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Álvaro Luiz Domingues Guimarães Filho

On Spinors, Twistors and Six-Dimensional General Relativity

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Prof. Bruno Geraldo Carneiro da Cunha
Supervisor
(DF-UFPE)

Prof. Carlos Alberto Batista da Silva Filho
Internal Examiner
(DF-UFPE)

Prof. Amílcar Rabelo de Queiroz
External Examiner
(UAF-UFCG)

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ABSTRACT

In this dissertation we review the article "On the Six-dimensional Kerr Theorem and Twistor Equation" [1], introducing advanced topics from general relativity along the way. We start by introducing how the concept of a spinor appears in physics from the simple requirement of Lorentz invariance on a field theory that would describe the electron. We then introduce ideas from representation theory so that we can approach the spinor concept from a more mathematical point of view and with that know its properties so we can apply the formalism as we wish. We then show the correspondence between tensors and spinors in four dimensions and rewrite General Relativity in terms of spinors, and we see that this allows us to simplify several equations in terms of the Newman-Penrose formalism. The concept of congruences is introduced, and its study in turn leads us to the Kerr Theorem and the notion of a twistor. Then, we move on to six-dimensional space-time, where we see that the correspondence between tensors and spinors in six dimensions is quite different than the one we described for the four-dimensional case, but group theory allows us to easily obtain the spinor form of the desired tensors. The Kerr Theorem is obtained in six dimensions, and its generalization to generic even dimensions is discussed. We then study the six-dimensional twistor equation and see that it imposes an algebraic constraint on the form of the Weyl tensor, and a family of examples that exhibits this property is discussed.

Keywords: General Relativity. Spinors. Kerr Theorem. Twistors.

RESUMO

Nessa dissertação, revisamos o artigo "On the Six-dimensional Kerr Theorem and Twistor Equation" [1], introduzindo tópicos avançados de Relatividade Geral ao longo do caminho. Começamos mostrando como o conceito de espinor aparece na física a partir da necessidade por invariância de Lorentz numa teoria de campo que então descreveria o elétron. Nós então introduzimos conceitos de teoria de representação para que possamos abordar espinores de um ponto de vista mais matemático, e com isso obter suas propriedades e aplicar o formalismo da maneira desejada. Mostramos então a correspondência entre tensores e espinores em quatro dimensões e reescrevemos a Relatividade Geral em termos de espinores, e vemos que isso nos permite simplificar diversas equações em termos do formalismo de Newman-Penrose. O conceito de congruências é introduzido, e seu estudo então nos leva ao teorema de Kerr e à noção de um twistor. Nós então seguimos para o espaço-tempo de seis dimensões, onde vemos que a correspondência entre tensores e espinores em seis dimensões é um tanto diferente da estudada no caso anterior, mas a teoria de grupo nos permite obter a forma espinorial do tensores desejados facilmente. O teorema de Kerr é obtido em seis dimensões , e sua generalização para dimensões pares genéricas é discutida. Estudamos então a equação do twistor em seis dimensões e vemos que ela impõe uma limitação algébrica na forma do tensor de Weyl, e encerramos com uma discussão sobre uma família de exemplos que exibe essa propriedade.

Palavras-chave: Relatividade Geral. Espinores. Teorema de Kerr. Twistors.

CONTENTS

1 INTRODUCTION 10

2 SPINORS FROM PHYSICS 12
2.1 The Dirac Equation 12
2.2 Group Theory 14
2.2.1 Lie Groups 15
2.2.2 Lie Algebra 15
2.3 SU(2) 16
2.3.1 The Generators of SU(2) 16
2.3.2 The $j = \frac{1}{2}$ representation of SU(2) 17
2.4 The Lorentz Group 18
2.4.1 Representations of the Lorentz Group 19
2.4.1.1 *The $(0, 0)$ Representation* 19
2.4.1.2 *The $(\frac{1}{2}$,0) Representation* 20
2.4.1.3 *The $(0, \frac{1}{2})$ Representation* 20
2.4.1.4 *The Relation Between Spinor Representations* 21
2.4.1.5 *The $(\frac{1}{2}, \frac{1}{2})$ Representation* 22
2.4.2 The Spinor on the Dirac equation 23
2.5 The n Dimensional Vector Representations 24
2.6 Clifford Algebra and the n Dimensional Spinor Representations . . . 24

3 SPINORS IN GENERAL RELATIVITY 27
3.1 Spinor Algebra 27
3.1.1 Spin Space 27
3.1.2 The connection between tensors and spinors 28
3.1.3 The Levi-Civita spinor 28
3.2 General Relativity with Spinors 30
3.2.1 The Covariant Derivative 30
3.2.2 The Riemann Tensor 31
3.2.3 The Ricci Tensor 32
3.2.4 The Ricci Scalar 33
3.2.5 Einstein Field Equations 33
3.3 Topics from General Relativity 33
3.3.1 The Newman-Penrose Formalism 33
3.3.2 Null Shear-Free Congruences 35
3.3.3 The Kerr Theorem and Twistors 37

4 SIX DIMENSIONAL KERR THEOREM 40
4.1 Spinors on Six Dimensions 40
4.1.1 Higher dimensional irreducible representations of SU(4) 41
4.1.2 Null Vectors 43
4.2 The Kerr Theorem in 6 dimensions 45
4.3 The Kerr Theorem in Generic Even Dimensions 47
4.4 Isotropic Subspaces 48
4.5 Six-Dimensional Twistors 48
4.6 Symmetries and Quaternions 50
4.7 The Kerr-Schild Class of Solutions 51

5 CONCLUSIONS 53

REFERENCES 54

1 INTRODUCTION

The field of General Relativity is over a 100 years old by now, and it showed us beautiful new physics, like gravitational waves and black holes, and also solved old mysteries such as the precession of the perihelion of Mercury as well as explaining facts about the large-scale universe through cosmology [2]. Despite its many successes, it was a difficult theory to deal with from the beginning, and it remains so, since in four dimensions it is described by a set of 10 non-linear partial differential equations describing the dynamics of the space-time itself. Obtaining new solutions corresponding to different physical systems would prove to be very difficult, and new mathematical tools would be needed to analyze the integrability aspects of the theory as well as solving the equations for given conditions. Many of these new tools were introduced by Roger Penrose, which in the 60's used spinors in the study of General Relativity [3] and also introduced the concept of a twistor [4] as a possible candidate of a way to quantize gravity [5]. Along with that, with the advent of String Theory, the interest in understanding the forces of nature in more dimensions than four rose [6, 7], which again poses the problem of choosing the appropriate tools to deal with the theory in this setting.

The objective of this dissertation is twofold: to present a review of [1] explaining the methods presented therein, and also to give a brief introduction to some methods and tools of General Relativity that are more advanced than those encountered in a first course, such as group theory, spinors, the Newman-Penrose tetrad formalism and twistors. Those are all well introduced in the great pioneering books by Roger Penrose and Wolfgang Rindler [8,9], but not always in a very comprehensive manner. So, for the reader interested in these tools, this dissertation also serves as a collection of references that trail the easiest path.

This work is divided into three chapters. With the objective of introducing the ideas presented on [1] in mind, we start with chapter 2 by introducing the concept of a spinor in four-dimensional space-time [10], first motivated from the physics point of view, as done by Dirac [11], and then the rest of the chapter is devoted to introducing spinors from the group theory point of view, following mostly [12] and [13], while also obtaining more general properties that would help us in the following chapters. We also make a rudimentary introduction to the ideas of representation theory along the way so that those not familiarized with group theory can keep up.

In chapter 3, we start to mix the ideas from chapter 2 with General Relativity. A correspondence between $SO(4)$ tensors and $SU(2) \otimes SU(2)$ is shown, and from that we are able to describe General Relativity in terms of spinors, where we obtain the spinor equivalent of the important tensorial quantities of the field. We finish the chapter talking about the Newman-Penrose formalism, which enables us to discuss how certain parameters should behave in order for us to have null shear free congruences, which in turn leads us to the Kerr Theorem for the first time, while also arriving at the concept of twistors, both central points of discussion of [1].

Finally, in chapter 4, we have the necessary tools to discuss [1] in its entirety. This time, we are talking about six-dimensional spaces, and akin to the previous chapters, we first discuss spinors, which this time comes from $SU(4)$, and their relation to $SO(6)$ tensors, and proceed to list the spinor equivalents of some tensors that will be of use. We then study how null vectors are represented by spinors, which leads us to the notion of isotropic spaces and, together with the Frobenius' Theorem, we arrive at the Kerr Theorem in 6 dimensions. This shows us a pattern of how to obtain the Kerr Theorem in generic even dimensions, and such generalization is discussed. As usual, the Kerr Theorem defines implicitly the notion of a Twistor, and we analyze how the twistor equation in six dimensions constrains algebraically the Weyl tensor, then finishes by discussing a family of examples with this property.

2 SPINORS FROM PHYSICS

Rotational symmetries always played an important role in physics, but in the last century we saw two major advances born out of this symmetry, the first was achieved by studying rotations in the Minkowski space, which led to the Lorentz Transformations and opened the field of Special Relativity. The second one came from the development of the spinor theory from Elie Cartan [10] and its introduction in physics by Paul Dirac [14]. In this chapter, we introduce the spinor concept first starting from the historical point of view in physics, and then by introducing it as an object that can be Lorentz transformed by a certain representation of SO(1,3). Group theory is briefly introduced so that we can make a more systematic discussion about rotations and symmetries in general, and different representations of the Lorentz group in four dimensions are studied.

2.1 The Dirac Equation

In 1926, Erwin Schrödinger published the paper "An undulatory theory of the mechanics of atoms and molecules" [15] where he introduced the now famous Schrödinger equation:

$$i\hbar\frac{\partial\varphi}{\partial t} = \hat{H}\varphi \tag{2.1.1}$$

Where ψ is the wave function, a scalar field that contains all the information about the system [16]. If we consider the free particle case, we get

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2\nabla^2\varphi}{2m}, \tag{2.1.2}$$

where we can see that space and time are clearly treated differently, and as consequence the equation is not Lorentz invariant.

One way we could try to obtain a relativistic version of the Schrödinger equation would be to use the relativistic expression for the energy, i.e.,

$$E = \sqrt{p^2c^2 + m^2c^4}, \tag{2.1.3}$$

and, by promoting the energy to the Hamiltonian operator and using it on (2.1.1), we have:

$$i\hbar\frac{\partial\varphi}{\partial t} = \sqrt{c^2\nabla^2 + m^2c^4}\varphi \tag{2.1.4}$$

Now, this equation still treats space and time differently, even though we are using the relativistic expression. One way to fix this is to square the operator by multiplying the whole equation by $\hat{H} = i\hbar\frac{\partial}{\partial t}$ and get

$$-\hbar^2\frac{\partial^2\varphi}{\partial t^2} = -\hbar^2\nabla^2\varphi + m^2c^4\varphi, \tag{2.1.5}$$

From now on we use units where $c = \hbar = G = 1$. We note that $-\partial_t^2 + \nabla^2 = \sum_\mu \partial_\mu \partial^\mu = \partial_\mu \partial^\mu = \partial^2$, where we use Einstein's sum convention, and the equation becomes:

$$(\partial^2 + m^2)\varphi = 0 \tag{2.1.6}$$

which is known as the Klein-Gordon (KG) equation and is indeed Lorentz invariant. According to Dirac [17], Schrödinger himself arrived at this relativistic version before publishing his famous equation, but found out that it didn't describe correctly the fine splitting of the hydrogen atom. Later, he realized that the non-relativistic limit of (2.1.1) would still be of use to describe the quantum phenomena known at the time.

As Schrödinger correctly guessed, the lack of success of the Klein-Gordon equation on describing the hydrogen atom, an electron on a Coulomb potential, was due to the intrinsic angular momentum of the electron, that was suggested to be $\frac{\hbar}{2}$ by George Uhlenbeck and Samuel Goudsmit in 1926 [18]. Indeed, the KG equation describes the dynamics of a particle associated with a scalar field, which means it has no intrinsic angular momentum, or spin, as it would become known.

What about the dynamics of a particle with non-zero spin? By 1928, Dirac tried a different approach on the search for a relativistic quantum mechanical equation [11]. Dirac's reasoning was that he wanted to find an equation that was of first order in both time and space derivatives,

$$i\frac{\partial \varphi}{\partial t} = \hat{H}\varphi, \tag{2.1.7}$$

where $\hat{H} = -i\alpha^i \partial_i + \alpha^0 m$ and $\alpha^1, \alpha^2, \alpha^3$ and α^0 are constant quantities. Now, if we regard these quantities as components of a vector, we see that $\alpha^i \partial_i$ defines a preferred direction in space, and the result would be clearly a non-relativistic equation. To avoid this problem, we regard the alphas as matrices ${\alpha^\mu}_{mn}$ where μ labels the different matrices ${\alpha^1}_{mn}, {\alpha^2}_{mn}, {\alpha^3}_{mn}$ and ${\alpha^0}_{mn}$ and the indices m and n labels the components of said matrices. Writing the full equation, we have

$$i\frac{\partial \varphi}{\partial t} = -i{\alpha^i}_{mn}\partial_i \varphi + {\alpha^0}_{mn} m\varphi, \tag{2.1.8}$$

where we see that, for our conditions to hold, φ cannot be a scalar field anymore, or we would have a scalar on the left side, and a matrix on the right side of the equation. Instead, φ is different object, which for now we will just treat as a multi component quantity and write

$$i\partial_t \Psi_m = -i{\alpha^i}_{mn}\partial_i \Psi_n + {\alpha^0}_{mn} m\Psi_n \tag{2.1.9}$$

Where we now write this quantity as Ψ_n to distinguish it from the scalar wave function φ. Dirac also imposed that, as a relativistic equation, this should also contain the relativistic energy relation $E^2 = p^2 + m^2$, so it should imply the Klein-Gordon equation somehow. This is done by "squaring" the Dirac hamiltonian (2.1.8), where we obtain

$$-\partial_t^2 \Psi = -\alpha^i \alpha^j \frac{\partial^2 \Psi}{\partial x^i \partial x^j} - im(\alpha^i \alpha^0 + \alpha^0 \alpha^i)\frac{\partial \Psi}{\partial x^i} + (\alpha^0)^2 m^2 \Psi \tag{2.1.10}$$

where we suppressed the matrix component indices and $\alpha^i\alpha^j$ is to be understood as the usual matrix product. This gives us the following restrictions on the alphas:

$$\alpha^{(i}\alpha^{j)} = \frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i) = \delta^{ij}\mathbb{I} \tag{2.1.11}$$

$$\alpha^i\alpha^0 + \alpha^0\alpha^i = 0 \tag{2.1.12}$$

$$(\alpha^0)^2 = \mathbb{I} \tag{2.1.13}$$

Where $\mathbb{I}$ is the identity matrix. With these conditions we get, by squaring the Dirac equation, the Klein-Gordon equation. Now, by making use of (2.1.13), we can multiply (2.1.9) by α^0 to get:

$$(\gamma^\mu\partial_\mu + m)\Psi = 0 \tag{2.1.14}$$

Where $\gamma^0 = -i\alpha^0$ and $\gamma^i = -i\alpha^0\alpha^i$ and we can write (2.1.11) and (2.1.12) in terms of γ^μ as

$$\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \eta^{\mu\nu}\mathbb{I}, \tag{2.1.15}$$

where $\eta^{\mu\nu}$ is the metric of flat space-time defined by the line element

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.1.16}$$

Equation (2.1.14) is finally a relativistic equation for the electron, now known as the Dirac equation. We also see that in order to obtain such an equation, we arrived at a new kind of multi-component quantity that we labeled Ψ_n. So, what's new about it, exactly? First, as we shall see, it is not a vector, since it does not transform as such under the usual Lorentz transformations. Ψ_n is called a spinor, a different object with a different Lorentz transformation which is closely related to (2.1.15), which we call Clifford Algebra [19].

