Pós-Graduação em Ciência da Computação

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A λ-Model with ∞-Groupoid Structure based in the λ-Model Scott's D_∞

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Recife
2020

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Dissertação apresentada ao Programa de Pós-Graduação em Ciências da Computação da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Ciências da Computação.

Área de Concentração: Teoria da Computação; Calculo Lambda e Matemática Construtiva.

Orientador: Prof. Dr. Ruy de Queiroz

Recife

2020

Catalogação na fonte
Bibliotecária Monick Raquel Silvestre da S. Portes, CRB4-1217

R625m Rivillas, Daniel Orlando Martinez
A λ-model with ∞-groupoid structure based in the Scott's λ-model D_∞ / Daniel Orlando Martinez Rivillas. – 2020.
44 f.: il., fig.

Orientador: Ruy José Guerra Barretto de Queiroz.
Dissertação (Mestrado) – Universidade Federal de Pernambuco. CIn, Ciência da Computação, Recife, 2020.
Inclui referências.

1. Teoria da computação. 2. Cálculo lambda. I. Queiroz, Ruy José Guerra Barretto de (orientador). II. Título.

004 CDD (23. ed.) UFPE - CCEN 2020 - 154

Daniel Orlando Martinez Rivillas

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Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Ciência da Computação da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Mestre em Ciência da Computação.

Aprovado em: 28/02/2020.

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Dedico o resultado deste esforço à memória dos meus pais

AGRADECIMENTOS

Antes de tudo, quero agradecer ao meu orientador, Ruy de Queiroz, por sua colaboração em todo o processo de aprendizado na área dos fundamentos matemáticos da computação e das orientações de pesquisa sobre a surpreendente conexão entre topologia algébrica e computação. Agradeço a toda a minha família: aos meus pais José Hilario e Luz Nelly que continuam me acompanhando na memória, aos meus irmãos Alex, Mao e Juancho que sempre estavam pendentes, apesar da distância, a minha esposa Clarena, que sempre me acompanhava mesmo até nos momentos mais difíceis durante toda a estadia neste mestrado, e aos outros parentes que de alguma forma me apoiaram nesta estadia. Muito obrigado a todos por fazer este trabalho possível.

ABSTRACT

The lambda calculus is a universal programming language that represents the functions computable from the point of view of the functions as a rule, that allow the evaluation of a function on any other function. This language can be seen as a theory, with certain pre-established axioms and inference rules, which can be represented by models. Dana Scott proposed the first non-trivial model of the extensional lambda calculus, known as D_∞, in order to represent the λ-terms as the typical functions of set theory, where it is not allowed to evaluate a function about itself. This thesis propose a construction of an ∞-groupoid from any lambda model endowed with a topology. We apply this construction for the particular case D_∞ and we observe that the Scott topology does not provide relevant information about of the relation between higher equivalences. This motivates the search for a new line of research focused on the exploration of λ-models with the structure of a non-trivial ∞-groupoid to generalize the proofs of term conversion (e.g., β-equality, η-equality) to higher proof in λ-calculus.

Keywords: Lambda calculus. Extensional lambda model. Infinity groupoid.

RESUMO

O cálculo lambda é uma linguagem de programação universal que representa as funções computáveis do ponto de vista das funções como regra, que permitem a avaliação de uma função em qualquer outra função. Essa linguagem pode ser vista como uma teoria, com certos axiomas e regras de inferência pré-estabelecidos, que podem ser representados por modelos. Dana Scott propôs o primeiro modelo não-trivial do cálculo lambda extensional, conhecido como D_∞, para representar os λ-termos como as funções típicas da teoria dos conjuntos, onde não é permitido avaliar uma função sobre si mesmo. Esta tese propõe a construção de um ∞-groupoid a partir de qualquer modelo lambda dotado de uma topologia. Aplicamos esta construção para o caso particular D_∞ e observamos que a topologia Scott não fornece informações relevantes sobre a relação entre equivalências superiores. Isso motiva uma nova linha de pesquisa focada na exploração de λ-modelos com a estrutura de um ∞-groupoide não trivial para generalizar as provas de conversão de termos (e.g., β-igualdade, η-equalidade) à provas do ordem superior em λ-calculus.

Palavras-chaves: Calculo lambda. Modelo extensional lambda. Groupoide infinito.

LISTA DE ABREVIATURAS E SIGLAS

c.p.o.	*complete partial order*
Caml	*Categorical Abstract Machine Language*
ML	*Metalanguage*
MLTT	*Martin-Löf Type Theory*
SML	*Standard Metalanguage*

LISTA DE SÍMBOLOS

$\mathbb{N}$	The numbers natural set
D, D', X, Y	Arbitrary sets
$a, b, \ldots x, y$	Members of these sets
f, g, F, G, ϕ, ψ	Functions
$\in$	Belongs
τ	Topology
$\langle D, \tau \rangle$	Topology space
$\bot$	The least element in D
$\sqsubseteq$	Partial ordering in D
$\bigsqcup X$	the least upper bound (supremum) of X
$\langle D, \sqsubseteq \rangle$	c.p.o.
$(D \to D')$	The the set of all functions from D to D'
$\mathfrak{D}$	Infinity groupoid generated by space D
$\pi_n(D, d)$	The n-group fundamental based in the point d in the space D
$\pi_\infty(D)$	The ∞-group fundamental based in the space D
$\Pi_\infty(D, a, b)$	The fundamental ∞-groupoid based in the points $a, b \in D$
$\Pi_\infty(D)$	The fundamental ∞-groupoid on topology space D

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SUMÁRIO

1 INTRODUCTION

In what follows we put in context the research problem in light of the related publications to the topic of this thesis, together with the objectives and organization of the work.

1.1 CONTEXTUALIZATION AND MOTIVATION

The lambda calculus is a programming language in which the computable functions are seen as rules instead of sets. Since the origin of Computer Science, lambda calculus has been widely used, since it constitutes the essence of functional programming languages *Call-by-Value* and the languages of the *Metalanguage* (ML) family, such as *Categorical Abstract Machine Language* (Caml), *Standard Metalanguage* (SML) and Haskell. It is also of great interest the study of lambda calculus from the point of view of the types, since it allows the detection of errors without the need to execute a given program. There are extensions of typed lambda calculus such as *Martin-Löf Type Theory* (MLTT), also known as *Intuitionistic Type Theory* or *Intentional Type Theory*, where unlike lambda calculus there are dependent types such as identity type $I_A(a, b)$, with a and b terms of a type A.

Under the so-called Curry-Howard isomorphism, the type $I_A(a, b)$ represents the proposition which says that a is equal to b in the type A, and its terms (if these exist) would be proofs of this equality. If there is a proof p of $I_A(a, b)$, this does not imply that $a = b$ in the primitive or extensional sense of equality, since it can happen that a and b are intentionally equal, but not necessarily extensionally equal. In the other direction, if $a = b$ this implies that there is an equal proof of $I_A(a, b)$. Thus, $I_A(a, b)$ is a weaker type of equality than primitive equality, but it can gather more information regarding the multiple proofs of the equality. The type $I_A(a, b)$ is known as a *propositional equality.*

Furthermore, let two proofs of equality p and q in $I_A(a, b)$, then we can consider the type $I_{I_A(a,b)}(p, q)$, and we can continue iterating indefinitely to obtain an infinite sequence of higher identity types, which carries an algebraic structure known as ∞-groupoid.

In Homotopy Type Theory (HoTT), the types of MLTT are interpreted as topological spaces, and proofs of identity p of $I_A(a, b)$ are seen as continuous paths from a to b. The proofs h of identity proofs in $I_{I_A(a,b)}(p, q)$ are interpreted as homotopies h from p to q, and so on, the fundamental ∞-groupoid $\Pi_\infty A(a, b)$ is obtained this way.

Since MLTT is a formalization of typed lambda calculus, it can also carry an algebraic ∞-groupoid structure; if the types are seen, not simply as sets, but as topological spaces, one has a rich mathematical structure to model complex phenomena such as computations. Our motivation is to study type-free lambda calculus from a model that allows us to clearly see the ∞-groupoid structure.

1.2 RELATED PUBLICATIONS

The system of MLTT with identity types (MARTIN-LÖF, 1975) was originally developed to give a formalization of proofs of identity statements. By studying the relationship between two elements of a certain type (i.e., two proofs of a proposition), the relation of two proofs of identity proofs, and so on, one could have the formal counterpart to such a hierarchical structure of globular sets. Later (HOFMANN; STREICHER, 1994) comes up with the idea of using higher order categories for the interpretation of MLTT, and later on (AWODEY; WARREN, 2009) manages to establish the connection between MLTT and algebraic topology, in the sense of which types of identity can be interpreted as equivalence of homotopies. This led (VOEVODSKY, 2010) to formulate the Univalence Axiom , which gives rise to Homotopy Type Theory (HoTT) (PROGRAM, 2013), where MLTT is interpreted in an ∞-groupoid (BERG; GARNER, 2011).

Since MLTT is an extension of simply-typed λ-calculus, it can also be seen as an ∞-groupoid in this topological interpretation given by (SCOTT, 1993). Also, for type-free λ-calculus, Dana Scott presented the model D_∞ in order to give an interpretation to λ-terms into the theory of ordered sets. This model is semantically rich in the sense that it is an ordered set with a topology. This allows for generating a λ-model $\mathfrak{D}_\infty$ with the structure of an ∞-groupoid and an operation of composition between cells based on the operation of path concatenation in the topology of D_∞.

1.3 RESEARCH PROBLEM

From the λ-model Scott's D_∞, is it possible to generate a λ-model with an ∞-groupoid structure for some composition operation between cells?.

1.4 OBJECTIVES

Objective of this work is to build the ∞-groupoid $\mathfrak{D}$ from a topological space D through higher groups homotopy (GREENBERG, 1967; HATCHER, 2001) and thus show how to calculate all higher groups generated by any *complete partial order* (c.p.o.), with the Scott topology (ACOSTA; RUBIO, (2002)). Finally, apply the construction of Chapter 3 for the particular case of the c.p.o. D_∞ to obtain an ∞-groupoid $\mathfrak{D}_\infty$, and prove that this is isomorphic to D_∞, which shows that $\mathfrak{D}_\infty$ is indeed an extensional λ-model.

1.5 WORK ORGANIZATION

The remaining sections of this thesis are structured as follows.

2 Preliminaries: introduces definitions, lemmas and basic theorems related to the problem that are used for the elaboration of this work.

3 **The extensional λ-model $\mathfrak{D}_\infty$**: shows the construction of the infinity groupoid from any topological space, through the use of higher fundamental groups and proves that the ∞-groupoid generated by D_∞ is also an extensional λ-model.

4 **Conclusions**: presents the final considerations on the main topics covered in this thesis, including the contributions achieved and indications of future work.

2 PRELIMINARIES

In this section we present some basic notions about infinite-groupoids, topological spaces, continuity, higher fundamental groups and extensional lambda models, to set up the groundwork for this thesis. All the proofs of results can be seen in the suggested references.

2.1 HIGHER GROUPS OF HOMOTOPY

Below we present the definition of topological space, continuity, group, higher group with some useful results.

Definition 2.1.1 (**Topological space**). *Let D be a set. A topology τ is a collection of subsets of D, which satisfies the following axioms*

1. $\emptyset, D \in \tau$
2. $\bigcap \mu \in \tau$*, for all finite sub-collection μ of τ,*
3. $\bigcup \mu \in \tau$*, for all sub-collection μ of τ,*

The ordered par (D, τ) is called topological space, and each set $A \in \tau$ we say it is an open of τ.

Definition 2.1.2 (**Continuous map**). *Let (D, τ) and (D', τ') be topological spaces. A map $f : (D, \tau) \to (D', \tau')$ is continuous if for every it $A \in \tau'$ satisfies $f^{-1}(A) \in \tau$, where $f^{-1}(A) := \{d \in D : f(d) \in A\}$.*

The pair (D, τ) will be called simply D.

Example 2.1.1 (**Euclidean topology**). *The usual topology on $\mathbb{R}$ is defined by: A is open if it is the union of open intervals.*

Example 2.1.2 (**Inherited topology**). *Take $\mathbb{R}$ with the usual topology. The topology inherited by $\mathbb{R}$ to interval $[0, 1]$, is defined as the collection of all intersections $A \cap [0, 1]$, such that A is a usual open in $\mathbb{R}$.*

Example 2.1.3 (**Quotient topology**). *Let (X, τ) be a topological space, Y a set and $f : X \to Y$ a surjective function. The quotient topology on Y is a collection*

$$\tau_Y^f = \{V \subseteq Y : f^{-1}(V) \in \tau\}.$$

Example 2.1.4 (**Box topology**). *Take $[0, 1]$ with the topology inherited by $\mathbb{R}$. The box topology on $[0, 1]^n$ is defined as the collection of all Cartesian products $A_1 \times A_2 \cdots \times A_n$, such that A_i is open in $[0, 1]$ for each $1 \leq i \leq n$.*

Definition 2.1.3 (**Compact space**)**.** *A topological space D is compact if, for any family of opens $\mathcal{C}$ such that $D \subseteq \bigcup \mathcal{C}$, there is a finite subfamily $\mathcal{F} \subseteq \mathcal{C}$ such that $D \subseteq \bigcup \mathcal{F}$.*

Example 2.1.5 (**Compact in $\mathbb{R}$**)**.** *For each $n \geq 1$. $[0,1]^n$ is a compact subspace of $\mathbb{R}$.*

Example 2.1.6 (**Open compact topology**)**.** *Given a topological space D. Let $C([0,1]^n, D)$ the set of all continuous functions from $[0,1]^n$ to D. Define the open compact topology on $C([0,1]^n, D)$ as the collection of all unions and finite intersections of sets*

$$S(K,U) = \{f \in C([0,1]^n, D) : f(K) \subseteq U\},$$

where K is a compact subspace of $[0,1]^n$ and U an open of D.

