UNIVERSIDADE FEDERAL DE PERNAMBUCO
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

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Irreducible classes and barycentric subdivision on triangle-free 3-connected matroids

Recife
2020

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Thesis presented to the Postgraduate Program in Mathematics of the Department of Mathematics of the Universidade Federal de Pernambuco as a partial requirement for obtaining the Doctor degree in math.

Área de Concentração: Combinatória

Orientador (a): Manoel José Machado Soares Lemos

Recife

2020

Catalogação na fonte
Bibliotecária Monick Raquel Silvestre da S. Portes, CRB4-1217

S237i Santos Filho, Jaime Cesar dos
Irreducible classes and barycentric subdivision on triangle-free 3 connected matroids / Jaime Cesar dos Santos Filho. – 2020.
75 f.: il., fig.

Orientador: Manoel José Machado Soares Lemos.
Tese (Doutorado) – Universidade Federal de Pernambuco. CCEN, Matemática, Recife, 2020.
Inclui referências.

1. Combinatória. 2. Matroides. I. Lemos, Manoel José Machado Soares (orientador). II. Título.

511.6 CDD (23. ed.) UFPE- CCEN 2020 - 193

JAIME CÉSAR DOS SANTOS FILHO

IRREDUCIBLE CLASSES AND BARYCENTRIC SUBDIVISION ON TRIANGLE-FREE 3-CONNECTED MATROIDS.

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

Aprovado em: 30/01/2020

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Dedicado a Maria de Lourdes e a Elaine, Lisie e Athos. Em memória de Krishna e Lindalva.

AGRADECIMENTOS

Agradeço a Manoel Lemos pela orientação doutoral, revisão e numerosas contribuições a este trabalho. Ele não têm nenhuma responsabilidade com relação aos erros de inglês que persintem nesta tese. Agradeço também à Universidade Federal de Pernambuco, em particular ao Departamento de Matemática, pelos mais de 10 anos de vivência e aprendizagem na instituição.

ABSTRACT

The 3-connected matroids, fundamental in matroid theory, have two families of *irreducible* matroids with respect to the operations of *deletion* and *contraction*. This result is known as Tutte's *Wheels and Whirls Theorem*, established in [11]. Lemos, in [4], considered seven reduction operations to classify the triangles-free 3-connected matroids, five in addition to the two considered by Tutte. The results obtained by Lemos generalize those obtained by Kriesell [2]. Considering only the first three reduction operations defined in [4], we prove that 4 local structures formed by squares and triads behave like *"building blocks"* for these families of irreducible. Subdividing the seventh reduction, we add another family of triangle-free 3-connected matoids: *diamantic matroids*. We have established, in a constructive way, that for each matroid in this family there is a unique *totally triangular matoid* associated. The construction of this one-to-one correspondence is based on the generalized parallel connection and passes through a matroid, unique up to isomorphisms, which corresponds to the barycentric subdivision in the case of graphic matroids.

Keywords: Matroids. 3-connectivity. Triangles. Triads. Squares.

RESUMO

As matroides 3-conexas, fundamentais na teoria das matroides, possuem duas família de *irredutíveis* com relação às operações de *deleção* e *contração*. Este resultado é conhecido como *Teorema da Roda e do Redemoinho* de Tutte [11]. Lemos, em [4], considerou sete operações de redução para classificar as matroides 3-conexas livre de triângulos irredutíveis, cinco além das duas consideradas por Tutte. Os resultados obtidos por Lemos generalizam os obtidos por Kriesell [2]. Considerando apenas as três primeiras operações de redução definidas em [4], provamos que 4 estruturas locais formadas por quadrados e triades se comportam como *"blocos construtores"* para estas famílias de irredutíveis. Subdividindo a sétima redução, acrescentamos mais uma família de matroides 3-conexas livre de triângulos irredutíveis: *diamantic matroids*, em inglês. Estabelecemos, de uma forma construtiva, que para cada matroide nesta família existe um única *matroide totalmente triangular* associada. A construção desta correspondência biunívoca é baseada na conexão em paralelo generalizada e passa por uma matroide, única a menos de isomorfismos, que corresponde a subdivisão baricêntrica no caso de matroides gráficas.

Palavras-chaves: Matroides. 3-conectividade. Triângulos. Triades. Quadrados.

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1 INTRODUCTION

1.1 THE ORIGIN OF THE PROBLEM

With rare exceptions, for matroid theory we use the notations and terminologies set in [10]. An edge of a 3-connected graph G is called *essential* if the 3-connection of G is destroyed both when the edge is deleted and when it is contracted to a single vertex. In paper *A theory of 3-connected graphs,* 1961, W. T. Tutte established the *Wheel Theorem* for graphs:

Theorem 1.1.1. *Let G be a 3-connected finite graph with at least 4 edges. Each edge $e \in E(G)$ is essential if and only if G is isomorphic to a wheel.*

The Wheel: Let C be a cycle with length $n \geq 3$. Adding a new vertex x incident to all vertices of C, we obtain a plane graph W_n with $V(W_n) = V(C) \cup \{x\}$ and $|E(W_n)| = 2n$. This graph is known as the *n-wheel.* We will also call *n-wheel* the cycle matroid $M(W_n)$ and will abuse of notation W_n to refer to both. The edges set $E(C)$ is a circuit-hyperplane of matroid W_n, called *rim* of W_n, and $E(W_n) - C$ is the *spokes set.*

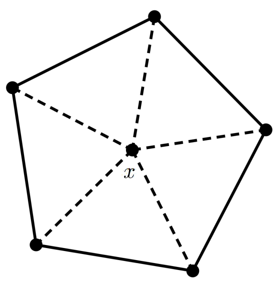

Figure 1 – W_5 denotes both this graph and its cycle matroid. Dashed edges are the spokes.

In 1966 Tutte established, in [11], the *Wheels and Whirls Theorem*, generalizing the previous theorem to matroids.

Theorem 1.1.2. *Let M be a non-empty 3-connected matroid. For each element $e \in E(M)$ we have that both $M \backslash e$ and M/e are not 3-connected if and only if M is isomorphic to either the cycle matroid of a wheel or isomorphic to a whirl.*

The Whirl: If W_n is a n-wheel, we denote by $\mathcal{W}_n$ the matroid obtained from W_n by relaxing its rim. The rank of $\mathcal{W}_n$ is the same of W_n, which is n. More information on the *relaxation operation*, see Oxley's [10].

In many situations, we must avoid triangles. The question of classifying triangle-free 3-connected graphs and matroids comes up. On the class of triangle-free 3-connected graphs, in [2], M. Kriesell considered the following reduction operations (in the sense that they decrease the number of edges of the graph without leaving the class of triangle-free 3-connected graphs):

i) Deletion of an edge that is not incident with any vertex of degree 3;

ii) Contraction of an edge that is not in a cycle with length 4;

iii) Suppose that e is an edge incident with just one vertex of degree 3, say v. Let f be another edge incident with v. Then we delete e and contract f;

iv) Suppose that e is an edge incident with two vertices of degree 3 , say v and v'. Let f and f' be edges, distinct from e, incidents on v and v', respectively. So we delete e and we contract $\{f, f'\}$;

v) Deletion of a degree-3 vertex;

vi) Contraction of the *peak* in a *cube fragment;* A *cube fragment* is a set F of four vertices of degree 3 in a graph G such that the graph obtained from $G|\left(F \cup N_G\left(F\right)\right)$ by adding a new vertex at $N_G\left(F\right)$ is a cube. Here, $N_G\left(F\right)$ denotes the set of neighbours of F in G. If F is a cube fragment then F contains exactly one vertex x not adjacent to any vertex in $N_G\left(F\right)$, which is called its *peak.*

In this context, a triangle-free 3-connected graph G is said to be irreducible, according to Kriesell, if in performing any of the above operations on elements of G we leave the class of triangle-free 3-conected graphs. Kriesell [2] states the following results:

Theorem 1.1.3. *Let G be a triangle-free 3-connected finite graph. If G is irreducible, then G is isomorphic to $K_{3,3}$ or to a double-wheel D_n. Moreover, all these reductions are necessary.*

Double-wheel: Let W_n be a n-wheel of even rank $n \geq 6$. There is just one 3-connected graph N with edges set $E\left(N\right) = E\left(W_n\right) \cup \{e\}$, for a new element e, such that N is triangle-free and $N/e = W_n$. The graph $\mathcal{D}_n = N \backslash e$ is called *double-wheel* with rank $n+1$. In [2], D_n is called by *biwheel.* By abuse of notation, we let D_n to refer to both the graph and the cycle matroid associated to D_n.

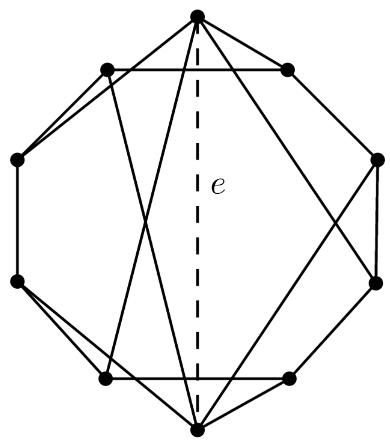

Figure 2 – D_8 denotes both this graph, deleting dashed edge e, and its cycle matroid.

1.2 A BRIEF RESUME OF LEMOS'S RESULTS ON TRIANGLE-FREE 3-CONNECTED MATROIDS

Circuits with 4 elements will be called by *squares* and cocircuits with 3 elements by *triads.* In order to extend Kriesell's result to matroids, Lemos considered, in [4], the following reductions on a triangle-free 3-connected matroid M:

First reduction: A triangle-free 3-connected matroid M is said *1-reducible* if there is an element e such that $M\backslash e$ is a triangle-free 3-connected matroid. Since the deletion of an element does not create triangles, $M\backslash e$ is triangle-free. Therefore, a matroid is said to be *1-irreducible* if the deletion of any element of the matroid destroys its 3-connectivity. In the case of M be graphic, this reduction corresponds to reduction *(i)* used by Kriesell.

Second reduction: A triangle-free 3-connected matroid M is said *2-reducible* if there is $e \in E(M)$ such that M/e is a triangle-free 3-connected matroid. Note that when e is in some square then M/e is not triangle-free. In the case of M be graphic, this reduction corresponds to reduction *(ii)* used by Kriesell.

Third reduction: A triangle-free 3-connected matroid M is said *3-reducible* when there are squares Q_1 and Q_2 such that $Q_1 \cap Q_2 = \{f\}$ and f belongs to a unique triad $T^* = \{e, f, g\}$ such that $M\backslash f/e$ is triangle-free 3-connected matroid. If M have a square Q that contains e and avoid f then this matroid will not be triangle-free. On graphs this reduction is more restrictive than the reduction *(iii)* of Kriesell since his does not require the existence of squares.

Fourth reduction: A triangle-free 3-connected matroid M is said *4-reducible* when has squares Q_1 and Q_2 such that $Q_1 \cap Q_2 = \{f\}$ and f is contained in exactly two triads $T^* = \{e, f, g\}$ and $T'^* = \{e', f, g'\}$, satisfying that $M\backslash f/\{e, e'\}$ is a triangle-free 3-connected matroid. The reduction *(iv)* of Kriesell resembles this reduction, except for the existence of the squares Q_1 and Q_2.

Fifth reduction: We say that a triangle-free 3-connected matroid M is *5-reducible* when M has squares Q_1 and Q_2 such that $Q_1 \cap Q_2 = \{f\}$, for some element f belonging to just three triads $T^* = \{e, f, g\}$, $T'^* = \{e', f, g'\}$ and $T''^* = \{e'', f, g''\}$ of M and N is a triangle-free 3-connected matroid, where N is obtained from $M \backslash f / \{e, e', e''\}$ after a $\Delta - Y$ operation along the triangle $\{g, g', g''\}$. As f belongs to three triads, this reduction is not necessary in graphic matroids. We will deal with $\Delta - Y$ operation in the chapter 6.

Sixth reduction: If there is a triad T^* and pairwise disjoint triads T_0^*, T_1^* and T_2^* such that $|T_i^* \cap T^*| = 1$, say $T^* \cap T_i^* = \{e_i\}$, $T_i^* - T^* = \{f_i, g_i\}$ and $Q_i = \{e_i, g_i, e_{i+1}, f_{i+1}\}$ is a square of M, for every $i \in \{1, 2, 3\}$ where the indices are taken modulus 3, and N is a triangle-free 3-connected matroid, where N i obtained from $M / \{g_0, g_1, g_2\} \backslash T^*$ after a $\Delta - Y$ operation along the triangle $\{f_0, f_1, f_2\}$, then we say that M is *6-reducible*.

Seventh reduction: We say that a triangle-free 3-connected matroid M is *7-reducible* when:

i) There are disjoint triads $T^* = \{e_0, e_1, e_2\}$ and $T'^* = \{e'_0, e'_1, e'_2\}$ such that $Q_0 = \{e_0, e_1, e'_0, e'_1\}$ and $Q_1 = \{e_1, e_2, e'_1, e'_2\}$ are squares of M and $M \backslash T^*$ is 3-connected; or

ii) There are pairwise different elements e_0, e_1, e_2, f_0, f_1, f_2, g_0, g_1 and g_2 of M such that $T^* = \{e_0, e_1, e_2\}$, $Q_i = \{e_i, g_i, e_{i+1}, f_{i+1}\}$ is a square of M and $Q_i - T^* \subseteq T_i^*$ for some triad T_i^* of M, for $i = \{0, 1, 2\}$, where the indices are taken modulus 3, and $M \backslash T^*$ is 3-connected;

When $I \subseteq \{1, 2, 3, 4, 5, 6, 7\}$, we say that a matroid M is *I-irreducible* provided M is a triangle-free 3-connected and is not i-reducible for all $i \in I$. We will use *abc...-irreducible* instead $\{a, b, c, ...\}$*-irreducible.* We will say that M is *irreducible* when is 1234567-irreducible.

Squares and triads plays a very important role in classifying irreducible classes of triangle-free 3-connected matroid. In the process of generalizing Kriesell's result to matroids, Lemos had to consider a possibility that does not occur in graphic matroids: a triad contained in a square.

Definition. A matroid M is said to be *semi-binary* provided $T^* \nsubseteq Q$ for every triad T^* and every square Q of M. Otherwise its said *non-semi-binary.*

The main result of [4] on semi-binary matroids is:

Theorem 1.2.1. *Suppose that M is semi-binary with at least 14 elements. Then M is irreducible if, and only if, M is isomorphic to the graphic matroid of a double-wheel, when it's rank is an odd integer exceeding 7, or to a matroid obtained from a triadic Mbius matroid deleting it's tip with even rank at least 8.*

Triadic Mbius matroids: For $n \geq 7$, there is just one 3-connected binary matroids N whose ground set $E(N) = E(W_n) \cup \{e\}$, for a new element e, such that N is triangle-free and $N/e = W_n$. When n is odd, the matroid N is called *triadic Mbius matroid* and the element e called *tip* of N. Mayhew, Royle and Whittle [6] denoted N by $_n$. Its rank is $n + 1$.

The previous theorem generalizes the Kriesell's Theorem 1.1.3 in the same way as the Tutte's Wheels and Whirls Theorem generalize the Wheels Theorem. The main results on non-semi-binary matroids of Lemos [4] is:

Theorem 1.2.2. *Suppose that M is a 1234-irreducible matroid with at least 10 elements. The matroid M is non-semi-binary if, and only if, M is isomorphic to an almost-double-wheel or an almost-double-whirl having rank at least 6.*

The *almost-double-wheel* and *almost-double-whirl* are non-semi-binary matroids defined and constructed by Lemos in Section 5 of the above-cited article.

Almost-double-wheel and Almost-double-whirl: Let $E = \{1, 2, ..., 2m, 2m + 1\}$ with $m \geq 5$. Then there are exactly two non-isomorphic triangle-free 3-connected matroids over E having:
i) $\{1, 2, 3, 4\}$ as a square; and
ii) for every $i \in \{1, 2, ..., m\}$, $\{2i - 1, 2i, 2i + 1\}$ as a triad; and
iii) for every $i \in \{2, 3, ..., m - 1\}$, $\{2i - 2, 2i - 1, 2i + 1, 2i + 2\}$ as a square;

The subset $I = \{i \in E : i \text{ is odd}\}$ is a circuit-hyperplane of one of theses matroids, say M. We say that M is an *almost-double-wheel* and the matroid obtained from M relaxing the circuit-hyperplane I is called an *almost-double-whirl.* Moreover, we have that $r(M) = m + 1$, I is a Hamiltonian circuit of M^* and $P = \{i \in E : i \text{ is even}\}$ is an independent-hyperplane of M.

Follows an auxiliary graph to illustrate these squares and triads:

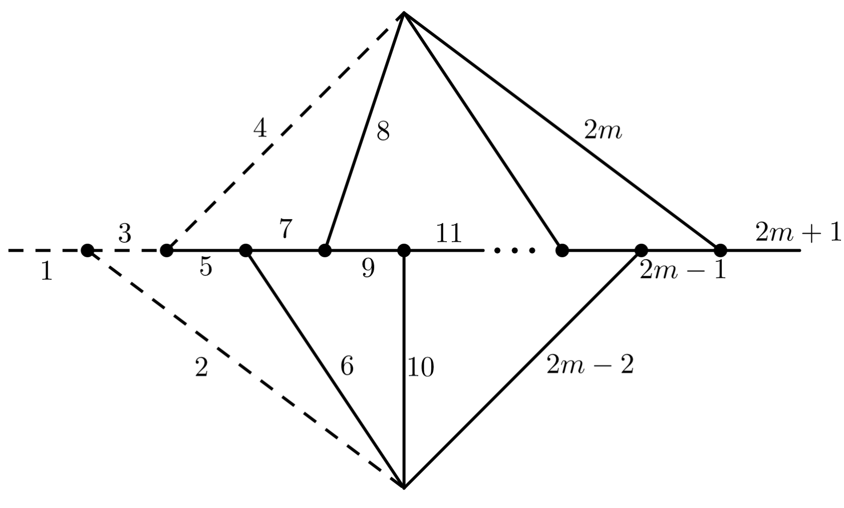

Figure 3 − The 3-set of edges incident with vertices of degree 3 illustrate the triads and the 4-set of dashed lines is a square containing the triad $\{1, 2, 3\}$.

Lemos realized that certain configurations of triads and squares are like building blocks for the irreducible matroids. These configurations of triads and squares have already appeared in the description of the reduction operations:

Sapphire

A *sapphire* with *nucleus* f is a union of two squares Q_1 and Q_2 such that $Q_1 \cap Q_2 = \{f\}$ and f is contained in two triads, say $T^* = \{e, f, g\}$ and $T'^* = \{e', f, g'\}$. Denoting $\{f_i\} = Q_i - (T^* \cup T'^*)$, for $i = 1$ and 2, we will say that the sapphire $S = Q_1 \cup Q_2$ is *pure* provided S is fullclosed, that is closed in both M and M^*, and $T''^* = \{f, f_1, f_2\}$ is also a triad of M. Note that a non-pure sapphire is the configuration of triads and squares where it is possible to apply the 4-reduction and the pure sapphire where it is possible to apply the 5-reduction. A pure sapphire does not occur in graphics matroids. A matroid M is called *sapphire-free* if has not sapphire. The squares Q_1 and Q_2 are the *faces* of sapphire S.

Sapphires enjoy an important role in the structure of the triangle-free 3-connected matroids. For example, all the irreducible classes that we have described so far are full of sapphires: $M(K_{3,3})$, double-wheels and triadic Mbius matroid, even deleting its tip, just like the almost-double-wheels and almost-double-whirls. All these matroids are made of sapphire.

In the main results of [5], Lemos prove that the 4-reduction can be avoided provided the number of families of irreducible matroids is increased by four:

Theorem 1.2.3. *Let M be a 123-irreducible matroid with at least 11 elements. If M is 4-reducible, then M is isomorphic to $M(G)$, where G is a ladder or a Mbius ladder graph, or M is isomorphic to a non-binary ladder or to a relaxed non-binary ladder.*

These four matroids cited in the above theorem was studied by Lemos in [5].

Ladder: For $n \geq 4$, the *ladder* L_n with $2n$ vertices is the graph illustrated in Figure 4, just like L_n denotes its cycle matroid.

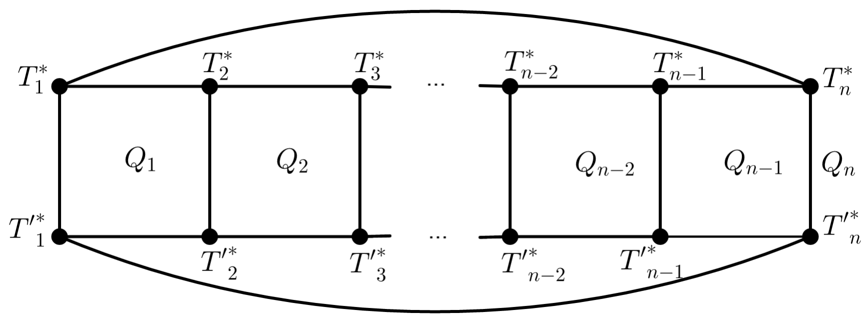

Figure 4 – Ladder L_n with $2n$ vertices. The 3-set of edges incident in a vertex of degree 3 are triads of cycle matroid associated to L_n.

Mbius Ladder: If in Figure 4, we delete the edges $T_1^*T_n^*$ and $T_1'^*T_n'^*$, and we add an edge incident with both $T_1'^*$ and T_n^* and other incident with T_1^* and $T_n'^*$ then we get the *Mbius ladder* $\mathcal{L}_n$ with $2n$ vertices.

Non-binary ladder and relaxed non-binary ladder: For $n \geq 4$, let G_n be the auxiliary graph displayed in Figure 5. Set

$$D = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$$

Then the *relaxed non-binary ladder* R_n *of rank* $2n$ is a matroid over $E(G_n)$ such that $\mathcal{C}(R_n) = \mathcal{C} \cup \mathcal{D}$, where

i) $C \in \mathcal{C}$ if and only if C is a circuit of the cycle matroid associated with G_n and $C \neq D \cup \{c_0, c_n\}$; and

ii) $C \in \mathcal{D}$ if and only if $C = E(T)$, where T is a tree of G_n such that: each leaf vertex of T is incident in G_n with c_0 or c_n, and every vertex incident with c_0 or c_n in G_n is a vertex of T.

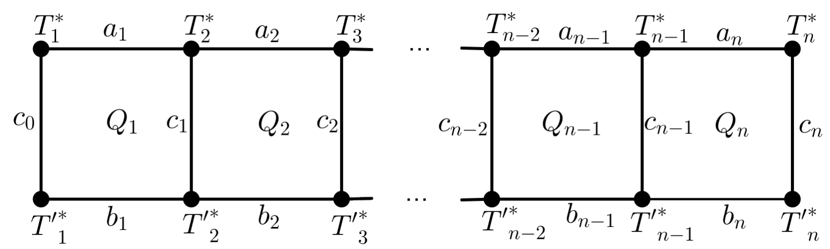

Figure 5 – Auxiliary graph G_n

The *2n*-set D is a basis of R_n. There is a matroid P_n over $E(G_n)$ such that

$$\mathcal{C}(P_n) = [\mathcal{C}(R_n) - \{D \cup c_i \,|\, 0 \leq i \leq n\}] - \{D\}$$

and R_n is obtained from P_n by relaxing the circuit-hyperplane D. We say that P_n is the *non-binary ladder of rank* $2n$.

The next result, due to Lemos [4], deals with sapphires in semi-binary 123-irreducible matroids:

Theorem 1.2.4. *Suppose that M is a semi-binary 123-irreducible matroid with at least 12 elements. If S is a sapphire whose nucleus belongs to 3 triads of M, then S is pure and M is 5-reducible.*

Rubies

Suppose that M has triads T^*, T_0^*, T_1^* and T_2^*, such that $T_i^* \cap T_j^* = \varnothing$, for $i \neq j$, and let's denote $T^* \cap T_i^* = \{e_i\}$, $T_i^* - T^* = \{f_i,\, g_i\}$ and $Q_i = \{e_i,\, g_i,\, e_{i+1},\, f_{i+1}\}$ is a square of M, where the indices are taken modulus 3. So $R = \bigcup_{i=0}^{2} T_i^*$ is called to *ruby* with *nucleus* T^*. We say that R is *pure* provided is fullclosed. Note that a ruby is the configuration of triads and squares where it is possible to apply the 6-reduction. Lemos verified that when M is 1-irreducible with at least 12 elements then the circuits of M contained in R are the graphic circuits of R. When M is graphic, a ruby R is the cube fragment referred to Kriesell.

The following result obtained by Lemos, Lemma 6.2 of [4], shows that rubies occur quite rigidly in triangle-free semi-binary 3-connected matroids:

Theorem 1.2.5. *Let M be a semi-binary 123-irreducible matroid with at least 14 elements. If R it is a ruby then R is pure and M is 6-reducible.*

According to Theorem 7.29 of [4]:

Theorem 1.2.6. *Suppose that M is a semi-binary 1234-irreducible matroid with at least 14 elements. If M has a sapphire, then*

i) M is isomorphic to the graphic matroid of a double-wheel, or to a matroid obtained from a triadic Mbius matroid deleting it's tip; or

ii) Every sapphire of M is pure or is contained in a pure ruby.

The Theorem 1.7 of [4] deals with the structures mentioned above:

Theorem 1.2.7. *Suppose that M is a 12347-irreducible matroid with at least 14 elements. Then:*

(i) M is isomorphic to an almost-double-wheel to an almost-double-whirl having rank at least 8;

(ii) M is isomorphic to the graphic matroid of a double-wheel, or to a matroid obtained from a triadic Mbius matroid deleting it's tip; or

(ii) M is (m, n)-triangular;

The family of *(m, n)*-triangular matroids is a huge class of triangle-free 3-connected matroids described by Lemos in [4]:

***(m, n)*-triangular matroid:** A matroid M is said to be *(m,n)-triangular*, for non-negative integers m and n such that $m + n \geq 2$, when M is obtained from a matroid N whose ground set is partitioned into $m+n$ triangles, say $T_1, \ldots, T_m, T_1', \ldots T_n'$, and whose simplification is 3-connected by

(i) adding an element e' in series with each element e of N;
(ii) for each $i \in \{1, \dots, m\}$, adding an element e_i such that, for every $e \in T_i$, $\{e_i, e, e'\}$ is a triad of M; and
(iii) for each $i \in \{1, \dots, n\}$, adding elements e_i, f_i, g_i such that $\{e_i, f_i, g_i\}$, $\{e_i, a_i, a'_i\}$, $\{f_i, b_i, b'_i\}$ and $\{g_i, c_i, c'_i\}$ are triads of M, where $T'_i = \{a_i, b_i, c_i\}$.