2.2 Group Theory

Last section we arrived at a different kind of object, which we called spinor, by trying to build a relativistic equation for the spin $\frac{\hbar}{2}$ particle. In four dimensions, the spinor in the Dirac equation has four components, but we know from experiment that an electron has 2 spin states, up and down. Where does those extra 2 components come from? In this section, we arrive at the spinor concept from a more mathematical point of view and also explain why we need a 4 component spinor to achieve Lorentz invariance in field equations. We will follow mainly the works of [12, 13]. See also [20, 21].

2.2.1 Lie Groups

A group $(G, \circ)$ is a set G, with a binary operation $\circ$ that satisfies the following axioms:

* Closure: $\forall\ g_1, g_2 \in G, g_1 \circ g_2 \in G$
* Associativity: $\forall\ g_1, g_2, g_3 \in G, g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$
* Existence of the identity element: $\exists e \in G$ such that $\forall$ g $\in$ G, $e \circ g = g = g \circ e$
* Existence of the inverse element: $\forall$ g $\in$ G $\exists g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$

For the purposes of this dissertation, we define a Lie group as a continuous group whose elements depend on one or more real parameters θ_i, with $i = 1, 2, 3...n$.

2.2.2 Lie Algebra

Lie group theory is the framework which deals with continuous symmetries, like the rotations of a circle. A useful concept emerges when we consider a transformation that is close to identity. If such a transformation is described by an element $g(\theta)$ of a Lie group, then our infinitesimal transformation is given by

$$g(\epsilon) = 1 + \epsilon X \tag{2.2.1}$$

where we regard ϵ as a really small number and X is called the generator of the transformation. By doing successive infinitesimal transformations we arrive at a finite transformation given by

$$h(\theta) = \lim_{N \to \infty} (1 + \frac{\theta}{N} X)^N = e^{\theta X} \tag{2.2.2}$$

and so, the generator of a given transformation is

$$X = \frac{dh(\theta)}{d\theta}|_{\theta=0} \tag{2.2.3}$$

A Lie algebra $\mathcal{G}$ is the set of generators Y together with an operation $[,]$ such that $e^{tY} \in G$. The relation between $\circ$ and $[,]$ is given by the Baker-Campbell-Hausdorff formula:

$$e^X \circ e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+...} \tag{2.2.4}$$

The operation $[,]$ is called Lie bracket, and for the groups considered here, is given simply by $[X, Y] = XY - YX$ which is called commutator.

2.3 SU(2)

The first Lie group we are interested in is the SU(2), which is the set of all unitary 2x2 matrices with unit determinant. We start by building a general 2x2 matrix

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.3.1}$$

where unitarity means that $U^\dagger = U^{-1}$, so:

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \tag{2.3.2}$$

Together with the requirement that $\det(U) = 1$ we have:

$$U = \begin{bmatrix} a & -c^* \\ c & a^* \end{bmatrix} \tag{2.3.3}$$

and $aa^* + cc^* = 1$ which leaves three free parameters. If we choose a and c to be real, we have $a^2 + c^2 = 1$ which can be parametrized as:

$$U = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{2.3.4}$$

which as we can see, is the rotation matrix in two dimensions. With the other two free parameters, SU(2) acts on vectors as the rotation group in three dimensions. In a more complete definition, a Lie group is also a manifold, and the $SU(2)$ group is the 3-sphere manifold, which we call S^3.

2.3.1 The Generators of SU(2)

To obtain the generators of SU(2) we work with the parameters set to unit and the convention of putting an "i" before the generators in each element of the group, so we write an arbitrary element of the group as e^{iJ_i}. We then have

$$U^\dagger U = e^{-iJ_i^\dagger} e^{iJ_i} = 1 = e^{-iJ_i^\dagger + iJ_i + \frac{1}{2}[-iJ_i^\dagger, iJ_i] \dots} \tag{2.3.5}$$

Which means $J_i^\dagger = J_i$. Now, for the condition $\det U = \det e^{iJ_i} = 1$ gets us:

$$\det(e^{iJ_i}) = e^{itr(J_i)} = 1 \tag{2.3.6}$$

So,

$$tr(J_i) = 0 \tag{2.3.7}$$

The generators of SU(2) are then hermitian matrices of null trace, the same properties of the Pauli matrices σ_i. We then define $J_i = \frac{1}{2}\sigma_i$ and the Lie algebra follows:

$$[J_i, J_j] = i\epsilon^{ijk} J_k \tag{2.3.8}$$

We now define the following operators

$$J^2 = J_1^2 + J_2^2 + J_3^2 \tag{2.3.9}$$

$$J_+ = J_1 + iJ_2 \tag{2.3.10}$$

$$J_- = J_1 - iJ_2 \tag{2.3.11}$$

where

$$[J_+, J_-] = 2J_3 \tag{2.3.12}$$

and identify J and $J_\pm$ as the total angular momentum and ladder operators from quantum mechanics, respectively (see [22] for a detailed treatment). J^2 is called the Casimir operator, which means that it commutes with every other generator of the group. The eigenvalues of a Casimir operator of a group labels its different representations which, in this case, will act on a $2j+1$ dimensional space, where j is a integer or half integer obtained via the relation:

$$J^2 \vec{v}_i = j(j+1)\vec{v}_i \tag{2.3.13}$$

where $\vec{v}_i$ are the eigenvectors of J_3, and j labels its largest eigenvalue. In Dirac notation, for example, we have

$$J_3 \left|j, m\right\rangle = m \left|j, m\right\rangle \tag{2.3.14}$$

and j is the maximum value for m.

2.3.2 The $j = \frac{1}{2}$ representation of SU(2)

We now treat the case where $j = \frac{1}{2}$, which will be important when we talk about representations of SO(1,3), the Lorentz Group. This representation acts on a space of dimension $2\frac{1}{2} + 1 = 2$, and J_3 will be given by:

$$J_3 = \begin{bmatrix} j & 0 \\ 0 & j-1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.3.15}$$

and we can find J_1 and J_2 by using that $J_1 = \frac{1}{2}(J_+ + J_-)$, $J_2 = \frac{1}{2i}(J_+ - J_-)$ and calculate its matrix elements:

$$[J_1]_{m'm} = \langle \frac{1}{2}, m'|\, J_1 \,|\frac{1}{2}, m\rangle\,, \tag{2.3.16}$$

$$[J_2]_{m'm} = \langle \frac{1}{2}, m' | \, J_2 \, | \frac{1}{2}, m \rangle \tag{2.3.17}$$

and we obtain:

$$J_1 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.3.18}$$

$$J_2 = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{2.3.19}$$

2.4 The Lorentz Group

The Lorentz group is the group of Lorentz transformations, defined as the set of transformations that do not change the inner product between two vectors in Minkowski space, i.e.

$$x'^{\alpha} \eta_{\alpha\beta} x'^{\beta} = x^{\mu} \eta_{\mu\nu} x^{\nu} \tag{2.4.1}$$

where x'^{μ} is a Lorentz transformed vector defined as $x'^{\alpha} = \Lambda^{\alpha}{}_{\mu} x^{\mu}$, so the defining condition of the Lorentz group becomes:

$$\Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \eta_{\alpha\beta} = \eta_{\mu\nu} \tag{2.4.2}$$

We may now consider the infinitesimal transformation case where $\Lambda^{\alpha}{}_{\mu} = \delta^{\alpha}{}_{\mu} + i\epsilon K^{\alpha}{}_{\mu}$, and by applying it in the definition, we obtain:

$$(\delta^{\alpha}_{\mu} + i\epsilon K^{\alpha}{}_{\mu})(\delta^{\beta}_{\nu} + i\epsilon K^{\beta}{}_{\nu})\eta_{\alpha\beta} = \eta_{\mu\nu} \Rightarrow K^{\alpha}{}_{\mu}\eta_{\alpha\nu} + K^{\beta}{}_{\nu}\eta_{\mu\beta} = 0 \tag{2.4.3}$$

Which means:

$$\begin{bmatrix} -a & e & i & m \\ -b & f & j & n \\ -c & g & k & o \\ -d & h & l & p \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ -e & -f & -g & -h \\ -i & -j & -k & -l \\ -m & -n & -o & -p \end{bmatrix} \tag{2.4.4}$$

And we obtain the general form of the generator

$$K = \begin{bmatrix} 0 & b & c & d \\ b & 0 & g & h \\ c & -g & 0 & l \\ d & -h & -l & 0 \end{bmatrix} \tag{2.4.5}$$

which has 6 parameters for the 3 boosts and 3 rotations, amounting to the total of 6 Lorentz transformations. We can now consider the following solutions for the generators of boosts:

$$K_1 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, K_3 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \tag{2.4.6}$$

and rotations:

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.4.7}$$

from which we can obtain the corresponding transformations by using

$$\Lambda = e^{i\vec{\theta}\cdot\vec{J}+i\vec{\phi}\cdot\vec{K}}. \tag{2.4.8}$$

Where Λ is a general Lorentz transformation. We can now calculate the corresponding Lie algebra, which gives us

$$[J_a, J_b] = i\epsilon_{abc}J_c, \tag{2.4.9}$$

$$[J_a, K_b] = i\epsilon_{abc}K_c, \tag{2.4.10}$$

$$[K_a, K_b] = -i\epsilon_{abc}J_c \tag{2.4.11}$$

We can now define the following combinations:

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) \tag{2.4.12}$$

which have the following algebra:

$$[N_a^+, N_b^+] = i\epsilon_{abc}N_c^+ \tag{2.4.13}$$

$$[N_a^-, N_b^-] = i\epsilon_{abc}N_c^- \tag{2.4.14}$$

$$[N_a^+, N_b^-] = 0 \tag{2.4.15}$$

This means that we have two copies of SU(2) in the Lorentz group, each characterized by its own j, which will label the different representations of the Lorentz group.

2.4.1 Representations of the Lorentz Group

As seen, the Lorentz Group is made up of two copies of SU(2), and as such will be characterized by the duplet (j, j'). We now investigate the cases where $(j, j') = \{(0,0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$.

2.4.1.1 *The* $(0,0)$ *Representation*

This is a trivial case made by 1x1 matrices, called the scalar representation. It describes objects which don't change under a Lorentz transformation.

2.4.1.2 *The ($\frac{1}{2}$,0) Representation*

In this representation, we only use N_i^+ and take N_i^- to be zero. This means that $N_i^- = \frac{1}{2}(J_i - iK_i) = 0 \Rightarrow J_i = iK_i$, so:

$$N_i^+ = \frac{1}{2}(J_i + iK_i) = \frac{1}{2}(iK_i + iK_i) \Rightarrow K_i = \frac{-i\sigma_i}{2}, \tag{2.4.16}$$

where we used the known relation $J_i = \frac{\sigma_i}{2}$ from SU(2). In this representation, the corresponding rotations and boosts are given by:

$$R_x(\theta_1) = e^{i\theta_1 J_1} = e^{i\theta_1 \frac{\sigma_1}{2}} = \begin{bmatrix} \cos(\frac{\theta_1}{2}) & i\sin(\frac{\theta_1}{2}) \\ i\sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_1}{2}) \end{bmatrix}, \tag{2.4.17}$$

$$R_y(\theta_2) = e^{i\theta_2 J_2} = e^{i\theta_2 \frac{\sigma_2}{2}} = \begin{bmatrix} \cos(\frac{\theta_2}{2}) & \sin(\frac{\theta_2}{2}) \\ -\sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2}) \end{bmatrix}, \tag{2.4.18}$$

$$R_z(\theta_3) = e^{i\theta_3 J_3} = e^{i\theta_3 \frac{\sigma_3}{2}} = \begin{bmatrix} e^{i\frac{\theta_3}{2}} & 0 \\ 0 & e^{-i\frac{\theta_3}{2}} \end{bmatrix}, \tag{2.4.19}$$

and

$$B_x(\phi_1) = e^{i\phi_1 K_1} = e^{\phi_1 \frac{\phi_1}{2}} = \begin{bmatrix} \cosh(\frac{\phi_1}{2}) & \sinh(\frac{\phi_1}{2}) \\ \sinh(\frac{\phi_1}{2}) & \cosh(\frac{\phi_1}{2}) \end{bmatrix}, \tag{2.4.20}$$

$$B_y(\phi_2) = e^{i\phi_2 K_2} = e^{\phi_1 \frac{\phi_1}{2}} = \begin{bmatrix} \cosh(\frac{\phi_2}{2}) & -i\sinh(\frac{\phi_2}{2}) \\ i\sinh(\frac{\phi_2}{2}) & \cosh(\frac{\phi_2}{2}) \end{bmatrix}, \tag{2.4.21}$$

$$B_z(\phi_3) = e^{i\phi_3 K_3} = e^{\phi_3 \frac{\phi_3}{2}} = \begin{bmatrix} e^{\frac{\phi_3}{2}} & 0 \\ 0 & e^{-\frac{\phi_3}{2}} \end{bmatrix}, \tag{2.4.22}$$

respectively.

2.4.1.3 *The* $(0, \frac{1}{2})$ *Representation*

This time, we only use N_i^- instead of N_i^-. This means that $N_i^+ = \frac{1}{2}(J_i - iK_i) = 0 \Rightarrow J_i = -iK_i$, so:

$$N_i^- = \frac{1}{2}(J_i - iK_i) = \frac{1}{2}(-iK_i - iK_i) \Rightarrow K_i = \frac{i\sigma_i}{2}, \tag{2.4.23}$$

where, again, we used that $J_i = \frac{\sigma_i}{2}$. In this representation, the corresponding rotations and boosts are given by:

$$R_x(\theta_1) = e^{i\theta_1 J_1} = e^{i\theta_1 \frac{\sigma_1}{2}} = \begin{bmatrix} \cos(\frac{\theta_1}{2}) & i\sin(\frac{\theta_1}{2}) \\ i\sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_1}{2}) \end{bmatrix}, \tag{2.4.24}$$

$$R_y(\theta_2) = e^{i\theta_2 J_2} = e^{i\theta_2 \frac{\sigma_2}{2}} = \begin{bmatrix} \cos(\frac{\theta_2}{2}) & \sin(\frac{\theta_2}{2}) \\ -\sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2}) \end{bmatrix}, \tag{2.4.25}$$

$$R_z(\theta_3) = e^{i\theta_3 J_3} = e^{i\theta_3 \frac{\sigma_3}{2}} = \begin{bmatrix} e^{i\frac{\theta_3}{2}} & 0 \\ 0 & e^{-i\frac{\theta_3}{2}} \end{bmatrix}, \tag{2.4.26}$$

and

$$B_x(\phi_1) = e^{i\phi_1 K_1} = e^{-\phi_1 \frac{\phi_1}{2}} = \begin{bmatrix} \cosh(\frac{\phi_1}{2}) & -\sinh(\frac{\phi_1}{2}) \\ -\sinh(\frac{\phi_1}{2}) & \cosh(\frac{\phi_1}{2}) \end{bmatrix}, \tag{2.4.27}$$

$$B_y(\phi_2) = e^{i\phi_2 K_2} = e^{-\phi_2 \frac{\phi_2}{2}} = \begin{bmatrix} \cosh(\frac{\phi_2}{2}) & i\sinh(\frac{\phi_2}{2}) \\ -i\sinh(\frac{\phi_2}{2}) & \cosh(\frac{\phi_2}{2}) \end{bmatrix}, \tag{2.4.28}$$

$$B_z(\phi_3) = e^{i\phi_3 K_3} = e^{-\phi_3 \frac{\phi_3}{2}} = \begin{bmatrix} e^{-\frac{\phi_3}{2}} & 0 \\ 0 & e^{\frac{\phi_3}{2}} \end{bmatrix}, \tag{2.4.29}$$

respectively.

