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ESDRAS BARBOSA DOS SANTOS

CONSERVED QUANTITIES FOR SPINNING TEST PARTICLES IN GENERAL RELATIVITY

Recife
2020

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Thesis presented to the graduation program of the Physics Department of Universidade Federal de Pernambuco as part of the duties to obtain the degree of Doctor of Philosophy in Physics.

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ABSTRACT

In 1937, Myron Mathisson initiated a research program with the aim of describing extended bodies in General Relativity, similar to what is done with rigid bodies in Classical Mechanics. In his theory, the internal structure of the bodies is described by using the so-called multipolar moments. When considering the spin of the particle beyond its translational motion, the motion of the particle is not necessarily geodesic anymore, being described by the so-called Mathisson-Papapetrou-Dixon (MPD) equations. In the early 1980s, R. Rudiger published two articles dedicated to study the most general conserved charges associated to the motion of particles with spin moving in a curved spacetime and described by the MPD equations. In particular, it was shown that, besides the well-known conserved quantity associated to Killing vectors, there is also another conserved quantity that is linear in the spin of the particle and which involves a Killing-Yano tensor. It was also shown in these papers that in order of this new scalar to be conserved, two obscure conditions involving the Killing-Yano tensor and the curvature tensor must be satisfied. In this thesis, we seek to shed light over these conditions, by trying delimitate the possible spacetimes in which these conditions are satisfied, i.e. in which this new conserved quantity is allowed to exist. In order to extract the content of the Rudiger conditions, some essential mathematical tools are introduced along the text. In this sense, we discuss about the so-called integrability conditions for the Killing-Yano tensors and also we introduce the Petrov classification, which is an important algebraic classification for the Weyl tensor valid in four-dimensional spacetimes. However, before discuss these tools, a revision concerning Mathisson's theory and the MPD equations is made.

Keywords: Extended bodies. General Relativity. Mathisson-Papapetrou-Dixon equations. Conserved charges. Killing-Yano tensor.

RESUMO

Em 1937, Myron Mathisson iniciou um programa de pesquisa com a finalidade de descrever corpos extensos em Relatividade Geral, de uma forma semelhante ao que é feito com corpos rígidos em Mecânica Clássica. Em sua teoria, a estrutura interna dos corpos é descrita através dos chamados momentos multipolares. Quando consideramos o spin da partícula além do seu movimento translacional, o movimento da partícula não é mais necessariamente geodésico, e passa a ser descrito pelas chamadas equações de Mathisson-Papapetrou-Dixon (MPD). No início da década de 1980, R. Rüdiger publicou dois artigos dedicados a estudar as cargas conservadas mais gerais associadas ao movimento de partículas com spin se movendo num espaço-tempo curvo e que são descritas pelas equações MPD. Em particular, foi mostrado que, além da bem conhecida quantidade conservada associada a vetores de Killing que essas partículas admitem, existe ainda uma outra quantidade conservada linear no spin da partícula e que envolve um tensor de Killing-Yano. Foi provado ainda nesses artigos que, para que esse novo escalar seja conservado, duas condições obscuras envolvendo o tensor de Killing-Yano e o tensor de curvatura devem ser satisfeitas. Nessa tese, buscamos esclarecer o conteúdo dessas duas condições, buscando delimitar os possíveis espaços-tempos em que essas condições são satisfeitas, isto é, em que a nova quantidade conservada é permitida existir. Com a finalidade de extrair o conteúdo das condições de Rüdiger, algumas ferramentas matemáticas essenciais são introduzidas ao longo do texto. Deste modo, discutimos sobre as chamadas condições de integrabilidade dos tensores de Killing-Yano e também introduzimos a classificação de Petrov, que é uma importante classificação algébrica para o tensor de Weyl válida em espaços-tempos em quatro dimensões. Contudo, antes de discutirmos essas ferramentas, uma revisão a respeito da teoria do Mathisson e das equações MPD é feita.

Palavras-chaves: Corpos extensos. Relatividade Geral. Equações de Mathisson-Papapetrou-Dixon. Cargas conservadas. Tensor de Killing-Yano.

SUMÁRIO

1 INTRODUCTION

The problem of describing the motion of an extended body is well-known in Newtonian mechanics. There, the description can be understood as the composition of two independent motions, namely, the translation of some representative point of the body and the rotation around this chosen point. So, one needs first to define the representative point of the body, which is usually taken as its center of mass, and then determine the orientation of the body. This can be implemented by introducing two coordinate systems: one fixed, which measures the coordinates of the reference point by means of three parameters, and another which is attained to the body, whose orientation in relation to the fixed system is described by three other independent parameters (Euler angles), such that the extended body is described by six degrees of freedom.

The idea of treating extended bodies within the framework of General Relativity arose since the beginning of Einstein's theory. One approach, developed by Einstein and others, considered bodies with comparable masses with the assumption that their velocities are small in comparison with the velocity of light [1]. Another approach, due to Mathisson and Papapetrou mainly, consisted of treating the problem supposing that one of the bodies possessed a mass much smaller than the other. In this latter approach, the Einstein field equations are used to determine the gravitational field generated by the more massive body, while another equation, namely $\nabla_\beta I^{\alpha\beta} = 0$, with $I^{\alpha\beta}$ being the energy-momentum tensor associated to the body with smallest mass, is used to determine its motion. In other words, the smallest body is considered as a test body. In this thesis, we will focus on the latter approach.

The motivation for studying this area is straightforward. Indeed, since the bodies found in the universe perform not only translational motion, but also a rotational one, the consideration of the internal motion of these bodies (such as the spin) in the general theory of relativity, certainly will modify some equations of the theory. In fact, as we will see later, when spin is considered, the motion of the body is not necessarily geodesic anymore. As usually is made, the bodies are modelled as point particles, so that considerations of their internal structure are neglected. Since general relativity is the best theory of gravitation we have up to now, it could be interesting include spin in order to make the equations of the theory more precise. This precision is required if we are interested in the correct description of some systems in the universe, such as binary systems, for example, or in the study of gravitational waves generated by these systems.

This work is structured as follows. A review of the theory which treats extended bodies in GR is made in chapters 2 and 3. In particular, chapter 2 is devoted to derive the equations of motion for a spinning test particle, the so-called Mathisson-Papapetrou-Dixon (MPD) equations, while in chapter 3 a discussion is made about the so-called spin supplementary conditions, which are conditions used to supplement the MPD equations, since, as we will see along this thesis, these equations are not sufficient to determine all necessary degrees of freedom. In chapter 4, we talk about Killing and Killing-Yano tensors, which are special tensors that generate the so-called hidden symmetries, and some important mathematical tools such as Petrov classification are introduced. Finally, in chapter 5, we solve the central problem of this thesis, concerning conserved quantities for the spinning test particle.

Specifically, we are interested in analysing the conserved quantity found by Rüdiger in Ref. [25], and how it can be useful in the problem of integrability of MPD equations. It is well-known how conserved quantities are useful in the integrability of the equations of motion. In fact, for a particle travelling along a geodesic path, the geodesic equation is not directly integrated. Instead, we make use of the conserved quantities associated with the Killing vectors that the spacetime possesses. In particular, in Kerr spacetime, the complete integrability of geodesic equation is achieved only if, besides the conserved quantities associated with Killing vectors, we use the conserved quantity associated with a Killing tensor of rank two admitted by the spacetime (actually, this Killing tensor comes from a more fundamental object called Killing-Yano tensor). In Rüdiger's work, it is found for the spinning particle a conserved quantity related with a Killing-Yano tensor. The main aim of this thesis is to study this conserved quantity, and to seek to answer the question if this conserved quantity could be or not useful for the integrability of MPD equations in some spacetimes.

2 MULTIPOLAR MOMENTS: TREATING EXTENDED BODIES IN GR

It is an implicit assumption of the matter coupling to Einstein's General Relativity (GR) that bodies moving in spacetime are treated as particles, i.e., objects without internal structure, described only by translational degrees of freedom. With this assumption, we can establish many known results of the theory, as for example the fact that free particles (particles subjected only to the gravitational force) move along geodesics. However, if we decide to relax this assumption and try to describe extended bodies, many interesting questions arise.

In particular, how can we treat extended bodies in GR? How does this change modify the results of the theory? Essentially, there are two ways to approach this problem. The first, due to Einstein, Infeld and others, treats the problem considering two bodies of comparable masses. In this approach, both bodies generate relevant gravitational fields, and the motion of each body should be considered with respect to the gravitational field generated by the other body. The second, due to Mathisson, Papapetrou, Dixon and others, describes the motion of bodies (called test particles) moving in an external gravitational field. In this approach, these bodies are supposed to have very small masses such that their gravitational fields can be neglected in comparison to the external gravitational field. In this thesis we will handle the problem of extended bodies from the point of view of the second approach.

In this chapter, we start introducing some fundamental concepts of GR which will be necessary along the thesis. Then, we describe Mathisson's formalism of multipolar moments which describes bodies of finite size in GR. In what follows we always work in a four-dimensional spacetime of signature $s = -2$ (Lorentzian manifold). Symmetrization and antisymmetrization of indices are denoted by round and square brackets around these indices, respectively. In particular, for some tensor $T_{\mu\nu}$ we have $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ and $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\mu\nu})$. In tha same way, $T_{[\alpha\beta\gamma]} = \frac{1}{3!}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha} - T_{\beta\alpha\gamma} - T_{\alpha\gamma\beta} - T_{\gamma\beta\alpha})$, and so on. Besides, $\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha{}_{\mu\beta} V^\beta$ denotes the covariant derivative of a vector field V^α, being $\Gamma^\alpha{}_{\mu\beta}$ the Christoffel symbol.

2.1 SOME BASIC CONCEPTS AND USEFUL RELATIONS IN GR

Let us start by defining the spacetime as a continuum formed by events, with an event being a point in space at a determined instant of time. It is assumed that for each event, we can always assign a set of four numbers, i.e. there is always a one-to-one mapping relating the spacetime with the set $\mathbb{R}^4$. In the language of differential geometry, it is said that the spacetime is a four-dimensional differential manifold.

In Einstein's gravitation theory, gravity is seen as curvature. The presence of matter and energy deforms the spacetime and gives its geometry. In the absence of curvature, we recover the flat spacetime of Special Relativity (Minkowski spacetime). In the flat spacetime, two covariant derivatives commute, i.e.,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\alpha = 0, \tag{2.1}$$

for any vector V^α. In the context of GR, however, it follows from differential geometry that the right-hand side of (2.1) does not vanish. Rather, in this case we have

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta, \tag{2.2}$$

which is known as *Ricci identity.* Of course, the formula (2.2) can be easily generalized for the application of the operator $2\nabla_{[\mu}\nabla_{\nu]}$ on a tensor of any type, in which case we only gain additional terms on the right-hand side of (2.2). The tensor $\boldsymbol{R}$ which appears in (2.2), is called Riemann tensor or curvature tensor, and its components with respect to a coordinate basis $\{\boldsymbol{\partial_\mu}\}$ are given by

$$R^\alpha{}_{\beta\gamma\delta} = 2\partial_{[\gamma|}\Gamma^\alpha{}_{\beta|\delta]} + 2\Gamma^\alpha{}_{\epsilon[\gamma|}\Gamma^\epsilon{}_{\beta|\delta]}, \tag{2.3}$$

where the notation above means antisymmetrization in the pair $\gamma\delta$ only, and $\Gamma^\alpha{}_{\beta\gamma}$ are the components of the Levi-Civita connection (Christoffel symbols). As its own name suggests, the curvature tensor describes the cuvature of the metric representing the spacetime (technically the curvature of the connection). In particular, in the flat spacetime, $\boldsymbol{R}$ vanishes at all points of the spacetime.

The curvature tensor possesses some symmetries that reduces significantly the number of its independent components. In particular, it satisfies the following relations:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}. \tag{2.4}$$

In addition, $\boldsymbol{R}$ also satisfies the identities [2]:

$$R_{\alpha[\beta\gamma\delta]} = 0\ , \tag{2.5}$$

$$\nabla_{[\mu} R_{\alpha\beta]\gamma\delta} = 0\ , \tag{2.6}$$

which are known as first and second Bianchi identities, respectively. A significant reduction is achieved in the degrees of freedom of the curvature if we consider the above symmetries. In fact, in a four-dimensional spacetime, the $4^4 = 256$ independent components of a general tensor with four indices reduces to only 20 if the tensor has the symmetries given in Eqs. (2.4) and (2.5).

An important fact that comes from the second Bianchi identity is that the Einstein tensor has null divergence. In fact, contracting $\nabla_{[\mu} R_{\alpha\beta]\gamma\delta} = 0$ twice with the metric tensor, we obtain

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) = 0 \Rightarrow \nabla_\mu G^{\mu\nu} = 0, \tag{2.7}$$

where we define the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, with $R_{\mu\nu}$ standing for the Ricci tensor ($R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$), R for the Ricci scalar and $g_{\mu\nu}$ for the metric tensor. Also, the curvature tensor can be decomposed into its irreducible blocks with respect to the action of the Lorentz group, which are given by the Weyl tensor $C_{\alpha\beta\gamma\delta}$, the traceless part of the Riemann tensor ($C^\alpha{}_{\beta\alpha\gamma} = 0$), the Ricci tensor and the Ricci scalar as follows:

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]\alpha} - \frac{1}{3} R g_{\alpha[\gamma} g_{\delta]\beta}, \tag{2.8}$$

where we define the Weyl tensor $C_{\alpha\beta\gamma\delta}$. Equation (2.8) means that from the 20 independent components of the Riemann tensor, 10 are encoded in $R_{\mu\nu}$ and 10 in the Weyl tensor. In addition, we define the traceless part $\Phi_{\mu\nu}$ of the Ricci tensor as $\Phi_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}$ satisfying the relation $g^{\alpha\beta}\Phi_{\alpha\beta} = 0$. In terms of the tensor \Φ, the above decomposition writes

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma} \Phi_{\delta]\beta} - g_{\beta[\gamma} \Phi_{\delta]\alpha} + \frac{1}{6} R g_{\alpha[\gamma} g_{\delta]\beta}, \tag{2.9}$$

a relation which will be useful in the fourth chapter.

A totally antisymmetric tensor $\boldsymbol{\omega}$ with p indices, i.e., $\omega_{a_1...a_p} = \omega_{[a_1...a_p]}$, is called a differential form of rank p, or simply a p-form. One can define some useful operations on differential forms, as its exterior derivative and the so-called Hodge duality.

The exterior derivative $\boldsymbol{d}$ of a p-form $\boldsymbol{\omega}$ is by definition a $p+1$-form defined by the relation

$$(\boldsymbol{d\omega})_{\mu a_1...a_p} = (p+1)\partial_{[\mu}\omega_{a_1...a_p]}. \tag{2.10}$$

In particular, the exterior derivative of a 0-form is the gradient $\partial_\mu \omega$. We can also define the exterior product $\wedge$ between two forms by

$$(\omega_1 \wedge \omega_2)_{\mu_1...\mu_{p+q}} = \frac{(p+q)!}{p!q!}(\omega_1)_{[\mu_1...\mu_p}(\omega_2)_{\mu_{p+1}...\mu_{p+q}]}, \tag{2.11}$$

with $\boldsymbol{\omega}_1$ being a p-form and $\boldsymbol{\omega}_2$ being a q-form.

In addition, in a four-dimensional manifold, a p-form $\boldsymbol{\omega}$ is always related to a $(4-p)$-form by the star operation:

$$\omega^{\star}_{a_1...a_{4-p}} = \frac{1}{p!}\epsilon^{b_1...b_p}{}_{a_1...a_{4-p}}\omega_{b_1...b_p}, \tag{2.12}$$

which is called *Hodge duality*, being $\boldsymbol{\epsilon}$ the Levi-Civita tensor.

Besides, throughout this thesis, the following formula giving the contraction between two Levi-Civita tensors will also be useful:

$$\epsilon^{a_1...a_p b_{p+1}...b_4}\epsilon_{a_1...a_p c_{p+1}...c_4} = -p!(4-p)!\delta^{[b_{p+1}}_{c_{p+1}}...\delta^{b_4]}_{c_4}. \tag{2.13}$$

2.2 DESCRIPTION OF THE METHOD

In this section, we will derive the set of equations that describe the motion of a spinning test particle in GR. The deduction follows the seminal paper *A New Mechanics of Material Systems* (translated title) of 1937 by Myron Mathisson [3]. In this article, the author proposes a method by which one can treat bodies of finite size in GR by the use of the so called *multipolar moments*. The idea to treat extended bodies by means of multipolar moments was first proposed by Mathisson to derive the equations of motion of a spinning particle. However, we point out that the same set of equations derived in [3] was also derived afterwards in different ways by other authors, including a Lagrangian formulation [4, 5, 7-9].

The central idea of Mathisson's paper was to introduce the concept of *gravitational skeleton* of a body, which consists in substituting the extended body by an infinite set of multipolar moments. Then, we obtain an integral equation from which all the dynamics of the extended body is extracted. In other words, the integral equation should determine the form and dynamics of the multipolar moments.

The equations of motion of a spinning test particle, here referred as Mathisson-Papapetrou-Dixon (MPD) equations, read

$$\begin{aligned} \dot{p}^{\alpha} &= -\frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}v^{\beta}S^{\gamma\delta}\,, \\ \dot{S}^{\alpha\beta} &= 2\,p^{[\alpha}v^{\beta]}\,, \end{aligned} \tag{2.14}$$

where p^{α}, v^{α} and $S^{\alpha\beta}$ are the momentum, velocity and spin of the particle, respectively, and dot means covariant differentiation in velocity direction, $S^{\alpha\beta}$ being antisymmetric ($S^{\alpha\beta} = -S^{\beta\alpha}$). In particular, $\dot{p}^{\alpha} = v^{\beta}\nabla_{\beta}p^{\alpha}$. We note that in this theory there are thirteen degrees of freedom, four from p^{α}, six from $S^{\alpha\beta}$ and three from v^{α}, once the velocity is supposed to satisfy the normalization condition $v^{\alpha}v_{\alpha} = 1$. However, the system of equations (2.14) has only ten equations, which tells us that such equations cannot fix all the degrees of freedom. So, the system needs to be supplemented by other three equations. This point will be discussed in the next chapter. In the present chapter, we focus on the derivation of the equations (2.14).

The MPD equations were derived by a variety of different methods. In [3], they are obtained by the use of multipole moments supposing that the particle moves in a background which is solution of the vacuum Einstein field equations. In [4], a non-covariant deduction is made starting from the conservation equation $\nabla_{\beta}I^{\alpha\beta} = 0$, with $I^{\mu\nu}$ being an energy-momentum tensor associated to the particle. In [5], the author simplifies the deduction made in [3] introducing the use of the Dirac delta function in this theory. A covariant formalism is proposed in [7] in which the covariance is maintained along the entire derivation. In addition, in [8] the equations are deduced through a canonical dynamic approach by making use of Poisson brackets in a phase space, whereas in [9] they are derived from a Lagrangian formulation. In the present thesis, we will derive the set of equations (2.14) following Mathisson's method, although without requiring that the background in which the particle moves needs to be an Einstein spacetime (spaces with Ricci tensor proportional to the metric, solutions of Einstein's equations in vacuum with a possible cosmological constant). At the end of this chapter, we also illustrate how one can obtain the MPD equations from a Lagrangian approach.

Our starting point is supposing that the dynamics of the gravitational field is described by the Einstein-Hilbert action, that is,

$$S_{EH} = \int \sqrt{-g} R d^4 x. \tag{2.15}$$

If we sum the above action with the action of the matter fields, then the application of the principle of least action leads to the Einstein field equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.16}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter fields and $\kappa = 8\pi G$, with G representing the Newtonian constant of gravitation. The assumption (2.15) is what properly defines GR, which is a theory that describes gravity at low energy levels (classical level). We note however that if we are interested in high energy effects, the action (2.15) is not sufficient anymore, and the use of other theories can be necessary, as is the case for example of the modified theory of gravity $f(R)$ [11], where the Ricci scalar R in action (2.15) is substituted by a function $f(R)$ of the Ricci scalar, and string theory [12]. Nevertheless, we note that these theories always should recover GR in low energy limits.

As we saw above, the Einstein tensor has zero divergence. From this fact it follows from (2.16) that $T_{\mu\nu}$ should also obey

$$\nabla_\beta T^{\alpha\beta} = 0, \tag{2.17}$$

i.e., $\boldsymbol{T}$ satisfies a conservation equation.

In what follows, the particle is supposed to move in an external gravitational field $g_{\alpha\beta}$ (which is called *background*) which is solution of the Einstein equation (2.16). The gravitational field generated by the moving particle is supposed to be very small in comparison with the field of the background. In other words, the energy-momentum tensor associated to the particle does not enter in the right-hand side of (2.16) for the computation of the metric of the background. A body with this feature is what we call a *test particle*.

