UNIVERSIDADE FEDERAL DE PERNAMBUCO
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

Thiago Yukio Tanaka

ALGUNS RESULTADOS DE CONTROLE PARA MODELOS PARABÓLICOS: controle insensibilizante e controlabilidade uniforme

Recife
2019

Thiago Yukio Tanaka

ALGUNS RESULTADOS DE CONTROLE PARA MODELOS PARABÓLICOS: controle insensibilizante e controlabilidade uniforme

Esta tese foi apresentada à Pós-Graduação em Matemática do Departamento de Matemática da Universidade Federal de Pernambuco como requisito parcial para obtenção do grau de Doutor em Matemática.

Área de Concentração: Análise.

Orientador: Prof. Dr. Maurício Cardoso Santos

Recife

2019

Catalogação na fonte
Bibliotecária Mariana de Souza Alves, CRB4-2106

T161a Tanaka, Thiago Yukio
Alguns resultados de controle para modelos parabólicos: controle insensibilizante e controlabilidade uniforme/ Thiago Yukio Tanaka – 2019.
74 f.

Orientador: Maurício Cardoso Santos.
Tese (Doutorado) – Universidade Federal de Pernambuco. CCEN Matemática, Recife, 2019.
Inclui referências e apêndices.

1. Análise (Matemática). 2. Controle Insensibilizante. 3. Controlabilidade Nula. I. Santos, Maurício Cardoso (orientador). II. Título.

515 CDD (23. ed.) UFPE- MEI 2019-089

THIAGO YUKIO TANAKA

ALGUNS RESULTADOS DE CONTROLE PARA MODELOS PARABÓLICOS: CONTROLE INSENSIBILIZANTE E CONTROLABILIDADE UNIFORME

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

Aprovado em: 26/02/2019

BANCA EXAMINADORA

Prof. Dr. Maurício Cardoso Santos (Orientador)
Universidade Federal da Paraíba

Prof. Dr. Felipe Wallison Chaves Silva (Examinador Interno)
Universidade Federal da Paraíba

Prof. Dr. Diego Araújo de Souza (Examinador Interno)
Universidade Federal de Pernambuco

Prof. Dr. Fágner Dias Araruna (Examinador Externo)
Universidade Federal da Paraíba

Prof.Dr. Ademir Fernando Pazoto (Examinador Externo)
Universidade Federal do Rio de Janeiro

AGRADECIMENTOS

Agradeço primeiramente a Deus, por ter me conduzido e ensinado a minha vida inteira.

Aos meus pais Jandira e Massayuki, e minha irmã Beatriz, por serem pilares de luz em minha vida.

À minha esposa Isabela Farias, por ter me incentivado, auxiliado e ter sido extremamente companheira.

Agradeço profundamente ao professor Maurício Cardoso Santos, que aceitou ser meu orientador de tese de doutorado já num período avançado de tempo, por ter confiado na minha capacidade, tendo proporcionado problemas interessantes e motivadores. Sem a atuação ativa do professor Maurício, este trabalho não seria possível.

Aos demais professores do departamento de matemática (DMAT) da UFPE que contribuíram com minha formação acadêmica, em especial ao professor Airton Castro que me auxiliou desde a graduação até o doutorado.

Agradeço, em especial, aos professores que me conduziram nos estudos de teoria de controle, Diego A. Souza, Roberto A. Capistrano Filho e Felipe W. Chaves-Silva, não apenas pela imensa contribuição neste trabalho, mas também pelas palavras de apoio.

Não poderia deixar de agradecer a todos os funcionários do DMAT.

Agradeço também a todos os meus amigos, por compartilharem em momentos finitos, incontáveis felicidades, dramas e histórias que carregarei por toda a vida.

Aos amigos de trabalho da UFRPE, por todo auxílio e todas as palavras de apoio para que esta tese tivesse um andamento favorável. Em especial aos que trabalharam comigo nas SEMATs e OPEMATs que organizamos juntos.

Aos amigos de mestrado, doutorado e todos os meus alunos. Em especial aos meus irmãos que conheci desde o mestrado na UFPE e que, por um acaso do destino, trabalham juntos comigo na UFRPE, Gilson Simões, Edgar Amorim, Lorena Brizza e Renato Teixeira, por todo o apoio e companheirismo.

Agradeço também aos membros que compuseram a banca avaliadora: Ademir Pazoto, Fagner Araruna, Diego Souza e Felipe Walisson, cujas contribuições melhoraram significativamente esta versão final da tese.

Ao CNPq pelo apoio financeiro.

Thiago Yukio Tanaka

RESUMO

O objetivo desta tese é apresentar alguns resultados recentes em teoria de controle para alguns problemas formulados por equações diferenciais parciais (EDPs). Inicialmente, estudaremos um problema de controle insensibilizante para uma equação do tipo Ginzburg-Landau com não linearidade cúbica. Neste tipo de problema, buscamos controles que tornem um dado funcional de energia insensível para pequenas alterações do dado inicial. É importante mencionar que Zhang M. e Liu X. (2014) provaram resultados de mesma natureza considerando funcionais energia que dependem apenas das soluções da equação não linear de Ginzburg-Landau. A novidade aqui é que estudaremos este problema considerando funcionais energia que dependem do gradiente das soluções, trazendo novas dificuldades do ponto de vista técnico. Em um segundo problema, tratamos da controlabilidade nula uniforme para uma família de sistemas parabólicos lineares com rápida difusão, tendo em sua estrutura termos de acoplamento de ordem zero ou um. Em tais sistemas, uma das equações se degeneram em uma equação elíptica quando o parâmetro de difusão converge a zero. É importante mencionar que Chaves-Silva *et al.* (2014) provaram resultados similares considerando apenas acoplamentos de ordem zero. Nosso objetivo é encontrar controles que sejam uniformemente limitados com respeito a esse parâmetro. As ferramentas principais para obtermos esses resultados propostos serão estimativas de Carleman. Utilizando estas estimativas, conseguiremos obter desigualdade de observabilidade adequadas que nos permitirão a construção de tais controles.

Palavras-chave: Controle Insensibilizante. Controlabilidade Nula. Desigualdade de Carleman. Equação de Ginzburg-Landau. Sistemas Parabólicos de Rápida Difusão.

ABSTRACT

The objective of this thesis is to present some recent results in control theory for some problems formulated by partial differential equations (PDEs). Initially, we will study an insensitizing control problem for a Ginzburg-Landau equation with cubic nonlinearity. In this kind of problem, we search for controls that make a given energy functional insensible to small perturbations of the initial data. It is important to mention that Zhang M. and Liu X. (2014) proved similar results considering energy functional that depend only on the solution of the nonlinear Ginzburg-Landau equation. The novelty here is that we study this problem considering energy functionals depending on the gradient of the solutions, bringing new difficulties from the technical point of view. In a second problem, we deal with the uniform null controllabiliy for a family of fast diffusion linear parabolic systems with with coupling terms of zero and first order in their structure. In these systems, one of the equations degenerates into an elliptic one when the diffusion parameter converges to zero. It is important to mention that Chaves-Silva *et al.* (2014) proved similar results considering only zero order coupling terms. Our goal is to find controls which are uniformly bounded with respect to this parameter. The main tools for obtaining these results will be Carleman estimates. Using these estimates we can prove suitable observability inequalities that will allow us to construct such controls.

Keywords: Insensitizing Controls. Null Controllability. Carleman Inequality. Ginzbug-Landau Equation. Fast Diffusion Parabolic Systems.

SUMÁRIO

1 INTRODUÇÃO

1.1 Organização desta tese

Para adequar a estrutura desta tese às normas da Associação Brasileira de Normas Técnicas (ABNT), este manuscrito está organizado da seguinte maneira: a Seção 1 traz uma introdução com conceitos básicos e históricos da Teoria de Controle e uma descrição sucinta dos problemas aqui abordados. Em seguida, expomos as Referências Bibliográficas utilizadas para o desenvolvimento deste trabalho de tese. Por fim, nos Apêndices A e B, dispomos os resultados obtidos em formato de artigo e em língua inglesa, sendo organizados da seguinte maneira:

- O Apêndice A está relacionado com a existência de controles insensibilizantes para a equação de Ginzburg-Landau com não linearidade cúbica;
- O Apêndice B trata sobre um resultado de controlabilidade nula uniforme para uma família de sistemas parabólicos de rápida difusão e acoplamentos de primeira ordem, nos quais uma das equações se degenera em uma equação elíptica quando o parâmetro de difusão converge a zero.

1.2 Alguns aspectos históricos da teoria de controle

É inegável na história das civilizações a busca por um entendimento de fenômenos naturais como meio para o desenvolvimento científico, tecnológico e socioeconômico. Uma vez que tais fenômenos são bem compreendidos, outra questão que surge naturalmente é se por meio de uma influência externa, isto é, algo que não acontece espontaneamente, é possível controlar a dinâmica de tais fenômenos afim de obter resultados desejados. Esses pensamentos, intrínsecos à natureza humana, nos levaram a considerar vários problemas de controle nas mais variadas áreas do conhecimento.