2.4.1.4 *The Relation Between Spinor Representations*

In the last two sections, we derived two representations of the Lorentz group, which defines rotations in the same way but differ by a minus sign on the boosts. It is now time to investigate the objects they act on, which are very similar, and the relations between them.

The $(\frac{1}{2}, 0)$ representation acts on a object called the left chiral spinor, defined as:

$$\psi_L = \psi_a = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{2.4.30}$$

while the $(0, \frac{1}{2})$ representation acts the right chiral spinor, defined as:

$$\psi^R = \psi^{\dot{a}} = \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix}. \tag{2.4.31}$$

Now, consider the matrix $i\sigma_2$. It is easy to check that $i\sigma_2$ is invariant under any Lorentz transformation in the spinor representations, be it $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$, that is:

$$i\sigma_2 = \Lambda_s^T i\sigma_2 \Lambda_s, \tag{2.4.32}$$

where Λ_s is a Lorentz transformation in the spinor representation. We can also define the following transformation:

$$\bar{\psi}_L = i\sigma_2(\psi_L)^*. \tag{2.4.33}$$

Now consider a boost in ψ_L, then we see that $\bar{\psi}_L$ transforms as:

$$\bar{\psi}'_L = i\sigma^2(\psi'_L)^* = i\sigma^2(e^{\frac{\phi_i\sigma^i}{2}}\psi_L)^* = e^{-\frac{\phi_i\sigma^i}{2}} i\sigma^2(\psi_L)^* = e^{-\frac{\phi_i\sigma^i}{2}}\bar{\psi}_L, \tag{2.4.34}$$

which is the transformation law for the $(0, \frac{1}{2})$ representation. In the same way, we will have:

$$\bar{\psi}^R = e^{\frac{\phi_i \sigma^i}{2}} \bar{\psi}^R. \quad (2.4.35)$$

The quantity that we defined as $i\sigma^2$ is, as we shall see next chapter, the Levi-Civita spinor, which plays a role similar to the metric tensor, in the sense that it defines an inner product in spin space.

2.4.1.5 The $(\frac{1}{2}, \frac{1}{2})$ Representation

We now turn our attention to the $(\frac{1}{2}, \frac{1}{2})$ representation, the last we will show here. Now that we have both copies of $SU(2)$ working at the same time, we will have a corresponding object that has two indices, where each transforms according to each copy of $SU(2)$. The components of such an object, say $X^{AA'}$, can be represented by a 2x2 matrix which we will choose to be hermitian, because as we will see, this matrices corresponds to our usual vectors, and by choosing them to be hermitian, we guarantee that our vectors will be real. So, we have:

$$X^{AA'} = \frac{1}{\sqrt{2}} \begin{bmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{bmatrix}, \quad (2.4.36)$$

where the choice of splitting the diagonal components in a sum will make itself clear in a few moments, and the factor of $\frac{1}{\sqrt{2}}$ is so that we have consistent notation with next chapter and the results we will obtain. Now, this can also be written in the form:

$$X^{AA'} = \frac{1}{\sqrt{2}}(x^0\mathbb{I} + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = x^\mu {\sigma_\mu}^{AA'}). \quad (2.4.37)$$

What we are seeing here as $X^{AA'}$ is the position vector x^μ represented in spin space, and the quantities that make the connection between them, ${\sigma_\mu}^{AA'}$, are called Van der Waerden symbols. To see that this is true, we must only show that $X^{AA'}$ transforms as a vector under a Lorentz transformation. That is [23]:

$$X' = L^\dagger X L, \quad (2.4.38)$$

or, in component form:

$$X'^{BB'} = \Lambda_A^B {\Lambda_{A'}}^{B'} X^{AA'}. \quad (2.4.39)$$

The matrix L is an element of $SL(2, C)$ the group of $2x2$ matrices with complex entries and unit determinant, which double covers the Lorentz group, which can be seen from the fact that L and $-L$ define the same transformation. Now, if we make these transformations infinitesimal, as in $\Lambda_A^B = (\mathbb{I} + \frac{i}{2}\theta_i\sigma^i - \frac{1}{2}\phi_i\sigma^i)_A^B$ and its complex conjugate, we obtain that:

$$X'^{BB'} = \frac{1}{\sqrt{2}} \begin{bmatrix} x'^0 + x'^3 & x'^1 + ix'^2 \\ x'^1 - ix'^2 & x'^0 + x'^3 \end{bmatrix}, \quad (2.4.40)$$

where

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} + \begin{bmatrix} 0 & i\theta_1 & i\theta_2 & i\theta_3 \\ i\theta_1 & 0 & i\phi_3 & -i\phi_2 \\ i\theta_2 & -i\phi_3 & 0 & i\phi_1 \\ i\theta_3 & i\phi_2 & -i\phi_1 & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{2.4.41}$$

which is exactly the Lorentz transformation law for vector components.

2.4.2 The Spinor on the Dirac equation

We can now ask ourselves again the question posed in the beginning: why the object that appears in the Dirac equation, which we called a spinor, has four entries? The answer to this is in the other two transformations not considered yet that satisfy (2.4.2). They are the parity Λ_p and time reversal Λ_t transformations, which were not treated in the same way as the other transformations because they are not connected to the identity, and are defined as:

$$\Lambda_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \Lambda_t = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.4.42}$$

which transform the generators in the following way:

$$\Lambda_p K_i (\Lambda_p)^T = -K_i, \tag{2.4.43}$$

$$\Lambda_p J_i (\Lambda_p)^T = J_i, \tag{2.4.44}$$

$$\Lambda_t K_i (\Lambda_t)^T = -K_i, \tag{2.4.45}$$

$$\Lambda_t J_i (\Lambda_t)^T = J_i, \tag{2.4.46}$$

and so:

$$N^{\pm} \iff N^{\mp} \tag{2.4.47}$$

where we see that the $(\frac{1}{2}, 0)$ representation is changed into the $(0, \frac{1}{2})$ representation and vice-versa. This means that in order to have a truly relativistic invariant equation, we need both representations together to build the four component spinor that we see in the Dirac equation that transforms according to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation.

2.5 The n Dimensional Vector Representations

In n dimensions, there will be $\frac{n(n-1)}{2}$ generators for the Lorentz group, no matter the signature, where we can obtain the finite transformations by exponentiating said generators. Instead of labelling them by a single index, we can encode these generators on a skew-symmetrical n dimensional matrix. So, $(J^{\mu\nu})^{ab}$ is the generator of the Lorentz transformation that can be either a boost or a rotation between the μ and ν directions and a, b labels the matrix components of such generators that will have the following form:

$$(J^{\mu\nu})^{ab} = -i(\eta^{\mu a}\eta^{\nu b} - \eta^{\nu a}\eta^{\mu b}), \tag{2.5.1}$$

which is great, because it works in any dimension or signature and the four dimensional case is easily recovered. The Lie algebra then can be proved to be:

$$[J^{\mu\nu}, J^{\rho\lambda}] = i(\eta^{\lambda\mu}J^{\rho\nu} - \eta^{\lambda\nu}J^{\rho\mu} - \eta^{\rho\mu}J^{\lambda\nu} + \eta^{\rho\nu}J^{\lambda\mu}), \tag{2.5.2}$$

and now we can obtain vector representations of the Lorentz group in any dimension, which can certainly be useful, given the interest in higher dimensions in theoretical physics.

2.6 Clifford Algebra and the n Dimensional Spinor Representations

Last section gave us a way to get vector representations of the Lorentz group in any dimension we want, but what about spinor representations? For that, we need to revisit an idea that we naturally arrived while trying to obtain the Dirac equation: Clifford Algebra. For the purposes of this work, it is sufficient to define a Clifford Algebra in n dimensions as the set of n matrices γ^μ that satisfy:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{I} \tag{2.6.1}$$

Now, what does this has to do with spinor representations of the Lorentz group, exactly? First, let us define the matrix:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{2.6.2}$$

The commutation relation of this set of matrices can be shown to be:

$$[S^{\mu\nu}, S^{\rho\lambda}] = i(\eta^{\lambda\mu}S^{\rho\nu} - \eta^{\lambda\nu}S^{\rho\mu} - \eta^{\rho\mu}S^{\lambda\nu} + \eta^{\rho\nu}S^{\lambda\mu}), \tag{2.6.3}$$

which is exactly the Lorentz algebra defined last section. Lets pick a specific solution of the Clifford Algebra to see how it works:

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{2.6.4}$$

where $\bar{\sigma}^\mu = (\mathbb{I}, -\sigma^1, -\sigma^2, -\sigma^3)$ and σ^i are the usual Pauli matrices, so that γ^μ are 4x4. That way, we have:

$$S^{\mu\nu} = \frac{i}{4}\begin{bmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^m u\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{bmatrix}. \tag{2.6.5}$$

From that, a general Lorentz transformation will be given by:

$$\Lambda = e^{i\omega_{\mu\nu}S^{\mu\nu}} \tag{2.6.6}$$

Knowing $S^{\mu\nu}$ amounts to knowing the commutation relations of the Pauli matrices, which are given by:

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ij}{}_k\sigma^k \tag{2.6.7}$$

and from that, it is easy to show that boosts and rotations generators will be given by:

$$K^i = \frac{i}{2}\begin{bmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{bmatrix}, \tag{2.6.8}$$

$$J^i = \frac{1}{2}\begin{bmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{bmatrix}, \tag{2.6.9}$$

where we make the identifications $S^{12} = J^3$, $S^{31} = J^2$, $S^{23} = J^1$ and $S^{0i} = K^i$. Now lets treat the Dirac spinor as a two component column matrix, where each component is itself a two component column matrix, similar to what we're doing with the generators here:

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{2.6.10}$$

and now let us check how finite Lorentz transformations act on them. The boosts and rotations will be given by

$$\boldsymbol{B}(\vec{\phi})\boldsymbol{\psi} = e^{i\phi_i K^i}\boldsymbol{\psi} = \begin{bmatrix} e^{i\phi_i\frac{\sigma^i}{2}} & 0 \\ 0 & e^{-i\phi_i\frac{\sigma^i}{2}} \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{2.6.11}$$

,

$$\boldsymbol{R}(\vec{\theta})\boldsymbol{\psi} = e^{i\theta_i J^i}\boldsymbol{\psi} = \begin{bmatrix} e^{i\theta_i\frac{\sigma^i}{2}} & 0 \\ 0 & e^{i\theta_i\frac{\sigma^i}{2}} \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{2.6.12}$$

which indeed is the spinor transformation law. Not only that, we also have explicitly that under boosts:

$$\psi_1' = e^{i\theta_i\frac{\sigma^i}{2}}\psi_1, \tag{2.6.13}$$

$$\psi_2' = e^{-i\theta_i\frac{\sigma^i}{2}}\psi_2, \tag{2.6.14}$$

and we see that $\psi_1 = \psi^R$ and $\psi_2 = \psi_L$. This is not the case in general, but it happened here because of our choice of the gamma matrices. This particular choice of gamma matrices gives us what is called the Chiral representation of the Lorentz group. For an arbitrary representation, we can also decompose the Dirac spinor into ψ_L and ψ^R, and to do that, we start by defining:

$$\gamma^4 = -\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.6.15}$$

which has the important properties

$$\{\gamma^4, \gamma^\mu\} = 0, \quad \forall\mu; \quad [S^{\mu\nu}, \gamma^4] = 0 \quad \forall\mu, \nu. \tag{2.6.16}$$

With that, we now define the operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma^4), \tag{2.6.17}$$

which obey $P_+P_+ = P_+$, $P_-P_- = P_-$ and $P_+P_- = P_-P_+ = 0$, which tells us that they are projection operators. It can be shown [13] that

$$P_+\boldsymbol{\psi}' = e^{i\omega_{\mu\nu}S^{\mu\nu}}P_+\boldsymbol{\psi} = \boldsymbol{\psi}'^R, \tag{2.6.18}$$

and analogously

$$P_-\boldsymbol{\psi}' = \boldsymbol{\psi}'_L, \tag{2.6.19}$$

so P_+ and P_- project $\boldsymbol{\psi}$ into ψ^R and ψ_L, respectively, and it works for any given representation.

With that, we showed the main basic elements of spinors as they appeared in quantum field theory. We are now set to tackle spinors from a more general point of view and apply it to general relativity.

3 SPINORS IN GENERAL RELATIVITY

Last Chapter we saw how the concept of spinor emerges as a requirement for Lorentz invariance in the spin $\frac{1}{2}$ field equations and investigated how the same concept appears when one studies different representations of the Lorentz Group. In this chapter, we are going to study the algebraic properties of the spinor in 4 dimensions, also called 2-spinor or spin vector. After that, we show how we can reformulate Einstein's Field Equations in terms of spinors. Our main goal here is to prove the Kerr theorem and with that make a brief discussion about twistors, and to be able to do that we first review some topics of General Relativity.

3.1 Spinor Algebra

In this section, we are going to study the spinor algebra and its properties. We are going to employ Penrose's abstract index notation when needed [8], where ξ^A refers not to the spinor's components, but to the spinor itself. When talking about components, we will denote it as $\xi^{\mathbf{A}}$.

3.1.1 Spin Space

Last chapter we arrived at four kinds of spinors and the relations between them, which we can exemplify here as $\{\xi^A, \xi_A, \xi^{A'}, \xi_{A'}\}$ and are related to each other by conjugation and the spinor metric. Spin vectors ξ^A are elements of the spin-space S^A, a two-dimensional vector space over the complex field $\mathbb{C}$ with a skew-symmetric, bilinear and non-degenerate inner product $[,]$.

For such space, we consider a basis, which we call dyad, $\{\mathbf{o}, \boldsymbol{\iota}\}$ such that $[\mathbf{o}, \boldsymbol{\iota}] = 1 = -[\boldsymbol{\iota}, \mathbf{o}]$. That way, we can write any spin-vector $\boldsymbol{\zeta}$ as:

$$\boldsymbol{\zeta} = \zeta^0 \boldsymbol{o} + \zeta^1 \boldsymbol{\iota}, \tag{3.1.1}$$

which means that the inner product $[,]$ between two spin-vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ is given by:

$$[\boldsymbol{\zeta}, \boldsymbol{\eta}] = \zeta^0 \eta^1 - \zeta^1 \eta^0 \tag{3.1.2}$$

as a consequence of bi-linearity and the choice of the basis.

We also have the following identity:

$$[\boldsymbol{\zeta}, \boldsymbol{\eta}]\boldsymbol{\tau} + [\boldsymbol{\eta}, \boldsymbol{\tau}]\boldsymbol{\zeta} + [\boldsymbol{\tau}, \boldsymbol{\zeta}]\boldsymbol{\eta} = 0, \tag{3.1.3}$$

which can be checked by writing explicitly their components:

$$(\zeta^0\eta^1 - \zeta^1\eta^0)(\tau^0, \tau^1) + (\eta^0\tau^1 - \eta^1\tau^0)(\zeta^0, \zeta^1) + (\tau^0\zeta^1 - \tau^1\zeta^0)(\eta^0, \eta^1) = 0 \tag{3.1.4}$$

which is obviously satisfied.