For the description of the extended body, we will proceed as follows. We assign for the extended body an energy-momentum tensor $I^{\mu\nu}$. Along its motion, the body describes a world tube in the spacetime, and we suppose that the tensor $I^{\mu\nu}$ vanishes outside this world tube. Within this world tube we select a line L of reference which we choose as the representative curve of the body's trajectory. The coordinates of the reference line are denoted by x_o^α and are functions of the proper time. With that scenario in mind, let us consider the integral in the spacetime

$$\int_V I^{\alpha\beta}\xi_{\alpha\beta}\sqrt{-g}\,d^4x, \tag{2.18}$$

for arbitrary symmetric tensor fields $\xi_{\alpha\beta}$ that vanish at infinity. Now, in any point P of the reference line L, we consider the Taylor expansion for the tensor fields $\boldsymbol{\xi}$

$$\xi_{\alpha\beta} = \xi_{\alpha\beta}|_o + \delta x^\gamma(\partial_\gamma\xi_{\alpha\beta})|_o + \frac{1}{2!}\delta x^\delta\delta x^\gamma(\partial_\delta\partial_\gamma\xi_{\alpha\beta})|_o + \ldots\,, \tag{2.19}$$

where we are using the compact notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\delta x^\alpha = x^\alpha - x_o^\alpha$. Substituting the expansion in the integral, we have

$$\int_L d\tau\,\Big(\xi_{\alpha\beta}\int d^3x\sqrt{-g}\,I^{\alpha\beta} + \partial_\gamma\xi_{\alpha\beta}\int d^3x\sqrt{-g}\,\delta x^\gamma I^{\alpha\beta} \\ + \frac{1}{2!}\partial_\delta\partial_\gamma\xi_{\alpha\beta}\int d^3x\sqrt{-g}\,\delta x^\delta\delta x^\gamma I^{\alpha\beta} + \ldots\Big), \tag{2.20}$$

which can be put into the form of the line integral

$$\int_L \Big(M^{\alpha\beta}\xi_{\alpha\beta} + M^{\gamma\alpha\beta}\partial_\gamma\xi_{\alpha\beta} + \frac{1}{2!}M^{\gamma\delta\alpha\beta}\partial_\gamma\partial_\delta\xi_{\alpha\beta} + \ldots\Big)\,d\tau, \tag{2.21}$$

with the identifications:

$$M^{\alpha\beta} = \int d^3x\sqrt{-g}\,I^{\alpha\beta},\; M^{\gamma\alpha\beta} = \int d^3x\sqrt{-g}\,\delta x^\gamma I^{\alpha\beta},\,\ldots \tag{2.22}$$

and so on. Now, we supplement in (2.21) the ordinary derivatives of $\xi_{\alpha\beta}$ in order to transform them into covariant derivatives. At the end, we arrive at an expression involving the tensor field $\xi_{\alpha\beta}$ and its covariant derivatives $\nabla_\gamma\xi_{\alpha\beta}$, $\nabla_\gamma\nabla_\delta\xi_{\alpha\beta}$ and so on, with each of these terms contracted with expressions involving the M's defined in (2.22), the components of the connection $\Gamma^\alpha_{\beta\gamma}$ and their partial derivatives. Identifying these expressions with quantities m's, we eventually arrive at

$$\int_L \Big(m^{\alpha\beta}\xi_{\alpha\beta} + m^{\gamma\alpha\beta}\nabla_\gamma\xi_{\alpha\beta} + \frac{1}{2!}m^{\gamma\delta\alpha\beta}\nabla_\gamma\nabla_\delta\xi_{\alpha\beta} + \ldots\Big)\,d\tau\,. \tag{2.23}$$

The tensors $m^{\alpha\beta}$, $m^{\gamma\alpha\beta}$, $m^{\gamma\delta\alpha\beta}$ etc... which are symmetric in α and β and in the remaining set of indices, are called *multipolar moments* of the extended body, while the particles described by these moments are called *multipolar particles.* These infinite set of moments is what Mathisson called the *gravitational skeleton* of the body. All the description of the body should be contained in these moments, and the description of the body motion reduces to obtaining the form and evolution of such moments. So, we have the relation

$$\int_V I^{\alpha\beta}\xi_{\alpha\beta}\sqrt{-g}\,d^4x = \int_L \left(m^{\alpha\beta}\xi_{\alpha\beta} + m^{\gamma\alpha\beta}\nabla_\gamma\xi_{\alpha\beta} + \frac{1}{2!}m^{\gamma\delta\alpha\beta}\nabla_\gamma\nabla_\delta\xi_{\alpha\beta} + ...\right) d\tau. \tag{2.24}$$

At this point, we postulate that the tensor $I^{\alpha\beta}$ obeys the conservation equation

$$\nabla_\beta I^{\alpha\beta} = 0, \tag{2.25}$$

and we suppose that the tensor field $\boldsymbol{\xi}$ is the derivative of an arbitrary vector field $\boldsymbol{\chi}$, that is,

$$\xi_{\alpha\beta} = \nabla_{(\alpha}\chi_{\beta)}. \tag{2.26}$$

Using (2.25) and (2.26), we can transform by Gauss theorem the integral of the left-hand side of (2.24) into a surface integral over the boundary ∂V of the volume V, which vanishes, so that we finally arrive at the equation:

$$\int_L \left(m^{\alpha\beta}\nabla_\alpha\chi_\beta + m^{\gamma\alpha\beta}\nabla_\gamma\nabla_\alpha\chi_\beta + \frac{1}{2!}m^{\gamma\delta\alpha\beta}\nabla_\gamma\nabla_\delta\nabla_\alpha\chi_\beta + ...\right) d\tau = 0. \tag{2.27}$$

Equation (2.27) is what Mathisson called the *variational equation of mechanics*, which is the central equation of Mathisson's work. All the dynamics of the multipolar moments should follow from that equation. Therefore, all the work from now on consists in extracting all the consequences of the variational equation (2.27).

2.2.1 The pole particle

We are using the infinite set of multipolar moments to describe an extended body. However, we observe that we do not need all the moments for the description of the body in practice. In fact, since we are working with test particles, some higher multipole moments can be neglected. As an initial example, one supposes that the particle is small enough in a way that it is sufficiently described by the first moment $m^{\alpha\beta}$ only. That is possible because from the integral expressions (2.22) of the multipolar moments, we note that higher order moments depend on higher powers of δx^{α}, in such a way that they can be neglected if the body is taken as a point particle. Later we will treat the more general case of the pole-dipole particle.

Let us illustrate in this section how we can determine the content of equation (2.27), i.e, how we can extract the form and evolution of multipolar moments. In this case we have the integral equation

$$\int_L m^{\alpha\beta}\nabla_\alpha\chi_\beta\, d\tau = 0. \tag{2.28}$$

At this point, we observe that the velocity v^{μ} of the test particle is a well defined quantity at each point of the reference world line L and as this line is parametrized by the proper time we should have the normalization condition $v^{\mu}v_{\mu} = 1$. Then, at each point of the world line L we split the degrees of freedom of the moment $m^{\alpha\beta}$ into degrees parallel and orthogonal to the velocity v^{μ}. So, we have the following decomposition into the three independent blocks [3]:

$$m^{\alpha\beta} = m\, v^{\alpha}v^{\beta} + 2\, m^{(\alpha}v^{\beta)} + \hat{m}^{\alpha\beta}, \tag{2.29}$$

where m is some scalar which depends on the proper time ($m = m(\tau)$) while m^{α} and $\hat{m}^{\alpha\beta}$ are tensors orthogonal to v^{α} ($m^{\alpha}v_{\alpha} = 0$ and $\hat{m}^{\alpha\beta}v_{\beta} = 0$), with $\hat{m}^{\alpha\beta}$ being symmetric ($\hat{m}^{\alpha\beta} = \hat{m}^{(\alpha\beta)}$). In fact, it is easy to see that all the degrees of freedom of the symmetric tensor $m^{\alpha\beta}$ are contained into the three blocks of the righ-hand side of (2.29). In particular, for the case we are interested (four dimensional spacetimes), $m^{\alpha\beta}$ contains ten degrees of freedom, while the first, second and third term of the right-hand side of (2.29) contain one, three and six degrees of freedom, respectively.

In the same way, the tensor $\nabla_\alpha\chi_\beta$ can be decomposed as follows

$$\nabla_\alpha\chi_\beta = \chi v_\alpha v_\beta + 2\,\tilde{\chi}_{(\alpha}v_{\beta)} + \hat{\chi}_{\alpha\beta}, \tag{2.30}$$

with $\tilde{\chi}_\alpha$ and $\hat{\chi}_{\alpha\beta}$ being orthogonal to velocity and $\hat{\boldsymbol{\chi}}$ being a symmetric tensor. In the decomposition (2.30), χ, $\tilde{\chi}_\alpha$ and $\hat{\chi}_{\alpha\beta}$ are completely arbitrary components, once the vector field itself χ_α (and so its derivative also) is arbitrary. Substituting the decomposition (2.29) into the integral equation (2.28), we obtain

$$\int_L m\, v^{\alpha}v^{\beta}\,\nabla_\alpha\chi_\beta\, d\tau + \int_L 2\, m^{(\alpha}v^{\beta)}\nabla_\alpha\chi_\beta\, d\tau + \int_L \hat{m}^{\alpha\beta}\nabla_\alpha\chi_\beta\, d\tau = 0. \tag{2.31}$$

This latter expression involves the integration of directional derivatives of $\boldsymbol{\chi}$ which are independent of each other, once it involves derivatives parallel and orthogonal to velocity. The independence of the three integrals above can be seen more clearly if we align the velocity v^α with the time direction of some Lorentz frame. In this frame, the first term involves only the time derivative of the time component of $\boldsymbol{\chi}$ ($\nabla_0\chi_0$), while the second involves derivatives like $\nabla_0\chi_i$ and $\nabla_i\chi_0$, and the third involves only spatial derivatives of spatial components of $\boldsymbol{\chi}$ ($\nabla_i\chi_j$). Then, it implies that each line integral of (2.31) should vanish separately. In particular, we should have

$$\int_L m^{(\alpha}v^{\beta)}\nabla_\alpha\chi_\beta\, d\tau = 0, \quad \text{and} \quad \int_L \hat{m}^{\alpha\beta}\nabla_\alpha\chi_\beta\, d\tau = 0. \tag{2.32}$$

Again, once $\nabla_\alpha\chi_\beta$ is arbitrary, the only way to guarantee that the above integrals always vanish is that all the coefficients of $\boldsymbol{\nabla\chi}$ vanish, that is,

$$m^{(\alpha}v^{\beta)} = 0, \quad \text{and} \quad \hat{m}^{\alpha\beta} = 0. \tag{2.33}$$

Contracting the first equation of (2.33) with v_β, we get $m^\alpha = 0$. Substituting these results into (2.31), we get

$$\int_L mv^\alpha v^\beta\nabla_\alpha\chi_\beta d\tau = \int_L \frac{d}{d\tau}(m\, v^\beta\chi_\beta)d\tau - \int_L v^\alpha\nabla_\alpha(m\, v^\beta)\chi_\beta d\tau = 0. \tag{2.34}$$

The first term vanishes because the field χ_μ is supposed be zero at infinity, and the second term implies finally

$$v^\beta\nabla_\beta(mv^\alpha) = 0, \tag{2.35}$$

Equation (2.35) is a constraint not only on the evolution of the pole particle but also on the evolution of the scalar m. In fact, contracting it with v_α, we get

$$\frac{dm}{d\tau} + m\, v_\alpha v^\beta\nabla_\beta v^\alpha = 0. \tag{2.36}$$

The second term clearly vanishes since $v^\alpha v_\alpha = 1$. Therefore, we conclude

$$\frac{dm}{d\tau} = 0, \tag{2.37}$$

which substituting into (2.35) produces the geodesic equation as we know

$$v^\beta\nabla_\beta v^\alpha = 0. \tag{2.38}$$

Therefore, the pole particle case tells us that the test particle should describe a geodesic equation with a parameter m being conserved along the motion of the particle. That was expected, since if we consider only the first moment we are in the case of point particle, so that the basic assumptions of GR should be recovered.

2.2.2 The pole-dipole particle

In the pole case, we saw that the geodesic equation is predicted by the method. Let us now analyse the more interesting case of the so called *pole-dipole approximation*, when the first two moments $m^{\alpha\beta}$ and $m^{\gamma\alpha\beta}$ are considered. Retaining these two multipolar moments, the variational equation reads

$$\int_L (m^{\alpha\beta}\nabla_\alpha\chi_\beta + m^{\gamma\alpha\beta}\nabla_\gamma\nabla_\alpha\chi_\beta)d\tau = 0. \tag{2.39}$$

In this case, in addition to decomposition (2.29), we need also to split the moment $m^{\gamma\alpha\beta}$ into components parallel and orthogonal to v^α. We start by observing that in this case the integral equation (2.39) has the property of remaining unchanged by the transformations

$$m^{\alpha\beta} \to m^{\alpha\beta} + \dot{n}^{\alpha\beta} \quad \text{and} \quad m^{\gamma\alpha\beta} \to m^{\gamma\alpha\beta} + v^\gamma n^{\alpha\beta}, \tag{2.40}$$

for any symmetric tensor $n^{\alpha\beta}$. It means that the variational equation (2.39) for the pole-dipole particle is not able to determine the multipole moments uniquely. For this reason, Mathisson imposes on the moment $m^{\gamma\alpha\beta}$ the supplementary condition

$$v_\gamma m^{\gamma\alpha\beta} = 0, \tag{2.41}$$

a choice that eliminates the arbitrariness in the determination of the multipolar moments. Then, considering this condition we have the following decomposition:

$$m^{\gamma\alpha\beta} = n^\gamma v^\alpha v^\beta + 2S^{(\alpha|\gamma}v^{|\beta)} + \tilde{m}^{\gamma\alpha\beta}, \tag{2.42}$$

where $n^\alpha v_\alpha = 0$, and $S^{\alpha\beta}$ and $\tilde{m}^{\gamma\alpha\beta}$ are orthogonal to v^α in all indices, with $\tilde{m}^{\gamma\alpha\beta}$ being symmetric in α and β. It is also easy to see that all the degrees of freedom present in the left-hand side of (2.42) are preserved in the right-hand side of the above decomposition. In fact, since one has $v_\gamma m^{\gamma\alpha\beta} = 0$, it follows that the multipolar moment $m^{\gamma\alpha\beta}$ has thirty degrees of freedom, while the first, second and third term of the right-hand side have three, nine and eighteen degrees of freedom, respectively.

In principle, terms involving the moments $m^{\alpha\beta}$ and $m^{\gamma\alpha\beta}$ can be analysed separately, since they are related to first and second derivatives of the vector field $\boldsymbol{\chi}$, which seems independent. However, we note that terms of second derivatives containing velocity vectors can be transformed into lower order derivative terms, such that these terms are not independent of the terms involving first derivatives of $\boldsymbol{\chi}$ in (2.39). Taking only the second derivative terms which are independent of the first derivative ones, we have

$$\int_L S^{\alpha\gamma} v^\beta \partial_\gamma \partial_\alpha \chi_\beta \, d\tau + \int_L \tilde{m}^{\gamma\alpha\beta} \partial_\gamma \partial_\alpha \chi_\beta \, d\tau = 0. \tag{2.43}$$

As the second derivatives of $\boldsymbol{\chi}$ are arbitrary, it implies that the symmetric part (in the indices α and γ) of its coefficients should vanish:

$$S^{(\alpha\gamma)} v^\beta = 0 \quad \text{and} \quad \tilde{m}^{(\gamma\alpha)\beta} = 0. \tag{2.44}$$

Contracting with v_β the first equation of (2.44) implies that the symmetric part of $S^{\alpha\beta}$ is zero, which means that $S^{\alpha\beta}$ is antisymmetric ($S^{\alpha\beta} = S^{[\alpha\beta]}$). The second equation of (2.44) establishes $\tilde{m}^{(\gamma\alpha)\beta} = 0$, which means that $\tilde{m}^{\gamma\alpha\beta}$ is antisymmetric in the first pair of indices. It is simple to prove by manipulation of indices that any tensor of three indices which is symmetric in the last pair and antisymmetric in the first pair is zero. So, we conclude that $\tilde{m}^{\gamma\alpha\beta} = 0$. Then, using the Ricci identity (2.2) and the symmetry properties of the Riemann tensor, we can obtain the relation:

$$\int_L (S^{\alpha\gamma} v^\beta + S^{\beta\gamma} v^\alpha) \nabla_\gamma \nabla_\alpha \chi_\beta \, d\tau = \frac{1}{2} \int_L \chi^\delta R_{\delta\beta\gamma\alpha} (S^{\gamma\alpha} v^\beta + 2 S^{\gamma\beta} v^\alpha) \, d\tau - \int_L \dot{S}^{\beta\gamma} \nabla_\gamma \chi_\beta \, d\tau \tag{2.45}$$

Now, we note that the last integral of (2.45) has the same structure of the terms involving the moment $m^{\alpha\beta}$ in the original expression (2.39) (all of them containing first derivatives of χ). Then, decomposing the antisymmetric tensor $\dot{S}^{\beta\gamma}$ as

$$\dot{S}^{\beta\gamma} = \tilde{S}^{\beta\gamma} + 2\, L^{[\beta} v^{\gamma]}, \tag{2.46}$$

with $\tilde{S}^{\alpha\beta}v_\beta = 0$ and $L^\alpha v_\alpha = 0$, and collecting all these terms together, we obtain the expression:

$$\int_L \tilde{S}^{\alpha\beta}\nabla_\alpha \chi_\beta \, d\tau + \int_L (L^\alpha + m^\alpha) v^\beta \nabla_\alpha \chi_\beta \, d\tau + \int_L \hat{m}^{\alpha\beta}\nabla_\alpha \chi_\beta \, d\tau = 0. \tag{2.47}$$

Again, this implies the vanishing of each of the coefficients of $\nabla\chi$. Then we have $\tilde{S}^{\alpha\beta} = 0$, $\hat{m}^{\alpha\beta} = 0$ and $(L^\alpha + m^\alpha)v^\beta = 0$, whose contraction with v_β leads to the result $L^\alpha + m^\alpha = 0$. Therefore, by contracting (2.46) with v_γ we conclude that $L^\alpha = v_\gamma \dot{S}^{\alpha\gamma}$, which allows us finally express (2.46) as

$$\dot{S}^{\alpha\beta} + v^\alpha v_\gamma \dot{S}^{\beta\gamma} - v^\beta v_\gamma \dot{S}^{\alpha\gamma} = 0. \tag{2.48}$$

The remaining terms of (2.39) read

$$\int_L \left[v^\nu \nabla_\nu (m\, v^\alpha) - \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} (S^{\gamma\delta} v^\beta + 2\, S^{\beta\delta} v^\gamma) - 2\dot{L}^\alpha \right] \chi_\alpha \, d\tau = 0, \tag{2.49}$$

which implies that the expression in square brackets is zero. Once that $L^\alpha = v_\gamma \dot{S}^{\alpha\gamma}$, we obtain

$$v^\lambda \nabla_\lambda (m\, v^\alpha + v_\beta \dot{S}^{\alpha\beta}) = -\frac{1}{2} R^\alpha{}_{\beta\gamma\delta}\, v^\beta S^{\gamma\delta}. \tag{2.50}$$

If we identify

$$p^\alpha = m\, v^\alpha + v_\beta\, \dot{S}^{\alpha\beta} \tag{2.51}$$

as the momentum of the spinning particle, the equations (2.50) and (2.48) finally become the MPD equations as presented in (2.14). This set of equations are the basic equations that govern the motion of a test particle with spin in GR.

A study of the pole-dipole particle, i.e., the particle which obeys the system of equations (2.14) will be the goal of the present work. However, of course we can proceed considering terms of higher order in the expansion. The general case can be found at [10]. For our purpose, it is sufficient to stop in the second multipole moment.

2.3 A LAGRANGIAN FORMULATION

Other approaches have also been proposed in literature in order to treat the spinning test particle by other means that do not use multipolar moments, as is the case of the lagrangian approach developed in [9, 13]. In [9], a lagrangian formulation for the spinning particle is made in the flat spacetime, and in [13] the treatment is extended for curved spacetimes with nonvanishing torsion. Let us illustrate the method for curved spacetimes endowed with a connection with vanishing torsion.

In the lagrangian method, the particle's equations of motion should arise from the variation of an action describing the particle with respect to all degrees of freedom of the system. So, one needs to say what are the degrees of freedom that will describe the particle. For a spinning particle, we need of course translational and rotational degrees of freedom. The momentum equation must emerge from the variation of the action with respect to the translational degrees of freedom, while the spin equation must emerge from the variation with respect to the rotational ones. For the description of the translational motion of the particle, we take the set of four coordinates $x^\mu(\tau)$ functions of the proper time, by means of which we can define the velocity of the test particle:

$$v^\mu = \frac{dx^\mu}{d\tau} = \dot{x}^\mu, \tag{2.52}$$

and for the description of the rotational motion, we take the tensor [9]:

$$\sigma^{\mu\nu} = \eta^{ab} e_a^\mu \, \dot{e}_b^\nu, \tag{2.53}$$

which is identified with the angular velocity of the particle. By the term angular velocity, we mean only a tensor which has the right properties to be the analog of the usual angular velocity ω^{ij} of a rotating frame, with the velocity of a point relative to the origin given by

$$dx^i/dt = -\omega^{ij} x_j. \tag{2.54}$$

In fact, it is natural to identify rotational degrees of freedom with antisymmetric tensors. One way to see this is by observing that the generators of the Lorentz group, in which are included spatial rotations, are formed by antisymmetric matrices.

The set of vectors $\{\boldsymbol{e}_a^\mu\}$ in (2.53) forms a tetrad (where greek indices denote tensorial indices and latin indices label the vectors of the tetrad) which is attached to the particle and obeys the relation $g_{\mu\nu} \, e_a^\mu \, e_b^\nu = \eta_{ab}$, with $\eta_{ab} = diag(+,-,-,-)$

being the Minkowski metric. In fact, it is easy to see that $\sigma^{\mu\nu}$ is antisymmetric ($\sigma^{\mu\nu} = \sigma^{[\mu\nu]}$), which means that it has only six degrees of freedom, which is expected for the angular velocity from a classical point of view (actually, it is expected only three degrees of freedom for ω^{ij}, which correspond to the three spatial rotations. A restriction on the degrees of freedom of $\sigma^{\mu\nu}$, or more precisely in $S^{\mu\nu}$, will be made when we study the spin supplementary conditions in the next chapter). In addition, we have the relation

$$\dot{e}_a^{\mu} = -\sigma^{\mu\nu} e_{a\nu}, \tag{2.55}$$

with $e_{a\nu} = g_{\mu\nu} e_a^{\mu}$, which is the same expression satisfied by the antisymmetric angular velocity tensor ω^{ij} for the classical spinning top (Eq. (2.54)), which reinforces the interpretation of $\sigma^{\mu\nu}$ as angular velocity.