Every *(m, n)*-triangular matroid is a semi-binary 12347-irreducible matroid such that is 5-reducible or 6-reducible provides $m \neq 0$ or $n \neq 0$ respectively.

1.3 DIAMANTIC MATROID AND TOTALLY TRIANGULAR MATROID: AN IDENTIFICATION THEOREM

Lemos described most of the 123-irreducible matroids. Grouping Lemos results, we have the following theorem.

Theorem 1.3.1. *Let M be a 123-irreducible matroid. Then:*
i) If M is non-semi-binary and 4-reducible with at least 11 elements then M is isomorphic to a non-binary ladder or to a relaxed non-binary ladder;
ii) If M is non-semi-binary and 4-irreducible with at least 10 elements then M is isomorphic to an almost-double-wheel or an almost-double-whirl having rank at least 6;
iii) If M is semi-binary and 4-reducible with at least 11 elements then M is isomorphic to $M(G)$, where G is a ladder or G is a Mbius ladder graph;
iv) If M is semi-binary and 4-irreducible with $|E(M)| \geq 14$ and:
iv.1) M is 567-irreducible, then M is isomorphic to the graphic matroid of a double-wheel, or to a matroid obtained from a triadic Mbius matroid deleting it's tip;
iv.2) M is 7-irreducible, then M is (m, n)-triangular.

Note that all non-semi-binary 123-irreducible matroids with at least 11 elements are described by Lemos, just like every semi-binary 4-reducible matroid. But not all semi-binary 1234-irreducibel matroids is described. In this work we will describe other class of semi-binary 1234-irreducible matroid.

For this purpose, we studied the effect of 7-reduction by dismembering it into two cases. In order not to change the nomenclature given by Lemos, we name these cases as *eighth reduction* and *ninth reduction*:

Eighth reduction: Let M be a triangle-free 3-connected matroid. If there are disjoint triads $T^* = \{e_0, e_1, e_2\}$ and $T'^* = \{e'_0, e'_1, e'_2\}$ such that $Q_0 = \{e_0, e_1, e'_0, e'_1\}$ and $Q_1 = \{e_1, e_2, e'_1, e'_2\}$ are squares of M and $M \backslash T^*$ is 3-connected, then M is said *8-reducible* and $M \backslash T^*$ a *8-reduction* of M. This reduction corresponds to the part *(i)* of the 7-reduction.

Ninth reduction: Let M be a triangle-free 3-connected matroid. Suppose that there are pairwise different elements e_0, e_1, e_2, f_0, f_1, f_2, g_0, g_1 and g_2 of M such that $T^* = \{e_0, e_1, e_2\}$ is a triad, $Q_i = \{e_i, g_i, e_{i+1}, f_{i+1}\}$ is a square of M and $Q_i - T^* \subseteq T^*_i$ for some triad T_i^* of M, for $i = \{0, 1, 2\}$, where the indices are taken modulus 3, and $M \backslash T^*$ is 3-connected. Then M is said 9-reducible. This reduction corresponds to the part *(ii)* of the 7-reduction.

For future gain in text simplicity, we will name these two structures described in the reductions above. Note that a matroid M is 7-irreducible if and only if is 89-irreducible.

Diamond: Let Q_i, $i \in \{0, 1, 2\}$, be squares as described in the ninth reduction. The union $D = \bigcup_{i=0}^{2} Q_i$ was called *diamond* by Lemos [4] . The triad

$$T^* = (Q_0 \cap Q_1) \cup (Q_0 \cap Q_2) \cup (Q_1 \cap Q_2)$$

is the *nucleus* of D. A diamond D is said *pure* when its nucleus T^* does not intersect another triad of M. A matroid M is said *diamond-free* if it has not diamond. The squares Q_i, $i \in \{0, 1, 2\}$, are the *faces* of D.

Emerald: Let T^* and T'^* be the triads, Q_0 and Q_1 the squares as described in the eighth reduction. Then

$$\mathcal{E} = T^* \cup T'^* = Q_0 \cup Q_1$$

will be called an *emerald*. When $Q_0 \triangle Q_1$ is a square, the emerald $\mathcal{E}$ is said *pure*. A matroid M is said *emerald-free* if it has not emerald. The squares Q_0 and Q_1 are the *faces* of $\mathcal{E}$.

On diamonds, Lemma 6.3 of [4] establish that:

Lemma 1.3.2. *Let M be a semi-binary 7-irreducible matroid. If D is a diamond with nucleus T^* then there is a triad T'^* distinct of T^* such that $T^* \cap T'^* \neq \varnothing$.*

With this results, if M is semi-binary and has a pure diamond D with nucleus T^* then $M \backslash T^*$ is 3-connected and M is 7-reducible. We will prove a similar result to this for emeralds in Lemma 3.2.3.

Our first result is a covering theorem to semi-binary 123-irreducible matroid. In Chapter 4 we will establish the following result:

Theorem 4.1.6. *Suppose that M is a semi-binary 123-irreducible matroid with at least 11 elements. Then each element of M belongs to a triad contained in a sapphire, an emerald or to a nucleus of a pure diamond.*

As corollary, we have:

Theorem 4.2.1. *Let M be a semi-binary 1234568-irreducible matroid with at least 11 elements. Then each element of M belongs to a nucleus of a pure diamond.*

Definition. A *diamantic matroid M* is an emerald-free 3-connected matroid such that each element of M belongs to a nucleus of a pure diamond.

We will see in Chapter 5 that Lemma 5.1.1 implies that there is no diamantic matroid with less than 12 elements. Diamantic matroids are 1234568-irreducible, only 9-reduction can be applied.

So, there is no coincidence in the fact that for each matroid cited in Theorem 1.3.1, its elements belongs to sapphires. As consequence, every 12389-irreducible matroid with at least 14 elements is described in Theorem 1.3.1.

To prove Theorem 4.1.6 it was necessary to develop certain configurations consisting by triads and squares, and this is done in Chapter 3.

And, in Chapter 6, we will prove the following identification theorem:

Theorem 6.3.7. *If M is a rank m diamantic matroid with $n \geq 4$ triads, $|E(M)| = 3n$, then $M^\flat$ is a totally triangular 3-connected matroid with rank $m - n$ and n triangles. Conversely, if M is a rank m totally triangular matroid with n triangles, $n \geq 4$, then $M^\sharp$ is a diamantic matroid with n triads, $\left|E\left(M^\sharp\right)\right| = 3n$, and $r\left(M^\sharp\right) = n + m$. Moreover, $\left(M^\flat\right)^\sharp = M$ and $\left(M^\sharp\right)^\flat = M$.*

Definition. A 3-connected matroid M is said *totally triangular* if:
i) Each element belongs to at least 2 triangles; and
ii) Every pair of triangles intersects in at most 1 element; and
iii) M has no triads.

The previous result was obtained through successive operations involving triads and triangles. These operations are based on the *generalized parallel connection* operation. This process involves the generalization of the *barycentric subdivision* operation for matroids.

Barycentric subdivision is a graph operation that involves the notion of vertex. Here resides the difficulty of extending it to matroids: there is no notion of vertex in matroid theory. We did this for the case where it involves triangles, circuits of length 3. Unfortunately, we cannot apply the same process to circuits with length $n \geq 4$.

2 PRELIMINARY RESULTS

As mentioned earlier, we are using terminologies and notations set in [10], with rare exceptions. One of these exceptions will be the *connectivity function*: If M denotes a matroid, its connectivity function is

$$\xi : 2^{E(M)} \longrightarrow \mathbb{N}$$

such that

$$\begin{aligned} \xi(X) &= r(X) + r(E(M) - X) - r(M) + 1 \\ &= r(X) + r^*(X) - |X| + 1 \end{aligned}$$

where r is the *rank function* of M and r^* of M^*. A subset $X \subseteq E(M)$ is a *k-separating set*, for $k \geq 1$, if

$$\xi(X) \leq k \leq \ min\{|X|, |E(M) - X|\}$$

As $\xi(X) = \xi(E(M) - X)$, we have that X is a k-separating set if and only if $E(M) - X$ is too. If X is a k-separating set, the partition $\{X, E(M) - X\}$ is said a *k-separation*. A k-separation $\{X, E(M) - X\}$ is *exact* if $\xi(X) = \xi(E(M) - X) = k$.

A matroid M is said *n-connected* if there is no k-separation for $k < n$

In this chapter we will discuss important results on 3-connected matroids. Are results that deal with circuits and cocircuits in 3-connected matroids and how the connectivity is affected by deletions or contractions of certain elements.

2.1 KNOWNS RESULTS ON 3-CONNECTED MATROIDS

We start with some key results on 3-connected matroids.

From Lemos [3], we use the following result:

Theorem 2.1.1. *Let M be a 3-connected matroid and C a circuit of M such that $M \backslash e$ is not 3-connected for all $e \in C$. Then there are at least two triads of M intersecting C.*

Most times we only need a weaker version of the above result, due to Oxley [7]:

Theorem. *Each circuit of a minimally 3-connected matroid meets at least two triads.*

The main result of Bixby [1]:

Theorem 2.1.2. *(Bixby's Theorem) If M is 3-connected and $e \in M$, then:*

i) Every 2-separation for $M \backslash e$ is trivial and so $co(M \backslash e)$ is 3-connected; or

ii) Every 2-separation for M/e is trivial and so $si(M/e)$ is 3-connected.

The Tuttes's Triangle Lemma:

Lemma 2.1.3. *(Tutte's Triangle Lemma) Let M be a 3-connected matroid having at least 4 elements and suppose that $\{e,\, f,\, g\}$ is a triangle of M such that neither $M\backslash e$ nor $M\backslash f$ is 3-connected. Then M has a triad that contains e and exactly one of f and g.*

Following are two auxiliary results that is widely used throughout this text.

From Oxley [8], we use the result:

Lemma 2.1.4. *Suppose that e and f are distinct elements of a n-connected matroid M with $|E(M)| \geq 2(n-1)$, $n \geq 2$. Assume that $M/e \setminus f$ is n-connected but $M \setminus f$ is not. Then M has a cocircuit with length n containing e and f.*

Lemma 2.1.5. *Suppose that T is a triangle and T^* is a triad of a 3-connected matroid M such that $\{e\} = T^* - T$. Then $si(M/e)$ is 3-connected.*

Demonstração. $co(M\backslash e)$ can not be 3-connected because the cosimplification of $M\backslash e$ involves the contraction of an element of $T \cap T^*$. Then Theorem 2.1.2 implies that $si(M/e)$ is 3-connected. □

The following Lemmas are in Section 2 of Lemos [4]:

Lemma 2.1.6. *Let M a 1-irreducible matroid with $|E(M)| \geq 7$. If Q_1 and Q_2 are different squares of M, then $|Q_1 \cap Q_2| \leq 2$.*

Lemma 2.1.7. *Suppose that M is a semi-binary 2-irreducible matroid. Then each coline of M has at most 3 elements.*

2.2 FULLCLOSURE OPERATOR AND SEQUENTIAL SEPARATION

The terminologies for *fullclosure operator* and *sequential separations* was introduced by Oxley, Semple and Whittlel [9]. Let M be a matroid. We define the *fullclosure operator* as the function

$$fcl_M : 2^{E(M)} \longrightarrow 2^{E(M)}$$

such that

$$fcl_M(X) = min\{Z \subseteq E(M) \mid X \subseteq Z = cl(Z) = cl^*(Z)\}$$

Where cl denotes the *closure operator* of M and cl^* of M^*. Note that $fcl_M(X) = fcl_{M^*}(X)$. We denote by $fcl(X)$ when it does not cause confusion.

One way of obtaining the fullclosure of a subset $X \subseteq E(M)$ is to take alternately *closure* and *coclosure* and so on until neither the closure nor the coclosure operator adds new elements. Consequently, the elements of $fcl(X) - X$ can be ordered

$$fcl(X) - X = \{x_1, \ldots, x_n\}$$

such that $x_i \in cl(X \cup \{x_1, \ldots, x_{i-1}\})$ or $x_i \in cl^*(X \cup \{x_1, \ldots, x_{i-1}\})$.

The following results holds for k-separating sets with $k \geq 1$, but our only interest is in the case $k = 2$ and 3:

Lemma 2.2.1. *(Lemma 3.1 - [9]) Let $\{X, Y\}$ be an exact k-separation for a matroid M.*

i) For $e \in Y$, the partition $\{X \cup e, Y - e\}$ is a k-separation iff $e \in cl(X)$ or $e \in cl^*(X)$;

ii) For $e \in Y$, the partition $\{X \cup e, Y - e\}$ is an exact k-separation iff

$$e \in [cl(X) \cap cl(Y - e)] \triangle [cl^*(X) \cap cl^*(Y - e)];$$

iii) The elements of $fcl(X) - X$ can be ordered $\{x_1, \ldots, x_n\}$ such that $X \cup \{x_1, \ldots, x_i\}$ is k-separating for all $i \in \{1, \ldots, n\}$.

Definition. A k-separation $\{X, Y\}$ for a matroid M is said *sequential* provided $fcl(X) = E(M)$ or $fcl(Y) = E(M)$. Otherwise, $\{X, Y\}$ is said *non-sequential.*

Example 2.2.2. A trivial 2-separation $\{X, Y\}$ for a connected matroid M is also sequential. Indeed, suppose that $|Y| = 2$, where $Y = \{y_1, y_2\}$. Since that M is connected, Y is a parallel or a series class. Hence $y_2 \in cl(y_1)$ or $y_2 \in cl^*(y_1)$. If $y_2 \in cl(y_1)$ then $y_2 \in cl(X)$, otherwise $y_2 \in cl^*(y_1)$ and $\xi(Y) = 1$, contradicting the connectivity of M. So we have that $Y \subseteq cl(X)$. By duality, if $y_2 \in cl^*(y_1)$ then $y_2 \in cl^*(X)$ and so $Y \subseteq cl^*(X)$.

Lemma 2.2.3. *Suppose that M is a triangle-free 3-connected matroid with at least* 5 *elements. Given $e \in E(M)$, every 2-separation for M/e is non-sequential.*

Lemma 2.2.4. *Suppose that M is a triangle-free 3-connected matroid and $e \in E(M)$. If $\{X, Y\}$ is a non-trivial 2-separation for $M \backslash e$ then $\{X, Y\}$ is non-sequential or e belongs to a coline with at least 4 elements. Moreover, when M is also semi-binary and 2-irreducible, $\{X, Y\}$ is non-sequential.*

2.3 FORCED SETS

This section is based on section 4 of Lemos [4].

Definition. Let M be a matroid with ground set $E(M)$. A subset $F \subseteq E(M)$ is *forced* provided, for every $e \in E(M) - F$ and 2-separation $\{X, Y\}$ for any $N \in \{M \backslash e, M/e\}$, there is $Z \in \{X, Y\}$ such that $F \subseteq fcl_N(Z)$.

Forced sets are not separated by 2-separations on matroids resulting from contraction or deletion of elements outside F. If F intersects both sets of a 2-separation, the elements of one can be pulled to the other using the closure and coclosure operators.

Example 2.3.1. If for every $W \subseteq F$, $F \subseteq fcl_M(W)$ or $F \subseteq fcl_M(F - W)$ then F is a forced set of M. Take $e \in E(M) - F$ and let $\{X, Y\}$ be a 2-separation of $N \in$

$\{M\backslash e,\, M/e\}$. We can suppose that $F \subseteq fcl_M(X \cap F)$ and denote $W = X \cap F$. We can take $F = W \cup \{x_1, \ldots, x_n\}$ such that $x_1 \in cl(W) \cup cl^*(W)$ and

$$x_i \in cl(W \cup \{x_1, \ldots, x_{i-1}\}) \cup cl^*(W \cup \{x_1, \ldots, x_{i-1}\})$$

for $i > 1$. Let C_i be a circuit or cocircuit contained in $W \cup \{x_1, \ldots, x_i\}$ and contains x_i, for $i \geq 1$. So C_i is a circuit or cocircuit of N, since $e \notin C_i$, and then $F \subseteq fcl_N(X)$.

Lemma 2.3.2. *(Lemma 4.1 - Lemos [4]) If F is a forced set of M and e is spanned by F in M or M^* then $F \cup e$ is a forced set in M.*

Demonstração. It follows from the fact that $fcl_M(F \cup e) = fcl_M(F)$. □

Lemma 2.3.3. *(Lemma 4.2 - Lemos [4]) Suppose that M is a triangle-free 3-connected matroid with at least 5 elements and F is a forced set of M. If $e \in cl^*(F) - F$ then M/e is 3-connected. Moreover, when M is 2-irreducible there is a square Q of M that contains e.*

Lemma 2.3.4. *(compare with Lemma 4.3 - Lemos [4]) Suppose that F is a forced set of a triangle-free 3-connected matroid M. If $e \in cl(F) - F$ then:*

i) Every 2-separation for $M\backslash e$ is trivial and so $co(M\backslash e)$ is 3-connected; or exclusively ii) e belongs to a coline with at least 4 elements.

Moreover:

iii) When (ii) occurs, e is spanned by F in M and M^*;

iv) When M is semi-binary and 2-irreducible, (i) happens.

Demonstração. Suppose that *(i)* is false. Theorem 2.1.2 implies that every 2-separation for M/e is trivial and so $si(M/e)$ is 3-connected. Since M is triangle-free $si(M/e) = M/e$. Let $\{X,\, Y\}$ be a non-trivial 2-separation for $M\backslash e$. By Lemma 2.2.4, $\{X,\, Y\}$ is non-sequential or e belongs to a coline with at least 4 elements.

Suppose that $\{X,\, Y\}$ is non-sequential. We can take $Z = fcl_{M\backslash e}(X)$ such that $F \subseteq Z$. So $\{Z,\, Y - Z\}$ is a 2-separation for $M\backslash e$ and $|Y - Z| \geq 3$, then $Z \cup e$ is a 2-separating set for M; a contradiction. Hence, there is a coline with at least 4 elements containing e.

Take L^* a coline containing e with at least 4 elements. $\{L^* - e,\, E(M) - L^*\}$ is a non-trivial 2-separation for $M\backslash e$.

When *(ii)* occurs, if C denote a circuit of M such that $e \in C \subseteq F \cup e$, there is a triad $T^* = \{e,\, f,\, g\} \subseteq L^*$ such that $f \in (T^* \cap C) - e$. There is $h \in L^* - T^*$ such that $T'^* = \{f,\, g,\, h\}$ is a triad of M and $f \in C$. Then g or h belongs to C, say h, and so $\{e,\, f,\, h\}$ is a triad of M intersecting F in at least f and h. Hence $e \in cl^*(F)$. *(iv)* is consequence of Lemma 2.1.7. □

Corollary 2.3.5. *If M is 1-irreducible, F a forced set of M and $e \in cl(F) - F$ then e belongs to a triad of M.*

Lemma 2.3.6. *(compare with Lemma 4.5 - Lemos [4]) Let M be a triangle-free 3-connected matroid, F a 3-separating and forced set of M. If there is a triad $T^* = \{e, f, g\}$ such that:*

i) T^* is the unique triad containing f and $T^* \cap F = \{e, f\}$;
ii) There is a square Q of M such that $\{e, f\} \subseteq Q \subseteq F$;
iii) Each elements of $Q - \{e, f\}$ belongs to a triad contained in F;
iv) Every square of M containing g avoids e; and
v) $co\,(M \backslash f)$ is 3-connected.

Then M is 2-reducible or 3-reducible.

Demonstração. By *(i)*, T^* is the unique triad that contains f and so $co\,(M\backslash f) = M\backslash f/g$. By *(v)*, $M\backslash f/g$ is 3-connected. The dual form of the Lemma 2.1.4 implies that M/g is 3-connected. So M is 2-reducible or there is a square Q' of M containing g. If M is 2-irreducible, by orthogonality and by *(iv),* we have that f belongs to Q' too. Suppose that $Q' - g \subseteq F$, then $g \in cl\,(F) \cap cl^*\,(F)$ and so $F \cup g$ is 2-separating for M; a contradiction. Hence there is $g' \in Q' - (F \cup g)$. If $|Q \cap Q'| = 2$, then there is a triad T'^* contained in F intersecting Q' and, since *(i),* we have that $g' \in F$; a contradiction. So $Q \cap Q' = \{f\}$ and M is 3-reducible. □

Lemma 2.3.7. *(compare with Lemma 4.6 - Lemos [4]) Let M be a triangle-free 3-connected matroid and F a 3-separating forced set of M. Suppose that $M|F$ is coloopless and $|E\,(M) - F| \geq 5$. If C^* is a cocircuit of M such that $|C^* - F| = 1$ and each element of $C^* \cap F$ belongs to a triad contained in F then M is (1 or 2 or 3)-reducible.*

Demonstração. Let e be the element in $C^* - F$. The Lemma 2.3.3 implies that M/e is 3-connected and M is 2-reducible or there is a square Q of M containing e. Suppose that M is 2-irreducible. By orthogonality with C^*, there are distinct elements x and y belongs to $Q \cap C^* \cap F$. So there are triads of M contained in F that contains x and y. If Q is contained in $F \cup e$, then e is spanned by F in both M and M^*, a contradiction. So $Q - F$ has just two elements, say $\{e, f\}$.

We have that $F' = F \cup \{e, f\}$ is a 3-separating set of M, provided $|E\,(M) - F'| \geq 3$, and forced. Moreover, f can't be spanned by $F \cup e$ in both M and M^*, otherwise F' will be 2-separating. Then Lemma 2.3.4 implies that every 2-separation for $M\backslash f$ is trivial and $co\,(M\backslash f)$ is 3-connected.

If $co\,(M\backslash f) = M\backslash f$, M is 1-reducible. Otherwise, there is a triad T^* containing f and this triad can't be contained in F', and so $|T^* \cap Q| = 2$. This triad do not intersect F: suppose that $T^* \cap Q = \{f, x\}$ and $x \neq e$. Since M is coloopless, F is dependent on $M|F$ that has a circuit containing x and, by orthogonality, T^* is contained in F'.

So $T^* \cap Q = \{f, e\}$ and T^* is the unique triad that contains f. Let g be the element in $T^* - Q$, hence $co\,(M\backslash f) = M\backslash f/g$. The Lemma 2.3.3 implies that there is a square Q' of M containing g. By orthogonality with T^*, e or f belongs to Q'. If $e \in Q'$, the orthogonality with C^* and the hypothesis that each element of $C^* \cap F$ belongs to a triad contained in F implies that $Q' \subseteq F' \cup g$ and so $F' \cup g$ is 2-separating for M, a contradiction. So $Q' \cap Q = \{f\}$ then M is 3-reducible, since every square containing g intersects Q only in f. $\square$

3 ON INTERSECTION OF SQUARES

In this section we will describe the behaviour of squares and triads in some specific configurations of intersection of squares. Sapphires, emeralds and pure diamonds will naturally appear in the study of these structures. The results of this section will be useful in the proof of the covering theorem, which we will see in the next chapter.

3.1 SQUARES HAVING JUST ONE ELEMENT IN COMMON

This section contains an auxiliary result that will be very useful in the course of this work.

Lemma 3.1.1. *Let M be a minimally 3-connected matroid with Q_1 and Q_2 squares of M such that $Q_1 \cap Q_2 = \{f\}$. Suppose that f belongs to a unique triad $T^* = \{e, f, g\}$. We can assume that $e \in Q_1$ and $g \in Q_2$. If M/e is 3-connected and if there is a triad T'^* containing $Q_2 - T^*$ then $co\,(M\backslash f) = M\backslash f/e$ is 3-connected.*

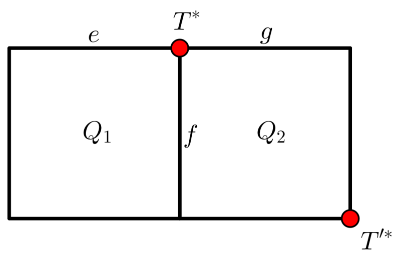

Figure 6 – Setup quoted in the following demonstration.

Demonstração. Suppose that $co\,(M\backslash f) = M\backslash f/e$ is not 3-connected. Denote by $\{x, y\} = Q_1 - T^*$. As T^* is the unique triad containing f, there is no triad of M/e that contains f. We have that $Q_1 - e = \{f, x, y\}$ is a triangle of M/e, M/e is 3-connected and $M\backslash f/e$ is not 3-connected. Tutte's Triangle Lemma implies that $M/e\backslash x$ and $M/e\backslash y$ are both 3-connected, otherwise there would be a triad of M/e containing f. Lemma 2.1.4 implies that there are triads containing $\{e, x\}$ and $\{e, y\}$, since $M\backslash x$ and $M\backslash y$ are not 3-connected. A possible representation of this configuration of triads and squares is done in the graph of Figure 6.

Bixby's Theorem implies that there is non-trivial (exact) 2-separation for $M\backslash f$, say $\{X, Y\}$. We will establish that $\{X, Y\}$ is non-sequential. Suppose that $\{X, Y\}$ is sequential. Lemma 2.2.1 implies that we can put an order on X or Y, say $Y = \{y_1, \ldots, y_{n-2}, y_{n-1}, y_n\}$, with $n \geq 3$, such that $\{y_{n-2}, y_{n-1}, y_n\}$ and $\{y_{n-1}, y_n\}$ are both 2-sparating set for

$M\backslash f$. Then $\{y_{n-1}, y_n\}$ is in a series class of $M\backslash f$ and so $\{y_{n-1}, y_n\} = \{e, g\}$. Since $\xi_{M/f}(\{y_{n-2}, e, g\}) = 2$ we have that

$$\begin{aligned}\xi_{M\backslash f}(\{y_{n-2}, e, g\}) &= r_{M\backslash f}(\{y_{n-2}, e, g\}) + r^*_{M\backslash f}(\{y_{n-2}, e, g\}) - 3 + 1 \\ &= r_M(\{y_{n-2}, e, g\}) + r^*_M(\{y_{n-2}, e, g, f\}) - 1 - 3 + 1 \\ &= r^*_M(\{y_{n-2}, e, g, f\})\end{aligned}$$

and so $\{y_{n-2}, e, g, \}$ is a triad of M. Hence $y_{n-2} \in Q_1 \cap Q_2$, because of orthogonality; a contradiction. Therefore $\{X, Y\}$ is non-sequential.