We denote the dual space of S^A as S_A, and denote complex conjugation with a bar, as in $\overline{S^A} = S^{A'}$ and $\overline{S_A} = S_{A'}$, which are the vector spaces of $\xi^A, \xi_A, \xi^{A'}$ and $\xi_{A'}$, respectively. So $\xi^A \in S^A, \xi_A \in S_A, \xi^{A'} \in S^{A'}$ and $\xi_{A'} \in S_{A'}$ are univalent spinors, or a (1, 0; 0, 0), (0, 0; 1, 0), (0, 1; 0, 0), (0, 0; 0, 1) spinors, respectively. The most general kind of spinor is a multilinear map $\xi^{A_1 \ldots A_p B'_1 \ldots B'_q}{}_{C_1 \ldots C_r D'_1 \ldots D'_s} : S^{A_1} \ldots \otimes S^{A_p} \otimes S^{B'_1} \ldots \otimes S^{B'_q} \otimes S_{C_1} \ldots \otimes S_{C_r} \otimes S_{D'_1} \ldots \otimes S_{D'_s} \to S$ where $\xi^{A_1 \ldots A_p B'_1 \ldots B'_q}{}_{C_1 \ldots C_r D'_1 \ldots D'_s} \in S^{A_1 \ldots A_p B'_1 \ldots B'_q}{}_{C_1 \ldots C_r D'_1 \ldots D'_s}$ is a (p, q; r, s) spinor, and (p, q; r, s) is called the spinor valence.

3.1.2 The connection between tensors and spinors

As stated, the purpose of this chapter is to show the spinor formalism of General Relativity, and for that to happen we need a way to write the analogous of tensorial quantities as elements of spin space, such as the metric, the curvature tensor etc. This is possible by using the van der Warden symbols:

$$\sigma^0{}_{AA'} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \sigma^2{}_{AA'} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \sigma^3{}_{AA'} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}, \sigma^4{}_{AA'} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{3.1.5}$$

which can be written in component form as $\sigma^{\boldsymbol{a}}{}_{AA'}$ with a vector index that can be lowered with the metric g_{ab}. As an example of its use, let's see the spinor version of position vector:

$$X^{AA'} = X^{\boldsymbol{a}}\sigma_{\boldsymbol{a}}{}^{AA'} = \begin{bmatrix}x^{00'} & x^{01'}\\ x^{10'} & x^{11'}\end{bmatrix} = \begin{bmatrix}x^0 + x^3 & x^1 + ix^2\\ x^1 - ix^2 & x^0 - x^3\end{bmatrix}, \tag{3.1.6}$$

and for a rank 2 tensor, we would have:

$$T_{AA'BB'} = T_{\boldsymbol{ab}}\sigma^{\boldsymbol{a}}{}_{AA'}\sigma^{\boldsymbol{b}}{}_{BB'}. \tag{3.1.7}$$

From now on, we will employ the convention of not showing the van der warden symbols explicitly, and follow the rule that each tensor index μ corresponds to a pair of spinor indices UU' wi, so:

$$T_{ab} = T_{AA'BB'} = T_{ABA'B'}, \tag{3.1.8}$$

where primed and unprimed indices can be interchanged at will.

3.1.3 The Levi-Civita spinor

Equation (3.1.1) defines the inner product in Spin space, which implies the existence of a bilinear, anti-symmetric form $\epsilon_{AB} \in S_{AB}$ such that:

$$[\boldsymbol{\zeta}, \boldsymbol{\eta}] = \zeta_B\eta^B = \epsilon_{AB}\zeta^A\eta^B, \tag{3.1.9}$$

where $\epsilon_{AB} = -\epsilon_{BA}$ is called the Levi-Civita spinor, and plays a role analogous to the metric tensor in spinor algebra. The same relation exists for $S^{A'}, S_A$ and $S_{A'}$, and we have:

$$\epsilon_{\boldsymbol{AB}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.1.10}$$

$$\epsilon^{\boldsymbol{AB}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.1.11}$$

and their complex conjugates. With that in mind, we can expand (3.1.4) as:

$$\epsilon_{AB}\zeta^A\eta^B\tau^C + \epsilon_{AB}\eta^A\tau^B\zeta^C + \epsilon_{AB}\tau^A\zeta^B\eta^C = 0 \tag{3.1.12}$$

which can be rewritten as:

$$(\epsilon_{AB}\epsilon_C{}^D + \epsilon_{BC}\epsilon_A{}^D + \epsilon_{CA}\epsilon_B{}^D)\zeta^A\eta^B\tau^C = 0 \tag{3.1.13}$$

where $\epsilon_A{}^B = -\epsilon^B{}_A = \epsilon_{AC}\epsilon^{CB}$. Now, since $\boldsymbol{\zeta}, \boldsymbol{\eta}$ and $\boldsymbol{\tau}$ are in general different from zero, we take the term between parenthesis to be zero. That way, we get the following important identities:

$$\epsilon_{AB}\epsilon_C{}^D + \epsilon_{BC}\epsilon_A{}^D + \epsilon_{CA}\epsilon_B{}^D = 0, \tag{3.1.14}$$

$$\epsilon_{AB}\epsilon_{CD} + \epsilon_{BC}\epsilon_{AD} + \epsilon_{CA}\epsilon_{BD} = 0, \tag{3.1.15}$$

$$\epsilon_{AB}\epsilon^{CD} = \epsilon_A{}^C\epsilon_B{}^D - \epsilon_B{}^C\epsilon_A{}^D, \tag{3.1.16}$$

and the complex conjugate of the three equations are also satisfied. Both (3.1.15) and (3.1.16) come from index lowering and raising from (3.1.14) and are very useful to calculate the spinorial form of tensors. Let's take g_{ab} as an example:

$$\begin{aligned} g_{ab} &= g_{(ab)} \\ &\Rightarrow g_{AA'BB'} = \frac{1}{2}(g_{AA'BB'} + g_{BB'AA'}) \\ &= \frac{1}{2}(g_{ABA'B'} + g_{ABB'A'}) + \frac{1}{2}(g_{BAB'A'} - g_{ABB'A'}) \\ &= g_{(AB)(B'A')} + g_{[AB][B'A']}. \end{aligned} \tag{3.1.17}$$

Where we interchanged primed and non-primed indices freely, since their order don't matter. Now, by applying (3.1.16) to $g_{[AB][B'A']}$, we get:

$$g_{AA'BB'} = g_{(AB)(B'A')} + \frac{1}{4}\epsilon_{AB}\epsilon_{A'B'}g_c{}^c \tag{3.1.18}$$

Which looks a lot like the usual decomposition of a symmetric tensor $T_{ab} = T_{(ab)}$ into a symmetric, trace free part $P_{(ab)}$ such that $P_a{}^a = 0$, and a part proportional to the metric:

$$T_{ab} = P_{ab} + \frac{1}{4}g_{ab}T_c{}^c \tag{3.1.19}$$

and indeed it is the same relation but in spinor form. For the metric, we have $P_{ab} = 0$ and so:

$$g_{ab} = g_{AA'BB'} = \epsilon_{AB}\epsilon_{A'B'} \tag{3.1.20}$$

which shows that ϵ indeed has a metric meaning beyond just raising or lowering spinor indices. By an analogous procedure, one can show that the electromagnetic field tensor $F_{ab} = -F_{ba}$ can be written as [24]

$$F_{ab} = F_{AA'BB'} = -F_{BB'AA'}, \tag{3.1.21}$$

which can be written as

$$F_{AA'BB'} = \frac{1}{2}(F_{ABA'B'} - F_{ABB'A'}) + \frac{1}{2}(F_{ABB'A'} - F_{BAB'A'}) = F_{AB[A'B']} + F_{[AB]B'A'} \tag{3.1.22}$$

and again, by applying (3.1.16), we have

$$F_{ab} = \frac{1}{2}\epsilon_{A'B'}F_{ABC'}{}^{C'} + \frac{1}{2}\epsilon_{AB}F_{C}{}^{C}{}_{B'A'}, \tag{3.1.23}$$

where from (3.1.21) we have that

$$2\phi_{AB} = F_{ABC'}{}^{C'} = F_{BAC'}{}^{C'}, \quad 2\bar{\phi}_{A'B'} = F_{C}{}^{C}{}_{B'A'} = F_{C}{}^{C}{}_{A'B'}, \tag{3.1.24}$$

and then we write

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB} \tag{3.1.25}$$

where $\phi_{AB} = \phi_{(AB)}$ is called the electromagnetic spinor. Note that it only has 3 independent components

$$\phi_0 = \phi_{00}, \qquad \phi_1 = \phi_{10} = \phi_{01}, \qquad \phi_2 = \phi_{22}, \tag{3.1.26}$$

such that [25]

$$\phi_{AB} = \phi_2 o_A o_B - 2\phi_1 o_{(A}\iota_{B)} + \phi_0 \iota_A \iota_B. \tag{3.1.27}$$

3.2 General Relativity with Spinors

We will now obtain the spinor form of the Einstein field equations. For that to happen, we need to obtain the spinor equivalents of the general relativity ingredients, that is, the covariant derivative, the Riemann tensor, the Ricci tensor and the Ricci scalar.

3.2.1 The Covariant Derivative

Since the covariant derivative is a tensorial quantity, in the sense that it transforms as a tensor, its spinorial components are given as usual by

$$\nabla_a \quad \longmapsto \quad \nabla_{AA'} = \sigma^a_{AA'}\nabla_a. \tag{3.2.1}$$

However, its action on spinors is not defined yet. Here, we define the covariant derivative ∇ of a $(p, q; r, s)$ spinor field as a map into a $(p, q; r+1, s+1)$ spinor field with the following properties [26]:

$$\begin{aligned} &\nabla_{AA'}\boldsymbol{\sigma} = 0, \qquad \nabla_{AA'}\epsilon_{CD} = 0, \\ &\nabla_{AA'}(\pi^B\kappa_C) = \kappa_C\nabla_{AA'}\pi^B + \pi^B\nabla_{AA'}\kappa_C, \quad \nabla_{AA'}\phi = \partial_{AA'}\phi. \end{aligned} \tag{3.2.2}$$

Also, it can be directly computed by the formula [27]

$$\nabla_{AA'}\kappa_B = \partial_{AA'}\kappa_B - \Gamma^C_{AA'B}\kappa_C, \tag{3.2.3}$$

where $\Gamma^C_{AA'B}$ is the spinor affine connection, related to Γ^c_{ab} by

$$\Gamma^C_{iB} = -\frac{\sigma_k^{CD}}{2}(\partial_i\sigma^k_{BD} + \sigma^j_{BD}\Gamma^k_{ij}). \tag{3.2.4}$$

3.2.2 The Riemann Tensor

Since our goal is to ultimately obtain the spinorial form of $G_{ab} = 8\pi T_{ab}$, lets start with the Riemann tensor. The thing is, as we noted in the case of the metric tensor, something like $R_{abcd} = R_{AA'BB'CC'DD'}$ is not quite the answer we are looking for. What we are truly interested in is "breaking" this tensor into smaller parts, that we call irreducible representations, by making use of its symmetries. For that, we could use group theoretical arguments straight away and obtain $R_{AA'BB'CC'DD'}$ in terms of such irreducible representations, and, indeed, this is what we are going to do next chapter on the six dimensional case, but here the relatively low dimension lends us more intuition, and we can obtain it using Penrose's kind of pedagogical method [8], [24]. As is known, the Riemann tensor has the following symmetries:

$$R_{abcd} = R_{[ab][cd]}, \tag{3.2.5}$$

$$R_{abcd} = R_{cdab}, \tag{3.2.6}$$

$$R_{a[bcd]} = 0. \tag{3.2.7}$$

As usual, we define:

$$R_{abcd} = R_{AA'BB'CC'DD'}. \tag{3.2.8}$$

Now, having the symmetries in mind and repeating an analysis analogous to the one we did for g_{ab}, we obtain:

$$R_{AA'BB'CC'DD'} = X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \bar{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}, \tag{3.2.9}$$

where $X_{ABCD} = \frac{1}{4}R_{ABE'}{}^{E'}{}_{CDF'}{}^{F'}$ and $\Phi_{ABC'D'} = \frac{1}{4}R_{ABE'}{}^{E'}{}_{F}{}^{F}{}_{C'D'}$ are called curvature spinors.

3.2.3 The Ricci Tensor

Now, the Ricci tensor is obtained from the Riemann tensor by $R_{ab} = R_{ac}{}^{c}{}_{b}$. For its spinor counterpart, this means:

$$R_{AA'BB'} = \epsilon^{BD}\epsilon^{B'D'}R_{AA'BB'CC'DD'} = X_{ACB}{}^{C}\epsilon_{A'B'} - \Phi_{ABB'A'} - \bar{\Phi}_{A'B'BA} + \overline{X}_{A'C'B'}{}^{C'}\epsilon_{AB} \tag{3.2.10}$$

Before proceeding any further, let's take a moment and analyze the curvature spinors symmetries from the symmetries of the Riemann tensor. From (3.2.5) we have:

$$X_{ABCD} = X_{(AB)(CD)}, \Phi_{ABC'D'} = \Phi_{(AB)(C'D')} \tag{3.2.11}$$

which from (3.2.6) we also get that $\mathbf{\Phi}$ is real.

Now, let's analyze an specific part of the Riemann tensor called the right dual, defined as:

$$R^{*}{}_{abcd} = \frac{1}{2}\epsilon_{cd}{}^{ef}R_{abef} \tag{3.2.12}$$

and we can also see that (3.2.7) can be written as:

$$R_{a[bcd]} = -\frac{1}{3}\epsilon_{hbcd}R^{*}{}_{ae}{}^{he} = 0 \tag{3.2.13}$$

where this implies that $R^{*}{}_{ae}{}^{he} = 0$. Then we have that:

$$R^{*}{}_{abcd} = \frac{i}{2}(\epsilon_{C}{}^{E}\epsilon_{D}{}^{F}\epsilon_{C'}{}^{F'}\epsilon_{D'}{}^{E'} - \epsilon_{C}{}^{F}\epsilon_{D}{}^{E}\epsilon_{C'}{}^{E'}\epsilon_{D'}{}^{F'})R_{AA'BB'EE'FF'} = iR_{AA'BB'CD'DC'}. \tag{3.2.14}$$

From (3.2.9) and paying attention to the symmetries we already know, we see that

$$R^{*}{}_{abcd} = -iX_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + i\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} - i\bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + i\bar{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}. \tag{3.2.15}$$

Now, by doing $R^{*}{}_{ae}{}^{he} = 0$, we obtain:

$$X_{ABC}{}^{B}\epsilon_{A'C'} = \bar{X}_{A'B'C'}{}^{B'}\epsilon_{AC} \tag{3.2.16}$$

Now, if we act on this equation with $\epsilon^{A'C'}\epsilon^{AC}$, we obtain

$$\bar{\Lambda} = \Lambda = \frac{1}{6}X_{AB}{}^{AB}, \tag{3.2.17}$$

which gives us that $X_{ABC}{}^{B} = 3\Lambda\epsilon_{AC}$ which also means $X_{AB}{}^{AB} = X_{[A|B|C]}{}^{B}$.