At this point, one takes the lagrangian depending on the velocities $\dot{x}^{\mu}$ and $\sigma^{\mu\nu}$ only, and depending on these variables by means of four scalars $I_1 = v^{\mu} v_{\mu}$, $I_2 = \sigma^{\mu\nu}\sigma_{\mu\nu}$, $I_3 = v^{\alpha}\sigma_{\alpha\beta}\sigma^{\beta\gamma}v_{\gamma}$ and $I_4 = \sigma_{\alpha\beta}\sigma^{\beta\gamma}\sigma_{\gamma\delta}\sigma^{\delta\alpha}$, which are the only relevant Poincaré invariants for this construction, as argued in [9]. Defining the lagrangian with the structure $\mathcal{L}(I_1, I_2, I_3, I_4)$, we define the momentum p^{μ} and the spin $S^{\mu\nu}$ of the particle as

$$p_{\mu} = \frac{\partial \mathcal{L}}{\partial v^{\mu}}, \quad S_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \sigma^{\mu\nu}}. \tag{2.56}$$

Therefore, we have the relations:

$$\begin{aligned} p^{\mu} &= 2\, v^{\mu} L_1 + 2\, \sigma^{\mu\alpha}\sigma_{\alpha\beta} v^{\beta} L_3\,, \\ S^{\mu\nu} &= 4\, \sigma^{\mu\nu} L_2 + 2(v^{\mu}\sigma^{\nu\alpha}v_{\alpha} - v^{\nu}\sigma^{\mu\alpha}v_{\alpha}) L_3 + 8\, \sigma^{\nu\alpha}\sigma_{\alpha\beta}\sigma^{\beta\mu} L_4\,, \end{aligned} \tag{2.57}$$

where we define $L_i = \frac{\partial \mathcal{L}}{\partial I_i}$. Defining the antisymmetric variation $\delta\theta^{\mu\nu} = \eta^{ab}\, e_a^{\mu}(\delta e_b^{\nu} + \Gamma^{\nu}_{\lambda\rho}\, e_b^{\lambda}\, \delta x^{\rho})$, one can obtain the formula [13,14]:

$$\delta\sigma^{\mu\nu} = (\delta\dot{\theta}^{\mu\nu}) + \sigma^{\mu\lambda}\delta\theta_{\lambda}^{\ \nu} - \delta\theta^{\mu\lambda}\sigma_{\lambda}^{\ \nu} - (R_{\alpha\beta}^{\ \ \mu\nu} v^{\beta} + \Gamma^{\mu}{}_{\alpha\beta}\sigma^{\beta\nu} + \Gamma^{\nu}{}_{\alpha\beta}\sigma^{\mu\beta})\delta x^{\alpha}. \tag{2.58}$$

Varying the action with respect to the fundamental variables that describe the test particle, we obtain

$$\delta\mathcal{S} = \int \left(\frac{\partial \mathcal{L}}{\partial v^\mu} \delta v^\mu + \frac{\partial \mathcal{L}}{\partial \sigma^{\mu\nu}} \delta \sigma^{\mu\nu} \right) d\tau$$
$$= \int \left[\left(\dot{p}_\alpha + \frac{1}{2} R_{\alpha\beta\gamma\delta} v^\beta S^{\gamma\delta} \right) \delta x^\alpha + (\dot{S}_{\mu\nu} + S_{\nu\alpha} \sigma^\alpha{}_\mu - S_{\mu\alpha}\sigma^\alpha{}_\nu) \delta\theta^{\mu\nu} \right.$$
$$\left. - \frac{d}{d\tau} \left(p_\mu \delta x^\mu + S_{\mu\nu} \delta\theta^{\mu\nu} \right) \right] d\tau. \quad (2.59)$$

Using the equation (2.57) along with some mathematical identities from appendix A of [9], it is possible to prove the identity $S^{\alpha[\mu}\sigma_\alpha{}^{\nu]} = -p^{[\mu}v^{\nu]}$. Imposing $\delta\mathcal{S} = 0$, and considering the arbitrary variations δx^μ and $\delta\theta^{\mu\nu}$ which vanish in the limits of integration, one eventually arrive at the MPD equations (2.14).

It is remarkable that we obtain the same set of equations from a completely different formalism, constructing a lagrangian where one requires only some simple reasonable hypotheses on it. That suggests that the MPD equations should be in fact the correct equations for the description of a spinning test body from a perspective of a relativistic gravitational theory. In the next section, we will shed light on the content of these equations.

As already pointed out, the MPD equations give only a subset of the necessary equations that we need to determine the momentum and the spin of the particle. These equations are not enough to close the system, so that they need to be supplemented by other three equations. In [3], Mathisson chose the condition $S^{\alpha\beta}v_\beta = 0$ to supplement his equations, and that choice is intimately related to the solutions he obtained for the MPD equations in flat spacetime, which are known as *Mathisson's helical motions*. However, in [5] Tulczyjew proposed $S^{\alpha\beta}p_\beta = 0$ as supplementary equations for (2.14), and showed that this condition leads to the expected equations for the motion of a spinning particle in flat spacetime (the helical motions do not appear). The equations that are used to supplement MPD equations are very important and are known in the literature as *spin supplementary conditions*. We will see that each condition has some special features. In particular, for the Mathisson condition, the scalar m is conserved, while in Tulczyjew condition the scalar μ^2, which is related to the norm of the momentum, is conserved. Depending on the type of problem we are interested in, it can be convenient to use one condition or the other. The next chapter is dedicated to explain these supplementary conditions, illustrate the differences existing between them and describe in what scenarios it could be adequate to use one condition or another.

3 SPIN SUPPLEMENTARY CONDITIONS

As anticipated in the previous chapter, Mathisson imposed the spin supplementary condition $S^{\alpha\beta}v_\beta = 0$ (a condition which comes from $v_\gamma m^{\gamma\alpha\beta} = 0$, see Eq. (2.41)) in order to make his equations determined. In fact, we saw that the MPD equations (2.14) by themselves comprise a total of ten equations for thirteen degrees of freedom, so that three other equations are necessary. These additional equations that must be imposed are the so-called spin supplementary conditions.

Besides, we mentioned that there is an infinite set of trajectories that describe the motion of the body satisfying the Mathisson supplementary condition, the famous Mathisson's helical solutions. However, it would be natural in flat spacetime to expect that a determined system leads to a unique trajectory, something that does not happen with the condition $S^{\alpha\beta}v_\beta = 0$, a fact which will be properly understood in this chapter. Some years later than Mathisson, Tulczyjew proposed as supplementary condition $S^{\alpha\beta}p_\beta = 0$ rather than that proposed in [3], and showed that this condition leads to a unique trajectory, in which the helical solutions do not appear [5]. These differences seem to lead us to conclude that different dynamics arise depending on what supplementary condition is chosen. Nevertheless, it will be showed that this conclusion is merely apparent, since both supplementary conditions describe the same body, only changing the point of view, that is, the observer [16].

As Mathisson's solutions predict particles that could move in trajectories with a radius of arbitrary size, they were considered *unphysical solutions* [17,18]. In this sense, these solutions need to be properly understood and their origins clarified. For this, it will be fundamental to discuss how one can define the center of mass of an extended body in relativity. We know from Newtonian mechanics the relevance of that special point of the body. In particular, an extended body is properly described if we know the dynamics of its center of mass and of its internal motion relative to the center of mass, such that it would be interesting if we could define something similar in relativity. In fact, many authors have discussed about how we can define the center of mass in relativity [19,20].

For the treatment of an extended body in relativity, we have seen that a body of finite size describes a world tube along its motion, representing the world lines of each point that forms the body, and that we chose a reference curve L inside this world tube. It will be discussed in this chapter that the point of the body which describes the curve L is precisely its center of mass when any of the supplementary conditions are used, i.e. for any condition of the type $S^{\alpha\beta}u_\beta = 0$, with u^β being a timelike vector [16]. It is fundamental, however, to keep in mind that in relativity the CM of an extended body is observer-dependent [20], such that does not exist a universal point that all the observers agree to be the CM of the body. Given this fact, we will see that the spin supplementary condition actually chooses the observer, which in turn measures the body's center of mass.

Although the supplementary conditions $S^{\alpha\beta}v_\beta = 0$ and $S^{\alpha\beta}p_\beta = 0$ are the best known, other conditions were also proposed [14,21]. In particular, in Ref. [14] a relaxed version of the Tulczyjew condition has been proposed to treat massless particles.

In addition, a discussion on massless particles will be made, by defending $S^{\alpha\beta}v_\beta = 0$ as an appropriate supplementary condition to treat the massless case, in agreement with precedent works where some authors concluded the same result [22,23].

In this section, we will elucidate some differences between these two supplementary conditions. From now on, the condition $S^{\alpha\beta}v_\beta = 0$ is occasionally referred to as *Mathisson-Pirani condition* (or simply MP condition) and $S^{\alpha\beta}p_\beta = 0$ as *Tulczyjew condition.*

3.1 THE TULCZYJEW SUPPLEMENTARY CONDITION: $S^{\alpha\beta}p_\beta = 0$

This section is devoted to explore the main properties of the condition $S^{\alpha\beta}p_\beta = 0$. Let us begin to understand the differences between Tulczyjew ($S^{\alpha\beta}p_\beta = 0$) and Mathisson-Pirani ($S^{\alpha\beta}v_\beta = 0$) conditions by studying the conserved quantities of the spinning test particle.

There are some scalars which are conserved in both supplementary conditions, as is the case of the quadratic spin $S^2 = S_{\mu\nu}S^{\mu\nu}$. In fact, contracting the spin equation of MPD equations (2.14) with $S_{\alpha\beta}$, we obtain

$$S_{\alpha\beta}\dot{S}^{\alpha\beta} = 2S_{\alpha\beta}p^{\alpha}v^{\beta} = 0 \quad \Rightarrow \quad \frac{D(S_{\alpha\beta}S^{\alpha\beta})}{D\tau} = 0, \tag{3.1}$$

where it was used $S^{\alpha\beta}u_{\beta} = 0$, with $u^{\alpha} = p^{\alpha}$ if the Tulczyjew condition is chosen and $u^{\alpha} = v^{\alpha}$ if the MP condition is chosen. Equation (3.1) tells us that the scalar $S_{\mu\nu}S^{\mu\nu}$ is conserved along the particle's trajectory. Besides, for both conditions, we have that the scalar

$$Q_{\boldsymbol{k}} = k_{\alpha}p^{\alpha} + \frac{1}{2}\nabla_{\alpha}k_{\beta}S^{\alpha\beta} \tag{3.2}$$

is conserved if $\boldsymbol{k}$ is a Killing vector field (KV) of the spacetime. In fact, to prove the conservation of (3.2) we only need the MPD equations, without mention of any supplementary condition. Explicitly, by deriving (3.2), we get

$$\dot{Q}_{\boldsymbol{k}} = \nabla_{\alpha}k_{\beta}v^{(\alpha}p^{\beta)} - \frac{1}{2}k_{\gamma}R^{\gamma}{}_{\mu\alpha\beta}v^{\mu}S^{\alpha\beta} + \frac{1}{2}v^{\mu}\nabla_{\mu}\nabla_{\alpha}k_{\beta}S^{\alpha\beta}, \tag{3.3}$$

where MPD equations have been used. We prove $\dot{Q}_{\boldsymbol{k}} = 0$ by considering that for a Killing vector field we have $\nabla_{(\mu}k_{\nu)} = 0$, which in turn implies $\nabla_{\mu}k_{\nu} = \nabla_{[\mu}k_{\nu]}$, and by using the identity $k_{\gamma}R^{\gamma}{}_{\mu\alpha\beta} = \nabla_{\mu}\nabla_{\alpha}k_{\beta}$, which is valid for any KV as a consequence of Ricci identity (2.2). The scalars $Q_{\boldsymbol{k}}$ are important conserved quantities whose Killing vectors associated with them give us information about the symmetries of the spacetime. We will return to this point and study this conserved charge in more detail in the next chapter.

Nevertheless, there are some scalars that are conserved just in one of the supplementary conditions used. In particular, the scalar $\mu^2 = p^{\alpha}p_{\alpha}$, which is the mass of the particle in the frame where the 3-momentum vanishes ($p^i = 0$), is conserved only if $S^{\alpha\beta}p_{\beta} = 0$ is considered. In fact, for the scalar $p_{\alpha}p^{\alpha}$, contracting the spin equation (2.14) with $\dot{p}_{\alpha}p_{\beta}$, we have

$$\dot{p}_{\alpha}p_{\beta}\dot{S}^{\alpha\beta} = m\dot{p}_{\alpha}p^{\alpha} - m\dot{p}_{\alpha}v^{\alpha}, \tag{3.4}$$

where $m = v_{\beta}p^{\beta}$. Since $\dot{p}_{\alpha}v^{\alpha} = 0$ (which can be seen immediately by contracting the momentum equation of (2.14) with v_{α}) and considering that $p_{\alpha}\dot{p}^{\alpha} = \frac{1}{2}\frac{D(p_{\alpha}p^{\alpha})}{D\tau}$, we have

$$\frac{D(p_{\alpha}p^{\alpha})}{D\tau} = \frac{2}{m}\dot{p}_{\alpha}p_{\beta}\dot{S}^{\alpha\beta} = 0, \tag{3.5}$$

where in the last equality the derivative of the condition $S^{\alpha\beta}p_\beta = 0$ and the antisymmetry of $S^{\alpha\beta}$ have been used.

An advantage of the condition $S^{\alpha\beta}p_\beta = 0$ over $S^{\alpha\beta}v_\beta = 0$ concerns the so-called momentum-velocity relation. The Tulczyjew condition has a very interesting and useful property that the actual degrees of freedom of the particle separate naturally, covariantly, in that condition. For example, the quantities that describe the particle are represented by v^α, p^α and $S^{\alpha\beta}$. However, these components are not independent of each other, since the MPD equations connect these variables. For the condition $S^{\alpha\beta}p_\beta = 0$, in addition to the fact that the spin tensor can be expressed as a function of the momentum p^α (as suggested by the supplementary condition), we also can get a covariant expression in which the velocity is written in terms of the momentum and the spin. That expression will be referred to from now on as *momentum-velocity relation*. The momentum-velocity relation can be obtained as follows. Contracting the spin equation of (2.14) with p_β and using the derivative of the condition $S^{\alpha\beta}p_\beta = 0$, we eventually arrive at

$$v^\alpha = \frac{m}{\mu^2}p^\alpha - \frac{1}{2\mu^2}S^{\alpha\beta}R_{\beta\gamma\rho\sigma}v^\gamma S^{\rho\sigma}\,, \tag{3.6}$$

where the momentum equation of (2.14) has been used. We rewrite the above equation in a compact form as

$$v^\alpha = a\,p^\alpha + D^\alpha{}_\gamma v^\gamma, \tag{3.7}$$

where we do the identifications

$$a = \frac{m}{\mu^2}\,, \quad \text{and} \quad D^\alpha{}_\gamma = -\frac{1}{2\mu^2}S^{\alpha\beta}R_{\beta\gamma\rho\sigma}S^{\rho\sigma}\,. \tag{3.8}$$

Inserting repeatedly the velocity expression (3.7) in the velocity of the right-hand side of the same expression, we obtain the series

$$v^\alpha = a\left[p^\alpha + D^\alpha{}_{\gamma_1}p^{\gamma_1} + D^\alpha{}_{\gamma_1}D^{\gamma_1}{}_{\gamma_2}p^{\gamma_2} + ...\right]. \tag{3.9}$$

Now, we note that the Tulczyjew supplementary condition allows us to express the spin tensor as follows:

$$S^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta}\tilde{S}_\gamma\,p_\delta\,, \tag{3.10}$$

with $\epsilon^{\alpha\beta\gamma\delta}$ standing for the Levi-Civita tensor. In fact, the solution for $S^{\alpha\beta}p_\beta = 0$ is given by $S^{\alpha\beta} = A^{\alpha\beta\gamma}p_\gamma$, for some completely antisymmetric tensor $\boldsymbol{A}$ ($A^{\alpha\beta\gamma} = A^{[\alpha\beta\gamma]}$). However, a completely antisymmetric tensor of rank 3 (or a 3-form, in the language of the differential forms) is always related in four-dimensional spaces to some vector $\tilde{S}^\mu$ by Hodge duality, i.e. the following relation is valid:

$$A^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma\delta}\tilde{S}_\delta, \tag{3.11}$$

from which follows the equation (3.10). We also note that the transformation $\tilde{S}_\gamma \to \tilde{S}_\gamma + \lambda p_\gamma$, for an arbitrary λ, does not change $S^{\alpha\beta}$ in Eq. (3.10), so that we could impose $\tilde{S}^\gamma p_\gamma = 0$ without loss of generality. Besides, one can prove that from (3.10) it follows the relation $S^{[\alpha\beta}S^{\mu]\nu} = 0$, which in turn we use to show that

$$D^\alpha{}_{\gamma_1}D^{\gamma_1}{}_{\gamma_2}p^{\gamma_2} = \frac{1}{2}D^\alpha{}_{\gamma_1}D^{\gamma_2}{}_{\gamma_2}p^{\gamma_1} = \frac{d}{2}D^\alpha{}_{\gamma_1}p^{\gamma_1}, \tag{3.12}$$

where $d = D^\alpha{}_\alpha$. From the latter relation, it follows that

$$D^\alpha{}_{\gamma_1}D^{\gamma_1}{}_{\gamma_2}D^{\gamma_2}{}_{\gamma_3}p^{\gamma_3} = \left(\frac{d}{2}\right)^2 D^\alpha{}_{\gamma_1}p^{\gamma_1}, \tag{3.13}$$

and so on. With these results, the series (3.9) is written as

$$v^\alpha = a\left(p^\alpha + D^\alpha{}_\beta p^\beta \sum_{n=0}^{\infty}\frac{d^n}{2^n}\right), \tag{3.14}$$

whose summation leads to

$$\begin{aligned} v^\alpha &= a\left(p^\alpha + \frac{D^\alpha{}_\gamma p^\gamma}{1-d/2}\right) \\ &= \frac{m}{\mu^2}\left(p^\alpha - \frac{2S^{\alpha\beta}R_{\beta\gamma\rho\sigma}p^\gamma S^{\rho\sigma}}{4\mu^2 - R_{\mu\nu\gamma\delta}S^{\mu\nu}S^{\gamma\delta}}\right), \end{aligned} \tag{3.15}$$

where in the last equality the definitions of a, d and $D^\alpha{}_\beta$ have been used. The momentum-velocity relation (3.15) was first obtained by Kunzle in [8]. We note that it could also be established projecting equation (3.6) onto an orthonormal Lorentz frame, solving the equations for the velocity components and then by recovering the covariance of the expression, as made in [8,25]. In the deduction presented here, however, no covariance breaking was necessary [26].

Now, as a final discussion, let us see in flat spacetime the solutions of MPD equations along with $S^{\alpha\beta}p_\beta = 0$ in order to shed some light on its content. We have in this scenario the following set of equations:

$$\dot{p}^\alpha = 0\,, \quad \dot{S}^{\alpha\beta} = p^\alpha v^\beta - p^\beta v^\alpha \quad \text{and} \quad S^{\alpha\beta}p_\beta = 0\,. \tag{3.16}$$

Then, we see immediately that the momentum p^α is conserved along the particle motion. From the momentum-velocity relation (3.15), we conclude that $v^\alpha = \frac{m}{\mu^2}\, p^\alpha$, such that in this scenario, Tulczyjew and MP conditions become the same. In other words, we have proved that condition $S^{\alpha\beta}p_\beta = 0$ implies condition $S^{\alpha\beta}v_\beta = 0$ in flat spacetime. We could also conclude this from Eq. (3.6). However, as we will see further, the contrary is not true. Contracting the momentum-velocity relation found with v_α leads us to $m = \mu$, which in turn allows us to express this relation simply as $v^\alpha = p^\alpha/\mu$. The substitution of this latter equation in the spin equation leads also to the conservation of the spin tensor ($\dot{S}^{\alpha\beta} = 0$). Integrating the momentum-velocity relation, we have

$$x^\alpha(\tau) = \frac{p^\alpha}{\mu}\tau + C^\alpha, \tag{3.17}$$

where the integration constants C^α are related to initial values of the position. Therefore, the solution of MPD equations with $S^{\mu\nu}p_\nu = 0$ tells us that the representative world line of the particle describes straight lines with a uniform rotational motion. This property of Tulczyjew condition of predicting a unique trajectory in flat spacetime is the reason why many authors defend it as the more natural choice for massive particles [5,7]. We point out that this uniqueness of the condition $S^{\mu\nu}p_\nu = 0$ is maintained in curved spacetimes [27,28]. In [28], the MPD equations are integrated numerically along with condition $S^{\mu\nu}p_\nu = 0$ in a Kerr spacetime. For maximally symmetric spaces with nontrivial curvature (de Sitter and anti-de Sitter), where the curvature tensor is written as $R_{\alpha\beta\mu\nu} = \kappa(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$, we can extract directly from Eq. (3.15) that $v^\alpha = \frac{m}{\mu^2}p^\alpha$, which in turn implies that $\dot{p}^\alpha = 0$ and $\dot{S}^{\alpha\beta} = 0$, so that the trajectories of spinning bodies are the geodesics of the de Sitter (or anti-de Sitter) space. Therefore, even in presence of spin, the MPD equations with Tulczyjew condition $S^{\alpha\beta}p_\beta = 0$ predicts no deviation from the geodesic trajectory.