A necessidade de modificar situações e aspectos do nosso planeta está diretamente ligada à nossa natureza investigativa e dominante. Na agricultura, por exemplo, a produção de alimentos é certamente um problema central desde as civilizações antigas, pois está diretamente ligada ao controle dos recursos hídricos, onde o controle da produção está ligado ao problema de obter os mesmos (ou melhores) resultados de colheita por uma ação que não pode ser espontânea como a espera de uma estação chuvosa. Há registros que comprovam que as técnicas de irrigação, como solução para os problemas de produção, datam de 4500 a.C. pelos assírios, caldeus e babilônios. Os registros dessas técnicas na China (2000 a.C.) e no Egito, Mesopotâmia e Índia (1000 a.C.) são mais divulgados por serem mais recentes e engenhosos.

Outro período importante que ressaltamos é a revolução industrial a partir do século XVIII. A resolução de problemas de controle aplicados a certos tipos de mecanismos permitiu um crescimento exponencial dos meios de produção, uma vez que as máquinas produzem muito mais quando comparadas aos seres humanos. Neste contexto, podemos citar a máquina a vapor que possuía um mecanismo de ajuste de pressão que permitia controlar a velocidade da máquina com o objetivo de torná-la aproximadamente constante.

Podemos citar também o período da segunda guerra mundial, de 1937 a 1945, como um período importante para a evolução da Teoria de Controle. Muitos dos dispositivos que usamos hoje em nossa vida diária tiveram suas origens como instrumentos militares, por exemplo, o equipamento de micro-ondas originalmente era associado a detecção de aeronaves. Mais ainda, neste período, havia uma busca para soluções de problemas de controle de monitoramento, sistemas eletrônicos, modelagem de esquadrões aéreos e mísseis balísticos.

As reflexões que surgiram após a segunda guerra deixaram claro que os modelos considerados naquele momento eram insuficientes para descrever os fenômenos complexos do mundo real, uma vez que já se sabia que os modelos mais eficientes eram indetermináveis ou não lineares, impossíveis de descrever com absoluta precisão, pois são afetados por um grande número de fatores externos. A partir da década de 1960, esse contexto de transição na modelagem não-linear e não-determinística reforça a necessidade de um uso rigoroso de ferramentas matemáticas na busca de soluções para problemas de controle, assim os métodos e teorias discutidos anteriormente passaram a fazer parte do que tem sido categorizada como a Teoria de Controle "clássica" e tudo o que tem sido e foi desenvolvido desde a década de 1960 é categorizado como a Teoria de Controle "moderna". A evolução da Teoria de Controle moderna se deve essencialmente à rigorosa formalização das ferramentas derivadas do Cálculo Diferencial e Integral que permitiu descrever diversos fenômenos físicos, químicos, biológicos, dentre outros, por meio de tais equações diferenciais.

Sobre as importantes contribuições que serviram de base e fundamento para a teoria de controle moderna, podemos destacar como protagonistas: o matemático e engenheiro húngaro, naturalizado norte-americano Rudolf Emil Kalman, co-inventor de técnicas de filtragem e aproximações algébricas para sistemas lineares; o matemático americano Richard Ernest Bellman pela invenção da programação dinâmica; e também o matemático russo Lev Semenovich Pontryagin, famoso pelo princípio do máximo para problemas de controle ótimo não lineares.

Em um contexto mais matemático, a controlabilidade para modelos governados por equações diferenciais tem sido objeto de intenso estudo nas últimas décadas. Podemos citar os trabalhos do matemático norte-americano Lawrence Markus (veja [1–3]) em que introduz o conceito de controlabilidade de sistemas descritos por equações diferenciais ordinárias (EDOs) e a controlabilidade de sistemas lineares em grupos de Lie. Sobre a controlabilidade de sistemas governados por equações diferenciais parciais (EDPs), citamos as contribuições do matemático David Lewis Russell (ver [4, 5]) que trata sobre a controlabilidade de sistemas parabólicos e hiperbólicos e também sobre controle e estabilização para sistemas de EDPs lineares. Suas publicações foram notáveis por conseguirem relacionar outras áreas das EDPs, como a teoria dos multiplicadores e a análise de Fourier não harmônica. Nos anos que se seguiram, vários matemáticos também se destacaram pela contribuição efetiva para resultados extremamente significativos no estudo da controlabilidade para sistemas de equações diferenciais.

Mais recentemente, em 1986, o matemático Jacques-Louis Lions introduziu em [6] um método sistemático e construtivo chamado Método de Unicidade Hilbertiana (HUM). Atualmente, o método é amplamente utilizado para obter resultados de controlabilidade estudando-se certas quantidades para os problemas duais dos problemas iniciais, neste contexto a técnica é resumida na obtenção de *desigualdades de observabilidade* para sistemas adjuntos associados, este método permitiu grandes avanços na teoria de controle.

Os problemas de controle para EDOs lineares são bem compreendidos (veja [7] e [8]). Até mesmo problemas não lineares estão em um estágio satisfatório de entendimento, isto é, são conhecidas quantidades razoáveis de condições para obter resultados de controlabilidade tanto no contexto local como global, como podemos ver em [9]. Se estamos lidando agora com EDPs, essa situação se torna mais delicada, mesmo no caso de sistemas lineares. Essas dificuldades podem ser vistas de acordo com cada EDP estudada, uma vez que cada operador diferencial associado induz propriedades diferentes nas soluções do sistema, por exemplo:

- Para EDPs hiperbólicas, como a equação da onda, há a problemática da propagação de singularidades com velocidade finita;
- Em se tratando de EDP dispersivas como a equação de Schrödinger e KdV, a propagação de informação ocorre em velocidade infinita em conjunto com um fraco efeito suavizante;
- Por fim, para problemas de EDPs parabólicas como a equação do calor ou de Stokes, tratam de problemas irreversíveis em tempo e com forte efeito suavizante.

Assim, dependendo de que tipo de equação estamos lidando, a estratégia para controlá-la pode ser completamente diferente, esse fato torna o problema de controlar os EDPs, em certo sentido, mais sofisticados comparados às EDOs.

Adiante, vamos definir em termos mais precisos sobre os conceitos de controle presentes na teoria. Alguns desses conceitos são referidos como *controle de equações diferenciais*.

1.3 Alguns conceitos matemáticos da teoria de controle

Um *sistema de controle* é uma equação de evolução (EDO ou EDP) que depende de dois parâmetros y_0 e u, descrito por meio da seguinte equação:

$$\begin{cases} \dfrac{d}{dt}y = f(t, y, u), \\ y(t_0) = y_0. \end{cases}$$

Aqui, $t \in [t_0, T]$ é a variável temporal, $T > t_0$ é um parâmetro real fixado, comumente chamado de "tempo final", a variável y é chamada de *variável de estado* e pertence ao *espaço dos estados* $\mathcal{Y}$, a variável u denota o *controle* e pertence ao espaço $\mathcal{U}_{ad}$ chamado de *espaço dos controles admissíveis*, a função f indica a forma que o controle u age sobre o sistema, se o problema vai ser uma equação diferencial ordinária ou parcial, e também se o problema vai ser linear ou não.

A problemática então é dada pelo seguinte questionamento: É possível encontrar uma função de controle $u \in \mathcal{U}_{ad}$ de tal forma que a função de estado y tenha um comportamento desejável (ou aceitável) em um determinado tempo final $t = T$? Problemas com essa característica são usualmente classificados como um *problema de controle* e dependendo do objetivo eles são categorizados em diferentes tipos. Definiremos com a devida precisão alguns problemas de controlabilidade que são comumente considerados.

Controlabilidade Exata: consiste em determinar a existência de um controle u que conduza a solução de um estado inicial a um estado final determinado. Assim, para cada dado inicial y_0 e dado final y_1, queremos garantir a existência de um controle u tal que

$$\begin{cases} \dfrac{d}{dt}y = f(t, y, u), & t \in [t_0, T], \\ y(t_0) = y_0, & y(T) = y_1. \end{cases}$$

Controlabilidade Nula: consiste em determinar a existência de um controle u que conduza a solução de um estado inicial ao estado final nulo. Mais precisamente, para cada dado inicial y_0, queremos garantir a existência de um controle u tal que

$$\begin{cases} \dfrac{d}{dt}y = f(t, y, u), & t \in [t_0, T], \\ y(t_0) = y_0, & y(T) = 0. \end{cases}$$

Controlabilidade Aproximada: este conceito é formulado para os casos em que estar "próximo" de um determinado objetivo é aceitável, ou seja, queremos conduzir o estado de modo para as proximidades de um determinado dado no tempo final. Mais precisamente, para cada dado inicial y_0, para $\epsilon > 0$ e um dado final y_1, queremos encontrar um controle u tal que

$$\begin{cases} \dfrac{d}{dt}y = f(t, y, u), & t \in [t_0, T], \\ y(t_0) = y_0, & ||y(T) - y_1||_{\mathcal{Y}} < \epsilon. \end{cases}$$

Controlabilidade exata para trajetórias: consiste em determinar a existência de um controle que conduza a solução, partindo de um estado inicial arbitrário, ao mesmo estado de uma determinada trajetória no tempo final; por uma trajetória entendemos uma solução não controlada do sistema. Mais precisamente, dada uma trajetória $\overline{y}$, queremos encontrar um controle u tal que

$$\begin{cases} \dfrac{d}{dt}y = f(t,y,u), & t \in [t_0, T], \\ y(t_0) = y_0, & y(T) = \overline{y}(T). \end{cases}$$

É fácil ver que a controlabilidade exata implica tanto na controlabilidade nula quanto na aproximada, mas a recíproca, em geral, não é verdadeira. Para sistemas que são reversíveis em tempo, a controlabilidade nula e exata são equivalentes. Sabe-se também que, se a função f dá origem a um problema linear, é possível mostrar que as noções de controlabilidade nula e controlabilidade exata para trajetórias são equivalentes.