Now we have what we want to finish the calculation of the Ricci tensor. By applying our knowledge of the curvature spinors to (3.2.10), we finally obtain

$$R_{ab} = R_{AA'BB'} = 6\Lambda\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'} \tag{3.2.18}$$

3.2.4 The Ricci Scalar

For the Ricci scalar, we must only take the trace of the Ricci tensor, that we already know the spinor equivalent. We have

$$R = R_a{}^a = R_{AA'}{}^{AA'} = 24\Lambda - 2\Phi_A{}^A{}_{A'}{}^{A'}, \tag{3.2.19}$$

but, as we can see, $\mathbf{\Phi}$ has vanishing trace, since $\Phi_A{}^A{}_{A'}{}^{A'} = \Phi_{ACA'C'}\epsilon^{AC}\epsilon^{A'C'} = \Phi_{(AC)(A'C')}\epsilon^{AC}\epsilon^{A'C'} = 0$. Then, the Ricci scalar is given by:

$$R = 24\Lambda \tag{3.2.20}$$

3.2.5 Einstein Field Equations

Now we can see what the spinor equivalent of the Einstein Field Equations look like. First, we have the Einstein tensor G_{ab} which is given by

$$G_{ab} = G_{AA'BB'} = -6\Lambda\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'} \tag{3.2.21}$$

and since $G_{ab} = 8\pi T_{ab} - \lambda g_{ab}$ where T_{ab} is the energy-momentum tensor and λ is the cosmological constant, we have

$$-6\Lambda\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'} = 8\pi T_{AA'BB'} - \lambda\epsilon_{AB}\epsilon_{A'B'}, \tag{3.2.22}$$

which, by contraction with $\epsilon^{AB}\epsilon^{A'B'}$ gives

$$\Lambda = \frac{\pi}{3}T_A{}^A{}_{A'}{}^{A'} + \frac{\lambda}{6} \tag{3.2.23}$$

and then, we finally have that the Einstein Field Equations are given by:

$$\Phi_{ABA'B'} = 4\pi(T_{ABA'B'} - \frac{1}{4}T_C{}^C{}_{C'}{}^{C'}\epsilon_{AB}\epsilon_{A'B'}) \tag{3.2.24}$$

3.3 Topics from General Relativity

In this section, we're going to talk shortly about some topics of General Relativity that will be useful in order to present and prove the Kerr Theorem.

3.3.1 The Newman-Penrose Formalism

The first of these topics is the Newman Penrose formalism, which is basically a way to study the relevant quantities of general relativity with a particular choice of basis. This basis is given by four null vectors $\boldsymbol{e_a} = (l_a, n_a, m_a, \bar{m}_a)$ called null tetrads, where m_a and $\bar{m}_a$ are complex conjugates of one another. We are using lower case latin letters for tetrad indices and greek letters for the usual tensor indices. Those vectors satisfy the following relations:

$$\boldsymbol{l}\cdot\boldsymbol{n} = -\boldsymbol{m}\cdot\bar{\boldsymbol{m}} = 1, \tag{3.3.1}$$

and every other inner product between them is null. That way, we can write the metric as:

$$\eta_{ab} = 2(l_{(a}n_{b)} - m_{(a}\bar{m}_{b)}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{3.3.2}$$

and we have

$$ds^2 = \eta_{ab}e^a e^b, \tag{3.3.3}$$

where $e^a = e^a{}_\mu dx^\mu$.

A vector which is written in terms of a coordinate basis can be expressed in a tetrad basis as:

$$V_a = e_a{}^\mu V_\mu = l_a V_0 + n_a V_1 + m_a V_2 + \bar{m}_a V_3. \tag{3.3.4}$$

In terms of the spin dyad, the tetrad vectors can be written as:

$$l^a = o^A\bar{o}^{A'}, n^a = \iota^A\bar{\iota}^{A'}, m^a = o^A\bar{\iota}^{A'}, \bar{m}^a = \iota^A\bar{o}^{A'}, \tag{3.3.5}$$

where clearly we still have $\boldsymbol{m} = \bar{\boldsymbol{m}}$, and $\{\boldsymbol{l}, \boldsymbol{n}\}$ are still real, while (3.3.5) is obviously satisfied.

In this formalism, we give special attention to the directional derivatives, which now have their own symbols, given by:

$$D = l^a\nabla_a, \Delta = n^a\nabla_a, \delta = m^a\nabla_a, \bar{\delta} = \bar{m}^a\nabla_a. \tag{3.3.6}$$

Now, since we can have $o^A\iota_A = c$ where c is a complex constant, we have that

$$o^A\bar{\nabla}\iota_A = \iota^A\bar{\nabla}o_A, \tag{3.3.7}$$

where $\bar{\nabla}$ is any of the directional derivatives defined previously. With that in mind, we can define a set of 12 scalars called connection coefficients defined as:

$$\begin{aligned} &o^A D o_A = \kappa, \quad o^A\Delta o_A = \tau, \quad o^A\delta o_A = \sigma, \\ &o^A\bar{\delta} o_A = \rho, \quad o^A D\iota_A = \epsilon, \quad o^A\Delta\iota_A = \gamma, \\ &o^A\delta\iota_A = \beta, \quad o^A\bar{\delta}\iota_A = \alpha, \quad \iota^A D\iota_A = \pi, \\ &\iota^A\Delta\iota_A = \nu, \quad \iota^A\delta\iota_A = \mu, \quad \iota^A\bar{\delta}\iota_A = \lambda. \end{aligned} \tag{3.3.8}$$

Now, these scalars satisfy a set of 18 coupled first order differential equations which also involve the Ricci and Weyl spinors, as well as the Ricci scalar, and this set is called The Newman-Penrose field equations. We will not go into further detail here, since they wont be used on this work, apart from commenting that they are quite useful on the study and search for solutions of the Einstein Field Equations. The equations and their applications can be checked in [25] and [28].

The equations that we are interested in are those differential equations satisfied by the null tetrad, the differential operators defined in (3.3.6) and the connection coefficients. Those are:

$$\begin{aligned} Dl &= (\epsilon+\bar{\epsilon})l - \bar{\kappa}m - \kappa\bar{m}, & \Delta l &= (\gamma+\bar{\gamma})l - \bar{\tau}m - \tau\bar{m}, \\ \delta l &= (\bar{\alpha}+\beta)l - \bar{\rho}m - \sigma\bar{m}, & Dn &= -(\epsilon+\bar{\epsilon})n + \pi m + \bar{\pi}\bar{m}, \\ \Delta n &= -(\gamma+\bar{\gamma})n + \nu m + \bar{\nu}\bar{m}, & \delta n &= -(\alpha+\bar{\beta})n + \mu m + \bar{\lambda}\bar{m}, \\ Dm &= \bar{\pi}l - \kappa n + (\epsilon-\bar{\epsilon})m, & \Delta m &= \bar{\nu}l - \tau n + (\gamma-\bar{\gamma})m, \\ \delta m &= \bar{\lambda}l - \sigma n + (\beta-\bar{\alpha})m, & \bar{\delta}m &= \bar{\mu}l - \rho n + (\alpha-\bar{\beta})m. \end{aligned} \tag{3.3.9}$$

The point of introducing the Newman-Penrose formalism here was so that we could define all the quantities needed to present this set of equations, which we will use on the next topic. As an example of the use of this formalism, the source-free Maxwell's Equations can be written as [28]

$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2, \tag{3.3.10}$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 - (\rho - 2\epsilon)\phi_2, \tag{3.3.11}$$

$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2, \tag{3.3.12}$$

$$\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2, \tag{3.3.13}$$

where ϕ_i are the ones defined on (3.1.26).

3.3.2 Null Shear-Free Congruences

The advantage of the NP formalism is that by making physical requirements about spacetime, we get certain constraints on the connection coefficients, and those will be true for any solution of the Einstein Field Equations that share the same requirements we did in the first place. Our requirement in this work will be that the spacetime admits a null shear-free congruence.

A null geodesic congruence Γ in a region U of a spacetime $\mathbb{M}$ is a set of null geodesics such that through each point of U passes one and only one null geodesic. We can define l^a as the tangent vector of the congruence, which obeys the geodesic equation:

$$Dl^a = 0 = o^A Do_A, \tag{3.3.14}$$

where we also see that $\kappa = 0$ and $\epsilon + \bar{\epsilon} = 0$. We also propagate the spin dyad along Γ by the conditions:

$$Do^A = D\iota^A = 0. \tag{3.3.15}$$

Now, lets define another vector $\boldsymbol{w}$ with the condition:

$$\mathcal{L}_l w^a = 0, \tag{3.3.16}$$

where $\mathcal{L}_l$ is the Lie derivative with respect to the vector field $\boldsymbol{l}$, and its action on another vector field ω^a is defined as [2]

$$\mathcal{L}_l \omega^a = l^b \nabla_b \omega^a - \omega^b \nabla_b l^a, \tag{3.3.17}$$

and then, condition (3.3.16) gives us

$$Dw^a = w^b \nabla_b l^a. \tag{3.3.18}$$

Now let's consider that at a point p, $\boldsymbol{w}$ and $\boldsymbol{l}$ are orthogonal. Then, we have that

$$D(l^a w_a) = l^a D w_a = l^a w^b \nabla_b l_a = \frac{1}{2} w^b \nabla_b (l^a l_a) = 0, \tag{3.3.19}$$

which guarantees that $\boldsymbol{w}$ is always orthogonal to Γ. With that, we can construct the vector $\boldsymbol{w}$ as a linear combination of the null tetrad vectors:

$$w^a = ul^a + \bar{z} m^a + z\bar{m}^a, \tag{3.3.20}$$

where n^a is absent since $l_a n^a = 1$ and we want $l_a w^a = 0$. Now we compute Dw^a expressing $\boldsymbol{w}$ with the spin dyad

$$Dw^a = w^b \nabla_b l^a \Rightarrow o^A \bar{o}^{A'} Du + o^A \bar{\iota}^{A'} D\bar{z} + \iota^A \bar{o}^{A'} Dz = \bar{z} o^A \delta \bar{o}^{A'} + \bar{z} \bar{p}^{A'} \delta o^A + z o^A \bar{\delta} \bar{o}^{A'} + z \bar{o}^{A'} \bar{\delta} o^A, \tag{3.3.21}$$

which, by contracting with $o_A \bar{\iota}_{A'}$ we get

$$-Dz = \bar{z} o_A \delta o^A - z o_A \bar{\delta} o^A, \tag{3.3.22}$$

and by substituting the connection coefficients defined last section, we get:

$$Dz = -\rho z - \sigma \bar{z} \tag{3.3.23}$$

where we used the fact that $o_A o^A = 0$ and so

$$\bar{\nabla}(o_A o^A) = o^A \bar{\nabla} o_A + o_A \bar{\nabla} o^A = 0 \Rightarrow o^A \bar{\nabla} o_A = -o_A \bar{\nabla} o^A. \tag{3.3.24}$$

We obtained an equation that tells us how z behaves along Γ, and this behavior is related to the connection coefficients σ and ρ and also to the complex conjugate of z. Considering that $z = x + iy$, lets analyze some cases.

$\rho = Re\{\rho\}, \sigma = 0$:
$$Dx = -\rho x, Dy = -\rho y, \tag{3.3.25}$$

$\rho = -i\omega, \sigma = 0$:
$$Dx = -\omega y, Dy = \omega x, \tag{3.3.26}$$

$\rho = 0, \sigma = Re\{\sigma\}$:
$$Dx = -\sigma x, Dy = \sigma y. \tag{3.3.27}$$

If we follow a circle in the plane defined by $\{\boldsymbol{m}, \bar{\boldsymbol{m}}\}$ along Γ, (3.3.25), (3.3.26) and (3.3.27) represent an isotropic expansion, a rotation and a shear, respectively, as can be seen by integrating those equations. Geometrically, the shearing of the circle means that it will turn into an ellipse.

The general case will then be a combination of these behaviors, but when studying solutions of the Einstein's Field Equations, we can start by making the requirement that $\sigma = 0$, for example, meaning that such solution will have shear-free geodesics.

3.3.3 The Kerr Theorem and Twistors

Let's now consider the case of the null shear-free congruence defined last section in Minkowski space. Since $o^A D o_A = o^A o^B \bar{o}^{B'} \nabla_{BB'} o_A = 0$ and $o^A \delta o_A = o^A o^B \bar{\iota}^{B'} \nabla_{BB'} o_A = 0$, we can write both conditions with the equation:

$$o^A o^B \nabla_{BB'} o_A = 0. \tag{3.3.28}$$

Now, let's write o^A in terms of a normalized constant dyad $\{\alpha^A, \beta^A\}$:

$$o^A = \lambda(\alpha^A - Y\beta^A) \tag{3.3.29}$$

where $\alpha^A = (1, 0)$ and $\beta^A = (0, 1)$. Now (3.3.28) reads

$$\nabla_{0A'} Y - Y \nabla_{1A'} Y = 0. \tag{3.3.30}$$

By using coordinates such that $ds^2 = dudv + dzd\bar{z}$, the equations become:

$$\frac{\partial Y}{\partial \bar{z}} - Y\frac{\partial Y}{\partial u} = 0, \qquad \frac{\partial Y}{\partial v} - Y\frac{\partial Y}{\partial z} = 0. \tag{3.3.31}$$

These equations can be solved by the method of characteristics [29], and their solutions are given implicitly by any analytic function of three variables satisfying

$$F(Y, u + \bar{z}Y, vY - z, Y) = 0, \tag{3.3.32}$$

which can also be written as

$$f(-ix^{A'A} o_A, o_B) = f(Z^\alpha) = 0, \tag{3.3.33}$$

where f is a homogeneous function such that:

$$F(a, b, c) = f(-ia, -ib, c, 1). \tag{3.3.34}$$

This is called the Kerr Theorem, which states that the general null shear free congruence in Minkowski space is defined by $f(Z^\alpha) = 0$, where f is an arbitrary homogeneous analytic function of four variables.

Even though our analysis was conducted on Minkowski space, the Kerr theorem allows other kinds of space time. Among the important solutions of (3.3.32) are a class of spaces known as Kerr-Schild, with metric defined as [30]

$$g_{ab} = \bar{g}_{ab} - 2Sl_a l_b, \tag{3.3.35}$$

where $\bar{g}_{ab}$ is a maximally symmetric space, S is a function and l_a is a null GSF 1-form. This class of metrics is given by the choice that (3.3.32) is given by

$$F = \phi(Y) + (q + pY)(u + \bar{z}Y) - (c + qY)(vY - z) = 0, \tag{3.3.36}$$

where p and c are real constants, q is a complex constant and $\phi(Y)$ is an arbitrary analytic function of Y. The function S is given in terms of Y by

$$S = -\frac{m}{2}P^{-3}(\rho + \bar{\rho}) \qquad P = pY\bar{Y} + q\bar{Y} + \bar{q}\bar{Y} + c. \tag{3.3.37}$$

One of the important solutions given by this class is that of the Kerr black hole. In its Kerr-Schild form, the metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{mr^3}{r^4 + a^2z^2}(dt + \frac{(rx + ay)dx + (ry - ax)dy}{r^2 + a^2} + \frac{z}{r}dz)^2, \tag{3.3.38}$$

with r defined implicitly by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \tag{3.3.39}$$

This solution is achieved by the particular choice

$$\phi = -iaY, \qquad q = 0, \qquad p = c = 2^{-\frac{1}{2}}, \tag{3.3.40}$$

and now m and a are identified with the mass and angular momentum of the black hole, respectively. For a proof of the Kerr Theorem on a Kerr-Schild background, see [31].