We write $C_{o.p}([0,1]^n, D)$ to emphasize that the set $C([0,1]^n, D)$ has the open compact topology.

Remark 2.1.1. *Let D be a topological space. A map $h : [0,1] \times [0,1]^n \to D$ is continuous iff the function $H : [0,1] \to C_{o.p}([0,1]^n, D)$ is continuous, where $H(t) := h_t$ and $h_t(s) = h(t,s)$ with $t \in [0,1]$ and $s \in [0,1]^n$.*

Definition 2.1.4 (**Continuous path**)**.** *Let D be a topological space and interval $[0,1]$ with the inherited topology by $\mathbb{R}$. Define a path f from point $a \in D$ to point $b \in D$, written as $f : a \rightsquigarrow b$, as the continuous map $f : [0,1] \to D$ such that $f(0) = a$ and $f(1) = b$.*

and define the product between paths $f : a \rightsquigarrow b$ and $g : b \rightsquigarrow c$, as the path

$$f * g := \begin{cases} f(2t) & \text{if } \ 0 \leq t \leq \frac{1}{2}, \\ g(2t-1) & \text{if } \ \frac{1}{2} \leq t \leq 1. \end{cases}$$

Definition 2.1.5 (**Homotopy between paths**)**.** *Let $f : a \rightsquigarrow b$ and $g : a \rightsquigarrow b$ be paths on space D. The paths f and g are homotopies, $f \simeq g$, if there is a map continuous $h : [0,1] \times [0,1] \to D$ such that*

(*a*) *$h(t,0) = a$ and $h(t,1) = b$, for all $t \in [0,1]$,*

(*b*) *$h(0;t) = f(t)$ and $h(1;t) = g(t)$, for all $t \in [0,1]$,*

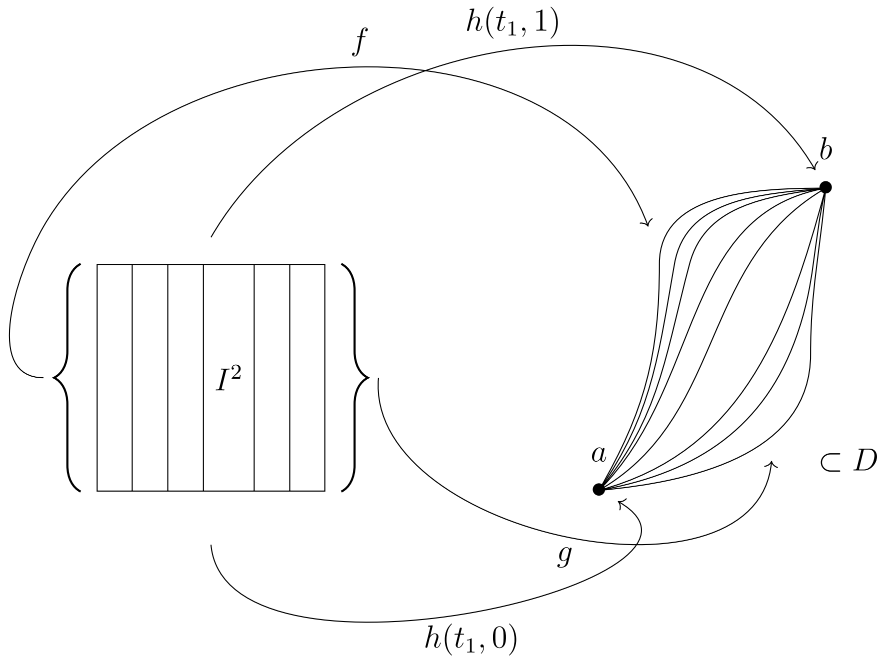
where $[0,1] \times [0,1] = [0,1]^2$ has the box topology (see Figure 1).

The homotopy is a relation of equivalence, where $[f]$ is homotopy class of f, i.e, the class of all the homotopy paths with f. The set of all homotopy classes, indicated by $\Pi_1(D)$, it is known as fundamental 1-groupoid, where the product between classes is defined naturally by $[f] * [g] := [f * g]$. A groupoid is category where the morphisms are isomorphism.

By the Remark 2.1.1, the homotopy h can be interpreted as $h : f \rightsquigarrow g$ into open compact topology $C_{o.c}([0,1], D)$. Thus, if we have two homotopies $h_1 : f \rightsquigarrow g$ and $h_2 : f \rightsquigarrow g$, we can talk about the homotopy of homotopies $h : h_1 \rightsquigarrow h_2$ into space $C_{o.c}([0,1]^2, D)$. This homotopy of homotopies is an equivalence relation that generates the class of homotopy classes $\Pi_2(D)$, known as the fundamental 2-groupoid.

We can keep iterating and get the infinite sequence of fundamental groupoids $\Pi_1(D)$, $\Pi_2(D)$, $\Pi_3(D), \ldots$, known as the fundamental ∞-groupoid, written as $\Pi_\infty(D)$. If we have two n-paths f and g such that $[f] = [g]$, so this fact induces an intentional equality relation $=_h$ between the n-paths f and g, i.e., $f =_h g$ and we say f and g are homotopically equivalent.

Figura 1 – Homotopy $h : [0,1] \times [0,1] \to D$



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Let us now generalize the notion of a continuous path from point a to point b, to n-paths based on points a and b in any space D.

Notation 2.1.1. *Take a map* $p : [0,1]^n \to D$*. Write* $p[s_r] : [0,1]^{n-1} \to D$ *for the map such that*

$$p[s_r](t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_n) := p(t_1, \ldots, s_r, \ldots, t_n),$$

$$(p[s_r])[k_s] := p[s_r, k_s],$$

and for a fixed $a \in [0,1]$*, write* $p[t_r = a] : [0,1]^{n-1} \to D$ *for the map such that*

$$p[t_r = a](t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_n) := p(t_1, \ldots, t_{r-1}, a, t_{r+1}, \ldots, t_n),$$

$$(p[t_r = a])[t_s = b] := p[t_r = a, t_s = b].$$

Definition 2.1.6 (n-path). *Let D be a topological space. For each $n \in \mathbb{N}$, define an n-path based in the points $a, b \in D$ as continuous function $p : [0,1]^n \to D$ such that*

$$p[t_n = 0](t_1, \ldots, t_{n-1}) = a \ \text{ and } \ p[t_n = 1](t_1, \ldots, t_{n-1}) = b,$$

for each $t_1, \ldots, t_{n-1} \in [0,1]$.

- *For each $r < n$, define the product $*_r$ of n-paths p and q as*
  1. *if $p[t_1 = 1] = q[t_1 = 0]$, define the n-path*

  $$(p *_{n-1} q)(t_1, \ldots, t_n) := \begin{cases} p(2t_1, t_2, \ldots, t_n) & \text{if } \ 0 \le t_1 \le \frac{1}{2}, \\ q(2t_1 - 1, t_2, \ldots, t_n) & \text{if } \ \frac{1}{2} \le t_1 \le 1. \end{cases}$$

  2. *if $r < n-1$ and $p[t_1 = 1, \ldots, t_{(n-1)-r} = 1] = q[t_1 = 0, \ldots, t_{(n-1)-r} = 0]$, define the n-path $p *_r q$ such that*

  $$(p \ *_r \ q)[t_1, \ldots, t_{(n-1)-r}] := p[t_1, \ldots, t_{(n-1)-r}] *_r \ q[t_1, \ldots, t_{(n-1)-r}].$$

- *For each n-path p, define the identity (n+1)-path as the constant path $c(p) : [0,1]^{n+1} \to D$ such that $(c(p))[t_1] = p$ for all $t_1 \in [0,1]$.*

We have that each (n+1)-path is an equivalence relation between two n-paths, since (n+1)-path $p : [0,1]^{n+1} \to D$ can be seen as a continuous path from n-path $p[t_1 = 0]$ to the n-path $p[t_1 = 1]$ into space $C_{o.c}([0,1]^n, D)$ (open compact topology), i.e., as the continuous function $P : [0,1] \longrightarrow C_{o.c}([0,1]^n, D)$ such $P(0) = p[t_1 = 0]$ and $P(1) = p[t_1 = 1]$.

Thus, the set of all equivalence classes on the set of n-paths along with the product of n-paths, it generates a groupoid; where the product between classes is defined naturally by $[p] *_r [q] := [p *_r q]$, which satisfies $[c^{n-r}(p[t_1 = 0, \ldots, t_{n-r} = 0])] *_r [p] = [p]$ and $[p] *_r [c^{n-r}(p[t_1 = 1, \ldots, t_{n-r} = 1])] = [p]$, and for each n-path p there is a $\bar{p}$ such that $\bar{p}[t_1, \ldots, t_{n-r}] = p[1 - t_1, \ldots, 1 - t_{n-r}]$, for which it holds that $[p] *_r [\bar{p}] = [c^{n-r}(p[t_1 = 0, \ldots, t_{n-r} = 0])]$ and $[\bar{p}] *_r [p] = [c^{n-r}(p[t_1 = 1, \ldots, t_{n-r} = 1])]$.

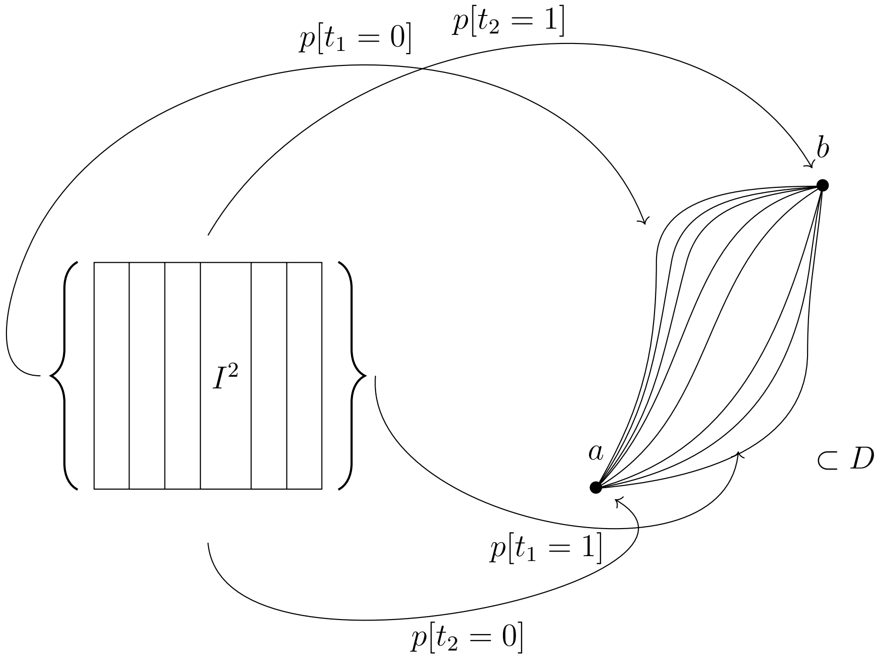
Let p and q be n-paths. If $[p] = [q]$, we write $p =_h q$ and we say p and q are equivalent higher homotopies. Write $\Pi_n(D, a, b)$ for the set of n-paths based at $a, b \in D$ governed by intensional equality $=_h$. In the literature the infinite sequence

$$\Pi_0(D, a, b), \Pi_1(D, a, b), \Pi_3(D, a, b), \ldots, \Pi_n(D, a, b), \ldots$$

is known as the fundamental ∞-groupoid based at $a, b \in D$, and $\Pi_\infty(D)$ is simply the fundamental ∞-groupoid for any $a, b \in D$.

Example 2.1.7. *For $n = 2$, any 2-path p would be homotopy between paths (1-paths) $p[t_1 = 0]$ and $p[t_1 = 1]$ as in the Figure 2. The r-product between the 2-paths p and q is given according to the cases:*

Figura 2 – 2-path (homotopy) $p : [0,1]^2 \to D$



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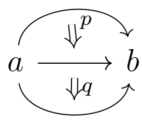
1. *If $p[t_1 = 1] = q[t_1 = 0]$, the 1-product is*

$$(p *_1 q)(t_1, t_2) := \begin{cases} p(2t_1, t_2) & \text{if } 0 \leq t_1 \leq \frac{1}{2}, \\ q(2t_1 - 1, t_2) & \text{if } \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

so $(p *_1 q)[t_1 = 0] = p[t_1 = 0]$ and $(p *_1 q)[t_1 = 1] = q[t_1 = 1]$, i.e., $(p *_1 q)$ is a homotopy from path $p[t_1 = 0]$ to path $q[t_1 = 1]$ as seen in Figure 3.

