\alpha}\text{, for } |z|\to 0. \tag{5.21}
$$

5.2 DISCRETE H THEORY FOR THE BACKGROUND VARIABLES

Our goal is find the image of the the H theory after the action of the mimetic map, constructing a discrete H theory for the background variables in terms of master equations instead of SDE and thus analyze the special cases $s=\frac{1}{2}$ and $s=1$.

First of all, we need to find the discrete version of the equation (5.8) finding the image of the maps $x^m\frac{d}{dx}\to E_m$ and $x^m\frac{d^2}{dx^2}\to H_m$. Defining

$$
\hat{H}_m\hat{f}_h(\nu):=\int_0^\infty\frac{dt}{\Gamma(\nu/h)}t^{\frac{\nu}{h}-1}e^{-t}(-ht)^m f''(-ht) \tag{5.22}
$$

and

$$
\hat{E}_m\hat{f}_h(\nu):=\int_0^\infty\frac{dt}{\Gamma(\nu/h)}t^{\frac{\nu}{h}-1}e^{-t}(-ht)^m f'(-ht), \tag{5.23}
$$

we obtain

$$
H_m\widehat{f}_h(x)=\widehat{x^m}_h\frac{\widehat{f}_h(x-mh)-2\widehat{f}_h(x-(m-1)h)+\widehat{f}_h(x-(m-2)h)}{h^2} \tag{5.24}
$$

and

$$E_m \widehat{f}_h(x) = \widehat{x^m}_h \frac{\widehat{f}_h(x-(m-1)h) - \widehat{f}_h(x-mh)}{h}. \tag{5.25}$$

Then, the discrete version of (5.8) will be

$$\begin{aligned} \dot{F}_{M_i}(t) = {} & \frac{k_i^2}{2} M_{i-1}^{2(1-s)} \widehat{M_i^{2s}} \left[F_{M_i-2s}(t) - 2F_{M_i-(2s-1)}(t) + F_{M_i-2(s-1)}(t) \right] \\ & + \gamma_i \{ M_i [F_{M_i}(t) - F_{M_i-1}(t)] - M_{i-1}(F_{M_i+1}(t) - F_{M_i}(t)) \} \\ & + (2s) k_i^2 M_{i-1}^{2(1-s)} \widehat{M_i^{2s-1}} [F_{M_i-2(s-1)}(t) - F_{M_i-(2s-1)}(t)] \\ & + [\gamma_i + (2s)(2s-1) \frac{k_i^2}{2} M_{i-1}^{2(1-s)} M_i^{2(s-1)}] F_{M_i}(t), \end{aligned} \tag{5.26}$$

for $h = 1$, $M_i = 0, 1, 2, \ldots$ and $\widehat{M_i^r} = \frac{\Gamma(M_i+1)}{\Gamma(M_i-r+1)}$.

5.2.0.1 *Case* $s = \frac{1}{2}$

For $s = \frac{1}{2}$, (5.26) becomes

$$\dot{F}_{M_i}(\tau) = (M_i + 2 - \beta_i) F_{M_i+1}(\tau) + u_i M_i F_{M_i-1}(\tau) - [(u_i+1) M_i + u_i + 1 - \beta_i] F_{M_i}(\tau), \tag{5.27}$$

where $u_i = 1 - \frac{\beta_i}{M_{i-1}} = 1 - 2\frac{\gamma_i}{k_i^2 M_{i-1}}$ and $\tau = \frac{k_i^2 M_{i-1}}{2} t$. That is just the discrete version of the equation (5.15), which is the master equation with stationary solution given by the negative binomial distribution. In fact, the discrete version of (5.9) is given by the negative binomial distribution

$$F_{s=\frac{1}{2}}(M_i \mid M_{i-1}) = \binom{M_i}{M_i + 1 - \beta_i} (1-u_i)^{\beta_i} u_i^{M_i + 1 - \beta_i}, \tag{5.28}$$

which satisfies the equation above. As mentioned in section 4.4, the F-P equation associated with the Pauli master equation is a particular case of the gamma distribution F-P equation (5.15). We note that for $\beta_i = 1$ (5.28) becomes the Pauli distribution, confirming the previous observation.

Been confirmed that our theory is consistent in the stationary case, we try to obtain the out of equilibrium solution of the master equation (5.27). To obtain that solution we just need to obtain the discrete version of (5.19) in the ϵ_i-dependence, which means, obtain the discrete version of

$$f(x) = \Lambda e^{-Cx} x^{\frac{\beta_i - 1}{2}} I_{\beta_i - 1}(\sqrt{\lambda^2 x}), \tag{5.29}$$

with

$$\Lambda \equiv \frac{C \exp\left\{-C\epsilon_i(t_0) e^{-\gamma(t-t_0)}\right\}}{\left[\epsilon_i(t_0) e^{-\gamma(t-t_0)}\right]^{\frac{\beta_i - 1}{2}}},$$

$$\lambda \equiv 2C\sqrt{e^{-\gamma_i(t-t_0)}\epsilon_i(t_0)}$$

and $x = \epsilon_i(t)$.

From the mimetic map,

$$\check{f}_h(\nu) = \int_0^\infty \frac{dt}{\Gamma(\frac{\nu}{h})} e^{-t} t^{\frac{\nu}{h}-1} e^{Cht} (-ht)^{\frac{\beta_i-1}{2}} I_{\beta_i-1}(i\sqrt{h\lambda^2 t}). \tag{5.30}$$

Using the identities

$$I_\alpha(z) = i^{-\alpha} J_\alpha(iz), \tag{5.31a}$$

$$J_\alpha(-z) = (-1)^\alpha J_\alpha(z) \tag{5.31b}$$

and (ERDéLYI, 1953)

$$G^{1,0}_{0,2}\left[{-- \atop a,b} \mid \frac{z}{4}\right] = \left(\frac{z}{4}\right)^{\frac{a+b}{2}} J_{a-b}(z^{1/2}), \tag{5.32a}$$

$$\int_0^\infty e^{-\omega x} x^{-\alpha} G^{m,n}_{p,q}\left[{\vec{a} \atop \vec{b}} \mid \eta x\right] dx = \omega^{\alpha-1} G^{m,n+1}_{p+1,q}\left[{\alpha,\{a_p\} \atop \vec{b}} \mid \frac{\eta}{\omega}\right], \tag{5.32b}$$

we have

$$\begin{aligned}\check{f}_h(\nu) &= \frac{\Lambda(-h)^{\frac{\beta_i-1}{2}}}{\Gamma(\frac{\nu}{h})} i^{\beta_i-1} \left(\frac{h\lambda^2}{4}\right)^{-\frac{\beta_i-1}{2}} \int_0^\infty \frac{dt}{\Gamma(\frac{\nu}{h})} e^{-(1-Ch)t} t^{\frac{\nu}{h}-1} G^{1,0}_{0,2}\left[{--- \atop \beta_i-1,0} \mid \frac{h\lambda^2}{4} t\right] \\ &= \frac{(-1)^{\beta_i-1}\Lambda}{\Gamma(\frac{\nu}{h})} \left(\frac{2}{\lambda}\right)^{\beta_i-1} G^{1,1}_{1,2}\left[{1-\frac{\nu}{h} \atop \beta_i-1,0} \mid \frac{h\lambda^2}{4(1-Ch)}\right].\end{aligned} \tag{5.33}$$

And from (ERDéLYI, 1953)

$$G^{1,1}_{1,2}\left[{a \atop b,0} \mid x\right] = \frac{\Gamma(1+b-a)}{\Gamma(1+b)} x^b {}_1F_1(1+b-a; 1+b; -x), \tag{5.34}$$

turns out that the discrete version of (5.29) is

$$\hat{f}_h(x) = \frac{h^{\beta_i-1}\Lambda}{\Gamma(\beta_i)} \frac{\Gamma(\frac{x}{h}+1)}{\Gamma(\frac{x}{h}+2-\beta_i)} \left(\frac{\lambda}{2}\right)^{\beta_i-1} (1-Ch)^{\frac{x}{h}+1-\beta_i} {}_1F_1\left(\beta_i - 1 - \frac{x}{h}; \beta_i; \frac{h\lambda^2}{4(Ch-1)}\right). \tag{5.35}$$

And thus, the discrete version of (5.19) is

$$\begin{aligned}F(M_i(t), t; M_{i0}, t_0 \mid M_{i-1}) = &\binom{M_i}{M_i - \beta_i + 1} e^{-C(t,t_0) M_{i0} e^{-\gamma(t-t_0)}} C(t,t_0)^{\beta_i} (1 - C(t,t_0))^{M_i - \beta_i + 1} \cdot \\ &\cdot {}_1F_1\left(\beta_i - 1 - M_i; \beta_i; \frac{C(t,t_0)^2}{4(C(t,t_0)-1)} e^{-\gamma(t-t_0)} M_{i0}\right),\end{aligned} \tag{5.36}$$

where $M_{i0} = M_i(t_0)$, ${}_1F_1(\alpha;\beta;x)$ is the confluent hypergeometric function and now

$$C(t,t_0) = \frac{\beta_i/M_{i-1}}{1-e^{-\gamma(t-t_0)}} = \frac{1-u_i}{1-e^{-\gamma(t-t_0)}}.$$

As a matter of fact, in the limit $t \to \infty$, we recover the result of (5.28).

Now only remains to confirm if such solution satisfy the master equation (5.27). Due the complexity of such expression, show it analytically is a bit cumbersome, so we choose to show it numerically. Note that from the definition of a conditional probability the master equation (5.27) is the same for $F(M_i(t), t; M_{i0}, t_0 \mid M_{i-1})$.

The derivative of (5.36) is

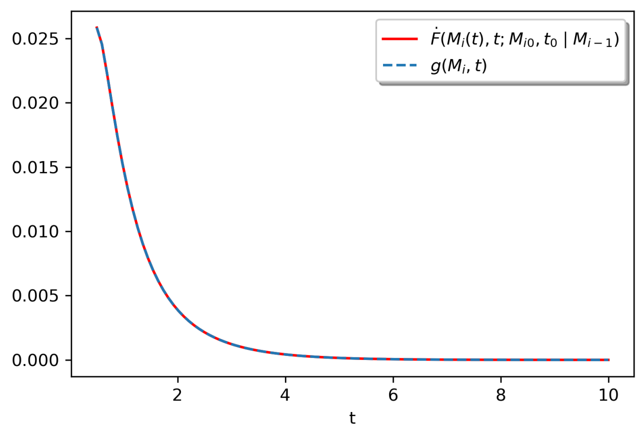
$$\begin{aligned} \dot{F}(M_i(t), t; M_{i0}, t_0 \mid M_{i-1}) = & \binom{M_i}{M_i-\beta_i+1} e^{-CM_{i0}e^{-\gamma(t-t_0)}} C^{\beta_i}(1-C)^{M_i-\beta_i+1} \cdot \\ & \cdot \left\{ \tfrac{\beta_i - M_i - 1}{\beta_i} \dot{x}\, {}_1F_1(\beta_i - M_i; \beta_i + 1; x) + \left[M_{i0}(\gamma C - \dot{C}) e^{-\gamma(t-t_0)} \right.\right. \\ & \left.\left. + \beta_i \tfrac{\dot{C}}{C} + (M_i + 1 - \beta_i) \tfrac{\dot{C}}{C-1} \right] {}_1F_1(\beta_i - M_i - 1; \beta_i; x) \right\}, \end{aligned} \tag{5.37}$$

with $x = \frac{C^2 e^{-\gamma(t-t_0)}}{C-1} M_{i0}$, $\dot{x} = \frac{Ce^{-\gamma(t-t_0)}}{C-1}\left(2\dot{C} - \frac{C\dot{C}}{C-1} - \gamma C\right) M_{i0}$ and $\dot{C} = \frac{-\gamma(1-u)e^{-\gamma(t-t_0)}}{[1-e^{-\gamma(t-t_0)}]^2}$. Also, denote the right hand side of (5.27) as

$$\begin{aligned} g(M_i, t) = \frac{k_i^2 M_{i-1}}{2} & \left[(M_i + 2 - \beta_i) F(M_i + 1, t; M_{i0} \mid M_{i-1}) + u_i M_i F(M_i - 1, t; M_{i0} \mid M_{i-1}) \right. \\ & \left. - [(u_i + 1) M_i + u_i + 1 - \beta_i] F(M_i, t; M_{i0} \mid M_{i-1}) \right] \end{aligned} \tag{5.38}$$

We can see in figure 8 that comparing the time evolution of these expressions, we have an excellent agreement, which deviation is of order of 10^{-15}.