Equation (3.3.33) also defines something called twistor, which we denoted by Z^α on (3.3.33). Here we will look only at its basics, for more complete accounts see [32], [33] and [9]. The twistor here is defined as the pair of spinors given by

$$Z^\alpha = (Z^1, Z^2, Z^3, Z^4) = (\omega^A, \pi_{A'}), \tag{3.3.41}$$

where

$$Z^1 = \omega^1, \qquad Z^2 = \omega^2, \quad Z^3 = \pi_{1'} \quad Z^4 = \pi_{2'}. \tag{3.3.42}$$

We can also define the dual twistor $W_\alpha = (\lambda_A, \mu^{A'})$ and the twistor conjugation where

$$\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'}) \qquad \bar{W}_\alpha = (\bar{\mu}^A, \bar{\lambda}_{A'}), \tag{3.3.43}$$

with that we can define an inner product in twistor space

$$Z^\alpha W_\alpha = \omega^A \lambda_A + \mu^{A'} \pi_{A'}. \tag{3.3.44}$$

Another way to define a twistor is given by the differential equation on a field $\omega^A(x)$

$$\nabla_{AA'} \omega^B = -i \delta_A^B \pi_{A'}, \tag{3.3.45}$$

which is known as the four dimensional twistor equation [9] and, in flat space, has the following solution:

$$\omega^A(x) = \omega^A - i x^{AA'} \pi_{A'}, \tag{3.3.46}$$

where $\omega^A(0) = \omega^A$ and $\pi_{A'}$ are the constant spinors that define Z^α. Now, let's consider the points of complexified Minkowski space such that $\omega^A(x) = 0$

$$\omega^A = i x^{AA'} \pi_{A'}, \tag{3.3.47}$$

this equation represents the set of points given by

$$x^{AA'} = x_0^{AA'} + \kappa^A \pi^{A'}, \tag{3.3.48}$$

with κ^A a varying spinor and $x_0^{AA'}$ a constant position spinor. This set of points defines a totally null complex 2-plane called an α-plane. Now, we define a null electromagnetic field as the one such that $F_{ab}F^{ab} = 0$, it will be self-dual if

$$F_{ab} = \phi_A \phi_B \epsilon_{A'B'}, \tag{3.3.49}$$

and anti-self dual if

$$F_{ab} = \bar{\phi}_{A'} \bar{\phi}_{B'} \epsilon_{AB}. \tag{3.3.50}$$

If we have a self-dual null Maxwell tensor of the form

$$F_{ab} = \phi o_{A'} o_{B'} \epsilon_{AB}, \tag{3.3.51}$$

we see from (3.1.26) that $\phi_0 = \phi_1 = 0$ and $\phi = \phi_2$ and so, from the Maxwell's equations, we get that $\sigma = \kappa = 0$, and so, $Dl^a = (\epsilon + \bar{\epsilon}) l^a$. This is called the Mariot-Robinson theorem, and it states that if F_{ab} is a null electromagnetic field, then its repeated null direction generates a null GSF congruence [25]. Since we can use a simple null bivector to describe a null plane [34], the Mariot-Robinson theorem connects the existence of null 2-planes and null GSF congruences.

4 SIX DIMENSIONAL KERR THEOREM

In this chapter, we are finally able to study the ideas from [1]. Here, we are concerned with six-dimensional spaces, and we will have a different approach to topics that we already encountered before. We will start by, again, establishing a correspondence between spinors and tensors, so that we can obtain the spinor version of all the quantities needed in our discussion. Here, we also introduce the idea of an isotropic subspace, e how its existence is related to the Kerr theorem. We will talk about twistors in six dimensions and how the twistor equation constrains the Weyl tensor algebraically, and then return to the Kerr-Schild family of metrics and show that it exhibits this property.

4.1 Spinors on Six Dimensions

So far, we have studied the spinorial formalism in 4 dimensions and noticed that there are two copies of SU(2) in the Lorentz Group. The key for studying spinors in six dimensions is noting that there is an isomorphism between SO(6) and SU(4), both having 15 generators. This can be seen from the fact that the Clifford algebra for $SO(2n)$ is made from 2n gamma matrices that are $2^n \times 2^n$ each [23]. So, analogously to (2.6.15), we define

$$\gamma^6 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5, \tag{4.1.1}$$

and with that

$$P_\pm = \frac{1}{2}(1 \pm \gamma^6), \tag{4.1.2}$$

which again decomposes the Dirac spinor $\boldsymbol{\psi}$ into its left chiral part and right chiral part, but this time the Dirac spinor has $2^3 = 8$ components, and so each part will have four components. That way, by applying $P_\pm$ on the transformation $\boldsymbol{\psi}' = e^{i\omega_{\mu\nu}S^{\mu\nu}}\boldsymbol{\psi}$ (see (2.6.2)) we get

$$P_\pm\boldsymbol{\psi}' = e^{i\omega_{\mu\nu}S^{\mu\nu}}P_\pm\boldsymbol{\psi} = (e^{i\omega_{\mu\nu}S^{\mu\nu}}P_\pm)(P_\pm\boldsymbol{\psi}), \tag{4.1.3}$$

where we used that $P_\pm P_\pm = P_\pm$. From that, we get

$$\psi'^R = (e^{i\omega_{\mu\nu}S^{\mu\nu}}P_+)\psi^R, \quad \psi'_L = (e^{i\omega_{\mu\nu}S^{\mu\nu}}P_-)\psi_L \tag{4.1.4}$$

This means that the general Lorentz transformation in each spinor space is given by

$$\Lambda_S = e^{i\omega_{\mu\nu}S^{\mu\nu}}P_\pm \tag{4.1.5}$$

which has 15 independent transformations and defines a set of 15 hermitian, unit determinant 4×4 matrices, which is exactly the definition of $SU(4)$. Now, instead of two component spinors as in SO(3,1), we will have four component spinors that transform according to:

$$\xi^A \Rightarrow \xi'^A = U^A{}_B \xi^B; \tag{4.1.6}$$

$$\zeta_A \Rightarrow \zeta'_A = \bar{U}_A{}^B \zeta_B, \tag{4.1.7}$$

where $\boldsymbol{U}$ is an $SU(4)$ matrix defined by (4.1.4) and $\bar{\boldsymbol{U}}$ its complex conjugate [23]. Note that there are no primed indices here, since that there's only one copy of $SU(4)$ in the isomorphism to $SO(6)$, where in the four dimensional case we have two copies of $SU(2)$ on the isomorphism with $SO(4)$. We denote the spinor representation by **4** and its complex conjugate by $\bar{\mathbf{4}}$, where the corresponding spinors have an upper index or a lower index, respectively.

We now pose the question: how do we go from vectors to spinors in six dimensions? To answer this, lets define the totally anti-symmetric symbols of SU(4), ϵ_{ABCD} and ϵ^{ABCD}, and see how they transform:

$$\epsilon_{ABCD} \Rightarrow \bar{U}_A{}^P \bar{U}_B{}^Q \bar{U}_C{}^R \bar{U}_D{}^S \epsilon_{PQRS} = \det U \epsilon_{ABCD} = \epsilon_{ABCD}, \tag{4.1.8}$$

where we used the defining condition for a SU matrix that $\det(U) = 1$. Now that we see that ϵ_{ABCD} is invariant under SU(4), and the same happens for ϵ^{ABCD}, then it might be related to the metric of the Euclidean space, which is invariant under SO(6). That way, the inner product must be defined as:

$$V^\mu g_{\mu\nu} V^\nu = \frac{1}{2} V^{AB} \epsilon_{ABCD} V^{CD}, \tag{4.1.9}$$

where $V^{AB} = V^{[AB]}$ is given by

$$V^{AB} = \Sigma^{AB}_\mu V^\mu. \tag{4.1.10}$$

Clearly, V^μ and V^{AB} share the same number of components, and since we also have that

$$V^{AB} = \frac{1}{2} \epsilon^{ABCD} V_{CD}, \tag{4.1.11}$$

$$V_{AB} = \frac{1}{2} \epsilon_{ABCD} V^{CD}, \tag{4.1.12}$$

then we can go from V^{AB} to V_{AB} by only making use of $\boldsymbol{\epsilon}$ and without complex conjugation. So, for both V^{AB} and V_{AB} we have the same representation which we denote by **6**.

4.1.1 Higher dimensional irreducible representations of SU(4)

It is also of interest to build higher dimensional irreducible representations of SU(4), irreps for short, if we want to express SO(6) tensors in the spinor formalism. For example, an antisymmetric tensor $T^{ab} = T^{[ab]}$ has 15 independent components, so we should look for a **15** representation in SU(4), and we can do this by taking direct products of smaller representations.

Equation (4.1.10) tells us that the corresponding SU(4) tensor of a SO(6) rank two tensor should have the form $T^{[AB][CD]} = \Sigma_a^{AB}\Sigma_b^{CD}T^{ab}$, which means that this tensor is in the $\mathbf{6}\otimes\mathbf{6}$ representation. Note, however, that $\mathbf{6}\otimes\mathbf{6} = \mathbf{36}$, so the idea is to "break" the **36** representation into a sum of irreducible representations [12].

As usual, we can write a tensor from SO(n) as a sum of its symmetric and anti-symmetric parts and its trace. For a general rank two tensor from SO(6), we have:

$$\underbrace{T^{ab}}_{\mathbf{36}} = \frac{1}{6}\underbrace{\delta^{ab}}_{\mathbf{1}} + \underbrace{A^{ab}}_{\mathbf{15}} + \underbrace{B^{ab}}_{\mathbf{20}}, \tag{4.1.13}$$

where $A^{ab} = T^{[ab]}$ and $B^{ab} = T^{(ab)}$, with $B^a{}_a = 0$. Then, we have:

$$\mathbf{6}\otimes\mathbf{6} = \mathbf{1}\oplus\mathbf{15}\oplus\mathbf{20}. \tag{4.1.14}$$

This is hardly striking news, and SU(4) didn't really have any special input on. For that special input, now consider the $\mathbf{4}\otimes\bar{\mathbf{4}}$ representation, which is made of objects of the form $T^A{}_B$. Again we are invited to take out its trace and, in the absence of further symmetries, we are left out with

$$\mathbf{4}\otimes\bar{\mathbf{4}} = \mathbf{1}\oplus\mathbf{15}. \tag{4.1.15}$$

The presence of **15** here means that in SU(4) an anti-symmetric tensor A^{ab} transforms as $A^A{}_B$, where $A^B{}_B = 0$. This means that $A^{ABCD} = A^{ab}$ can be written as

$$A^{ABCD} = A^{[A}{}_E\epsilon^{B]ECD} - A^{[C}{}_E\epsilon^{D]EAB}, \tag{4.1.16}$$

which can also be expressed by:

$$A^A{}_B = \frac{1}{4}A^{ACDE}\epsilon_{CDEB}. \tag{4.1.17}$$

This comes from the requirement that $A^{ABCD} = A^{[AB][CD]} = A^{[CD|AB]} = A^{[AB|CD]}$ and the fact that $\boldsymbol{\epsilon}$ is invariant under $SU(4)$. The spinor representation of $B^{ab} = B^{(ab)}$ will be given by [35]

$$B^{ab} = B^{AB}{}_{CD}, \qquad B^{AB}{}_{CB} = 0. \tag{4.1.18}$$

This is the basics of building higher dimensional irreducible representations for $SU(4)$. For higher representations, the idea is the same: explore the symmetries and take out the trace to express the tensor in its irreducible parts. The larger the dimension is, the harder it gets, so the use of Young Tableaux [36] is usually employed. For further details into SU(4) and its representations we refer to [37] and [38]

$\mathbf{4} \otimes \mathbf{4}$	$=$	$\mathbf{6} \oplus \mathbf{10}$
$\mathbf{4} \otimes \bar{\mathbf{4}}$	$=$	$\mathbf{1} \oplus \mathbf{15}$
$\mathbf{6} \otimes \bar{\mathbf{6}}$	$=$	$\mathbf{1} \oplus \mathbf{15} \oplus \mathbf{20}$
$\mathbf{10} \otimes \bar{\mathbf{10}}$	$=$	$\mathbf{1} \oplus \mathbf{15} \oplus \mathbf{84}$
$\mathbf{15} \otimes \bar{\mathbf{15}}$	$=$	$\mathbf{1} \oplus \mathbf{15} \oplus \mathbf{15} \oplus \mathbf{20} \oplus \mathbf{45} \oplus \bar{\mathbf{45}} \oplus \mathbf{84}$

Table 1 – Some useful irreps of SU(4)

As one example of interest, lets obtain the spinorial equivalent of the Riemann tensor from Table 1. The Riemann tensor can be defined by:

$$(\nabla_{AB}\nabla_{CD} - \nabla_{CD}\nabla_{AB})\kappa^E = \mathbb{R}_{ABCD}{}^E{}_F\kappa^F \tag{4.1.19}$$

where we note that $\mathbb{R}$ has the same symmetries in $\{ABCD\}$ as an antisymmetric tensor A_{ab} has on its four spinorial indices. So, from (4.1.17), we can write $\mathbb{R}$ as

$$R_{AB}{}^{CD} = \frac{1}{4}\epsilon^{EFGC}\mathbb{R}_{AEFGB}{}^D. \tag{4.1.20}$$

Before going further, note that one of the irreducible parts of the Riemann tensor is the Weyl tensor, which on n dimensions has $C_n = \frac{n}{12}(n+1)(n+2)(n-3)$ components [39]. So, for $n = 6$, the Weyl tensor is in the **84** representation, then we must look for representations that have **84** as one of its irreducible parts. On the other hand, the Riemann tensor has $R_n = \frac{n^2}{12}(n^2-1)$ components [40], which gives $R_6 = 105$ in six dimensions, so by taking a look at 1 we see that $\mathbf{10} \otimes \bar{\mathbf{10}}$ is too small for $\boldsymbol{R}$ to fit in, so we go for $\mathbf{15} \otimes \mathbf{15}$, where we see that $\mathbf{1} \oplus \mathbf{20} \oplus \mathbf{84}$ add up to **105** as we need, being the trace, the antisymmetric and symmetric parts of $R_{AB}{}^{CD}$, respectively.

$$R_{AB}{}^{CD} = \Lambda(\delta^C_A\delta^D_B - 4\delta^D_A\delta^C_B) + \Phi_{AB}{}^{CD} + \Psi_{AB}{}^{CD}, \tag{4.1.21}$$

where Λ, as in the four dimensional case, is related to the Ricci scalar, $\Phi_{AB}{}^{CD} = \Phi_{[AB]}{}^{[CD]}$ with $\Phi_{AB}{}^{AD} = 0$ is the spinor analogue of the Ricci tensor, and $\Psi_{AB}{}^{CD} = \Psi_{(AB)}{}^{(CD)}$ with $\Psi_{AB}{}^{AD} = 0$ is the spinor analogue of the Weyl tensor.

As a quick wrap up, Table 2 has the spinor analogues of the $SO(6)$ tensors we will consider in this work [35].