Figura 3 – The product $p *_1 q$



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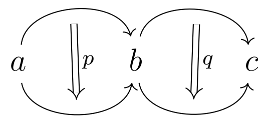
2. *If $p[t_2 = 1] = q[t_2 = 0]$, the 0-product is given by*

$$(p *_0 q)(t_1, t_2) := (p[t_1] *_0 q[t_1])(t_2) := \begin{cases} p[t_1](2t_2) & \text{if } 0 \leq t_2 \leq \frac{1}{2}, \\ q[t_1](2t_2 - 1) & \text{if } \frac{1}{2} \leq t_2 \leq 1. \end{cases}$$

then $(p *_0 q)[t_1 = 0] = p[t_1 = 0] *_0 q[t_1 = 0]$ and $(p *_0 q)[t_1 = 1] = p[t_1 = 1] *_0 q[t_1 = 1]$, i.e., $(p *_0 q)$ is a homotopy from path $p[t_1 = 0] *_0 q[t_1 = 0]$ to path $p[t_1 = 1] *_0 q[t_1 = 1]$ as in Figure 4.

Figura 4 – The product $p *_0 q$



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Next we define closed n-paths based at some $d_0 \in D$, with the purpose of building the fundamental n-group $\pi_n(D, d_0)$.

Definition 2.1.7 (Closed n-path). *Let D be a topological space. An n-path based on $d_0 \in D$ is a continuous map $\sigma : [0,1]^n \to D$ which sends border of $[0,1]^n$ to d_0.*

It defines the product of closed n-paths, $\alpha * \beta = \gamma$, as the n-path

$$\gamma(t_1, \dots, t_n) = \begin{cases} \alpha(2t_1, t_2, \dots, t_n) & \text{if } \ 0 \le t_1 \le \frac{1}{2}, \\ \beta(2t_1 - 1, t_2, \dots, t_n) & \text{if } \ \frac{1}{2} \le t_1 \le 1. \end{cases}$$

Definition 2.1.8 (Homotopic n-paths). *Two n-paths α, β are homotopic in d_0, $\alpha \simeq \beta$, if there exists a continuous map $H : [0,1] \times [0,1]^n \to D$, such that*

(*a*) $H(0; t_1, \dots, t_n) = \alpha(t_1, \dots, t_n)$, *for* $(t_1, \dots, t_n) \in [0,1]^n$,

(*b*) $H(1; t_1, \dots, t_n) = \beta(t_1, \dots, t_n)$, *for* $(t_1, \dots, t_n) \in [0,1]^n$,

(*c*) $H(s; t_1, \dots, t_n) = d_0$, *for each* $s \in [0,1]$ *and each* $(t_1, \dots, t_n) \in \partial([0,1]^n)$.

This homotopy is an equivalence relation, and one writes $[\sigma]$ for the class of homotopic n-paths to the n-path σ. The set of all homotopy classes is denoted by $\pi_n(D, d_0)$, where the product between classes is defined in the natural way $[\alpha] * [\beta] := [\alpha * \beta]$.

Definition 2.1.9 (Group). *A group is an ordered pair $(G, *)$, where G is a set and $* : G \times G \to G$ is a binary operation, which satisfies the following axioms*

1. *(associativity) $(a * b) * c = a * (b * c)$ for each $a, b, c \in G$,*
2. *(identity element) there exits an element $e \in G$, such that $a * e = e * a = a$ for all $a \in G$,*
3. *(inverse) there is an element $\bar{a} \in G$ for each $a \in G$, such that $a * \bar{a} = \bar{a} * a = e$.*

Theorem 2.1.1. *$\pi_n(D, d_0)$ is a group, called the fundamental group of D on d_0 of dimension n.*

The identity element of the group $\pi_n(D, d_0)$ is the homotopy class of the constant n-path $c^n(d_0) : [0,1]^n \to D$, i.e., $c^n(d_0)(t_1, \dots, t_n) = d_0$ for all $t_1, \dots, t_n$ in $[0,1]$.

Definition 2.1.10 (**Homotopic functions**). *Let D be a topological space. Two continuous functions $f, g : D \to D$ are homotopic, $f \simeq g$, if there exists a continuous map $H : D \times [0,1] \to D$ such that $H(d,0) = f(d)$ and $H(d,1) = g(d)$ for all $d \in D$. H is called a homotopy between continuous functions.*

Definition 2.1.11 (**Contractible space**). *A topological space D is said to be contractible if there exists a constant function $f_c : D \to D$, $f_c(d) = c$ for each $d \in D$, homotopic to the identity function $I_D : D \to D$. The homotopy $H : f_c \simeq I_D$ is called a contraction.*

Theorem 2.1.2. *If D is contractible, then $\pi_n(D, d_0) = \{[c^n(d_0)]\}$ for each $n \geq 0$.*

Remark 2.1.2. *Let D be contractible. For each $t_1, \ldots, t_n \in [0,1]$ we have*

$$c^n(d_0)(t_1, t_2, \ldots, t_n) = e_n(t_1)(t_2)\cdots(t_n) = d_0,$$

where $e_n : [0,1] \to \Pi_{n-1}(D, d_0)$ is the constant path in e_{n-1} defined by

$$\begin{aligned} e_0 &= d_0, \\ e_n &= c_{e_{n-1}}, \end{aligned}$$

then $\pi_n(D, d) \cong \{e_n\}$ (group isomorphisms).

2.2 STRICT ∞-GROUPOIDS

In the literature we can find two types of ∞-groupoids: strict and weak ∞-groupoids (LEINSTER, 2003). For this thesis we shall only work with the first type. It is well known that every strict ∞-groupoid is weak. It is usual to call a weak ∞-groupoid just ∞-groupoid.

Definition 2.2.1 (**∞-globular set**). *An ∞-globular set D is a diagram*

$$\cdots \rightrightarrows^s_t D_n \rightrightarrows^s_t D_{n-1} \rightrightarrows^s_t \cdots \rightrightarrows^s_t D_1 \rightrightarrows^s_t D_0,$$

of sets and functions such that

$$s(s(d)) = s(t(d)), \quad t(s(d)) = t(t(d)),$$

for all $n \geq 2$ and $d \in D_n$.

Definition 2.2.2. *Let D be a globular set and $n \in \mathbb{N}$. For each $0 \leq p < n$ define the relation into $D_n \times D_n$ as the set*

$$D_n \times_{D_p} D_n = \{(d', d) \in D_n \times D_n : t^{n-p}(d) = s^{n-p}(d')\}.$$

Definition 2.2.3 (**strict ∞-groupoid**). *Let $n \geq 2$ be a natural number. A strict ∞-groupoid is an ∞-globular set D equipped with*

- *a function* $\circ_p : D_n \times_{D_p} D_n \to D_n$ *for each* $0 \le p < n$*, where* $\circ_p(d', d) := d' \circ d$ *and call it a composite of* d' *and* d*,*
- *a function* $i : D_n \to D_{n+1}$ *for each* $n \ge 0$*, where* $i(d) := 1_d$ *and call it the identity on* d*,*

satisfying the following axioms:

(*a*) (*sources and targets of composites*) *if* $0 \le p < n$ *and* $(d', d) \in D_n \times_{D_p} D_n$ *then*

$$\begin{aligned} s(d' \circ_p d) &= s(d) && \text{and} \quad t(d' \circ_p d) = t(d') && if\ p = n-1, \\ s(d' \circ_p d) &= s(d') \circ_p s(d) && \text{and} \quad t(d' \circ d) = t(d') \circ_p t(d) && if\ p \le n-2, \end{aligned}$$

(*b*) (*sources and targets of identities*) *if* $0 \le p < n$ *and* $d \in D_n$ *then* $s(1_d) = d = t(1_d)$*,*

(*c*) (*associativity*) *if* $0 \le p < n$ *and* $d, d', d'' \in D_n$ *with* $(d'', d'), (d', d) \in D_n \times_{D_p} D_n$ *then*

$$(d'' \circ_p d') \circ_p d = d'' \circ_p (d' \circ_p d),$$

(*d*) (*identities*) *if* $0 \le p < n$ *and* $d \in D_n$ *then*

$$i^{n-p}(t^{n-p}(d)) \circ_p d = d = d \circ_p i^{n-p}(s^{n-p}(d)),$$

(*e*) (*binary interchange*) *if* $0 \le q < p < n$ *and* $d, d', e, e' \in D_n \times_{D_q} D_n$ *with*

$$(e', e), (d', d) \in D_n \times_{D_p} D_n, \quad (e', d'), (e, d) \in D_n \times_{D_q} D_n,$$

then

$$(e' \circ_p e) \circ_q (d' \circ_p d) = (e' \circ_q d') \circ_p (e \circ_q d),$$

(*f*) (*nullary interchange*) *if* $0 \le q < p < n$ *and* $d, d' \in D_p \times_{D_q} D_p$*, then* $1_{d'} \circ_q 1_d = 1_{d' \circ_q d}$*.*

(*g*) (*inverse*) *if* $0 \le p < n$ *and* $d \in D_n$ *then exist* $\bar{d} \in D_n$ *with* $s^{n-p}(\bar{d}) = t^{n-p}(d)$*,* $t^{n-p}(\bar{d}) = s^{n-p}(d)$ *such that*

$$\bar{d} \circ_p d = i^{n-p}(s^{n-p}(d)), \quad d \circ_p \bar{d} = i^{n-p}(t^{n-p}(d)).$$

If $d \in D_n$, we say that d is an n-cell or an n-isomorphism from some $a \in D_{n-1}$ to some $b \in D_{n-1}$. Or, in other words, we say that a and b are n-equivalent if there is an n-isomorphism between a and b.

For example in the fundamental ∞-groupoid $\Pi_\infty(D)$, the n-isomorphisms are the n-paths class $[f]$ in $\Pi_n(D)$. Even thought the n-paths $f : a \rightsquigarrow b$ may not satisfy the properties of a strict ∞-groupoid, the n-paths class $[f]$ does satisfy them. Thus $\Pi_\infty(D)$ is not a strict ∞-groupoid strict, but it is a weak ∞-groupoid.

Notation 2.2.1. *Write $(a \simeq_n b)$ the set of all n-isomorphisms between a and b. Note that this set can be empty.*

Since $\simeq_n$ is an equivalence relation and the set $(a \simeq_n b)$ can have cardinality greater than one, we can see $\simeq_n$ as an intensional equality $=_n$, i.e., the equivalence $a \simeq_n b$ can be seen as the intensional equal $a =_{int} b$, which motivates a more precise definition of intensional equality between n-cells.

Definition 2.2.4 (Extensional and Intensional equality). *Two n-morphisms a and b are intentionally equal, $a =_{int} b$, if there is $d \in (a \simeq_n b)$. Two n-morphisms a and b are extensionally equal if $a = b$ in the primitive sense of equality.*

Note that if $a = b$ then $a =_{int} b$, since $1_a \in (a \simeq_n b)$. The converse does not always hold. For example, for the fundamental groupoid $\Pi_1\{0,1\}$, where $\{0,1\}$ has the topology $\{\{0,1\}, \emptyset\}$, we have that $0 =_{int} 1$, but $0 \neq 1$.

2.3 EXTENSIONAL λ-THEORIES AND λ-MODELS

Next we present a brief introduction to the theories and extensional lambda models. The proofs of the theorems can be consulted in (HINDLEY; SELDIN, 2008).

Definition 2.3.1 ($\boldsymbol{\lambda}$-terms). *Suppose that there is an infinite sequence of expressions called variables, and a sequence of expressions called atomic constants, different form the variables. The set of expressions called λ-terms is defined inductively as follows*

(*a*) *all variables and atomic constants are λ-terms (called atoms);*

(*b*) *if M and N are any λ-terms, then (MN) is a λ-term (called application);*

(*c*) *if M is any λ-term and x is any variable, then $(\lambda x.M)$ is λ-term (called an abstraction).*

Write $FV(M)$ for the set of free variables of M i.e. all the variables of M that are not bound by λ, and $[N/x]M$ the term that results from replacing x with N into M.

Definition 2.3.2 ($\boldsymbol{\alpha}$-conversion). *Let P be a λ-term and assume it contains terms of form $\lambda x.M$. If $y \notin FV(M)$ and we replace in P any term $\lambda x.M$ by $\lambda y.[y/x]M$, and the result is Q, we say P is α-converts to Q and we write $P \equiv_\alpha Q$.*

Definition 2.3.3 ($\boldsymbol{\beta\eta}$-contracting). *Let P a λ-term. If P contains a term of form $(\lambda x.M)N$ (called β-redex) or of form $\lambda x.Mx$ such that $x \notin FV(M)$ (called η-redex) and we replace in P the term $(\lambda x.M)N$ by $[N/x]M$ (β-contracts) or $\lambda x.Mx$ by M (η-contracts), and the result is P', we say that P $\beta\eta$-contracts to P' and we write it as $P \rhd_{1\beta\eta} P'$.*

Definition 2.3.4 ($\beta\eta$-equality). *We say that P is $\beta\eta$-equal to Q, $P =_{\beta\eta} Q$ if there exist $P_1, P_2, \ldots, P_n$ such that*

$$(\forall i \leq n-1)(P_i \rhd_{1\beta\eta} P_{i+1} \text{ or } P_{i+1} \rhd_{1\beta\eta} P_i \text{ or } P_i \equiv_\alpha P_{i+1}),$$

where $P_1 = P$ and $P_n = Q$.