Figure 8 – Numerical confirmation of the solution (5.36) with $M_i = 10$, $M_{i0} = 1$, $M_{i-1} = 6$, $\gamma_i = 1$, $k_i = \sqrt{2}$ and $\beta_i = 1$.



Source: the author (2021).

5.2.0.2 Case $s = 1$

For $s = 1$, the F-P equation is

$$\partial_t \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1}) = \gamma_i \partial_i \left[(\epsilon_i - \epsilon_{i-1}) \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1}) \right] + \frac{k_i^2}{2} \partial_i^2 \left[\epsilon_i^2 \mathcal{F}(\epsilon_i(t), t|\epsilon_{i-1}) \right]. \tag{5.39}$$

And from (5.26), its discrete version is

$$\begin{aligned} \dot{F}_{M_i}(\tau) = &- [2M_i(M_i - 1) + (\beta_i + 4)M_i] F_{M_i-1}(\tau) - \beta_i M_{i-1} F_{M_i+1}(\tau) \\ &+ [M_i(M_i - 1) + \beta_i(M_i + M_{i-1} + 1) + 2(2M_i + 1)] F_{M_i}(\tau) \\ &+ M_i(M_i - 1) F_{M_i-2}(\tau). \end{aligned} \tag{5.40}$$

It is not trivial to find the stationary solution of such equation by inspection, as was done before. So, we obtain the discrete version of the inverse-gamma distribution (5.10), which we expect to be the stationary solution of (5.40). From (5.10), we have

$$F_{s=1}(M_i \mid M_{i-1}) = \lim_{\substack{\nu \to -M_i \\ \epsilon_{i-1} \to M_{i-1} \\ h \to 1}} \frac{(\beta_i \epsilon_{i-1})^{\beta_i+1}}{\Gamma(\beta_i+1)} \int_0^\infty \frac{dt}{\Gamma(\nu/h)} e^{-t} t^{\frac{\nu}{h}-1} (-ht)^{-(\beta_i+2)} e^{\frac{\beta_i \epsilon_{i-1}}{ht}}. \tag{5.41}$$

To solve the integral appearing in (5.41) we make use of the Meijer G-function relations. Consider the following identities of the G-function:

$$e^z = G_{0,1}^{1,0} \left[{}^{-}_{0} \mid -z \right], \tag{5.42a}$$

$$G_{p,q}^{m,n} \left[{}^{\{a_p\}}_{\{b_q\}} \mid z \right] = G_{q,p}^{n,m} \left[{}^{\{1-b_q\}}_{\{1-a_p\}} \mid z^{-1} \right] \tag{5.42b}$$

and

$$\int_0^\infty e^{-\omega x} x^{-\alpha} G_{p,q}^{m,n} \left[{}^{\vec{a}}_{\vec{b}} \mid \eta x \right] dx = \omega^{\alpha-1} G_{p+1,q}^{m,n+1} \left[{}^{\alpha, \{a_p\}}_{\vec{b}} \mid \frac{\eta}{\omega} \right]. \tag{5.42c}$$

With that we can rewrite (5.41) as

$$F_{s=1}(M_i \mid M_{i-1}) = \lim_{\substack{\nu \to -M_i \\ \epsilon_i \to M_{i-1} \\ h \to 1}} \frac{(\beta_i \epsilon_{i-1})^{\beta_i+1}}{\Gamma(\beta_i+1)} \frac{(-h)^{-(\beta_i+2)}}{\Gamma\left(\frac{\nu}{h}\right)} G_{2,0}^{0,1} \left[{}^{-\frac{\nu}{h}+\beta_i+3, 1}_{------} \mid \frac{-h}{\beta_i \epsilon_{i-1}} \right] \tag{5.43}$$

or

$$F_{s=1}(M_i \mid M_{i-1}) = (\beta_i M_{i-1})^{\beta_i+1} \frac{\Gamma(M_i+1)}{\Gamma(\beta_i+1)} G_{2,0}^{0,1} \left[{}^{M_i+\beta_i+3, 1}_{------} \mid \frac{1}{\beta_i M_{i-1}} \right]. \tag{5.44}$$

One can easily check, in a similar manner as was done for the case $s = \frac{1}{2}$, that (5.44) satisfy (5.40) as its stationary solution.

With this results, we can see that a discrete version of continuous stochastic processes in terms of discrete stochastic processes can be constructed using the mimetic map in the probabilistic description of such processes. What is done here for the H theory can be exploited in other stochastic models, as it will be done in the next chapter with stochastic epidemic models.

5.3 DISCRETE H THEORY FOR THE OBSERVABLE VARIABLE

The description of the H theory involves probability distributions that does not satisfy the central limit theorem, which clearly can be seen by the distributions found describing the background and observable variables in section 5.1. Here we show how such distributions emerge by constructing a discrete H theory for the observable variable for the case $s = \frac{1}{2}$ (MACêDO, 2020) using the construction use to prove the CLT.

Consider the procedure done to verify that the Pauli distribution violates the CLT in subsection 4.2.2, but now for its generalization, the negative binomial distribution

$$P_N = \binom{N+\nu-1}{N}\frac{a^N}{(1+a)^{N+\nu}};\ \nu \geqslant 1. \tag{5.45}$$

The probability generating function (4.43) associated with it is

$$F(z) = \frac{1}{[1-a(z-1)]^{\nu}}, \tag{5.46}$$

which gives the mean $\langle N\rangle = \nu a$. Such a mean diverges if ν is large, violating the CLT. As mentioned in subsection 4.2.1, the averaged characteristic function can be obtained from $F(z)$ by analogy, changing z by $\Phi(k)$. Then,

$$\chi_Y(k) = \frac{1}{[1-a(\Phi(k)-1)]^{\nu}} \approx \frac{1}{[1+\frac{\sigma^2k^2}{2\nu}]^{\nu}}, \tag{5.47}$$

for $\Phi(k) \approx 1 - \frac{\sigma^2}{2\langle N\rangle}k^2$ and $Y = \sum_{i=1}^n Z_i$ with $Z_i = \frac{N_i - \langle N\rangle}{\sqrt{\langle N\rangle}}$ and N_i following the negative binomial distribution. Then, using

$$\mathcal{P}_Y(y) = \int \frac{dk}{2\pi}e^{-iky}\chi_Y(k),$$

and the identities (GRADSHTEYN; RYZHIK, 2007)

$$\int_{-\infty}^{\infty} e^{-iky}\left(1+\alpha^2k^2\right)^{-\nu}dk = \frac{2^{\frac{3}{2}-\nu}\sqrt{\pi}}{\Gamma(\nu)}\frac{y^{\nu-\frac{1}{2}}}{\alpha^{\nu+\frac{1}{2}}}K_{\frac{1}{2}-\nu}\left(\frac{y}{\alpha}\right) \tag{5.48}$$

and (ERDéLYI, 1953)

$$G^{2,0}_{0,2}\left[{}^{-}_{a,b} \mid x\right] = 2x^{\frac{1}{2}(a+b)}K_{a-b}(2\sqrt{x}). \tag{5.49}$$

we obtain

$$\mathcal{P}_Y(y) = \sqrt{\frac{\nu}{2\pi\sigma^2}} \frac{1}{\Gamma(\nu)} G^{2,0}_{0,2}\left[{}^{---}_{0,\nu-\frac{1}{2}} \mid \frac{\nu y^2}{2\sigma^2}\right], \tag{5.50}$$

confirming the violation of the CLT.

Comparing the result above with the expression in (5.13) with $n = 1$, we see those are identical, apart of a change of parameters. So the discrete variables which compose the observable X in the case $n = 1$, in the CLT sense, follow the negative binomial distribution. Starting from the PDF (5.13) and performing the inverse procedure, we can generalize it for any scale n.

Consider the probability distribution of X for the case $s = \frac{1}{2}$, given in (5.13). From

$$\chi_X(k) = 2\int_0^\infty \cos(kx)\mathcal{P}_n(x)dx$$

we get the associated averaged characteristic function. To do so, we rewrite it using (5.3) and the identities

$$\int_0^\infty e^{-\lambda x} x^{-\alpha} G^{m,n}_{p,q}\left[{}^{\vec{a}}_{\vec{b}} \mid \eta x\right] dx = \lambda^{\alpha-1} G^{n+1,m}_{q,p+1}\left[{}^{\vec{1}-\vec{b}}_{1-\alpha,\vec{1}-\vec{a}} \mid \frac{\lambda}{\eta}\right] \tag{5.51}$$

and

$$x^\sigma G^{m,n}_{p,q}\left[{}^{\vec{a}}_{\vec{b}} \mid x\right] = G^{m,n}_{p,q}\left[{}^{\vec{a}+\vec{\sigma}}_{\sigma\vec{b}} \mid x\right], \tag{5.52}$$

to obtain

$$\chi_X(k) = \sqrt{\frac{\omega}{2\pi\epsilon_0}} \frac{1}{\Gamma(\vec{\beta})} G^{1,n}_{n,1}\left[{}^{\vec{1}-\vec{\beta}}_{0} \mid \frac{\epsilon_0}{2\omega}k^2\right]. \tag{5.53}$$

Using the identity

$$\frac{1}{\Gamma(\nu)} G^{1,1}_{1,1}\left[{}^{1-\nu}_{0} \mid x\right] = (1+x)^{-\nu}, \tag{5.54}$$

for $n = 1$, we have the same expression of (5.47) for the negative binomial distribution, if we change

$$\frac{\omega}{\epsilon_0} \to \frac{\tilde{\nu}}{\sigma^2}$$

with $\tilde{\nu} = \prod_{j=1}^n \nu_j$, and $\vec{\beta} \to \vec{\nu}$. In that case we get

$$\chi_X(k) \approx \sqrt{\frac{\tilde{\nu}}{2\pi\sigma^2}} \frac{1}{\Gamma(\vec{\nu})} G^{1,n}_{n,1}\left[{}^{\vec{1}-\vec{\nu}}_{0} \mid -a(\Phi(k)-1)\right]. \tag{5.55}$$

Thus changing $\Phi(k)$ by z, we obtain

$$F(z) = \sqrt{\frac{\tilde{\nu}}{2\pi\sigma^2}} \frac{1}{\Gamma(\vec{\nu})} G^{1,n}_{n,1}\left[{}^{\vec{1}-\vec{\nu}}_{0} \mid a(1-z)\right]. \tag{5.56}$$

And using

$$G^{1,p}_{p,q+1}\left[{}^{\vec{1}-\vec{a}}_{0,\vec{1}-\vec{b}} \mid z\right] = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}\,{}_pF_q(\vec{a};\vec{b};-x), \tag{5.57}$$

we can simplify those as

$$\chi_X(k) = \sqrt{\frac{\tilde{\nu}}{2\pi\sigma^2}}\,{}_nF_0\left(\vec{\nu}; a(\Phi(k)-1)\right) \tag{5.58}$$

and

$$F(z) = \sqrt{\frac{\tilde{\nu}}{2\pi\sigma^2}}\,{}_nF_0\left(\vec{\nu}; a(z-1)\right). \tag{5.59}$$

With $F(z)$ we can now get the probability distribution for the random variable N for any number of scales n. From (5.56), using

$$G^{m,n}_{p,q}\left[{}^{\vec{a}}_{\vec{b}} \mid \lambda z\right] = \lambda^{b_1}\sum_{r=0}^{\infty}\frac{(1-\lambda)^r}{r!}G^{m,n}_{p,q}\left[{}^{\vec{a}}_{b_1+r,b_2,\ldots,b_q} \mid z\right], \tag{5.60}$$

we get

$$P_n(N) = \sqrt{\frac{\tilde{\nu}}{2\pi\sigma^2}}\frac{1}{\Gamma(\vec{\nu})N!}G^{1,n}_{n,1}\left[{}^{\vec{1}-\vec{\nu}}_{N} \mid a\right], \tag{5.61}$$

which is the discrete version of (5.13) in the CLT sense. Also, using the power series of ${}_nF_0(\vec{\nu}; a(z-1))$ we get

$$\langle N\rangle = -\frac{d}{dz}F(1-z)\bigg|_{z=0} = a\tilde{\nu}. \tag{5.62}$$

Changing back for the β notation, we have

$$P_n(N) = \sqrt{\frac{\omega}{2\pi\varepsilon_0}}\frac{1}{\Gamma(\vec{\beta})N!}G^{1,n}_{n,1}\left[{}^{\vec{1}-\vec{\beta}}_{N} \mid \frac{\langle N\rangle}{\omega}\right] \tag{5.63a}$$

with

$$\langle N\rangle = a\omega. \tag{5.63b}$$

That is r random variables $\{N_i\}_{i=1}^{r}$ following the above probability distribution are composed to obtain the observable variable $X = \sum_{i=1}^{r}\frac{N_i-\langle N\rangle}{\sqrt{r}}$ which follows the probability distribution (5.13) of the continuous H theory.