Denote by T^*_x and T^*_y the triads of M that contains $\{e, x\}$ and $\{e, y\}$, respectively. We can suppose that $|X \cap T^*_x| \geq 2$ and that X is fullclosed. If $y \in X$ then f belongs to $cl(X)$ and this is a contradiction. So $\left|T^*_y \cap Y\right| \geq 2$ and then $T^*_y \neq T^*_x$. Since e is in series with g, we have that $g \in X$ and so $|T'^* \cap Y| \geq 2$. Therefore $T'^* \cup \{e, g\} \subseteq cl^*(Y)$, $Y \cup T'^* \cup \{e, g\}$ is a 2-separating set for $M\backslash f$ and $f \in cl(Y \cup T'^* \cup \{e, g\})$; a contradiction. Thus $co(M\backslash f) = M\backslash f/e$ is 3-connected. □

3.2 SQUARES HAVING TWO ELEMENTS IN COMMON

In this section, we will establish results about the local structure of the union of two squares having two elements in common.

Lemma 3.2.1. *Let T^* be a triad of a semi-binary matroid M that intersects $Q_1 \cup Q_2$, where Q_1 and Q_2 are squares of M such that $|Q_1 \cap Q_2| = 2$. Then*

i) $T^* \cap F \in \{Q_1 - Q_2, Q_2 - Q_1, Q_1 \cap Q_2\}$; or

ii) $|T^* \cap (Q_1 - Q_2)| = |T^* \cap (Q_2 - Q_1)| = |T^* \cap (Q_1 \cap Q_2)| = 1$.

Demonstração. Since M is semi-binary, we have that $|T^* \cap Q_i| \in \{0, 2\}$, for $i \in \{1, 2\}$. If $T^* \cap Q_1 = \emptyset$ then $T^* \cap F = Q_2 - Q_1$. Similarly if $T^* \cap Q_2 = \emptyset$. If both intersections are non-empty, the orthogonality implies that $T^* \cap F = Q_1 \cap Q_2$ or *(ii)* occurs. □

Definition. In the previous lemma, when (i) happens we say that T^* is *type-1* with respect to $Q_1 \cup Q_2$. Otherwise, when *(ii)* happens we say that T^* is *type-2* with respect to $Q_1 \cup Q_2$.

Lemma 3.2.2. *Let M be a semi-binary triangle-free 3-connected matroid with at least 9 elements. If Q_1 and Q_2 are squares of M such that $|Q_1 \cap Q_2| = 2$ and $Q_1 \cup Q_2$ contains at least two triads of M, then $Q_1 \cup Q_2$ is a 3-separating forced set of M.*

Demonstração. Let T^* and T'^* be distinct triads contained in $F = Q_1 \cup Q_2$. Since we can delete one element of each square, which is not contained in the other square, without reducing the rank of F, then we have to $r(F) \leq 4$. The same occurs with the triads, without reducing the corank of F, and so $r^*(F) \leq 4$. Hence

$$\xi(F) = r(F) + r^*(F) - |F| + 1 \leq 8 - 6 + 1 = 3$$

and $|E(M) - F| \geq 3$. The 3-connectivity of M implies that the above inequality is in fact an equality, otherwise we have to $\xi(F) \leq 2$ and $|E(M) - F| \leq 1$ contradicting the fact of $|E(M)| \geq 9$. Therefore F is a 3-separating set with $r(F) = r^*(F) = 4$.

If $|T^* \cap T'^*| = 2$ then $L^* = T^* \cup T'^*$ is contained in a coline. In this case, L^* intersects one of the squares and so this square contain a triad; a contradiction.

If $|T^* \cap T'^*| = 1$ then F is a forced set of M, since for each subset $W \subseteq F$ we have that $F \subseteq fcl(W)$ or $F \subseteq fcl(F - W)$ (see Example 2.3.1).

So we can assume that $T^* \cap T'^* = \emptyset$, that is $Q_1 \cup Q_2$ is an emerald. Suppose that F is a non-forced set of M. There are $e \in E(M) - F$, $N \in \{M \setminus e, M/e\}$ and $\{X, Y\}$ 2-separation for N such that $F \nsubseteq fcl_N(X)$ and $F \nsubseteq fcl_N(Y)$. If $|X \cap F| \geq 4$ then $F \subseteq fcl_M(X \cap F) \subseteq fcl_N(X)$. So we can take $|X \cap F| = |Y \cap F| = 3$ and, moreover, $fcl_M(X \cap F) \cap F = X \cap F$ and the same holds for Y.

If $X \cap F$ intersect T^* or T'^* only in 2 elements then $F \subseteq fcl_M(X \cap F)$, a contradiction. So we have that $\{X \cap F, Y \cap F\} = \{T^*, T'^*\}$. As $\{T^*, T'^*\}$ is not 2-separation for $M|F$, it follows that $M|F \neq N|F$. Hence $N = M/e$.

As $3 \leq r(N|F) \leq r(M|F) = 4$, we have

$$\xi_{N|F}(T^*) = \{ r_{N|F}(T^*) + r_{N|F}(T'^*) - 3 + 1,$$

if $r(N|F) = 3$ $\qquad = \{ r_M(T^* \cup e) + r_M(T'^* \cup e)$

$r_{N|F}(T^*) + r_{N|F}(T'^*) - 4 + 1,$

then $e \in cl_M(T^*) \cup cl_M(T'^*)$, and so $T^* \cup e$ or $T'^* \cup e$ is a square of M; a contradiction because M is semi-binary. Thus F is a forced set of M. □

Follows a sufficient condition to a matroid M be 8-reducible.

Lemma 3.2.3. *Let M be a semi-binary 123-irreducible matroid with at least 11 elements. If T^* and T'^* are triads of M such that $T^* \cup T'^*$ is an emerald then $M \backslash T'^*$ is 3-connected and M is 8-reducible.*

Demonstração. Denote by $T^* = \{e_0, e_1, e_2\}$ and $T'^* = \{e'_0, e'_1, e'_2\}$ triads of M such that $Q_0 = \{e_0, e_1, e'_0, e'_1\}$ and $Q_1 = \{e_1, e_2, e'_1, e'_2\}$ are squares of M. Suppose that $M \backslash T'^*$ is not 3-connected. Let $\{X, Y\}$ be a k-separation for $M \backslash T'^*$, with $k = 1$ or $k = 2$. We can suppose that $|X \cap T^*| \geq 2$. If $|Y| = 1$ then $k = 1$ and Y is a coloop of M/T'^* and $T'^* \cup Y$ is a coline. A contradiction since M is semi-binary and 2-irreducible. Thus $|Y| \geq 2$.

Let's denote $M' = M\backslash (T'^{*} - e'_0)$. Since

$$\begin{aligned}
k = \xi_{M\backslash T'^{*}}(X) &= r_{M\backslash T'^{*}}(X) + r^{*}_{M\backslash T'^{*}}(X) - |X| + 1 \\
(|X \cap T^{*}| \geq 2) &\geq \xi_{M\backslash T'^{*}}(X \cup T^{*}) = \xi_{M'\backslash e'_0}(X \cup T^{*}) \\
(e'_0 \text{ is a coloop in } M') &= r_{M'}(X \cup T^{*}) + r^{*}_{M'}(X \cup T^{*} \cup e'_0) - \cancel{r^{*}_{M'}(e'_0)} - |X \cup T'^{*}| + 1 \\
&\geq r_{M'}(X \cup T^{*} \cup e'_0) + r^{*}_{M'}(X \cup T^{*} \cup e'_0) - |X \cup T^{*} \cup e'_0| + 1 \\
&= \xi_{M'}(X \cup T^{*} \cup e'_0)
\end{aligned}$$

We have that $\{X \cup T^{*}, Y - T^{*}\}$ is a k-separation for $M\backslash T'^{*}$ or $|Y - T^{*}| < k \leq 2$.

Since $\{X \cup T^{*}, Y - T^{*}\}$ is a k-separation for $M\backslash T'^{*}$, and $r_M(T^{*} \cup e'_0) = 4$, we have that $F \subseteq fcl_M(X \cup T^{*} \cup e'_0)$ and so

$$\begin{aligned}
k &= \xi_{M'}(X \cup T^{*} \cup e'_0) \\
&= r_{M'}(X \cup T^{*} \cup e'_0) + r^{*}_{M'}(X \cup T^{*} \cup e'_0) - |X \cup T^{*} \cup e'_0| + 1 \\
&\geq r_M(X \cup T^{*} \cup T'^{*}) + r^{*}_M(X \cup T^{*} \cup T'^{*}) - |X \cup T^{*} \cup T'^{*}| + 1 \\
&= \xi_M(X \cup T^{*} \cup T'^{*})
\end{aligned}$$

Therefore $\{X \cup T^{*} \cup T'^{*}, Y - T^{*}\}$ is a k-separation for M, which is a contradiction. So $|Y - T^{*}| = 1$, $T^{*} \nsubseteq X$ and $|Y| = 2$. Taking $Y = \{e_0, g\}$ with $g \notin F$, there is a cocircuit D^{*} of M such that $Y = D^{*} - T'^{*}$. Hence $D^{*} - F = \{g\}$. Since F is a 3-separating forced set and $|E(M) - F| \geq 5$, then Lemma 2.3.7 implies that M is (1 or 2 or 3)-reducible; a contradiction. Thus $M\backslash T'^{*}$ is 3-connected and M is 8-reducible. □

From now on, until the end of this section, M denotes a semi-binary 1238-irreducible matroid with at least 11 elements, Q_1 and Q_2 are squares of M such that $|Q_1 \cap Q_2| = 2$ and $Q_1 \cup Q_2$ is not an emerald. Denote by $F = Q_1 \cup Q_2$.

Lemma 3.2.4. *F contains at most one type-2 triad.*

Demonstração. Suppose, for contradiction, that T^{*}_0 and T^{*}_1 are both type-2 triads for F. Lemma 3.2.2 implies that F is a 3-separating forced set and the previous lemma implies that $|T^{*}_0 \cap T^{*}_1| = 1$. Denote by f the element in $F - (T^{*}_0 \cup T^{*}_1)$. Since M is 1-irreducible, $M\backslash f$ is not 3-connected.

Sub-lemma 3.2.4.1. *Every 2-separation for $M\backslash f$ is trivial.*

Suppose that $\{X, Y\}$ is a non-trivial 2-separation for $M\backslash f$. By Lemma 2.2.4, $\{X, Y\}$ is non-sequential. We can take X such that $|X \cap T^{*}_i| \geq 2$ for some $i \in \{0, 1\}$, say $i = 0$, $|X|$ is maximum. By Lemma 2.2.1, we can assume that X is fullclosed. Therefore $T^{*}_0 \subseteq X$, $\{X, Y\}$ is a 2-separation for $M\backslash f$ and $|Y| \geq 3$. If $|X \cap T^{*}_1| \geq 2$ the same happens with T^{*}_1 and $T^{*}_1 \subseteq X$, implying that f is spanned by X in M, then $X \cup f$ will be a 2-separating set for M. So $T^{*}_1 - T^{*}_0 \subseteq Y$. Denote by e the element in common to T^{*}_0 and T^{*}_1. We have that

$$\begin{aligned} \xi_{M\backslash\{e,f\}}(X-e) &= r_{M\backslash\{e,f\}}(X-e)+r^*_{M\backslash\{e,f\}}(X-e)-|X-e|+1 \\ &= r_{M\backslash f}(X-e)+r^*_{M\backslash f}(X)-r^*_{M\backslash f}(e)-|X|+2 \\ \left(e\notin cl_{M\backslash f}(X-e)\right) &= r_{M\backslash f}(X)-1+r^*_{M\backslash f}(X)-1-|X|+2 \\ &= \xi_{M\backslash f}(X)-1=1 \end{aligned}$$

Therefore $\{X-e,\, Y\}$ is a 1-separation for $M\setminus\{e,f\}$. As T^*_0 and T^*_1 are both type-2 triads, we have that

$$\{e,f\}\in\{Q_1-Q_2,\, Q_2-Q_1,\, Q_1\cap Q_2\}$$

There is no circuit of $M\backslash f$ intersecting both $X-e$ and Y, then $\{e,f\}=Q_1\cap Q_2$ and $Q_1\triangle Q_2$ is not a circuit of M. So $(Q_1\triangle Q_2)\cup e$ and $(Q_1\triangle Q_2)\cup f$ are both circuits of M. we have that $\{X-e, Y\cup e\}$ is a 2-separation for $M\backslash f$ and $T^*_1\subseteq Y\cup e$. By the choice of $\{X,Y\}$, we have that $|X|\geq|Y\cup e|$, and so

$$2\,|X-e|+1\geq|X|+|Y|+1=|E\,(M)|\geq 11$$

and then $|X-e|\geq 5$.

We have that

$$\begin{aligned} 2 \quad = \quad \xi_{M\backslash f}(X) &= r_{M\backslash f}(X)+r^*_{M\backslash f}(X)-|X|+1 \\ &= r_{M\backslash f}(X-e)+r^*_{M\backslash f}(X-e)-|X-e|+1 \\ &= r_M(X-e)+r^*_M((X-e)\cup f)-|X-e| \\ &= \{\,\xi_M(X-e), \end{aligned}$$

$$f\notin cl_{M^*}(X-e)$$

$$\xi_M(X-e)-1,$$

Since M is 3-connected, we have to $f\in cl_{M^*}(X-e)$ and $\xi_M(X-e)=3$. Take

$$\{q_i\}=Q_i-Y\cup\{e,f\}\subseteq T^*_0$$

We have that $\{X-\{e,\, q_1\}\,, Y\cup\{e,\, f,\, q_1\}\}$ is a 3-separation for M. Then $\xi_M(Y\cup\{e,f,q_1,q_2\})=2$, provided q_2 is spanned by $Y\cup\{e,f,q_1\}$ in both M and M^*. Therefore

$$\{X-\{e,\, q_1,\, q_2\}\,, Y\cup\{e,\, f,\, q_1,\, q_2\}\}$$

is a 2-separation for M, which is a contradiction.

Sub-lemma 3.2.4.2. *There is a unique triad T^* containing f and $T^*\cap F=\{e,f\}$.*

By previous lemma, every 2-sparation for $M\backslash f$ is trivial and so $co\,(M\backslash f)$ is 3-connected. Then there is a triad T^* containing f and T^* can not be of type-2, because of Lemma

2.1.7. Therefore $T^* \cap F = \{e, f\}$, because of Lemma 3.2.1, and there is no other triad that contains f.

Let's denote $T^* = \{e,\, f,\, g\}$.

Sub-lemma 3.2.4.3. *Every square containing g avoids e.*

Suppose that Q is a square of M such that $\{e,\, g\} \subseteq Q$. Since $e \in T_0^* \cap T_1^*$ there is $e_i \in T_i^* - e$, $i \in \{0,\, 1\}$, such that $Q = \{e_0,\, e_1,\, e,\, g\}$. Therefore $g \in cl_M(F) \cap cl_{M^*}(F)$ and hence $F \cup g$ is a 2-separating set for M; a contradiction.

Applying Lemma 2.3.6, M is (2 or 3)-reducible; a contradiction. □

Lemma 3.2.5. *$F = Q_1 \cup Q_2$ is a union of triads of type-1 such that $Q_3 = Q_1 \triangle Q_2$ is also a square of M.*

Demonstração. We will prove this result in some sub-lemmas.

Sub-lemma 3.2.5.1. *There are at least 3 triads that meets F.*

Suppose false. The Theorem 2.1.1 implies that there are two triads T^* and T'^* intersecting F and one of them must be a type-1 triad. If F intersects just two triads and both are of type-2, then at least one circuit contained in F intersects just one triad; a contradiction. Then we can assume that T^* is a type-1 and T'^* is a type-2 triads. Moreover, also by Theorem 2.1.1, $T^* \cap F = Q_1 \cap Q_2$. Then $Q_3 = Q_1 \triangle Q_2$ is a square of M, provided the exchange axiom for circuits and the orthogonality with T^*. The contradiction occurs since T'^* is the unique triad intersecting Q_3. Therefore, there are at least 3 triads that meets F.

Sub-lemma 3.2.5.2. *There are type-1 triads, T_1^* and T_2^* such that $T_1^* \cap F = Q_1 - Q_2$ and $T_2^* \cap F = Q_2 - Q_1$.*

We have that at least 3 triads intersects F and at most one of them is a type-2 triad. So we have at least two type-1 triads T_1^* and T_2^*. If $T_1^* \cap F = Q_1 - Q_2$ and $T_2^* \cap F = Q_2 - Q_1$, then the problem is solved. Otherwise, we can assume that $T_1^* \cap F = Q_1 - Q_2$ and $T_2^* \cap F = Q_2 \cap Q_1$. Then $Q_3 = Q_1 \triangle Q_2$ is a square such that $F = Q_2 \cup Q_3$, $T_1^* = Q_3 - Q_2$ and $T_2^* = Q_2 - Q_3$. Now, just replace Q_1 by Q_3.

Sub-lemma 3.2.5.3. *There is a type-1 triad, T_3^*, such that $T_3^* = Q_1 \cap Q_2$.*

Suppose false. There are just 3 triads intersecting F, T_1^*, T_2^* and T^*, such that T^* is a type-2 triad. There is $g \in F - T^*$ such that g does not belongs to any triad of M. So $co(M \backslash g) = M \backslash g$ is not 3-connected and has non-trivial 2-separations. Choose $\{X,\, Y\}$ 2-separation for $M \backslash g$ such that

$$n = |\{i \in \{1, 2\} \mid T_i^* \cap F \subseteq X \text{ or } T_i^* \cap F \subseteq Y\}|$$

is maximum. Lemma 2.2.4 implies that $\{X,\, Y\}$ is non-sequential. If $n < 2$, we can assume that $T_1^* \cap F \nsubseteq X$ and $T_1^* \cap F \nsubseteq Y$, moreover that $|T_1^* \cap X| = 2$. Then $\{X \cup T_1^*,\, Y - T_1^*\}$

is a 2-separation for $M\backslash g$, contradicting the maximality of the choice of $\{X,\, Y\}$. Therefore $n = 2$.

If $T_1^* \cap F \subseteq X$ and $T_2^* \cap F \subseteq X$, so $T^* \subseteq cl_{M\backslash g}(X)$ and then $\{X \cup T^*,\, Y - T^*\}$ is a 2-separation for $M\backslash g$ such that $g \in cl_M(X \cup T^*)$, a contradiction. Then we can assume that $T_1^* \cap F \subseteq X$, $T_2^* \cap F \subseteq Y$ and, without lost of generality, $|X \cap T^*| = 2$. Therefore $X \cup T^*$ is 2-separating for $M\backslash g$ and $g \in cl_M(X \cup T^*)$, a contradiction. □

We will denote T_i^*, for $i \in \{1, 2, 3\}$, the type-1 triads for F such that $T_i^* \cap F = \{f_i,\, g_i\}$ and $\{e_i\} = T_i^* - F$. Also, $Q_3 = Q_1 \triangle Q_2$ is a square of M.

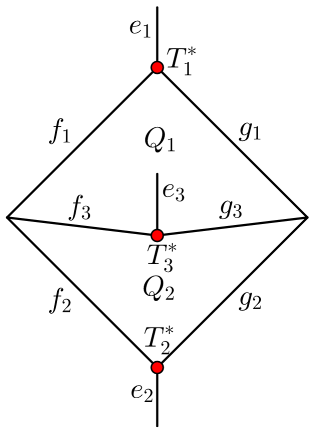

Figure 7 – Type-1 triads intersecting $Q_1 \cup Q_2$.

Lemma 3.2.6. *For $i \in \{1, 2, 3\}$, M/e_i is 3-connected.*

Demonstração. By symmetry, it is sufficient to establish the result for $i = 1$. Suppose, for contradiction, that M/e_1 is not 3-connected. Let $\{X,\, Y\}$ be a (exact) 2-separation for M/e_1. By Lemma 2.2.3, $\{X,\, Y\}$ is non-sequential. We can assume that $|X \cap T_2^*| \geq 2$ and that X is fullclosed, because Lemma 2.2.1, and so $T_2^* \subseteq X$. As e_1 can not be spanned by X or Y in M^*, otherwise we would have a 2-separation for M, we have $T_1^* \cap X \neq \varnothing$ and $T_1^* \cap Y \neq \varnothing$. Then $|Q_3 \cap X| \geq 3$ and $Q_3 \subseteq X$, since X is fullclosed; a contradiction provided $(T_1^* \cap Y) \subseteq Q_3$. □

Lemma 3.2.7. *For each $i \in \{1,\, 2,\, 3\}$ there are squares Q_i' containing $T_i^* - F$ such that:*

i) $|Q_i' \cap F| = 1$; or

ii) for some $j \in \{1,\, 2,\, 3\} - \{i\}$, $T_i^* \cup T_j^*$ is an emerald with $Q_i' \subseteq T_i^* \cup T_j^*$;

Moreover, if (ii) does not occur, then $\left|Q_i' \cap Q_j'\right| \leq 1$ for each 2-subset $\{i,\, j\} \subseteq \{1,\, 2,\, 3\}$.

Demonstração. Since M is 2-irreducible, there are triangles T_i in M/e_i such that $Q_i' = T_i \cup e_i$ is a square of M. Assume that $|Q_1' \cap F| > 1$. As $|Q_1' \cap T_1^*| = 2$, provided M is semi-binary, we have that $|Q_1' \cap T_2^*| = 2$ or $|Q_1' \cap T_3^*| = 2$. Without lost of generality, we can assume $|Q'_1 \cap T^*_2| = 2$. In this case we have $T^*_2 - F \subseteq Q'_1$, otherwise $|Q'_1 \cap Q_3| = 3$, which is a contradiction because of Lemma 2.1.6. Then $|Q'_1 \cap Q_3| = 2$ and T^*_1 and T^*_2 are both type-2 triads for $Q'_1 \cup Q_3$, and so $Q'_1 \cup Q_3$ is an emerald.

Suppose that (ii) does not occur. Assume, by contradiction, that $|Q'_1 \cap Q'_2| = 2$. As (ii) does not occur, we have that $|Q'_i \cap F| = 1$. Then $Q'_1 \cap Q'_2 = Q'_i - T^*_i$, for $i = 1$ and $i = 2$. As T^*_1 and T^*_2 are both type-1 triads for $Q'_1 \cup Q'_2$, we have that there is a triad T'^* such that $T'^* \cap (Q'_1 \cup Q'_2) = Q'_1 \cap Q'_2$. Then $Q'_1 \triangle Q'_2$ is a square of M such that $|(Q'_1 \triangle Q'_2) \cap Q_3| = 2$. So T^*_1 and T^*_2 are both type-2 triads for $(Q'_1 \triangle Q'_2) \cup Q_3$; a contradiction. □

Lemma 3.2.8. *For each $i \in \{1, 2, 3\}$, if $|Q_i \cap F| = 1$ then T^*_i is the unique triad of M that contains $Q_i \cap Q'_i$.*

Demonstração. Suppose that T^* is another triad, different to T^*_1, such that $Q_1 \cap Q'_1 \subseteq T^*$. So $T^* \cap F \neq \emptyset$ and $T^* \neq T^*_1$, then T^* is type-2 with respect F and then $|Q'_1 \cap F| > 1$; a contradiction.

□

Let's take $\{e_i\} = T^*_i - F$, $\{f_i\} = Q_i \cap Q'_i$ e $\{g_i\} = T^*_i - \{e_i, f_i\}$, for $i = 1, 2$ and 3. Thus:

Lemma 3.2.9. *For each $i \in \{1, 2, 3\}$, $co\,(M \setminus f_i) = M \setminus f_i/e_i$ is 3-connected and there are squares Q''_i of M such that $Q''_i \cap T^*_i = T^*_i - f_i = \{e_i, g_i\}$ where $Q''_i \cap F = \{g_i\}$ and $Q''_i \cap Q'_i = \{e_i\}$. Moreover, T^*_i is the unique triad containing g_i.*

Demonstração. We can apply Lemma 3.1.1 to the squares Q_i and Q'_i, concluding that $co\,(M \setminus f_i) = M \setminus f_i/e_i$ is 3-connected. As M is 3-irreducible, there is a triangle T in $M \setminus f_i/e_i$ such that $Q''_i = T \cup e_i$ is a square of $M \setminus f_i$. By orthogonality with T^*_i, we have that $g_i \in Q''_i$. If T'^* is another triad containing g_i then T'^* is a type-2 triad for $Q_1 \cup Q_2$, and so Q''_i intersect Q_1 or Q_2 in two elements. Because of the configuration of the triads, this is impossible. □

Corollary 3.2.10. *F is contained in a union of three type-1 triads and no other triad intersects F.*

If there is a triad other than T^*_i, say T'^*_i, that contains e_i then $Q'_i \cup Q''_1$ is a sapphire with nucleus e_i. The case where T^*_i is the unique triad containing e_i will be studied in the next section. In this case $Q_i \cup Q'_i \cup Q''_i$ is a pure diamond with nucleus T^*_i.

The union of squares having two elements in common has the following configuration:

Lemma 3.2.11. *Let M be a semi-binary 123-irreducible matroid with $|E\,(M)| \geq 11$. Let Q_1 and Q_2 be squares of M such that $|Q_1 \cap Q_2| = 2$. Then $F = Q_1 \cup Q_2$ is an emerald, and M is 8-reducible, or there are just 3 triads T^*_i, $i \in \{1, 2, 3\}$, intersecting F such that*

$$\{T^*_1 \cap F, T^*_2 \cap F, T^*_3 \cap F\} = \{Q_1 - Q_2, Q_2 - Q_1, Q_1 \cap Q_2\}$$

$Q_3 = Q_1 \triangle Q_2$ is a square of M and for each $i \in \{1, 2, 3\}$ we have that:

i) $T_i^* \cup T_j^*$ is an emerald, for some $j \in \{1, 2, 3\} - \{i\}$; or

ii) T_i^* is a nucleus of a pure diamond D such that $D \cap F \in \{Q_1, Q_2, Q_3\}$; or

iii) $T_i^* - F$ is a nucleus of a sapphire S such that $S \cap F = T_i^* \cap F$.