Além dos problemas de controlabilidade definidos anteriormente, existem dois tipos de problemas de controle que são de interesse para os propósitos desta tese.

Controles insensibilizantes: introduzido por J. L. Lions em [10] e [11], o estudo de controles insensibilizantes tem o objetivo de encontrar controles que tornem um determinado funcional de energia, chamado de *sentinela*, insensível para pequenas perturbações do dado inicial. O termo sentinela provém do termo utilizado em francês *sentinelles*, que pode ser traduzido também como "observadores", pois ainda que não percebam variações do dado inicial, elas de uma certa maneira "observam" a evolução do sistema. Mais precisamente, modificamos ligeiramente o sistema de controle, perturbando o dado inicial,

$$\begin{cases} \dfrac{d}{dt}y = f(t,y,u), \\ y(t_0) = y_0 + \tau \hat{y_0}. \end{cases}$$

O parâmetro τ é um número real suficientemente pequeno e $\hat{y_0}$ é um dado desconhecido, porém com norma unitária. Seja J um funcional diferenciável definido no conjunto das soluções do sistema acima. Dizemos que o controle u insensibiliza o funcional $J(\tau, y)$ se a seguinte condição for satisfeita:

$$\frac{\partial}{\partial \tau} J\left(y(x,t;u,\tau)\right)\bigg|_{\tau=0} = 0.$$

Aqui, $y(x,t;u,\tau)$ é a respectiva solução com controle u e parâmetro $\tau > 0$.

Há inúmeras possibilidades de escolhas para a sentinela (funcional acima) e cada escolha gera um problema diferente de controle insensibilizante. Uma escolha bastante comum, que aparece em muitos fenômenos físicos, é considerar J como uma norma L^2 local da solução, ou seja,

$$J(\tau; y) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |y(x,t)|^2 dx dt.$$

Nesse caso, essa quantidade pode ser interpretada, dependendo do sistema em questão, como a energia cinética ou potencial quando observada numa região $\mathcal{O}$ do domínio, num intervalo de tempo entre $t = 0$ até $t = T$. Uma vez feita esta escolha, pode ser provado que problemas de controles insensibilizantes podem ser reduzidos a um problema de controlabilidade nula parcial para um sistema em cascata em que os termos de acoplamento dependem da sentinela que é considerada.

Controlabilidade nula uniforme: outro problema de controle interessante é o estudo de controles uniformes, que consistem em controlar um determinado sistema a partir da controlabilidade de sistemas dependendo de um parâmetro e que se degeneram no sistema desejado quando tal parâmetro converge para zero. Mais precisamente, dado um parâmetro $\epsilon > 0$, considere a seguinte família de sistemas exatamente controláveis, dependendo do parâmetro $\epsilon > 0$:

$$\begin{cases} M(\epsilon) \cdot \dfrac{d}{dt} y^\epsilon = f(t, y^\epsilon, u^\epsilon), \\ y^\epsilon(t_0) = y_0^\epsilon. \end{cases}$$

Aqui, $M(\epsilon)$ é uma matriz que indica como o parâmetro ϵ está inserido no sistema, $y^\epsilon \in \mathcal{Y}^\epsilon \subset \mathcal{Y}$ e $u^\epsilon \in \mathcal{U}_{ad}$. A família de sistemas acima pode ser vista como uma aproximação do seguinte sistema de controle

$$\begin{cases} M(0) \cdot \dfrac{d}{dt} y = f(t, y, u), \\ y(t_0) = y. \end{cases}$$

Dado um estado final y_T e $\epsilon > 0$, existe um controle u^ϵ tal que $y^\epsilon(T) = y_T$. O problema de controlabilidade uniforme consiste em construir uma família de controles $\{u^\epsilon\}_\epsilon$ que são uniformes com respeito a ϵ, ou seja,

$$||u^\epsilon||_{\mathcal{U}_{ad}} \leq C,$$

em que C não depende de ϵ. Com isso é esperado a obtenção de um controle u para o sistema limite como um limite fraco da sequência $\{u^\epsilon\}$ em $\mathcal{U}_{ad}$ quando $\epsilon \to 0$. Um problema similar pode ser reformulado em termos de controlabilidade nula ou aproximada para tais sistemas.

1.4 Descrição dos resultados

Descreveremos agora os problemas específicos de controle estudados nesta tese.

APÊNDICE A - Um problema de controles insensibilizantes para a equação de Ginzburg-Landau

O Apêndice A consiste em uma versão mais detalhada do artigo [12] publicado na revista *Journal of Optimization Theory and Applications* (JOTA), e teve participação ativa de mais dois pesquisadores, o Prof. Maurício Cardoso Santos da Universidade Federal da Paraíba e o Prof. Diego Araujo de Souza da Universidade Federal de Pernambuco.

Neste trabalho tratamos sobre a existência de controles insensibilizantes para a equação de Ginzburg-Landau com não linearidade cúbica, veja Seção 1.3. para uma explicação geral sobre problemas deste tipo. Sendo mais preciso, sejam N um número natural, $\Omega \subset \mathbb{R}^N$ um domínio limitado cuja fronteira, denotada por $\partial\Omega$, é suficientemente regular, $T > 0$ representa o tempo final, ω e $\mathcal{O}$ dois subconjuntos abertos e não vazios de Ω que serão chamados respectivamente de *domínio de controle* e *domínio de observação.*

Usando as notações $Q_T = \Omega\times]0,T[$, $q_T = \omega\times]0,T[$ e $\Sigma_T = \partial\Omega\times]0,T[$, considere a seguinte equação diferencial não linear conhecida como equação de *Ginzburg-Landau*,

$$\begin{cases} y_t - (1+ia)\Delta y + Ry - (1+ib)|y|^2 y = v1_\omega + f, & \text{em} \quad Q_T, \\ y = 0, & \text{sobre} \quad \Sigma_T, \\ y(0) = y_0 + \tau\widehat{y}_0, & \text{em} \quad \Omega. \end{cases} \tag{1.1}$$

A equação (1.1) foi primeiramente considerada em estudos relacionados a sistemas de reação e difusão que, como próprio nome indica, representam fenômenos em que a reação e difusão de reagentes em um determinado meio ocorrem simultaneamente. É também um modelo que descreve o comportamento de fluxos confinados em que não há movimento entre as extremidades do meio, como por exemplo o fluxo de sangue nas artérias e veias, fenômenos conhecidos como fluxos de Poiseuille. Esta equação também é uma equação bastante importante na teoria ondulatória, considerada em modelagens de problemas que descrevem desde fenômenos de ondas não lineares até transição de fase de segunda ordem, supercondutividade, superfluidade e condensado de Bose-Einstein. É portanto uma equação bastante conhecida na comunidade acadêmica, em virtude da modelagens desses diversos fenômenos.

Em (1.1), v denota a variável de controle, y é a variável de estado $a, b, R \in \mathbb{R}$, y_0 e f são funções conhecidas. Nesse sistema de controle, o dado inicial é parcialente conhecido, uma vez que apesar de conhecermos y_0, temos que:

- $\widehat{y}_0 \in L^2(\Omega)$ é uma função desconhecida tal que $||\widehat{y}_0||_{L^2(\Omega)} = 1$;
- $\tau \in \mathbb{R}$ é um parâmetro desconhecido suficientemente pequeno.

Problemas de controles insensibilizantes consistem em determinar a existência de um controle que torne insensível para pequenas variações do dado inicial, um determinado funcional de energia J, chamado de *sentinela*, que depende da variável de estado ou de alguma de suas derivadas. A sentinela observa uma determinada quantidade na região $\mathcal{O}$ em um intervalo de tempo de $t = 0$ até $t = T$. Aqui, os dados são tomados de tal maneira que o funcional J esteja bem definido. Nesse âmbito, dizer que o controle v insensibiliza o funcional J pode ser quantificado da seguinte maneira,

$$\frac{\partial}{\partial \tau} J(\tau, v)\bigg|_{\tau=0} = 0, \quad \forall \ \hat{y}_0 \in L^2(\Omega) \ \text{ com } \ ||\hat{y}_0||_{L^2(\Omega)} = 1. \tag{1.2}$$

Há então uma infinidade de problemas de controlabilidade dessa natureza, uma vez que diversos fatores podem ser considerados nesse estudo: o sistema de controle pode ser governado por uma equação linear, semilinear ou não linear; se tem condições de fronteira de Dirichlet ou Neumann, homogênea ou não; a sentinela pode vir a depender do estado ou de uma de suas derivadas (de alguma ordem, espacial ou temporal); controles distribuídos internamente ou na fronteira, dentre outros.