Confirming this result, consider the particular case $n = 0$. Then,

$$P_0(N) = \frac{1}{N!}G^{1,0}_{0,1}\left[{}^{-}_{N} \mid \langle N\rangle\right] = \frac{\langle N\rangle^N}{N!}e^{-\langle N\rangle}, \tag{5.64}$$

which is the Poisson distribution. As was seen in 4.2.2, such distribution satisfies the CLT, converging to the Gaussian distribution, which is the probability distribution of X at the large scale $n = 0$ (see equation (5.2)). And taking $n = 1$, we have

$$P_1(N) = \sqrt{\frac{\omega}{2\pi\epsilon_0}}\binom{N+\beta-1}{N}\frac{a^N}{(1+a)^{N+\beta}} \tag{5.65}$$

and $\langle N \rangle = a\beta$. Such a distribution violates the CLT as was saw above. In fact, from (5.63b) and once the β_j parameters can assume any positive value, we see that only the case $n = 0$ satisfies the CLT because the mean will be the finite parameter a.

Thus, with the results presented in this and previous section, we see that a discrete version of the H theory can be constructed. For $n = 0$ we recover the Gaussianity assumed for the large scale and for $n > 0$ we have the non-Gaussianity observed in the theory and experiments.

Although not clear, may exists a connection between the discretization process in terms of the mimetic map and in the CLT sense which possibly can be found if a discrete version of the Ito calculus is constructed, together with a stochastic mimetic map connecting both continuous and discrete Ito calculus.

6 EPIDEMIC MODELS

In this chapter we present the SIR stochastic epidemic model as a possible application of the mimetic map in stochastic models used to model disease outbreaks. Here we present how the three stochastic approaches can be directly connected using the mimetic map and the discrete calculus in a similar as was done for the H theory in the previous chapter. In the section 6.1, we present the deterministic SIR epidemic model in terms of the Kernack-McKendrick equations, which describes the epidemic spread in terms of a compartmental Petri net model. We present how the parameters indicates the existence of an epidemic and in what conditions the system can reach a disease-free or an endemic equilibrium. Also a parametric solution of the Kernack-McKendrick equations is presented. In section 6.2 we present how a stochastic epidemic model can be constructed, starting from the deterministic one, and the three approaches used to described it. Then we end the chapter showing how these three approaches can be connected using the discrete calculus and the mimetic map.

6.1 DETERMINISTIC SUSCEPTIBLE-INFECTED-REMOVED (SIR) MODEL

For any transmissive disease the individuals of a population, which are not immune to the virus (or bacteria) which is causing the disease, are susceptible to acquire it. After the contamination by the virus, exists a time interval of latency where the virus incubates, and then becomes active; that is when the individuals is notably infected by the disease. After the time interval of infection, the individuals could die or be recovered of the disease, becoming immune to the virus. After some time the immunity of the individual change due to some response of the body or because the virus suffered a mutation, making the individual susceptible again and restarting the cycle. An epidemic consists of cycles like this occurring in a large number of individuals of some population.

Such complicated dynamics could be mathematically modelled by a **Petri Net** (BAEZ; BIAMONTE, 2019), which consists of a network describing the dynamics of a general model, similar to the one above, in terms of compartments. As examples could be the predator-prey dynamics between animals, a chemical reaction or even the creation-annihilation of quantum particles. Here we concentrate in the epidemic model described above.

The Petri nets could be mathematically subdivided in two points of views which are the rate (deterministic) equations and the master (stochastic) equations, presented previously. Those points of views are directly related once the deterministic emerges as an averaged limit of the stochastic one, for a large population.

6.1.1 The Kernack-McKendrick Equations

In 1927 (KERNACK; MCKENDRICK, 1927) was published a mathematical model to describe the spread of contagious diseases in a population, known as Kernack-McKendrick equations, which was capable not only to foresee the number of infected and recovered individuals at a given instant of time but also indicates how many individuals would be recovered at the end of the epidemic, when all the infected got recovered. Such equations predicted such values according to numerical parameters associated which describes the intensity of the interaction between individuals and the individuals reaction to the disease.

If we set S as the number of susceptible, I as the number of infected, R as the number of recovered and N as the number of total individuals of the population, the **Kernack-McKendrick equations** (K-M) are given by

$$\dot{S}(t) = -\frac{\beta}{N}S(t)I(t), \tag{6.1a}$$

$$\dot{I}(t) = \frac{\beta}{N}S(t)I(t) - \gamma I(t), \tag{6.1b}$$

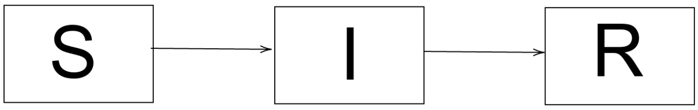
$$\dot{R}(t) = \gamma I(t). \tag{6.1c}$$

Here $\beta > 0$ is the **Transmission rate**, which is inversely proportional to the time of contact between susceptible and infected individuals, and $\gamma > 0$ is the **Recovery rate**, which is inversely proportional to the time of infection of the infected individual. These constants indicates the proportion of individuals which pass from one compartment of individuals to the other, per unit of time.

In that case, the equations in (6.1) describes the dynamics mentioned above, assuming the time of incubation of the virus is insignificant. Thus, the compartmental description of the system is also called **Susceptible-Infected-Individual (SIR) Model** and is represented by the net scheme of figure 9, or by figure 10 in terms of a Petri net.

The K-M equations can be obtained via the Petri net formalism. Observe figure 10 and consider a system of k compartments and l transitions, e.g., infection, recovery, with the

Figure 9 – SIR compartmental model scheme.



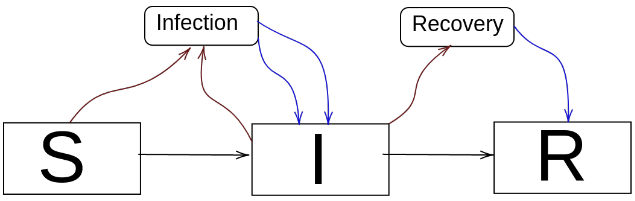
Source: the author (2021).

number of individuals in each compartment given by $\{x_i\}_{i=1}^{k}$. For the jth transition, $m_i^{(j)}$ inputs and $n_i^{(j)}$ outputs of the ith compartment occurs with rate $r_j > 0$. Then, the dynamics of the system is described by ODEs of the form (BAEZ; BIAMONTE, 2019)

$$\frac{dx_i}{dt} = \sum_{j=1}^{l} r_j (n_i^{(j)} - m_i^{(j)}) x_1^{m_1^{(j)}} \ldots x_k^{m_k^{(j)}}. \tag{6.2}$$

Such equations are known as **Rate Equations** In the SIR model the compartments are the susceptible, infected and recovered individuals and the transitions are the process of infection and recovery. From the equation above is easy to see that we obtain (6.1) if $r_1 = \frac{\beta}{N}$ and $r_2 = \gamma$, with $x_1 = S$, $x_2 = I$ and $x_3 = R$.

Figure 10 – Petri net scheme of the SIR model.



Source: the author (2021).

For such a system of ODEs have connection with the real world, the initial conditions should satisfy

$$S(0) > 0,\ I(0) > 0 \text{ and } R(0) \geqslant 0, \tag{6.3}$$

otherwise there is no way of a disease spread. Also, observe that summing the equations in (6.1), we have

$$\dot{S}(t) + \dot{I}(t) + \dot{R}(t) = 0 \tag{6.4}$$

or just

$$S(t) + I(t) + R(t) = const. = N; \ \ \forall t \in [0, \infty), \tag{6.5}$$

once the total number of individuals of the population should be the same, since we disregard the births and deaths of the population in this model. Observe that the equation (6.5) eliminates one of the compartmental variables, say $R(t)$, been necessary to solve the differential equations only for $S(t)$ and $I(t)$.

We can see that the dynamics of the epidemic can be characterized by the values of the rate parameters. In (6.1b), if $\frac{S(0)}{N} > \frac{\gamma}{\beta}$, $\dot{I}(0) > 0$ ensures that the number of infected will increase, causing an epidemic. Otherwise, if $\frac{S(0)}{N} < \frac{\gamma}{\beta}$, $\dot{I}(0) < 0$ and the number of infected decreases to zero monotonically. That impel us to define a quantity, called **Basic Reproduction Number**, by

$$\mathrm{R}_0 \equiv \frac{\beta}{\gamma}\frac{S(0)}{N}, \tag{6.6}$$

or just

$$\mathrm{R}_0 = \frac{\beta}{\gamma}, \tag{6.7}$$

if we set the initial condition as $S(0) \approx N$. Such condition means that a very small number of infected individuals is inserted on the population at time $t = 0$, which is what we will consider from now on. The quantity R_0 denotes the average number of second infected cases as a result of one infected case in a population of only susceptible individuals. Such quantity will be of great importance as we will see.

However (6.1) have its modelling value, it excludes many feature of a disease transmission in the real world. For instance, the lost of immunity, the time of incubation or the effect of vaccination are not taken into account here. Those are elements which can be added to the scheme in figure 9, see (KEELING; ROHANI, 2008). Thus, we can extend the SIR model a bit to include birth and death of the population. In that case, the Kernack-McKendrick equations becomes

$$\dot{S}(t) = -\frac{\beta}{N}S(t)I(t) + \mu(I(t) + R(t)), \tag{6.8a}$$

$$\dot{I}(t) = \frac{\beta}{N}S(t)I(t) - (\mu + \gamma)I(t), \tag{6.8b}$$

$$\dot{R}(t) = \gamma I(t) - \mu R(t), \tag{6.8c}$$

where $\mu > 0$ denotes the **Birth and Death rates**, which are made equal to maintain the total population size N constant, such that $S(t) + R(t) + I(t) = N$. We call it the **SIR Model with birth/death**, which possess reproduction number given by

$$\mathrm{R}_0 = \frac{\beta}{\mu + \gamma}\frac{S(0)}{N} \approx \frac{\beta}{\mu + \gamma}. \tag{6.9}$$

Once the population is constant and the number of susceptibles only decrease, after a long time an equilibrium should be reached, such that

$$\lim_{t\to\infty}(S(t), I(t), R(t)) = (S_\infty, I_\infty, N - S_\infty - I_\infty) \tag{6.10}$$

and

$$\lim_{t\to\infty}(\dot{S}(t), \dot{I}(t), \dot{R}(t)) = (0, 0, 0). \tag{6.11}$$

Thus, we can obtain the stationary solutions of the system solving

$$\begin{cases} \frac{\beta}{N} S_\infty I_\infty - \mu(I_\infty + R_\infty) = 0 \\ \frac{\beta}{N} S_\infty I_\infty - (\mu + \gamma) I_\infty = 0 \\ \gamma I_\infty - \mu R_\infty = 0. \end{cases} \tag{6.12}$$

It will gives

$$(S_\infty, I_\infty, R_\infty) = (N, 0, 0). \tag{6.13}$$

or

$$(S_\infty, I_\infty, R_\infty) = \left(\frac{N}{\mathrm{R}_0}, \frac{\mu N}{\mu + \gamma}(1 - \mathrm{R}_0^{-1}), N - \frac{N}{\mathrm{R}_0} - \frac{\mu N}{\mu + \gamma}(1 - \mathrm{R}_0^{-1})\right). \tag{6.14}$$

The first solution is known as the **Disease-free Equilibrium** and the second is the **Endemic Equilibrium**. Particularly for the second solution, we see that $\mathrm{R}_0 > 1$ once $0 \leqslant S, I, R \leqslant N$. Indeed, can be shown (KEELING; ROHANI, 2008) that if $\mathrm{R}_0 \leqslant 1$ we have a disease-free equilibrium and if $\mathrm{R}_0 > 1$ we have an endemic equilibrium.