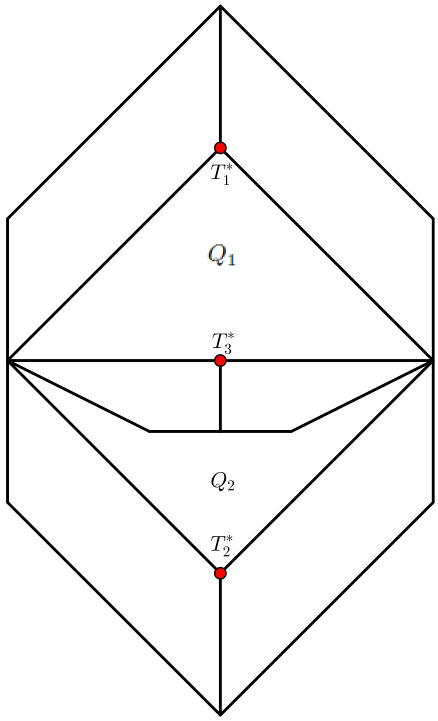

Figure 8 – Squares having two elements in common in matroids semi-binary 1238-irreducible

3.3 LAPPING PURE DIAMONDS

In the previous section we studied the configuration in which two squares intersects in two elements. We have established that, in a semi-binary 123-irreducible matroid, there are only 3 possibilities for developing of this configuration: sapphires, pure diamonds and emeralds. Similar result occurs with pure diamonds.

Our goal in this section will describe the surrounding structure of a diamond in a similar way to the union of squares having two elements in common. The main result of this section is:

Lemma 3.3.1. *Let M be a semi-binary 123-irreducible matroid with $|E(M)| \geq 11$. Let $D = Q_0 \cup Q_1 \cup Q_2$ be a pure diamond with nucleus T^* and triads T_i^*, $i \in \{0, 1, 2\}$, such that $T_i^* \cap D = Q_i - (Q_j \cup Q_k)$ with $\{i, j, k\} = \{0, 1, 2\}$. Then for each $i \in \{0, 1, 2\}$:*

i) $T_i^* \cup T^*$ is an emerald; or

ii) T_i^* is a nucleus of a pure diamond D_i such that $D_i \cap D = Q_i$; or

iii) $T_i^* - D$ is a nucleus of a sapphire S_i such that $S_i \cap D = T_i^* \cap D$.

We will prove this result with a sequence of lemmas. In this section, M will denote a semi-binary 123-irreducible matroid with $|E(M)| \geq 11$. Let Q_0, Q_1 and Q_2 be squares of M such that $D = Q_0 \cup Q_1 \cup Q_2$ is a pure diamond with nucleus T^*. For every $i \in \{0, 1, 2\}$, there is a triad T_i^* of M such that $T_i^* \cap D = Q_i - T^*$. We have that T_i^* is the unique triad of M that intersect $Q_i - T^*$, otherwise if another triad $T_i'^*$ intersect Q_i then $T_i^* \cup T_i'^*$ is contained in a coline, a contradiction provided M is semi-binary and 2-irreducible. See Lemma 2.1.7.

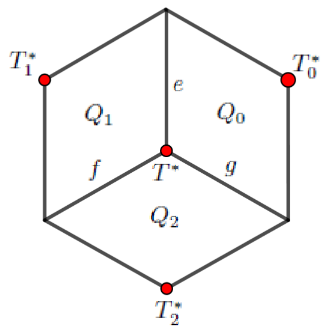

Figure 9 – Pure diamond.

From now until the end of this section, we will set the labels above for the cited triads.

Lemma 3.3.2. *If e belongs to nucleus of a pure diamond then M/e is 3-connected.*

Demonstração. Take $e \in T^*$. Suppose that the result is false and consider $\{X, Y\}$ a 2-separation for M/e. By Lemma 2.2.3, $\{X, Y\}$ is a non-sequential 2-separation. Since neither X nor Y can to span e in M^*, we have that $T^* \cap X \neq \varnothing$ and $T^* \cap Y \neq \varnothing$. Denote by $T^* \cap X = \{f\}$ and $T^* \cap Y = \{g\}$. We have that $\{f, g\} \cup (T_i^* \cap D)$ is a face of diamond D for some $i \in \{0, 1, 2\}$, say $i = 2$.

There are $Z \in \{X, Y\}$ and a fixed 2-subset $\{i, j\} \subseteq \{0, 1, 2\}$ such that $|T_i^* \cap Z| \geq 2$ and $\left|T_j^* \cap Z\right| \geq 2$. Without lost of generality, we can assume that $Z = X$ and that X is a fullclosed set in M/e. Therefore $T_i^* \cup T_j^* \subseteq X$. If $T_2^* \subseteq X$ then $g \in cl_{M/e}(T_2^* \cup f) \subseteq cl_{M/e}(X) = X$ and hence $e \in cl_M^*(X)$, a contradiction. Thus we must have to $T_0^* \cup T_1^* \subseteq X$. In this case, $g \in cl_{M/e}(T_0^* \cup T_1^*) \subseteq X$ and then $e \in cl_M^*(X)$; a contradiction

□

Lemma 3.3.3. *Let Q_i be a face of diamond D, and $e \in Q_i \cap T^*$. If we denote by $\{e_i\} = T_i^* - Q_i$, then $M/\{e, e_i\}$ is 3-connected or $T^* \cup T_i^*$ is an emerald.*

Demonstração. By previous lemma, M/e is 3-connected. We have that $T_i = Q_i - e$ is a triangle and T_i^* is a triad of M/e. If there is a square Q of M that contains both e and e_i, then $Q \cup Q_i$ is a union of squares having two elements in common with T^* and T_i^* type-2 triads for $Q \cup Q_i$, so Lemma 3.2.11 implies that $T^* \cup T_i^*$ is an emerald. Otherwise, Lemma 2.1.5 implies that $si\,(M/\{e,\, e_i\}) = M/\{e,\, e_i\}$ is 3-connected. □

Lemma 3.3.4. *Let's denote* $\{e_i\} = T_i^* - D$. *If* $T^* \cup T_i^*$ *is not an emerald then* M/e_i *is 3-connected.*

Demonstração. Suppose that M/e_i is not 3-connected. Since $(M/e_i)/e$ is 3-connected, because of previous lemma, then e belongs to a series class of M/e_1 that is a series class of M too; a contradiction. □

Lemma 3.3.5. *If* M/e_i *is 3-connected then there are squares* Q_i' *and* Q_i'' *of* M *such that* $\{e_i\} = Q_i' \cap Q_i''$, $|Q_i \cap Q_i'| = 1$ *and* $|Q_i \cap Q_i''| = 1$.

Demonstração. As M is 2-irreducible, there are a triangle T_i in M/e_i. Then $Q_i' = T_i \cup e_i$ is a square of M. Since M is semi-binary, Q_i' contains just another one element of T_i^*. The configuration of triads and the Lemma 2.1.6 prevents that Q_i' intersects the nucleus T^*. Let's denote $\{f_i\} = Q_i \cap Q_i'$. Lemma 3.1.1 applying to Q_i and Q_i' implies that $co\,(M\backslash f_i) = M\backslash f_i/e_i$ is 3-connected. So there is a square Q_i'' such that $Q_i'' \cap T_i^* = \{e_i,\, g_i\}$, where $\{g_i\} = T_i^* - \{e_i,\, f_i\}$, provided M is 3-irreducible. □

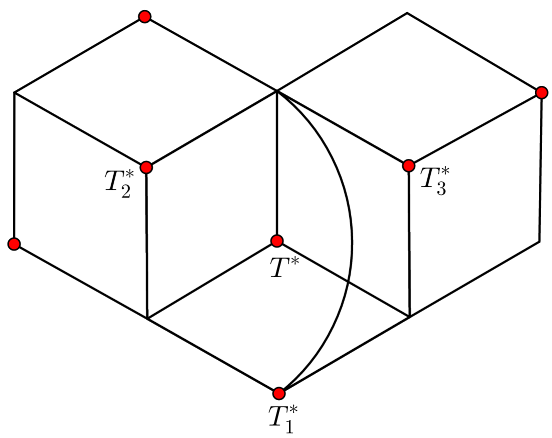

Figure 10 – Lapped diamond D. Here we have displayed the 3 possibilities: $T_1^* \cup T^*$ is an emerald, T_2^* is a nucleus of a pure diamond and $T_3^* - D$ is a nucleus of a sapphire.

4 AN IRREDUCIBLE CLASS

The main result of this chapter is a covering theorem: each element in a triangle-free semi-binary 123-irreducible matroid M, with at least 11 elements, belongs to a sapphire, pure diamond or an emerald. Thus, sapphires, pure diamonds and emeralds are the "building blocks" of a semi-binary 123-irreduclible matroid with at least 11 elements.

In this chapter M denotes a semi-binary 123-irreducible matroid with at least 11 elements. We denote by $\mathcal{F}$ the union of every sapphires, pure diamonds and emeralds of M. Our goal is to show that $\mathcal{F} = E(M)$.

4.1 A COVERING THEOREM

Lemma 4.1.1. *Let Q be a square of M and T^* a triad such that $T^* \cap Q \neq \emptyset$. If $e \in Q - T^*$, then:*

i) M/e is not 3-connected. In this case there is a triad containing e; or

ii) M/e is 3-connected and there is a square Q' such that $T^* \subseteq Q \cup Q'$ and $|Q \cap Q'| = 1$; or

iii) M/e is 3-connected and there is a square Q' such that $T^* \subseteq Q \cup Q'$ and $Q \cup Q'$ is an emerald.

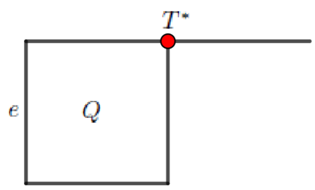

Figure 11 – $e \in Q - T^*$.

Demonstração. Suppose that M/e is not 3-connected. Since $si(M/e) = M/e$, Bixby's Theorem implies that $co(M\backslash e)$ is 3-connected. Therefore $co(M\backslash e) \neq M\backslash e$ and e belongs to a triad of M.

Suppose that M/e is 3-connected. We will denote by f the unique element in $T^* - Q$. Lemma 2.1.5 implies that $si(M/\{e, f\})$ is 3-connected.

If $si(M/\{e, f\}) \neq M/\{e, f\}$ then the contraction of f created a parallel class in $M/\{e, f\}$ and so f belongs to a triangle T' of M/e. In this case, $Q' = T' \cup e$ is a square of M containing e and f. By orthogonality with T^*, we have to Q' intersect Q in a element different than e. So $|Q \cap Q'| = 2$ and then T^* is contained in $Q \cup Q'$. Because of Lemma 3.2.11, we have that $Q \cup Q'$ is an emerald.

If $si\,(M/\{e,\, f\}) = M/\{e,\, f\}$ we have that M/f is 3-connected too. Consequently, there is a square Q' of M containing f and avoiding e, and so $T^* \subseteq Q \cup Q'$. Lemma 3.2.11 implies that $|Q \cap Q'| = 1$ or $Q \cup Q'$ is an emerald. □

Now, we will establish that every square of M is contained in $\mathcal{F}$. It will be done in the next two lemmas.

Lemma 4.1.2. *Let Q be a square of M. Suppose that T_1^* and T_2^* are distinct triads of M such that $T_1^* \cap T_2^* \cap Q \neq \emptyset$ then $Q \subseteq \mathcal{F}$.*

Demonstração. Suppose, by contradiction, that Q is a square of M such that $Q \nsubseteq \mathcal{F}$ and $T_1^* \cap T_2^* \cap Q \neq \emptyset$ for some triads T_1^* and T_2^* of M. Denote by g the unique element in $T_1^* \cap T_2^* \cap Q$. If there is a square $Q' \neq Q$ such that $g \in Q'$ then Lemma 3.2.11 implies that $|Q \cap Q'| = 1$, because of triads, and so $Q \cup Q'$ is a sapphire. Therefore $Q \subseteq \mathcal{F}$; a contradiction.

So Q is the unique square that contains g. We denote by $T_i^* = \{g,\, f_i,\, g_i\}$, $i \in \{1,\, 2\}$, and $Q = \{e,\, f_1,\, f_2,\, g\}$ where $e \notin T_1^* \cup T_2^*$. Under these notations, we have that:

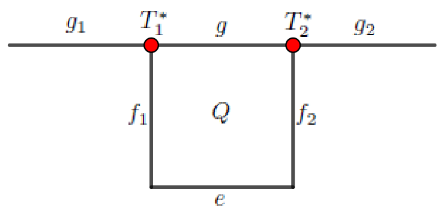

Figure 12 – Q is the unique square that contains $T_1^* \cap T_2^*$.

Sub-lemma 4.1.2.1. *M/e is not 3-connected.*

Suppose, by contradiction, that M/e is 3-connected. Since $Q \nsubseteq \mathcal{F}$, the previous lemma implies that there are squares Q_1 and Q_2 such that $T_i^* \subseteq Q \cup Q_i$ and $Q \cap Q_i = \{f_i\}$, for $i \in \{1,\, 2\}$. We have that T_i^* is the unique triad that contains f_i, otherwise $Q \cup Q_i$ is a sapphire. Since Q is the unique square that contains g and its also contains f_i, we have that $co\,(M\backslash f_i) = M\backslash f_i/g$ is triangle-free and then is not 3-connected because M is 3-irreducible. We have that $Q - e$ and T_2^* are triangle and triad, respectively, in M/e that is 3-connected. The dual form of Lemma 2.1.5 implies that $co\,(M/e\backslash f_1) = M/\{e,\, g\}\backslash f_1$ is 3-connected. Since $co\,(M\backslash f_1) = M\backslash f_1/g$ is not 3-connected, we have that e is in a series class of $M\backslash f_1/g$. This is a contradiction because the unique series class of $M\backslash f_1$ is $\{g,\, g_1\}$ and so $M\backslash f_1/g$ have not series classes. Therefore M/e is not 3-connected.

Sub-lemma 4.1.2.2 *There is a triad T_3^* such that $T_3^* \cap Q = \{e,\, g\}$.*

As M/e is not 3-connected, Lemma 4.1.1 implies that there is a triad T_3^* that contains e. If $g \notin T_3^*$ then its contains f_i for some $i \in \{1,\, 2\}$. Suppose, without lost of generality, that $f_1 \in T_3^*$. Since $si\,(M/e) = M/e$ is not 3-connected, Bixby's Theorem implies that

$co\,(M\backslash e)$ is 3-connected. Since $\{f_1\} = T_1^* \cap T_3^* \cap Q$ and $Q \nsubseteq \mathcal{F}$, we have that Q is the unique square that contains f_1 and then $\{f_2\} = Q - (T_1^* \cup T_3^*)$ plays the role of e in the previously sub-lemma. Hence $M/f_2 = si\,(M/f_2)$ is not 3-connected and Bixby's Theorem implies that $co\,(M\backslash f_2)$ is 3-connected. Then M/g_2 and M/g are both 3-connected, because of the dual form of Tutte's Triangle Lemma. Therefore M has a square Q' that contains g_2 and so $Q \cap Q' = \{f_2\}$, since that Q is a unique squared that contains g. Then T_2^* is the unique triad that contains f_2 and so $co\,(M\backslash f_2) = M\backslash f_2/g_2$ is 3-connected. As M is 3-irreducible, there is a square Q'' that contains g_2 and avoids f_2. By orthogonality with T_2^*, we have that $g \in Q''$; a contradiction. Therefore $T_3^* \cap Q = \{e, g\}$.

Denote by f_3 the unique element in $T_3^* - Q$. We have to M/f_3 and M/g are both 3-connected, because of the dual form of Tutte's Triangle Lemma. There is a square Q' of M such that $f_3 \in Q'$ and Lemma 3.2.11 implies that $\{e\} = Q \cap Q'$, since Q is the unique square that contains g. Therefore T_3^* is the unique triad that contains e, otherwise $Q \cup Q'$ is a sapphire. Since $si\,(M/e) = M/e$ is not 3-connected, Bixby's Theorem implies that $co\,(M\backslash e) = M\backslash e/f_3$ is 3-connected. Then there is a triangle T'' in $M\backslash e/f_3$ and so $Q'' = T'' \cup f_3$ is a square of M that avoid e. By orthogonality with T_3^* we have that $g \in Q''$, that is a contradiction. Conclusion: $|T_1^* \cap T_2^* \cap Q| = 0$. □

Lemma 4.1.3. *If Q is a square of M such that $Q \cap T_1^* \cap T_2^* = \varnothing$ for every pair of distinct triads T_1^* and T_2^* then $Q \subseteq \mathcal{F}$. As consequence of the previous lemma, every square of M is contained in $\mathcal{F}$.*

Demonstração. Because of Theorem 2.1.1, we have that there are triads $T_i^* = \{e_i, f_i, g_i\}$, for $i \in \{1, 2\}$, such that $Q = \{f_1, g_1, f_2, g_2\}$ and $\{e_i\} = T_i^* - Q$. By hypothesis T_i^* is the unique triad that contains f_i or g_i. As consequence of Tutte's Triangle Lemma, there is $x_i \in T_i^* \cap Q$ such that M/x_i is 3-connected, for $i = 1$ and 2. We can assume that $x_i = f_i$.

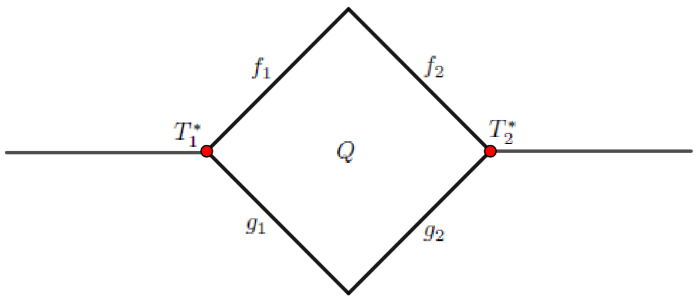

Figure 13 – Unique triads intersecting Q.

Because of Lemma 2.1.5, for $\{i, j\} = \{1, 2\}$, we have that $si\,(M/f_i/e_j)$ is 3-connected.

If $si\,(M/f_i/e_j) \neq M/f_i/e_j$, there is a parallel class in $M/f_i/e_j$ caused by contraction of e_j and so there is a square Q' that contains $\{f_i, e_j\}$. In this case, by orthogonality with

T_j^*, we have $|Q \cap Q'| = 2$ and so $e_1 \neq e_2$. In this case, $Q \cup Q'$ is an emerald, because of Lemma 3.2.11, and $Q \subseteq \mathcal{F}$.

Suppose that $si\,(M/f_i/e_j) = M/f_i/e_j$. Then $M/f_i/e_j$ is 3-connected and so M/e_j is 3-connected too. By 2-irreducibility of M, there is a square Q' containing e_j. If $e_1 = e_2$ we have that Q' intersect Q in two elements, because of the orthogonality with the triads, and so $Q \cup Q'$ is an emerald and $Q \subseteq \mathcal{F}$. If $e_1 \neq e_2$, Lemma 4.1.1 implies that there are squares Q_1 and Q_2 such that $T_i^* \subseteq Q \cup Q_i$, for $i \in \{1, 2\}$. We can assume that $|Q \cap Q_i| = 1$, otherwise $Q \cup Q_i$ is an emerald. Because of Lemma 3.1.1, we have that $co\,(M \backslash (Q \cap Q_i)) = M \backslash (Q \cap Q_i)\,/e_i$ is 3-connected. Since M is 3-irreducible, there are squares Q_1' and Q_2' such that Q_i' contains e_i and avoid the element in $Q \cap Q_i$. Thus, with the labels we are using , $T_i^* \cap Q_i' = T_i^* - Q \cap Q_i$ and $Q_i \cap Q_i' = \{e_i\}$. If T_i^* is the unique triad that contains e_i, then $Q \cup Q_i \cup Q_i'$ is a pure diamond with nucleus T_i^*. Otherwise, $Q_i \cup Q_i'$ is a sapphire with nucleus e_i. Anyway, in this configuration, $Q \subseteq T_1^* \cup T_2^* \subseteq \mathcal{F}$.

If there are a square of M such that is not contained in $\mathcal{F}$, we have a contradiction because of the previous lemma.

□

Lemma 4.1.4. *Every triad of M is contained in $\mathcal{F}$.*

Demonstração. Suppose that the result fails. Let $T^* = \{e,\, f,\, g\}$ be a triad of M such that $T^* \nsubseteq \mathcal{F}$. Then there is a element in T^*, say e, such that M/e is not 3-connected, otherwise every element in T^* belongs to a square and by the previous lemma, $T^* \subseteq \mathcal{F}$. By dual form of Tutte's Triangle Lemma, we have that M/f and M/g are both 3-connected. So there are squares of M containing f and g, and then $\{f,\, g\} \subseteq \mathcal{F}$. Since e can not belongs to any square of M, we have that there is a square Q of M such that $T^* \cap Q = \{f,\, g\}$ and $\{e\} = T^* - Q$. Denote by $Q = \{f,\, g,\, x,\, y\}$.

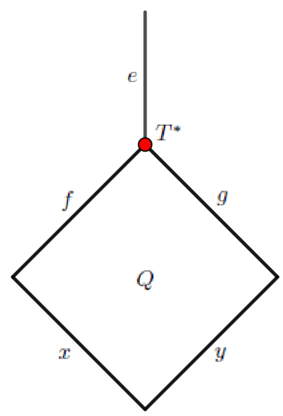

Figure 14 – Square intersecting T^*.

By Lemma 4.1.1 we have that M/x and M/y are both not 3-connected and both the elements belongs to triads of M. The dual form of Tutte's Triangle Lemma implies that there is no triad of M containing $\{x,\, y\}$.

Let T_1^* be a triad containing x. Since $y \notin T_1^*$, we have that T_1^* intersects T^* because of orthogonality with Q. Without lost of generality, we can suppose that $T_1^* \cap T^* = \{f\}$. As $x \in T_1^*$, the dual form of Tutte's Triangle Lemma implies that M/g_1 is 3-connected, where $\{g_1\} = T_1^* - Q$. As M/g_1 is 3-connected, there is a square Q' containing g_1. Since $T_1^* \subseteq Q \cup Q'$ and $|T_1^* \cap T^*| = 1$, Lemma 3.2.11 implies that $|Q \cap Q'| = 1$ otherwise $Q \cup Q'$ is an emerald containing T_1^* and $M \backslash T_1^*$ is not 3-connected, since $|T_1^* \cap T^*| = 1$, contradicting Lemma 3.2.3. Therefore $Q \cap Q' = \{x\}$.

Note that every square of M that contains g_1 avoids f. Indeed, if $\widetilde{Q}$ is a square that contains $\{f,\, g_1\}$ then $g \in \widetilde{Q}$, because $e \notin \widetilde{Q}$. We have that $T_1^* \subseteq \widetilde{Q} \cup Q$, $\left|\widetilde{Q} \cap Q\right| = 2$ and $T_1^* \cap T^* = \{f\}$. Lemma 3.2.11 implies that $\widetilde{Q} \cup Q$ is an emerald containing T_1^* and $M \backslash T_1^*$ is not 3-connected, contradicting Lemma 3.2.3.

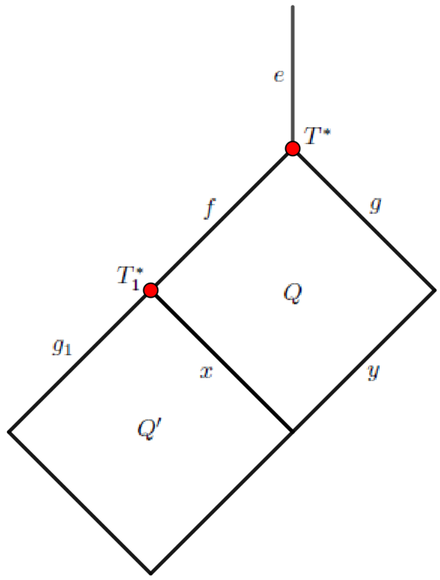

Figure 15 – Graphic representation of a square that intersects T^*, with labels given in the above description.

If T_1^* is the unique triad that contains x, then $co\,(M \backslash x) = M \backslash x/f$. Since every square that contains g_1 avoids f, the Bixby's Theorem implies that $co\,(M \backslash x)$ is 3-connected and so M is 3-reducible; a contradiction. Therefore, there is another triad T_2^* that contains x and so $Q \cup Q'$ is as sapphire with nucleus x. Because of $\{x,\, y\} \nsubseteq T_2^*$, we have that g belongs to a triad T_2^* and we will to label $T_2^* = \{x,\, g,\, f_2\}$, where $f_2 \in Q' - Q$.

Analogously, let T_3^* be a triad that contains y. There is $u \in \{f,\, g\}$ such that $T_3^* = \{y,\, u,\, f_3\}$ where $\{f_3\} = T_3^* - Q$. Denote by $\{v\} = \{f,\, g\} - \{u\}$. There is a square Q'' of M such that $Q \cap Q'' = \{y\}$, and every square that contains f_3 avoids u. Then there is another triad T_4^* that contains y, otherwise M would be 3-reducible. Then $Q \cup Q''$ is a sapphire with nucleus y and $T_4^* = \{y,\, v,\, g_4\}$ where $\{g_4\} = T_4^* - Q$.

We will denote $\{w'\} = Q' - \bigcup_{i=1}^{4} T_i^*$ and $\{w''\} = Q'' - \bigcup_{i=1}^{4} T_i^*$. There is no problem if $w' = w''$. To reinforce, we have the following labels: $Q = \{f,\, g,\, x,\, y\}$, $Q' = \{x,\, g_1,\, f_2,\, w'\}$

and $Q'' = \{y,\ f_3,\ g_4,\ w''\}$ are squares of M, with f, f_2, f_3, g, g_1, g_4, x, y and w' pairwise different. We have that $T_1^* = \{f,\ x,\ g_1\}$, $T_2^* = \{g,\ x,\ f_2\}$, $T_3^* = \{u,\ y,\ f_3\}$ and $T_4^* = \{v,\ y,\ g_4\}$ are triads, where $u = f$ and $v = g$, or $u = g$ and $v = f$. In the figure below, we show a representation of a possible configuration.