3.2 THE MATHISSON-PIRANI SUPPLEMENTARY CONDITION: $S^{\alpha\beta}v_\beta = 0$

Let us see now the differences that MP condition $S^{\alpha\beta}v_\beta = 0$ has in relation to $S^{\alpha\beta}p_\beta = 0$. First, we note that the scalar $m = v^\alpha p_\alpha$, which is the mass of the particle in the frame where the 3-velocity is zero ($v^i = 0$), i.e. the proper mass, is conserved if $S^{\alpha\beta}v_\beta = 0$ is considered, whereas $\mu = \sqrt{p_\alpha p^\alpha}$ is not conserved anymore. In fact, for the scalar $m = v_\mu p^\mu$, we have

$$\dot{m} = \frac{D(v_\alpha p^\alpha)}{D\tau} = \dot{v}_\alpha p^\alpha + v_\alpha \dot{p}^\alpha. \tag{3.18}$$

Contracting the momentum equation of (2.14) with v_α, we obtain immediately $v_\alpha \dot{p}^\alpha = 0$, and contracting the spin equation of (2.14) with $\dot{v}_\mu v_\nu$, we get

$$\dot{v}_\alpha p^\alpha = \dot{v}_\alpha v_\alpha \dot{S}^{\alpha\beta} + m\dot{v}_\alpha v^\alpha. \tag{3.19}$$

Once $v_\alpha v^\alpha = 1$, it follows that $\dot{v}_\alpha v^\alpha = 0$, so that substituting in (3.18) we finally obtain

$$\dot{m} = \dot{v}_\alpha v_\beta \dot{S}^{\alpha\beta} = 0, \tag{3.20}$$

when the derivative of the condition $S^{\alpha\beta}v_\beta = 0$ is used.

Now, concerning a relation of the type (3.15), we point out that it was derived a momentum-velocity relation for the Mathisson-Pirani condition [21]. The relation the authors found turns out to be equivalent to the restriction $S^{\alpha\beta}v_\beta = 0$, and it was used to integrate numerically the MPD equations. However, since the expression found in [21] is equivalent to MP condition, the substitution of $S^{\mu\nu}$ expression (Eq. (3.22) given below) in that relation of course provides a trivial identity, as pointed out by the authors themselves, such that the expression cannot be used to conduct a similar investigation of that made in [25] in order to find the most general linear conserved charge in MP condition. Nevertheless, we observe that it is possible at least to obtain an equation for the derivative of the velocity in terms of the momentum and spin in MP condition, as shown in [27].

Up to now, we are discussing only massive particles, for which the equation $v_\alpha v^\alpha = 1$ is valid. However, it is worth mentioning here the massless case. This case is interesting once the condition $S^{\alpha\beta}v_\beta = 0$ is perhaps the appropriate condition for the description of massless particles. A discussion about what are the best equations that supplement the MPD equations for massless particles can be found in [22,23]. In [22] the MPD equations are solved using $S^{\alpha\beta}v_\beta = 0$ and $v^\alpha v_\alpha = 0$ to supplement (2.14) and then it is showed that $m = v_\alpha p^\alpha = 0$ is satisfied. In addition, it was also shown that a massless particle satisfying this set of equations necessarily follows a null geodesic $v^\beta \nabla_\beta v^\alpha = 0$. In [23], in turn, the authors adopted MP condition along with $m = v_\alpha p^\alpha = 0$ as the adequate supplementary conditions for (2.14) in the massless case and also considered the possibility of $v_\alpha v^\alpha \neq 0$. We point out that these latter equations are perhaps the appropriate equations which supplement (2.14). In fact, if we consider MPD equations along with conditions $S^{\alpha\beta}v_\beta = 0$ and $m = v_\alpha p^\alpha = 0$, it is straightforward to show that the charge

$$Q = \hat{k}_\mu p^\mu + \frac{1}{2}\nabla_\mu \hat{k}_\nu S^{\mu\nu} \tag{3.21}$$

is conserved, with $\hat{\boldsymbol{k}}$ being a conformal Killing vector (CKV), that is, it satisfies the equation $\nabla_{(\alpha}\hat{k}_{\beta)} = f\, g_{\alpha\beta}$, with $f = \frac{1}{4}\nabla^\alpha \hat{k}_\alpha$. In fact, from the case of a spinless particle we know that CKV's are related to conserved charges in null geodesics, in the same way as KV's are related to conserved charges along arbitrary geodesics. Namely, for spinless particles, the scalar $k_\mu p^\mu$ is conserved along arbitrary geodesics (with $\boldsymbol{k}$ a KV), while $\hat{k}_\mu p^\mu$ is conserved along null geodesics (with $\hat{\boldsymbol{k}}$ a CKV).

At this point, as we made for the condition $S^{\alpha\beta}p_\beta = 0$, it is important to make a discussion concerning the solutions of MPD equations in flat spacetime with MP condition. First, we note that for the case $S^{\alpha\beta}v_\beta = 0$, we have similarly the following expression for the spin tensor:

$$S^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}\Sigma_\mu v_\nu, \tag{3.22}$$

for some vector Σ^μ. For the Mathisson-Pirani condition, we have the set of equations:

$$\dot{p}^\alpha = 0\,, \quad \dot{S}^{\alpha\beta} = p^\alpha v^\beta - p^\beta v^\alpha \quad \text{and} \quad S^{\alpha\beta}v_\beta = 0\,. \tag{3.23}$$

Considering the spin tensor as given by (3.22) and taking the derivative of this expression, we have:

$$\dot{S}^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}\dot{\Sigma}_\mu v_\nu + \epsilon^{\alpha\beta\mu\nu}\Sigma_\mu \dot{v}_\nu, \tag{3.24}$$

since the Levi-Civita tensor is covariantly constant ($\nabla_\mu \epsilon_{\alpha\beta\gamma\delta} = 0$). Using the spin equation of MPD and contracting it with $\epsilon_{\alpha\beta\gamma\delta}v^\gamma$, we get the dynamic equation for the spin vector $\boldsymbol{\Sigma}$:

$$\dot{\Sigma}^{\delta} = -\dot{v}^{\gamma}\Sigma_{\gamma}v^{\delta}, \tag{3.25}$$

where we have used $v^{\gamma}\Sigma_{\gamma} = 0$. The orthogonality between $\boldsymbol{v}$ and $\boldsymbol{\Sigma}$ can be established from the spin tensor equation (3.22), once the Σ_{μ} in that expression can be decomposed as $\Sigma_{\mu} = \Sigma_{\mu}^{||} + \Sigma_{\mu}^{\perp}$, with $\Sigma_{\mu}^{||}$ and $\Sigma_{\mu}^{\perp}$ being parallel and orthogonal to velocity, respectively. Since $\Sigma_{\mu}^{||} \propto v_{\mu}$, this component does not contribute to the spin tensor $S^{\alpha\beta}$, once we will have something like $\epsilon^{\alpha\beta\mu\nu}v_{\mu}v_{\nu}$, which is automatically zero due to the simultaneous symmetry and antisymmetry in the indices μ and ν. Then, we can take $\Sigma_{\mu}^{||} = 0$ without loss of generality, which amounts to the validity of $v^{\alpha}\Sigma_{\alpha} = 0$. Besides, expression (3.25) can be further simplified by proving that $\dot{v}^{\alpha}\Sigma_{\alpha} = 0$. This can be done as follows. Contracting again the spin equation with v_{β} and using the derivative of $S^{\alpha\beta}v_{\beta} = 0$, we have for the momentum:

$$p^{\alpha} = mv^{\alpha} - \dot{v}_{\beta}S^{\alpha\beta}. \tag{3.26}$$

Of course, from the spin equation (3.22) it follows that $S^{\alpha\beta}\Sigma_{\beta} = 0$. Using this latter relation and contracting (3.26) with Σ_{α}, we conclude that $\Sigma_{\alpha}p^{\alpha} = 0$. Now, by deriving the momentum equation (3.26), using the set of equations (3.23) and contracting the resulting expression with Σ_{α}, we prove finally that $\Sigma_{\alpha}\dot{v}^{\alpha} = 0$, which then implies that the spin vector is parallel transported ($\dot{\Sigma}^{\alpha} = 0$). With this latter result, equation (3.24) becomes

$$\dot{S}^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}\Sigma_{\mu}\dot{v}_{\nu}. \tag{3.27}$$

Now, substituting the spin equation $\dot{S}^{\alpha\beta} = 2\,p^{[\alpha}v^{\beta]}$ into Eq. (3.27) and then by contracting the resulting expression with $\epsilon_{\alpha\beta\mu\nu}\Sigma^{\nu}$, we arrive at the dynamic equation for v^{α}:

$$\dot{v}^{\alpha} = -\frac{1}{\Sigma^2}\epsilon^{\alpha\beta\mu\nu}\Sigma_{\beta}p_{\mu}v_{\nu}, \tag{3.28}$$

where we define $\Sigma^2 = \Sigma^{\mu}\Sigma_{\mu}$, which is constant. At this point, we take the particular frame where the 3-momentum of the spinning particle vanishes ($p^i = 0$), that is,

$$p^{\alpha} = (\mu, \vec{0}). \tag{3.29}$$

This amounts to align the momentum p^{α} with the time direction of a Lorentz frame, which can always be accomplished once the momentum is timelike. From the orthogonality relation $\Sigma^\alpha p_\alpha = 0$, it follows that the time component of Σ^α is zero, i.e, it has the form $\Sigma^\alpha = (0, \vec{\Sigma})$. By means of a rotation, we can align the vector $\vec{\Sigma}$ with the z-direction such that $\Sigma^\alpha = (0, 0, 0, \Sigma)$, which in turn implies that the z-component v^3 of the velocity is zero, since $\Sigma^\mu v_\mu = 0$. With these simplifications and taking the velocity as $v^\alpha = (v^0, v^1, v^2, 0)$, equation (3.28) becomes

$$\dot{v}^0 = 0, \quad \dot{v}^1 = -\frac{\mu}{\Sigma} v^2, \quad \dot{v}^2 = \frac{\mu}{\Sigma} v^1. \tag{3.30}$$

The first equation provides v^0 constant, and the last two coupled equations can be separated such that each component obeys

$$\ddot{v}^j + \omega^2 v^j = 0\,, \tag{3.31}$$

with $\omega = \mu/\Sigma$ and $j = 1, 2$. The solutions of (3.31) are given by

$$v^1 = \dot{x}(\tau) = A\,sin(\omega\tau) + B\,cos(\omega\tau)\,, \tag{3.32}$$
$$v^2 = \dot{y}(\tau) = A'\,sin(\omega\tau) + B'\,cos(\omega\tau)\,. \tag{3.33}$$

By integrating them and imposing the restrictions given by Eq. (3.30), we get:

$$x(\tau) = -A\,cos(\omega\tau) + B\,sin(\omega\tau) + C\,, \tag{3.34}$$
$$y(\tau) = -B\,cos(\omega\tau) - A\,sin(\omega\tau) + C'\,, \tag{3.35}$$

with A, B, C and C' being integration constants, and where the redefinitions $A/\omega \to A$, $B/\omega \to B$ have been made. This can be put into the form

$$(x(\tau) - C)^2 + (y(\tau) - C')^2 = A^2 + B^2\,, \tag{3.36}$$

which is the equation of a circumference with center at $(x_c, y_c) = (C, C')$ and radius $r = (A^2 + B^2)^{1/2}$. Therefore, the general solution of the problem in the frame $p^i = 0$ is

$$x^\alpha(\tau) = (\gamma\tau, x(\tau), y(\tau), z_0). \tag{3.37}$$

The solution (3.37) describes the so-called Mathisson's helical motions, a point that follows a circular path while the time evolves. We obtain for the radius of the circumference:

$$r = \frac{\gamma^2 v \Sigma}{m}. \tag{3.38}$$

Once that γ can take arbitrary values, these solutions were considered as *unphysical* [29]. However, a more recent work defends that this consideration is due to a misconception [16]. In fact, it was defended in the literature that the circular motions predicted by the condition $S^{\alpha\beta}v_\beta = 0$ were not contained in a finite radius, which supported the wrong interpretation. However, as pointed out in [16], one can prove that all the trajectories are contained in a radius with finite size given by $r_{max} = S_{r=0}/\mu$, with $S_{r=0}$ being the magnitude of the spin vector for the trajectory with $r = 0$. This interpretation is totally consistent with Moller's work [20].

Therefore, the solution of MPD equations along with MP supplementary condition in flat spacetime predicts many possible curves which can be followed by the spinning particle, that is, there is an infinite set of solutions obeying $S^{\alpha\beta}v_\beta = 0$. That non-uniqueness can be explained as follows, which gives the physical interpretation of the supplementary conditions. The fact is that in relativity, unlike Newtonian mechanics, the center of mass of a body is observer-dependent, such that it is necessary to specify the frame in which the center of mass is calculated. A condition of type $S^{\alpha\beta}u_\beta = 0$ always selects as reference curve the worldline of the center of mass of the body. The difference between the supplementary conditions is the observer which measures the center of mass. The condition $S^{\alpha\beta}v_\beta = 0$ describes the center of mass as measured by observers who see the body's center of mass not in the center of the rotating body. Then, this observer sees the center of mass displaced relative to the observer who sees the center of mass in the center. Thus, depending on the relative velocity between the observers, they will measure different trajectories, each one referring to the center of mass observed, which is precisely the many trajectories predicted by the solution (3.37). This interpretation explains why the condition $S^{\alpha\beta}p_\beta = 0$ gives just one solution, since this condition, according to the mentioned interpretation, means that the center of mass is at rest in relation to the observer which moves with 4-velocity $v^\alpha \propto p^\alpha$ constant in flat spacetime, and it corresponds to the unique point that does not rotate $(r = 0)$.

Summing up, the Tulczyjew condition $S^{\mu\nu}p_\nu = 0$ seems to be the appropriate condition for massive particles, since as we have seen it leads to a unique trajectory for the spinning particle in flat and curved spacetimes, while the MP condition $S^{\mu\nu}v_\nu = 0$ seems to be adequate to handle massless particles, once among other reasons it leads to the correct conserved charge involving CKV's.

3.3 SUPPLEMENTARY CONDITIONS AND THE CENTER OF MASS OF THE BODY

Now, we use $S^{\alpha\beta}u_\beta = 0$, with u_β any timelike unity vector to show that the supplementary conditions always select the curve which represents the center of mass of the body as seen by an observer which moves with velocity u^α. There are many works which discuss the problem of defining the center of mass in relativity, as [19,20]. In special relativity, the momentum and spin of an extended body are given by the integral relations:

$$p^\alpha = \int_\sigma T^{\alpha\beta} d\sigma_\beta \quad \text{and} \quad S^{\alpha\beta} = \int_\sigma (x^{[\alpha} - z^{[\alpha}(\tau))T^{\beta]\gamma} d\sigma_\gamma, \tag{3.39}$$

where σ is a 3-dimensional spacelike hypersurface orthogonal to u^α which generates the body and $z^\alpha(\tau)$ is a point over the reference curve L. The conditions of the kind $S^{\alpha\beta}u_\beta = 0$ are the appropriate conditions to supplement MPD equations since they determine the center of mass in the rest frame of an observer which moves with velocity u^α. In fact, in that frame we have $u^i = 0$, which implies $S^{\alpha\beta}u_\beta = S^{i0}u_0$. For the calculation of S^{i0} we have

$$S^{i0} = \int (x^i - z^i(\tau))T^{0\gamma} d\sigma_\gamma - \int (x^0 - z^0(\tau))T^{i\gamma} d\sigma_\gamma. \tag{3.40}$$

Since the integration is performed in the hypersurface $u_\alpha(x^\alpha - z^\alpha(\tau)) = 0$ (or simply $x^0 = z^0(\tau)$) orthogonal to $\boldsymbol{u}$, it follows that the second integral above is zero. Then, we get

$$S^{i0} = \int x^i T^{0\gamma} d\sigma_\gamma - z^i(\tau)m(u), \tag{3.41}$$

with $m(u)$ the mass as measured in this reference frame. The first term of the above equation is by definition $m(u)\, x^i_{CM}(u)$, with $x^i_{CM}(u)$ the coordinates of the mass center in the frame, since $d\sigma_\gamma$ has only the zero component $d\sigma_0$ and the component T^{00} of the energy-momentum tensor is related to the mass of the body. Then, we obtain

$$x^i_{CM}(u) - z^i = -\frac{S^{\alpha\beta}u_\beta}{m(u)}, \tag{3.42}$$

where $m(u) = \mu$ for $S^{\alpha\beta}p_\beta = 0$ and $m(u) = m$ for $S^{\alpha\beta}v_\beta = 0$. Then, establishing $S^{\alpha\beta}u_\beta = 0$ gives that $z^i(\tau)$ is the center of mass of the body, as stated earlier.

In the next chapter, we will learn more about symmetries of the spacetime and the so called hidden symmetries. The generators of these symmetries, that is, Killing vectors, and Killing and Killing-Yano tensors, will also be properly discussed.

4 KILLING-YANO TENSORS AND THEIR INTEGRABILITY CONDITIONS

We have seen in section 3.1 that Tulczyjew supplementary condition $S^{\mu\nu}p_\nu = 0$ has, among other features, the property of admitting a covariant relation expressing velocity in terms of the momentum and the spin of the particle, given in Eq. (3.15) and called momentum-velocity relation. This property was explored by Rüdiger in [25] in order to obtain the most general conserved linear charge which can be constructed in the pole-dipole approximation. Of course, the conserved scalar (3.2) should emerge from Rudiger's method, as it actually does. Nevertheless, besides the scalar (3.2) it was also found by the author that the scalar

$$Q_Y = Y^{\star}_{\mu\nu} S^{\mu\nu}, \tag{4.1}$$

where star means Hodge duality, is conserved if $\boldsymbol{Y}$ is a Killing-Yano tensor (KY), i.e. an antisymmetric tensor ($Y_{\mu\nu} = Y_{[\mu\nu]}$) which satisfies $\nabla_{(\alpha}Y_{\mu)\nu} = 0$. So, Q_Y needs Y to exist. Quadratic conserved quantities, in turn, as S^2 and $p_\alpha p^\alpha$, should emerge from an investigation of the most general conserved charge which is quadratic in momentum and spin, an investigation which was carried out by the same author in a subsequent paper [30]. In this chapter, we will understand Rüdiger's work in detail, since the problem tackled in this thesis is based on it.

The interesting thing about Rüdiger's work is precisely the appearance of the new conserved charge (4.1) involving a Killing-Yano tensor. Killing-Yano tensors, together with Killing tensors are the antisymmetric and symmetric generalizations of Killing vectors, respectively, and they are related to symmetries of the phase space of the geodesic motion, in the same way that Killing vectors are related to symmetries of the spacetime [31]. For that reason these symmetries are sometimes referred to as *hidden symmetries*. Nevertheless, the interesting thing about spacetimes which admit the existence of a KY tensor is that it can be showed that these spacetimes satisfy a set of algebraic equations known as *integrability conditions*, which involve the curvature tensor. Therefore, if a spacetime admits a KY tensor, these conditions on the curvature necessarily need to be satisfied. In this chapter we will learn about these objects and their integrability conditions, once they will be essential to the problem we consider in this thesis.

4.1 KILLING AND KILLING-YANO TENSORS

A *conformal Killing-Yano tensor* (CKY) of rank p is a totally antisymmetric tensor $Y_{\alpha_1\alpha_2\ldots\alpha_p} = Y_{[\alpha_1\alpha_2\ldots\alpha_p]}$, i.e., a p-form, satisfying the equation [32]

$$\nabla_\mu Y_{\alpha_1\alpha_2\ldots\alpha_p} = \nabla_{[\mu} Y_{\alpha_1\alpha_2\ldots\alpha_p]} + 2\, g_{\mu[\alpha_1} \tilde{Y}_{\alpha_2\ldots\alpha_p]}, \tag{4.2}$$

with $\tilde{\boldsymbol{Y}}$ being related to the divergence of $\boldsymbol{Y}$. In fact, by tracing expression (4.2) we obtain

$$\tilde{Y}_{\alpha_2\ldots\alpha_p} = \frac{p}{2(5-p)} \nabla^\mu Y_{\mu\alpha_2\ldots\alpha_p}. \tag{4.3}$$

Equation (4.2) can also be displayed in an equivalent way by the expression

$$\nabla_{(\mu} Y_{\alpha_1)\alpha_2\ldots\alpha_p} = g_{\mu[\alpha_1} \tilde{Y}_{\alpha_2\ldots\alpha_p]} + g_{\alpha_1[\mu} \tilde{Y}_{\alpha_2\ldots\alpha_p]}. \tag{4.4}$$

As one can see from the definition (4.2) of a CKY, the derivative of this object is formed by two terms, one involving its exterior derivative and other its interior derivative (divergence). From this definition one can define two important objects, the Killing-Yano tensor (KY) and the closed conformal Killing-Yano tensor (CCKY). A KY is by definition a CKY with zero divergence ($\tilde{\boldsymbol{Y}} = 0$), which amounts to having zero at the right-hand side of (4.4). Then, a KY is a p-form which obeys the equation

$$\nabla_{(\mu} Y_{\alpha_1)\alpha_2\ldots\alpha_p} = 0. \tag{4.5}$$

A KY can be seen as the antisymmetric generalization of a KV. In fact, we see that for $p = 1$ the above equation satisfies the Killing equation.