6.1.2 Parametric Solutions of the Kernack-McKendrick Equations

The SIR and SIR with birth/death models can be easily solved numerically, as can be seen in figures 11 and 12. To plot the solutions bellow, we rewrite the compartmental variables as

$$(s(t), i(t), r(t)) = \frac{1}{N}(S(t), I(t), R(t)),$$

turning the equations (6.1) into

$$\dot{s}(t) = -\beta s(t) i(t), \tag{6.15a}$$

$$\dot{i}(t) = \beta s(t) i(t) - \gamma i(t), \tag{6.15b}$$

$$\dot{r}(t) = \gamma i(t), \tag{6.15c}$$

and (6.8) into

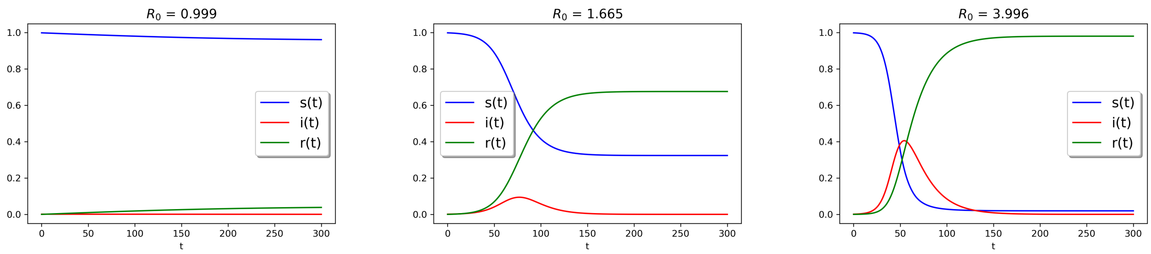
$$\dot{s}(t) = -\beta s(t)i(t) + \mu(i(t) + r(t)), \tag{6.16a}$$

$$\dot{i}(t) = \beta s(t)i(t) - (\mu + \gamma)i(t), \tag{6.16b}$$

$$\dot{r}(t) = \gamma i(t) - \mu r(t), \tag{6.16c}$$

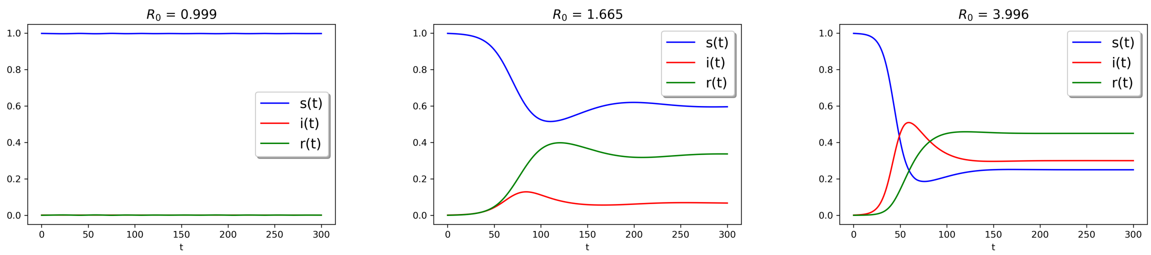
with (6.5) becoming $s(t) + i(t) + r(t) = 1$. Due to the simplicity of those equations, let us stick with the expressions above until otherwise necessary.

Figure 11 – Numerical solutions for $s(t)$ (blue), $i(t)$ (red) and $r(t)$ (green) of the SIR model. Left: $\beta = 0.2$ and $\gamma = 0.2$. Middle: $\beta = 0.2$ and $\gamma = 0.12$. Right: $\beta = 0.2$, $\gamma = 0.05$.



Source: the author (2021).

Figure 12 – Numerical solutions for $s(t)$ (blue), $i(t)$ (red) and $r(t)$ (green) of the SIR with birth/death model, with equal birth and death rates. Left: $\beta = 0.2$, $\gamma = 0.1$ and $\mu = 0.1$. Middle: $\beta = 0.2$, $\gamma = 0.1$ and $\mu = 0.02$. Right: $\beta = 0.2$, $\gamma = 0.03$ and $\mu = 0.02$.



Source: the author (2021).

In figure 11 we observe that a significant number of infected individuals only appears when $\mathrm{R}_0 > 1$, decreasing fast to zero after some time. Here the value of R_0 only influence the maximum value of infected individuals. Yet, in figure 12 we see that the increase of R_0 affects the maximum value as well as the final value of infected individuals, causing the endemic equilibrium when $\mathrm{R}_0 > 1$.

Although numerical solutions have its own value, we can have a better interpretation of the model by showing how exactly it depends on its parameters (β, γ and μ). Therefore, we can decouple the differential equations in (6.1) and (6.8) to solve those analytically, in terms of independent parameters related to the time variable (HARKO; LOBO; MAK, 2014). Let us show it first for the SIR model, without birth/death.

Consider the equations (6.1) and (6.5). Deriving (6.1a), we get a term with $\dot{i}(t)$, allowing to use (6.1b) to obtain

$$\frac{\ddot{s}}{s} - \left(\frac{\dot{s}}{s}\right)^2 + \gamma\frac{\dot{s}}{s} - \beta\dot{s} = 0. \tag{6.17}$$

And using (6.1a) in (6.1c) we have

$$\dot{r} = -\frac{\gamma}{\beta}\frac{\dot{s}}{s}. \tag{6.18}$$

Assuming that $r(t) = r(s(t))$, we can solve it to obtain

$$s(t) = s(0)e^{-\frac{\beta}{\gamma}r(t)}, \tag{6.19}$$

with $s(0) = e^{\frac{\beta}{\gamma}r(0)}$.

Thus, differentiating (6.18) and (6.19) and putting it all in (6.17) we get

$$\ddot{r} = \beta s(0)\dot{r}e^{-\frac{\beta}{\gamma}r(t)} - \gamma\dot{r}. \tag{6.20}$$

Thus, solving (6.17) we have the solution of (6.15), using (6.19) and $s(t) + i(t) + r(t) = 1$.

To do that, we make the following change of variables:

$$u = e^{-\frac{\gamma}{\beta}r(t)} \text{ and } \xi = \frac{dt}{du}, \tag{6.21}$$

transforming (6.20) in

$$\frac{d\xi}{du} + \frac{1}{u}\xi = (\gamma - \beta s(0)u)\xi^2, \tag{6.22}$$

which is a Bernoulli differential equation with solution

$$\xi(u) = \frac{1}{u(C - \gamma\ln u + \beta s(0)u)}, \tag{6.23}$$

for a constant of integration C. Then,

$$t - t_0 = \int_{u_0}^{u} \frac{du'}{u'(C - \gamma\ln u' + \beta s(0)u')}, \tag{6.24}$$

where $t_0 = 0$ and $u_0 = u(0) = e^{-\frac{\gamma}{\beta}r(0)}$.

Thus, solving the above integral we can find the expression $t(u)$, or $u(t)$, such that

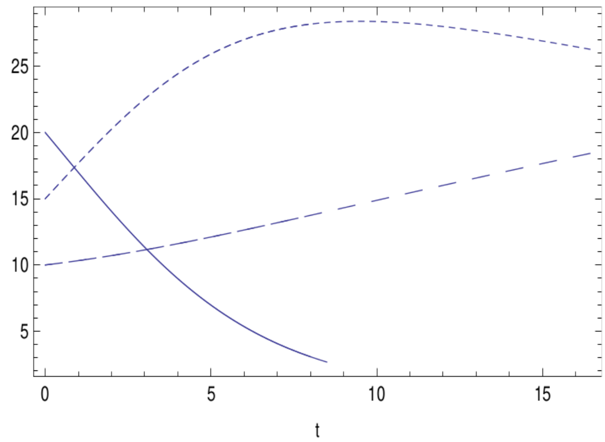
$$s(t) = s(0)u(t), \tag{6.25a}$$

$$i(t) = \frac{\gamma}{\beta}\ln u(t) - s(0)u(t) + 1, \tag{6.25b}$$

$$r(t) = \frac{\gamma}{\beta}\ln u(t). \tag{6.25c}$$

Thus we have the parametric analytic solutions for the SIR model in terms of a parameter u, given by (6.25) together with (6.24). As can be seen comparing figures 13 and 14, the plot of the parametric solution agrees perfectly with the numeric solution. In this plot the $S(t)$, $I(t)$ and $R(t)$ has been used again.

Figure 13 – Parametric solutions for S(t) (solid curve), I(t) (dotted curve) and R(t) (dashed curve) of the SIR model. Here $\beta = 0.01N = 0.45$ and $\gamma = 0.02$, and the initial conditions are $S(0) = 20$, $I(0) = 15$, $R(0) = 10$.



Source: (HARKO; LOBO; MAK, 2014).

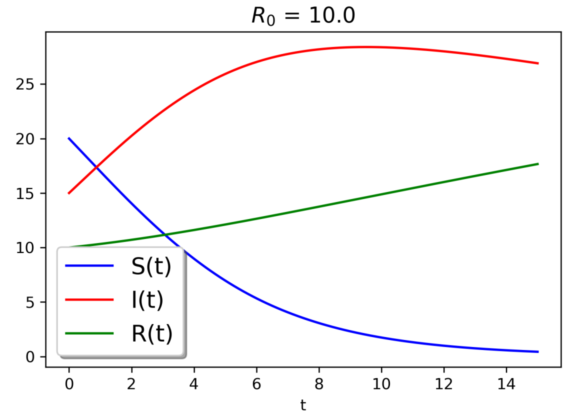
A parametric solution can also be obtained for the SIR model with equal birth/death rates. In that case, what we obtain is the Abel differential equation of first kind given by

$$\frac{dv}{d\zeta} = \left(a + \frac{b}{\zeta}\right)v^3 + \left(c + \frac{\mu}{\zeta}\right)v^2, \tag{6.26}$$

where

$$\zeta = \dot{r} + \mu r \quad \text{and} \quad v = \zeta\frac{dt}{d\zeta}, \tag{6.27}$$

Figure 14 – Numerical solutions for S(t) (blue), I(t) (red) and R(t) (green) of the SIR model. Here $\beta = 0.01N = 0.45$ and $\gamma = 0.02$, and the initial conditions are $S(0) = 20$, $I(0) = 15$, $R(0) = 10$.