The equality $=_{\beta\eta}$ can be axiomatized in an extensional theory of $\lambda\beta\eta$ as seen below.

Definition 2.3.5 ($\lambda\beta\eta$-theory of equality). *The extensional λ-theory consists of the all λ-terms and a relation symbol $=$, where the λ-formulas are just $M = N$ for all λ-terms M and N. The axioms are the cases α, β and ρ below, for all λ-terms M, N an d all variables x, y. The rules are μ, ν, ξ, τ and σ below. Axiom-schemas:*

(α) $\lambda x.M = \lambda y.[y/x]M$ *if* $y \notin FV(M)$;

(β) $(\lambda x.M)N = [N/x]M$;

(η) $\lambda x.Mx = M$ *if* $x \notin FV(M)$;

(ρ) $M = M$,

Rules of inference:

(μ) $\dfrac{M = M'}{NM = NM'}$; (τ) $\dfrac{M = N \quad N = P}{M = P}$;

(ν) $\dfrac{M = M'}{MN = M'N}$; (ρ) $\dfrac{M = N}{N = M}$;

(ξ) $\dfrac{M = M'}{\lambda x.M = \lambda x.M'}$.

Note that the axiom β defines the equality relation $=_\beta$ between λ-terms. This β-equality generates the theory called $\lambda\beta$, here to simplify notation we call it just λ-theory.

Definition 2.3.6 (Extensional λ-models). *An extensional λ-model is a triple $\langle D, \bullet, [\![\,]\!]\rangle$, where D is a set, $\bullet : D \times D \to D$ is a binary operation and $[\![\,]\!]$ is a mapping which assigns, to λ-term M and each valuation $\rho : Var \to D$, a element $[\![M]\!]_\rho$ of D such that*

(a) $[\![x]\!] = \rho(x)$;

(b) $[\![PQ]\!]_\rho = [\![P]\!]_\rho \bullet [\![Q]\!]_\rho$;

(c) $[\![\lambda x.P]\!]_\rho \bullet d = [\![P]\!]_{[d/x]\rho}$ *for all* $d \in D$;

(d) $[\![M]\!]_\rho = [\![M]\!]_\sigma$ *if* $\rho(x) = \sigma(x)$ *for* $x \in FV(M)$;

(e) $[\![\lambda x.M]\!]_\rho = [\![\lambda y.[y/x]M]\!]_\rho$ *if* $y \notin FV(M)$;

(*f*) *if* $(\forall d \in D)\left([\![P]\!]_{[d/x]\rho} = [\![Q]\!]_{[d/x]\rho}\right)$, *then* $[\![\lambda x.P]\!]_\rho = [\![\lambda x.Q]\!]_\rho$;

(*g*) $[\![\lambda x.Mx]\!]_\rho = [\![M]\!]_\rho$ *if* $x \notin FV(M)$,

where $[d/x]\rho$ *means: replace* $\rho(x)$ *with* d *in the interpretation of* λ*-term in question.*

Definition 2.3.7 (**Combinatory logic terms, or CL-terms**). *Suppose that there is an infinite sequence of expressions called variables, and a sequence of expressions called atomics constants, different form the variables. The set of expressions called CL-terms is defined inductively as follows*

(*a*) *all variables and atomic constants, including* $\boldsymbol{I}$, $\boldsymbol{K}$, $\boldsymbol{S}$, *are CL-terms;*

(*b*) *if* X *and* Y *are CL-terms, then so is* (XY).

Definition 2.3.8 (**Weak reduction**). *Let* P *be a CL-term. If* P *contains a term of the form* $\boldsymbol{I}X$, $\boldsymbol{K}XY$ *or* $\boldsymbol{S}XYZ$ *(called* β*-redex) and we replace into* P *the term* $\boldsymbol{I}X$ *by* X, $\boldsymbol{K}XY$ *by* X *or* $\boldsymbol{S}XYZ$ *by* $XZ(YZ)$, *and the result is* P', *we say that* P *(weakly) contracts to* P' *and we write it as* $P \rhd_{1w} P'$. *If there is a finite sequence of contractions from* P *to* P', *we write* $P \rhd_w P'$.

Definition 2.3.9 (**Weak equality**). *We say that* P *is weakly equal to* Q, $P =_w Q$ *if there exist* $P_1, P_2, \ldots, P_n$ *such that*

$$(\forall i \leq n-1)(P_i \rhd_{1w} P_{i+1} \text{ or } P_{i+1} \rhd_{1w} P_i),$$

where $P_1 = P$ *and* $P_n = Q$.

Definition 2.3.10 (**Abstraction**). *For every CL-term* M *and every variable* x, *a CL-term written* $[x].M$ *is defined by induction on* M, *thus:*

(*a*) $[x].M := \boldsymbol{K}M$ *if* $x \notin FV(M)$;

(*b*) $[x].x := \boldsymbol{I}$;

(*c*) $[x].Ux := U$ *if* $x \notin FV(U)$;

(*d*) $[x].UV := \boldsymbol{S}([x].U)([x].V)$ *if neither (a) or (c) applies.*

Similar to the lambda calculus abstraction, the combinatorial logic abstraction satisfies the analogous property to β-contraction, i.e.,

$$([x].M)N \rhd_w [N/x]M.$$

Definition 2.3.11. *For all variables* $x_1, \ldots, x_n$ *(not necessarily distinct),*

$$[x_1, \ldots, x_n].M = [x_1].([x_2].(\ldots([x_n].M)\ldots)).$$

The equality $=_w$ can be axiomatized in an CLw-theory of as seen below.

Definition 2.3.12 (*CLw*, or theory of weak equality). *The extensional CL_w-theory consist of the all CL-terms and a relation symbol $=$, where the CL-formulas are just $M = N$ for Four axiom-schemes below, for all CL-terms X, Y and Z. The rules are μ, ν, τ and σ below. Axioms-schemas:*

($\boldsymbol{I}$) $\boldsymbol{I}X = X$;

($\boldsymbol{K}$) $\boldsymbol{K}XY = X$;

($\boldsymbol{S}$) $\boldsymbol{S}XYZ = XZ(YZ)$;

(ρ) $X = X$,

Rules of inference:

(μ) $\dfrac{X = X'}{ZX = ZX'}$; (τ) $\dfrac{X = Y \quad Y = Z}{X = Z}$;

(ν) $\dfrac{X = X'}{XZ = X'Z}$; (σ) $\dfrac{X = Y}{Y = X}$;

Definition 2.3.13 (Extensional combinatory algebra). *A combinatory algebra is a pair $\langle D, \bullet\rangle$ such that there are elements $k, s \in D$ which satisfy*

(a) $(\forall a, b \in D)$ $k \bullet a \bullet b = a$;

(b) $(\forall a, b, c \in D)$ $s \bullet a \bullet b \bullet c = a \bullet c \bullet (b \bullet c)$;

and it is an extensional combinatory algebra if it also meets

(c) $(\forall a, b \in D)$ *if* $a \bullet c = b \bullet c$ *for all* $c \in D$, *then* $a = b$.

A model of CLw is a quintuple $\langle D, \bullet, i, k, s\rangle$ such that $\langle D, \bullet\rangle$ is a combinatory algebra and k and s satisfies (a), (b), and $i = s \bullet k \bullet k$.

Definition 2.3.14 (Interpretation of CL-terms). *Let $\langle D, \bullet, i, k, s\rangle$ a model of CLw and $\rho : Var \to D$ a valuation of variables. Using ρ, we assign to every CLw-term X a member of D called its interpretation or $[\![X]\!]_\rho$, thus:*

(a) $[\![X]\!]_\rho := \rho(x)$;

(b) $[\![\boldsymbol{I}]\!] := i, \qquad [\![\boldsymbol{K}]\!] := k, \qquad [\![\boldsymbol{S}]\!] := s$;

(c) $[\![XY]\!]_\rho := [\![X]\!]_\rho \bullet [\![Y]\!]_\rho$.

To simplify we write $[\![X]\!]$ instead of $[\![X]\!]_\rho$.

Definition 2.3.15 (Combinatory complete). *An par $\langle D, \bullet \rangle$ is called combinatorially complete if: for all sequence $x_1, \ldots, x_n$ of distinct variables, and every combination X of $x_1, \ldots, x_n$ only, there exist $a \in D$ such that, for all $d_1, d_2, \ldots, d_n$,*

$$a \bullet d_1 \bullet \cdots \bullet d_n = [\![X]\!]_{[d_1/x_1]\cdots[d_n/x_n]\rho},$$

where ρ is arbitrary.

With respect to abstraction in CL we will have to make sure that interpretation of $[x].M$ satisfies, for all $d \in D$

$$[\![[x].M]\!] \bullet d = [\![M]\!]_{[d/x]}.$$

Theorem 2.3.1. *$\langle D, \bullet \rangle$ is a combinatory algebra, then it is combinatorially complete.*

Proof. Let X a combination of variables $x_1, \ldots, x_n$, we have that the formula

$$([x_1, \ldots, x_n]X)x_1 \ldots \cdot x_n = X$$

is true in $\langle D, \bullet, i, k, s \rangle$, if we take $a = [\![[x_1, \ldots, x_n]X]\!]$ it get the equality of Definition 2.3.15, thus $\langle D, \bullet \rangle$ is a combinatorially complete. □

Note that if $\langle D, \bullet \rangle$ is complete combinatory algebra and P is a combination of variables only, then there is a unique $a \in D$ such that $a \bullet d = [\![P]\!]_{[d/x]\rho}$ for each $d \in D$. Thus we can interpret abstraction as $[\![\lambda x.P]\!] = a$, which meets (c) and (f) of Definition 2.3.6.

Theorem 2.3.2. *If $\langle D, \bullet \rangle$ a extensional combinatory algebra, then $\langle D, \bullet, [\![\,]\!] \rangle$ is a extensional λ-model, where $[\![\,]\!]$ is defined for each evaluation $\rho : Var \to D$ by*

(*a*) $[\![x]\!]_\rho = \rho(x)$,

(*b*) $[\![PQ]\!]_\rho = [\![P]\!]_\rho \bullet [\![Q]\!]_\rho$,

(*c*) $[\![\lambda x.P]\!]_\rho = a$, *where* $(\forall d \in D)(a \bullet d = [\![P]\!]_{[d/x]\rho})$.

Proof. We have that $\langle D, \bullet, [\![\,]\!] \rangle$ satisfies (a), (b) and (c) of Definition 2.3.6, so we must prove from (d) to (g) in this same definition.

(*d*) By induction on M we have:

  (*i*) if $M = x$ and $\rho(x) = \sigma(x)$ is clear that $[\![x]\!]_\rho = [\![x]\!]_\sigma$,

  (*ii*) If $M = PQ$, so $[\![PQ]\!]_\rho = [\![P]\!]_\rho \bullet [\![Q]\!]_\rho$ and $[\![PQ]\!]_\sigma = [\![P]\!]_\sigma \bullet [\![Q]\!]_\sigma$, by induction hypothesis we have $[\![P]\!]_\rho = [\![P]\!]_\sigma$ and $[\![Q]\!]_\rho = [\![Q]\!]_\sigma$, thus $[\![PQ]\!]_\rho = [\![PQ]\!]_\sigma$.

  (*iii*) If $M = \lambda x.P$, then for each $d \in D$ it hold $[\![\lambda x.P]\!]_\rho \bullet d = [\![P]\!]_{[d/x]\rho}$ and $[\![\lambda x.P]\!]_\sigma \bullet d = [\![P]\!]_{[d/x]\sigma}$, by induction hypothesis $[\![P]\!]_{[d/x]\rho} = [\![P]\!]_{[d/x]\sigma}$ for all $d \in D$. Since $\langle D, \bullet \rangle$ is an extensional combinatory algebra, then $[\![\lambda x.P]\!]_\rho = [\![\lambda x.P]\!]_\sigma$.

Thus $[\![M]\!]_\rho = [\![M]\!]_\sigma$ if $\rho(x) = \sigma(x)$ for each $x \in FV(M)$.

(e) For each $d \in D$ we have

$$[\![\lambda y.[y/x]M]\!]_\rho \bullet d = [\![[y/x]M]\!]_{[d/y]\rho} = [\![M]\!]_{[d/x]\rho} = [\![\lambda x.M]\!]_\rho \bullet d$$

where $y \notin FV(M)$ So by extensionality $[\![\lambda y.[y/x]M]\!]_\rho = [\![\lambda x.M]\!]_\rho$.