A generic p-form $\boldsymbol{\omega}$ is said to be closed if its exterior derivative is zero, that is, $\boldsymbol{d\omega} = 0$. Then, a closed CKY, or CCKY, is an object which obeys (4.2) with vanishing exterior derivative $\nabla_{[\mu} Y_{\alpha_1\alpha_2\ldots\alpha_p]} = 0$, that is,

$$\nabla_\mu Y_{\alpha_1\alpha_2\ldots\alpha_p} = 2\, g_{\mu[\alpha_1} \tilde{Y}_{\alpha_2\ldots\alpha_p]}. \tag{4.6}$$

It can be proved that the Hodge dual of a KY is a CCKY and vice versa. In fact, for the Hodge duality we have

$$(Y^\star)_{\alpha_1\ldots\alpha_{n-p}} = \frac{1}{p!} \epsilon^{\beta_1\ldots\beta_p}{}_{\alpha_1\ldots\alpha_{n-p}} Y_{\beta_1\ldots\beta p}. \tag{4.7}$$

Supposing that $Y^\star$ is a KY, it obeys, from (4.2), the relation

$$\nabla_\mu (Y^\star)_{\beta_1\ldots\beta_p} = \nabla_{[\mu}(Y^\star)_{\beta_1\ldots\beta_p]}. \tag{4.8}$$

Contracting this latter expression with $\frac{1}{p!}\epsilon^{\beta_1\ldots\beta_p}{}_{\alpha_1\ldots\alpha_{n-p}}$ and using (4.7) we get

$$\nabla_\mu Y_{\alpha_1\ldots\alpha_{n-p}} = \frac{1}{p!(n-p)!}\epsilon^{\beta_1\ldots\beta_p}{}_{\alpha_1\ldots\alpha_{n-p}}\epsilon^{\tilde{\alpha}_1\ldots\tilde{\alpha}_{n-p}}{}_{[\beta_1\ldots\beta_p}\nabla_{\mu]}Y_{\tilde{\alpha}_1\ldots\tilde{\alpha}_{n-p}}. \tag{4.9}$$

Developing the antisymmetrization of the right-hand side, we obtain finally that Y satisfies the CCKY equation (4.6). Similarly, we can prove the converse, that is, by starting from a tensor $Y^\star$ obeying the CCKY equation, prove that its Hodge dual $(Y^\star)^\star = Y$ obeys a KY equation.

In addition, one can also prove that the Hodge dual of a CKY is also a CKY. In other words, under Hodge duality, the exterior derivative becomes interior derivative and vice versa, a fact which explains this latter result and the previous one. As a remark, we point out that the CKY equation (4.2) can also be put into a more compact form by using the language of differential forms [33]. This approach has the advantage that the fact that two CKY's are related via Hodge duality, in the same way as a KY and a CCKY, it can be shown more easily in this notation. However, this language will not be adopted here.

Another important object we can define is the *Killing tensor* (KT). A KT of rank p is by definition the symmetric generalization of a KV, that is, a completely symmetric tensor $K_{\alpha_1\ldots\alpha_p} = K_{(\alpha_1\ldots\alpha_p)}$ which satisfies the equation

$$\nabla_{(\mu}K_{\alpha_1\ldots\alpha_p)} = 0. \tag{4.10}$$

In fact, we see immediately that a KT of rank $p = 1$ satisfies a Killing equation. Furthermore, we have the important property that, given a KY, its square is a KT, that is, the tensor $\boldsymbol{K}$ defined by the contraction

$$K_{\alpha\beta} = Y_{\alpha\alpha_2\ldots\alpha_p}Y_\beta{}^{\alpha_2\ldots\alpha_p} \tag{4.11}$$

is a rank 2 Killing tensor if $\boldsymbol{Y}$ is a Killing-Yano tensor of any order p.

The interesting point about KY's and KT's is that they are related to conserved quantities for the motion of a free particle [31]. It is widely known that for the motion of a spinless particle in GR, the Killing vectors are related to conserved quantities along the geodesic motion. Indeed, if $\boldsymbol{k}$ is a Killing vector field, then the scalar $k_\mu p^\mu$ is a constant of motion for the particle. This result is very important once it is closely related to the problem of integrating the geodesic equation in curved spacetimes. In fact, even in spacetimes with many symmetries, as Schwarzschild spacetime, the geodesic equation is not integrated directly. Rather, one uses the Killing vector fields of the spacetime, which generates the *isometries*, in order to use the conserved charges associated to these symmetries and then integrate the geodesic equation. In the same way as Killing vectors, Killing tensors are also related to conserved quantities for the geodesic motion. In fact, if we consider scalars of type $C = K_{\mu\nu}p^\mu p^\nu$, for some tensor $K_{\mu\nu}$, we have by derivation in the direction of motion:

$$\dot{C} = v^\alpha \nabla_\alpha K_{\mu\nu}\, p^\mu p^\nu + K_{\mu\nu}\dot{p}^\mu p^\nu + K_{\mu\nu}p^\mu \dot{p}^\nu. \tag{4.12}$$

Imposing $\dot{C} = 0$, and since the motion is geodesic, i.e. $p^\mu \propto v^\mu$ and $\dot{v}^\mu = 0$, then the last two terms above vanish and we conclude that $\dot{C} \propto \nabla_\alpha K_{\mu\nu} v^\alpha v^\mu v^\nu = 0$, which is satisfied for an arbitrary geodesic curve only if $\nabla_{(\alpha} K_{\mu\nu)} = 0$, i.e., $\boldsymbol{K}$ is a rank 2 Killing tensor field. For instance, the metric tensor $g_{\mu\nu}$ is a trivial example of a rank 2 Killing tensor, once $\nabla_\alpha g_{\mu\nu} = 0$. This KT is related to the conserved quantity $g_{\mu\nu}p^\mu p^\nu = p_\mu p^\mu$ for the spinless particle. In the same way, if a spacetime admits a KY, we can form a KT by means of (4.11), so that we can always generate a conserved scalar with the structure $K_{\mu\nu}p^\mu p^\nu$.

Therefore, KY's and KT's are related to conserved quantities even for the geodesic motion, and not only for the case of a particle with spin. We note, however, the particularity of the Rüdiger's result (4.1), since for the geodesic motion the generators of hidden symmetries (in this case, Killing tensors) appear in conserved quantities quadratic in the momenta, while for the spinning particle the Killing-Yano tensor appears already in the linear case.

After this presentation, we are ready to investigate the integrability conditions of these objects. In particular, we will focuses our attention on the Killing-Yano tensor, which is the relevant object which appears in the conserved charge we are interested in. However, before discussing about integrability conditions, it is necessary to introduce an important classification which will be very relevant to subsequent topics, the Petrov classification.

4.2 THE PETROV CLASSIFICATION

In four-dimensional spacetimes, there is an important classification which allows us to classify the spacetimes in six different types. This classification is known as Petrov classification, and it is an algebraic classification for the curvature tensor, or more precisely, the Weyl tensor.

Petrov classification can be presented by a variety of different approaches [34]. In this thesis, however, we will present only one of them, which classify the Weyl tensor according to which components of this tensor can be made simultaneously zero by transforming a null tetrad frame under the action of Lorentz group. We point out that, as proposed originally by Petrov [35] and further developed by Penrose [36], the classification is valid only in four dimensional spacetimes. However, in [34,37] can be found a discussion on how this classification can be generalized to higher dimensions.

The Weyl tensor can be defined as the traceless part of the curvature tensor. Then, it obeys the same algebraic identities of the curvature tensor, with the additional property of being traceless, i.e., it satisfies

$$C_{\alpha\beta\mu\nu} = C_{[\alpha\beta][\mu\nu]}, \quad C_{\alpha[\beta\mu\nu]} = 0 \quad \text{and} \quad C^{\mu}{}_{\alpha\mu\beta} = 0. \tag{4.13}$$

As pointed out in chapter 2, the Weyl tensor in four dimensions has only ten independent components. It is possible to accommodate these ten components in five complex scalars by using a null tetrad frame, which we denote by $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$. A null tetrad frame is a set of four vectors $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$, where $\boldsymbol{\ell}$ and $\boldsymbol{n}$ are real vector fields, whereas $\boldsymbol{m}$ is complex with $\bar{\boldsymbol{m}}$ being its complex conjugate. By definition, all the tetrad vectors are null and the only nonvanishing inner products in this frame are the following:

$$\ell^{\mu}\, n_{\mu} = 1 \quad \text{and} \quad m^{\mu}\, \bar{m}_{\mu} = -1\,. \tag{4.14}$$

For instance, if $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a Lorentz frame, with their inner products yielding the Minkowski metric $\eta_{\mu\nu}$, then

$$\boldsymbol{\ell} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_0 + \boldsymbol{e}_1\right) \ , \ \ \boldsymbol{n} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_0 - \boldsymbol{e}_1\right) \ ,$$

$$\boldsymbol{m} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_2 + i\boldsymbol{e}_3\right) \ , \ \ \bar{\boldsymbol{m}} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_2 - i\boldsymbol{e}_3\right) \ ,$$

is a null tetrad frame.

The complex scalars, which are called Weyl scalars, are defined as projections of the Weyl tensor in the vectors of the tetrad in the following way:

$$\begin{gathered}\Psi_0 \equiv C_{\alpha\beta\mu\nu}\ell^\alpha m^\beta \ell^\mu m^\nu;\ \Psi_1 \equiv C_{\alpha\beta\mu\nu}\ell^\alpha n^\beta \ell^\mu m^\nu;\ \Psi_2 \equiv C_{\alpha\beta\mu\nu}\ell^\alpha m^\beta \bar{m}^\mu n^\nu; \\ \Psi_3 \equiv C_{\alpha\beta\mu\nu}\ell^\alpha n^\beta \bar{m}^\mu n^\nu;\ \Psi_4 \equiv C_{\alpha\beta\mu\nu}n^\alpha \bar{m}^\beta n^\mu \bar{m}^\nu.\end{gathered} \tag{4.15}$$

In this language, the spacetimes are classified depending on which Weyl scalars can be made zero.

In addition, the Weyl tensor can be expressed in terms of the null tetrad frame introduced. In order to present the expression, let us introduce the following notation:

$$\{\ell_\alpha n_\beta m_\mu \bar{m}_\nu\} \equiv 4\ell_{[\alpha}n_{\beta]}m_{[\mu}\bar{m}_{\nu]} + 4m_{[\alpha}\bar{m}_{\beta]}\ell_{[\mu}n_{\nu]}. \tag{4.16}$$

The Weyl tensor is then expressed as

$$\begin{gathered}C_{\alpha\beta\mu\nu} = \tfrac{1}{2}(\Psi_2 + \bar{\Psi}_2)[\{\ell_\alpha n_\beta \ell_\mu n_\nu\} + \{m_\alpha \bar{m}_\beta m_\mu \bar{m}_\nu\}] - \tfrac{1}{2}(\Psi_2 - \bar{\Psi}_2)\{\ell_\alpha n_\beta m_\mu \bar{m}_\nu\} \\ +\bar{\Psi}_0\{n_\alpha \bar{m}_\beta n_\mu \bar{m}_\nu\} + \bar{\Psi}_4\{\ell_\alpha m_\beta \ell_\mu m_\nu\} - \bar{\Psi}_2\{\ell_\alpha m_\beta n_\mu \bar{m}_\nu\} \\ +\bar{\Psi}_1[\{\ell_\alpha n_\beta n_\mu \bar{m}_\nu\} + \{n_\alpha \bar{m}_\beta \bar{m}_\mu m_\nu\}] + \bar{\Psi}_3[\{\ell_\alpha m_\beta m_\mu \bar{m}_\nu\} - \{\ell_\alpha n_\beta \ell_\mu m_\nu\}] + c.c.\,,\end{gathered} \tag{4.17}$$

where *c.c.* means the complex conjugate of all previous terms.

An important feature on Petrov classification is that it is local, i.e. the Petrov type of the Weyl tensor can change from point to point of the spacetime. However, it is observed that in general the Petrov type is kept invariant throughout the spacetime, at least for most of the known spaces, and it is thanks to this property that Petrov classification becomes useful to distinguish spacetimes. For example, black hole solutions as Schwarzschild and Kerr spacetimes are Petrov type D in all points of the manifold, while solutions as plane gravitational waves are type N. In particular, according with the definition presented in the table below of Petrov classification in terms of Weyl scalars, it means that in these spacetimes we can always find a null tetrad which allows us to vanish all the Weyl scalars (in all points of the manifold) except Ψ_2 and Ψ_4, respectively. The following table summarizes which Weyl scalars can be made zero depending on the corresponding Petrov type.

Petrov Type	**Vanishing Weyl Scalars**
I	$\Psi_0\,,\ \Psi_4$
II	$\Psi_0\,,\ \Psi_1\,,\ \Psi_4$
III	$\Psi_0\,,\ \Psi_1,\ \Psi_2\,,\ \Psi_4$
D	$\Psi_0\,,\ \Psi_1,\ \Psi_3\,,\ \Psi_4$
N	$\Psi_0\,,\ \Psi_1,\ \Psi_2\,,\ \Psi_3$
O	$\Psi_0\,,\ \Psi_1,\ \Psi_2\,,\ \Psi_3,\ \Psi_4$

Tabela 1 – Petrov types and its relation with the possibility of annihilating the Weyl scalars by a suitable choice of null tetrad frame. Note that the type O means a conformally flat spacetime, i.e. the Weyl tensor is identically zero in such a case.

Lorentz transformations can be understood in terms of rotations in the tetrad frame. We can distinguish all possible rotations in a null tetrad frame in three different types. They are [34,38]

$$
\begin{aligned}
(i) \quad \boldsymbol{\ell} \;&\to \boldsymbol{\ell}; \boldsymbol{n} \to \boldsymbol{n} + w\boldsymbol{m} + \bar{w}\bar{\boldsymbol{m}} + |w|^2\boldsymbol{\ell}; \boldsymbol{m} \to \boldsymbol{m} + \bar{w}\boldsymbol{\ell}; \bar{\boldsymbol{m}} \to \bar{\boldsymbol{m}} + w\boldsymbol{\ell}\,,\\
(ii) \quad \boldsymbol{\ell} \;&\to \boldsymbol{\ell} + |z|^2\boldsymbol{n} + \bar{z}\boldsymbol{m} + z\bar{\boldsymbol{m}}; \boldsymbol{n} \to \boldsymbol{n}; \boldsymbol{m} \to \boldsymbol{m} + z\boldsymbol{n}; \bar{\boldsymbol{m}} \to \bar{\boldsymbol{m}} + \bar{z}\boldsymbol{n}\,,\\
(iii) \quad \boldsymbol{\ell} \;&\to \lambda\,\boldsymbol{\ell};\; \boldsymbol{n} \to \lambda^{-1}\boldsymbol{n};\; \boldsymbol{m} \to e^{i\theta}\boldsymbol{m}; \bar{\boldsymbol{m}} \to e^{-i\theta}\bar{\boldsymbol{m}}\,,
\end{aligned}
$$

with λ and θ being real numbers, z and w being complex numbers, and where (i) means rotation around the null direction $\boldsymbol{\ell}$, (ii) rotation around the null direction $\boldsymbol{n}$ and (iii) a Lorentz boost. Performing the transformation (ii), the Weyl scalars transform in the following way

$$\Psi_0 \to \Psi_0' = \Psi_0 + 4z\Psi_1 + 6z^2\Psi_2 + 4z^3\Psi_3 + z^4\Psi_4\,;$$

$$\Psi_1 \to \Psi_1' = \tfrac{1}{4}\tfrac{d}{dz}\Psi_0'; \quad \Psi_2 \to \Psi_2' = \tfrac{1}{3}\tfrac{d}{dz}\Psi_1'\,; \tag{4.18}$$

$$\Psi_3 \to \Psi_3' = \tfrac{1}{2}\tfrac{d}{dz}\Psi_2'; \quad \Psi_4 \to \Psi_4' = \tfrac{1}{2}\tfrac{d}{dz}\Psi_3' = \Psi_4.$$

Establishing $\Psi_0' = 0$, we will have a fourth order polynomial equation in the variable z, which possesses four roots that we designate by $\{z_1, z_2, z_3, z_4\}$. It means that is always possible to annihilate Ψ_0 by means of a convenient choice of the frame. The different Petrov types of the Weyl tensor are then established depending on the characteristic of these four roots. For instance, we can have a situation where the four roots z_i $(i = 1, 2, 3, 4)$ are distinct, in which case the roots do not possess degeneracy, or cases where some of the roots coincide, in which case is said that the roots possess degeneracy of order r, with r being the number of equal roots. Each one of these possible situations will be associated with a Petrov type.

Let us illustrate how this association can be done using as example the case where two of the roots coincide, i.e., are of the kind $z_1 = z_2 \neq z_3 = z_4$. Firstly, we call attention to the important mathematical fact that in the equation (4.18) the transformed Weyl scalars are always the derivative of the precedent Weyl scalar. This important fact will allow us to establish the different Petrov types. In the proposed case, the polynomial Ψ_0' can be expressed as $\Psi_0' = \Psi_4(z - z_1)^2(z - z_3)^2$ $(= 0)$. The remaining transformed Weyl scalars can be obtained by derivation in relation to z, and they read

$$\Psi_1' = \tfrac{1}{2}\Psi_4(z - z_1)(z - z_3)(2z - z_1 - z_3)\,;$$

$$\Psi_2' = \tfrac{1}{3}\Psi_4\left[(z - z_1)(z - z_3) + \tfrac{1}{2}(2z - z_1 - z_3)^2\right]\,; \tag{4.19}$$

$$\Psi_3' = \tfrac{1}{2}\Psi_4(2z - z_1 - z_3); \quad \Psi_4' = \Psi_4.$$

Choosing $z = z_1$ for example, we see that the scalars Ψ_0 and Ψ_1 become zero simultaneously. Similarly, if after this we perform a transformation of the type (i), one proves that the scalars Ψ_3 and Ψ_4 also vanish, while Ψ_0 and Ψ_1 are kept invariant, i.e., equal to zero. Then, the situation where two of the roots coincide admits just $\Psi_2 \neq 0$ after Lorentz transformations, i.e., it is identified as the Petrov type D case. Similarly, we show that the each of the remaining situations involving the four roots of the polynomial equation $\Psi'_0 = 0$ is related to a specific Petrov type of the Weyl tensor. This classification will be useful in calculations made in the next chapter.

4.3 INTEGRABILITY CONDITIONS

Suppose that Z^α is a covariantly constant vector field, i.e., it satisfies $\nabla_\alpha Z_\beta = 0$. A vector field with this property, due to the Ricci identity (2.2), satisfies the equation

$$2\nabla_{[\mu}\nabla_{\nu]} Z^\alpha = R^\alpha{}_{\beta\mu\nu} Z^\beta = 0\,.$$

The latter equation is said to be an integrability condition for the existence of a constant vector field. Thus, if the curvature of a spacetime is such that there exists no vector field T^α satisfying $R_{\alpha\beta\mu\nu}T^\alpha = 0$, then we can already state that no covariantly constant vector field exists, without having to try to integrate the differential equation $\nabla_\alpha Z_\beta = 0$ for a generic vector field Z^α. In the same way, if one wants that a KY exists in a spacetime, some integrability conditions should be satisfied. For instance, concerning KY's of rank two, $Y_{\mu\nu}$, the following constraints must hold [39-41]:

$$0 = R^\alpha{}_{(\mu} Y_{\nu)\alpha}\,, \tag{4.20}$$

$$0 = C_{\alpha\beta[\mu}{}^{\sigma} Y_{\nu]\sigma} + C_{\mu\nu[\alpha}{}^{\sigma} Y_{\beta]\sigma}\,, \tag{4.21}$$

where $R^\alpha{}_\mu$ stands for the Ricci tensor whereas $C_{\mu\nu\alpha\beta}$ denotes the Weyl tensor. Thus, the curvature of the spacetime must obey some algebraic restrictions if a spacetime admits a KY.