Source: the author (2021).

with the constants

$$a = \beta\left(\frac{\mu}{\gamma} + 1\right) \;,\; b = \mu(\mu + \gamma - \beta N) \;\text{ and }\; c = \frac{\beta}{\gamma}. \tag{6.28}$$

Since the ODE above do not have a closed solution, we approximate it by an iterative solution. Making the change of variable

$$\Xi = \ln v, \tag{6.29}$$

(6.27) becomes

$$\frac{d\Xi}{d\zeta} = \left(a + \frac{b}{\zeta}\right) e^{2\Xi} + \left(c + \frac{\mu}{\zeta}\right) e^{\Xi}. \tag{6.30}$$

The iterative method consists of expands the factors e^{Ξ} as

$$e^{\Xi} = 1 + \Xi + \frac{(\Xi)^2}{2!} + \frac{(\Xi)^3}{3!} + \ldots,$$

retain the first order term and obtain the approximated solution Ξ_1 of the approximated version of the differential equation, namely

$$\frac{d\Xi_1}{d\zeta} = \left(2a + c + \frac{2b + \mu}{\zeta}\right) \Xi_1 + a + c + \frac{b + \mu}{\zeta}. \tag{6.31}$$

After that, one do the same retaining the high order terms and assuming the validity of the previous order differential equation solution, obtaining in general

$$\frac{d\Xi_n}{d\zeta} = \left(2a + c + \frac{2b + \mu}{\zeta}\right) \Xi_n + a + c + \frac{b + \mu}{\zeta} + \sum_{k=2}^{\infty} \left[2^k \left(a + \frac{b}{\zeta}\right) + \left(c + \frac{\mu}{\zeta}\right)\right] \frac{\Xi_{n-1}^k}{k!}, \tag{6.32}$$

for $n > 1$. The solution for the above differential equation is given by

$$\Xi_n(\zeta) = \Xi_0(\zeta) + e^{(2a+c)\zeta}\zeta^{2b+\mu}\int e^{-(2a+c)\zeta}\zeta^{-(2b+\mu)}\sum_{k=2}^{\infty}\left[2^k\left(a+\frac{b}{\zeta}\right)+\left(c+\frac{\mu}{\zeta}\right)\right]\frac{\Xi_{n-1}^k}{k!}d\zeta. \tag{6.33}$$

The general solution of (6.30) is obtained taking the limit

$$\Xi(\zeta) = \lim_{n\to\infty}\Xi_n(\zeta). \tag{6.34}$$

With that, the parametric solutions of (6.16) is given by

$$s(\zeta) = \Lambda(\zeta)\left[\int_1^{\zeta}\frac{\mu e^{\Xi(\chi)}\Lambda(\chi)}{\chi}d\chi - \int_1^{\gamma i(0)}\frac{\mu e^{\Xi(\chi)}\Lambda(\chi)}{\chi}d\chi + s(0)\Lambda(\gamma i(0))\right], \tag{6.35a}$$

$$i(\zeta) = \frac{\zeta}{\gamma}, \tag{6.35b}$$

$$r(\zeta) = \Psi(\zeta)^{-1}\left[\int_1^{\zeta}e^{\Xi(\chi)}\Psi(\chi)d\chi - \int_1^{\gamma i(0)}e^{\Xi(\chi)}\Psi(\chi)d\chi + r(0)\Psi(\gamma i(0))\right] \tag{6.35c}$$

with $\Lambda(x) \equiv e^{-\int_1^x \frac{e^{\Xi(\eta)}(c\eta+\mu)}{\eta}d\eta}$ and $\Psi(x) \equiv e^{\mu\int_1^x \frac{e^{\Psi(\eta)}}{\eta}d\eta}$.

Therefore, we have analytic parametric solutions for the SIR model with birth/death rates in terms of the parameter ζ, considering the relation (6.27). However, notice how complicated became the solution just adding the contribution of the birth/death rate μ to the model, imposing limits to which situations that kind of solution could be obtained.

In what follows we pass to the stochastic description of the SIR model and how the discrete calculus could be used to connect the stochastic approaches.

6.2 STOCHASTIC SIR MODEL

In the previous section was presented the deterministic approach for the SIR model, which well describes the main features of the model. However, such approach gives good results only when the total size of the population is large, ignoring possible errors in the estimation of the rates and the possible fluctuations of the compartmental variables (BRITTON, 2010), making the deterministic approach an idealization. To correct this, we interpret the compartmental variables not as functions but rather as random variables (ALLEN, 2008).

Although not presented here, the disease-free/endemic equilibrium conditions can also be described in the stochastic SIR formulation.

6.2.1 Discrete Time Markov Chain

Consider a discrete stochastic process $\{X_t\}_{t\in T}$ for $T = \Delta t\mathbb{N}_0$, with $\Delta t < \infty$, and $X_t \in \mathbb{Z}$. This stochastic process is defined in such a way that in a time interval Δt, only the transitions

$$X_t \to X_{t+\Delta t} = X_t - 1 \ , \quad X_t \to X_{t+\Delta t} = X_t \quad \text{or} \quad X_t \to X_{t+\Delta t} = X_t + 1$$

are allowed. In that case, the probability distribution of X_t is defined by

$$P_n(t) := Prob(X_t = n), \tag{6.36}$$

with conditional probability given by the Markov assumption (4.52)

$$P(n', t + \Delta t|n, t) = Prob(X_{t+\Delta t} = n'|X_t = n), \tag{6.37}$$

where n' can only be $n-1$, n or $n+1$. Such a construction defines a **Discrete Time Markov Chain** (DTMC).

Since it is a Markovian process, we can represent the dynamics of the probability distribution as a law of total probability for a birth-death stochastic process given by

$$P_n(t + \Delta t) = \zeta_{n+1}\Delta t P_{n+1}(t) + \gamma_{n-1}\Delta t P_{n-1}(t) + [1 - (\zeta_n + \gamma_n)\Delta t]P_n(t), \tag{6.38}$$

where the coefficients are proportional to the conditional probabilities by

$$P(n', t + \Delta t|n, t) = \begin{cases} \gamma_n \Delta t, & n' = n + 1 \\ \zeta_n \Delta t, & n' = n - 1 \\ 1 - (\gamma_n + \zeta_n)\Delta t, & n' = n. \end{cases} \tag{6.39}$$

To accept it, just remember that the coefficients ζ and γ appearing in the master equations can be interpreted as probabilities per unit of time, as discussed in subsection 4.4.1.2. And since the sum of all probabilities is equal to one, the case $n' = n$ can only have this form. Notice that to assure that these probabilities are well defined, Δt should be chosen such that

$$\max\{(\gamma_n + \zeta_n)\Delta t\} \leqslant 1, \ \forall n.$$

Observe that equation (6.38) can be viewed actually as a difference equation, revealing that the DTMC can be seen as a discrete case of a differential equation.

Thus, to construct a **DTMC SIR Model** we consider the discrete random variables S_t, I_t and R_t, which represent the number of susceptible, infected and recovered individuals of a population, respectively. And since the total size of the population N is finite, we have

$$S_t, I_t, R_t \in \{0, 1, \ldots, N\} \tag{6.40}$$

such that

$$S_t + I_t + R_t = N. \tag{6.41}$$

Which means that the probability distributions can be defined in terms of S_t and I_t only, as a bivariate probability distributions given by

$$P_{s,i}(t) = Prob(S_t = s, I_t = i) \tag{6.42}$$

and

$$P(s', i', t + \Delta t | s, i, t) = Prob(S_{t+\Delta t} = s', I_{t+\Delta t} = i' | S_t = s, I_t = i). \tag{6.43}$$

Then the conditional probabilities are defined by

$$P(s', i', t + \Delta t | s, i, t) = \begin{cases} \frac{\beta}{N} is\Delta t, & (s', i') = (s - 1, i + 1) \\ \gamma i\Delta t, & (s', i') = (s, i - 1) \\ \mu i\Delta t, & (s', i') = (s + 1, i - 1) \\ \mu(N - s - i)\Delta t, & (s', i') = (s + 1, i) \\ 1 - \left[\frac{\beta}{N} is + \gamma i + \mu(N - s)\right]\Delta t, & (s', i') = (s, i) \end{cases} \tag{6.44}$$

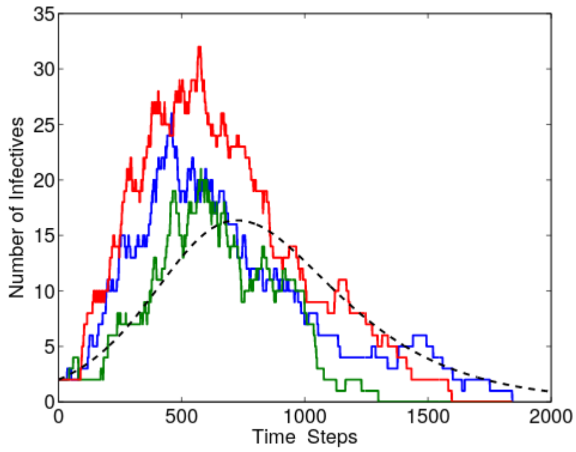
From the Petri nets point of view it is easy to see how those coefficients can be obtained (BAEZ; BIAMONTE, 2019). Once the rates defined in (6.1) are interpreted as a frequency of individuals passing from one compartment to the other per unit time, the probability of a transition can be given by the rate times the time interval Δt which the transition occurs times the total number of ways the transition can occur. For instance, the probability for the transition $(s, i) \to (s - 1, i + 1)$ occurs is $\frac{\beta}{N}\Delta t$, but it occurs in SI ways, once an infected individual appears from the interaction of one infected and one susceptible. Thus, the conditional probability is $\frac{\beta}{N} SI\Delta t$. The same reasoning can be followed for the other conditional probabilities.

Then, the dynamics of the probability distribution (6.42) can be described by

$$\begin{aligned} P_{s,i}(t+\Delta t) =& \frac{\beta}{N}(i-1)(s+1)\Delta t P_{s+1,i-1}(t) + \gamma(i+1)\Delta t P_{s,i+1}(t) \\ &+ \mu(i+1)\Delta t P_{s-1,i+1}(t) + \mu(N-s+1-i)\Delta t P_{s-1,i}(t) \\ &+ \left\{1 - \left[\frac{\beta}{N} is + \gamma i + \mu(N-s)\right]\Delta t\right\} P_{s,i}(t). \end{aligned} \tag{6.45}$$

In the figure 15 we see how the trajectories of the stochastic process, in average, agree with the deterministic solution. Also, we see that the discrete random variable makes the trajectories discontinuous.

Figure 15 – Trajectories of the number of infected individuals of the DTMC SIR model and solution of the the number of infected individuals of the deterministic SIR model (dashed curve). Here $\Delta t = 0.01$, $N = 100$, $\beta = 1.0$, $\mu = 0$ and $\gamma = 0.5$, with initial conditions $S_0 = 98$, $I_0 = 2$ and $R_0 = 0$. The reproduction number is $\mathrm{R}_0 = 1.96$.



Source: (ALLEN, 2008).

6.2.2 Continuous Time Markov Chain

The **Continuous Time Markov Chain** (CTMC) approach distinguishes from the DTMC only by the definition of the time variable, defining a discrete stochastic process $\{X_t\}_{t\in T}$ with $T = [0, \infty)$, instead of $\Delta t \mathbb{N}_0$. As a consequence, the conditional probability is defined by

$$P(n', t_{j+1}|n, t_j) = Prob(X_{j+1} = n'|X_j = n), \tag{6.46}$$

denoting $X_j = X_{t_j}$, for a time partition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n < t_{n+1} < \ldots$. The difference $t_n - t_{n-1} = \Delta t$, is now assumed to be an infinitesimal quantity. Thus, taking the limit of $\Delta t \to 0$, the law of total probability becomes a birth-death master equation given by

$$\dot{P}_n(t) = \zeta_{n+1}P_{n+1}(t) + \gamma_{n-1}P_{n-1}(t) - (\zeta_n + \gamma_n)P_n(t), \tag{6.47}$$

for

$$\dot{P}_n(t) = \lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t}. \tag{6.48}$$

Now we have the conditional probabilities as infinitesimal quantities given by

$$P(n', t_{j+1}|n, t_j) = \begin{cases} \gamma_n \Delta t + O(\Delta t), & n' = n+1 \\ \zeta_n \Delta t + O(\Delta t), & n' = n-1 \\ 1 - (\gamma_n + \zeta_n)\Delta t + O(\Delta t), & n' = n; \end{cases} \tag{6.49}$$

such that $\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0$.