Uma vez cientes das diversas possibilidades acerca de problemas de controles insensibilizantes, apresentaremos alguns resultados já conhecidos:

- Problemas de controles insensibilizantes foram introduzidos por J.-L. Lions em [10] e [11];
- Em [13] os autores flexibilizaram a definição em (1.2) com uma noção de controle insensibilizante aproximado, também conhecido como controle ϵ-insensibilizante e o problema considerado foi para uma equação do calor não homogênea;
- Estudando uma equação do calor semilinear, *de Tereza* provou em [14] que não era esperado obter existência de controles insensibilizantes para dado inicial qualquer. A demonstração de um resultado negativo foi feita por construção de uma solução que não satisfazia a desigualdade de observabilidade;
- Em [15] e [16] os autores generalizaram os resultados em [14] para dois casos: para uma equação do calor com não linearidade superlinear e para uma equação do calor com não linearidade dependendo do estado e do gradiente do estado;
- Em [17] os autores investigaram para quais dados iniciais eram possíveis obter resultados positivos e negativos;
- Em [18], [19] e [20] a problemática foi considerada para outros tipos de equação como a equação de Navier-Stokes e sistemas de Boussinesq;
- Acerca das sentinelas, a grande maioria dos resultados consideram o funcional J como uma norma L^2 do estado, ver [10, 11, 13–17]. Em [21], o autor considerou uma sentinela com a norma L^2 do gradiente do estado para uma equação do calor linear. Já em [22] foi considerado um sistema de Stokes com funcional dependendo da norma L^2 do rotacional do estado.

Em se tratando da equação de Ginzburg-Landau, poucos resultados são conhecidos. Em [23] os autores provam um resultado de controlabilidade nula para uma classe de equações de Ginzburg-Landau não lineares e em [24] os autores resolvem um problema de controles insensibilizantes com sentinela dependendo da norma L^2 do estado.

Inspirados nos artigos descritos acima, em especial, [22], [23] e [24], nos propusemos a estudar o seguinte problema: provar a existência de controles insensibilizantes para a equação de Ginzburg-Landau com a não linearidade cúbica, com sentinela dependendo do gradiente do estado, isto é,

$$J(\tau, v) = \frac{1}{2} \int_0^T \int_{\mathcal{O}} |\nabla y(x,t;\tau,v)|^2 dxdt. \tag{1.3}$$

A dificuldade em considerar sentinelas como em (1.3) está no fato de que a existência de controles insensibilizantes se reduz a provar um resultado de controlabilidade para um sistema de equações de Ginzburg-Landau que estão acopladas por meio de um termo de segunda ordem, portanto diversos problemas técnicos surgem e novas ferramentas são necessárias. Neste trabalho provaremos uma nova desigualdade de Carleman que ajudará a lidar com o acoplamento de segunda ordem, esta desigualdade por si só já representa uma grande contribuição para a área de controle de equações diferenciais parciais.

Assim, em linhas gerais, o problema a ser considerado no Apêndice A é o seguinte:

PROBLEMA 1. Para cada f em um espaço de função apropriado e cada dado inicial $y_0 \in L^2(\Omega)$, será que existe um par (y, v) solução de (1.1), que insensibilize o funcional (1.3) no sentido dado por (1.2)?

APÊNDICE B - Controlabilidade nula uniforme para sistemas parabólicos de rápida difusão com termos de acoplamento de primeira ordem

O Apêndice B é um trabalho em conjunto com outros três pesquisadores: o Prof. Maurício Cardoso Santos e o Prof. Felipe Wallison Chaves Silva ambos da Universidade Federal da Paraíba, e o Prof. Diego Araujo de Souza da Universidade Federal de Pernambuco.

Neste trabalho tratamos sobre a existência de controles uniformes, com respeito a um dado parâmetro, para uma família de sistemas parabólicos lineares de rápida difusão. Veja Seção 1.3. para uma explicação geral sobre problemas deste tipo. Sejam N um número natural, $\Omega \subset \mathbb{R}^N$ um subconjunto conexo e limitado cuja fronteira, denotada por $\partial\Omega$, é suficientemnte regular. Sejam também $T > 0$ chamado de tempo final, ω_1 e ω_2 subconjuntos não vazio de Ω que serão chamados de *domínios de controle*.

Usando a notação $Q_T = \Omega \times]0, T[$ e $\Sigma_T = \partial\Omega \times]0, T[$, definimos o seguinte sistema parabólico de rápida difusão com termos de acoplamento de primeira ordem

$$\begin{cases} y_t^\epsilon - \Delta y^\epsilon = a_{1,1}y^\epsilon + A_{1,1} \cdot \nabla y^\epsilon + a_{1,2}z^\epsilon + A_{1,2} \cdot \nabla z^\epsilon + f_1^\epsilon 1_{\omega_1}, & \text{em } Q_T, \\ \epsilon z_t^\epsilon - \Delta z^\epsilon = a_{2,1}z^\epsilon + A_{2,1} \cdot \nabla z^\epsilon + a_{2,2}y^\epsilon + A_{2,2} \cdot \nabla y^\epsilon + f_2^\epsilon 1_{\omega_2}, & \text{em } Q_T, \\ y^\epsilon = z^\epsilon = 0, & \text{sobre } \Sigma_T, \\ y^\epsilon|_{t=0} = y_0, z^\epsilon|_{t=0} = z_0, & \text{em } \Omega. \end{cases} \tag{1.4}$$

Uma das motivações para considerar sistemas como (1.4) é que estes podem ser vistos como aproximações para sistemas do tipo

$$\begin{cases} y_t - \Delta y = a_{1,1}y + A_{1,1} \cdot \nabla y + a_{1,2}z + A_{1,2} \cdot \nabla z + f_1 1_{\omega_1}, & \text{em } Q_T \\ -\Delta z = a_{2,1}z + A_{2,1} \cdot \nabla z + a_{2,2}y + A_{2,2} \cdot \nabla y + f_2 1_{\omega_2}, & \text{em } Q_T, \\ y = z = 0, & \text{sobre } \Sigma_T, \\ y|_{t=0} = y_0, & \text{em } \Omega. \end{cases} \tag{1.5}$$

O sistema dado por (1.5) modela diversos fenômenos tais como modelo de agregação que descrevem o comportamento de crescimento de determinadas estruturas tais como células ou também podem descrever reações químicas de reagentes com diferentes concentrações.

Nos sistemas acima, $\epsilon > 0$ é um parâmetro fixo (que vai se degenerar a zero no caso limite), as variáveis de estado são y e z, f_1 e f_2 denotarão os controles e $\{a_{i,j}\}_{i,j=1}^2$ são funções de Q_T em $\mathbb{R}$ e $\{A_{i,j}\}_{i,j=1}^2$ são campos de Q_T em $\mathbb{R}^N$.

Problemas de controlabilidade uniforme consistem em determinar controles que conduzam as soluções da equação em questão à configuração final desejada, com controles que sejam uniformemente limitados com respeito a um determinado parâmetro. Nesta parte do trabalho, queremos provar um resultado de controlabilidade nula para o sistema (1.4) em que os controles f_1^ϵ e f_2^ϵ são uniformemente limitados com respeito ao parâmetro ϵ, na realidade, o caso interessante a ser estudado é aquele em que o único controle ativo é o da primeira equação, i.e., $f_2^\epsilon \equiv 0$, como veremos ao longo do Apêndice B. É importante mencionar que uma vez provado o resultado de controlabilidade nula uniforme para o sistema (1.4), poderemos, por um processo de passagem ao limite nas soluções fracas do sistema (1.4), provar a controlabilidade nula do sistema parabólico-elíptico (1.5).

Essa problemática de aproximar equações ou sistemas de equações com propriedades intrínsecas diferentes, decorrentes do operador que governa a equação, já foi considerada em diferentes contextos, dentre os quais podemos destacar os seguintes trabalhos:

- Em [25] os autores estudaram o caso de uma equação de onda amortecida convergindo para um equação do calor;
- O artigo [21] contém resultado de controlabilidade nula uniforme para uma equação de transporte e difusão convergindo para uma equação do transporte;

- Com relação a controlabilidade uniforme de sistemas governados por equações parabólicas e convergindo para um sistema parabólico-elíptico, destacamos [26] como sendo um dos trabalhos pioneiros.

Em se tratando da controlabilidade nula uniforme para o sistema (1.4), citamos [27] no qual os autores provam um resultado de controlabilidade nula uniforme para um sistema do tipo (1.4), porém com acoplamentos de ordem zero, ou seja, no caso em que $A_{1,1} = A_{1,2} = A_{2,1} = A_{2,2} = 0$.