(f) If each $d \in D$ we have $[\![P]\!]_{[d/x]\rho} = [\![Q]\!]_{[d/x]\rho}$, then

$$[\![\lambda x.P]\!]_\rho \bullet d = [\![P]\!]_{[d/x]\rho} = [\![Q]\!]_{[d/x]\rho} = [\![\lambda x.Q]\!]_\rho \bullet d$$

for all $b \in D$. By extensionality $[\![\lambda x.P]\!]_\rho = [\![\lambda x.Q]\!]_\rho$.

(g) If $x \notin FV(M)$, for each $d \in D$ it has

$$[\![\lambda x.Mx]\!]_\rho \bullet d = [\![Mx]\!]_{[d/x]\rho} = [\![M]\!]_\rho \bullet d.$$

Thus $[\![\lambda x.Mx]\!]_\rho = [\![M]\!]_\rho$.

Therefore $\langle D, \bullet, [\![\,]\!]\rangle$ is an extensional λ-model. □

2.4 THE λ-MODEL SCOTT'S D_∞

Below is a summary of the construction of the D_∞ λ-model through c.p.o.'s. The proofs of the results can be seen in (HINDLEY; SELDIN, 2008) and (BARENGREGT, 1984).

Definition 2.4.1 (Partial order and least element). *A partial order is a pair $\langle D, \sqsubseteq\rangle$, where D is a set and $\sqsubseteq$ is a binary relation on D, which satisfies for each $a, b, c \in D$*

(a) *(reflexivity) $a \sqsubseteq a$;*

(b) *(anti-symmetry) $a \sqsubseteq b$ and $b \sqsubseteq a$, then $a = b$;*

(c) *(transitivity) $a \sqsubseteq b$ and $b \sqsubseteq c$, then $a \sqsubseteq c$.*

If there is $\perp \in D$, such that

$$(\forall d \in D)\ \perp \sqsubseteq d,$$

$\perp$ is called the least element of D, or bottom.

Definition 2.4.2 (Least upper bound l.u.b.). *Let $\langle D, \sqsubseteq\rangle$ be a partial order and let $X \subseteq D$. An upper bound (u.b.) of X is any $b \in D$ such that*

(a) $(\forall a \in X)\ a \sqsubseteq b$.

The least upper bound (l.u.b.) or supremum of X is called $\bigsqcup X$, it is an upper bound b of X such that (*b*) $(\forall c \in D)$ (*if c is an u.b. of X, then* $b \sqsubseteq c$).

Definition 2.4.3 (**Directed set**)**.** *Let* $\langle D, \sqsubseteq \rangle$ *be a partially ordered set. A non-empty subset* $X \subset D$ *is said to be directed if for all* $a, b \in X$, *there exists* $c \in X$ *such that* $a \sqsubseteq c$ *and* $b \sqsubseteq c$.

Definition 2.4.4 (**Complete partial orders c.p.o.'s**)**.** *A c.p.o. is a partially ordered set* $(D, \sqsubseteq)$ *such that*

(*a*) *D has a least element (called* $\perp$*),*

(*b*) *every directed subset* $X \subset D$ *has l.u.b. (called* $\bigsqcup X$*).*

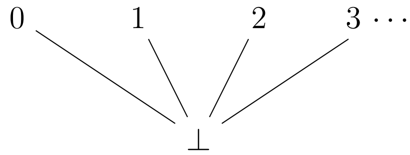
The pair $(D, \sqsubseteq)$ *will be called D.*

Definition 2.4.5 (**The set** $\mathbb{N}^{+}$)**.** *Choose any object* $\perp \notin \mathbb{N}$, *and define* $\mathbb{N}^{+} = \mathbb{N} \cup \{\perp\}$. *For all* $a, b \in \mathbb{N}^{+}$, *define*

$$a \sqsubseteq b \Longleftrightarrow (a = \perp \text{ and } b \in \mathbb{N}) \text{ or } a = b.$$

Clearly $\mathbb{N}^{+}$ is a c.p.o., since every directed subset is finite so has l.u.b and $\perp$ is the least element as seen in Figure 5.

Figura 5 – The c.p.o. $\mathbb{N}^{+}$



Sink: Adapted of (HINDLEY; SELDIN, 2008)

On other hand, every c.p.o. has a topology called the Scott topology, which we define below.

Definition 2.4.6 (**Final and inaccessible set**)**.** *Let D a c.p.o. and* $A \subseteq D$. *The set A is final if it satisfies*

$$a \in A \text{ and } a \sqsubseteq b \Longrightarrow b \in A,$$

and A is inaccessible by directedness if for every directed subset X of D,

$$\bigsqcup X \in A \Longrightarrow X \cap A \neq \emptyset.$$

Definition 2.4.7 (**Scott topology**)**.** *Let D be a c.p.o. The Scott topology is defined as follows*

$$\sigma = \{A \subseteq D : A \text{ is final and inaccessible by directedness}\}.$$

Definition 2.4.8 (**The function-set $[D \to D']$**). *For c.p.o.s D and D', define $[D \to D']$ to be set of all continuous functions from D to D' in Scott's topology. For $\phi, \psi \in [D \to D']$, define*

$$\phi \sqsubseteq \psi \iff (\forall d \in D)(\phi(d) \sqsubseteq' \psi(d)).$$

Remark 2.4.1. *There is a result that says, if D and D' c.p.o's, then $[D \to D']$ is a c.p.o. (see proof in (HINDLEY; SELDIN, 2008)). It would allow from a c.p.o. initially generate inductively an infinite sequence of c.p.o's as below in the Definition 2.4.10.*

Definition 2.4.9 (**Projections**). *Let D and D' be c.p.o's. A projection from D' to D is a pair $\langle \phi, \psi \rangle$ of functions, where $\phi \in [D \to D']$ and $\psi \in [D' \to D]$, such that*

$$\psi \circ \phi = I_D, \quad \phi \circ \psi \sqsubseteq I_{D'}.$$

In this case we say that $D \preceq D'$.

Next we define the increasing sequence of c.p.o's $D_0 \prec D_1 \prec D_2 \prec \cdots$ from the initial c.p.o. $D_0 = \mathbb{N}^+$.

Definition 2.4.10 (**The sequence $D_0, D_1, D_2, \ldots$**). *For each $n \geq 0$, define D_n by recursion*

$$D_0 := \mathbb{N}^+,$$
$$D_{n+1} := [D_n \to D_n].$$

The $\sqsubseteq$-relation on D_n will be called just '$\sqsubseteq$'. The least element of D_n will be called $\perp_n$.

By Remark 2.4.1, every D_n is a *complete partial order* (c.p.o.)

Below we define from the initial projection $\langle \phi_0, \psi_0 \rangle$ the projection $\langle \phi_n, \psi_n \rangle$ between two arbitrary consecutive c.p.o's D_n and D_{n+1} in order to define the c.p.o. D_∞, which results from inverse limit

$$D_0 \longleftarrow_{\psi_0} D_1 \longleftarrow_{\psi_1} \cdots \longleftarrow_{\psi_{n-1}} D_n \longleftarrow_{\psi_n} D_{n+1} \longleftarrow_{\psi_{n+1}} \cdots$$

Definition 2.4.11 (**Projection from D_{n+1} to D_n**). *For every $n \geq 0$ define the projection $\langle \phi_n, \psi_n \rangle$ from D_{n+1} to D_n by the recursion*

$$\phi_0(d) := \boldsymbol{\lambda} a \in D_0.d, \qquad \psi_0(g) := g(\perp_0),$$
$$\phi_{n+1}(d) := \phi_n \circ d \circ \psi_n, \qquad \psi_{n+1}(g) := \psi_n \circ g \circ \phi_n,$$

where $\boldsymbol{\lambda} a \in D_0.d \in [D_0 \to D_0]$ is the constant function to $d \in D_0$.

Definition 2.4.12 (**Construction of D_∞**). *We define D_∞ to be the set of all infinite sequences*

$$d = \langle d_0, d_1, d_2, \ldots \rangle,$$

such that $d_n \in D_n$ *and* $\psi_n(d_{n+1}) = d_n$*, for all* $n \geq 0$*.*

A relation $\sqsubseteq$ *on* D_∞ *is defined by*

$$d \sqsubseteq d' \iff (\forall n \geq 0)(d_n \sqsubseteq d'_n).$$

In (BARENGREGT, 1984) D_∞ coincides with the inverse limit of its projections sequence $\psi_0, \psi_1, \psi_2, \ldots$ in the category of c.p.o's. Also D_∞ is a c.p.o., since the last element is given by

$$\bot = \langle \bot_0, \bot_1, \bot_2, \ldots \rangle,$$

where each $\bot_n$ is the least element from c.p.o D_n, and for each directed set $X \subseteq D_\infty$ the l.u.b is given by

$$\bigsqcup X = \langle \bigsqcup X_0, \bigsqcup X_1, \bigsqcup X_2, \ldots \rangle,$$

where $X_n = \{d_n : d \in X\}$, since that X is directed so each X_n also is directed.

Next we define from consecutive projections $\langle \phi_n, \psi_n \rangle$ the projection $\langle \phi_{m,n}, \phi_{n,m} \rangle$ between two arbitrary c.p.o's D_m and D_n in order to define for each $n \geq 0$ the projection $\langle \phi_{n,\infty}, \phi_{\infty,n} \rangle$ from D_∞ to D_n and with it define the application operation

$$\bullet : D_\infty \times D_\infty \to D_\infty.$$

Definition 2.4.13. *For all pairs* $m, n \geq 0$*, a map* $\phi_{m,n} : D_m \to D_n$ *is defined by*

$$\phi_{m,n} = \begin{cases} \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_{m+1} \circ \phi_m & \textit{if } m < n, \\ I_{D_m} & \textit{if } m = n, \\ \psi_n \circ \psi_{n+1} \circ \cdots \circ \psi_{m-2} \circ \psi_{m-1} & \textit{if } m > n. \end{cases}$$

Let $m < n$. The pair $\langle \phi_{m,n}, \phi_{n,m} \rangle$ is a projection from D_n to D_m.

Definition 2.4.14 (Projection from D_∞ to D_n). *The projection* $\langle \phi_{n,\infty}, \phi_{\infty,n} \rangle$ *from* D_∞ *to* D_n *is defined by*

$$\phi_{\infty,n}(d) := d_n,$$
$$\phi_{n,\infty}(a) := \langle \phi_{n,0}(a), \phi_{n,1}(a), \phi_{n,2}(a), \ldots \rangle .$$

For all $a, b \in D_\infty$ *and for each* $n \geq 1$ *it holds that*

$$\phi_{n,\infty}(a_{n+1}(b_n)) \sqsubseteq \phi_{n+1,\infty}(a_{n+2}(b_{n+1})).$$

Thus the set $\{\phi_{n,\infty}(a_{n+1}(b_n)) : n \geq 0\}$ is an increasing sequence, then it is a directed set; hence it has l.u.b. which allows the definition of application operator in D_∞.

Definition 2.4.15 (Application in D_∞). *Let* $a, b \in D_\infty$*. Define the application of* a *to* b *by*

$$a \bullet b := \bigsqcup_{n \geq 0} \phi_{n,\infty}(a_{n+1}(b_n)).$$

Remark 2.4.2. *$\langle D_\infty, \bullet\rangle$ is an extensional combinatory algebra. Consider the combinator*

$$k = \langle \bot_0, I_{D_0}, k_2, k_3, k_4, \ldots\rangle,$$

where $k_{n+2} = \boldsymbol{\lambda} a \in D_{n+1}.\boldsymbol{\lambda} b \in D_n.\psi_{n+1}(a)$ for each $n \geq 0$. Then one has that $k \in D_\infty$ and meets

$$k \bullet a \bullet b = a,$$

and consider the other combinator

$$s = \langle \bot_0, I_{D_0}, \psi_2(s_3), s_3, s_4, \ldots\rangle,$$

where $s_{n+3} = \boldsymbol{\lambda} a \in D_{n+2}.\boldsymbol{\lambda} b \in D_{n+1}.\boldsymbol{\lambda} c \in D_n.a(\phi_n(c))(b(c))$ for each $n \geq 0$. Its combinator belongs to D_∞ and satisfies

$$s \bullet a \bullet b \bullet c = a \bullet c \bullet (b \bullet c).$$

Therefore, by the Remark 2.4.2 and Theorem 2.3.2 we have $\langle D_\infty, \bullet, [\![\,]\!]\rangle$ is an extensional λ-model.

In (BARENGREGT, 1984) can be seen another way to prove that D_∞ is a λ-model through the result: If D is a c.p.o. for which there exists a projection $\langle F, G\rangle$ from $[D \to D]$ to D such that $F \circ G = I_{[D\to D]}$, then the triple $\langle D, \bullet, [\![\,]\!]\rangle$ defined for every assignment $\rho : Var \to D$ by

(a) $a \bullet b := F(a)(b)$ for each $a, b \in D$,

(b) $[\![x]\!]_\rho := \rho(x)$,

(c) $[\![PQ]\!]_\rho := [\![P]\!]_\rho \bullet [\![Q]\!]_\rho$,

(d) $[\![\lambda x.P]\!]_\rho := G(\boldsymbol{\lambda} d \in D.[\![P]\!]_{[d/x]_\rho})$,

is a λ-model. Also if $G \circ F = I_D$, i.e., $G = F^{-1}$ and $D \cong [D \to D]$, then $\langle D, \bullet, [\![\,]\!]\rangle$ is an extensional λ-model.