By this approach the stochastic **CTMC SIR Model** have conditional probabilities given by

$$P(s', i', t+\Delta t|s, i, t) = \begin{cases} \frac{\beta}{N} is\Delta t + O(\Delta t), & (s', i') = (s-1, i+1) \\ \gamma i \Delta t + O(\Delta t), & (s', i') = (s, i-1) \\ \mu i \Delta t + O(\Delta t), & (s', i') = (s+1, i-1) \\ \mu(N-s-i)\Delta t + O(\Delta t), & (s', i') = (s+1, i) \\ 1 - \left[\frac{\beta}{N} is + \gamma i + \mu(N-s)\right]\Delta t + O(\Delta t), & (s', i') = (s, i); \end{cases} \tag{6.50}$$

resulting in the master equation

$$\begin{aligned} \dot{P}_{s,i}(t) =& \frac{\beta}{N}(i-1)(s+1)P_{s+1,i-1}(t) + \gamma(i+1)P_{s,i+1}(t) \\ &+ \mu(i+1)P_{s-1,i+1}(t) + \mu(N-s+1-i)P_{s-1,i}(t) \\ &- \left[\frac{\beta}{N} is + \gamma i + \mu(N-s)\right] P_{s,i}(t). \end{aligned} \tag{6.51}$$

6.2.3 Stochastic Differential Equations for Epidemic Models

In this last approach we define a SDE of the stochastic process given by $\{X_t\}_{t\in T}$, for $T = [0, \infty)$ and $X_t \in \mathbb{R}$, been time and random variable both continuous. Thus, the probabilities are given by the PDF

$$Prob(a \leqslant X_t \leqslant b) = \int_a^b \mathcal{P}(x, t)dx \tag{6.52}$$

and the conditional PDF

$$Prob(X_{j+1} \leqslant x'|X_j \leqslant x) = \int_{-\infty}^{x'} \int_{-\infty}^{x} \mathcal{P}(z, t_{j+1}|y, t_j) dy dz. \tag{6.53}$$

To determine the F-P equation satisfied by the PDF above, we deduce the SDE estimating it by the Euler-Maruyama method (ALLEN, 2017), which is used to solve SDEs numerically, applied to the CTMC model. Thus, consider the random variable $\Delta X = X_{t+\Delta t} - X_t$, that has a Gaussian distribution for small values of Δt. In the CTMC approach it will possess mean and variance given by

$$\langle \Delta X \rangle = (\gamma_n - \zeta_n)\Delta t + O(\Delta t) \tag{6.54}$$

and

$$\sigma^2_{\Delta X} = (\gamma_n + \zeta_n)\Delta t + O(\Delta t). \tag{6.55}$$

The Euler-Maruyama method consists in write the SDE of X_t as (ALLEN, 2017)

$$\Delta X \approx \langle \Delta X \rangle \Delta t + \sigma_{\Delta X} \eta \sqrt{\Delta t}, \tag{6.56}$$

where η is a random variable with normal distribution. In the limit of $\Delta t \to 0$, we obtain the SDE

$$dX = (\gamma(x) - \zeta(x))dt + \sqrt{(\gamma(x) + \zeta(x))} dW(t), \tag{6.57}$$

where γ_n and ζ_n became $\gamma(x)$ and $\zeta(x)$, passing from discrete to continuous random variables. Therefore, we can obtain the F-P equation for $\mathcal{P}(x,t)$ as

$$\partial_t \mathcal{P}(x,t) = -\partial_x[(\gamma(x) - \zeta(x))\mathcal{P}(x,t)] + \frac{1}{2}\partial_x^2[(\gamma(x) + \zeta(x))\mathcal{P}(x,t)]. \tag{6.58}$$

To construct a SIR model based in SDEs, we consider a bivariate PDF associated with the continuous random variables S_t and I_t, such that

$$Prob(s_0 \leqslant S_t \leqslant s_n, i_0 \leqslant I_t \leqslant i_n) = \int_{s_0}^{s_n} \int_{i_0}^{i_n} \mathcal{P}(s, i, t) ds di \tag{6.59}$$

with conditional PDF

$$Prob(S_{j+1} \leqslant s', I_{j+1} \leqslant i'|S_j \leqslant s, I_j \leqslant i) = \int_0^{s'} \int_0^{i'} \int_0^{s} \int_0^{i} \mathcal{P}(x', y', t_{j+1}|x, y, t_j) dy dx dy' dx'. \tag{6.60}$$

Now we should have here a pair of SDEs to describe the evolution of the random variables and, instead of the variance, we need to estimate the covariance matrix. Thus, for the random vector $\Delta\vec{X} = (\Delta S, \Delta I)^T = (S_{t+\Delta t} - S_t, I_{t+\Delta t} - I_t)^T$ we have

$$\left\langle \Delta\vec{X} \right\rangle = \begin{pmatrix} -\frac{\beta}{N} SI + \mu(N - S) \\ \frac{\beta}{N} SI - (\mu + \gamma) I \end{pmatrix} \Delta t \tag{6.61}$$

and

$$\left\langle \Delta\vec{X}, \Delta\vec{X} \right\rangle = \begin{pmatrix} \frac{\beta}{N}SI + \mu(N-S) & -\frac{\beta}{N}SI - \mu I \\ -\frac{\beta}{N}SI - \mu I & \frac{\beta}{N}SI + (\mu+\gamma)I \end{pmatrix} \Delta t. \tag{6.62}$$

The Euler-Maruyama method gives

$$\Delta\vec{X} \approx \left\langle \Delta\vec{X} \right\rangle + B\,\vec{\eta}, \tag{6.63}$$

for the random vector $\vec{\eta} = (\eta_1, \eta_2)^T$ with normal distributed random variables η_1 and η_2, and the 2×2 matrix B satisfying the equation

$$BB^T = \left\langle \Delta\vec{X}, \Delta\vec{X} \right\rangle.$$

Then the SDEs for the SIR model with birth/death are

$$dS = \left[-\frac{\beta}{N}SI + \mu(N-S) \right] dt + B_{11}dW_1(t) + B_{12}dW_2(t) \tag{6.64}$$

and

$$dI = \left[ ]\frac{\beta}{N}SI - (\mu+\gamma)I \right] dt + B_{21}dW_1(t) + B_{22}dW_2(t), \tag{6.65}$$

where $dW_1(t)$ and $dW_2(t))$ are independent Wiener processes.

Since can be complicated to obtain B for the SIR model with birth/death, we set $\mu = 0$ to stick with the SIR model without birth/death. Making easier to obtain the SDEs (ALLEN, 2017)

$$dS = -\frac{\beta}{N}SIdt - \sqrt{\frac{\beta}{N}SI}dW_1(t) \tag{6.66}$$

and

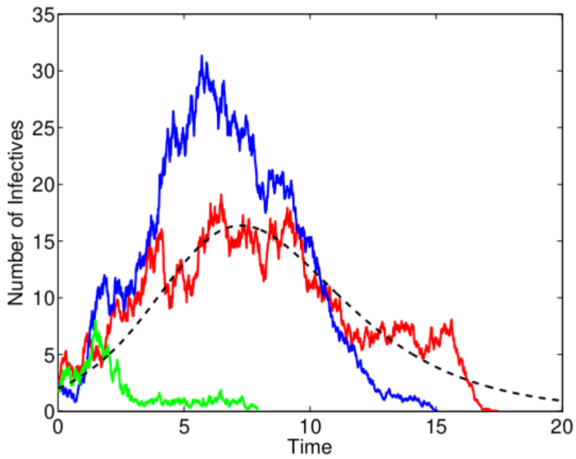
$$dI = \left(\frac{\beta}{N}SI - \gamma I \right) dt + \sqrt{\frac{\beta}{N}SI}dW_1(t) - \sqrt{\gamma I}dW_2(t). \tag{6.67}$$

Using Ito's formula for n random variables (subsection 4.6.2.3), we obtain

$$\begin{aligned} \partial_t \mathcal{P}(s,i,t) =& \partial_s \left(\frac{\beta}{N}si\mathcal{P}(s,i,t) \right) - \partial_i \left[\left(\frac{\beta}{N}si - \gamma i \right) \mathcal{P}(s,i,t) \right] \\ &+ \frac{1}{2}\partial_s^2 \left(\frac{\beta}{N}si\mathcal{P}(s,i,t) \right) + \frac{1}{2}\partial_i^2 \left[\left(\frac{\beta}{N}si + \gamma i \right) \mathcal{P}(s,i,t) \right] \\ &- \partial_s\partial_i \left(\frac{\beta}{N}si\mathcal{P}(s,i,t) \right). \end{aligned} \tag{6.68}$$

Again, we see in figure 16 how the stochastic trajectories agrees, in average, with the deterministic solution. Also, observe that the stochastic trajectories are continuous, which now occurs due to the continuous stochastic variable assumption of the SDE approach.

Figure 16 – Trajectories of the number of infected individuals of the stochastic differential equations of the SIR model and solution of the the number of infected individuals of the deterministic SIR model (dashed curve). Here $\Delta t = 0.01$, $N = 100$, $\beta = 1.0$, $\mu = 0$ and $\gamma = 0.5$, with initial conditions $S_0 = 98$, $I_0 = 2$ and $R_0 = 0$. The reproduction number is $\mathrm{R}_0 = 1.96$.



Source: (ALLEN, 2008).

6.2.4 The Role of Discretization Connecting these Approaches

The dynamics of the DTMC and the CTMC processes are described by the equations (6.38) and (6.47), respectively. It is easy to see that the transition from one equation to the other can be done following the same procedure made for the construction of the master equation on section 4.3. The inverse path can be done if we apply the mimetic map in the term $\dot{P}_n(t)$ of (6.47) to get

$$\dot{P}_n(t) \to D_t^{\Delta} P_n(t), \tag{6.69}$$

with $h = \Delta t$.

We should expect that a similar procedure can be applied to connect the master equation of the CTMC and the F-P of the SDE approach. As a matter of fact, (ALLEN, 2008) shows that (6.47) can be rewritten as

$$\begin{aligned} \dot{P}_n(t) = &- \frac{[(\gamma_{n+1} - \zeta_{n+1})P_{n+1}(t) - (\gamma_{n-1} - \zeta_{n-1})P_{n-1}(t)]}{2\Delta n} \\ &+ \frac{1}{2}\frac{[(\gamma_{n+1} + \zeta_{n+1})P_{n+1}(t) - 2(\gamma_n + \zeta_n)P_n(t) + (\gamma_{n-1} + \zeta_{n-1})P_{n-1}(t)]}{(\Delta n)^2} \end{aligned} \tag{6.70}$$

or, in difference operator notation, as

$$\dot{P}_n(t) = - D_n[(\gamma_n - \zeta_n)P_n(t)]\big|_{h=2\Delta n} + D_n^2[(\gamma_n + \zeta_n)P_n(t)]\big|_{h=\Delta n}, \tag{6.71}$$

where $D_x(\widehat{f}_h(x))\Big|_{h=\Delta x} = \frac{\widehat{f}_{\Delta x}(x+\Delta x)-\widehat{f}_{\Delta x}(x)}{\Delta x}$. Then, taking the limit $\Delta n \to 0$, we obtain (6.58). In that case the sequence $P_n(t)$ becomes the function $\mathcal{P}(x,t)$ and the coefficients γ_n and ζ_n become the functions $\gamma(x)$ and $\zeta(x)$.

Once again, the connection between discrete and continuous stochastic processes is made possible by the mimetic map and discrete calculus. We observe that if the mimetic map was directly applied to the F-P equation (6.58) of the SDE approach we would not obtain the CTMC master equation due to the stochastic terms of the F-P equation, coming from the stochastic terms of the SDE. It may occurs because the probabilistic feature of the system can not be captured by the mimetic map, which discretizes the deterministic infinitesimal calculus. The situation becomes even more complicated if the SDE SIR model is considered once it involves two random variables with constraints caused by the physical behavior of the system. This situation could be overcome by the construction of a mimetic map which captures such feature, a stochastic mimetic map, which would be a generalization of the current mimetic map.