4.1.2 Null Vectors

As in the four-dimensional case, we can write a vector as a tensor product of two spinors. First, lets choose a spinor basis κ_i^A normalized such that:

$$\epsilon_{ABCD}\kappa_1^A\kappa_2^B\kappa_3^C\kappa_4^D = 1. \tag{4.1.22}$$

Then, we can expand any vector as a combination of totally antisymmetric tensor products of $\boldsymbol{\kappa_i}$:

$$\boldsymbol{V} = \alpha_1\boldsymbol{\kappa}_1 \wedge \boldsymbol{\kappa}_2 + \alpha_2\boldsymbol{\kappa}_1 \wedge \boldsymbol{\kappa}_3 + \alpha_3\boldsymbol{\kappa}_1 \wedge \boldsymbol{\kappa}_4 + \alpha_4\boldsymbol{\kappa}_2 \wedge \boldsymbol{\kappa}_3 + \alpha_5\boldsymbol{\kappa}_2 \wedge \boldsymbol{\kappa}_4 + \alpha_6\boldsymbol{\kappa}_3 \wedge \boldsymbol{\kappa}_4. \tag{4.1.23}$$

SO(6) Tensor	Corresponding SU(4) Tensor
V^{μ}	$V^{AB} = V^{[AB]}$
$A^{\mu\nu} = A^{[\mu\nu]}$	$A^{A}{}_{B}$; $A^{B}{}_{B} = 0$
$B^{\mu\nu} = B^{(\mu\nu)}$	$B^{AB}{}_{CD}$; $B^{AB}{}_{CB} = 0$
$T^{\mu\nu\rho} = T^{[\mu\nu\rho]}$	$\tau^{ABC}{}_{D} = T^{A[B}\delta^{C]}_{D} + \frac{1}{2}\bar{T}_{DE}\epsilon^{EABC}$; $\tau^{ABC}{}_{D} = \frac{1}{12}T^{CBAEFG}\epsilon_{EFGD}$
$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu}$; $R_{\mu[\nu\rho\sigma]} = 0$	$R_{AB}{}^{CD}$; $R_{AB}{}^{AD} = 0$; $R_{AB}{}^{CA} = -15\Lambda\delta^{C}_{B}$
$C_{\mu\nu\rho\sigma} = -C_{\nu\mu\rho\sigma} = -C_{\mu\nu\sigma\rho} = C_{\rho\sigma\mu\nu}$; $C_{\mu[\nu\rho\sigma]} = 0; C_{\mu\nu\rho}{}^{\nu} = 0$	$C_{AB}{}^{CD} = C_{(AB)}{}^{(CD)}$; $C_{AB}{}^{AD} = 0$

Table 2 – SO(6) tensors and its SU(4) equivalents

Now, note that if the expansion does not involve all of the basis spinors $\boldsymbol{\kappa}_i$, then the vector will necessarily be null. Let's build a vector using only $\{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}_3\}$ as an example:

$$V^{AB} = \alpha_1 \kappa_1^A \wedge \kappa_2^B + \alpha_2 \kappa_1^A \wedge \kappa_3^B + \alpha_3 \kappa_2^A \wedge \kappa_3^B. \tag{4.1.24}$$

Now, we calculate $V_{AB} = \frac{1}{2}\epsilon_{ABCD}V^{CD}$. From (4.1.22), we have:

$$V_{AB} = \alpha_1 \epsilon_{AB12} + \alpha_2 \epsilon_{AB13} + \alpha_3 \epsilon_{AB23}, \tag{4.1.25}$$

then, we have the inner product given by:

$$V_\mu V^\mu = \frac{1}{2} V_{AB} V^{AB} = 0, \tag{4.1.26}$$

because of the total antisymmetry of ϵ_{ABCD}. We can also recast (4.1.24) as:

$$V^{AB} = (\kappa_1 + \frac{\alpha_3}{\alpha_2}\kappa_2) \wedge (\alpha_1 \kappa_2 + \alpha_2 \kappa_3) = \chi^{[A}\xi^{B]}, \tag{4.1.27}$$

so as in the four dimensional case, a null vector can be expressed as the outer product of spinors. Now, consider a general spinor given by:

$$\kappa^A = a\kappa_1^A + b\kappa_2^A + c\kappa_3^A + d\kappa_4^A. \tag{4.1.28}$$

We can build a set of null vectors by doing outer products of this spinor with the basis spinors in the following way:

$$V_i^{AB} = \kappa^{[A}\kappa_i^{B]}. \tag{4.1.29}$$

Even though there are four elements in the spinor basis, there are only 3 linearly independent vectors defined by (4.1.29) as can be easily checked from

$$V_1^{AB} = -\frac{b}{a}V_2^{AB} - \frac{c}{a}V_3^{AB} - \frac{d}{a}V_4^{AB}. \tag{4.1.30}$$

Note that every vector in this subspace is a linear combination of the null vectors that generate it, and since a linear combination of a null vector is also a null vector, this null subspace is called isotropic 3-plane, and it's relation to the Kerr theorem will be discussed up next. The spinor κ^A is then associated with the null subspace generated by the set of three vectors $\{\boldsymbol{V}_i\}$, and a spinor with such property is called pure spinor [19, 41, 42].

4.2 The Kerr Theorem in 6 dimensions

Last chapter we obtained the Kerr Theorem by studying the congruences of a submanifold in Minkowski space. This time, we approach it differently, and first we need to talk about distributions and integrability.

Consider a n-dimensional Manifold M endowed with a metric g, an m-dimensional distribution [38,43] is a map that associates to every point $p \in M$ an m-dimensional vector space $U_p \in T_pM$. If a set of vector fields $\{\boldsymbol{V}_i\}$ span the vector space $U_p \; \forall \; p \in M$, we say that they generate the distribution, which is said to be integrable when there exists a family of submanifolds of M such that their tangent spaces are U_p.

Now, given a set of m vector fields $\{\boldsymbol{V}_i\}$, then the tangent space can be described by $\{\boldsymbol{V}_i, \boldsymbol{W}_j\}$, where $\{\boldsymbol{W}_j\}$ is a set of $(n-m)$ vector fields. Also, we have the dual frame $\{\boldsymbol{\vartheta}^i, \boldsymbol{\omega}^j\}$ with the relations $\boldsymbol{\vartheta}^i \cdot \boldsymbol{V}_j = \boldsymbol{\omega}^i \cdot \boldsymbol{W}_j = \delta^i_j$ and $\boldsymbol{\vartheta}^i \cdot \boldsymbol{W}_j = \boldsymbol{\omega}^i \cdot \boldsymbol{V}_j = 0$. The Frobenius theorem says that the distribution generated by the set $\{\boldsymbol{V}_i\}$ is integrable if, and only if:

$$d\boldsymbol{\omega}^k \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge ... \wedge \boldsymbol{\omega}^{(n-m)} = 0 \quad \forall k \in \{1, ..., n-m\}. \tag{4.2.1}$$

The Frobenius theorem as presented here is in its dual formulation. For a more complete account, see [43].

Now, back to the case at hand, let's see how the Kerr theorem, defined before as an implicit solution to all NSF congruences in flat space, relates to the problem of existence of integrable null distributions. We begin with flat 6-dimensional space with signature $(-, +, +, +, +, +)$ and coordinates given by:

$$ds^2 = dudv + dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2. \tag{4.2.2}$$

Then we start with the following fields of null 1-forms [44]:

$$e_0 = du + \bar{Y}^i dz_i + Y^i d\bar{z}_i - Y_i \bar{Y}^i dv, \qquad e_i = dz_i - \bar{Y}_i dv. \tag{4.2.3}$$

According to the Frobenius theorem, these null 1-forms will determine an integrable distribution if they satisfy (4.2.1). Then, the condition that $de_0 \wedge e_0 \wedge e_1 \wedge e_2 = 0$ gives us the following equations:

$$\partial_v Y^1 + Y^2 \partial_2 Y^1 + Y^1 \partial_1 Y^1 - Y^1 \bar{Y}^1 \partial_u Y^1 - Y^1 \bar{Y}^2 \partial_u Y^2 + \bar{Y}^1 \bar{\partial}_1 Y^1 + \bar{Y}^2 \bar{\partial}_1 Y^2 = 0, \tag{4.2.4}$$

$$\partial_v Y^2 + Y^2 \partial_2 Y^2 + Y^1 \partial_1 Y^2 - Y^2 \bar{Y}^1 \partial_u Y^1 - Y^2 \bar{Y}^2 \partial_u Y^2 + \bar{Y}^1 \bar{\partial}_2 Y^1 + \bar{Y}^2 \bar{\partial}_2 Y^2 = 0, \tag{4.2.5}$$

$$\bar{\partial}_2 Y^1 - \bar{\partial}_1 Y^2 + Y^1 \partial_u Y^2 - Y^2 \partial_u Y^1 = 0, \tag{4.2.6}$$

$$\begin{aligned}&Y^2(\partial_v Y^1 + \bar{Y}^2\bar{\partial}_1 Y^2 - Y^1\partial_2 Y^2 + Y^2\partial_2 Y^1 + \bar{Y}^1\bar{\partial}_1 Y^1)\\ &- Y^1(\partial_v Y^2 + \bar{Y}^1\bar{\partial}_2 Y^1 + Y^1\partial_1 Y^2 - Y^2\partial_1 Y^1 + \bar{Y}^2\bar{\partial}_2 Y^2) = 0,\end{aligned} \tag{4.2.7}$$

and the conditions $de_i \wedge e_0 \wedge e_1 \wedge e_2 = 0$ give:

$$\bar{\partial}_1 Y^i - Y^1\partial_u Y^i = 0, \tag{4.2.8}$$

$$\bar{\partial}_2 Y^i - Y^2\partial_u Y^i = 0, \tag{4.2.9}$$

$$Y^2\bar{\partial}_1 Y^i - Y^1\bar{\partial}_2 Y^i = 0. \tag{4.2.10}$$

These equations can be solved, but first they must be decoupled. Note that (4.2.8) and (4.2.9) imply (4.2.6) and (4.2.10). We also see that if we multiply (4.2.8) by $\bar{Y}^1$ and set $i = 1$, we get:

$$\bar{Y}^1\bar{\partial}_1 Y^1 - \bar{Y}^1 Y^1\partial_u Y^1 = 0, \tag{4.2.11}$$

which gets rid of two terms in (4.2.4). If we multiply (4.2.8) by $\bar{Y}^2$ and set $i = 2$ we get

$$\bar{Y}^2\bar{\partial}_1 Y^2 - \bar{Y}^2 Y^1\partial_u Y^2 = 0, \tag{4.2.12}$$

which gets rid of another two terms in (4.2.4). By doing the same thing between (4.2.5) and (4.2.9), we get the decoupled equations given by:

$$(\bar{\partial}_i - Y^i\partial_u)Y^j = 0, \qquad (\partial_v + Y^i\partial_i)Y^j = 0, \tag{4.2.13}$$

for $i, j = 1, 2$, and their complex conjugate. We recognize the resemblance of these equations with (3.3.31), and indeed they can also be solved by the method of characteristics [29], whose solution is given implicitly by:

$$F(Y^1, Y^2, vY^1 - z_1, vY^2 - z_2, u + \bar{z}_1 Y^1 + \bar{z}_2 Y^2) = 0, \tag{4.2.14}$$

where F is an general function of five complex arguments. To obtain the spinoral interpretation as we did last chapter, let us introduce the spinors:

$$\kappa^A = \begin{bmatrix} Y^1 \\ Y^2 \\ 0 \\ 1 \end{bmatrix} \qquad \bar{\kappa}^A = \begin{bmatrix} \bar{Y}^2 \\ -\bar{Y}^1 \\ 1 \\ 0 \end{bmatrix}, \tag{4.2.15}$$

which, as seen, generate the null vector $k^a = \kappa^{[A}\bar{\kappa}^{B]}$, and $\bar{\kappa}^A$ is related to κ^A by:

$$\bar{\kappa}^A = B^{AB}\kappa^*_B, \qquad B = 1 \otimes i\sigma^2. \tag{4.2.16}$$

where B amounts to a choice of a real slice of $SU(4)$, which in turn gives $k^a = \kappa^{[A}\bar{\kappa}^{B]}$ as a real vector. Now, we introduce the position vector $x_{AB} = \Sigma^{\mu}_{AB}x_{\mu}$ in spinor space given by:

$$x_{AB} = \begin{bmatrix} 0 & v & -\bar{z}_1 & -z_2 \\ -v & 0 & -\bar{z}_2 & z_1 \\ \bar{z}_1 & \bar{z}_2 & 0 & u \\ z_2 & -z_1 & -u & 0 \end{bmatrix}, \tag{4.2.17}$$

so we can write (4.2.14) as:

$$F(\kappa^A, \xi_B) = 0, \tag{4.2.18}$$

where $\xi_A = x_{AB}\kappa^B$. Because of the antisymmetry of x_{AB}, we see that (4.2.18) needs only to be defined for $\kappa^A\xi_A = 0$. Again, this defines implicitly the notion of a twistor $Z^{\alpha} = (\kappa^A, \xi_B)$, and its equation will be investigated shortly.

4.3 The Kerr Theorem in Generic Even Dimensions

Lets now make a more generic discussion about the Kerr theorem. A flat, $n = 2m$-dimensional Lorentzian metric can be expressed by null coordinates as

$$ds^2 = dudv + \sum_{i=1}^{m-1} dz_i d\bar{z}_i. \tag{4.3.1}$$

In such manifold, we will have the null 1-forms that determine an integrable distribution defined in a analogous way

$$e_0 = du + \bar{Y}^i dz_i + Y^i d\bar{z}_i - Y_i\bar{Y}^i dv, \quad e_i = dz_i - \bar{Y}_i dv, \quad i = 1, 2, ..., m-1, \tag{4.3.2}$$

where we see that a $2m$-dimensional space will have at maximum a m dimensional null integrable subspace defined by the m null 1-forms, and as such are called maximally isotropic subspaces. (4.3.2) together with the Frobenius theorem will now lead to the differential equations

$$(\bar{\partial}_i - Y^i\partial_u)Y^j = 0, \quad (\partial_v + Y^i\partial_i)Y^j = 0, \quad i, j = 1, ..., m-1 \tag{4.3.3}$$

and their complex conjugates. Their solutions will, again, be given implicitly by a homogeneous function of $2m-1$ complex arguments [29]

$$F(\{Y^i\}, \{vY^i - z_i\}, u\bar{z}_iY^i) = 0. \tag{4.3.4}$$

Note that, for the four dimensional case, we have now arrived at this result from two apparently distinct starting points that are actually equivalent, that is, the requirement that $\kappa = \sigma = 0$ determines the existence of (4.3.2) that spam an integrable distribution and vice versa. Defining and working with the Newman-Penrose spin coefficient method in higher dimensions can be cumbersome, since there'll be a lot more directional derivatives, more spinors in the basis, and consequently, more coefficients, so starting with the null forms defining the distribution is more reasonable.

4.4 Isotropic Subspaces

Now, we show how we can define an n-dimensional isotropic subspace through a simple null n-form [38]. In the four-dimensional case, for example, the maximally isotropic subspace would be a 2-plane defined by $\boldsymbol{B} = e_0 \wedge e_1$. In six dimensions, however, we have three different 2-planes defined by the 2-forms $\boldsymbol{B}_1 = e_0 \wedge e_1$, $\boldsymbol{B}_2 = e_0 \wedge e_2$ and $\boldsymbol{B}_3 = e_1 \wedge e_2$ and one 3-plane defined by $\boldsymbol{T} = e_0 \wedge e_1 \wedge e_2$, let's see how these are expressed in spinorial language in a discussion that follows [35].

For a one-dimensional isotropic subspace, we would need only a null vector to fully describe it, so this space is said to be generated by $V^\mu = \xi^{[A}\eta^{B]}$ as described on our section about null vectors in six dimensions.

For a two dimensional isotropic subspace, we need two null vectors $\{V_1^\mu, V_2^\mu\}$ satisfying $V_{1\mu}V_1^\mu = V_{2\mu}V_2^\mu = V_{1\mu}V_2^\mu = 0$ so that $B^{\mu\nu} = B^{[\mu\nu]} = V_1^{[\mu}V_2^{\nu]}$. In spinorial language this would read

$$B^{\mu\nu} = B^{ABCD} = \frac{1}{2}(\xi^{[A}\eta^{B]}\xi^{[C}\kappa^{D]} - \xi^{[C}\eta^{D]}\xi^{[A}\kappa^{B]}), \tag{4.4.1}$$

but since a anti-symmetric bivector [45] can be expressed by $B^A{}_B = \frac{1}{4}B^{AEFG}\epsilon_{EFGB}$ where $B^A{}_A = 0$, we obtain

$$B^A{}_B = \frac{1}{8}\xi^A(\epsilon_{BCDE}\xi^C\eta^D\kappa^E), \tag{4.4.2}$$

where we write $\gamma_B = \frac{1}{8}\epsilon_{BCDE}\xi^C\eta^D\kappa^E$. For a three dimensional isotropic subspace, we will need three vectors $\{V_1^\mu, V_2^\mu, V_3^\mu\}$ satisfying $V_{i\mu}V_j^\mu = 0, \forall i,j = 1,2,3$ so that $T^{\mu\nu\rho} = T^{[\mu\nu\rho]} = V_1^{[\mu}V_2^\nu V_3^{\rho]}$. Since the three vector will be equivalent to the pair $(T^{AB}, \bar{T}_{AB})$ from Table (2), there will be two ways to achieve this:

1) $V_1^\mu = \xi^{[A}\eta^{B]}$, $V_2^\mu = \xi^{[A}\kappa^{B]}$ and $V_3^\mu = \xi^{[A}\chi^{B]}$, where

$$(T^{AB}, \bar{T}_{AB}) = (\xi^A\xi^B, 0), \tag{4.4.3}$$

2) $V_1^\mu = \xi^{[A}\eta^{B]}$, $V_2^\mu = \xi^{[A}\kappa^{B]}$ and $V_3^\mu = \eta^{[A}\kappa^{B]}$, where

$$(T^{AB}, \bar{T}_{AB}) = (0, \gamma_A\gamma_B), \tag{4.4.4}$$

and $\gamma_A = \epsilon_{ABCD}\xi^B\eta^C\kappa^D$. T^{ABC} as determined by (4.4.3) is a self-dual tensor, while (4.4.4) is an anti self-dual tensor.