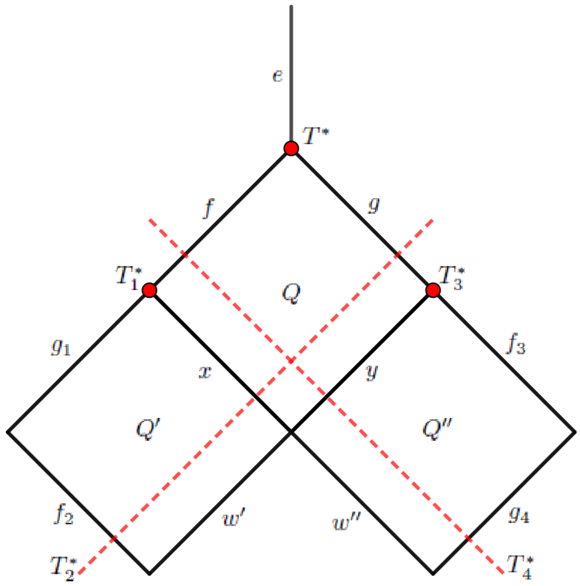

Figure 16 – Representation with labels given in the above description.

We will to show that the above configuration of square and triads implies that for every 2-sparation $\{X,\ Y\}$ of M/e, X or Y span e in M^*; a contradiction.

Denote by $F = Q \cup Q' \cup Q''$. Take $\{X,\ Y\}$ 2-separation for M/e. By Lemma 2.2.3 we have that $\{X,\ Y\}$ is non-sequential. We can suppose that X is fullclosed and $|X \cap F| \geq 5$, since $9 \leq |F| \leq 10$. We have that e can not be spanned by X nor Y in M^*, so we can assume without lost of generality that $f \in X$ and $g \in Y$. If $|T_1^* \cap Y| = 2$, then $Y \cup T_1^* = Y \cup f$ is a 2-separating set for M/e, and so $Y \cup \{e,\ f\}$ is a 2-separating set for M; a contradiction. Hence $|T_1^* \cap X| \geq 2$ and so $T_1^* \subseteq X$. If $f_2 \in X$ then $|X \cap T_2| \geq 2$ and then $g \in X$, and this is a contradiction. Hence $f_2 \in Y$ and so $w' \in Y$, otherwise $Q' - f_2 \subseteq X$. Therefore, we have that $x \in cl_M^*(Y)$ and so $Y \cup x$ is a 2-separating set for M/e. As $Q' - g_1 \subseteq Y \cup x$ we have that $Y \cup \{x,\ g_1\}$ is also a 2-separating set for M/e. Since $Q' \subseteq Y \cup \{x,\ g_1\}$, we have that $f \in cl_M^*(Y \cup \{x,\ g_1\})$ and then $Y \cup \{x,\ g_1,\ f\}$ is a 2-separating set for M/e. This is a contradiction because $e \in cl_M^*(Y \cup \{x,\ g_1,\ f\})$. □

Lemma 4.1.5. *Every element of M belongs to a triad.*

Demonstração. Let e be an element of M. Suppose, by contradiction, that there is no triad that contains e. Then $co(M \backslash e) = M \backslash e$ is not 3-connected and so Bixby's Theorem implies that $si(M/e) = M/e$ is 3-connected. Thus e belongs to a square Q of M. By Lemma 4.1.3 we have that $e \in \mathcal{F}$ and so, by the fact of that there is no triad that contains e, we can suppose e belongs to a sapphire $S = Q \cup Q'$ and there are triads T_1^*

and T_2^* contained in S such the $e \notin T_1^* \cup T_2^*$. If we denote $F = S - e$, we have that for every $W \subseteq F$, $F \subseteq fcl_M(W)$ or $F \subseteq fcl_M(F - W)$, then F is a forced set of M according to the Example 2.3.1. Lemma 2.3.4 implies that $co(M \backslash e)$ is 3-connected, contradicting the claim above. □

Last lemma establish that each element in a semi-binary 123-irreducible matroid, with at least 11 elements, belongs to a triad. In Theorem 5.2.1 we will see that this results holds even when the matroid has less then 11 elements.

Theorem 4.1.6. *Suppose that M is a semi-binary 123-irreducible matroid with at least 11 elements. Then each element of M belongs to a triad contained in a sapphire, an emerald or to a nucleus of a pure diamond.*

As consequence, if we add to the last theorem Lemmas 1.3.2 and 3.2.3 then every 1237-irreducible matroid with at least 11 elements is described in Theorem 1.3.1.

4.2 AVOIDING THE 9TH REDUCTION

If M is an emerald-free 3-connected matroid, with $|E(M)| \geq 11$, such that each element of M belongs to a nucleus of a pure diamond then M is a sapphire-free semi-binary 1238-irreducible matroid.

Theorem 4.2.1. *Let M be a semi-binary 123-irreducible matroid, with at least 11 elements, without sapphires nor emeralds. Then each element of M belongs to a nucleus of a pure diamond.*

Definition. A *diamantic matroid* M is an emerald-free 3-connected matroid such that each element of M belongs to a nucleus of a pure diamond.

We will see in the next chapter that Lemma 5.1.1 implies that there is no diamantic matroid with less than 12 elements. Diamantic matroids are 1234568-irreducible, only 9-reduction can be applied.

Later, in Chapter 6, we will establish a one-to-one correspondence between diamantic and totally triangular matroids. Diamantic matroids are a huge class of semi-binary 123-irreducible matroids.

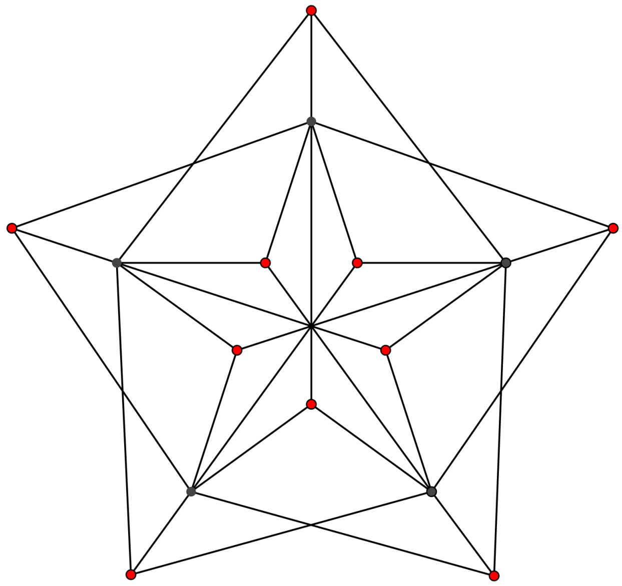

Figure 17 – The cycle matroid of this graph, having 10 vertices of degree 3 and 5 of degree 6, is a diamantic matroid with 10 triads. The triads are illustrated by vertices of degree 3.

Pure diamonds are graphical structures, in the sense that the circuits contained therein are the circuits of a graph. This local behaviour of pure diamonds is not necessarily global: diamantic matroids need not be graphically representable.

5 ON TRIANGLE-FREE 3-CONNECTED MATROIDS WITH FEW ELEMENTS

Before we study further the diamantic matroids, we will establish two results on triangle-free 3-connected matroids with few elements. The first of them is an auxiliary result that implies that there is no diamantic matroid with less then 12 elements. The latter, along with Lemma 4.1.5, shows that every element in a semi-binary 123-irreducible matroid belongs to a triad.

5.1 ON PURE DIAMONDS

Lemma 5.1.1. *If M is a 123-irreducible matroid with at most 10 elements then M has not a pure diamond.*

Demonstração. Suppose that $D = Q_1 \cup Q_2 \cup Q_3$ is a pure diamond with nucleus T^*. As M has at most 10 elements, we have that $T_i^* - Q_i = \{e\}$ for every $i \in \{1, 2, 3\}$ and $|E(M)| = 10$. We will adopt the following labels described in the graphic representation below.

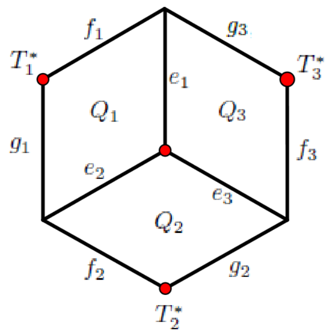

Figure 18 – Graphic representation of a pure diamond.

Suppose that M/e is not 3-connected. Let $\{X, Y\}$ be a 2-separation for M/e. Since $e \notin cl_M^*(X) \cup cl_M^*(Y)$, we can assume that $\{f_1, f_2, f_3\} \subseteq X$, $\{g_1, g_2, g_3\} \subseteq Y$ and $|T^* \cap X| \geq 2$. Therefore, there is $g_i \in cl_{M/e}(X)$ for some $i \in \{1, 2, 3\}$ and hence $\{X \cup g_i, Y - g_i\}$ is a 2-separation for M/e, and this is a contradiction because $f_i \in X$.

So, M/e is 3-connected. There is a square Q containing e, because M is 2-irreducible. We can assume that $Q = \{e, f_1, f_2, f_3\}$, by orthogonality with T_i^*. Applying Lemma 3.1.1 to $Q \cup Q_1$ we have that $co(M \backslash f_1) = M \backslash f_1/e$ is 3-connected. Since M is 3-irreducible, there is a square Q' containing $\{e, g_1\}$. If $|Q \cap Q'| = 1$ then $Q \cup Q'$ is a sapphire with nucleus e; a contradiction. Therefore we can assume that $f_3 \in Q'$ and hence $Q' = \{e, g_1, g_2, f_3\}$. As $Q \cap Q' = \{e, f_3\} \subseteq T_3^*$, we have that $Q \triangle Q' = \{f_1, f_2, g_1, g_2\}$ is a square of M.

There is a circuit of M contained in $((Q \triangle Q') \cup Q_2) - f_2 = \{e_2,\, e_3,\, f_1,\, g_1,\, g_2\}$. Because of orthogonality with T_2^* we have that $\{e_2,\, e_3,\, f_1,\, g_1\}$ is a square; a contradiction with Lemma 2.1.6. □

Corollary 5.1.2. *There is no diamantic matroid with less than 12 elements.*

5.2 TRIADS COVERS THE MATROID

This section is all dedicated to proving the following result:

Theorem 5.2.1. *If M is a semi-binary 123-irreducible matroid with at most 10 elements then each elements of M belongs to a triad.*

We will prove by contradiction. Take $e \in E(M)$ and suppose that there is no triad containing e. Then $co(M \backslash e) = M \backslash e$ is not 3-connected because M is 1-irreducible. Bixby's Theorem implies that $si(M/e) = M/e$ is 3-connected, hence there is a square Q containing e, because M is 2-irreducible. Theorem 2.1.1 implies that there is two triads T_1^* and T_2^* intersecting Q. Since every triads avoids e, Lemma 2.1.7 implies that $|T_1^* \cap T_2^*| = 1$. Denote by $T_i^* = \{e_i,\, f_i,\, g\}$ where g is the unique element in $T_1^* \cap T_2^*$ and $Q = \{e,\, f_1,\, f_2,\, g\}$. As consequence we have that $|E(M)| \geq 6$. Follows a graphic representation for such configuration.

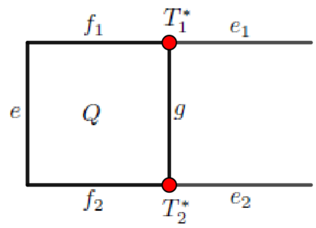

Figure 19 – Element that do not belongs to a triad.

Lemma 5.2.2. We have that $r(T_1^* \cup T_2^*) \geq 4$ and $|E(M)| \geq 7$. Moreover, if $T_1^* \cup T_2^*$ is independent then $|E(M)| \geq 8$.

Demonstração. We will prove this lemma by contradiction. If $r(T_1^* \cup T_2^*) < 4$ then $r(T_1^* \cup T_2^*) = 3$, because M is triangle-free. There is a circuit $C \subseteq T_1^* \cup T_2^*$ and $|C| \leq 4$. Since M is a triangle-free 3-connected matroid we have that C is a square of M. Because M is semi-binary, $g \notin C$ and $C = (T_1^* \cup T_2^*) - g$. As $g \in cl_M(C)$, there is a circuit (a square) contained in $T_1^* \cup T_2^*$ such that contains g; a contradiction. Therefore $r(T_1^* \cup T_2^*) \geq 4$.

Suppose that $|E(M)| = 6$. Then $r^*(M) = |E(M)| - r(M) \leq 6 - 4 = 2$, contradicting the fact of every 3-set containing e is coindependent in M. Note that if $r(T_1^* \cup T_2^*) = 5$ then $|E(M)| \geq 8$. □

Now we can apply Lemma 2.1.6. We have so:

Corollary 5.2.3. *Different squares of M intersects in at most 2 elements.*

Lemma 5.2.4. *Every square that contains e_i avoids e, for $i \in \{1, 2\}$.*

Demonstração. Suppose that Q_1 is a square of M containing $\{e, e_1\}$. By orthogonality with T_1^*, we have that $|Q \cap Q_1| = 2$. We have two possibilities, $Q \cap Q_1 = \{e, f_1\}$ or $Q \cap Q_1 = \{e, g\}$.

Assume that $Q \cap Q_1 = \{e, f_1\}$. Denote by g_1 the element in $Q_1 - (T_1^* \cup Q)$. By orthogonality, $g_1 \notin T_2^*$.

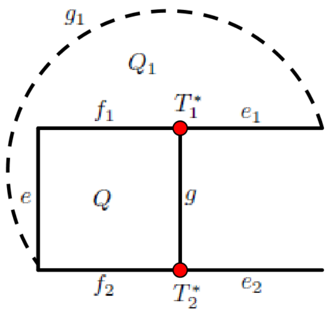

Figure 20 – Square containing e and e_1.

Lemma 2.1.1 implies that there is another triad T_3^* intersecting Q_1. Since $e \notin T_3^*$ and $\{e_1, f_1\} \nsubseteq T_3^*$, we have that $g_1 \in T_3^*$. If $\{X, Y\}$ is a 2-separation for $M\backslash e$ then $\{X, Y\}$ is non-trivial, because there is no triad containing e, and is non-sequential, because of Lemma 2.2.4. We can suppose that X intersects two of above cited triads in at least 2 elements each. Therefore e belongs to $cl_M\left(fcl_{M\backslash e}(X)\right)$ and this is a contradiction, since $\left\{fcl_{M\backslash e}(X), Y - fcl_{M\backslash e}(X)\right\}$ is a 2-separation for $M\backslash e$ as we saw in Lemma 2.2.1.

As consequence, Q is the unique square that contains $\{e, f_1\}$ and $\{e, f_2\}$.

Then $Q \cap Q_1 = \{e, g\}$ and $Q_1 = \{e, e_1, e_2, g\}$, because of orthogonality with T_1^* and T_2^*. We denote by $F = Q \cup Q_1$.

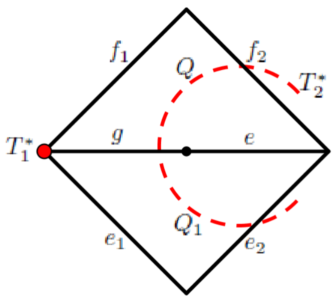

Figure 21 – Another possibility to square.

Sub-Lemma 5.2.4.1. *There is no other triad that intersects F.*

If T^* is another triad that intersects F then T^* is a type-1 triad for F, since T^* is not contained in F because of Lemma 2.1.7. We can assume that $T^* \cap F = Q - Q_1$, without lost of generality. Let $\{X, Y\}$ be a 2-separation for $M\backslash e$. Then $\{X, Y\}$ is non-trivial and non-sequential 2-separation. We can suppose that X intersects two of above cited triads in at least 2 elements each. Therefore e belongs to $cl_M\left(fcl_{M\backslash e}(X)\right)$ and this is a contradiction, since $\left\{fcl_{M\backslash e}(X), Y - fcl_{M\backslash e}(X)\right\}$ is a 2-separation for $M\backslash e$ as we saw in Lemma 2.2.1. This sub-lemma ends.

As $|E(M)| \geq 7$, there is an element $x \in E(M) - F$.

Sub-Lemma 5.2.4.2. *There is a triad T^* containing x.*

Otherwise, there is a square Q_2 containing x just like Figure 21, with x playing role of e. Since M has at most 10 elements, the triads that intersects Q_2 also intersects F. Therefore, by previous sub-proposition, these triads are the same triads contained in F. This implies that $g \in Q_2$ and so $Q_2 = \{e_1, f_2, g, x\}$ or $Q_2 = \{e_2, f_1, g, x\}$, because M is sapphire-free. We can assume that $Q_2 = \{e_1, f_2, g, x\}$.

If $\{X, Y\}$ is a 2-separation for $M\backslash e$ then $\{X, Y\}$ is non-trivial and non-sequential, and we can suppose that $|T_1^* \cap X| \geq 2$. So $T_1^* \subseteq fcl_{M\backslash e}(X)$ and then $\{e_2, f_2, x\} \subseteq Y$ (f_2 because of Q, e_2 because of Q_1 and x because of $\{e_1, g\} \subseteq Q_2$). Hence $\{g, f_1\} \subseteq fcl_{M\backslash e}(Y)$ and $e \in cl_M\left(fcl_{M\backslash e}(Y)\right)$; a contradiction. This prove the sub-lemma.

So, there is a triad T^* containing x such that $T^* \cap F = \emptyset$. Let's denote $T^* = \{x, y, z\}$. Hence $|E(M)| \geq 9$ and F is a 3-separating forced set, by Lemma 3.2.2. Because of dual version of Tutte's Triangle Lemma, we can suppose that M/x is 3-connected. As M is 2-irreducible, there is a square Q_2 containing x. We can suppose that $Q_2 \cap T^* = \{x, y\}$, because of orthogonality. Since $|E(M)| \leq 10$, we have that Q_2 intersects F. We must be $|Q_2 \cap F| = 2$. If $|Q_2 \cap F| \neq 2$ then $Q_2 \cap F = \{e\}$, because each other element belongs to a triad contained in F. Therefore $Q_2 = \{e, x, y, w\}$ where $w \notin F \cup \{z\}$. There is a triad T'^* containing w and x or y. Since each coline has at most 3 elements, $z \notin T'^*$ and so T'^* intersects F; a contradiction. Therefore $|Q_2 \cap F| = 2$.

Since $|Q_2 \cap F| = 2$, we have that Q_2 intersects F in a pair of elements belonging to a same triad. Then $e \notin Q_2$ and $g \notin Q_2$, by orthogonality. So $Q_2 = \{x, y, e_i, f_i\}$ for $i = 1$ or $i = 2$. We can assume that $Q_2 = \{x, y, e_1, f_1\}$. Since M/e_1 or M/f_1 is 3-connected, Lemma 2.1.5 implies that $si(M/\{e_1, z\})$ or $si(M/\{f_1, z\})$ is 3-connected. We can assume, without lost of generality, that $si(M/\{e_1, z\})$ is 3-connected. If $si(M/\{e_1, z\}) \neq M/\{e_1, z\}$ then there is a square Q' containing $\{e_1, z\}$ and then x or y belongs to Q'. Since $g \notin Q'$, because of orthogonality with T_2^*, we have that $f_1 \in Q'$ and then $|Q' \cap Q_2| = 3$; a contradiction. Therefore $si(M/\{e_1, z\}) = M/\{e_1, z\}$.

As $si(M/\{e_1, z\}) = M/\{e_1, z\}$ is 3-connected, we have that M/z is 3-connected too and there is a square Q_3 containing z and avoid e_1. If Q_3 intersects T_1^* then

$$Q_3 \in \{\{f_1, g, x, z\}, \{f_1, g, y, z\}\}$$

and this contradicts the orthogonality with T_2^*.

So $Q_3 \cap T_1^* = \emptyset$ and $Q_2 \cap Q_3 \in \{\{x\}, \{y\}\}$. Since M is sapphire-free, T^* is the unique triad containing $Q_2 \cap Q_3$. We have that Q_3 intersects F. So, as we have seen, $|Q_3 \cap F| = 2$ and then $Q_3 \cap F = \{e_2, f_2\}$.

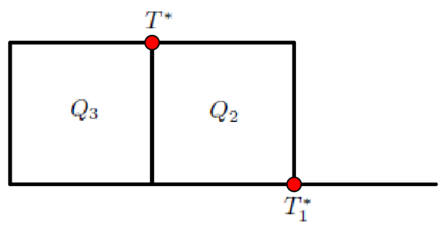

Figure 22 – Intersection of new squares.

Lemma 3.1.1 implies that $co\left(M \backslash (Q_2 \cap Q_3)\right) = M \backslash (Q_2 \cap Q_3)/z$ is 3-connected. Since M is 3-irreducible, there is a square Q_4 such that

$$Q_4 \cap T^* = T^* - (Q_2 \cap Q_3)$$

Since $|E(M)| \leq 10$, we have that Q_4 intersects F and so $Q_4 \cap F = \{e_i, f_i\}$ for some $i \in \{1, 2\}$; a contradiction. □

As consequence of Lemma 5.2.4, we have that

$$si\left(M/\{e, e_i\}\right) = M/\{e, e_i\}$$

that is 3-connected, because of Lemma 2.1.5. Then M/e_i is 3-connected. Since M is 2-irreducible, there is a square Q_i containing e_i.

Lemma 5.2.5. *Every square containing e_i avoids g too, for $i = 1$ and $i = 2$.*

Demonstração. Let Q_1 be a square containing e and g. Then $e_2 \notin Q_1$, because M is sapphire-free. By orthogonality, $f_2 \in Q_1$ and so $\{e_1, f_2, g\} \subseteq Q_1$. Therefore $Q_1 \cap Q = \{f_2, g\} \subseteq T_2^*$ and so

$$Q_3 = Q_1 \triangle Q = \{e, e_1, f_1, x\}$$

is a square of M, where $x \notin T_1^* \cup T_2^* \cup Q$. Therefore, there is a triad T_3^* containing $\{x, e_1\}$ or $\{x, f_1\}$. Let $\{X, Y\}$ be a 2 separation for $M \backslash e$, then $\{X, Y\}$ is non-trivial and non-sequential. We can assume that $|X \cap T_i^*| \geq 2$ for two indices $i \in \{1, 2, 3\}$. Therefore $e \in cl_M\left(fcl_{M\backslash e}(X)\right)$, contradicting the 3-connectivity of M. Then $Q_1 \cap T_1^* = \{e_1, f_1\}$. □

Thus, we have two possibilities:

P1) $|Q \cap Q_i| = 2$, in this case $Q_1 = Q_2 = \{e_1, e_2, f_1, f_2\}$; or

P2) $|Q \cap Q_1| = |Q \cap Q_2| = 1$, in this case $Q_i \cap Q = \{f_i\}$ and $Q_i \cap T_j^* = \emptyset$ for $\{i, j\} = \{1, 2\}$;

Let's see that both lead to contradictions:

P1) If $Q_1 = \{e_1, e_2, f_1, f_2\}$ is a square of M, then there is no other triad that intersects $Q \cup Q_1$. Otherwise, with similar arguments to those presented in Sub-Lemma 5.2.4.1, we came to the conclusion that for each 2-separating set X for $M\backslash e$, $e \in cl_M\left(fcl_{M\backslash e}(X)\right)$ or $e \in cl_M\left(fcl_{M\backslash e}(E(M)-X)\right)$; a contradiction. Follows a graphic representation of squares Q and Q_1

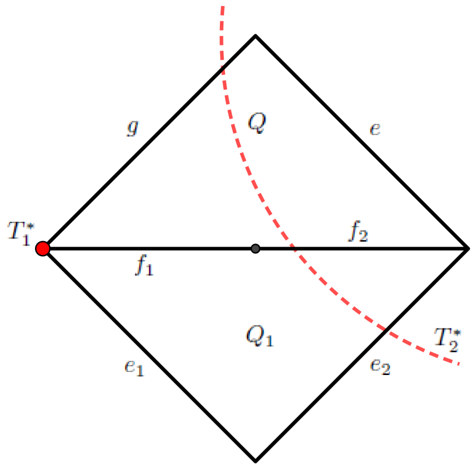

Figure 23 – Possibility 1

Let's denote $F = Q \cup Q_1$. There is an element $x \notin F$ because $|E(M)| \geq 7$. Suppose that x do not belongs to any triad of M. Thus there is a square Q' containing x, just like Figure 20 with x in the place of e. Since M has at most 10 elements, we have that $g \in Q'$ and so $|Q' \cap Q| = 2$ since M is sapphire-free. Then

$$Q' \in \{\{g,\, e_1,\, f_2,\, x\},\, \{g,\, e_2,\, f_1,\, x\}\}$$

We can assume that $Q' = \{g, e_1, f_2, x\}$. If $\{X, Y\}$ is a 2-separation for $M\backslash e$, then $\{X, Y\}$ is non-trivial and non-sequential. We can suppose that X intersects T_1^* in at least two elements. Then $f_2 \notin fcl_{M\backslash e}(X)$ and $\{e_2, x\} \subseteq Y$. Therefore $e \in cl_M\left(fcl_{M\backslash e}(Y)\right)$; a contradiction. Hence x belongs to a triad M and this triad do not intersects F.

Denote by $T^* = \{x, y, z\}$. Then $|E(M)| \geq 9$ and so F is 3-separating forced set of M, because of Lemma 3.2.2. We can suppose that M/x is 3-connected, because of dual version of Tutte's Triangle Lemma. As M is 2-irreducible, there is a square Q_2 containing x. We can suppose that $Q_2 \cap T^* = \{x, y\}$, because of orthogonality. Since $|E(M)| \leq 10$, we have that Q_2 intersects F. We must be $|Q_2 \cap F| = 2$, otherwise $Q_2 \cap F = \{e\}$ because each other element belongs to a triad contained in F. Therefore $Q_2 = \{e, x, y, w\}$ where $w \notin F \cup \{z\}$. There is a triad T'^* containing w and x or y. Since each coline has at most 3 elements, $z \notin T'^*$ and so T'^* intersects F; a contradiction. Therefore $|Q_2 \cap F| = 2$.