Dado o exposto, nesta parte da tese estamos interessados em resolver um problema do seguinte tipo:

PROBLEMA 2. Para cada dados iniciais $y_0, z_0 \in L^2(\Omega)$, é possível obter controles $f_i = f_i(\epsilon)$, $i = 1$ ou $i = 2$, associados com as soluções (y^ϵ, z^ϵ) de (1.4) que satisfazem $(y^\epsilon(T), z^\epsilon(T)) = (0, 0)$, e tais que f_i são uniformemente limitados com respeito ao parâmetro degenerativo ϵ quando este converge para zero? Mais ainda, podemos obter um resultado de controlabilidade nula uniforme para (1.4) com apenas um controle ativo?

Uma vez que temos uma resposta positiva para este problema, é possível, por um procedimento de passagem ao limite com soluções fracas, deduzir um resultado de controle nulo para (1.5).

REFERÊNCIAS

1 MARKUS, L. Controllability of Nonlinear Processes. *Journal of the Society for Industrial and Applied Mathematics Series A Control*, Society for Industrial and Applied Mathematics, v. 3, n. 1, p. 78–90, 1 1965. ISSN 0887-4603. Disponível em: <http://epubs.siam.org/doi/10.1137/0303008>.

2 LEE, E. B.; MARKUS, L. *Foundations of Optimal Control Theory,*. 1967. Disponível em: <https://apps.dtic.mil/docs/citations/AD0663614>.

3 MARKUS, L. Controllability of multi-trajectories on Lie groups. In: . Springer, Berlin, Heidelberg, 1981. p. 250–265. Disponível em: <http://link.springer.com/10.1007/BFb0091918>.

4 RUSSELL, D. L. A Unified Boundary Controllability Theory for Hyperbolic and Parabolic Partial Differential Equations. *Studies in Applied Mathematics*, John Wiley & Sons, Ltd (10.1111), v. 52, n. 3, p. 189–211, 9 1973. ISSN 00222526. Disponível em: <http://doi.wiley.com/10.1002/sapm1973523189>.

5 RUSSELL, D. L. Controllability and Stabilizability Theory for Linear Partial Differential Equations: Recent Progress and Open Questions. *SIAM Review*, Society for Industrial and Applied Mathematics, v. 20, n. 4, p. 639–739, 10 1978. ISSN 0036-1445. Disponível em: <http://epubs.siam.org/doi/10.1137/1020095>.

6 LIONS, J. *Problemes aux limites non homogenes et applications.* Paris: Dunod, 1968. Disponível em: <https://www.worldcat.org/title/problemes-aux-limites-non-homogenes-et-applications-vol-1/oclc/25525823>.

7 KALMAN, R. On the general theory of control systems. *IRE Transactions on Automatic Control*, v. 4, n. 3, p. 110–110, 12 1959. ISSN 0096-199X. Disponível em: <http://ieeexplore.ieee.org/document/1104873/>.

8 LASALLE, J. P.; CESARI, L.; LEFSCHETZ, S. *Contributions to the theory of nonlinear oscillations. Volume V.* [s.n.], 2016. ISBN 9781400882649. Disponível em: <https://books.google.com.br/books?hl=en&lr=&id=-tHfCwAAQBAJ&oi=fnd&pg=PA1&dq=The+time+optimal+control+problem&ots=oN6jG0qzGW&sig=THuTOngkCQkVKr_zVMyrEc9k1Rs#v=onepage&q=The\%20time\%20optimal\%20control\%20problem&f=false>.

9 CORON, J.-M. *Control and nonlinearity.* American Mathematical Society, 2007. 426 p. ISBN 9780821849187. Disponível em: <https://books.google.com.br/books?hl=en&lr=&id=R5h9X2yYKqkC&oi=fnd&pg=PR9&dq=Control+and+nonlinearity&ots=H8d7d08-7l&sig=89Xd0aZFuJioQVzTTEq1S-8T7x8#v=onepage&q=Control\%20and\%20nonlinearity&f=false>.

10 LIONS, J. *Quelques notions dans l'analyse et le contrôle de systemesa données incompletes.* [S.l.: s.n.], 1990. 4354 p.

11 LIONS, J. L. *Sentinelles pour les systemès distribués à données incomplètes* . [S.l.]: Masson, 1992. ISBN 2225826978.

12 SANTOS, M. C.; TANAKA, T. Y. An Insensitizing Control Problem for the Ginzburg–Landau Equation. *Journal of Optimization Theory and Applications*, Springer Science and Business Media LLC, v. 183, n. 2, p. 440–470, 11 2019. ISSN 0022-3239.

13 BODART, O.; FABRE, C. Controls Insensitizing the Norm of the Solution of a Semilinear Heat-Equation. *Journal of Mathematical Analysis and Applications*, Academic Press, v. 195, n. 3, p. 658–683, 11 1995. ISSN 0022-247X. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0022247X85713820>.

14 TERESA, L. d. Insensitizing controls for a semilinear heat equation. *Communications in Partial Differential Equations*, Marcel Dekker, Inc., v. 25, n. 1-2, p. 39–72, 1 2000. ISSN 0360-5302. Disponível em: <http://www.tandfonline.com/doi/abs/10.1080/03605300008821507>.

15 BODART, O.; GONZÁLEZ-BURGOS, M.; PÉREZ-GARCÍA, R. Existence of Insensitizing Controls for a Semilinear Heat Equation with a Superlinear Nonlinearity. *Communications in Partial Differential Equations*, Taylor & Francis Group, v. 29, n. 7-8, p. 1017–1050, 1 2004. ISSN 0360-5302. Disponível em: <http://www.tandfonline.com/doi/abs/10.1081/PDE-200033749>.

16 BODART, O.; GONZÁLEZ-BURGOS, M.; PÉREZ-GARCíA, R. Insensitizing controls for a heat equation with a nonlinear term involving the state and the gradient. *Nonlinear Analysis: Theory, Methods & Applications*, v. 57, n. 5-6, p. 687–711, 5 2004. ISSN 0362546X. Disponível em: <https://linkinghub.elsevier.com/retrieve/pii/S0362546X04000793>.

17 TERESA, L. D.; ZUAZUA, E. Identification of the class of initial data for the insensitizing control of the heat equation. *PURE AND APPLIED ANALYSIS*, v. 8, n. 1, 2009. Disponível em: <http://verso.mat.uam.es/web/ezuazua/documentos_public/archivos/publicaciones/insterezua.pdf>.

18 GUEYE, M. Insensitizing controls for the Navier–Stokes equations. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, v. 30, n. 5, p. 825–844, 9 2013. ISSN 02941449. Disponível em: <https://linkinghub.elsevier.com/retrieve/pii/S029414491200087X>.

19 CARREÑO, N.; GUERRERO, S.; GUEYE, M. Insensitizing controls with two vanishing components for the three-dimensional Boussinesq system. *ESAIM: Control, Optimisation and Calculus of Variations*, EDP Sciences, v. 21, n. 1, p. 73–100, 1 2015. ISSN 1292-8119. Disponível em: <http://www.esaim-cocv.org/10.1051/cocv/2014020>.

20 CARREÑO, N. Insensitizing controls for the Boussinesq system with no control on the temperature equation. *Advances in Differential Equations*, Khayyam Pub, v. 22, n. 3/4, p. 235–258, 1996. ISSN 1079-9389. Disponível em: <https://projecteuclid.org/euclid.ade/1487386868>.

21 GUERRERO, S.; LEBEAU, G. Singular Optimal Control for a Transport-Diffusion Equation. *Communications in Partial Differential Equations*, Taylor & Francis Group, v. 32, n. 12, p. 1813–1836, 12 2007. ISSN 0360-5302. Disponível em: <http://www.tandfonline.com/doi/abs/10.1080/03605300701743756>.

22 GUERRERO, S. Null Controllability of Some Systems of Two Parabolic Equations with One Control Force. *SIAM Journal on Control and Optimization*, Society for Industrial and Applied Mathematics, v. 46, n. 2, p. 379–394, 1 2007. ISSN 0363-0129. Disponível em: <http://epubs.siam.org/doi/10.1137/060653135>.

23 ROSIER, L.; ZHANG, B.-Y. Null controllability of the complex Ginzburg–Landau equation. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, v. 26, n. 2, p. 649–673, 2 2009. ISSN 02941449. Disponível em: <https://linkinghub.elsevier.com/retrieve/pii/S0294144908000395>.

24 ZHANG, M.; LIU, X. Insensitizing controls for a class of nonlinear Ginzburg-Landau equations. *Science China Mathematics*, Science China Press, v. 57, n. 12, p. 2635–2648, 12 2014. ISSN 1674-7283. Disponível em: <http://link.springer.com/10.1007/s11425-014-4837-8>.

25 LÓPEZ, A.; ZHANG, X.; ZUAZUA, E. Null controllability of the heat equation as singular limit of the exact controllability of dissipative wave equations. *Journal de Mathématiques Pures et Appliquées*, Elsevier Masson, v. 79, n. 8, p. 741–808, 10 2000. ISSN 0021-7824. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0021782499001440>.

26 CHAVES-SILVA, F. W.; BENDAHMANE, M. Uniform Null Controllability for a Degenerating Reaction-Diffusion System Approximating a Simplified Cardiac Model. *SIAM Journal on Control and Optimization*, Society for Industrial and Applied Mathematics, v. 53, n. 6, p. 3483–3502, 1 2015. ISSN 0363-0129. Disponível em: <http://epubs.siam.org/doi/10.1137/110836766>.