So in the particular case of D_∞, it holds that $D_\infty \cong [D_\infty \to D_\infty]$ where the isomorphism between c.p.o's $F : D_\infty \to [D_\infty \to D_\infty]$ is given for each $a \in D_\infty$ by

$$F(a) = \boldsymbol{\lambda} b \in D_\infty.a \bullet b,$$

whose inverse $F^{-1} : [D_\infty \to D_\infty] \to D_\infty$ corresponds to

$$F^{-1} = \boldsymbol{\lambda} f \in [D_\infty \to D_\infty].\bigsqcup_{n\geq 0} \phi_{n,\infty}(\boldsymbol{\lambda} a \in D_n.(\phi_{\infty,n} \circ f \circ \phi_{n,\infty})(a)).$$

Therefore $\langle D_\infty, \bullet, [\![\,]\!]\rangle$ is an extensional λ-model.

3 THE EXTENSIONAL λ-MODEL $\mathfrak{D}_\infty$

Next we show the construction of the infinity-groupoid from any topological space, through the use of higher fundamental groups. Finally this construction is applied for the particular case D_∞ to generate the λ-model $\mathfrak{D}_\infty$ with ∞-groupoid structure.

3.1 THE ∞-GROUPOID $\mathfrak{D}_\infty$ GENERATED BY AN ARBITRARY TOPOLOGICAL SPACE D

Definition 3.1.1. *Let D a topological space. For each $n \in \mathbb{N}$, define the ∞-globular set $\mathfrak{D}$*

$$\cdots \rightrightarrows_t^s \mathfrak{D}_n \rightrightarrows_t^s \mathfrak{D}_{n-1} \rightrightarrows_t^s \cdots \rightrightarrows_t^s \mathfrak{D}_1 \rightrightarrows_t^s \mathfrak{D}_0,$$

as follows

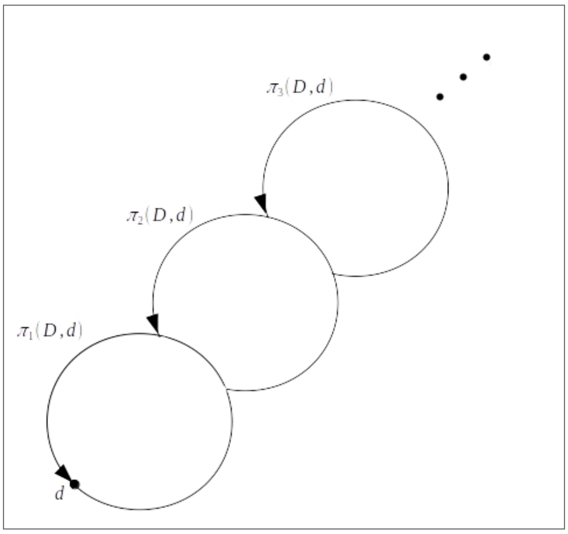
$$\mathfrak{D}_n := \{\pi_n(D,d) : d \in D\},$$

where $\pi_0(D,d) := d$, $\pi_n(D,d)$ is the fundamental group of dimension $n \geq 1$ and

$$s(\pi_{n+1}(D,d)) = t(\pi_{n+1}(D,d)) := \pi_n(D,d).$$

Remark 3.1.1. *Clearly $\mathfrak{D}$ is an ∞-globular set, since $s = t$, i.e., every morphism is an automorphism, then $s \circ s = s \circ t$ and $t \circ t = t \circ s$, see Figure 6.*

Figura 6 – ∞-Globular set $\mathfrak{D}$.



Sink: The author (2020)

Remark 3.1.2. *For each $n \in \mathbb{N}$, the following holds:*

$$\pi_n(D,d) = \pi_n(D,d') \iff d = d'.$$

The fact that D is connected by paths, and thus $\pi_n(D,d) \cong \pi_n(D,d')$ (isomorphic groups), it does not imply that they are equal.

Notation 3.1.1. *For each $n \in \mathbb{N}$, write*

$$\mathfrak{D}_{n+1}(d_0) := \{\mathfrak{d} \in \mathfrak{D}_{n+1} : s(\mathfrak{d}) = t(\mathfrak{d}) = \pi_n(D,d_0)\}.$$

Proposition 3.1.1. *For each $n \in \mathbb{N}$, the following holds*

$$\mathfrak{D}_{n+1}(d_0) = \{\pi_{n+1}(D,d_0)\}.$$

Proof. Let $\mathfrak{d} \in \mathfrak{D}_{n+1}(d_0)$, then there is $d \in D$ such that $\mathfrak{d} = \pi_{n+1}(D,d)$. By Definition 3.1.1 $s(\mathfrak{d}) = t(\mathfrak{d}) = \pi_n(D,d)$. Since $\mathfrak{d} \in \mathfrak{D}_{n+1}(d_0)$, then $\pi_n(D,d) = \pi_n(D,d_0)$, so $d = d_0$ by Remark 3.1.2, thus $\mathfrak{d} = \pi_{n+1}(D,d_0)$. □

Definition 3.1.2 (Diagonal). *Let D a set. Define the diagonal on $D \times D$ as*

$$Diag(D \times D) := \{(d,d') \in D \times D : d = d'\}$$

Lemma 3.1.1. *For any natural number $n \geq 1$ and for each $0 \leq p < n$,*

$$\mathfrak{D}_n \times_{\mathfrak{D}_p} \mathfrak{D}_n = Diag(\mathfrak{D}_n \times \mathfrak{D}_n).$$

Proof.

$$\begin{aligned}
\mathfrak{D}_n \times_{\mathfrak{D}_p} \mathfrak{D}_n &= \{(\mathfrak{d},\mathfrak{d}') \in \mathfrak{D}_n \times \mathfrak{D}_n : t^{n-p}(\mathfrak{d}) = s^{n-p}(\mathfrak{d}')\} \\
&= \{(\mathfrak{d},\mathfrak{d}') \in \mathfrak{D}_n \times \mathfrak{D}_n : \pi_{n-(n-p)}(D,d) = \pi_{n-(n-p)}(D,d')\} \\
&= \{(\mathfrak{d},\mathfrak{d}') \in \mathfrak{D}_n \times \mathfrak{D}_n : \pi_p(D,d) = \pi_p(D,d')\} \\
&= \{(\mathfrak{d},\mathfrak{d}') \in \mathfrak{D}_n \times \mathfrak{D}_n : d = d'\} \\
&= \{(\mathfrak{d},\mathfrak{d}') \in \mathfrak{D}_n \times \mathfrak{D}_n : \mathfrak{d} = \mathfrak{d}'\} \\
&= Diag(\mathfrak{D}_n \times \mathfrak{D}_n).
\end{aligned}$$

□

Lemma 3.1.1 indicates that it is enough to define the composition for pairs $(\mathfrak{d},\mathfrak{d}) \in \mathfrak{D}_n \times \mathfrak{D}_n$.

Definition 3.1.3 (Composition). *For each $0 \leq p < n$, define the composition of $\mathfrak{d} \in \mathfrak{D}_n$ with itself by*

$$\mathfrak{d} \circ_p \mathfrak{d} := \{x *_p y : x, y \in \mathfrak{d}\}$$

where $*_p$ is the paths concatenation operator.

Lemma 3.1.2. *For all $n \geq 1$, $0 \leq p < n$ e $\mathfrak{d} \in \mathfrak{D}_n$,*

$$\mathfrak{d} \circ_p \mathfrak{d} = \mathfrak{d}.$$

Proof. Since $(\mathfrak{d}, *_p)$ is a group, it is clear that $\mathfrak{d} \circ_p \mathfrak{d} \subseteq \mathfrak{d}$. On the other hand, if $x \in \mathfrak{d}$ then $x = x *_p e \in \mathfrak{d} \circ_p \mathfrak{d}$, where e is the identity element of the group $(\mathfrak{d}, *_p)$. Thus $\mathfrak{d} \circ_p \mathfrak{d} = \mathfrak{d}$. □

Definition 3.1.4 (Identity). *For each $n \in \mathbb{N}$ e $\mathfrak{d} = \pi_n(D, d) \in \mathfrak{D}_n$, define the identity function $i : \mathfrak{D}_n \to \mathfrak{D}_{n+1}$, $i(\mathfrak{d}) = 1_{\mathfrak{d}}$ as follows*

$$1_{\mathfrak{d}} := \pi_{n+1}(D, d).$$

Theorem 3.1.1. *$\mathfrak{D}$ is an ∞-groupoid.*

Proof. Let $n \geq 1$, $0 \leq p < n$. For the axioms related to composition of morphisms, Lemma 3.1.1 allows us to verify them by the composition of $\mathfrak{d} = \pi_n(D, d) \in \mathfrak{D}_n$ with itself, then

(a) (sources and targets of composites) by Lemma 3.1.2 and Lemma 3.1.3 we have

$$s(\mathfrak{d} \circ_p \mathfrak{d}) = s(\mathfrak{d}) = s(\mathfrak{d}) \circ_p s(\mathfrak{d}),$$
$$t(\mathfrak{d} \circ_p \mathfrak{d}) = t(\mathfrak{d}) = t(\mathfrak{d}) \circ_p t(\mathfrak{d}),$$

(b) (sources and targets od identities) by Definition 3.1.3 and Definition 3.1.4

$$s(1_{\mathfrak{d}}) = s(\pi_{n+1}(D, d)) = \pi_n(D, d) = \mathfrak{d},$$
$$t(1_{\mathfrak{d}}) = t(\pi_{n+1}(D, d)) = \pi_n(D, d) = \mathfrak{d},$$

(c) (associativity) by Lemma 3.1.2

$$(\mathfrak{d} \circ_p \mathfrak{d}) \circ_p \mathfrak{d} = \mathfrak{d} \circ_p \mathfrak{d} = \mathfrak{d} \circ_p (\mathfrak{d} \circ_p \mathfrak{d}),$$

(d) (identities) by Definition 3.1.3, Definition 3.1.4 and Lemma 3.1.2

$$i^{n-p}(t^{n-p}(\mathfrak{d})) \circ_p \mathfrak{d} = i^{n-p}(\pi_p(D, d)) \circ_p \mathfrak{d} = \pi_{p+n-p}(D, \mathfrak{d}) \circ_p \mathfrak{d} = \mathfrak{d} \circ_p \mathfrak{d} = \mathfrak{d},$$
$$\mathfrak{d} \circ_p i^{n-p}(s^{n-p}(\mathfrak{d})) = \mathfrak{d} \circ_p i^{n-p}(\pi_p(D, d)) = \mathfrak{d} \circ_p \pi_{p+n-p}(D, \mathfrak{d}) = \mathfrak{d} \circ_p \mathfrak{d} = \mathfrak{d},$$

(e) (binary interchange) let $0 \leq q < p < n$, by Lemma 3.1.2

$$(\mathfrak{d} \circ_p \mathfrak{d}) \circ_q (\mathfrak{d} \circ_p \mathfrak{d}) = \mathfrak{d} \circ_q \mathfrak{d} = (\mathfrak{d} \circ_q \mathfrak{d}) \circ_p (\mathfrak{d} \circ_q \mathfrak{d}),$$

(f) (nullary interchange) let $0 \leq q < p < n$, by Lemma 3.1.2

$$1_{\mathfrak{d}} \circ_q 1_{\mathfrak{d}} = 1_{\mathfrak{d}} = 1_{\mathfrak{d} \circ_q \mathfrak{d}},$$

(g) (inverse) by Lemma 3.1.2 and (d)

$$\mathfrak{d} \circ_p \mathfrak{d} = \mathfrak{d} = i^{n-p}(t^{n-p}(\mathfrak{d})) = i^{n-p}(s^{n-p}(\mathfrak{d})),$$

thus $\mathfrak{d}$ is the inverse of itself.

□

Next we define $\mathfrak{D}_\infty$ as a set in the sense of ZFC set theory.

Definition 3.1.5 (The set $\mathfrak{D}_\infty$). *Define $\mathfrak{D}_\infty$ as the set of all the infinite sequences*

$$\mathfrak{d} := \langle \mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2, \ldots \rangle,$$

such that $\mathfrak{d}_n \in \mathfrak{D}_n$ e $s(\mathfrak{d}_{n+1}) = t(\mathfrak{d}_{n+1}) = \mathfrak{d}_n$, for each $n \in \mathbb{N}$.

Equality in $\mathfrak{D}_\infty$ is defined as

$$\mathfrak{a} = \mathfrak{b} \Longleftrightarrow \mathfrak{a}_n = \mathfrak{b}_n,$$

for all $n \in \mathbb{N}$.