Since $|Q_2 \cap F| = 2$, we have that $e \notin Q_2$ and $g \notin Q_2$. Then $Q_2 = \{x, y, e_i, f_i\}$ for $i = 1$ or $i = 2$. We can assume that $Q_2 = \{x, y, e_1, f_1\}$. Since M/e_1 is 3-connected, Lemma 2.1.5 implies that $si(M/\{e_1, z\})$ is 3-connected. If $si(M/\{e_1, z\}) \neq M/\{e_1, z\}$ then there is a square Q' containing $\{e_1, z\}$ and then x or y belongs to Q'. Since $g \notin Q'$, because of orthogonality with T_2^*, we have that $f_1 \in Q'$ and then $|Q' \cap Q_2| = 3$; a contradiction.

Therefore $si\left(M/\left\{e_1,\, z\right\}\right) = M/\left\{e_1,\, z\right\}$ is 3-connected. With arguments similar to those given after Sub-Lemma 5.2.4.2, we ends case P1.

P2) $|Q \cap Q_1| = |Q \cap Q_2| = 1$, in this case $Q_i \cap Q = \{f_i\}$ and $Q_i \cap T_j^* = \emptyset$ for $\{i,\, j\} = \{1,\, 2\}$. There are elements x_i and y_i, for $i \in \{1,\, 2\}$, such that $Q_i - (T_1^* \cup T_2^* \cup Q) = \{x_i,\, y_i\}$. Follows a graphic representation of squares above referred.

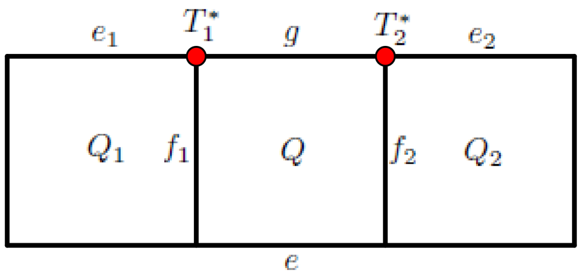

Figure 24 – Possibility 2.

As M is sapphire-free, T_i^* is the unique triad containing f_i. Since M/e_1 is 3-connected and Q is a square of M/e_1 that intersects just one triad, Theorem 2.1.1 implies that $M/e_1\backslash x$ is 3-connected for some $x \in Q$. As f_2 and g belongs to T_2^*, a triad of M/e_1, then $x \in \{e,\, f_1\}$. Because of $M\backslash e$ is not 3-connected, $M\backslash e/e_1$ is not 3-connected, otherwise there would be a triad containing e on M. Hence $M\backslash f_1/e_1 = co\,(M\backslash f_1)$ is 3-connected. Since M is 3-irreducible, there is a square of M containing $\{e_1,\, g\}$, contradicting the previous lemma.

Theorem 5.2.6. *If M is a semi-binary 123-irreducible 3-connected matroid the each element belongs to a triad.*

6 GENERALIZED PARALLEL CONNECTION ON 3-CONNECTED MATROIDS: AN IDENTIFICATION THEOREM

Going back to the case of irreducible classes, we will establish that for each diamantic matroid M with m triads there is a unique totally triangular matroid N, up to isomorphism, with m triangles such that:

i) There is an amalgam B of M and N;

ii) For each triad T^* of M, there is a unique triangle Y of N such that Y surrounds T^* in B.

The process of constructing this correspondence uses the generalized parallel connection and its properties.

6.1 GENERALIZED PARALLEL CONNECTION AND $\Delta - Y$ EXCHANGE

The reference to this subsection is the Oxley's book [10], sections 11.4 and 11.5, where we find much more results on parallel connection. For the sake of clarity of the text and in order to avoid using the book to recall some knowns results about connection in parallel, we put this first section.

6.1.1 Generalized parallel connection

Let M_1 and M_2 be two matroids with ground sets E_1 and E_2, respectively. We will denote by r_i and cl_i the rank function and closure operator of M_i, for $i = 1$ and 2. Analogously, we denote by to r_i^* and cl_i^* rank function and closure operator of M_i^*. Suppose that $M_1|T = M_2|T$, where $T = E_1 \cap E_2$. We denote $N = M_1|T = M_2|T$.

Definition. An *amalgam* of M_1 and M_2 is a matroid M with ground set $E = E_1 \cup E_2$ such that $M|E_i = M_i$. The *free amalgam* of M_1 and M_2 is the amalgam M such that for any other amalgam M', every independent set in M' is independent in M.

First, there may not be an amalgam of M_1 and M_2. Even when there is an amalgam, there may not be the free amalgam. See Example 11.4.1 of Oxley [10].

By sub-modularity of rank function, for every amalgam M of M_1 and M_1 we have that:

$$r_M(X) \leq r_1(X \cap E_1) + r_2(X \cap E_2) - r_N(X \cap T)$$

for all $X \subseteq E$.

Definition. The free amalgam M of M_1 and M_2 is called *proper amalgam* if for every $X \subseteq E$,

$$r_M(X) = min\{r_1(Y \cap E_1) + r_2(Y \cap E_2) - r_N(Y \cap T) : X \subseteq Y \subseteq E\}$$

Some free amalgams are not proper, as Example 11.4.4 of Oxley [10] shows.

Definition. A closed set Z of a matroid M is called *modular flat* if for every closed set $X \subseteq E(M)$,

$$r_M(X \cup Z) = r_M(X) + r_M(Z) - r_M(X \cap Z)$$

A matroid M is called *modular* if every closed set $X \subseteq E(X)$ is a modular flat.

Theorem 6.1.1. *(Theorem 11.4.10, Oxley [10]) If either of the following conditions holds, then the proper amalgam of M_1 and M_2 exists:*

i) T is a modular flat of M_1;

ii) N is a modular matroid.

Definition. When $E(si(N))$ is a modular flat of $si(M_1)$, the proper amalgam is called the *generalized parallel connection of M_1 and M_2 across T*, and will be denoted by $P_T(M_1, M_2)$. The existence of $P_T(M_1, M_2)$ is guaranteed by previous theorem.

If $M = P_T(M_1, M_2)$ and, for $X \subseteq E$, we denote by $[X]_i = X \cup cl_i(X \cap E_i)$, then we have that:

$$\{ r_M(X) = r_1([X]_2 \cap E_1) + r_2([X]_1 \cap E_2) - r_N(([X]_1 \cup [X]_2) \cap T)\; cl_M(X) = cl_1([X]_2 \cap E_1) \cup cl_2([X]_1 \tag{6.1}$$

Lemma 6.1.2. *Let $P_T(M_1, M_2)$ be the generalized parallel connection of n-connected matroids M_1 and M_2 with $min\{|E_1|, |E_2|\} \geq 2n-2$, $n \geq 2$. Then $P_T(M_1, M_2)$ is n-connected if and only if $|T| \geq n-1$.*

Proposition 6.1.3. *(Proposition 11.4.14, Oxley [10]) The generalized parallel connection has the following properties:*

i) If $si(N)$ is a modular flat in $si(M_2)$ as well as in $si(M_1)$, then $P_T(M_1, M_2) = P_T(M_2, M_1)$;

ii) The ground set of $si(M_2)$ is a modular flat of the $si(P_T(M_1, M_2))$;

iii) If $e \in E_1 - T$, then $P_T(M_1, M_2)\backslash e = P_T(M_1\backslash e, M_2)$;

iv) If $e \in E_1 - cl_1(T)$, then $P_T(M_1, M_2)/e = P_T(M_1/e, M_2)$;

v) If $e \in E_2 - T$, then $P_T(M_1, M_2)\backslash e = P_T(M_1, M_2\backslash e)$;

vi) If $e \in E_2 - cl_2(T)$, then $P_T(M_1, M_2)/e = P_T(M_1, M_2/e)$;

vii) If $e \in T$, then $P_T(M_1, M_2)/e = P_{T/e}(M_1/e, M_2/e)$;

viii) $P_T(M_1, M_2)/T$ is the direct sum $(M_1/T) \oplus (M_2/T)$;

ix) The classes of graphic, regular, binary and ternary matroids are all closed under the operation of generalized parallel connection (see Proposition 11.4.18 of [10]).

6.1.2 $\Delta - Y$ exchange in 3-connected matroids

Let $T = \{e_1,\, e_2,\, e_3\}$ be a triangle of matroid M. Denote by W_3 a 3-wheel with ground set $\{e_1,\, e_2,\, e_3,\, y_1,\, y_2,\, y_3\}$, spokes set $Y = \{y_1,\, y_2,\, y_3\}$ and rim T such that $Y \cap E(M) = \emptyset$. As $M|T = W_3|T$ and $T = E(si(N))$ is a modular flat in $W_3 = si(W_3)$, Theorem 6.1.1 implies that the parallel connection $P_T(W_3,\, M)$ is well defined.

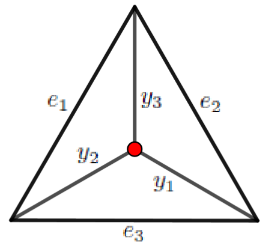

Figure 25 – 3-wheel with rim $T = \{e_1,\, e_2,\, e_3\}$ and spokes set $Y = \{y_1,\, y_2,\, y_3\}$.

Here, we have an important feature of the 3-wheel: if $n \geq 4$ then the rim of W_n is not a modular flat in W_n.

When M is 3-connected, Lemma 6.1.2 implies that $P_T(W_3,\, M)$ is 3-connected.

As T is a coindependent in M, we have that:

Lemma 6.1.4. *(Lemma 11.5.6, Oxley [10]) The set Y is a triad of $P_T(W_3,\, M)\backslash T$;*

Analogously, if M is a 3-connected matroid such that $T = \{e_1,\, e_2,\, e_3\}$ is a triad of M then $P_T(W_3,\, M^*)$ is well defined.

Definition. Let M be a 3-connected matroid and T a triangle of M. The *delta-wye exchange on T* is defined as the matroid $\Delta_T(M)$ obtained from $P_T(W_3,\, M)\backslash T$ after relabel each y_i by e_i. Note that $E(\Delta_T(M)) = E(M)$ and T is a triad of $\Delta_T(M)$. With the fixed labels, the matroid $\Delta_T(M)$ is uniquely determined. In case of T^* be a triad of M, instead a triangle, the *wye-delta exchange on T^** is defined as the matroid $\nabla_{T^*}(M) = (\Delta_{T^*}(M^*))^*$.

Some properties of delta-wye exchange is describe in the following proposition:

Proposition 6.1.5. *Let M be a matroid and T be a triangle of M. Then:*

i) $r(\Delta_T(M)) = r(M) + 1$;

ii) When M is 3-connected, $co(\Delta_T(M))$ is 3-connected;

iii) $\Delta_T(M)\backslash T = M\backslash T$ and $\Delta_T(M)/T = M/T$;

iv) For all $i \in \{1,\, 2,\, 3\}$, $\Delta_T(M)\backslash e_i/(T - e_i) = M/e_i\backslash(T - e_i)$;

v) $\nabla_T(\Delta_T(M)) = M$ *(in case of* T *be a triad of* M*, instead a triangle, we have* $\Delta_T(\nabla_T(M)) = M$*);*
vi) If $x \in E(M) - cl^*_M(T)$ *then* $\Delta_T(M) \backslash x = \Delta_T(M \backslash x)$*;*
vii) If $x \in E(M) - cl_M(T)$ *then* $\Delta_T(M)/x = \Delta_T(M/x)$*;*
viii) A subset of $E(M)$ *is a basis of* $\Delta_T(M)$ *if and only if is a member of one of the following sets:*

$\{\ \{T \cup B\ :\ B \textit{ is a basis of } M/T,\ \textit{or} \{(T - e_i) \cup B\ :\ B \textit{ is a basis of } M/e_i \backslash (T - e_i), \textit{ for all } i \in \{1,\ 2$

For this and more contents about the above properties see [10], pages 452 to 456.

6.2 TRIANGULATION AROUND A TRIAD

Definition. Let M be a matroid, T^* a triad and Y a triangle of M. We say that Y *surrounds* T^* if $M|(T^* \cup Y)$ is isomorphic to a cycle matroid of a 3-wheel.

There is a simple way to put a triangle around a triad T^* on a 3-connected matroid M: denoting by W_3 a 3-wheel with spokes set T^* and rim Y, such that $Y \cap E(M) = \emptyset$, we have that

$$\blacktriangle_{T^*}(M) = si\left[P_Y\left(W_3,\ P^*_{T^*}\left(W^*_3,\ M^*\right)/T^*\right)\right]$$

is a 3-connected matroid with ground set $E(M) \cup Y$, having Y and T^* as triangle and triad, respectively. We have that $\blacktriangle_{T^*}(M) \backslash Y = M$ and Y surrounds T^*.

In this section we will study the circuits of $\blacktriangle_{T^*}(M)$, its triads and how $\blacktriangle_{T^*}(M)$ behaves according to T^*. Moreover, when M is semi-binary then $\blacktriangle_{T^*}(M)$ is also semi-binary. In the end of this section we will describe the triads of $\blacktriangle_{T^*}(M)$ and what happens when T^* is contained in an emerald. These results will be required for the construction described in the next section.

Follows a graphic representation:

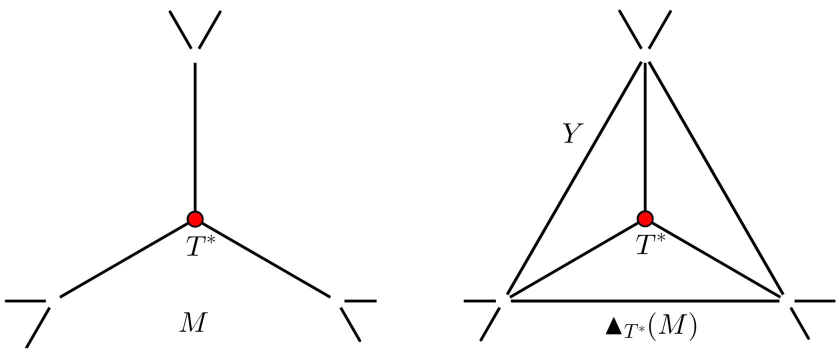

Figure 26 – Putting a triangle Y around $T*$.

Let M be a 3-connected matroid and $T^* = \{e_1, e_2, e_3\}$ a triad of M. We will denote by W_3 a 3-wheel with ground set $E(W_3) = \{e_1, e_2, e_3, y_1, y_2, y_3\}$, spokes set $E(W_3) \cap E(M) = T^*$ and rim $Y = \{y_1, y_2, y_3\}$. We take the labels of Y such that $\{y_i, e_j, e_k\}$ is a triangle of W_3 for $\{i, j, k\} = \{1, 2, 3\}$.

We have that T^* is independent in M, T^* is a modular flat of W_3^* and $si(W_3^*|T^*) = W_3^*|T^*$. Theorem 6.1.1 implies that $P_{T^*}^*(W_3^*, M^*)$ is well defined and we have the following properties:

Proposition 6.2.1. *Properties of* $P_{T^*}^*(W_3^*, M^*)$*:*

i) $E(P_{T^*}^*(W_3^*, M^*)) = E(M) \cup E(W_3) = E(M) \cup Y$*;*

ii) $r(P_{T^*}(W_3^*, M^*)) = r(M^*) + 1$*, and so* $r(P_{T^*}^*(W_3^*, M^*)) = r(M) + 2$*;*

iii) Because of Proposition 6.1.3(iii) we have that

$$P_{T^*}^*(W_3^*, M^*)/Y = [P_{T^*}(W_3^*, M^*)\backslash Y]^* = [P_{T^*}(W_3^*\backslash Y, M^*)]^* = [M^*]^* = M$$

iv) Lemma 6.1.2 implies that $P_{T^*}^*(W_3^*, M^*)$ *is 3-connected;*

v) Lemma 6.1.4 implies that Y *is a triangle of* $P_{T^*}^*(W_3^*, M^*)/T^*$*;*

vi) Because of Proposition 6.1.5(i) we have that $r(P_{T^*}^*(W_3^*, M^*)/T^*) = r(M) - 1$*;*

vii) Proposition 6.1.5(ii) implies that $si(P_{T^*}^*(W_3^*, M^*)/T^*)$ *is 3-connected;*

viii) For indices $\{i, j, k\} = \{1, 2, 3\}$*, we have that* $\{y_i, y_j, e_k\}$ *is a triangle of* W_3^* *and so is a triangle of* $P_{T^*}(W_3^*, M^*)$*. Therefore it is a triad of* $P_{T^*}^*(W_3^*, M^*)$*, just like* T^**.*

First, we will study the circuits of $P_{T^*}^*(W_3^*, M^*)$ a little further and how they relate to the circuits of M.

Lemma 6.2.2. *Y is a triangle of* $P_{T^*}^*(W_3^*, M^*)$*. Moreover, Y is the unique circuit of* $P_{T^*}^*(W_3^*, M^*)$ *contained in* $E(W_3)$ *(see Figure 27).*

Demonstração. First, we will prove that Y is a triangle of $P_{T^*}^*(W_3^*, M^*)$. Since $P_{T^*}^*(W_3^*, M^*)$ is 3-connected, its sufficient to show that $r_{P_{T^*}^*}(Y) = 2$, where $r_{P_{T^*}^*}$ is the rank function of $P_{T^*}^*(W_3^*, M^*)$. If $r_{P_{T^*}}$ denotes the rank function of $P_{T^*}(W_3^*, M^*)$, then we have that

$$r_{P_{T^*}^*}(Y) = |Y| + r_{P_{T^*}}(E - Y) - (r(M^*) + 1) = r_{P_{T^*}}(E - Y) - r(M^*) + 2$$

where $E = E(M) \cup E(W_3)$. As $E - Y = E(M)$ and

$$\{ \ [E - Y_{W_3^*} = (E - Y) \cup cl_{W_3^*}((E - Y) \cap E(W_3)) = E(M), \ [E - Y]_{M^*} = (E - Y) \cup cl_{M^*}((E - Y) \cap$$

then, by (6.1), we have that

$$\begin{aligned} r_{P_{T^*}}(E - Y) &= r_{W_3^*}(E(M) \cap E(W)) + r_{M^*}(E(M)) - r_{M^*|T^*}(T^*) \\ &= 2 + r(M^*) - 2 \\ &= r(M^*) \end{aligned}$$

therefore $r_{P^*_{T^*}}(Y) = 2$, and so Y is a triangle of $P^*_{T^*}(W^*_3,\, M^*)$.

Now, suppose that C is a circuit of $P^*_{T^*}(W^*_3,\, M^*)$, different to Y, contained in $E(W_3)$. Since $Y \nsubseteq C$, we have that $|C| \leq 5$. If $T^* \subseteq C$ then $C = T^*$, otherwise $C - T^*$ is a loop or a parallel class of $P^*_{T^*}(W^*_3,\, M^*)/T^*$ contained in Y; a contradiction because Y is a triangle of $P^*_{T^*}(W^*_3,\, M^*)/T^*$. If $C = T^*$, we have that C is a triangle and triad of $P^*_{T^*}(W^*_3,\, M^*)$, contradicting its 3-connectivity. Therefore $T^* \nsubseteq C$. Since T^* and Y are both not contained in C, we have that $C \cap Y \neq \emptyset$ and $C \cap T^* \neq \emptyset$. Then $C - T^*$ is a loop or a parallel class of $P^*_{T^*}(W^*_3,\, M^*)/T^*$ contained in Y; a contradiction. □

Lemma 6.2.3. *If C is a circuit of $P^*_{T^*}(W^*_3,\, M^*)$ contained in $E(M)$ then C is a circuit of $M\backslash T^*$.*

Demonstração. As $T^*_i = \{e_i,\, y_j,\, y_k\}$ is a triad of $P^*_{T^*}(W^*_3,\, M^*)$ for $\{i,\, j,\, k\} = \{1,\, 2,\, 3\}$, by Proposition 6.2.1 *(viii)*, we have that $C \cap T^* = \emptyset$. Since C do not intersect T^*, C is a circuit of $P^*_{T^*}(W^*_3,\, M^*)\backslash T^*$. Proposition 6.1.3 *(viii)* implies that $P^*_{T^*}(W^*_3,\, M^*)\backslash T^* = (W_3\backslash T^*) \oplus (M\backslash T^*)$. Therefore C is a circuit of $M\backslash T^*$and then its a circuit of M. □

Proposition 6.1.3 *(viii)* implies that $P^*_{T^*}(W^*_3,\, M^*)\backslash T^* = (W_3\backslash T^*) \oplus (M\backslash T^*)$ and so the previous lemma implies

$$\mathcal{C}(M\backslash T^*) = \{C \in \mathcal{C}(P^*_{T^*}(W^*_3,\, M^*)) \mid C \subseteq E(M)\}$$

What happens with circuits of M that intersects T^*?

Lemma 6.2.4. *Let C be a circuit of M such that $C \cap T^* \neq \emptyset$.*

i) If $C \cap T^* = \{e_i,\, e_j\}$, then $C \cup y_k$ is a circuit of $P^*_{T^*}(W^*_3,\, M^*)$, where $\{i,\, j,\, k\} = \{1,\, 2,\, 3\}$. By circuit elimination axiom $C \cup \{y_i,\, y_j\}$ is a circuit of $P^*_{T^*}(W^*_3,\, M^*)$ too;
ii) If $T^* \subseteq C$, then $C \cup \{y_i,\, y_j\}$ is a circuit of $P^*_{T^*}(W^*_3,\, M^*)$, for every 2-subset $\{i,\, j\} \subseteq \{1,\, 2,\, 3\}$;

Demonstração. We will adopt the symbols $r_{P_{T^*}}$ and $r_{P^*_{T^*}}$ for the rank functions of $P_{T^*}(W^*_3,\, M^*)$ and $P^*_{T^*}(W^*_3,\, M^*)$ respectively. Denote by $T^*_k = \{y_i,\, y_j,\, e_k\}$, for $\{i,\, j,\, k\} = \{1,\, 2,\, 3\}$. Remember that T^*_k is a triad of $P^*_{T^*}(W^*_3,\, M^*)$ for $k = 1$, 2 and 3.

Suppose that $C \cap T^* = \{e_1,\, e_2\}$. Denote by $X = C \cup y_3$. In this case

$$[E - X]_{W^*_3} = (E - X) \cup cl_{W^*_3}(\{y_1,\, y_2,\, e_3\}) = E - X$$

and

$$[E - X]_{M^*} = (E - X) \cup cl_{M^*}(E(M) - C) = E - X$$

By (6.1) we have that

$$r_{P_{T^*}}(E - X) = r_{W^*_3}(\{y_1,\, y_2,\, e_3\}) + r_{M^*}((E(M) - C)) - r_{M^*|T^*}(e_3)$$

and so $r_{P_{T^*}}(E-X) = r_{M^*}((E(M)-C))+1 = r(M^*)$, because $E(M)-C$ is a hyperplane of M^*. Therefore $r_{P^*_{T^*}}(X) = |X| + r_{P_{T^*}}(E-X) - r(M^*) - 1 = |X| - 1$ and hence X is dependent in $P^*_{T^*}(W^*_3, M^*)$. Now we will prove that X is a circuit.

If X is not a circuit of $P^*_{T^*}(W^*_3, M^*)$, there is a circuit $Z \subsetneq X$. Because of orthogonality with T^*_1 and T^*_2, we have that C can not be a circuit of $P^*_{T^*}(W^*_3, M^*)$, since $C \cap Y = \emptyset$. Therefore $y_3 \in Z$. The orthogonality with T^*_1 and T^*_2 implies that $\{e_1, e_2\} \subseteq Z$. Then $Z - y_3$ is a circuit of $P^*_{T^*}(W^*_3, M^*)/y_3$. Proposition 6.1.3(iii) implies that $P_{T^*}(W^*_3, M^*)\backslash y_3 = P_{T^*}(W^*_3\backslash y_3, M^*)$. Then $E-(Z-y_3)$ is a hyperplane of $P_{T^*}(W^*_3\backslash y_3, M^*)$.

Denote by r' the rank function of $P_{T^*}(W^*_3\backslash y_3, M^*)$. From (6.1) we have that

$$\begin{aligned} r'(E-(Z-y_3)) &= r_{W^*_3\backslash y_3}([E-(Z-y_3)]_{M^*} \cap (E(W_3)-y_3)) \\ &\quad + r_{M^*}\left([E-(Z-y_3)]_{W^*_3\backslash y_3} \cap (E(M))\right) \\ &\quad - r_{M^*|T^*}\left(\left([E-(Z-y_3)]_{M^*} \cup [E-(Z-y_3)]_{W^*_3\backslash y_3}\right) \cap T^*\right) \end{aligned}$$

where

$$\{ \ [E-(Z-y_3)_{M^*} = (E-(Z-y_3)) \cup cl_{M^*}((E-(Z-y_3)) \cap E(M)), \text{and } [E-(Z-y_3)]_{W^*_3\backslash y_3} = (E$$

Since $(E-(Z-y_3)) \cap E(M) = E(M)-Z$ and $E(M)-C \subsetneq E(M)-Z$, we have that

$$cl_{M^*}((E-(Z-y_3)) \cap E(M)) = E(M)$$

and so $[E-(Z-y_3)]_{M^*} = E$. Because $(E-(Z-y_3)) \cap (E(W^*_3)-y_3) = \{y_1, y_2, e_3\}$ is a triangle of $W^*_3\backslash y_3$, we have that $[E-(Z-y_3)]_{W^*_3\backslash y_3} = E-(Z-y_3)$. Therefore

$$\begin{aligned} r'(E-(Z-y_3)) &= r_{W^*_3}(E \cap (E(W_3)-y_3)) + r_{M^*}((E-(Z-y_3)) \cap (E(M))) \\ &\quad - r_{W^*_3}((E \cup (E-(Z-y_3))) \cap T^*) \\ &= 3 + r_{M^*}(E(M)-(Z-y_3)) - 2 \\ &= r(M^*)+1 \\ &= r(P_{T^*}(W^*_3\backslash y_3, M^*)) \end{aligned}$$

contradicting the fact that $E-(Z-y_3)$ is a hyperplane of $P_{T^*}(W^*_3\backslash y_3, M^*)$. Therefore $C \cup y_3$ is a circuit of $P^*_{T^*}(W^*_3, M^*)$.