27 CHAVES-SILVA, F. W.; GUERRERO, S.; PUEL, J. P. Controllability of fast diffusion coupled parabolic systems. v. 4, n. 4, p. 465–479, 2014. Disponível em: <https://www.ljll.math.upmc.fr/guerrero/publis/MCSS-FC-JPP-2014.pdf>.

28 GUERRERO, S. Controllability of systems of Stokes equations with one control force: existence of insensitizing controls. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, Elsevier Masson, v. 24, n. 6, p. 1029–1054, 11 2007. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0294144906001168>.

29 FURSIKOV, A. *Controllability of evolution equations*. Seoul: Seoul National University, 1996. Disponível em: <https://www.worldcat.org/title/controllability-of-evolution-equations/oclc/37991544>.

30 ALEKSEEV, V. M. V. M.; TIKHOMIROV, V. M. V. M.; FOMIN, S. V. S. V. *Optimal control*. Consultants Bureau, 1987. 309 p. ISBN 9781461575511. Disponível em: <https://books.google.com.br/books?hl=en&lr=&id=cPgICAAAQBAJ&oi=fnd&pg=PA1&ots=XBHa7GAYt7&sig=IjIjkgopIEi7ye2NxEKZMUkiT1o&redir_esc=y#v=onepage&q&f=false>.

31 FERNÁNDEZ-CARA, E. Null controllability of the semilinear heat equation. *ESAIM: Control, Optimisation and Calculus of Variations*, v. 2, p. 87–103, 1997. Disponível em: <http://www.numdam.org/item/COCV_1997___2___87_0/>.

32 CAZENAVE, T.; HARAUX, A. *An introduction to semilinear evolution equations*. New York: Oxford Lectures Series in Mathematics and its Applications, v.13, The Clarendon Press, Oxford University Press on Demand, 1998. v. 13.

33 CHAVES-SILVA, F. W.; GUERRERO, S. A uniform controllability result for the Keller–Segel system. *Asymptotic Analysis*, IOS Press, v. 92, n. 3-4, p. 313–338, 1 2015. ISSN 0921-7134. Disponível em: <https://content.iospress.com/articles/asymptotic-analysis/asy1282>.

34 CARREÑO, N.; GUERRERO, S. Uniform null controllability of a linear KdV equation using two controls. *Journal of Mathematical Analysis and Applications*, Academic Press, v. 457, n. 1, p. 922–943, 1 2018. ISSN 0022-247X. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0022247X17307965>.

35 LEBEAU, G.; ROBBIANO, L. Contrôle exact de l'équation de la chaleur. *Communications in Partial Differential Equations*, v. 20, n. 1-2, p. 335–356, 1995. ISSN 15324133.

36 HORSTMANN, D. *Keller-Segel model in chemotaxis and its consequences*. 2003. Disponível em: <https://www.semanticscholar.org/paper/Keller-Segel-model-in-chemotaxis-and-its-Horstmann/fa7c582f19be185de11ca7fb0526aeebde183822>.

37 LIONS, J.-L. J.-L. *Some methode [i.e. methods] in the mathematical analysis of systems and their control*. [S.l.]: Science Press, 1981. 542 p. ISBN 0677602006.

38 CHOW, J. H. Preservation of controllability in linear time-invariant perturbed systems†. *International Journal of Control*, v. 25, n. 5, p. 697–704, 1977. ISSN 13665820.

39 DUPREZ, M.; LISSY, P. Indirect controllability of some linear parabolic systems of m equations with m - 1 controls involving coupling terms of zero or first order. *Journal de Mathématiques Pures et Appliquées*, Elsevier Masson, v. 106, n. 5, p. 905–934, 11 2016. ISSN 0021-7824. Disponível em: <https://www.sciencedirect.com/science/article/pii/S002178241630023X>.

40 KHODJA, F. A.; BENABDALLAH, A.; GONZÁLEZ-BURGOS, M.; TERESA, L. de. New phenomena for the null controllability of parabolic systems: Minimal time and geometrical dependence. *Journal of Mathematical Analysis and Applications*, Academic Press, v. 444, n. 2, p. 1071–1113, 12 2016. ISSN 0022-247X. Disponível em: <https://www.sciencedirect.com/science/article/pii/S0022247X16302955>.

APÊNDICE A – AN INSENSITIZING CONTROL PROBLEM FOR THE GINZBURG-LANDAU EQUATION

Abstract

In this paper, we prove the existence of insensitizing controls for the nonlinear Ginzburg-Landau equation. Here, we have a partially unknown initial data, and the problem consists in finding controls such that a specific functional is insensitive for small perturbations of the initial data. In general, the problem of finding controls with this property is equivalent to prove a partial null controllability result for an optimality system of cascade type. The novelty here is that we consider functionals depending on the gradient of the state, which leads to a null controllability problem for a system with second-order coupling terms. To manage coupling terms of this order, we need a new Carleman estimate for the solutions of the corresponding adjoint system.

Keywords: Ginzburg-Landau equation, Carleman estimates, Insensitizing controls, Null controllability.

AMS Classification: 93C20, 93B05, 93B07, 93C41, 35K40

Introduction

This paper deals with an insensitizing control problem for a nonlinear Ginzburg-Landau equation. The initial data is partially unknown, in the sense it is a small perturbation of a given data, and we want to make some measurements of the state by means of some cost functionals. Here, we want to solve the problem of finding controls acting on the system which turns the functionals insensitive for the small variations of the data. This kind of problem was originally addressed by J.-L. Lions, see for instance [10] and [11]. After these works, the existence of insensitizing controls was studied in different contexts. For instance, in [13], an *approximated insensitizing problem* was considered for the nonhomogeneous heat equation. In [14], the author proves that, under some suitable conditions, the existence of insensitizing controls may or may not hold, which indicates that this kind of problem cannot be solved for every initial data. This result was generalized in [15] and [16] to nonlinearities with certain superlinear growth and nonlinear terms depending on the state and its gradient. Regarding the class of initial data that can be insensitized, the results in [17] give different results of positive and negative nature. In [22], the author considers a functional involving the gradient of the state for a linear heat system, while in [28], it is treated the case of the curl for the Stokes system. In [19], [18] and [20], the authors study the existence of insensitizing controls for the Navier-Stokes equation and the Boussinesq system.

Concerning the Ginzburg-Landau equation, few results are known. We refer to [23], where the authors prove the null controllability for a class of nonlinear Ginzburg-Landau equations. Concerning insensitizing control problems, we can cite [24], in which measurements depending only on the state itself are considered. In this paper, the novelty is that we consider functionals depending on the gradient of the state, and we search for controls which turns these functionals insensitive. In general, this kind of problem is equivalent to a partial null controllability result for a nonlinear cascade system, and the kind of functional affects the way the equations are coupled. For the fuctionals considered here, the coupling term is a second order derivative of the state, and because of this, we can not proceed in the same way as in [24].

The strategy adopted here is based on duality arguments. We prove a suitable observability inequalities for the solutions of an adjoint system, where the main tools are Carleman estimates. We prove a Carleman estimate with right-hand side in Sobolev space of negative order, and this will be the key point to deal with the second order coupling.

This work is divided as follows: In Section A, we present in more details the problem and its difficulties. Section A is dedicated to prove some Carleman estimates. In Section A, we prove controllability results for the linear and nonlinear cases. In Section A, we give some related open questions, and in Section A we make some conclusions. For completeness, at the end of this paper, we have an Appendix about the existence of solutions for the systems considered here.

Statement of the Problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary $\partial\Omega$ is regular enough. Let $T > 0$, ω (called *control domain*) and $\mathcal{O}$ (called *observation domain*) be two nonempty subsets of Ω. We denote by $Q_T = \Omega\times]0,T[$, $q_T = \omega\times]0,T[$, $\Sigma_T = \partial\Omega\times]0,T[$, $\mathcal{O}_T = \mathcal{O}\times]0,T[$, and by $\nu(x)$ the outward unitary normal to Ω at the point $x \in \partial\Omega$. Unless otherwise stated, the functions that appears on this work are complex-valued.

This work deals with an insensitizing control problem for the nonlinear Ginzburg-Landau equation

$$\begin{cases} y_t - (1+ia)\Delta y + Ry - (1+ib)|y|^2 y = v1_\omega + f, & \text{in} \quad Q_T, \\ y = 0, & \text{on} \quad \Sigma_T, \\ y|_{t=0} = y_0 + \tau\widehat{y}_0, & \text{in} \quad \Omega. \end{cases} \tag{A.1}$$

Here, v stands for the control variable, y is the state variable, $a, b, R \in \mathbb{R}$, y_0 and f are given data, τ is an unknown (small) real number and $\widehat{y}_0$ is an unknown function.

The complex Ginzburg-Landau equation was first derived in the studies of Poiseuille flows and reaction-diffusion systems. Nowadays, it is used to describe phenomena from nonlinear waves to second order phase transition, superconductivity, superfluidity, and Bose-Einstein condensation.