4.5 Six-Dimensional Twistors

Here, we discuss a few general facts about twistors in six dimensions in comparison with the four-dimensional counterpart and its relation to integrability of 3 planes. We start by remarking that, as in the four-dimensional case, for a flat spacetime the twistor equation [46]

$$\nabla_{AB}\kappa^{C} = \pi_{[A}\delta^{C}_{B]} \tag{4.5.1}$$

has the solution $\kappa^A = \psi^A + x^{AB}\pi_B$ which defines the object we call a twistor $Z^\alpha = (\kappa^A, \pi_B)$ with an inner product defined by

$$Z^\alpha W_\alpha = \kappa^A\lambda_A + \pi_B\xi^B \tag{4.5.2}$$

for a twistor $W^\alpha = (\xi^B, \lambda_A)$, in complete analogy with the four dimensional case.

Now, as we saw last section, the subspace generated by the 3 vector $T_{abc} = T_{[abc]}$, which is completely determined by the pair $(T^{AB}, \tilde{T}^{AB})$, is an self-dual isotropic 3-plane when $\tilde{T}^{AB} = 0$ and $T^{AB} = \kappa^A\kappa^B$. From the Frobenius' theorem, this will determine an integrable distribution if:

$$\nabla_{AB}T^{AB} = \nabla_{AB}\kappa^A\kappa^C = 0 \Rightarrow \kappa^A\nabla_{AB}\kappa^C = -\kappa^C\nabla_{AB}\kappa^A, \tag{4.5.3}$$

which can be solved by the six-dimensional twistor equation (4.5.1) for $\pi_A\kappa^A = 0$, as can be seen from

$$\kappa^A\nabla_{AB}\kappa^C = \kappa^A\pi_{[A}\delta^C_{B]} = -\frac{1}{2}\kappa^C\pi_B = -\kappa^C\nabla_{AB}\kappa^A. \tag{4.5.4}$$

Now, the Ricci identity (4.1.19) gives an integrability condition for κ^A:

$$(\nabla_{AB}\nabla_{CD} - \nabla_{CD}\nabla_{AB})\kappa^E = \mathbb{R}_{ABCDF}{}^{E}\kappa^F, \tag{4.5.5}$$

and, by contracting with ϵ^{BCDG}, we get

$$\epsilon^{BCDG}(\nabla_{AB}\nabla_{CD} - \nabla_{CD}\nabla_{AB})\kappa^E = R_{AF}{}^{GE}\kappa^F \tag{4.5.6}$$

where, as we defined previously, $R_{AB}{}^{CD} = \epsilon^{FGHC}\mathbb{R}_{AFGHB}{}^{D}$. Now, by using that (4.5.1), we have

$$\begin{aligned} &\epsilon^{BCDG}(\nabla_{AB}\pi_{[C}\delta^E_{D]} - \nabla_{CD}\pi_{[A}\delta^E_{B]}) = \\ &= \frac{\epsilon^{BCEG}}{2}\nabla_{AB}\pi_C - \frac{\epsilon^{BEDG}}{2}\nabla_{AB}\pi_D - \frac{\epsilon^{ECDG}}{2}\nabla_{CD}\pi_A + \frac{\epsilon^{BCDG}}{2}\delta^E_A\nabla_{CD}\pi_B \end{aligned}, \tag{4.5.7}$$

which leads to

$$\epsilon^{BCEG}\nabla_{AB}\pi_C - \epsilon^{BEDG}\nabla_{AB}\pi_D - \epsilon^{ECDG}\nabla_{CD}\pi_A + \epsilon^{BCDG}\delta^E_A\nabla_{CD}\pi_B = 2R_{AF}{}^{GE}\kappa^F. \tag{4.5.8}$$

Now, by contracting A and E, we will have $R_{AF}{}^{GA}$ on the right side, which by (4.1.21) we know to be

$$\epsilon^{BCAG}\nabla_{AB}\pi_C - \epsilon^{BADG}\nabla_{AB}\pi_D - \epsilon^{ACDG}\nabla_{CD}\pi_A + 4\epsilon^{BCDG}\nabla_{CD}\pi_B = -30\Lambda\delta^G_F\kappa^F, \tag{4.5.9}$$

which, by assigning the right letters to the summed indices, we get:

$$\epsilon^{ABCD}\nabla_{AB}\pi_C = -6\Lambda\kappa^D. \tag{4.5.10}$$

The anti-symmetric part of the left hand side gives us

$$\nabla_{AB}\pi_C = -\Lambda\epsilon_{ABCD}\kappa^D. \tag{4.5.11}$$

which we can contract it with ϵ^{ABEF}, we obtain

$$\nabla^{AB}\pi_C = -4\Lambda\delta_C^{[A}\kappa^{B]}. \tag{4.5.12}$$

Now, by taking the symmetric traceless part of the EG indexes in (4.5.8) we get an algebraic condition on the Weyl spinor

$$\Psi_{AF}{}^{EG}\kappa^F = 0, \tag{4.5.13}$$

and by doing an analogous analysis to π_C and using (4.5.12), we obtain the similar condition

$$\Psi_{AF}{}^{EG}\pi_G = 0. \tag{4.5.14}$$

With a more unifying view, we see the Weyl spinor $\Psi_{AB}{}^{CD}$ as an operator that has the twistor $Z^\alpha = (\kappa^A, \pi_C)$ as an eigenvector with zero eigenvalue. Spinors with the properties (4.5.13) and (4.5.14) are called principal spinors.

4.6 Symmetries and Quaternions

As seen, the Kerr Theorem defines an integrable 3-plane generated by the spinors $\{\kappa^A, \zeta_B\}$. Since our study of integrability is concerned with the 3-plane, we can ask ourselves what is the set of transformations that leave the plane invariant, and that would be

$$\kappa^A \to \alpha\kappa^A + \beta\bar{\zeta}^A, \quad \zeta_B \to \alpha\zeta_B + \beta\bar{\kappa}_B. \tag{4.6.1}$$

Since a linear transformation on the generators of the plane defines the same plane. This transformation can be understood in terms of quaternions, so lets first review the properties of its algebra $\mathbb{H}$. A quaternion [47] $q \in \mathbb{H}$ is a set of four real numbers $\{q_0, q_1, q_2, q_3\}$ written in the form

$$q = q_0 + q_1 i + q_2 j + q_3 k, \tag{4.6.2}$$

where $\{i, j, k\}$ are Hamilton's imaginary numbers that obey

$$i^2 = j^2 = k^2 = ijk = -1. \tag{4.6.3}$$

Now, a set of four complex numbers $\{z_1, z_2, z_3, z_4\}$, we can embed them into a pair a quaternions by doing

$$(z_1, z_2, z_3, z_4) \to (z_1 + jz_2, z_3 + jz_4) \tag{4.6.4}$$

which is a quaternion since $z_i = z_i' + iz_i''$, where $\{z_i', z_i''\} \in \mathbb{R}$, so

$$(z_1 + jz_2, z_3 + jz_4) = (z_1' + iz_1'' + jz_2' - kz_2'', z_3' + iz_3'' + jz_4' - kz_4'') \tag{4.6.5}$$

so indeed we have $(z_1 + jz_2, z_3 + jz_4) = (a, b)$ where $\{a, b\} \in \mathbb{H}$. This allows us to see symmetry (4.6.1) as a right multiplication by a quaternion $q = \alpha + \beta j$ on the spinors (κ^A, ζ_B), so that

$$\begin{bmatrix} \kappa^A \\ \zeta_B \end{bmatrix} = \begin{bmatrix} \alpha\kappa^A + \beta\bar{\zeta}^A \\ \alpha\zeta_B + \beta\bar{\kappa}_B \end{bmatrix}. \tag{4.6.6}$$

So, in six dimensions an integrable 3-plane corresponds to multiple spinors related to each other by (4.6.6) through the choice a quaternionic line. This means that the Kerr theorem defines a function F on $\mathbb{HP}^1$, the one dimensional quaternionic projective space [48].

4.7 The Kerr-Schild Class of Solutions

An important family of spaces satisfying the Kerr theorem, and consequently allowing for a principal spinor, is given by the Kerr-Schild spaces, defined by the metric

$$g_{ab} = \bar{g}_{ab} - 2Sk_a k_b, \tag{4.7.1}$$

where $\bar{g}_{ab}$ is a maximally symmetric, S is a function of the coordinates satisfying $DS = k^a\nabla_a S = 0$ and k^a is a geodesic, shear-free null vector field with respect to $\bar{g}_{ab}$. Those informations are actually not stated beforehand, but obtained through the field equations [49, 50]. Now, defining the covariant derivatives associated with each metric $\nabla_a g_{bc} = \bar{\nabla}_a \bar{g}_{bc} = 0$, their relation is given by the relative connection C^c_{ab} between the derivatives defined as

$$(\bar{\nabla}_a - \nabla_a)\omega_b = C^c_{ab}\omega_c \tag{4.7.2}$$

and given by [40]

$$C^c_{ab} = \frac{1}{2}g^{cd}(\bar{\nabla}_a g_{bd} + \bar{\nabla}_b g_{ad} - \bar{\nabla}_d g_{ab}), \tag{4.7.3}$$

which, by inserting the metric (4.7.1) and using the defined properties of S and k_a, we get

$$C^c_{ab} = (\bar{\nabla}_a S)k^c k_b + (\bar{\nabla}_b S)k^c k_a - (\bar{\nabla}^c S)k_a k_b + 2S(DS)k^c k_a k_b. \tag{4.7.4}$$

With that, we use that Ricci identity to compute the relation between the Riemann tensors of each metric

$$\begin{aligned} R_{abc}{}^d\omega_d &= (\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c = \\ &= (\nabla_a(\bar{\nabla}_b\omega_c - C^d_{bc}\omega_d) - \nabla_b(\bar{\nabla}_a\omega_c - C^d_{ac}\omega_d)) \\ &= (\bar{R}_{abc}{}^d - \bar{\nabla}_a C^d_{bc}\bar{\nabla}_b C^d_{ac} + C^e_{ac}C^d_{eb} - C^e_{bc}C^d_{ea})\omega_d \end{aligned} \tag{4.7.5}$$

where $\bar{R}_{abc}{}^d$ is maximally symmetric, so:

$$\bar{R}_{abcd} = \frac{\bar{R}}{30}(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{4.7.6}$$

So, by lowering 'd' on the Riemann tensor, applying the properties, and using (4.7.4), we get

$$R_{abcd} = \bar{R}_{abcd} - \frac{R}{15} S(k_{[a}\bar{g}_{b]c}k_d - k_{[a}\bar{g}_{b]d}k_c) - 2k_{[a}(\bar{\nabla}_{b]}\bar{\nabla}_c S)k_d - 2k_{[a}(\bar{\nabla}_{b]}\bar{\nabla}_d S)k_c. \quad (4.7.7)$$

Now, the Weyl tensor in 6 dimensions is defined as [39], [2]

$$C_{abcd} = R_{abcd} + \frac{1}{4}(R_{ad}g_{bc} - R_{ac}g_{bd} + R_{dc}g_{ab} - R_{bd}g_{ac}) + \frac{R}{30}(g_{ac}g_{bd} - g_{ad}g_{bc}), \quad (4.7.8)$$

with that, we compute $C_{abcd}k^b k^d$ and obtain

$$C_{abcd}k^b k^d = R_{abcd}k^c k^d - \frac{R}{30}k_a k_c, \quad (4.7.9)$$

and from (4.7.7) we see that $R_{abcd}k^b k^d = \bar{R}_{abcd}k^b k^d = \frac{R}{30}k_a k_c$, so

$$C_{abcd}k^b k^d = 0. \quad (4.7.10)$$

Now, the Weyl spinor is given by

$$\Psi_{AE}{}^{IJ} = C_{[AB][CD][EF][GH]}\epsilon^{BCDI}\epsilon^{FGHJ}, \quad (4.7.11)$$

so, in spinorial language, (4.7.10) translates to

$$\Psi_{AE}{}^{IJ}\kappa^{[A}\bar{\kappa}^{B]}\kappa^{[E}\bar{\kappa}^{F]} = 0, \quad (4.7.12)$$

where we wrote $k^b = \kappa^{[A}\bar{\kappa}^{B]}$. If we complete $\{\kappa^A, \bar{\kappa}^B\}$ to a basis such that $\epsilon^{ABCD} = \kappa^{[A}\bar{\kappa}^{B}(\pi^*)^{C}(\bar{\pi}^*)^{D]}$, then we also have

$$\Psi_{AE}{}^{IJ} = \Psi_{AE}{}^{IJ}\pi_{[I}\bar{\pi}_{B]}\pi_{[J}\bar{\pi}_{F]}, \quad (4.7.13)$$

these two equations constrain the Weyl tensor so that $\Psi_{AE}{}^{IJ}\kappa^E = 0$, as we concluded last section.

5 CONCLUSIONS

As seen, the study of spinors makes for an important tool in general relativity and can lead to a variety of techniques that can help us discover new solutions of Einstein's field equations or to better analyze existing ones. However, the spinor approach doesn't generalize trivially to higher dimensions, and a more careful study of symmetries of the space considered in terms of representation theory is necessary, so the spinor formulation of general relativity will have different subtleties in each different dimensions, but besides that, the Kerr theorem can be generalized to arbitrary even dimensions [51].

The study of integrable planes, both in the four-dimensional and the six-dimensional case, led us to the concept of a twistor, and its equation was showed to restrict the Weyl tensor, a property shared by the family of solutions known as Kerr-Schild spaces. This was, however, a somewhat trivial example which is already well-known in the literature, and we hope to address other cases in the future. The choice of studying physics in six dimensions could be seen as kind of arbitrary, but since the latter half of the last century, higher dimensional physics was shown to be an invaluable laboratory for theoretical physics and, in particular, the study of integrability in six dimensions is interesting because it can be applied to lower-dimensional, conformally flat spaces such as AdS_5, where it will be seen as embedded in a higher-dimensional space. An important case is the one where we have a $(4,2)$ signature, so the symmetry group is given by $SO(4,2)$, which can be seen as the conformal group for four-dimensional Minkowski space, and twistor methods find their way on applications to the AdS/CFT correspondence [52].

The twistor programme was used by Penrose to describe solutions of wave equations and of massless fields in twistor space by making use of what we call the Penrose transform [9], so we want to check if this idea can be used in six dimensions and how it connects to these lower-dimensional embedded spaces. That being said, we worked in a $(5,1)$ signature, but the results could be generalized by working with a complexified space and then taking the suitable real slices by imposing complex conjugate relations between the coordinates [38].

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