A general treatment of the integrability conditions valid for CKY's of any order is given in [41]. However, we are interested in understanding the conserved scalar (4.1) found by Rudiger in [25]. Therefore, from now on, it is sufficient to investigate the integrability conditions associated to KY's of rank $p = 2$. Considering this simplification, let us illustrate how to obtain the integrability condition (4.21). For this purpose, we start from the more general equation of a conformal KY of rank 2, which according to (4.4) reads

$$\nabla_\nu Y_{\alpha\beta} + \nabla_\alpha Y_{\nu\beta} = 2\, g_{\nu[\alpha}\tilde{Y}_{\beta]} + 2\, g_{\alpha[\nu}\tilde{Y}_{\beta]}. \tag{4.22}$$

The integrability conditions for a CKY are algebraic conditions satisfied by the curvature tensor necessary for its existence, that is, if we suppose that a spacetime admits the existence of a CKY, then its curvature tensor necessarily should satisfy some algebraic conditions. Then, starting from (4.22) and since we are looking for an algebraic expression, we need to eliminate all the derivatives of the CKY $\boldsymbol{Y}$. For this purpose, we proceed as follows. Deriving the above relation and by making cyclic permutations of the indices $(\mu\nu\alpha)$, we get

$$\begin{aligned} \nabla_\mu\nabla_\nu Y_{\alpha\beta} + \nabla_\mu\nabla_\alpha Y_{\nu\beta} &= 2\, h_{\mu[\beta}g_{\alpha]\nu} + 2\, h_{\mu[\beta}g_{\nu]\alpha}, \\ -\nabla_\alpha\nabla_\mu Y_{\nu\beta} - \nabla_\alpha\nabla_\nu Y_{\mu\beta} &= -2\, h_{\alpha[\beta}g_{\nu]\mu} - 2\, h_{\alpha[\beta}g_{\mu]\nu}, \\ \nabla_\nu\nabla_\alpha Y_{\mu\beta} + \nabla_\nu\nabla_\mu Y_{\alpha\beta} &= 2\, h_{\nu[\beta}g_{\mu]\alpha} + 2\, h_{\nu[\beta}g_{\alpha]\mu}, \end{aligned} \tag{4.23}$$

where the second equation was multiplied by -1 and we define $h_{\alpha\beta} = \nabla_\alpha\tilde{Y}_\beta$. Summing the three equations above and using the Ricci identity (2.2), we arrive at

$$\begin{aligned} 2\nabla_\mu\nabla_\nu Y_{\alpha\beta} = {} & R_\alpha{}^\gamma{}_{\mu\nu}Y_{\gamma\beta} + R_\beta{}^\gamma{}_{\mu\nu}Y_{\alpha\gamma} - R_\nu{}^\gamma{}_{\mu\alpha}Y_{\gamma\beta} - R_\beta{}^\gamma{}_{\mu\alpha}Y_{\nu\gamma} - R_\mu{}^\gamma{}_{\nu\alpha}Y_{\gamma\beta} \\ & - R_\beta{}^\gamma{}_{\nu\alpha}Y_{\mu\gamma} + 2\, h_{\mu[\beta}g_{\alpha]\nu} + 2\, h_{\mu[\beta}g_{\nu]\alpha} - 2\, h_{\alpha[\beta}g_{\nu]\mu} - 2\, h_{\alpha[\beta}g_{\mu]\nu} \\ & + 2\, h_{\nu[\beta}g_{\mu]\alpha} + 2\, h_{\nu[\beta}g_{\alpha]\mu}. \end{aligned} \tag{4.24}$$

Now, starting from (4.2) for $p = 2$, expanding its anti-symmetrization symbols, and then deriving with ∇_μ the resulting expression we get

$$2\,\nabla_\mu\nabla_\nu Y_{\alpha\beta} = 3g_{\nu\alpha}h_{\mu\beta} - 3g_{\nu\beta}h_{\mu\alpha} - \nabla_\mu\nabla_\beta Y_{\nu\alpha} + \nabla_\mu\nabla_\alpha Y_{\beta\nu}. \tag{4.25}$$

Using equation (4.24) to build explicit expressions for $\nabla_\mu\nabla_\beta Y_{\nu\alpha}$ and $\nabla_\mu\nabla_\alpha Y_{\beta\nu}$, and then substituting the results in the equation above, we obtain

$$2\,\nabla_\mu\nabla_\nu Y_{\alpha\beta} = -3R^\gamma{}_{\mu[\nu\alpha}Y_{\beta]\gamma} + 2\left(g_{\nu\alpha}h_{\mu\beta} - g_{\nu\beta}h_{\mu\alpha} - 3g_{\mu[\nu}h_{\alpha\beta]}\right). \tag{4.26}$$

Since we have established Eq. (4.24) by using Ricci identity and Eq. (4.25) by merely index manipulation, then we have two independent equations for the term

$2\,\nabla_\mu \nabla_\nu Y_{\alpha\beta}$. Thus, equating the right-hand side of (4.24) and (4.26), we get, after some algebra,

$$R^\gamma{}_{\mu\alpha\nu} Y_{\beta\gamma} + R^\gamma{}_{\nu\beta\mu} Y_{\alpha\gamma} + R^\gamma{}_{\alpha\mu\beta} Y_{\nu\gamma} + R^\gamma{}_{\beta\nu\alpha} Y_{\mu\gamma} + 2g_{\mu\alpha}\nabla_{(\nu}\tilde{Y}_{\beta)} + 2g_{\nu\beta}\nabla_{(\mu}\tilde{Y}_{\alpha)} - 2g_{\mu\nu}\nabla_{(\alpha}\tilde{Y}_{\beta)} - 2g_{\alpha\beta}\nabla_{(\mu}\tilde{Y}_{\nu)} = 0. \quad (4.27)$$

Equation (4.27) is still dependent on derivatives of $\boldsymbol{Y}$ through $\tilde{\boldsymbol{Y}}$, such that it should be eliminated in order to arrive at the integrability condition. This can be made by contracting (4.27) with $g^{\mu\nu}$, which allows us to express the derivative $\nabla_{(\alpha}\tilde{Y}_{\beta)}$ only in terms of the product of the CKY with the Ricci tensor, that is,

$$\nabla_{(\alpha}\tilde{Y}_{\beta)} = \frac{1}{2} R^\gamma{}_{(\alpha} Y_{\beta)\gamma}. \quad (4.28)$$

Substituting Eq. (4.28) into Eq. (4.27), and expressing the curvature tensor in terms of the Weyl tensor, eventually leads to

$$C_{\alpha\beta[\mu}{}^\gamma Y_{\nu]\gamma} + C_{\mu\nu[\alpha}{}^\gamma Y_{\beta]\gamma} = 0, \quad (4.29)$$

which is the integrability condition for the existence of a conformal Killing-Yano tensor.

We can now work out the consequences of the integrability condition (4.20) for the Ricci tensor. In the next chapter we will be more interested in the traceless part of the Ricci tensor, which is defined by

$$\Phi_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}\, R\, g_{\mu\nu}$$

As already mentioned in section 1.1, a spacetime is called an Einstein spacetime whenever its Ricci tensor is proportional to the metric, which is equivalent to say that $\boldsymbol{\Phi}$ vanishes. Note that, in terms of the null tetrad frame, the traceless condition $g^{\alpha\beta}\Phi_{\alpha\beta} = 0$ implies

$$g^{\alpha\beta}\Phi_{\alpha\beta} = (2\ell^{(\alpha} n^{\beta)} - 2m^{(\alpha}\bar{m}^{\beta)})\Phi_{\alpha\beta} = 0 \quad \Rightarrow \quad \Phi_{\ell n} = \Phi_{m\bar{m}}\,, \quad (4.30)$$

where the compact notation $\Phi_{\alpha\beta}\ell^\alpha n^\beta = \Phi_{\ell n}$ and $\Phi_{\alpha\beta} m^\alpha \bar{m}^\beta = \Phi_{m\bar{m}}$ has been used.

At this point, we need to digress on bivectors and their classification. By definition, a bivector $\boldsymbol{B}$ at a point x of a manifold is any antisymmetric tensor of second order. If its components are $B_{\mu\nu}(= B_{[\mu\nu]})$, then its Hodge dual is defined as follows:

$$B^{\star}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}B^{\alpha\beta}. \tag{4.31}$$

A nonzero bivector can be of two algebraic types: null or non-null. A null bivector is such that the contractions $B^{\mu\nu}B_{\mu\nu}$ and $B^{\mu\nu}B^{\star}_{\mu\nu}$ are both zero. Otherwise, it is called a non-null bivector. Once introduced a null tetrad frame $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$, it is possible to express the two algebraic types of the bivector in terms of the tetrad vectors. It turns out that given a real bivector $\boldsymbol{B}$ one can always find a null frame in which the bivector is written in one of the following forms depending on its algebraic type [42]:

$$\begin{cases} \text{Null Bivector: } \boldsymbol{B} = \boldsymbol{\ell} \wedge (\boldsymbol{m} + \bar{\boldsymbol{m}}) \\ \\ \text{Non-Null Bivector: } \boldsymbol{B} = f\,\boldsymbol{\ell} \wedge \boldsymbol{n} + ih\,\boldsymbol{m} \wedge \bar{\boldsymbol{m}}\,, \end{cases} \tag{4.32}$$

where f and h are real functions that cannot both be identically zero. We can see this, proceeding in a similar way we did on Petrov classification. In terms of a null tetrad, we define the components of a bivector $B_{\mu\nu}$ as $B_0 = B_{\mu\nu}\ell^{\mu}m^{\nu}$, $B_1 = B_{\mu\nu}\ell^{\mu}n^{\nu}$, $B_2 = B_{\mu\nu}m^{\mu}\bar{m}^{\nu}$ and $B_3 = B_{\mu\nu}n^{\mu}\bar{m}^{\nu}$. By means of a rotation around the null vector n, these components transforms in the following way:

$$\begin{aligned} B'_0 &= B_0 + z(B_1 - B_2) - z^2 B_3; \quad B'_1 = \frac{1}{2}\left(\frac{dB'_0}{dz} + \frac{d\bar{B}'_0}{d\bar{z}}\right) \\ B'_2 &= \frac{1}{2}\left(\frac{d\bar{B}'_0}{d\bar{z}} - \frac{dB'_0}{dz}\right); \qquad B'_3 = B_3. \end{aligned} \tag{4.33}$$

Establishing $B'_0 = 0$, we have a polynomial equation of second order in z (at first, two different roots $\{z_1, z_2\}$). We classify the bivector $B_{\mu\nu}$ depending on which scalars can be made zero simultaneously. In particular, when we can write $B'_0 = -B_3(z - z_1)^2$ (case $z_1 = z_2$), Eq. (4.33) implies that only $B_3 \neq 0$, in which case the bivector can be written as $\boldsymbol{B} = \boldsymbol{\ell} \wedge (\boldsymbol{m} + \bar{\boldsymbol{m}})$ (null bivector).

Now, assuming that the KY is a null bivector, i.e., it can be written as $\boldsymbol{Y} = \boldsymbol{\ell} \wedge (\boldsymbol{m} + \bar{\boldsymbol{m}})$, for some null tetrad frame, then inserting this form into Eq. (4.20), and finally contracting the free indices of this equation with the vectors of the null tetrad we eventually conclude that

$$\boldsymbol{Y}\ \text{Null:}\begin{cases}\Phi_{\ell\ell}=\Phi_{\ell m}=\Phi_{\ell\bar{m}}=\Phi_{nm}+\Phi_{n\bar{m}}=0\,,\\ \Phi_{mm}=\Phi_{\bar{m}\bar{m}}=-2\Phi_{\ell n}\,.\end{cases} \tag{4.34}$$

Analogously, assuming that the KY is non-null and writing it in the standard form given in Eq. (4.32), it follows that the integrability condition (4.20) implies

$$\boldsymbol{Y}\ \text{Non-Null:}\begin{cases}\Phi_{\ell m}=\Phi_{\ell\bar{m}}=\Phi_{nm}=\Phi_{n\bar{m}}=0\,,\\ \begin{cases} f\neq 0\ \Rightarrow\ \Phi_{\ell\ell}=\Phi_{nn}=0\\ h\neq 0\ \Rightarrow\ \Phi_{mm}=\Phi_{\bar{m}\bar{m}}=0\,.\end{cases}\end{cases} \tag{4.35}$$

Thus, for a generic non-null KY, i.e. when the real functions f and h appearing in the standard form of Eq. (4.32) are both nonvanishing, we have that $\Phi_{\ell\ell}$, Φ_{nn}, Φ_{mm}, and $\Phi_{\bar{m}\bar{m}}$ all vanish. However, if h vanishes then we cannot assert that $\Phi_{mm}=\Phi_{\bar{m}\bar{m}}=0$, whereas if f vanishes the integrability condition does not imply $\Phi_{\ell\ell}=\Phi_{nn}=0$. We recall that the functions f and h cannot vanish simultaneously, otherwise the KY tensor would be trivial.

In addition, we see that the integrability condition (4.29) involves the Weyl tensor, so that it must imply some restrictions on this tensor. In fact, it is possible to prove that Eq.(4.29) implies that the following conclusion holds [41]:

$$\begin{cases}\text{Null KY: }\Psi_0=\Psi_1=\Psi_2=\Psi_3=0\,,\\ \\ \text{Non-Null KY: }\Psi_0=\Psi_1=\Psi_3=\Psi_4=0\,.\end{cases} \tag{4.36}$$

Indeed, let us take as an example the case where the bivector $\boldsymbol{Y}$ is null. Inserting $\boldsymbol{Y}=\boldsymbol{\ell}\wedge(\boldsymbol{m}+\bar{\boldsymbol{m}})$ into Eq.(4.29), we get

$$\begin{aligned} C_{\alpha\beta\mu}{}^{\gamma}(\ell_{[\nu}m_{\gamma]}+\ell_{[\nu}\bar{m}_{\gamma]})-C_{\alpha\beta\nu}{}^{\gamma}(\ell_{[\mu}m_{\gamma]}+\ell_{[\mu}\bar{m}_{\gamma]})\\ +C_{\mu\nu\alpha}{}^{\gamma}(\ell_{[\beta}m_{\gamma]}+\ell_{[\beta}\bar{m}_{\gamma]})-C_{\mu\nu\beta}{}^{\gamma}(\ell_{[\alpha}m_{\gamma]}+\ell_{[\alpha}\bar{m}_{\gamma]})=0\,. \end{aligned} \tag{4.37}$$

Contracting the above equation with $\ell^{\alpha}m^{\beta}\ell^{\mu}n^{\nu}$, we obtain $C_{\ell m\ell m}=\Psi_0=0$. Similarly, we can take other contractions such that we eventually arrive at (4.36).

Therefore, from the algebraic type of the Weyl tensor it is possible to rule out the existence of a KY of rank two. For example, if we suppose that a spacetime is Petrov type III, then it cannot admit a rank two KY. This statement can be done prior to any attempt of integrating the KY equation. In the next chapter we shall use this tool along with other two conditions, that are required in order to guarantee that the scalar $Q_{\boldsymbol{Y}} = Y^{\star}_{\alpha\beta}S^{\alpha\beta}$ is conserved along a solution of MPD equations, and conclude that very few spacetimes allow this conserved charge. In particular, we will prove that this scalar is useless for Einstein spacetimes.

4.4 RÜDIGER'S CONSERVED CHARGE

In this section, we follow the steps adopted by Rüdiger in Ref. [25] in order to obtain the most general conserved scalar for the spinning free particle that is linear in its linear and angular momentum. The most general scalar of this type is given by

$$Q = k_\mu p^\mu + L_{\mu\nu}S^{\mu\nu}\,, \tag{4.38}$$

for some tensors k_μ and $L_{\mu\nu} = L_{[\mu\nu]}$. Now, let us impose that Q is conserved along the particle's trajectory described by MPD equations and then verify what conditions this requirement implies for the tensors $\boldsymbol{k}$ and $\boldsymbol{L}$. Derivating (4.38) along the trajectory of the particle, we obtain

$$\dot{Q} = v^\alpha\left(\nabla_\alpha k_\mu p^\mu - \frac{1}{2}k_\mu R^\mu{}_{\alpha\beta\gamma}S^{\beta\gamma} + \nabla_\alpha L_{\mu\nu}S^{\mu\nu} + 2\,L_{\mu\alpha}p^\mu\right) = 0, \tag{4.39}$$

where the MPD equations have been used. Now, it is desirable to express the latter relation only in terms of the independent degrees of freedom of the spinning particle. This can be accomplished by substituting v^α by p^α and $S^{\alpha\beta}$ using (3.15). Doing this and after some algebra, we arrive at the following equation:

$$\begin{aligned}
\dot{Q} &= 4\mu^2\nabla_\alpha k_\beta p^\alpha p^\beta - (2\mu^2 k_\mu R^\mu{}_{\alpha\beta\gamma}S^{\beta\gamma}p^\alpha - 4\mu^2 p^\alpha\nabla_\alpha L_{\mu\nu}S^{\mu\nu}) \\
&- (R_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma}\nabla_\alpha k_\beta p^\alpha p^\beta + 2S^{\alpha\beta}R_{\beta\nu\rho\sigma}p^\nu S^{\rho\sigma}\nabla_\alpha k_\mu p^\mu \\
&+ 4S^{\alpha\beta}R_{\beta\nu\rho\sigma}p^\nu S^{\rho\sigma}L_{\mu\alpha}p^\mu) \\
&+ \Big(\frac{1}{2}k_\delta R^\delta{}_{\alpha\beta\gamma}S^{\beta\gamma}p^\alpha R_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma} - R_{\alpha\beta\rho\sigma}S^{\alpha\beta}S^{\rho\sigma}p^\gamma\nabla_\gamma L_{\mu\nu}S^{\mu\nu} \\
&+ S^{\alpha\gamma}R_{\gamma\nu\rho\sigma}p^\nu S^{\rho\sigma}k_\mu R^\mu{}_{\alpha\beta\gamma}S^{\beta\gamma} - 2S^{\alpha\gamma}R_{\gamma\delta\rho\sigma}p^\delta S^{\rho\sigma}\nabla_\alpha L_{\mu\nu}S^{\mu\nu}\Big) = 0,
\end{aligned} \tag{4.40}$$

In order to extract the conditions necessary to guarantee that Q is conserved, we need to replace $S^{\alpha\beta}$ by the spin vector $\tilde{S}^\alpha$ in the expression above, since $S^{\alpha\beta}$ and

p^α are not independent variables, once they must obey $S^{\alpha\beta}p_\beta = 0$. Since p^μ and $\tilde{S}^\mu$ are completely arbitrary, we see that the above equation implies four conditions on the tensors $\boldsymbol{k}$ and $\boldsymbol{L}$, once Eq. (4.40) has terms containing four different orders in these variables. These terms are independent and they need to vanish separately. The first condition, which involves only terms quadratic in momentum implies

$$\nabla_\alpha k_\beta p^\alpha p^\beta = 0\,. \tag{4.41}$$

Since the momentum p^α is arbitrary, it implies simply that $\nabla_{(\alpha}k_{\beta)} = 0$, i.e., $\boldsymbol{k}$ is a Killing vector field. Once it has been established, we can use in subsequent calculations the identity $k_\mu R^\mu{}_{\alpha\beta\gamma} = \nabla_\alpha\nabla_\beta k_\gamma$, which is valid for an arbitrary KV.

The second condition that arises from equation (4.40) is formed by terms which are quadratic in $\boldsymbol{p}$ and linear in $\tilde{\boldsymbol{S}}$, and it reads

$$\epsilon^{\mu\nu\gamma}{}_\alpha\nabla_\beta\left(L_{\mu\nu} - \frac{1}{2}\nabla_\mu k_\nu\right)p^\alpha p^\beta\tilde{S}_\gamma = 0, \tag{4.42}$$

where the fact of $\boldsymbol{k}$ being a KV has been used. Since $\boldsymbol{p}$ and $\tilde{\boldsymbol{S}}$ are completely arbitrary, it follows from (4.42) the condition

$$\epsilon^{\mu\nu\gamma}{}_{(\alpha}\nabla_{\beta)}\left(L_{\mu\nu} - \frac{1}{2}\nabla_\mu k_\nu\right) = 0. \tag{4.43}$$

Now, defining the tensor

$$Y_{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}\left(L_{\mu\nu} - \frac{1}{2}\nabla_\mu k_\nu\right), \tag{4.44}$$

it follows from Eq. (4.43) that $\boldsymbol{Y}$ obeys the equation $\nabla_{(\alpha}Y_{\beta)\nu} = 0$, i.e., it must be a Killing-Yano tensor. In other words, the second condition states that the tensor $\boldsymbol{L}$ appearing in the conserved charge should be written as $L_{\mu\nu} = \frac{1}{2}\nabla_\mu k_\nu + Y^\star_{\mu\nu}$, with $\boldsymbol{Y}^\star$ being the Hodge dual of the Killing-Yano $\boldsymbol{Y}$, that is, a CCKY.