Let us now present the problem under consideration. Let $J : \mathbb{R} \times L^2(q_T) \to \mathbb{R}$ (called *sentinel functional*) defined by:

$$J(\tau, v) := \frac{1}{2} \iint_{\mathcal{O}_T} |\nabla y(x,t;\tau,v)|^2 dxdt, \tag{A.2}$$

where $y = y(x,t;\tau,v)$ is the corresponding solution of (A.1) associated to τ and control v. The data will be taken in such a way that the functional J is well-defined.

Let us now define the insensitizing control problem:

Definition A.1. Let $y_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$. We say that a control v insensitizes the functional J if

$$\frac{\partial}{\partial \tau} J(\tau, v)\bigg|_{\tau=0} = 0, \quad \forall\, \widehat{y}_0 \in L^2(\Omega) \text{ with } \|\widehat{y}_0\|_{L^2(\Omega)} = 1, \tag{A.3}$$

i. e., J does not detect small pertubations of the initial data y_0.

Notice that, Definition A.1 means that we want J to be locally insensitive to the perturbation $\tau\hat{y}_0$.

The main goal of this paper is to prove the existence of controls v which insensitizes the functional J, which is given by the next result.

Theorem A.2. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y_0 \equiv 0$. There exists a constant $C > 0$ and $\delta > 0$ such that for any f satisfying*

$$\|e^{C/t} f\|_{L^2(Q_T)} \leq \delta,$$

one can find a control $v \in L^2(q_T)$ which insensitizes the functional J in the sense of Definition A.1.

The smallness of f is related to the fact that here we apply a local inversion argument, that is, we first study this problem when a linearized form of equation (A.1) is considered, and then we apply a local inversion mapping theorem.

It is standard that the insensitizing control problems can be reformulated as a partial null controllability problem for a nonlinear system associated to (A.1). Due to the choice of J we have made (see (A.2)), our problem is reduced to find $v \in L^2(Q_T)$ such that $z|_{t=0} = 0$, where (y, z) is the solution of the following optimality system

$$\begin{cases} y_t - (1+ia)\Delta y + Ry - (1+ib)|y|^2 y = v1_\omega + f, & \text{in} \quad Q_T, \\ -z_t - (1-ia)\Delta z + Rz - (1-ib)\overline{y}^2\overline{z} - 2(1-ib)|y|^2 z = \nabla\cdot(\nabla y 1_{\mathcal{O}}), & \text{in} \quad Q_T, \\ y = z = 0, & \text{on} \quad \Sigma_T, \\ y|_{t=0} = y_0, z|_{t=T} = 0, & \text{in} \quad \Omega. \end{cases} \tag{A.4}$$

In fact, system (A.4) is obtained as we compute (A.3). Note that

$$\frac{\partial}{\partial\tau}J(\tau,v)\bigg|_{\tau=0} = \frac{\partial}{\partial\tau}\left(\frac{1}{2}\int\int_{\mathcal{O}_T}|\nabla y(x,t;\tau,v)|^2dxdt\right)\bigg|_{\tau=0} = \int_0^T\int_\Omega \frac{\partial}{\partial\tau}y\bigg|_{\tau=0}\nabla\cdot(\nabla\overline{y}1_{\mathcal{O}})\,dxdt,$$

and at this point, we can see that the equation for variable z given by $(A.4)_2$ is exactly the adjoint problem for the derivative of the equation $(A.1)_1$ with respect to τ at $\tau = 0$, and also with right-hand side equals $\nabla\cdot(\nabla y 1_{\mathcal{O}})$. Now, with the coupling equation $(A.4)_2$, we get that

$$\frac{\partial}{\partial\tau}J(\tau,v)\bigg|_{\tau=0} = \int_\Omega \hat{y_0}\overline{z(0)}dx.$$

For the insensitizing condition be true for all $\hat{y_0} \in L^2(\Omega)$, one must have $z(0) = 0 \in \Omega$, and we conclude that (A.3) is equivalent to obtain the partial null controllability.

This motivates the next theorem.

Theorem A.3. *Assume that $\omega\cap\mathcal{O}\neq\emptyset$, $N\leq 2$ and the initial data $y_0\equiv 0$. Then, there exists positive constants C and δ, depending on $a, b, R, \omega, \Omega, \mathcal{O}$ and T, such that for any f satisfying*

$$\|e^{C/t}f\|_{L^2(Q_T)} \leq \delta,$$

one can find a control $v \in L^2(Q_T)$ such that the corresponding solution (y, z) of (A.4) *satisfies $z|_{t=0} = 0$ in Ω.*

In [24], the authors solved an insensitizing control problem for functionals depending only on a local L^2 norm of the state. The novelty here is that we consider functionals like (A.2), depending on the gradient of the state. A consequence of this is that the optimality system (A.4) has a second order coupling term, this introduces some extra difficulties in order to solve the controllability problem given in Theorem A.3.

We remark that Theorem A.3 cannot be proved for any initial data, that is why we assume $y_0 = 0$ in Theorems A.2 and A.3. We can cite [14] where the author proves this fact for the heat equation. For the equation considered here, the proof is very similar to the one in [14], and for this reason, we will omit it. It consists in using the fundamental solution to construct an explicit solution where the observability inequality does not hold.

As mentioned before, to find controls insensitizing the functional J, that is, to prove Theorem A.2, it is sufficient to prove the partial null controllability result given in Theorem A.3. Therefore, from now on, we will concentrate on proving Theorem A.3.

The ideas on how we are going to prove Theorem A.3 are the following. We first consider a linear system of the form

$$\begin{cases} y_t - (1+ia)\Delta y + Ry = v1_\omega + f^0, & \text{in } Q_T, \\ -z_t - (1-ia)\Delta z + Rz = \nabla\cdot(\nabla y 1_{\mathcal{O}}) + f^1, & \text{in } Q_T, \\ y = z = 0, & \text{on } \Sigma_T, \\ y|_{t=0} = y_0, z|_{t=T} = 0, & \text{in } \Omega, \end{cases} \tag{A.5}$$

and we prove that, for any (f^0, f^1) in a suitable weighted space, there exist $v \in L^2(q_T)$ such that $z|_{t=0} = 0$. In order to do so, we will prove a new Carleman estimate for an associated adjoint state. Then, we combine this linear result with an inverse mapping theorem, which proves Theorem A.3. The linear result is given by the following

Theorem A.4. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$, and the initial data $y_0 \equiv 0$. Then, there exists a positive constant C, depending on $a, R, \omega, \Omega, \mathcal{O}$ and T, such that for any f^0 and f^1 in a suitable weighted space $e^{C/t}L^2(Q_T)$, one can find a control $v \in L^2(Q_T)$ such that the corresponding solution (y,z) of* (A.5) *satisfies $z|_{t=0} = 0$ in Ω.*

Remark A.0.1. In Theorem A.3 we have assumed a restriction on the space dimension ($N \leq 2$), this restriction is considered in order to deal with the nonlinear terms, notice that in Theorem A.4 this assumption is not needed. The main reason for this is that the coupling term $\nabla\cdot(\nabla y 1_{\mathcal{O}})$ generates a lack of regularity. If we replace the functional (A.2) for

$$J(\tau, v) := \frac{1}{2}\iint_{Q_T} \Lambda_{\mathcal{O}} |\nabla y(x,t;\tau,v)|^2 dxdt, \tag{A.6}$$

with $\Lambda_{\mathcal{O}} \in C^\infty(\Omega)$ and $Supp\,\Lambda_{\mathcal{O}} \subset \mathcal{O}$, and considering f sufficiently regular, assumptions over the dimension can be weakened.

Carleman Estimates

This section is dedicated to prove some Carleman estimates. Here we prove that for any (φ, ψ) solution of

$$\begin{cases} -\varphi_t - (1-ia)\Delta\varphi + R\varphi = \nabla\cdot(\nabla\psi 1_{\mathcal{O}}) + g^0, & \text{in } \quad Q_T, \\ \psi_t - (1+ia)\Delta\psi + R\psi = g^1, & \text{in } \quad Q_T, \\ \varphi = \psi = 0, & \text{on } \quad \Sigma_T, \\ \psi|_{t=0} = \psi_0, \varphi|_{t=T} = 0, & \text{in } \quad \Omega, \end{cases} \tag{A.7}$$

the following observability inequality holds

$$\iint_{Q_T} \rho_1(t)(|\varphi|^2 + |\nabla\varphi|^2 + |\Delta\psi|^2)\, dxdt \\ \leq C\iint_{q_T} \rho_2(t)|\varphi|^2\, dxdt + \iint_{Q_T} \rho_3(t)(|g^0|^2 + |\nabla g^1|^2)\, dxdt, \tag{A.8}$$

where, for $i = 1, 2, 3$, the weight functions ρ_i blows up at t $= 0$ and $C > 0$ only depends on Ω, ω, $\mathcal{O}$ and T.

The proof of the observability inequality (A.8) is based on global Carleman estimates. In order to establish these estimates, we must introduce some weight functions which are standard in this kind of approach. Initially, we prove Carleman estimates for single equations (see Lemmas A.5 and A.6), after that we prove a Carleman estimate for the adjoint system (A.7) (see Proposition A.7).