Since Y is a triangle of $P^*_{T^*}(W^*_3, M^*)$ and $Y \cap (C \cup y_3) = \{y_3\}$, we have that there is a circuit D of $P^*_{T^*}(W^*_3, M^*)$ contained in $C \cup \{y_1, y_2\}$. Lemma 6.2.3 implies that $D \cap \{y_1, y_2\} \neq \emptyset$, otherwise the orthogonality implies that $D \cap T^* = \emptyset$ and then D is a circuit of M contained in C; a contradiction. The orthogonality with T^*_3 and the fact of $e_3 \notin D$ implies that $\{y_1, y_2\} \subseteq D$. Therefore $D - \{y_1, y_2\}$ and C are both circuits of $P^*_{T^*}(W^*_3, M^*)/Y$, and then $D - \{y_1, y_2\} = C$. Consequently $C \cup \{y_1, y_2\}$ is a circuit of $P^*_{T^*}(W^*_3, M^*)$ too.

From now on, we will suppose that $T^* \subseteq C$.

First, we will prove that for every $i \in \{1, 2, 3\}$ we have that $C \cup y_k$ is an independent of $P^*_{T^*}(W^*_3, M^*)$. Denote by $X = C \cup y_3$, for example. In this case, we have that

$$\{ \;\; [E - X_{W^*_3} = (E - X) \cup cl_{W^*_3}(\{y_1, y_2\}), \text{and} \, [E - X]_{M^*} = (E - X) \cup cl_{M^*}(E(M) - C)$$

Then $cl_{W^*_3}(\{y_1, y_2\}) = \{y_1, y_2, e_3\}$ and $cl_{M^*}(E(M) - C) = E(M) - C$, because $E(M) - C$ is a hyperplane of M^*. Therefore $[E - X]_{W^*_3} = (E - C) \cup e_3$ and $[E - X]_{M^*} = E - X$. Hence

$$r_{P_{T^*}}(E - X) = r_{W^*_3}(\{y_1, y_2\}) + r_{M^*}((E(M) - C) \cup e_3) - r_{M^*|T^*}(e_3)$$

and so $r_{P_{T^*}}(E - X) = r_{M^*}((E(M) - C) \cup e) + 1 = r(M^*) + 1$. Then

$$r_{P^*_{T^*}}(X) = |X| + r_{P_{T^*}}(E - X) - r(M^*) - 1 = |X| + (r(M^*) + 1) - r(M^*) - 1 = |X|$$

Consequently, $X = C \cup y_3$ is an independent of $P^*_{T^*}(W^*_3, M^*)$. More generally, $C \cup y_i$ is an independent of $P^*_{T^*}(W^*_3, M^*)$.

Now, we will prove that $C \cup \{y_i, y_j\}$ is a dependent of $P^*_{T^*}(W^*_3, M^*)$. Take $X = C \cup \{y_1, y_2\}$. we have that

$$\{ \;\; [E - X_{W^*_3} = (E - X) \cup cl_{W^*_3}(\{y_3\}), \text{and} \, [E - X]_{M^*} = (E - X) \cup cl_{M^*}(E(M) - C)$$

Then $cl_{W^*_3}(\{y_1, y_2\}) = \{y_3\}$ and $cl_{M^*}(E(M) - C) = E(M) - C$. Therefore $[E - X]_{W^*_3} = E - X$ and $[E - X]_{M^*} = E - X$. Hence

$$r_{P_{T^*}}(E - X) = r_{W^*_3}(\{y_3\}) + r_{M^*}((E(M) - C)) - r_{M^*|T^*}(\emptyset)$$

and so $r_{P_{T^*}}(E - X) = r_{M^*}((E(M) - C) \cup e) + 1 = r(M^*)$. Then

$$r_{P^*_{T^*}}(X) = |X| + r_{P_{T^*}}(E - X) - r(M^*) - 1 = |X| - 1$$

Consequently, $X = C \cup \{y_1, y_2\}$ is a dependent of $P^*_{T^*}(W^*_3, M^*)$. Hence, there is a circuit D contained in $C \cup \{y_1, y_2\}$. Suppose, by contradiction, that $D \subsetneq C \cup \{y_1, y_2\}$. Since $C \cup y_1$ and $C \cup y_2$ are both independents of $P^*_{T^*}(W^*_3, M^*)$, we have that $\{y_1, y_2\} \subseteq D$. As $y_3 \notin D$, the orthogonality with T^*_1 and T^*_2 implies that $\{e_1, e_2\} \subseteq D$. Therefore $D = (C \cup \{y_1, y_2\}) - e_3$. There is a circuit D' contained in $(D \cup Y) - y_1$. The orthogonality with T^*_3 implies that $y_2 \notin D'$ and so $D' = (C - e_3) \cup y_3$, otherwise $D' \subseteq E(M) - T^*$ and this is a contradiction. Then $D' = C - e_3$ is a circuit of $P^*_{T^*}(W^*_3, M^*)/Y = M$; a contradiction. □

Corollary 6.2.5. *Let C be a circuit of $P^*_{T^*}(W^*_3,\, M^*)$. Then for $\{i,\, j,\, k\} = \{1,\, 2,\, 3\}$:*

i) $C = Y$, a triangle of $P^*_{T^*}(W^*_3,\, M^*)$; or

ii) $C \cap Y = \{y_i,\, y_j\}$. Then $C - Y$ is a circuit of M intersecting T^* in two or three elements; or

iii) $C \cap Y = \{y_i\}$. Then $C - Y$ is a circuit of M intersecting T^* in $\{e_j,\, e_k\}$ and $e_i \notin C - Y$; or

iv) $C \cap Y = \emptyset$ and then C is a circuit or $M \backslash T^*$.

Corollary 6.2.6. *There is no square of $P^*_{T^*}(W^*_3,\, M^*)$ containing T^*. Consequently, $P^*_{T^*}(W^*_3,\, M^*)\,/T^*$ is not 3-connected if and only if it has parallel classes. Moreover, if M is semi-binary then $P^*_{T^*}(W^*_3,\, M^*)$ is semi-binary.*

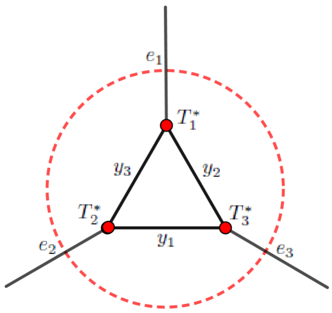

Figure 27 – Graphic representation of local structure of $P^*_{T^*}(W^*_3,\, M^*)$ around T^*, highlighting the triads that intersects T^*.

Consider the matroid

$$P_Y\left(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*\right)$$

We have that:

- Y is a triangle of $P_Y\left(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*\right)$;
- $r\left(P_Y\left(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*\right)\right) = r(W_3) + r\left(P^*_{T^*}(W^*_3,\, M^*)\,/T^*\right) - r(W_3|Y) = r(M)$;
- Because of Proposition 6.1.5*(v)*, we have that $P_Y\left(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*\right)\backslash Y = M$;

Lemma 6.1.2 implies that if $P^*_{T^*}(W^*_3,\, M^*)\,/T^*$ is 3-connected then $P_Y\left(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*\right)$ is 3-connected too.

Now, we will look for the parallel classes of $P^*_{T^*}(W^*_3,\, M^*)\,/T^*$. Note that $\nabla_{T^*}(M)$ is obtained from $P^*_{T^*}(W^*_3,\, M^*)\,/T^*$ changing y_i to e_i.

Lemma 6.2.7. *A 2-set $\{x,\, y\}$ is a parallel class of $P^*_{T^*}(W^*_3,\, M^*)\,/T^*$ if and only if there is a square Q of $P^*_{T^*}(W^*_3,\, M^*)$ such that $Q \cap T^* = \{e_i,\, e_j\}$, $\{x,\, y\} \subseteq Q$, $x \in E(M) - E(W_3)$ and $y = y_k$, for $\{i,\, j,\, k\} = \{1,\, 2,\, 3\}$.*

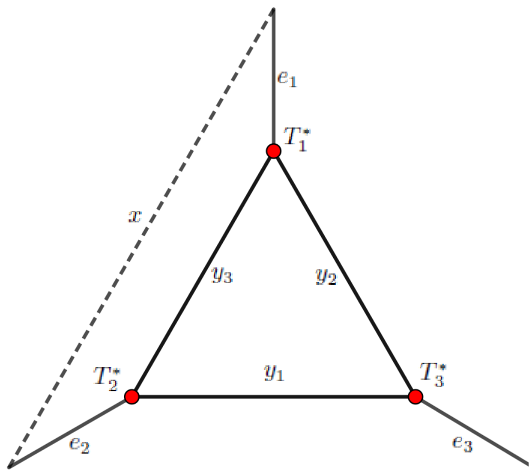

Figure 28 – $\{x,\, y_3\}$ is a parallel class of $P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*$.

Demonstração. If Q is a square of $P^*_{T^*}\left(W_3^*,\, M^*\right)$ such that $Q\cap T^* = \{e_1,\, e_2\}$, $\{x,\, y\}\subseteq Q$ and $y = y_3$, then $Q - T^* = \{x,\, y\}$ is a a parallel class of $P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*$.

There is no triangle of $P^*_{T^*}\left(W_3^*,\, M^*\right)$ that intersect T^* in just one element, because of orthogonality with T^*. If T is a triangle of $P^*_{T^*}\left(W_3^*,\, M^*\right)$ that intersect T^* in two element, the orthogonality with $T_i^* = \{e_i,\, y_j,\, y_k\}$, $\{i,\, j,\, k\} = \{1,\, 2,\, 3\}$, implies that $T\subseteq E\left(W_3\right)$, contradicting Lemma 6.2.2.

Suppose that C is a circuit of $P^*_{T^*}\left(W_3^*,\, M^*\right)$ with five elements such that $T^*\subseteq C$. Because of orthogonality with $T_i^* = \{e_i,\, y_j,\, y_k\}$, $\{i,\, j,\, k\} = \{1,\, 2,\, 3\}$, we have that $C = \{e_1,\, e_2,\, e_3,\, y_i,\, y_j\}$, for some 2-subset $\{i,\, j\}\subseteq\{1,\, 2,\, 3\}$, and so $C\subseteq E\left(W_3\right)$, contradicting Lemma 6.2.2.

We have that if $\{x,\, y\}$ is a parallel class of $P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*$ then there are a square Q of $P^*_{T^*}\left(W_3^*,\, M^*\right)$ such that intersects T^* in just two elements, say $\{e_1,\, e_2\}$. Since Q is not contained in $E\left(W_3\right)$, there is $x\in E\left(M\right) - T^*$ such that $x\in Q$. The orthogonality with T_1^* and T_2^* implies that $y_3\in Q$. □

Corollary 6.2.8. *$P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*$ has a parallel class if and only if there is a triangle T of M such that $T\cap T^*\neq\emptyset$*

Lemma 6.2.9. *$P_Y\left(W_3,\, P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*\right)$ is not 3-connected if and only if there is a triangle of M that intersects T^*, and si$\left(P_Y\left(W_3,\, P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*\right)\right)$ is 3-connected.*

Demonstração. If $P_Y\left(W_3,\, P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*\right)$ is not 3-connected then $P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*$ is not 3-connected and so there is a triangle T of M such that. If there is a triangle T of M such that $T\cap T^*\neq\emptyset$ then $P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*$ has a parallel class that is a parallel class of

$$P_Y\left(W_3,\, P^*_{T^*}\left(W_3^*,\, M^*\right)/T^*\right)$$

too. □

Corollary 6.2.10. *If M is a triangle-free 3-connected matroid then $P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)$ is 3-connected for each triad T^* of M.*

Definition. Let M be a 3-connected matroid and T^* a triad of M. Take W_3 a 3-wheel with spokes set T^* and rim Y, such that $Y \cap E\left(M\right) = \emptyset$. We define a *triangulation around* T^* as the matroid

$$\blacktriangle_{T^*}\left(M\right) = si\left[P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\right] \tag{6.2}$$

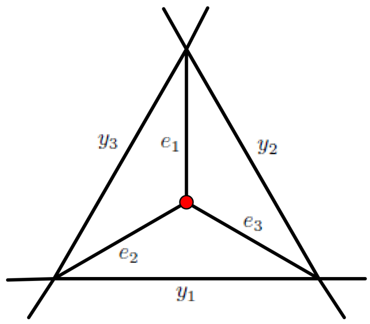

Figure 29 – Triangulation around T^*.

When M has a triangle T intersecting T^*, there is a parallel class on $P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)$ containing the element on $T - T^*$ and a element of Y (see $\{x,\, y_3\}$ in Figure 28). In this case, we will treat this class representative as an element of $E\left(M\right)$ or $E\left(W_3\right)$ conveniently, for the sake of textual fluidity. With this caution warned, it now makes sense write $\blacktriangle_{T^*}\left(M\right) | \left(T^* \cup Y\right) = W_3$. If M has a triangle surrounding T^* then $\blacktriangle_{T^*}\left(M\right) = M$.

Proposition 6.2.11. *Let $\blacktriangle_{T^*}\left(M\right)$ be a triangulation around a triad T^* of M with ground set $E\left(\blacktriangle_{T^*}\left(M\right)\right) = E\left(M\right) \cup Y$, where Y is the triangle surrounds T^* on $\blacktriangle_{T^*}\left(M\right)$. Then $\blacktriangle_{T^*}\left(M\right)$ is 3-connected and:*

i) $\blacktriangle_{T^*}\left(M\right) | E\left(M\right) = M$;

ii) $r\left(\blacktriangle_{T^*}\left(M\right)\right) = r\left(M\right)$;

iii) $\blacktriangle_{T^*}\left(M\right) \backslash T^*$ is 3-connected and $r\left(\blacktriangle_{T^*}\left(M\right) \backslash T^*\right) = r\left(M\right) - 1$;

iv) If N is a 3-connected matroid with ground set $E = E\left(M\right) \cup Y$ such that $N|E\left(M\right) = M$, T^* is a triad of N and Y is a triangle surrounding T^* on N, then $N = \blacktriangle_{T^*}\left(M\right)$;

v) If T^* is a nucleus of a pure diamond of M then every element of Y belongs to at least two triangles of $\blacktriangle_{T^*}\left(M\right) \backslash T^*$.

Demonstração. *i)* If we denote by

$$\widehat{Y} = Y - \{y \in Y \,|\, y \text{ belongs to a parallel class of } P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\}$$

then

$$\begin{aligned}
\blacktriangle_{T^*}(M)\backslash\widehat{Y} &= si\left[P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\right]\backslash\widehat{Y} \\
&= \left[P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\backslash\left(Y-\widehat{Y}\right)\right]\backslash\widehat{Y} \\
&= P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\backslash Y \\
&= \Delta_{T^*}\left(\nabla_{T^*}(M)\right) \\
\text{(by Proposition 6.1.5(v))} &= M
\end{aligned}$$

ii) $r\left(\blacktriangle_{T^*}(M)\right) = r\left(P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\right) = r(M)$.

iii)

$$\begin{aligned}
\blacktriangle_{T^*}(M)\backslash T^* &= si\left[P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\right]\backslash T^* \\
&= si\left[P_Y\left(W_3,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\backslash T^*\right] \\
\text{(Proposition 6.1.3}(iii)\text{)} &= si\left[P_Y\left(W_3\backslash T^*,\, P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right)\right] \\
&= si\left[P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right]
\end{aligned}$$

that is 3-connected because of Proposition 6.1.5(*ii*), and $r\left(si\left[P^*_{T^*}\left(W^*_3,\, M^*\right)/T^*\right]\right) = r(M) - 1$;

iv) Since Y is a triangle surrounding T^*on N, we have that

$$N = \blacktriangle_{T^*}(N) = \blacktriangle_{T^*}(N|E(M)) = \blacktriangle_{T^*}(M)$$

v) Take $y \in Y$. Since T^* is a nucleus of a pure diamond, there is a square Q of M such that $(Q \cap T^*) \cup \{y\}$ is a triangle of $\blacktriangle_{T^*}(M)$. Therefore $(Q - T^*) \cup \{y\}$ is a triangle, because of circuit elimination axiom and orthogonality. The self Y is the other triangle containing y on $\blacktriangle_{T^*}(M)\backslash T^*$. □

Lemma 6.2.12. *Let M be a 3-connected matroid and T^* a triad of M. Denote by Y the triangle surround T^* on $\blacktriangle_{T^*}(M)$. Suppose that for each triangle T of M that intersects T^* we have that $T - T^*$ do not belongs to any triad of M (equivalently, $M\backslash(T - T^*)$ is 3-connected). Then there is no triad of $\blacktriangle_{T^*}(M)$ intersecting Y.*

Demonstração. Suppose that T'^* is a triad of $\blacktriangle_{T^*}(M)$ that intersects Y. Since $\blacktriangle_{T^*}(M)\,|\,(T^* \cup Y)$ is a 3-wheel with spokes set T^* and rim Y, we have that there is $e \in T^*$ and a 2-set $\{y_1,\, y_2\} \subseteq Y$ such that $T'^* = \{e,\, y_1,\, y_2\}$. As T'^* is not contained in $E(M)$, we have that $\{y_1,\, y_2\} \nsubseteq E(M)$. If y_1 and y_2 are both not containing in $E(M)$ then there is a base B of M such that $e \notin B$. As $r\left(\blacktriangle_{T^*}(M)\right) = r(M)$, we have that B is a base of $\blacktriangle_{T^*}(M)$ that do not intersects T'^*; a contradiction. So we can assume that $y_1 \in E(M)$. Since M is 3-connected, there is a base B of M that do not intersects $\{e,\, y_1\}$, and then B is a base of $\blacktriangle_{T^*}(M)$ that do not intersects T'^*; also a contradiction. □

Lemma 6.2.13. *Let M be a 3-connected matroid and T^* a triad of M. Suppose that for each triangle T of M that intersects T^*, $M\backslash(T-T^*)$ is 3-connected. Then the triads of $\blacktriangle_{T^*}(M)$ are T^* and the triads of M that do not intersects T^*. The triads of M that intersects T^*, different of it self, are destroyed.*

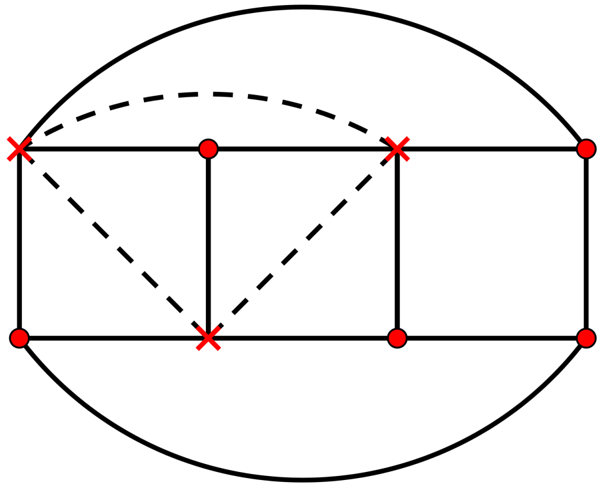

Figure 30 – Triads destroyed on triangulation around a triad in $L_4 = D_6$.

Demonstração. Denote by Y the triangle surround T^* on $\blacktriangle_{T^*}(M)$ and take

$$\widehat{Y} = Y - \{y \in Y \,:\, y \text{ belongs to a parallel class of } P_Y(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*)\}$$

By (6.2), we have that

$$\blacktriangle_{T^*}(M) = P_Y(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*) \setminus \left(Y - \widehat{Y}\right)$$

Lemma 6.2.2 implies that T^* is a triangle of $P^*_Y(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*)$ that do not intersects $Y-\widehat{Y}$, since each element of $Y-\widehat{Y}$ belongs to a triangle of M that intersects T^*. Therefore T^* is a triangle of $\blacktriangle^*_{T^*}(M)$.

Take T'^* a triad of M such that $T'^* \cap T^* = \emptyset$. Therefore T'^* is a triangle of $M^*\backslash T^*$. Since

$$\begin{aligned}
\blacktriangle^*_{T^*}(M)\backslash Y &= \left[P^*_Y(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*)\,/\left(Y-\widehat{Y}\right)\right]\backslash Y \\
&= \left[P^*_Y(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*)\backslash Y\right]/\left(Y-\widehat{Y}\right) \\
&= \left[P_Y(W_3,\, P^*_{T^*}(W^*_3,\, M^*)\,/T^*)\,/Y\right]^*/\left(Y-\widehat{Y}\right) \\
\text{(Proposition 6.1.3 (viii))} \quad &= \left[(W_3/Y) \oplus (P^*_{T^*}(W^*_3,\, M^*)\,/T^*/Y)\right]^*/\left(Y-\widehat{Y}\right) \\
&= \left[(W^*_3\backslash Y) \oplus (P_{T^*}(W^*_3,\, M^*) \setminus (T^* \cup Y))\right]/\left(Y-\widehat{Y}\right) \\
&= \left[(W^*_3\backslash Y) \oplus (M^*\backslash T^*)\right]/\left(Y-\widehat{Y}\right)
\end{aligned}$$

and T'^* do not intersects $Y-\widehat{Y}$, we have that T'^* is a triangle of $\blacktriangle^*_{T^*}(M)\backslash Y$, and so of $\blacktriangle^*_{T^*}(M)$. If $T'^* \neq T^*$ is a triad of M such that $T'^* \cap T^* \neq \emptyset$ then T'^* is not a triad of $\blacktriangle^*_{T^*}(M)$ because of Y surrounds T^*. □

Lemma 6.2.14. *Let M be a 3-connected matroid such that M has a triangle Y surrounding a triad T^*. If a subset $X \subseteq Y$ do not intersects any triad of M then $M\backslash X$ is 3-connected and $M = \blacktriangle_{T^*}(M\backslash Y)$.*

Demonstração. Suppose that $X \neq \emptyset$ and take $x_1 \in X$. Since $si\,(M/x_1)$ is not 3-connected, because has a parallel class, Bixby's Theorem implies that $co\,(M\backslash x_1) = M\backslash x_1$ is 3-connected. If $X - x_1 \neq \emptyset$, take $x_2 \in X - x_1$. There is a triangle of M containing x_2 that avoids x_1 and intersects T^*, then this triangle is also a triangle of $M\backslash x_1$. So $si\,[(\blacktriangle_{T^*}(M)\backslash y_1)/y_2]$ is not 3-connected, provide involves a deletion of one element of T^*. Therefore $co\,((M\backslash x_1)\backslash x_2) = M\backslash\{x_1,\, x_2\}$ is 3-connected. Analogously to the case $X = Y$. Because of Proposition 6.2.11(*iv*), $M = \blacktriangle_{T^*}(M\backslash Y)$. □

Corollary 6.2.15. *If M is a triangle-free 3-connected matroid and $\blacktriangle_{T^*}(M)$ is a triangulation around a triad T^* of M, then $\blacktriangle_{T^*}(M)\backslash X$ is 3-connected for every $X \subseteq Y$, the triangle surrounding T^* on $\blacktriangle_{T^*}(M)$.*

Now, in the end of this section, we will look at how triangulation relates to emeralds:

Lemma 6.2.16. *Let M be a 3-connected matroid with T^* and T'^* different triads of M. Then $\blacktriangle_{T^*}(M) = \blacktriangle_{T'^*}(M)$ if and only if $T^* \cup T'^*$ is a pure emerald of M. Moreover, if $T^* \cup T'^*$ is a non-pure emerald then $\blacktriangle_{T^*}(M)$ is non-semi-binary.*

Demonstração. If $\blacktriangle_{T^*}(M) = \blacktriangle_{T'^*}(M)$ then there is a triangle Y surrounding both T^* and T'^*. Circuit elimination axiom implies that $T^* \cup T'^*$ is a pure emerald. If $T^* \cup T'^*$ is a pure emerald of M and Y is the triangle surrounding T^* on $\blacktriangle_{T^*}(M)$, then circuit elimination axiom implies that Y also surrounds T'^*. Therefore $\blacktriangle_{T^*}(M) = \blacktriangle_{T'^*}(M)$.

Now, suppose that $T^* \cup T'^*$ is a non-pure emerald. Then there is squares Q and Q' of M such that $T^* \cup T'^* = Q \cup Q'$ and $Q \triangle Q'$ is not a squares of M. By circuit elimination axiom, there is a circuit contained in $Q \cup Q' - e$ for each $e \in Q \cap Q'$. Since there is no triangle contained in $Q \triangle Q'$, because of triads, we have that $Q \cup Q' - e$ is a circuit of M for each $e \in Q \cap Q'$. Take $e \in Q \cap Q' \cap T^*$. Then $Q \cup Q' - e = T'^* \cup (T^* - e)$ is a circuit of M, and so of $\blacktriangle_{T^*}(M)$. If $Y = \{x,\, y,\, z\}$ is the triangle surrounding T^* then, by circuit elimination axiom, $\{x\} \cup (Q \cap T'^*)$ and $\{y\} \cup (Q' \cap T'^*)$ are triangles of $\blacktriangle_{T^*}(M)$. Since Y do not surrounds T'^*, there is no triangle containing z contained in $T'^* \cup \{z\}$, otherwise this triangle would be intersects $\{x\} \cup (Q \cap T'^*)$ or $\{y\} \cup (Q' \cap T'^*)$ in two elements.

We have that $T'^* \cup (T^* - e)$ and $(T^* - e) \cup \{z\}$ are both circuits of $\blacktriangle_{T^*}(M)$. Circuit elimination axiom and orthogonality implies that $T'^* \cup \{z\}$ is a circuit of $\blacktriangle_{T^*}(M)$. □

6.3 AN IDENTIFICATION THEOREM: ASSOCIATED TRIANGULAR MATROID

As first application of the triangulation around a triad operation, we will establish that the family of 3-connected diamantics matroids is in a one-to-one correspondence with the family of 3-connected totally triangular matroids.

Definition. A 3-connected matroid M is said *triangular* if each element belongs to at least 2 triangles. A triangular matroid is said *totally triangular* if:
i) Every pair of triangles intersects in at most 1 element; and
ii) M has no triads.

Note that M is totally triangular if and only if M^* is a triangle-free 3-connected matroid such that each element belongs to at least 2 triads and every pair of triads of intersects in at most 1 element.

Example 6.3.1. A 3-wheel is a triangular 3-connected matroid such that each element belongs to a two triads, therefore a 3-wheel is a non-totally triangular matroid. The Fano matroid F_7 is an example of a totally triangular matroid.