Proposition 3.1.2. *$\mathfrak{d} \in \mathfrak{D}_\infty$ if and only if there exists $d \in D$ for all $n \in \mathbb{N}$, such that $\mathfrak{d}_n = \pi_n(D, d)$.*

Proof. Let $\mathfrak{d} \in \mathfrak{D}_\infty$, then $\mathfrak{d}_0 \in \mathfrak{D}_0$, i.e., $\mathfrak{d}_0 = \pi_0(D, d)$ for some $d \in D$. Suppose that $\mathfrak{d}_n = \pi_n(D, d)$ and we will prove by induction that $\mathfrak{d}_{n+1} = \pi_{n+1}(D, d)$. Since $\mathfrak{d} \in \mathfrak{D}_\infty$, by induction hypothesis $s(\mathfrak{d}_{n+1}) = t(\mathfrak{d}_{n+1}) = \mathfrak{d}_n = \pi_n(D, d)$. By Proposition 3.1.1 we have $\mathfrak{d}_{n+1} = \pi_{n+1}(D, d)$. On the other hand, if $\mathfrak{d}_n = \pi_n(D, d)$ for every $n \in \mathbb{N}$, then $s(\mathfrak{d}_{n+1}) = t(\mathfrak{d}_{n+1}) = \pi_n(D, d) = \mathfrak{d}_n$ for all $n \in \mathbb{N}$, thus $\mathfrak{d} \in \mathfrak{D}_\infty$. □

Proposition 3.1.3. *Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{D}_\infty$, such that $\mathfrak{a}_n = \pi_n(D, a)$ and $\mathfrak{b}_n = \pi_n(D, b)$ for all $n \in \mathbb{N}$, then*

$$\mathfrak{a} = \mathfrak{b} \Longleftrightarrow \mathfrak{a}_0 = \mathfrak{b}_0.$$

Proof. If $\mathfrak{a} = \mathfrak{b}$, by Definition 3.1.5 we have $\mathfrak{a}_0 = \mathfrak{b}_0$. If $a = \mathfrak{a}_0 = \mathfrak{b}_0 = b$, clearly $\mathfrak{a}_n = \pi_n(D, a) = \pi_n(D, b) = \mathfrak{b}_n$. □

3.2 HIGHER FUNDAMENTAL GROUPS OF A C.P.O.

Next we show that every higher fundamental groupoids on any c.p.o. are trivial, particularly those generated by D_∞.

Lemma 3.2.1. *Let D be a c.p.o. with the Scott topology. If $A \neq D$ is an open, then $\bot \notin A$.*

Proof. Let A be an open from D. Suppose $\bot \in A$. Since D is a c.p.o, then for all $d \in D$ we have $\bot \sqsubseteq d$. Since A is final, $\bot \in A$ and $\bot \sqsubseteq d$, then $d \in A$. Thus $A = D$, which is a contradiction. □

Theorem 3.2.1. *If D is a c.p.o. with the Scott topology, then $\pi_n(D, d) = \{[c^n(d)]\}$ for all $d \in D$ and $n \geq 0$.*

Proof. By Theorem 2.1.2, it is enough to check that D is contractible, i.e., one has to check that for the identity function $I_D : D \to D$ there exists some constant function $f_c : D \to D$ such that $f_c \simeq I_D$. Consider the map $H : D \times [0,1] \to D$ defined by

$$H(x,t) = \begin{cases} \bot & \text{if } t = 0, \\ x & \text{if } t \in (0,1] \end{cases}$$

and let us show that H is a contraction from D. Clearly $H(\cdot, 0) = f_\bot(\cdot)$ and $H(\cdot, t) = I_D(\cdot)$ if $t \in (0,1]$. Now take any open $A \neq D$, then

$$\begin{aligned} H^{-1}(A) &= \{(x,t) \in D \times [0,1] : H(x,t) \in A\} \\ &= (\{x \in D : H(x,0) = \bot \in A\} \times \{0\}) \cup (\{x \in D : x \in A\} \times (0,1]) \\ &= (\emptyset \times \{0\}) \cup (A \times (0,1]) \quad \text{(by Lemma 3.2.1 } \bot \notin A) \\ &= A \times (0,1], \end{aligned}$$

which is an open from $D \times [0,1]$, then H is continuous. Thus H is a contraction from D. □

Definition 3.2.1 (**Parallel paths**)**.** *Two n-paths p and q based $a, b \in D$ are parallel if for each $r = 1 \ldots, n-1$ satisfies*

1. $p[t_1 = 0, \ldots, t_r = 0] = q[t_1 = 0, \ldots, t_r = 0]$,
2. $p[t_1 = 1, \ldots, t_r = 1] = q[t_1 = 1, \ldots, t_r = 1]$.

Corollary 3.2.1. *If p and q are parallel n-paths based $a, b \in D_\infty$, then $p =_h q$.*

Proof. Since p and q are parallel, we have

$$p[t_1 = 1, \ldots t_{n-1} = 1] = q[t_1 = 1, \ldots, t_{n-1} = 1] = \bar{q}[t_1 = 0, \ldots, t_{n-1} = 0],$$

by definition of product

$$(p *_0 \bar{q})[t_1, \ldots, t_{n-1}] = p[t_1, \ldots, t_{n-1}] *_0 \bar{q}[t_1, \ldots, t_{n-1}]$$

where

$$\begin{aligned} (p *_0 \bar{q})[t_1, \ldots, t_{n-1}](0) &= p[t_1, \ldots, t_{n-1}](0) = a, \\ (p *_0 \bar{q})[t_1, \ldots, t_{n-1}](1) &= \bar{q}[t_1, \ldots, t_{n-1}](1) = a, \end{aligned}$$

for each $t_1, \ldots, t_{n-1} \in [0,1]$.

Then $(p *_0 \bar{q})[t_1, \ldots, t_{n-1}]$ is a 1-path closed in a for all $t_1, \ldots, t_{n-1} \in [0,1]$. By Theorem 3.2.1, it has $(p *_0 \bar{q})[t_1, \ldots, t_{n-1}] =_h c(a) = c^n(a)[t_1, \ldots, t_{n-1}]$ for each $t_1, \ldots, t_{n-1} \in [0,1]$, where $c(a)$ is the constant path in a, i.e., $c(a)(t) = a$ for each $t \in [0,1]$, and $c^n(a)[t_1, \ldots, t_r] = c^{n-r}(a)$. Thus $p *_0 \bar{q} =_h c^n(a)$, so it $p =_h q$. □

Notation 3.2.1. *Write* $\Pi_n(D_\infty, a, b)$ *for the n-groupoid (weak) of* n*-paths based at* $a, b \in D_\infty$*. And write* $\Pi_\infty(D_\infty, a, b)$ *for the globular set*

$$\cdots \rightrightarrows_t^s \Pi_n(D_\infty, a, b) \rightrightarrows_t^s \Pi_{n-1}(D_\infty, a, b) \rightrightarrows_t^s \cdots \rightrightarrows_t^s \Pi_1(D_\infty, a, b) \rightrightarrows_t^s \Pi_0(D_\infty, a, b),$$

where $s(p) := p[t_1 = 0]$ *and* $t(p) := p[t_1 = 1]$*.*

Therefore, by the Corollary 3.2.1 the n-groupoid $\Pi_n(D_\infty, a, b)$ is trivial for each $n \geq 0$, i.e., under the intensional equal $=_h$ only there is one parallel n-path which inhabits into $\Pi_n(D_\infty, a, b)$. Since D_∞ is connected by paths, given $a', b' \in D_\infty$ holds that $\Pi_n(D_\infty, a, b) \cong \Pi_n(D_\infty, a', b')$. Thus any n-groupoid $\Pi_n(D_\infty, a, b)$ can be written simply as $\Pi_n(D_\infty)$.

On other hand, note that if $a = b$ then the n-groupoid $\Pi_n(D_\infty, a, b) = \Pi_n(D_\infty, a, a) := \Pi_n(D_\infty, a)$ is contained in the n-group $\pi_n(D_\infty, a)$ for each $n \geq 0$. So the fundamental ∞-groupoid $\Pi_\infty(D_\infty, a)$ contains to the fundamental ∞-group $\pi_\infty(D_\infty, a)$.

3.3 THE λ MODEL $\mathfrak{D}_\infty$ AND ITS FUNDAMENTAL ∞-GROUPOID

According to Definition 3.1.5, let $\mathfrak{D}_\infty$ be the ∞-groupoid generated by the c.p.o. D_∞ with the Scott topology. By Proposition 3.1.2 we have that for each $\mathfrak{d} \in \mathfrak{D}_\infty$ there is $d \in D_\infty$ such

$$\begin{aligned}
\mathfrak{d} &= \langle \pi_0(D_\infty, d), \pi_1(D_\infty, d), \pi_2(D_\infty, d), \ldots \rangle \\
&= \langle \{d\}, \{[c(d)]\}, \{[c^2(d)]\}, \ldots \rangle \\
&\cong \langle [d], [c(d)], [c^2(d)], \ldots, \rangle \in \pi_\infty(D_\infty, d) \;\; \text{(by Definition 3.1.5)}.
\end{aligned}$$

Therefore $\mathfrak{D}_\infty(d) \cong \pi_\infty(D_\infty, d)$ (isomorphism of groups). Other interpretation by Definition 2.4.12, Theorem 3.2.1 and Remark 2.1.2, we have that $\pi_0(D_\infty, d) = d = \langle d_0, d_1, d_2, \ldots \rangle$, $\pi_1(D_\infty, d) \cong \{c_d\}$, $\pi_2(D_\infty, d) \cong \{c_{c_d}\}, \ldots$. Thus, according to Definition 3.1.5 each $\mathfrak{d} \in \mathfrak{D}_\infty$ can be seen as the infinite matrix

$$\mathfrak{d} \cong \langle d, c_d, c_{c_d}, \ldots \rangle := \begin{pmatrix} d_0 & c_{d_0} & c_{c_{d_0}} & \cdots \\ d_1 & c_{d_1} & c_{c_{d_1}} & \cdots \\ d_2 & c_{d_2} & c_{c_{d_2}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Definition 3.3.1 (**Application in** $\mathfrak{D}_\infty$)**.** *For* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{D}_\infty$ *such that* $\mathfrak{a}_n = \pi_n(D_\infty, a)$ *and* $\mathfrak{b}_n = \pi_n(D_\infty, b)$*, define the product* $\pi_n(D_\infty, a)$ *with* $\pi_n(D_\infty, b)$ *as*

$$\pi_n(D_\infty, a) \bullet \pi_n(D_\infty, b) := \pi_n(D_\infty, a \bullet b),$$

so the application of $\mathfrak{a}$ *to* $\mathfrak{b}$ *in* $\mathfrak{D}_\infty$ *as the infinite sequence*

$$\mathfrak{a} \bullet \mathfrak{b} = \langle \mathfrak{a}_0 \bullet \mathfrak{b}_0, \mathfrak{a}_1 \bullet \mathfrak{b}_1, \mathfrak{a}_2 \bullet \mathfrak{b}_2, \ldots \rangle = \langle a \bullet b, \pi_1(D_\infty, a \bullet b), \pi_2(D_\infty, a \bullet b), \ldots \rangle.$$

Theorem 3.3.1. $\langle \mathfrak{D}_\infty, \bullet \rangle \cong \langle D_\infty, \bullet \rangle$. *So* $\langle \mathfrak{D}_\infty, \bullet \rangle$ *is an extensional* λ*-model.*

Proof. It is enough to show that the mapping $F : \langle D_\infty, \bullet \rangle \to \langle \mathfrak{D}_\infty, \bullet \rangle$ such that $(F(a))_n = \pi_n(D_\infty, a)$ for each $n \in \mathbb{N}$, is an isomorphism.

$$\begin{aligned}(F(a) \bullet F(b))_n &= (F(a))_n \bullet (F(b))_n \\ &= \pi_n(D_\infty, a) \bullet \pi_n(D_\infty, b) \\ &= \pi_n(D_\infty, a \bullet b) \\ &= (F(a \bullet b))_n ,\end{aligned}$$

for all $n \in \mathbb{N}$. This is $F(a) \bullet F(b) = F(a \bullet b)$.

F is injective. Let $F(a) = F(b)$, by Proposition 3.1.3 we have

$$a = (F(a))_0 = (F(b))_0 = b.$$

F is surjective. Let $\mathfrak{a} \in \mathfrak{D}_\infty$, then we have $\mathfrak{a}_0 = a$ for some $a \in D_\infty$. Thus $F(a) = \mathfrak{a}$. □

Next we will study the topological structure of $\mathfrak{D}_\infty$ and its relation with D_∞.

Definition 3.3.2 (Partial order in $\mathfrak{D}_\infty$). *For each* $\mathfrak{a}$ *and* $\mathfrak{b}$ *in* $\mathfrak{D}_\infty$ *define the partial order in* $\mathfrak{D}_\infty$ *as*

$$\mathfrak{a} \sqsubseteq \mathfrak{b} \iff a \sqsubseteq b,$$

where $\mathfrak{a}_n = \pi_n(D_\infty, a)$ *and* $\mathfrak{b}_n = \pi_n(D_\infty, b)$ *for each* $n \geq 0$.