Finally, two other conditions can still be extracted from Eq. (4.40), one being of fourth order in $\boldsymbol{p}$ and second order in $\tilde{\boldsymbol{S}}$, and the other being of fourth order in $\boldsymbol{p}$ and third order in $\tilde{\boldsymbol{S}}$. These conditions can be put in the form:

$$({}^\star R^{\star\kappa(\alpha\beta)}{}_{(\mu}g_{\nu\rho} + {}^\star R^{\star\kappa}{}_{(\mu\nu}{}^{(\alpha}\delta^{\beta)}_\rho)Y^\star_{\sigma)\kappa} = 0 \tag{4.45}$$

and

$$(R^{\star(\alpha\beta}{}_\mu g_{\nu\rho} + R^{\star\kappa}_{\mu\nu}{}^{(\alpha}\delta^\beta_\rho)\nabla_\kappa Y^{\gamma)}_\sigma = 0, \tag{4.46}$$

where we introduce the double Hodge dual as

$$ {}^{\star}R^{\star}_{\alpha\beta\gamma\delta} = \frac{1}{4}\epsilon_{\alpha\beta\alpha'\beta'}R^{\alpha'\beta'\gamma'\delta'}\epsilon_{\gamma\delta\gamma'\delta'}\,, \tag{4.47} $$

and

$$ R^{\star}_{\alpha\beta\mu\nu} = \frac{1}{2}\epsilon_{\alpha\beta\alpha'\beta'}R^{\alpha'\beta'}{}_{\mu\nu}. $$

As pointed out by Rüdiger in Ref. [25], Eqs. (4.45) and (4.46) can be further simplified. Indeed, after some algebra, one can prove that they are equivalent to the following two constraints, respectively:

$$ {}^{\star}R^{\star\sigma(\alpha\beta)}{}_{(\mu}Y^{\star}_{\nu)\sigma} - \frac{1}{6}G^{\sigma(\alpha}\delta^{\beta)}_{(\mu}Y^{\star}_{\nu)\sigma} + \frac{1}{4}{}^{\star}R^{\star\rho\sigma(\alpha}{}_{(\mu}\delta^{\beta)}_{\nu)}Y^{\star}_{\rho\sigma} = 0\,, \tag{4.48} $$

$$ J^{(\alpha}R^{\star\beta}{}_{(\mu\nu)}{}^{\gamma)} - J^{\kappa}R^{\star}_{\kappa(\mu\nu)}{}^{(\alpha}g^{\beta\gamma)} - J^{\kappa}R^{\star}_{\kappa}{}^{(\alpha\beta}{}_{(\mu}\delta^{\gamma)}_{\nu)} = 0\,, \tag{4.49} $$

where J^{α} is the divergence of $\boldsymbol{Y}^{\star}$, namely $J_{\beta} = \nabla^{\alpha}Y^{\star}_{\alpha\beta}$, whereas $G_{\mu\nu}$ stands for the Einstein tensor. Let us illustrate how we can obtain the simplified versions given above. For Eq. (4.45), we note first that we can take only three independent traces: the first involving a contraction of one up index with one down (α with μ for example), the second involving a contraction between two down indices (μ with ν for example) and the third being the contraction with the both up indices (α with β). The second and third traces are given by

$$ \begin{aligned} 4\Phi_{(\mu}{}^{\lambda}\delta^{(\alpha}_{\nu)}Y^{\star\beta)}{}_{\lambda} - 6\Phi^{(\alpha}{}_{(\mu}Y^{\star\beta)}{}_{\nu)} - 7g^{\alpha\beta}\Phi_{(\mu}{}^{\lambda}Y^{\star}_{\nu)\lambda} - g_{\mu\nu}\Phi^{\lambda(\alpha}Y^{\star\beta)}{}_{\lambda} + 2\delta^{(\alpha}_{(\mu}\Phi^{\beta)\lambda}Y^{\star}_{\nu)\lambda} \\ - 14C^{\lambda(\alpha\beta)}{}_{(\mu}Y^{\star}_{\nu)\lambda} - 3Y^{\star}_{\kappa\lambda}C^{\kappa\lambda(\alpha}{}_{(\mu}\delta^{\beta)}_{\nu)} + 2Y^{\star\lambda(\alpha}C^{\beta)}{}_{(\mu\nu)\lambda} = 0 \end{aligned} \tag{4.50} $$

and

$$ \begin{aligned} g_{\nu(\alpha}\Phi_{\beta)}{}^{\lambda}Y^{\star}_{\mu\lambda} + g_{\nu(\alpha|}\Phi_{\mu}{}^{\lambda}Y^{\star}_{|\beta)\lambda} + g_{\mu(\alpha}\Phi_{\beta)}{}^{\lambda}Y^{\star}_{\nu\lambda} + g_{\mu(\alpha|}\Phi_{\nu}{}^{\lambda}Y^{\star}_{|\beta)\lambda} \\ + g_{\mu\nu}\Phi_{(\alpha}{}^{\lambda}Y^{\star}_{\beta)\lambda} + g_{\alpha\beta}\Phi_{(\mu}{}^{\lambda}Y^{\star}_{\nu)\lambda} = 0, \end{aligned} \tag{4.51} $$

respectively. Combining the traces (4.50) and (4.51), we arrive at the simplified equation (4.48). Then, it proves that Eq. (4.45) implies Eq. (4.48). To prove now the equivalence between them, we need only to show the contrary, i.e., that Eq. (4.48) also implies Eq. (4.45). This is made by substituting the first trace along with Eq. (4.48) into Eq. (4.45) and showing that this leads to the trivial identity $0 = 0$. Therefore, (4.45) and (4.48) are equivalent expressions, and then all the content of (4.45) is contained in (4.48). Similarly, one can prove the equivalence between (4.46) and (4.49). Hereafter, the third and fourth conditions (which we will call sometimes *additional conditions*) that make Q constant are taken to be Eq.'s (4.48) and (4.49). (We call the attention for the different sign in the middle term of Eq. (4.48) in comparison with Eq. (4.9) of Ref. [25], where in the latter the sign in front of the fraction $1/6$ is positive, although the correct sign is negative, as written here. Indeed, as noted by Rüdiger himself, the additional conditions should be identically satisfied for maximally symmetric spacetimes, i.e., spacetimes whose curvature tensor has the structure

$$R_{\alpha\beta\mu\nu} = \kappa(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}), \tag{4.52}$$

which is not accomplished if the sign is positive. Thus, there must have been a typo at this point in Ref. [25]).

Summing up, assuming Tulczyjew supplementary condition $S^{\alpha\beta}p_\beta = 0$, we have proved that the most general conserved charge for MPD equations that is linear in momenta is given by

$$Q = \left(k_\mu p^\mu + \frac{1}{2}\nabla_\mu k_\nu S^{\mu\nu}\right) + Y^\star_{\mu\nu}S^{\mu\nu}, \tag{4.53}$$

where $\boldsymbol{k}$ is a Killing vector and $\boldsymbol{Y}$ is a rank two KY tensor. In addition, the constraints (4.48) and (4.49) must hold. Since $\boldsymbol{k}$ and $\boldsymbol{Y}$ are totally independent from each other and the latter constraints do not depend on $\boldsymbol{k}$,

$$Q_{\boldsymbol{k}} = k_\mu p^\mu + \frac{1}{2}\nabla_\mu k_\nu S^{\mu\nu} \text{ and } Q_{\boldsymbol{Y}} = Y^\star_{\mu\nu}S^{\mu\nu}\,,$$

are independently conserved. Indeed, it is widely known that $Q_{\boldsymbol{k}}$ is conserved for any Killing vector $\boldsymbol{k}$, as proved in section 3.1. The important result of Ref. [25] is that the scalar $Q_{\boldsymbol{Y}}$ is conserved as long as $\boldsymbol{Y}$ is a KY and conditions (4.48) and (4.49) hold. The problem is that the latter conditions are quite obscure and have not been tackled in the literature so far. The main goal of the present thesis is to shed light over the meaning of these constraints and determine the scenarios in which the conserved charge $Q_{\boldsymbol{Y}}$ is allowed to exist.

5 THE CONSERVED CHARGE Q_Y

This chapter aims to investigate if the additional conserved quantity (4.1), found by Rüdiger in Ref. [25], can be useful in spacetimes of physical interest. In other words, we plan to answer the question: which spacetimes admit a Killing-Yano tensor satisfying the additional conditions (4.48) and (4.49)? The problem of delimiting the possible spacetimes is a very interesting question since conserved charges reveal to be very useful in the integrability of the equations of motion of the particle. For example, in the integrability of the geodesic equation in Kerr spacetime for the general case (i.e., for orbits not necessarily contained in the equatorial plane), the conserved charges associated to the Killing vectors are not sufficient to integrate the trajectory of the particle, which is accomplished just if we also consider the conserved charge associated to the Killing-Yano tensor admitted by the Kerr geometry (more precisely, the conserved charge associated to the Killing tensor, which can be constructed by the square of a KY, as discussed in chapter 3) [43,44]. Therefore, if the Rüdiger's new constant exists in physically interesting spacetimes, it can be useful in the integrability of the MPD equations, as will be explained further in this chapter.

We have seen in the previous chapter that four conditions arise when we require that the scalar Q given by (4.38) be conserved. The first two conditions have very simple interpretations, namely that $\boldsymbol{k}$ must be a KV and $\boldsymbol{Y}$ must be a KY tensor, which does not happen with the last two ones. The point in this chapter is to use some mathematical tools introduced along previous chapters in order to shed light on the content of the two somewhat obscure conditions (4.48) and (4.49).

This chapter is based on our published article [26].

5.1 SPACETIMES ALLOWING THE CONSERVED CHARGE

We will use the decomposition of the Riemann tensor as given by (2.9), which writes the curvature in terms of the Weyl tensor $\boldsymbol{C}$, the traceless part of the Ricci tensor $\boldsymbol{\Phi}$ and the Ricci scalar R. We repeat the decomposition here for convenience of the reader:

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma}\Phi_{\delta]\beta} - g_{\beta[\gamma}\Phi_{\delta]\alpha} + \frac{R}{6} g_{\alpha[\gamma} g_{\delta]\beta} \,. \tag{5.1}$$

In particular, the spacetime is said to be maximally symmetric if $C_{\alpha\beta\mu\nu}$ and $\Phi_{\mu\nu}$ vanish simultaneously. We observe that each of the irreducible blocks of the above decomposition have a simple transformation with respect to the double Hodge dual, such that taking the star operation twice, we get

$$^{\star}R^{\star}_{\alpha\beta\gamma\delta} = -C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma}\Phi_{\delta]\beta} - g_{\beta[\gamma}\Phi_{\delta]\alpha} - \frac{R}{6} g_{\alpha[\gamma} g_{\delta]\beta} \,.$$

Then, using this expression along with $G_{\alpha\beta} = \Phi_{\alpha\beta} - \frac{R}{4} g_{\alpha\beta}$, it follows that the condition (4.48) can be written as

$$\begin{aligned} & C^{\kappa(\alpha\beta)}{}_{(\gamma} Y^{\star}_{\delta)\kappa} + \frac{1}{4} Y^{\star}_{\epsilon\kappa} C^{\epsilon\kappa(\alpha}{}_{(\gamma}\delta^{\beta)}_{\delta)} - \frac{1}{2}\Phi^{(\alpha}{}_{(\gamma} Y^{\star}_{\delta)}{}^{\beta)} \\ & + \frac{1}{2} g^{\alpha\beta}\Phi_{(\gamma}{}^{\kappa} Y^{\star}_{\delta)\kappa} - \frac{1}{12}\delta^{(\alpha}_{(\gamma}\Phi^{\beta)\kappa} Y^{\star}_{\delta)\kappa} - \frac{1}{4}\Phi_{(\gamma}{}^{\kappa}\delta^{(\alpha}_{\delta)} Y^{\star\beta)}{}_{\kappa} = 0. \end{aligned} \tag{5.2}$$

In the same way, condition (4.49) is written as

$$\begin{aligned} & J^{\kappa} C^{\star}_{\kappa(\mu\nu)}{}^{(\alpha} g^{\beta\gamma)} - J^{(\alpha} C^{\star\beta}{}_{(\mu\nu)}{}^{\gamma)} + \frac{1}{2} J^{\kappa}\epsilon_{(\mu|}{}^{(\alpha|}{}_{\kappa}{}^{\delta}\Phi_{\delta|\nu)} g^{|\beta\gamma)} \\ & \qquad + J^{\kappa} C^{\star}_{\kappa}{}^{(\alpha\beta}{}_{(\mu}\delta^{\gamma)}_{\nu)} - \frac{1}{2} J^{\kappa}\epsilon^{\delta}{}_{(\mu|\kappa}{}^{(\alpha}\Phi_{\delta}{}^{\beta}\delta^{\gamma)}_{|\nu)} = 0. \end{aligned} \tag{5.3}$$

In the sequel, we will clarify the content of Eqs. (5.2) and (5.3) using a null tetrad frame and the Petrov classification. For this, let us consider the two possible algebraic forms of the Killing-Yano tensor $\boldsymbol{Y}$, namely null and non-null. These possibilities will be considered separately in what follows.

5.1.1 Null Killing-Yano tensor

In what follows we will consider that the KY tensor is a null bivector, so that there exists a null frame such that $\boldsymbol{Y} = \boldsymbol{\ell} \wedge (\boldsymbol{m} + \bar{\boldsymbol{m}})$, so that its Hodge dual is $\widetilde{\boldsymbol{Y}} = i\boldsymbol{\ell} \wedge (\boldsymbol{m} - \bar{\boldsymbol{m}})$. In this case the integrability condition of the KY implies that Weyl tensor is of Petrov type N (or more special, namely O), i.e., the only Weyl scalar that can be different from zero is Ψ_4, as explained in the previous section. Hence, the Weyl tensor can be written as:

$$C_{\mu\nu\alpha\beta} = 4\Psi_4\, \ell_{[\mu} m_{\nu]} \ell_{[\alpha} m_{\beta]} + 4\bar{\Psi}_4\, \ell_{[\mu}\bar{m}_{\nu]} \ell_{[\alpha}\bar{m}_{\beta]} \,, \tag{5.4}$$

where $\bar{\Psi}_4$ stands for the complex conjugate of Ψ_4. In addition, several components of the traceless part of the Ricci tensor vanish, in accordance with Eq. (4.34). The only components that can, in principle, be different from zero are

$$\Phi_{nn}\,,\ \Phi_{nm}\,,\ \Phi_{n\bar{m}}\,,\ \Phi_{mm}\,,\ \Phi_{\bar{m}\bar{m}}\,,\ \Phi_{\ell n}\,,\ \Phi_{m\bar{m}}\,.$$

In addition, the following constraints must hold:

$$\begin{cases} \Phi_{n\bar{m}} = -\Phi_{nm}\,, \\ \Phi_{mm} = \Phi_{\bar{m}\bar{m}} = -2\Phi_{m\bar{m}} = -2\Phi_{\ell n}\,, \end{cases} \tag{5.5}$$

Thus, just three degrees of freedom are left for $\Phi_{\alpha\beta}$, namely Φ_{nn}, Φ_{nm} and $\Phi_{\ell n}$. Now, contracting Eq. (5.2) with $n_\alpha n_\beta m^\gamma m^\delta$ and $m_\alpha m_\beta m^\gamma n^\delta$ leads to $\Phi_{nm} = 0$ and $\Phi_{mm} = 0$, respectively. Then, taking Eq. (5.5) into consideration, it follows that $\Phi_{n\bar{m}}$, $\Phi_{\bar{m}\bar{m}}$, $\Phi_{\ell n}$, and $\Phi_{m\bar{m}}$ are also zero. Hence, the only component of $\Phi_{\alpha\beta}$ that can be different from zero is Φ_{nn}. Finally, contracting Eq. (5.2) with $n_\alpha n_\beta n^\gamma m^\delta$, we obtain

$$\Psi_4 + \frac{1}{2}\Phi_{nn} = 0. \tag{5.6}$$

Therefore, Φ_{nn} vanishes if, and only if, Ψ_4 vanishes. Thus, if either Φ_{nn} or Ψ_4 vanish then the spacetime is maximally symmetric, since in this case $\boldsymbol{C}$ and $\boldsymbol{\Phi}$ both vanish, in which case the conserved quantity $Q_{\boldsymbol{Y}}$ is useless, once the Killing vectors are sufficient for the integrability of MPD equations [45]. In particular, if the spacetime is Einstein, i.e., if $\Phi_{\alpha\beta}$ vanish identically, then Ψ_4 vanishes and we have the trivial case.

Concerning the condition (5.3), contracting it with $m_\alpha m_\beta m_\gamma n^\mu \bar{m}^\nu$, we obtain $J_\ell \Psi_4 = 0$, where it has been used that Ψ_4 is real, which is a consequence of Eq. (5.6). Similarly, contracting with $n_\alpha n_\beta n_\gamma \bar{m}^\mu \bar{m}^\nu$, $n_\alpha m_\beta n_\gamma \ell^\mu m^\nu$ and $m_\alpha m_\beta m_\gamma n^\mu n^\nu$ implies that $J_n \Psi_4 = 0$, $(J_m + J_{\bar{m}})\Psi_4 = 0$ and $J_m \Psi_4 = 0$, respectively. Therefore, the constraint (5.3) leads to

$$J_\alpha \Psi_4 = 0\,, \tag{5.7}$$

meaning that either $\Psi_4 = 0$, which again lead to the trivial case of a maximally symmetric spacetime, or $J_\alpha = 0$, which means that $\boldsymbol{Y}$ is covariantly constant. Indeed, the KY equation can equivalently be written as $\nabla_\alpha Y_{\mu\nu} = \nabla_{[\alpha} Y_{\mu\nu]}$. Thus, if J_α vanishes it follows that $\nabla^\alpha Y^\star_{\alpha\beta} = 0$, which is equivalent to the condition $\nabla_{[\alpha} Y_{\mu\nu]} = 0$, which implies that $\boldsymbol{Y}$ is covariantly constant.

However, if $\boldsymbol{Y}$ is covariantly constant so is its Hodge dual $\boldsymbol{Y}^\star$. Particularly, this implies that $\boldsymbol{Y}^\star$ is also a KY, so that it is reasonable to suppose that the scalar $Q_{\boldsymbol{Y}^\star}$ is conserved, although this is not a necessary requirement as it is independent from the requirement that $Q_{\boldsymbol{Y}}$ is conserved. Nevertheless, if besides the conservation of $Q_{\boldsymbol{Y}}$ we also assume that $Q_{\boldsymbol{Y}^\star}$ is conserved, it follows that the condition (5.2) must also hold if we replace $\boldsymbol{Y}$ by $\boldsymbol{Y}^\star$. Performing this replacement and then contracting Eq. (5.2) with $n_\alpha n_\beta n^\gamma \bar{m}^\delta$, we arrive at the equation:

$$\Psi_4 - \frac{1}{2}\Phi_{nn} = 0. \tag{5.8}$$

Composing Eqs. (5.6) and (5.8) lead us to the conclusion that Ψ_4 and $\Phi_{nn} = 0$, which then imply that the spacetime is maximally symmetric, in which case the conserved charges are useless. This can be understood as a consequence of the fact that in these spacetimes the number of independent Killing vectors is ten, which lead to ten conserved charges of type $Q_{\boldsymbol{k}}$, which in turn are enough to obtain expressions for the ten unknowns $\boldsymbol{p}$ and $\boldsymbol{S}$ in terms of the initial conditions of the particle. In addition, since we are using the supplementary condition $S^{\alpha\beta} p_\beta = 0$, we can use Eq. (3.15) to obtain also an expression for the velocity $\boldsymbol{v}$ of the particle, which can in principle be integrated in order to give the trajectory of the particle. In fact, in Ref. [45], the complete integrability of the MPD equations was explicitly obtained for the de Sitter spacetime. Thus, we can say that the conservation of $Q_{\boldsymbol{Y}}$ is somehow trivial for maximally symmetric spacetimes. At the end of this chapter, we will seek for some non-trivial examples where the conditions (4.48) and (4.49) hold.

Summing up, in order for the conserved charge $Q_{\boldsymbol{Y}}$ be nontrivial for the case of a KY tensor whose algebraic type is null, the Weyl tensor must be Petrov type N and the only component of $\Phi_{\alpha\beta}$ that can be different from zero is Φ_{nn}. In addition, the KY must be covariantly constant, which is a very restrictive hypothesis, since a very narrow class of spaces admit covariantly constant bivectors [46,47]. Due to the latter fact, it follows that $\boldsymbol{Y}^\star$ is also a KY. If we further impose that $Q_{\boldsymbol{Y}^\star}$ is conserved, in addition to $Q_{\boldsymbol{Y}}$, we conclude that the spacetime is maximally symmetric and the conserved charges are useless.

5.1.2 Non-null Killing-Yano tensor

Now, let us assume that the Killing-Yano tensor is non-null, which means that there exists some null frame such that $\boldsymbol{Y} = f\,\boldsymbol{\ell}\wedge\boldsymbol{n} + ih\,\boldsymbol{m}\wedge\bar{\boldsymbol{m}}$, where f and h are real functions that cannot vanish simultaneously. The Hodge dual of the KY is then given by $\boldsymbol{Y}^\star = h\,\boldsymbol{\ell}\wedge\boldsymbol{n} - if\,\boldsymbol{m}\wedge\bar{\boldsymbol{m}}$. As discussed in Sec.4.3, in this case the integrability condition of the KY implies that the Weyl tensor is Petrov type D, that is, the only Weyl scalar that can be different from zero is Ψ_2, so that, from Eq. (4.17), the Weyl tensor can be written as follows:

$$\begin{aligned} C_{\mu\nu\alpha\beta} &= (\Psi_2 + \bar{\Psi}_2)\left(\ell_{[\mu}n_{\nu]}\ell_{[\alpha}n_{\beta]} + m_{[\mu}\bar{m}_{\nu]}m_{[\alpha}\bar{m}_{\beta]}\right) \\ &\quad - (\Psi_2 - \bar{\Psi}_2)\left(\ell_{[\mu}n_{\nu]}m_{[\alpha}\bar{m}_{\beta]} + m_{[\mu}\bar{m}_{\nu]}\ell_{[\alpha}n_{\beta]}\right) \\ &\quad - \Psi_2\left(\ell_{[\mu}m_{\nu]}n_{[\alpha}\bar{m}_{\beta]} + n_{[\mu}\bar{m}_{\nu]}\ell_{[\alpha}m_{\beta]}\right) \\ &\quad - \bar{\Psi}_2\left(\ell_{[\mu}\bar{m}_{\nu]}n_{[\alpha}m_{\beta]} + n_{[\mu}m_{\nu]}\ell_{[\alpha}\bar{m}_{\beta]}\right)\,. \end{aligned} \tag{5.9}$$

In addition, the following components of the trace-free part of the Ricci tensor must vanish due to the fact that $\boldsymbol{Y}$ is a KY, as discussed in chapter 4:

$$\Phi_{\ell m} = \Phi_{\ell\bar{m}} = \Phi_{nm} = \Phi_{n\bar{m}} = 0\,. \tag{5.10}$$

Now, taking Eqs. (5.9) and (5.10) into consideration, we are ready to analyse Eq. (5.2), which is necessary for $Q_{\boldsymbol{Y}}$ to be conserved. Contracting (5.2) with $n_\alpha n_\beta n^\gamma \ell^\delta$, $\ell_\alpha \ell_\beta \ell^\gamma n^\delta$, $m_\alpha m_\beta m^\gamma \bar{m}^\delta$, and $\bar{m}_\alpha \bar{m}_\beta \bar{m}^\gamma m^\delta$ we obtain, respectively:

$$\Phi_{nn} = 0\,,\ \Phi_{\ell\ell} = 0\,,\ \Phi_{mm} = 0\,,\ \Phi_{\bar{m}\bar{m}} = 0\,. \tag{5.11}$$

Since the trace-free condition obeyed by $\boldsymbol{\Phi}$ means that $\Phi_{\ell n} = \Phi_{m\bar{m}}$, it follows that both components $\Phi_{\ell n}$ and $\Phi_{m\bar{m}}$ represent the same degree of freedom. Thus, from Eqs. (5.10) and (5.11) one concludes that only one degree of freedom of $\boldsymbol{\Phi}$ can be different from zero, which we take as $\Phi_{\ell n}$.