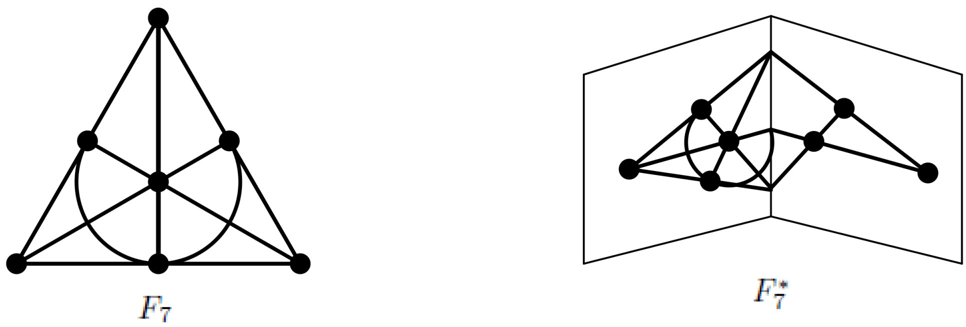

Figure 31 – Geometric representation to F_7 and its dual.

Let M be a triangle-free 3-connected matroid such that each element belongs to a unique triad. If we denote by

$$n = \frac{|E(M)|}{3}$$

then we can put an order on family of triads of M,

$$\{T_1^*, T_2^*, \ldots, T_n^*\}$$

For each $k \in \{1, 2, \ldots, n\}$, we will denote by W_3^k the 3-wheel with ground set $E\left(W_3^k\right) = T_k^* \cup Y_k$ such that T_k^* is the spokes set of W_3^k, Y_k the rim of W_3^k and $Y_k \cap (E(M)\, i \neq k \bigcup Y_i) = \emptyset$.

Denote by $M_1 = \blacktriangle_{T_1^*}(M)$ and for each $k > 1$, we define

$$M_k = \blacktriangle_{T_k^*}(M_{k-1})$$

By Lemma 6.2.13, we have that M_k has the same triads of M for each $k \in \{1,\ 2,\ \ldots,\ n\}$. Lemma 6.2.9 implies that M_k is 3-connected for every k.

For each $k \in \{1,\ 2,\ \ldots,\ n\}$, consider the following sets

$$\widehat{Y_k} = Y_k - \left\{ y \in Y_k \,:\, y \text{ belongs to a parallel class of } P_{Y_k}\left(W_3^k,\ P^*_{T_k^*}\left(W_3^{k*},\ M^*_{k-1}\right)/T_k^*\right)\right\}$$

Note that $\widehat{Y_1} = Y_1$. We have that $E(M_k) = E(M) \cup \left(\bigcup_{i=1}^{k} \widehat{Y_i}\right)$.

Lemma 6.3.2. *For every $k \in \{1,\ 2,\ \ldots,\ n\}$, M_k is a 3-connected matroid such that:*

i) $M_k \backslash \widehat{Y_k} = M_{k-1}$, and so $M_k \backslash \bigcup_{i=1}^{k} \widehat{Y_i} = M$;

ii) $r(M_k) = r(M)$;

iii) $M_k \backslash \bigcup_{i=1}^{k} T_i^*$ is a 3-connected matroid with rank $r(M) - k$;

iv) If every triad of M is a nucleus of a pure diamond then every element of $\bigcup_{i=1}^{k} Y_i$ belongs to at least two triangles of $M_k \backslash \bigcup_{i=1}^{k} T_i^*$;

v) The triads of M_k are the same triads of M;

Demonstração. Items (i), (ii) and (iii) are consequence of Proposition 6.2.11 (i), (ii) and (iii), applied k times.

Item (iv): case $k = 1$ is consequence of Proposition 6.2.11 (v). Suppose valid for $k - 1$. Take $y \in \bigcup_{i=1}^{k} Y_k$. If $y \in Y_j$ for $j < k$ then y belongs to at least two triangles of M_{k-1}, and so of M_k. If $y \in \widehat{Y_k}$, then we can apply the same arguments as Proposition 6.2.11 (v).

Item (v) is consequence of 6.2.13, since each element of M belongs to a unique triad. □

Note that M_n is a triangular 3-connected matroid with $r(M_n) = r(M)$ such that $M_n | E(M) = M$. When every triad of M is a nucleus of a pure diamond then $M_n \backslash \bigcup_{i=1}^{n} T_i^*$ is a 3-connected triangular matroid without triads, with rank $r(M) - n$. Moreover, if every emerald of M is pure (for example when M is diamantic) then $M_n \backslash \bigcup_{i=1}^{n} T_i^*$ is a totally triangular matroid (see Lemma 6.2.16).

Lemma 6.3.3. *M_n does not depend on the order of the triads.*

Demonstração. Take $\sigma : \{1,\ 2,\ ...,\ n\} \longrightarrow \{1,\ 2,\ ...,\ n\}$ a permutation. Denote by

$$T_i'^* = T^*_{\sigma(i)} = T^*_{\sigma_i}$$

then $\{T_1'^*,\ T_2'^*,\ ...,\ T_n'^*\}$ is another ordination for the triads. As W_3^k denotes a 3-wheel with spokes set T_k^* and rim Y_k, then M_n has ground set $E(M_n) = E(M) \cup \left(\bigcup_{i=1}^{n} Y_i\right)$. For each $k \in \{1, ..., n\}$, take $W_3'^k$ a copy of a 3-wheel with spokes set $T_k'^* = T_{\sigma_k}^*$ and rim Y_k', such that

$$Y_k' \cap \left(E\left(M_n\right) \cup \left(\bigcup_{i \neq k} Y_i' \right) \right) = \emptyset$$

We denote by

$$\{\, M'_1 = \blacktriangle_{T_1'^*}\left(M\right), M_k' = \blacktriangle_{T_k'^*}\left(M_{k-1}'\right), \text{for each } k > 1.$$

We will to prove that $M_n' = M_n$, up to relabel of the triangles that surrounds the triads.

By Lemma 6.2.13 M_n and M_n' has the same triads of M. For each $k \in \{1, ..., n\}$ we have that

$$M_n = \blacktriangle_{T_k'^*}\left(M_n \backslash Y_{\sigma_k}\right)$$

because of Lemma 6.2.14. So, we have that

$$\begin{aligned} M_n &= \blacktriangle_{T_1'^*}\left(M_n \backslash Y_{\sigma_1}\right) \\ &= \blacktriangle_{T_2'^*}\left(\blacktriangle_{T_1'^*}\left(M_n \backslash Y_{\sigma_1}\right) \backslash Y_{\sigma_2}\right) \end{aligned}$$

If we denote by

$$\widehat{Y_{\sigma_2}} = Y_{\sigma_2} - Y_{\sigma_1}$$

then we have that

$$\begin{aligned} \blacktriangle_{T_1'^*}\left(M_n \backslash Y_{\sigma_1}\right) \backslash Y_{\sigma_2} &= si\left[P_{Y_1'}\left(W_3'^1, P_{T_1'^*}^*\left(\left[W_3'^1\right]^*, \left[M_n \backslash Y_{\sigma_1}\right]^*\right) / T_1'^*\right)\right] \backslash Y_{\sigma_2} \\ \text{(Corollary 6.2.8)} &= P_{Y_1'}\left(W_3'^1, P_{T_1'^*}^*\left(\left[W_3'^1\right]^*, \left[M_n \backslash Y_{\sigma_1}\right]^*\right) / T_1'^*\right) \backslash Y_{\sigma_2} \\ \text{(Proposition 6.1.3}(v)) &= P_{Y_1'}\left(W_3'^1, P_{T_1'^*}^*\left(\left[W_3'^1\right]^*, \left[M_n \backslash Y_{\sigma_1}\right]^*\right) / T_1'^* \backslash \widehat{Y_{\sigma_2}}\right) \backslash \left(Y_{\sigma_2} - \widehat{Y_{\sigma_2}}\right) \\ \text{(Proposition 6.1.3}(vi)) &= P_{Y_1'}\left(W_3'^1, P_{T_1'^*}^*\left(\left[W_3'^1\right]^*, \left[M_n \backslash \bigcup_{i=1}^{2} Y_{\sigma_i}\right]^*\right) / T_1'^*\right) \backslash \left(Y_{\sigma_2} - \widehat{Y_{\sigma_2}}\right) \\ &= \blacktriangle_{T_1'^*}\left(M_n \backslash \bigcup_{i=1}^{2} Y_{\sigma_i}\right) \backslash \left(Y_{\sigma_2} - \widehat{Y_{\sigma_2}}\right) \end{aligned}$$

Since there is no triangle of $M_n \backslash Y_{\sigma_1}$ intersecting $T_1'^*$, we can apply Corollary 6.2.8 in the second equality.

For fourth equality, we have to show that $\widehat{Y_{\sigma_2}} \cap cl_{M_n^*/Y_{\sigma_1}}\left(T_1'^*\right) = \emptyset$. Suppose that $y \in \widehat{Y_{\sigma_2}} \cap cl_{M_n^*/Y_{\sigma_1}}\left(T_1'^*\right)$, then

$$2 = r_{M_n^*/Y_{\sigma_1}}\left(T_1'^* \cup \{y\}\right) = r_{M_n^*}\left(T_1'^* \cup \{y\} \cup Y_{\sigma_1}\right) - r_{M_n^*}\left(Y_{\sigma_1}\right)$$

and $r_{M_n^*}\left(T_1'^* \cup \{y\} \cup Y_{\sigma_1}\right) = 5$. So, for some $e \in T_1'^*$, we have that $\left(T_1'^* - e\right) \cup \{y\} \cup Y_{\sigma_1}$ is a circuit of M_n^* that contains $y \in Y_{\sigma_2}$ but do not intersects $T_2'^*$. A contradiction since Y_{σ_2} surrounds $T_2'^*$.

Therefore

$$\begin{aligned} M_n &= \blacktriangle_{T_2'^*}\left(\blacktriangle_{T_1'^*}\left(M_n \backslash \left(\bigcup_{i=1}^{2} Y_{\sigma_i}\right)\right) \backslash \left(Y_{\sigma_2} - \widehat{Y_{\sigma_2}}\right)\right) \\ &= \blacktriangle_{T_3'^*}\left(\blacktriangle_{T_2'^*}\left(\blacktriangle_{T_1'^*}\left(M_n \backslash \left(\bigcup_{i=1}^{2} Y_{\sigma_i}\right)\right) \backslash \left(Y_{\sigma_2} - \widehat{Y_{\sigma_2}}\right)\right) \backslash Y_{\sigma_3}\right) \end{aligned}$$

and, if we take

$$\widehat{Y_{\sigma_3}} = Y_{\sigma_3} - \bigcup_{i=1}^{2} Y_{\sigma_i}$$

then we have that

$$M_n = \blacktriangle_{T_3'^*}\left(\blacktriangle_{T_2'^*}\left(\blacktriangle_{T_1'^*}\left(M_n\backslash\left(\bigcup_{i=1}^{3} Y_{\sigma_i}\right)\right)\backslash\left(Y_{\sigma_2}-\widehat{Y_{\sigma_2}}\right)\right)\backslash\left(Y_{\sigma_3}-\widehat{Y_{\sigma_3}}\right)\right)$$

and so on, until

$$M_n = \blacktriangle_{T_n'^*}\left(\blacktriangle_{T_{n-1}'^*}\left(...\blacktriangle_{T_2'^*}\left(\blacktriangle_{T_1'^*}\left(M_n\backslash\left(\bigcup_{i=1}^{n} Y_{\sigma_i}\right)\right)\backslash\left(Y_{\sigma_2}-\widehat{Y_{\sigma_2}}\right)...\right)\backslash\left(Y_{\sigma_{n-1}}-\widehat{Y_{\sigma_{n-1}}}\right)\right)\backslash\left(Y_{\sigma_n}-\widehat{Y_{\sigma_n}}\right)\right)$$

By previous lemma, we have that $M_n\backslash\left(\bigcup_{i=1}^{n} Y_{\sigma_i}\right) = M$. Then

$$\blacktriangle_{T_1'^*}\left(M_n\backslash\left(\bigcup_{i=1}^{n} Y_{\sigma_i}\right)\right)\backslash\left(Y_{\sigma_2}-\widehat{Y_{\sigma_2}}\right) = \blacktriangle_{T_1'^*}(M)$$

and

$$\blacktriangle_{T_2'^*}\left(\blacktriangle_{T_1'^*}\left(M_n\backslash\left(\bigcup_{i=1}^{n} Y_{\sigma_i}\right)\right)\backslash\left(Y_{\sigma_2}-\widehat{Y_{\sigma_2}}\right)\right)\backslash\left(Y_{\sigma_3}-\widehat{Y_{\sigma_3}}\right) = \blacktriangle_{T_2'^*}\left(\blacktriangle_{T_1'^*}(M)\right)$$

Continuing this, we obtain that

$$M_n = \blacktriangle_{T_n'^*}\left(\blacktriangle_{T_{n-1}'^*}\left(...\blacktriangle_{T_2'^*}\left(\blacktriangle_{T_1'^*}(M)\right)\right)\right) = M_n'$$

□

Because of previous result, we will denote M_n by $\mathbb{B}(M)$.

Denote by

- $\Im^n = \{M$: 3-connected with n triads and each element belongs to a nucleus of a pure diamon
- $\wp^n = \{M \in \Im^n$: M is a diamantic matroid$\}$;
- $\mathcal{T}^n$ the class of 3-connected triangular matroids without triads having n triangles;
- $\mathcal{Y}^n$ the class of totally triangular matroids having n triangles;

Because of Lemma 5.1.1, we have that $\wp^n = \emptyset$ for $n < 4$. Note that $\mathcal{Y}^n = \emptyset$ for $n < 4$.

Denoting by τ^* the set of triads of M, previous lemma implies that $\flat : \Im^n \longrightarrow \mathcal{T}^n$ such that

$$\flat(M) = M^\flat = \mathbb{B}(M)\backslash T^* \in \tau^* \bigcup T^*$$

is a function, unless re-labelling the triangles. Note that $\flat|_{\wp^n}$ is injective and $\flat(\wp^n) \subseteq \mathcal{Y}^n$. This function is not injective over $\Im^n$ because of the pure emeralds. If $M \in \Im^n$ and has a non-pure emerald then $M^\flat \notin \mathcal{Y}^n$.

Definition. For $M \in \wp^n$, we say that $M^\flat$ is the *triangular matroid associated* to M.

Example 6.3.4. If M denotes the cycle matroid of graph described in Figure 17 then $M^\flat = M(K_5)$ is its associated triangular matroid, where K_5 denotes the complete graph with 5 vertices.

Definition. If M is a diamantic matroid then $\mathbb{B}(M)$ is an amalgam of M and $M^\flat$ having the same triads of M such that each triad has a unique triangle surround it. $\mathbb{B}(M)$ is the *barycentric subdivision* of M.

Now, we will to show that $\flat|_{\wp^n}$ is a bijection of $\wp^n$ on $\mathcal{Y}^n$. From now until the end of this section, M will denotes a triangular 3-connected matroid. Denote by

$$\{Y_1, Y_2, \ldots, Y_n\}$$

the family of triangles of M.

For each $k \in \{1, 2, ..., n\}$, we take W_3^k as the 3-wheel with spokes set T_k^* and rim Y_k, where

$$T_k^* \cap (E(M)\, i \neq k \cup T_i^*) = \emptyset$$

We define $M^1 = P_{Y_1}(W_3^1, M)$ and for $k > 1$,

$$M^k = P_{Y_k}\left(W_3^k, M^{k-1}\right)$$

By Lemma 6.2.2, T_1^* is a triad of M^1. Since Y_2 is a triangle of M, is also a triangle of M^1. Therefore Y_k is a triangle of M^{k-1}, and so M^k is well defined for every $k \in \{1, 2, ..., n\}$. Lemma 6.2.2 implies that T_k^* is a triad of M^k, for every k.

Lemma 6.1.4 implies that T_k^* is a triangle of $P_{Y_k}^*\left(W_3^k, M^{k-1}\right)/Y_k = M^{k*}/Y_k$, then is a triad of $M^k \backslash Y_k$. We have that

$$M^k \backslash T_k^* = P_{Y_k}\left(W_3^k, M^{k-1}\right) \backslash T_k^* = M^{k-1}$$

therefore $M^k \backslash \bigcup_{i=1}^{k} T_i^* = M$.

We have that $M^1 \backslash Y_1 = P_{Y_1}(W_3^1, M) \backslash Y_1$ is 3-connected iff $P_{Y_1}^*(W_3^1, M)/Y_1$ is 3-connected. Proposition 6.1.5 (ii) implies that $si\left(P_{Y_1}^*(W_3^1, M)/Y_1\right)$ is 3-connected and $P_{Y_1}^*(W_3^1, M)/Y_1$ has no loop. Therefore Lemma 6.2.7 implies that $M^1 \backslash Y_1$ is 3-connected if, and only if, there is no triad of M intersecting Y_1.

Lemma 6.3.5. *Suppose that $M \in \mathcal{T}^n$, then $M^k \backslash \bigcup_{i=1}^{k} Y_i$ is a 3-connected matroid and has $\{T_1^*, \ldots, T_k^*\}$ as the set of triads, for every $k \in \{1, 2, ..., n\}$.*

Demonstração. For every k, we have that $M^k = P_{Y_k}\left(W_3^k, M\right)$ is 3-connected. Lemma 6.2.2 implies that T_k^* is the unique cocircuit of M^k containing in $E\left(W_3^k\right)$. Now, we will to show that $\{T_1^*, T_2^*, ..., T_k^*\}$ are the triads of M^k.

For $k=1$, if $T^* \neq T_1^*$ is a triad of M^1 then we have that T^* intersects T_1^* or $T^* \cap T_1^* = \emptyset$ and both are prohibited by Corollary 6.2.5. Therefore, T_1^* is the unique triad of M^1.

Suppose that the triads of M^{k-1} are $\left\{T_1^*, T_2^*, ..., T_{k-1}^*\right\}$. Take T^* a triad of M^k. Since $M^k = P_{Y_k}\left(W_3^k, M\right)$, Corollary 6.2.5 implies that $T^* = T_k^*$ or $T^* \cap T_k^* = \emptyset$, and hence T^* is a triad of M^{k-1}/Y_k. So, in last case $T^* \in \left\{T_1^*, T_2^*, ..., T_{k-1}^*\right\}$.

Since the triads of M^k are $\{T_1^*, T_2^*, ..., T_k^*\}$, for each k, we have that $M^k \backslash Y_k$ is 3-connected because of Corollary 6.2.8. Moreover, if we denote by $\widehat{Y_1} = Y_1$ and for $k > 1$

$$\widehat{Y_k} = Y_k - \bigcup_{i=1}^{k-1} \widehat{Y_i}$$

then we have that $\overset{k}{\underset{i=1}{\cup}} \widehat{Y_i} = \overset{k}{\underset{i=1}{\cup}} Y_i$ and

$$M^k \backslash \overset{k}{\underset{i=1}{\cup}} \widehat{Y_i} \quad = \quad \left(M^k \backslash \widehat{Y_k}\right) \backslash \overset{k-1}{\underset{i=1}{\cup}} \widehat{Y_i}$$

Since M^k is a 3-connected matroid and Y_k is a triangle surrounding T_k^* on M^k such that Y_k do not intersects any triad of M^k, then Lemma 6.2.14 implies that $M^k \backslash \widehat{Y_k}$ is 3-connected. We have that Y_{k-1} is a triangle surrounding T_{k-1}^* on $M^k \backslash \widehat{Y_k}$. Lemma 6.2.14 implies that $M^k \backslash \widehat{Y_k} \backslash \widehat{Y_{k-1}}$ is 3-connected. Continuing with this precedence, we obtain that $M^k \backslash \overset{k}{\underset{i=1}{\cup}} Y_i$ is 3-connected. □

If $M \in \mathcal{T}^n$ then $M^n \backslash \overset{n}{\underset{i=1}{\cup}} Y_i$ is a 3-connected matroid such that each element belongs to a unique triad.

Lemma 6.3.6. *M^n does not depend on the order of the triangles.*

Demonstração. Take $\sigma : \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., n\}$ a permutation. Denote by

$$Y_i' = Y_{\sigma(i)} = Y_{\sigma_i}$$

then $\{Y_1', Y_2', ..., Y_n'\}$ is another ordination for the triangles. If W_3^k denotes a 3-wheel with spokes set T_k^* and rim Y_k, then M^n has ground set $E(M^n) = E(M) \cup \left(\overset{n}{\underset{i=1}{\cup}} T_i^*\right)$. For each $k \in \{1, ..., n\}$, take $W_3'^k$ a copy of a 3-wheel with spokes set $T_k'^*$ and rim $Y_k' = Y_{\sigma_k}$, such that

$$T_k'^* \cap \left(E(M^n) \cup \left(\bigcup_{i \neq k} T_i'^*\right)\right) = \emptyset$$

We denote by

$$\{ M'^1 = P_{Y_1'}\left(W_3'^1, M\right), M'^k = P_{Y_k'}\left(W_3'^k, M'^{k-1}\right), \text{for each } k > 1.$$

We will to prove that $M'^n = M^n$, unless label of the triads surrounded by triangles. Note that for every k, $T_k'^* \cup T_{\sigma_k}^*$ is a pure emerald of $P_{Y_k'}\left(W_3'^k, M^n\right)$. Then

$$\begin{aligned} M^n &= P_{Y_k'}\left(W_3'^k, M^n\right) \backslash T_k'^* \\ &= P_{Y_k'}\left(W_3'^k, M^n\right) \backslash T_{\sigma_k}^* \end{aligned}$$

and $T^*_{\sigma_k} \subseteq E(M^n) - Y'_k$. Then

$$M^n = P_{Y'_k}\left(W_3'^k,\, M^n \backslash T_k'^*\right)$$

We have that

$$M^n = P_{Y'_2}\left(W_3'^2,\, M^n\right)\backslash T_2'^* = P_{Y'_2}\left(W_3'^2,\, P_{Y'_1}\left(W_3'^1,\, M^n \backslash \left(T^*_{\sigma_1} \cup T^*_{\sigma_2}\right)\right)\right)$$

Going on until

$$\begin{aligned} M^n &= P_{Y'_n}\left(W_3'^n,\, P_{Y'_{n-1}}\left(W_3'^{n-1}, \ldots P_{Y'_1}\left(W_3'^1,\, M^n \backslash \left(\bigcup_{i=1}^{n} T^*_{\sigma_i}\right)\right)\right)\right) \\ &= M'^n \end{aligned}$$

since $M^n \backslash \left(\bigcup_{i=1}^{n} T^*_{\sigma_i}\right) = M$. □

Denoting by τ the set of triangles of a M, we have a function $\sharp : \mathcal{Y}^n \longrightarrow \wp^n$ such that

$$\sharp(M) = M^\sharp = M^n \backslash Y \in \tau \bigcup Y$$

is injective, unless re-labelling the triads.

Definition. Given $M \in \mathcal{Y}^n$, we have that $M^\sharp$ is semi-binary (Corollary 6.2.6) and 123-irreducible. We say that $\sharp(M) = M^\sharp$ is the *diamantic matroid associated* to M.

Theorem 6.3.7. *If M is a rank m 3-connected diamantic matroid with $n \geq 4$ triads, $|E(M)| = 3n$, then $M^\flat$ is a totally triangular 3-connected matroid with rank $m - n$ and n triangles. Conversely, if M is a rank m totally triangular 3-connected matroid with n triangles, $n \geq 4$, then $M^\sharp$ is a 3-connected diamantic matroid with n triads, $\left|E\left(M^\sharp\right)\right| = 3n$, and $r\left(M^\sharp\right) = n + m$. Moreover, $\left(M^\flat\right)^\sharp = M$ and $\left(M^\sharp\right)^\flat = M$.*

Demonstração. Take $M \in \wp^n$. Then

$$\begin{aligned} \left(M^\flat\right)^\sharp &= \left(M^\flat\right)^n \backslash \bigcup_{i=1}^{n} Y_i \\ &= \left(M_n \backslash \bigcup_{i=1}^{n} T_i^*\right)^n \backslash \bigcup_{i=1}^{n} Y_i \end{aligned}$$

Note that

$$\left(M_n \backslash \bigcup_{i=1}^{n} T_i^*\right)^1 = P_{Y_1}\left(W_3^1,\, M_n \backslash \bigcup_{i=1}^{n} T_i^*\right) = M_n \backslash \bigcup_{i=2}^{n} T_i^*$$

and

$$\left(M_n \backslash \bigcup_{i=1}^{n} T_i^*\right)^2 = P_{Y_2}\left(W_3^2,\, M_n \backslash \bigcup_{i=2}^{n} T_i^*\right) = M_n \backslash \bigcup_{i=3}^{n} T_i^*$$

and so on until

$$\left(M_n \backslash \bigcup_{i=1}^{n} T_i^*\right)^n = M_n$$

Therefore

$$\left(M^\flat\right)^\sharp = M_n \backslash \bigcup_{i=1}^{n} Y_i = M$$

Analogously we can check that $\left(M^\sharp\right)^\flat = M$, for $M \in \mathcal{Y}^n$. □

As consequence, if M is a totally triangular matroid then $M^n = \mathbb{B}\left(M^\sharp\right)$ and so:

Definition. If M and N are diamantic and totally triangular matroids associated each others, then $\mathbb{B}(M) = \mathbb{B}(N) = \mathbb{B}(M, N)$ denotes its *barycentric subdivision* of M or N.

Example 6.3.8. The Fano matroid F_7 is a rank 3 totally triangular matroid with 7 triangles such that different triangles intersects in exactly one element. Then $(F_7)^\sharp$ is a 3-connected diamantic matroid with $\left|E\left[(F_7)^\sharp\right]\right| = 21$ and $r\left((F_7)^\sharp\right) = 10$. We have that $\left(F_7^-\right)^\sharp$ is a diamantic matroid, with $\left|E\left[\left(F_7^-\right)^\sharp\right]\right| = 18$ and $r\left(\left(F_7^-\right)^\sharp\right) = 9$.

Example 6.3.9. For $n \geq 4$, the dual of Ladder, L_n^*, and dual of Mbius Ladder, $\mathcal{L}_n^*$, are two non-isomorphic totally triangular matroids. We have that $\sharp(L_n^*)$ and $\sharp(\mathcal{L}_n^*)$ are two non-isomorphic diamantic matroids with $6n$ elements and rank $3n + 1$.

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