Such a map would generate a discrete version of the stochastic calculus presented in section 4.6. It would be responsible to map stochastic differential equations onto stochastic difference equations, allowing the obtainment of the Master equations from those, analogously to the F-P equations obtained via Ito's formula.

7 CONCLUSIONS

The connection between a infinitesimal and a discrete substratum to construct mathematical models of Nature seems of great relevance to narrow the connection between the mathematical and experimental/numerical descriptions. Thus, it was proposed in this dissertation how it can be achieved via an integral transform, called mimetic map, connecting all the structure of the infinitesimal calculus with a discrete calculus which preserves such a structure, presenting applications in mathematics and physical stochastic processes.

In conclusion, we saw along the dissertation that the space of functions can be mapped onto a space of discrete functions via an integral transform, making it possible to obtain the discrete h-calculus from the infinitesimal calculus in a very systematic way. With that it was possible not only to obtain the already known discrete calculus, but also to generalize it providing a direct connection between functions and differential equations to discrete functions and difference equations, respectively. It is worth mentioning that only with this systematic procedure it was possible to obtain discrete versions of the incomplete gamma, the complex confluent hypergeometric functions and some orthogonal polynomials. Plus, due to the mimetic map it was shown that it is possible to construct discrete versions of integral transforms, relating it to already known discrete transforms, like the h-Laplace and the Z transform, and also obtaining new discrete integral transforms, like the h-Mellin transform. It was also shown that the sequences and difference equations can be generalized to include steps depending on a parameter h, where the standard objects appears in the particular case $h = 1$. It can be seen by how the method to solve difference equations via generating function is extended to the h-generating function.

The application of the mimetic map in the physical context proved to be useful as a tool to relate discrete and continuous stochastic processes due to the mapping of F-P equations onto master equations and vice-versa. It was capable to present how the gamma distribution emerges as a continuum limit of the negative binomial distribution, or more general, it was possible to construct a discrete version of the description of the background variable of the H theory described in terms of master equations instead of F-P equations. Motivated by that, once the central limit theorem consists of a continuum limit of composed discrete random variables, we could construct a discrete H theory for the observable variable, from where the standard H theory emerges in the central limit sense. Also, applications of the mimetic map in stochastic epidemic models are foreseen once we can move from one of the approaches (DTMC, CTMC, SDE) to the other via the mimetic map and the discrete calculus.

Therefore the discrete calculus, and specially the mimetic map, provides theoretical and numerical tools to a deeper understanding of many areas of science. In the numerical tools context, the discretization procedure provides a new way to look to continuous quantities for simulations, giving the evolution of a PDF of a continuous variable in terms of a probability distribution of a discrete one, for instance. Or, equivalently, the numerical solution of a F-P equation given in terms of the numerical solution of a master equation.

In the discretization via mimetic map of the F-P equations of the case $s = 1$ (5.39) of the H theory and of the stochastic epidemic model (6.68) we expected that a birth-death master equation would be obtained which did not happen (see (6.51)). A possibility is that a discrete Ito calculus could provide a set of tools to obtain discrete SDEs from the standard one, associating master equations to those, instead of F-P equations. Such a discrete Ito calculus could also be used to obtain a discrete version of a stochastic variational method (KOIDE; KODAMA; TSUSHIMA, 2015), similarly to the discrete variational method mentioned in the introduction of the dissertation (DESBRUN et al., 2005).

Many applications of the mimetic map will be explored by us in a near future in many topics of mathematics, as in discrete analogues of mathematical objects such as monodromy (IWASAKI et al., 1991), the construction of a discrete Clifford algebra (SCHEPPER; SOMMEN; VOORDE, 2009) and further applications in number theory will be explored, which may lead to connections with statistical mechanics (KNAUF, 1999). In computational science, useful algorithms for finite difference methods can be developed or the current ones can be updated.

In physics, we will use it to connect continuum models to lattice models, like the Schramm-Loewner evolution (CARDY, 2005) and the Hubbard model, it can have applications in the renormalization group (PANZER, 2015) and the corresponding regularization procedure. Also some applications in quantum mechanics will be explored, connecting systems described by continuous observables (continuous eigenvalues) with the ones described by discrete observables (discrete eigenvalues).

REFERENCES

ABBOTT, D.; DAVIS, B.; PHILLIPS, N.; ESHRAGHIAN, K. Simple derivation of the thermal noise formula using window-limited fourier transforms and other conundrums. *IEEE Transactions on Education*, v. 39, n. 1, p. 1–13, 1996.

ABRAMOWITZ, I. A. S. M. *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables,*. USA: Dover Publications, Inc., 1974. ISBN 0486612724.

ALLEN, L. J. A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis. *Infectious Disease Modelling*, v. 2, n. 2, p. 128–142, 2017. ISSN 2468-0427. Disponível em: <https://www.sciencedirect.com/science/article/pii/S2468042716300495>.

ALLEN, L. J. S. *An Introduction to Stochastic Epidemic Models*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008. 81–130 p. ISBN 978-3-540-78911-6. Disponível em: <https://doi.org/10.1007/978-3-540-78911-6_3>.

APPLEBAUM, D. *Lévy Processes and Stochastic Calculus*. 2. ed. [S.l.]: Cambridge University Press, 2009. (Cambridge Studies in Advanced Mathematics).

BAEZ, J. C.; BIAMONTE, J. *Quantum Techniques for Stochastic Mechanics*. 2019.

BALACHANDRAN, A.; BIMONTE, G.; ERCOLESSI, E.; TEOTONIO-SOBRINHO, P. Finite approximations to quantum physics: Quantum points and their bundles. *Nuclear Physics B*, Elsevier BV, v. 418, n. 3, p. 477–493, May 1994. ISSN 0550-3213. Disponível em: <http://dx.doi.org/10.1016/0550-3213(94)90527-4>.

BALACHANDRAN, A.; BIMONTE, G.; ERCOLESSI, E.; LANDI, G.; LIZZI, F.; SPARANO, G.; TEOTONIO-SOBRINHO, P. Finite quantum physics and noncommutative geometry. *Nuclear Physics B - Proceedings Supplements*, Elsevier BV, v. 37, n. 3, p. 20–45, Jan 1995. ISSN 0920-5632. Disponível em: <http://dx.doi.org/10.1016/0920-5632(94)00787-V>.

BASSALO, J.; CATTANI, M. *Cálculo exterior*. [S.l.]: Editora Livraria da Física, 2009. ISBN 9788578610265.

BEALS, R.; SZMIGIELSKI, J. Meijer g-functions: a gentle introduction. *Notices of the American Mathematical Society*, v. 60, p. 866–873, 2013.

BECK, C. Dynamical foundations of nonextensive statistical mechanics. *Phys. Rev. Lett.*, American Physical Society, v. 87, p. 180601, Oct 2001. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevLett.87.180601>.

BOHNER, M.; GUSEINOV, G. S. The h-laplace and q-laplace transforms. *Journal of Mathematical Analysis and Applications*, v. 365, n. 1, p. 75–92, 2010. ISSN 0022-247X. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0022247X09008178>.

BRITTON, T. Stochastic epidemic models: A survey. *Mathematical Biosciences*, v. 225, n. 1, p. 24–35, 2010. ISSN 0025-5564. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0025556410000143>.

BROWN, e. a. R. Xxvii. a brief account of microscopical observations made in the months of june, july and august 1827, on the particles contained in the pollen of plants;

and on the general existence of active molecules in organic and inorganic bodies. *The Philosophical Magazine*, Taylor & Francis, v. 4, n. 21, p. 161–173, 1828. Disponível em: <https://doi.org/10.1080/14786442808674769>.

BYWATER, J. *Physiological Fragments: To Which are Added Supplementary Observations, to Shew that Vital Energies are of the Same Nature, and Both Derived from Solar Light*. [S.l.]: London: R. Hunter, 1824. 127-128 p.

CARDY, J. Sle for theoretical physicists. *Annals of Physics*, Elsevier BV, v. 318, n. 1, p. 81–118, Jul 2005. ISSN 0003-4916. Disponível em: <http://dx.doi.org/10.1016/j.aop.2005.04.001>.

COX, J. C.; INGERSOLL, J. E.; ROSS, S. A. A theory of the term structure of interest rates. *Econometrica*, [Wiley, Econometric Society], v. 53, n. 2, p. 385–407, 1985. ISSN 00129682, 14680262. Disponível em: <http://www.jstor.org/stable/1911242>.

DENNERY, P.; KRZYWICKI, A. *Mathematics for Physicists*. [S.l.]: Dover Publications, 1996.

DESBRUN, M.; HIRANI, A. N.; LEOK, M.; MARSDEN, J. E. Discrete exterior calculus. *arXiv preprint math/0508341*, 2005.

DURRETT, R. *Probability: Theory and Examples*. 5. ed. [S.l.]: Cambridge University Press, 2019. (Cambridge Series in Statistical and Probabilistic Mathematics).

ERDéLYI, e. a. A. *Higher Transcendetal Functions*. [S.l.]: McGraw-Hill Book Company, 1953.

FELLER, W. Two singular diffusion problems. *Annals of Mathematics*, Annals of Mathematics, v. 54, n. 1, p. 173–182, 1951. ISSN 0003486X. Disponível em: <http://www.jstor.org/stable/1969318>.

FRISCH, U.; KOLMOGOROV, A. N. *Turbulence: The Legacy of A. N. Kolmogorov*. [S.l.]: Cambridge University Press, 1995. ISBN 9780521457132.

GARDINER, C. W. *Handbook of stochastic methods for physics, chemistry and the natural sciences*. 3. ed. Berlin: Springer-Verlag, 2004. v. 13. (Springer Series in Synergetics, v. 13). ISBN 3-540-20882-8.

GEERDT, M. Jan ingenhousz, or why don't we know who discovered photosynthesis? In: . [s.n.], 2007. Disponível em: <http://philsci-archive.pitt.edu/3790/>.

GILLETTE, A. *Notes on Discrete Exterior Calculus*. [S.l.], 2009.

GONZáLEZ I. R. R., e. a. Coexistence of turbulence-like and glassy behaviours in a photonic system. *Scientific Reports*, v. 8, p. 180601, Nov 2018. Disponível em: <https://doi.org/10.1038/s41598-018-35434-z>.

GOODRICH, C. *Discrete Fractional Calculus*. [S.l.]: Springer, 2016.

GRADSHTEYN, I. S.; RYZHIK, I. M. *Table of integrals, series, and products*. 7. ed. [S.l.]: Elsevier/Academic Press, Amsterdam, 2007.

GRADY, L. J.; POLIMENI, J. R. *Discrete Calculus: Applied Analysis on Graphs for Computational Science*. 1st. ed. [S.l.]: Springer Publishing Company, Incorporated, 2010. ISBN 1849962898.

HAAG, G. *Modelling with the Master Equation: Solution Methods and Applications in Social and Natural Sciences*. [S.l.]: Springer Nature, 2017. ISBN 978-3-319-60299-8.

HARKO, T.; LOBO, F. S.; MAK, M. Exact analytical solutions of the susceptible-infected-recovered (sir) epidemic model and of the sir model with equal death and birth rates. *Applied Mathematics and Computation*, Elsevier BV, v. 236, p. 184–194, Jun 2014. ISSN 0096-3003. Disponível em: <http://dx.doi.org/10.1016/j.amc.2014.03.030>.

HASSLER, U. *Stochastic Processes and Calculus*. Springer, 2016. (Springer Texts in Business and Economics, 978-3-319-23428-1). ISBN ARRAY(0x52d0b778). Disponível em: <https://ideas.repec.org/b/spr/sptbec/978-3-319-23428-1.html>.

HOOFT, G. 't. Quantum gravity: A fundamental problem and some radical ideas. In: ______. *Recent Developments in Gravitation: Cargèse 1978*. Boston, MA: Springer US, 1979. p. 323–345. ISBN 978-1-4613-2955-8. Disponível em: <https://doi.org/10.1007/978-1-4613-2955-8_8>.