Then, contracting Eq. (5.2) with $n_\alpha n_\beta \ell^\gamma \ell^\delta$ and $m_\alpha m_\beta \bar{m}^\gamma \bar{m}^\delta$, we arrive at the following relations, respectively:

$$\begin{aligned} h\,Re\{\Psi_2\} + f\,Im\{\Psi_2\} + \frac{1}{3}h\,\Phi_{n\ell} &= 0\,, \\ -f\,Re\{\Psi_2\} + h\,Im\{\Psi_2\} + \frac{1}{3}f\,\Phi_{m\bar{m}} &= 0\,. \end{aligned} \tag{5.12}$$

Finally, using $\Phi_{n\ell} = \Phi_{m\bar{m}}$, we conclude that

$$\Psi_2 = \frac{1}{3}\,\frac{f - ih}{f + ih}\,\Phi_{n\ell}\,. \tag{5.13}$$

Thus, if the spacetime is Einstein, i.e., if $\Phi_{\alpha\beta} = 0$, then Ψ_2 vanishes. Therefore, since in this case Ψ_2 is the unique Weyl scalar that can be different from zero, then we conclude that the whole Weyl tensor vanishes. Hence, if the spacetime is Einstein it will also be conformally flat (Weyl tensor zero) and these two conditions means that the spacetime is maximally symmetric, so that the conserved quantity $Q_{\boldsymbol{Y}}$ is trivial.

Now, considering Eq. (5.3), one eventually arrive at

$$J_\alpha\,\Psi_2 = 0\,,$$

where Eq. (5.13) has been used. Hence, either the space is maximally symmetric (if $\Psi_2 = 0$, which then implies $\Phi_{\alpha\beta} = 0$), or the KY is covariantly constant (if $J_\alpha = 0$). Thus, the only non-trivial case in which $Q_{\boldsymbol{Y}}$ is conserved for a non-null KY is when this tensor is covariantly constant, the Weyl tensor is Petrov type D and with the only nonvanishing components of $\Phi_{\alpha\beta}$ being $\Phi_{\ell n} = \Phi_{m\bar{m}}$. Furthermore, the relation between Ψ_2 and $\Phi_{\ell n}$ given in Eq. (5.13) must hold. These are quite restrictive conditions.

Now, since $\boldsymbol{Y}$ is covariantly constant, it follows that its Hodge dual is also a KY. Then, we can require that $Q_{\boldsymbol{Y}^\star}$ is also conserved along the trajectories described by the MPD equations, but it is important to keep in mind that this actually is an independent requirement. Comparing the expressions for $\boldsymbol{Y}$ and $\boldsymbol{Y}^\star$,

$$\begin{cases} \boldsymbol{Y} = f\,\boldsymbol{\ell}\wedge\boldsymbol{n} + ih\,\boldsymbol{m}\wedge\bar{\boldsymbol{m}} \\ \boldsymbol{Y}^\star = h\,\boldsymbol{\ell}\wedge\boldsymbol{n} - if\,\boldsymbol{m}\wedge\bar{\boldsymbol{m}} \end{cases},$$

we note that $\boldsymbol{Y}^\star$ can be obtained from $\boldsymbol{Y}$ by making the changes $f \to h$ and $h \to -f$. Thus, since Eq. (5.13) must hold in order to guarantee that $Q_{\boldsymbol{Y}}$ is conserved, it follows that the analogous condition

$$\Psi_2 = \frac{1}{3}\,\frac{h + if}{h - if}\,\Phi_{n\ell} \tag{5.14}$$

must hold in order to assure the conservation of $Q_{\boldsymbol{Y}^\star}$. Hence, assuming that the scalars $Q_{\boldsymbol{Y}}$ and $Q_{\boldsymbol{Y}^\star}$ are both conserved, it follows that Eqs. (5.13) and (5.14) are satisfied simultaneously. Equating both expressions for Ψ_2 and assuming that $\Phi_{\ell n} \neq 0$, so that the spacetime is nontrivial, lead us to the condition

$$\frac{h+if}{h-if} = \frac{f-ih}{f+ih} \;\Rightarrow\; f^2 + h^2 = 0\,.$$

Since f and h are real functions, the unique solution for the latter constraint turns out to be the trivial one, $f = h = 0$, which is unacceptable, since by hypothesis $\boldsymbol{Y}$ is a nonvanishing KY. Thus, we conclude that the only case in which $Q_{\boldsymbol{Y}}$ and $Q_{\boldsymbol{Y}^\star}$ are both conserved is when $\Phi_{\ell n} = 0$, which then implies $\Psi_2 = 0$. This means that the spacetime is maximally symmetric and, therefore, the conserved scalars of interest have no applicability.

Therefore, we have proved that, when the conditions (4.48) and (4.49) are analysed together, they provide very restrictive impositions, such that just a very narrow class of spacetimes obey them. In particular, we have mentioned that for maximally symmetric spacetimes, i.e., Minkowski, de Sitter and anti-de Sitter spacetimes, the conserved charge $Q_{\boldsymbol{Y}}$ is useless.

5.1.3 Physical restrictions by energy conditions

As we have seen, the integrability conditions for the KY along with the additional conditions required for $Q_{\boldsymbol{Y}}$ be conserved impose huge restrictions over the Weyl and Ricci tensors. In the present subsection, we are interested in physical restrictions that may come from the so-called *energy conditions* [48]. Energy conditions are inequalities satisfied by reasonable energy-momentum tensors. Then, in this subsection we shall make use of Einstein's equation in order to convert the restrictions we have over the Ricci tensor onto restrictions over the energy-momentum tensor of the background. More precisely, we shall analyse the implications of the weak energy condition (WEC) over the Ricci tensor. In what follows, we will assume that the spacetime is not maximally symmetric, which means that we are requiring that just $Q_{\boldsymbol{Y}}$ is conserved, while $Q_{\boldsymbol{Y}^\star}$ is not a conserved scalar, otherwise $\Phi_{\mu\nu}$ would vanish identically, as discussed in the previous section. In addition, we also assume that the KY exists.

In suitable units, Einstein's equation reads $G_{\mu\nu} = T_{\mu\nu}$, where $T_{\mu\nu}$ is the energy-momentum tensor of the background matter. This can be equivalently written as

$$T_{\mu\nu} = \Phi_{\mu\nu} - \frac{R}{4} g_{\mu\nu}\,.$$

The weak energy condition then amounts to the constraint $T_{\mu\nu}Z^\mu Z^\nu \geq 0$ for any timelike vector field Z^μ, which means that the energy density of the matter is not negative as measured by an arbitrary observer. Writing the vector field $\boldsymbol{Z}$ in terms of the null tetrad frame we have

$$\boldsymbol{Z} = Z_n\,\boldsymbol{\ell} + Z_\ell\,\boldsymbol{n} - \, Z_{\bar{m}}\,\boldsymbol{m} - \, Z_m\,\bar{\boldsymbol{m}}.$$

The WEC then reads

$$\Phi_{\mu\nu}Z^\mu Z^\nu - \frac{R}{2}\left(Z_n Z_\ell - Z_m Z_{\bar{m}}\right) \geq 0\,, \tag{5.15}$$

for any vector $\boldsymbol{Z}$ such that $Z_n Z_\ell > Z_m Z_{\bar{m}}$. Since most of the components of $\Phi_{\mu\nu}$ vanish when $Q_{\boldsymbol{Y}}$ is conserved, the above restriction becomes simpler to be analysed. In what follows let us consider the two possible algebraic types of the KY separately.

When the KY is null, the only component of $\Phi_{\mu\nu}$ that can be different from zero is Φ_{nn}, so that Eq. (5.15) becomes

$$\Phi_{nn} Z_\ell Z_\ell - \frac{R}{2}\left(Z_n Z_\ell - Z_{\bar{m}} Z_m\right) \geq 0\,.$$

Defining $\zeta \equiv \left(Z_n Z_\ell - Z_{\bar{m}} Z_m\right)/(Z_\ell^2)$, it follows that the timelike condition reads $\zeta > 0$, so that the WEC becomes

$$\Phi_{nn} \geq \frac{R}{2}\zeta\,, \quad \text{for all } \zeta > 0\,.$$

This is possible only if $\Phi_{nn} \geq 0$ and $R \leq 0$. Thus, besides the geometrical restrictions found in subsection 5.1.1, there exists the physical restriction that the Ricci scalar cannot be positive whereas the component Φ_{nn} cannot be negative. Otherwise the background spacetime is not generated by a physically reasonable matter.

Now, let us consider that the KY has a non-null algebraic type, in which case the only components of $\Phi_{\mu\nu}$ that can be different from zero are $\Phi_{\ell n} = \Phi_{m\bar{m}}$, so that Eq. (5.15) becomes

$$\Phi_{\ell n}(Z_\ell Z_n + Z_{\bar{m}} Z_m) - \frac{R}{4}\left(Z_n Z_\ell - Z_{\bar{m}} Z_m\right) \geq 0\,.$$

Since the timelike condition for $\boldsymbol{Z}$ reads

$$Z_n Z_\ell > Z_m Z_{\bar{m}} = |Z_m|^2 ,$$

it follows that $Z_n Z_\ell$ is positive and, therefore, defining $\xi \equiv (Z_n Z_\ell - |Z_m|^2)/(Z_n Z_\ell + |Z_m|^2)$, it follows that ξ is positive, so that the WEC for spacetimes with conserved $Q_{\boldsymbol{Y}}$ for a non-null KY is given by

$$\Phi_{\ell n} \geq \frac{R}{4}\,\xi\,, \quad \text{for all } \xi > 0\ .$$

This, in turn, implies that $\Phi_{\ell n}$ cannot be negative and the Ricci scalar cannot be positive.

5.2 LOOKING FOR EXPLICIT EXAMPLES

The aim of the present section is to find non-trivial examples of spacetimes obeying the several restrictions necessary to assure the conservation of the scalar $Q_{\boldsymbol{Y}}$. We shall start considering the case in which the KY is null and after this we consider the non-null case.

5.2.1 An example with a null KY

As argued in Sec. 4.3, when the algebraic type of the KY is null the Weyl tensor must be Petrov type N. A well known class of type N spacetimes is given by the so-called *pp*−wave metrics [49]. The *pp*−wave spacetimes (or plane-fronted waves with parallel rays) are a family of solutions of Einstein's field equations whose metric, in some coordinate system, is given by

$$ds^2 = 2F(u,z,\bar{z})\,du^2 + 2\,dudr - 2\,dz\,d\bar{z}\,, \tag{5.16}$$

where u and r are real coordinates, whereas z is a complex coordinate with $\bar{z}$ being its complex conjugate. The function F is an arbitrary real function of the coordinates u, z and $\bar{z}$. These spacetimes can also be defined more elegantly through a coordinate-free definition by establishing that they are all spacetimes admitting a covariantly constant null vector field $\boldsymbol{V}$, i.e., there is a null V^α such that

$$\nabla_\alpha V_\beta = 0. \tag{5.17}$$

A null tetrad frame for the spacetime (5.16) is given by

$$\boldsymbol{\ell} = \partial_r\,,\ \boldsymbol{n} = \partial_u - F\partial_r\,,\ \boldsymbol{m} = \partial_z\,,\ \bar{\boldsymbol{m}} = \partial_{\bar{z}}\,. \tag{5.18}$$

Indeed, it is easy to check that all the inner products (4.14) are satisfied by (5.18). The null vector field $\boldsymbol{\ell}$ is the covariantly constant vector that characterizes the *pp*−wave spacetime. In the null frame (5.18), the unique Weyl scalar that is different from zero is

$$\Psi_4 = -\partial_{\bar{z}}\partial_{\bar{z}}F\,, \tag{5.19}$$

whereas the only component of the Ricci tensor that is different from zero, in this null frame, is

$$R_{nn} = \Phi_{nn} = 2\partial_z\partial_{\bar{z}}F\,.$$

The null bivector $\boldsymbol{Y} = \boldsymbol{\ell}\wedge(\boldsymbol{m}+\bar{\boldsymbol{m}})$ is covariantly constant and, therefore, is also a KY. Thus, out of the restrictions necessary in order to be $Q_{\boldsymbol{Y}}$ conserved, the only that remains to be imposed is the one given in Eq. (5.6), namely $\Psi_4 + \frac{1}{2}\Phi_{nn} = 0$. Imposing the latter equation, we arrive at the partial differential equation

$$\partial_{\bar{z}}\partial_{\bar{z}}F(u,z,\bar{z}) = \partial_z\partial_{\bar{z}}F(u,z,\bar{z}), \tag{5.20}$$

whose general solution is given by

$$F(u,z,\bar{z}) = F_1(u,z+\bar{z}) + F_2(u,z)\,, \tag{5.21}$$

where F_1 and F_2 are general real functions of their arguments. In addition, we observe that if we take the complex conjugate of Eq. (5.6), it follows that Ψ_4 must be a real function. Indeed, since the Ricci tensor and the null vector $\boldsymbol{n}$ are real, it follows that the complex conjugate of (5.6) should produce $\bar{\Psi}_4 + \frac{1}{2}\Phi_{nn} = 0$, whose comparison with (5.6) lead to $\Psi_4 = \bar{\Psi}_4$. Therefore, from this reality condition and using Eq. (5.19), it follows that

$$\Psi_4 = \bar{\Psi}_4 \ \Rightarrow\ \partial_{\bar{z}}\partial_{\bar{z}}F = \partial_z\partial_z F\,.$$

This condition, along with Eq. (5.21) implies that the function F must have the form

$$F(u,z,\bar{z}) = F_3(u,z+\bar{z})\,, \tag{5.22}$$

where F_3 is an arbitrary real function of u and $z+\bar{z}$. This choice of function F leads to the most general $pp-$wave spacetime such that the scalar

$$Q_{\backslash Y} = S^{\mu\nu} Y^{\star}_{\mu\nu} = 2i\left(S_{\ell m} - S_{\ell\bar{m}}\right) = 2i\left(S_{rz} - S_{r\bar{z}}\right)$$

is conserved along the trajectories of the MPD equations, where in the last equality it has been used that $\boldsymbol{Y}$ is the bivector $\partial_r \wedge (\partial_z + \partial_{\bar{z}})$.

However, it turns out that the bivector $\boldsymbol{Y}^{\star} = i\partial_r \wedge (\partial_z - \partial_{\bar{z}})$ is also a KY (actually, it is covariantly constant). Imposing the scalar $Q_{\boldsymbol{Y}^{\star}}$ to be conserved we would find from Rüdiger's conditions that the function F appearing in the line element should have the form

$$F(u,z,\bar{z}) = F_4(u, z-\bar{z})\,. \tag{5.23}$$

Note that Eqs. (5.22) and (5.23) hold simultaneously only if F is a function of u alone, $F=F(u)$, in which case the spacetime would be maximally symmetric, in agreement with what has been obtained in Sec. 5.1.1 when the constancy of $Q_{\boldsymbol{Y}}$ and $Q_{\boldsymbol{Y}^{\star}}$ are imposed simultaneously.

5.2.2 Looking for an example with a non-null KY

Unfortunately, it is not known in literature the most general metric whose associated Weyl tensor is Petrov type D and admitting a covariantly constant bivector $\boldsymbol{Y}$. However, it is available the most general type D metric possessing a KY and two commuting Killing vector fields [50]. These spacetimes are physically relevant since every star in the universe, once it reaches the equilibrium regime, should generate a stationary and axisymmetric gravitational field, which means that its associated metric possess the two Killing vectors ∂_t and ∂_φ. In particular, Kerr metric is a member of this class of spacetimes. As presented in Ref. [50], the most general metric possessing these properties is given by

$$ds^2 = S(x,y)\left[\frac{A_2(y)\Delta_2}{(x^2+y^2)^2}(dt + x^2 d\varphi^2)^2 - \frac{dy^2}{\Delta_2} - \frac{A_1(x)\Delta_1}{(x^2+y^2)^2}(dt - y^2 d\varphi^2)^2 - \frac{dx^2}{\Delta_1}\right], \tag{5.24}$$

where Δ_1 and Δ_2 are arbitrary functions whereas $A_1(x)$, $A_2(y)$ and $S(x,y)$ are functions given by

$$\begin{aligned} A_1(x) =& \frac{x^2}{(b_1x^2+\eta_1)(b_2x^2+\eta_2)}\,, \\ A_2(y) =& \frac{y^2}{(\eta_1-b_1y^2)(b_2y^2-\eta_2)}\,, \\ S(x,y) =& \frac{b_3x^2+\eta_3}{b_1x^2+\eta_1}+\frac{b_3y^2-\eta_3}{\eta_1-b_1y^2}\,, \end{aligned}$$

with the b's and η's being arbitrary constants. For the metric (5.24), we define the following null tetrad frame:

$$\begin{aligned} \boldsymbol{\ell} &= \frac{1}{\sqrt{2S\Delta_2}}\left(\frac{y^2}{\sqrt{A_2}}\partial_t+\frac{1}{\sqrt{A_2}}\partial_\varphi-\Delta_2\partial_y\right)\,, \\ \boldsymbol{n} &= \frac{1}{\sqrt{2S\Delta_2}}\left(\frac{y^2}{\sqrt{A_2}}\partial_t+\frac{1}{\sqrt{A_2}}\partial_\varphi+\Delta_2\partial_y\right)\,, \\ \boldsymbol{m} &= \frac{1}{\sqrt{2S\Delta_1}}\left(\frac{x^2}{\sqrt{A_1}}\partial_t-\frac{1}{\sqrt{A_1}}\partial_\varphi+i\,\Delta_1\partial_x\right)\,, \\ \bar{\boldsymbol{m}} &= \frac{1}{\sqrt{2S\Delta_1}}\left(\frac{x^2}{\sqrt{A_1}}\partial_t-\frac{1}{\sqrt{A_1}}\partial_\varphi-i\,\Delta_1\partial_x\right)\,, \end{aligned}$$

such that $g_{\alpha\beta}=2\ell_{(\alpha}n_{\beta)}-2m_{(\alpha}\bar{m}_{\beta)}$. The KY is given by

$$\boldsymbol{Y}=f(x)\boldsymbol{\ell}\wedge\boldsymbol{n}+ih(y)\boldsymbol{m}\wedge\bar{\boldsymbol{m}}\,,$$

where

$$f(x)=-\sqrt{\frac{b_2x^2+\eta_2}{b_1x^2+\eta_1}}\quad\text{and}\quad h(y)=\sqrt{\frac{b_2y^2-\eta_2}{\eta_1-b_1y^2}}$$

Using this frame it follows that the only Weyl scalar that is different from zero is Ψ_2, whereas the components of $\Phi_{\mu\nu}$ all vanish apart from $\Phi_{\ell n}$ and $\Phi_{m\bar{m}}$, with $\Phi_{\ell n}=\Phi_{m\bar{m}}$ due to the traceless condition for $\boldsymbol{\Phi}$. Then, the only constraints that remain to be imposed in order to assure that $Q_{\boldsymbol{Y}}$ is conserved along the solutions of the MPD equations are Eq. (5.13), which connects Ψ_2 and $\Phi_{\ell n}$, and the requirement that $\boldsymbol{Y}$ must be covariantly constant. In particular, imposing the latter constraint we find that either $b_3/\eta_3=b_1/\eta_1$ or $b_2/\eta_2=b_1/\eta_1$. If we assume that $b_3/\eta_3=b_1/\eta_1$, it leads to a vanishing function $S(x, y)$ and, therefore, a vanishing metric. This is the reason why we eliminate this possibility. Thus, we must have $b_1/\eta_1 = b_2/\eta_2$. However, in this case either A_1 or A_2 becomes negative, so that the signature ceases to be Lorentzian, i.e., the spacetime is nonphysical. Thus, for the broad class of spacetimes considered in this case, there exists no example in which the scalar $Q_{\boldsymbol{Y}}$ is conserved along the solutions of the MPD equations.

6 CONCLUSIONS AND PERSPECTIVES

In this work, we were interested in studying conserved quantities for spinning particles whose motion is described by the MPD equations. We demonstrated that the conserved charge found by Rüdiger has no applicability for the majority of the physically interesting spacetimes. We have shown that Rüdiger's additional conditions imply that the spacetimes admitting the conserved charge $Q_{\boldsymbol{Y}} = Y^{\star}_{\mu\nu} S^{\mu\nu}$ must possess a covariantly constant Killing-Yano tensor, a fact that certainly decreases significantly the spacetimes allowed. In the search of explicit examples of spacetimes satisfying all the conditions necessary in order to make $Q_{\boldsymbol{Y}}$ conserved, we have found a specific subclass of spacetimes within the *pp*-wave spacetimes, where the function F which characterizes the spacetime in some coordinate system (see Eq. (5.16)) must have a specific form. In other words, even for the *pp*-wave spacetimes (a very specific class of spacetimes), we have found that $Q_{\boldsymbol{Y}}$ is not conserved (does not satisfy all Rüdiger's conditions), except if the function F has a specific form, which restricts even more this class of spacetimes. This leads us to the conclusion that the new charge $Q_{\boldsymbol{Y}}$ can hardly be helpful for the integrability of MPD equations.

Besides these mathematical restrictions, we have shown that we can impose further restrictions on the spacetime coming from a physical origin, by means of the so-called energy conditions.

In addition, we have seen that if we make an additional requirement, namely, that the scalar $Q_{\boldsymbol{Y}^{\star}}$ be also conserved, the conclusion is even more restrictive, implying that the unique spacetimes satisfying Rüdiger conditions are maximally symmetric spacetimes. We note, however, that even without imposing this additional requirement, the restrictions on the spacetime are already very strong.

As perspective, we can explore conserved charges for massless spinning particles. As mentioned in chapter 3, for massless particles, we were able to find a conserved charge involving a conformal Killing vector in this theory. This charge can be helpful in the integrability of MPD equations for the massless case. In fact, it is known that conformally flat spacetimes possess the maximal number of conformal Killing vector fields. For four-dimensional spacetimes, this number is fifteen, which in principle allow for the complete integrability of MPD equations. As an example, by counting degrees of freedom, we know that it is possible in principle to achieve the complete integrability of MPD equations studying massless particles in FLRW spacetime, which is an interesting space with applications in Cosmology.

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