Theorem 3.3.2. $\langle \mathfrak{D}_\infty, \sqsubseteq \rangle \cong \langle D_\infty, \sqsubseteq \rangle$. *So* $\langle \mathfrak{D}_\infty, \sqsubseteq \rangle$ *is a c.p.o. and the quotient topology on* $\mathfrak{D}_\infty$ *is exactly the Scott topology.*

Proof. By the bijection $F : D_\infty \to \mathfrak{D}_\infty$ of proof Theorem 2.3.2 one can easily see it is the isomorphism between both partial orders. So $\langle \mathfrak{D}_\infty, \sqsubseteq \rangle$ is a c.p.o., since $F(\perp)$ is the least element in $\mathfrak{D}_\infty$, and give a directed set X in $\mathfrak{D}_\infty$ we have that $F^{-1}(X)$ is directed in D_∞ then $\bigsqcup X = F(\bigsqcup F^{-1}(X))$. On the other hand, the quotient topology on $\mathfrak{D}_\infty$ of map $F : \langle D_\infty, \tau_{Scott} \rangle \to \mathfrak{D}_\infty$ is given by

$$\tau^F_{D_\infty} = \{X \subseteq \mathfrak{D}_\infty : F^{-1}(X) \in \tau_{Scott}\}.$$

Clearly the Scott topology τ'_{Scott} is contained in $\tau^F_{D_\infty}$, since the map $F : \langle D_\infty, \tau_{Scott} \rangle \to \langle \mathfrak{D}_\infty, \tau'_{Scott} \rangle$ is continuous, i.e., $F(\bigsqcup X) = \bigsqcup F(X)$ for each directed set X in D_∞. Given $X \in \tau^F_{D_\infty}$, so $F^{-1}(X)$ is final and inaccessible by directedness, thus $F(F^{-1}(X)) = X$ is final and inaccessible by directedness. Therefore $\tau^F_{D_\infty} = \tau'_{Scott}$. □

3.4 INTERPRETATION OF β-EQUALITY PROOFS IN D_∞

In λ-calculus we have that two λ-terms M and N are β-equal, $M =_\beta N$, if there is a sequence of λ-terms N_1, N_2,...,N_n such that

$$(\forall i \leq n-1)(N_i \rhd_{1\beta} N_{i+1} \ \ or \ \ N_{i+1} \rhd_{1\beta} N_i \ \ or \ \ N_i \equiv_\alpha N_{i+1}),$$

where $N_1 = M$ and $N_n = N$. Thus the equality of the theory $\lambda\beta$ can be seen as an intensional equality, in the sense that the chain

$$M = N_0 =_\beta N_1 =_\beta \cdots =_\beta N_n = N,$$

it would be a proof P of equality $M =_\beta N$, which can be interpreted in some topological model $\langle D, \bullet, [\![\,]\!]\rangle$ as a continuous path $p : [\![M]\!] \rightsquigarrow [\![N]\!]$ which passes through the intermediate points $[\![N_1]\!], [\![N_2]\!] \ldots, [\![N_{n-1}]\!]$. Then we could ask ourselves if given two 1-proofs P and Q of equality $M =_\beta N$, in space D is there a homotopy (2-path) between the paths $p := [\![P]\!]$ and $q := [\![Q]\!]$?. Now if we have some intensional definition (with relation to D) of D-equality between the proofs of equality P and Q such that its interpretation into D is a homotopy from p to q, we could ask again if for the 2-proofs F and G of the equality $P =_D Q$ is there a homotopy of homotopies (3-path) between the homotopies $f := [\![F]\!]$ and $g := [\![G]\!]$? And so on, we can continue asking with the purpose of forming from model topology D a ∞-groupoid structure in λ-calculus.

We have that D_∞ is an topological λ-model, but its topological structure does not allow to capture relevant information about equality between higher proofs at λ-calculus, since the ∞-groupoid generated by D_∞ is trivial. The reason is that any proof P of equality $M =_\beta N$ given by the chain

$$P : M = N_0 =_\beta N_1 =_\beta \cdots =_\beta N_n = N,$$

would be interpreted by some path $p : [\![M]\!] \rightsquigarrow [\![N]\!]$ that passes through the intermediate points $[\![N_1]\!], \ldots, [\![N_{n-1}]\!]$, but all these points are equal in space D_∞, i.e.,

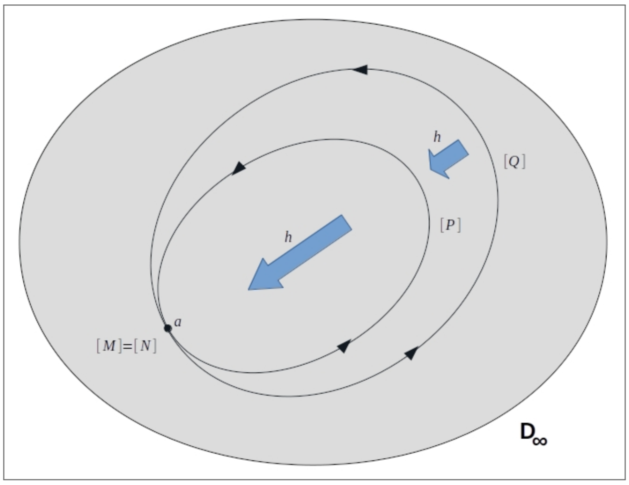
$$p : [\![M]\!] = [\![N_0]\!] = [\![N_1]\!] = \cdots = [\![N_n]\!] = [\![N]\!].$$

Therefore the interpretation of proof P is some closed path $p : [\![M]\!] \rightsquigarrow [\![M]\!]$, we wrote such interpretation as $[\![P]\!] := p$. Since $\pi_1(D_\infty, [\![M]\!])$ is trivial by Theorem 3.2.1, then p is homotopically equal to the constant path $c([\![M]\!])$, i.e., $p =_h c([\![M]\!])$. Now given any other proof

$$Q : M = N'_0 =_\beta N'_1 =_\beta \cdots =_\beta N'_n = N,$$

of equality $M =_\beta N$, with interpretation $[\![Q]\!] = q$, by Corollary 3.2.1 we have $p =_h q$, so we could assert that $P =_{D_\infty} Q$ (see Figure 7). Thus the class of all the proofs of any equality is trivial with respect to D_∞.

Figura 7 – Interpretation of equal proofs $P, Q : (M =_\beta N)$ on D_∞.



Sink: The author (2020)

If we continue at the next level, i.e., given any 2-proofs F and G of equality $P =_{D_\infty} Q$, we have that the interpretation of F and G is given for some pair of homotopies (2-paths) $f, g : p \rightsquigarrow q$, where $p, q : [\![M]\!] \rightsquigarrow [\![M]\!]$, by Corollary 3.2.1 it has $f =_h g$ thus $F =_{D_\infty} G$. Therefore the class of all 2-proofs of any proof equality $P =_{D_\infty} Q$ is also trivial.

A better way to study the intentionality of equality $=_\beta$ would be to set aside the set equality of the Definition 2.3.6 and opt rather for homotopic models (or some homotopy variation) defined below.

Definition 3.4.1 (**Homotopic λ-model**). *A homotopic λ-model is a triple $\langle D, \bullet, [\![\,]\!]\rangle$, where D is a topological space, $\bullet : D \times D \to D$ is a binary operation and $[\![\,]\!]$ is a mapping which assigns, to λ-term M and each assignment $\rho : Var \to D$, an element $[\![M]\!]_\rho$ of D such that*

(*a*) $[\![x]\!] = \rho(x)$;

(*b*) $[\![PQ]\!]_\rho = [\![P]\!]_\rho \bullet [\![Q]\!]_\rho$;

(*c*) $[\![\lambda x.P]\!]_\rho \bullet d =_h [\![P]\!]_{[d/x]\rho}$ *for all* $d \in D$;

(*d*) $[\![M]\!]_\rho = [\![M]\!]_\sigma$ *if* $\rho(x) = \sigma(x)$ *for* $x \in FV(M)$;

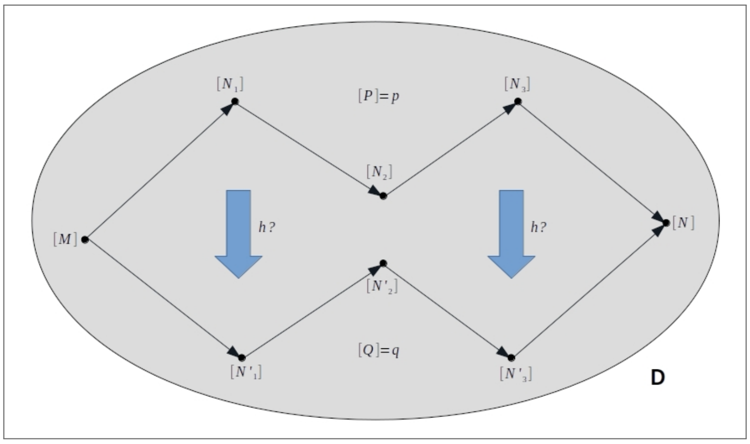
(*e*) $[\![\lambda x.M]\!]_\rho = [\![\lambda y.[y/x]M]\!]_\rho$ *if* $y \notin FV(M)$;

(*f*) *if* $(\forall d \in D)\left([\![P]\!]_{[d/x]\rho} = [\![Q]\!]_{[d/x]\rho}\right)$, *then* $[\![\lambda x.P]\!]_\rho = [\![\lambda x.Q]\!]_\rho$.

The homotopic model $\langle D, \bullet, [\![\,]\!]\rangle$ is an extensional homotopic model if it satisfies the additional property: $[\![\lambda x.Mx]\!]_\rho =_h [\![M]\!]_\rho$ with $x \notin FV(M)$.

To solve the triviality problem of proofs interpretation on D_∞, we would have to propose a λ-homotopic model D with another topology, for which must there exist two proofs $P, Q : (M =_\beta N)$ whose interpretations are not homotopically equal, $p \neq_h q$ (of course there must also be different equality proofs whose interpretations are homotopically equal), as can be seen in the Figure 8. It would allow us to capture more information about the multiple β-contractions and reverse β-contractions of an equality proof than a traditional model based on extensional equality between sets.

Figura 8 – Proofs $P, Q : (M =_\beta N)$ on an homotopic λ-model D.



Sink: The author (2020)

On other hand, if we forget the homotopies between continuous paths in D_∞ (or $\mathfrak{D}_\infty$) and consider simply extensional equality between functions, we could define the interpretation of the equality proof $P : M = N_0 =_\beta \cdots =_\beta N_n = N$ as a concatenation of continuous paths

$$p := r_1 * r_2 * r_3 * \cdots * r_n,$$

where each continuous path $r_i : [0,1] \to D_\infty$ is given by

$$r_i(t) := \begin{cases} a & \text{if } t \in [0, 1/2) \cup (1/2, 1], \\ \perp & \text{if } t = 1/2 \end{cases}$$

with $a = [\![M]\!] = [\![N_1]\!] = \cdots = [\![N_n]\!] = [\![N]\!]$. We write the interpretation of P in D_∞ as $[\![P]\!] := p$ and each r_i is called a *time period*, thus we say that p consists of n *time periods* and is written as $t(p) = n$.

Now if we have another proof of equality $Q : M = N'_0 =_{1\beta} \cdots =_{1\beta} N'_m = N$ whose interpretation would be

$$q := r_1 * r_2 * r_3 * \cdots * r_m,$$

where it is clear that $t(q) = m$. So we say that the equality proofs $P, Q : M =_\beta N$ are equal according to model D_∞, noted by if their respective interpretations are equal in the traditional sense of set theory, i.e.,

$$P =_{D_\infty} Q \Longleftrightarrow p = q,$$

thus we would have

$$P =_{D_\infty} Q \Longleftrightarrow t(p) = t(q).$$

Thus we have that two proofs P and Q of $M =_\beta N$ are "equal" if they require at same time to complete the proof or else, if P and Q are sequences of the same length.

Although the extensional equality $p = q$ manages to capture information about the length of the P and Q proofs in λ-calculus, the nature of its extensionality does not allow to capture more information about the 2-proofs of proof equality: $P =_{D_\infty} Q$, since there is only one canonical way to prove $P =_{D_\infty} Q$, so the generated ∞-groupoid by $=_{D_\infty}$ it would be trivial for 2-equality, 3-equality and so on.

4 CONCLUSIONS

This section aims to present the final considerations on the main topics covered in this thesis, including the contributions achieved and indications for future work.

4.1 CONTRIBUTIONS

Starting from any topological space that models λ-calculus, we propose a method to build an ∞-groupoid that also models λ-calculus. This construction was applied to a particular c.p.o. with the Scott topology, resulting in a constant cell infinite sequences set, where each cell sequence is naturally isomorphic to a constant higher paths infinite matrix.

4.2 FUTURE WORKS

The next step would be to see the implications of the ∞-groupoid constructed in this work, in the theory of extensional λ-calculus, particularly in its syntax; in what refers to the "equality" between two equality proofs of two λ-terms, "equality" between proofs of equality proofs, and so on.

We can also apply this construction of ∞-groupoids for other lambda models that are endowed with a topological structure, to see their consequences in the syntax of "equality" proofs in lambda calculus.

With a little more difficulty, we could try to build homotopic λ-model in order to avoid trivialities in the fundamental ∞-groupoid associated with the topology of the model, which would allow to capture relevant information about the higher equality proof in the λ-calculus syntax.

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