INGEN-HOUSZ, J. *Vermischte schrifen physisch medicinischen inhalts*. [S.l.]: Wappler, Vienna, 1784. 123-126 p.

IWASAKI, K.; KIMURA, H.; SHIMOMURA, S.; YOSHIDA, M. From gauss to painlevé. In: . [S.l.]: Springer, 1991.

IZADI, F. A. *Discrete Calculus by Analogy*. [S.l.]: Bentham Science Publishers, 2018.

KAC, M. Random walk and the theory of brownian motion. *The American Mathematical Monthly*, v. 54, n. 7, p. 369–91, 1947.

KAC, V.; CHEUNG, P. *Quantum Calculus*. Springer New York, 2001. (Universitext). ISBN 9780387953410. Disponível em: <https://books.google.com.br/books?id=mK8HXHSTbB0C>.

KAMPEN, N. V. *Stochastic Processes in Physics and Chemistry (North-Holland Personal Library)*. [S.l.]: North Holland, 2007.

KEELING, M. J.; ROHANI, P. *Modeling Infectious Diseases in Humans and Animals*. Princeton University Press, 2008. ISBN 9780691116174. Disponível em: <http://www.jstor.org/stable/j.ctvcm4gk0>.

KELLEY, W.; PETERSON, A. *Difference Equations: An Introduction with Applications*. Elsevier Science, 2001. ISBN 9780124033306. Disponível em: <https://books.google.com.br/books?id=D7cF_pn7NhoC>.

KERNACK, W. O.; MCKENDRICK, A. G. A contribution to the mathematical theory of epidemics. *Proceedings of The Royal Society A: Mathematical, Physical and Engineering Sciences*, v. 115, p. 700–721, 1927.

KESSLER, D. P.; GREENKORN, R. A. *Momentum,Heat, and Mass Transfer Fundamentals*. [S.l.]: Marcel Dekker, 1999. ISBN 0824719727.

KHAN, M. A.; NAJMI, M. Discrete analogue of cauchy's integral formula. *Pro Mathematica*, v. 13, n. 25-26, p. 87–93, feb. 1999. Disponível em: <https://revistas.pucp.edu.pe/index.php/promathematica/article/view/8152>.

KILBAS, M. S. A. A. *H-Transforms: Theory and Applications*. [S.l.]: CRC Press Company, 2004.

KLAUBER, R. D. *Student Friendly Quantum Field Theory:: Basic Principles and Quantum Electrodynamics*. Fairfield, Iowa: Sandtrove Press, 2013. ISBN 978-0-9845139-4-9.

KNAUF, A. Number theory, dynamical systems and statistical mechanics. *Reviews in Mathematical Physics*, v. 11, n. 08, p. 1027–1060, 1999. Disponível em: <https://doi.org/10.1142/S0129055X99000325>.

KNILL, O. If archimedes would have known functions. *CoRR*, abs/1403.5821, 2014. Disponível em: <http://arxiv.org/abs/1403.5821>.

KOGUT, J. B. The lattice gauge theory approach to quantum chromodynamics. *Rev. Mod. Phys.*, American Physical Society, v. 55, p. 775–836, Jul 1983. Disponível em: <https://link.aps.org/doi/10.1103/RevModPhys.55.775>.

KOIDE, T.; KODAMA, T.; TSUSHIMA, K. Unified description of classical and quantum behaviours in a variational principle. *Journal of Physics: Conference Series*, IOP Publishing, v. 626, p. 012055, Jul 2015. ISSN 1742-6596. Disponível em: <http://dx.doi.org/10.1088/1742-6596/626/1/012055>.

KOLMOGOROV, A. Foundations of the theory of probability. In: . [S.l.: s.n.], 1950.

KOLMOGOROV, A. N. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high reynolds number. *Journal of Fluid Mechanics*, Cambridge University Press, v. 13, n. 1, p. 82–85, 1962.

LEMONS, D. S.; GYTHIEL, A. Paul Langevin's 1908 paper "On the Theory of Brownian Motion" ["Sur la théorie du mouvement brownien," C. R. Acad. Sci. (Paris) 146, 530-533 (1908)]. *American Journal of Physics*, v. 65, n. 11, p. 1079–1081, nov. 1997.

LETOKHOV, V. S. Generation of Light by a Scattering Medium with Negative Resonance Absorption. *Soviet Journal of Experimental and Theoretical Physics*, v. 26, p. 835, Apr 1968.

MACêDO, A. M. S. *A Teoria H e o Teorema Central do Limite*. [S.l.: s.n.], 2020.

MACêDO, A. M. S.; GONZáLEZ, I. R. R.; SALAZAR, D. S. P.; VASCONCELOS, G. L. Universality classes of fluctuation dynamics in hierarchical complex systems. *Phys. Rev. E*, American Physical Society, v. 95, p. 032315, 2017.

MARSDEN, J. E.; WEST, M. Discrete mechanics and variational integrators. *Acta Numerica*, Cambridge University Press, v. 10, p. 357–514, 2001.

MATHAI, e. a. A. M. *The H-Function: Theory and Applications*. [S.l.]: Springer-Verlag New York, 2010.

MERCAT, C. Discrete riemann surfaces and the ising model. *Communications in Mathematical Physics*, Springer Science and Business Media LLC, v. 218, n. 1, p. 177–216, Apr 2001. ISSN 1432-0916. Disponível em: <http://dx.doi.org/10.1007/s002200000348>.

NICOLELIS, M. *O Verdadeiro Criador de Tudo.* [S.l.]: Planeta do Brasil, 2020.

PANZER, E. Renormalization, hopf algebras and mellin transforms. *Contemporary Mathematics*, American Mathematical Society, p. 169–202, 2015. ISSN 1098-3627. Disponível em: <http://dx.doi.org/10.1090/conm/648/13003>.

PERRIN, J. *Brownian movement and molecular reality*. [S.l.]: Courier Corporation, 2013.

REICHL, L.; LUSCOMBE, J. *A Modern Course in Statistical Physics, 2nd Edition*. [S.l.: s.n.], 1998. v. 67. 1285-1287 p.

RISKEN, H.; HAKEN, H. *The Fokker-Planck Equation: Methods of Solution and Applications*. 2. ed. [S.l.]: Springer, 1989.

SALAZAR, D. S. P.; LIRA, S. A. Exactly solvable nonequilibrium langevin relaxation of a trapped nanoparticle. *Journal of Physics A: Mathematical and Theoretical*, IOP Publishing, v. 49, n. 46, p. 465001, oct 2016. Disponível em: <https://doi.org/10.1088/1751-8113/49/46/465001>.

SALAZAR, D. S. P.; VASCONCELOS, G. L. Stochastic dynamical model of intermittency in fully developed turbulence. *Physical Review E*, American Physical Society (APS), v. 82, n. 4, Oct 2010. ISSN 1550-2376. Disponível em: <http://dx.doi.org/10.1103/PhysRevE.82.047301>.

SALINAS, S. *Introdução a Física Estatística Vol. 09*. [S.l.]: EDUSP, 1997.

SCHEPPER, H. D.; SOMMEN, F.; VOORDE, L. V. de. A Basic Framework for Discrete Clifford Analysis. *Experimental Mathematics*, A K Peters, Ltd., v. 18, n. 4, p. 385 – 395, 2009. Disponível em: <https://doi.org/>.

SILVA, J. M.; LIMA, J. Quatro abordagens para o movimento browniano (four approaches to the brownian motion). *Revista Brasileira de Ensino de Física [online]*, v. 29, n. 1, p. 25–35, 2007. Disponível em: <http://doi.org/10.1590/S1806-11172007000100007>.

SMIRNOV, S. Discrete complex analysis and probability. 2010.

SMOLUCHOWSKI, M. von. Zur kinetischen theorie der brownschen molekularbewegung und der suspensionen. *Annalen der Physik*, v. 326, n. 14, p. 756–780, 1906. Disponível em: <https://onlinelibrary.wiley.com/doi/abs/10.1002/andp.19063261405>.

SORKIN, R. D. A finitary substitute for continuous topology. *Int. J. Theor. Phys.*, v. 30, p. 923–948, 1991.

UFUOMA, O.; IKHILE, A. On simplicial polytopic numbers. *Asian Research Journal of Mathematics*, p. 1–20, 06 2019.

VASCONCELOS, G. L.; SALAZAR, D. S. P.; MACÊDO, A. M. S. Maximum entropy approach to h-theory: Statistical mechanics of hierarchical systems. *Phys. Rev. E*, American Physical Society, v. 97, p. 022104, Feb 2018. Disponível em: <https://link.aps.org/doi/10.1103/PhysRevE.97.022104>.

APPENDIX A – MEIJER AND FOX FUNCTIONS

A.1 DEFINITION OF THE MEIJER G FUNCTION

The Meijer G function is a generalization of the generalized hypergeometric function. We can define the Fox G-function by the Mellin-Barnes representation as (ERDéLYI, 1953)

$$G_{p,q}^{m,n}\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix} \mid z\right] := \int_{\gamma} \frac{d\eta}{2\pi i} \tilde{G}_{p,q}^{m,n}\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix} \mid \eta\right] z^{-\eta}, \tag{A.1a}$$

where

$$\tilde{G}_{p,q}^{m,n}\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix} \mid \eta\right] := \frac{\left[\prod_{j=1}^{m}\Gamma(b_j+\eta)\right]\left[\prod_{j=1}^{n}\Gamma(1-a_j-\eta)\right]}{\left[\prod_{j=m+1}^{q}\Gamma(1-b_j-\eta)\right]\left[\prod_{j=n+1}^{p}\Gamma(a_j+\eta)\right]}. \tag{A.1b}$$

The indices m, n, p, q are such that $0 \leqslant m \leqslant q$ and $0 \leqslant n \leqslant p$, the parameters a_j and b_j are both real or complex and the contour of integration γ is choose in such a way the poles of $\Gamma(b_j + \eta)$ and $\Gamma(1 - a_j - \eta)$ are separated from each other. An excellent discussion about the construction of the Meijer G-function and its relation with the generalized hypergeometric function can be found in (BEALS; SZMIGIELSKI, 2013).

A.2 DEFINITION OF THE FOX H FUNCTION

The Fox H-function is a generalization of the Meijer G-function. We define the Fox H-function by the Mellin-Barnes representation as (MATHAI, 2010) (KILBAS, 2004)

$$H_{p,q}^{m,n}\left[\begin{smallmatrix}(\vec{a},\vec{A})\\ (\vec{b},\vec{B})\end{smallmatrix} \mid z\right] := \frac{1}{2\pi i}\int_{\gamma} \tilde{H}_{p,q}^{m,n}\left[\begin{smallmatrix}(\vec{a},\vec{A})\\ (\vec{b},\vec{B})\end{smallmatrix} \mid \eta\right] z^{-\eta} d\eta, \tag{A.2}$$

where

$$\tilde{H}_{p,q}^{m,n}\left[\begin{smallmatrix}(\vec{a},\vec{A})\\ (\vec{b},\vec{B})\end{smallmatrix} \mid \eta\right] := \frac{\left[\prod_{j=1}^{m}\Gamma(b_j+B_j\eta)\right]\left[\prod_{j=1}^{n}\Gamma(1-a_j-A_j\eta)\right]}{\left[\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j)\eta)\right]\left[\prod_{j=n+1}^{p}\Gamma(a_j+A_j\eta)\right]}. \tag{A.3}$$

The indices m, n, p, q are such that $0 \leqslant m \leqslant q$ and $0 \leqslant n \leqslant p$, just like the Meijer G-function defined previously. The parameters a_j and b_j are both real or complex numbers and the A_j and B_j are real positive numbers. The contour of integration γ is choose in such a way the poles of $\Gamma(b_j + B_j\eta)$ and $\Gamma(1 - a_j - A_j\eta)$ are separated from each other.