Assume that $\omega \cap \mathcal{O} \neq \emptyset$, and let $\tilde{\omega} \subset \Omega$ be an open and nonempty subset satisfying $\tilde{\omega} \subset \omega \cap \mathcal{O}$. As usual, when dealing with Carleman estimates, we define the function $\eta^0 \in C^2(\overline{\Omega})$ such that $\eta^0 > 0$ in Ω, $\eta^0 = 0$ over $\partial\Omega$, and $|\nabla \eta^0| \geq C > 0$ in $\overline{\Omega \setminus \tilde{\omega}}$. Given any real parameters $s > 1$ and $\lambda > 1$, we denote by

$$\begin{aligned} \xi(t,x) &= \frac{e^{\lambda \eta^0(x)}}{t(T-t)}, \quad \alpha(t,x) = \frac{e^{2\lambda \|\eta^0\|_\infty} - e^{\lambda \eta^0(x)}}{t(T-t)}, \\ \hat{\xi}(t) &= \max_{x \in \overline{\Omega}} \xi(x,t), \quad \hat{\alpha}(t) = \min_{x \in \overline{\Omega}} \alpha(x,t), \\ \xi^*(t) &= \min_{x \in \overline{\Omega}} \xi(x,t), \quad \alpha^*(t) = \max_{x \in \overline{\Omega}} \alpha(x,t). \end{aligned} \tag{A.9}$$

For $i, j = 1, \ldots, N$, the following estimates for the weight functions in (A.9) hold:

$$\begin{aligned} |\alpha_t| + |\xi_t| &\leq CT\xi^2, \\ |\partial_{x_i} \alpha| + |\partial_{x_i} \xi| &\leq C\lambda \xi, \\ |\partial^2_{x_i x_j} \alpha| + |\partial^2_{x_i x_j} \xi| &\leq C\lambda^2 \xi. \end{aligned} \tag{A.10}$$

More details concerning weight functions of this kind and how they are applied in the controllability of parabolic equations can be found in [29].

The first result needed in this paper is the following Carleman estimate. For a proof, see [23].

Lemma A.5. *Let $a \in \mathbb{R}$, $h \in L^2(Q_T)$, $\zeta^T \in L^2(\Omega)$, and let ζ the solution of*

$$\begin{cases} \zeta_t + (1 - ia)\Delta\zeta = h, & in \quad Q_T, \\ \zeta = 0, & on \quad \Sigma_T, \\ \zeta|_{t=T} = \zeta^T, & in \quad \Omega. \end{cases}$$

There exists a positive constant λ_1, such that for any $\lambda \geq \lambda_1$ one can find two positive constants $C = C(\lambda)$ and $s_1 = s_1(\lambda)$, such that for any $s \geq s_1$ the following inequality holds

$$\begin{aligned} s^{-1} \iint_{Q_T} \xi^{-1} e^{-2s\alpha} (|\zeta_t|^2 + |\Delta\zeta|^2) dxdt + s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha} |\zeta|^2 dxdt \\ + s\lambda^2 \iint_{Q_T} \xi e^{-2s\alpha} |\nabla\zeta|^2 dxdt \\ \leq C \left(\iint_{Q_T} e^{-2s\alpha} |h|^2 dxdt + s^3\lambda^4 \iint_{q_T} e^{-2s\alpha} \xi^3 |\zeta|^2 dxdt \right). \end{aligned} \tag{A.11}$$

In order to deal with the second order coupling term in the right-hand side of (A.7), we will prove a new Carleman estimate, given by the following lemma.

Lemma A.6. *Let $a \in \mathbb{R}$, $h^0 \in L^2(Q_T)$, $h^1 \in [L^2(Q_T)]^N$, $\phi^T \in L^2(\Omega)$, and let ϕ the solution of*

$$\begin{cases} \phi_t + (1-ia)\Delta\phi = h^0 + \nabla\cdot h^1, & in \quad Q_T, \\ \phi = 0, & on \quad \Sigma_T, \\ \phi|_{t=T} = \phi^T, & in \quad \Omega. \end{cases} \tag{A.12}$$

There exists a positive constant λ_1, such that for any $\lambda \geq \lambda_1$, one can find two positive constants $C = C(\lambda)$ and $s_2 = s_2(\lambda)$, such that for any $s \geq s_2$, the following inequality holds

$$\begin{aligned} s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\phi|^2 dxdt + s\lambda^2 \iint_{Q_T} \xi e^{-2s\alpha}|\nabla\phi|^2 dxdt \\ \leq C\left(\iint_{Q_T} e^{-2s\alpha}|h^0|^2 dxdt + s^2\lambda^2 \iint_{Q_T} \xi^2 e^{-2s\alpha}|h^1|^2 dxdt \right. \\ \left. + s^3\lambda^4 \iint_{q_T} \xi^3 e^{-2s\alpha}|\phi|^2 dxdt\right). \end{aligned} \tag{A.13}$$

Proof. If $\phi \in L^2(Q_T)$ is the solution of (A.12), it satisfies

$$\Re \iint_{Q_T} \phi\overline{g} dxdt = -\Re \int_\Omega \phi^T \bar{z}|_{t=T}\, dx + \Re \iint_{Q_T} h^0 \overline{z} dxdt - \Re \iint_{Q_T} h^1 \cdot \nabla\overline{z} dxdt, \tag{A.14}$$

for every $g \in L^2(Q_T)$, where z is the solution of

$$\begin{cases} -z_t + (1+ia)\Delta z = g, & \text{in} \quad Q_T, \\ z = 0, & \text{on} \quad \Sigma_T, \\ z_{|t=0} = 0, & \text{in} \quad \Omega. \end{cases}$$

Above, the notation $\Re$ stands for the real part of a complex number.

Consider now the following null controllability problem: find a pair (w, u) such that

$$\begin{cases} -w_t + (1+ia)\Delta w = s^3\lambda^4\xi^3 e^{-2s\alpha}\phi + u1_\omega, & \text{in} \quad Q_T, \\ w = 0, & \text{on} \quad \Sigma_T, \\ w_{|t=0} = w_{|t=T} = 0, & \text{in} \quad \Omega. \end{cases} \tag{A.15}$$

In order to solve the null controllability problem (A.15), we are going to use the Lax-Milgram Theorem. Indeed, define the space

$$X_0 = \{p \in C^\infty(\overline{Q_T});\ p = 0 \ \text{ on } \Sigma_T\},$$

the operator $L = \partial_t + (1-ia)\Delta$, and the bilinear form $(\cdot\,,\cdot)_X : X_0 \times X_0 \to \mathbb{R}$ given by

$$(p_1, p_2)_X = \Re\left(\iint_{Q_T} e^{-2s\alpha} L(p_1)\overline{L(p_2)}\, dxdt + s^3\lambda^4 \iint_{q_T} e^{-2s\alpha}\xi^3 p_1\overline{p_2}\, dxdt\right).$$

Define also the linear form $l : X_0 \to \mathbb{R}$ by

$$l(p) = s^3\lambda^4\Re \iint_{Q_T} \xi^3 e^{-2s\alpha}\phi\overline{p}\, dxdt.$$

If s and λ are large enough, we have by (A.11) that $\|\cdot\|_X := (\cdot,\cdot)_X^{1/2}$ defines a norm on the space X_0. Let X be the completion of X_0 with the norm $\|\cdot\|_X$, then X is a Hilbert space with inner product given by $(\cdot,\cdot)_X$. Moreover, combining Cauchy-Schwarz inequality and Carleman estimate (A.11), we obtain that

$$|l(p)| \leq C\left(s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\phi|^2 dxdt\right)^{1/2} \|p\|_X^{1/2}, \tag{A.16}$$

which means that l is a bounded linear operator in X_0. By Hahn-Banach Theorem, we can extend $l : X \to \mathbb{R}$ to a bounded linear and continuous operator defined over X. Using Lax-Milgram Theorem, we see that there exists a unique solution $\widehat{p} \in X$ such that

$$(\widehat{p}, p)_X = l(p), \quad \forall p \in X. \tag{A.17}$$

We remark that $\widehat{p}$, the solution of (A.17), is exactly the weak solution of the following system, which is of fourth order in space and second order in time

$$\begin{cases} L^*(e^{-2s\alpha}L\widehat{p}) + s^3\lambda^4\xi^3 e^{-2s\alpha}\widehat{p}1_\omega = s^3\lambda^4\xi^3e^{-2s\alpha}\phi, & \text{in} \quad Q_T, \\ \widehat{p} = e^{-2s\alpha}L\widehat{p} = 0, & \text{on} \quad \Sigma_T, \\ (e^{-2s\alpha}L\widehat{p})|_{t=0} = (e^{-2s\alpha}L\widehat{p})|_{t=T} = 0, & \text{in} \quad \Omega, \end{cases}$$

where $L^* = -\partial_t + (1+ia)\Delta$ is the formal adjoint of L.

If we set

$$w = e^{-2s\alpha}L\widehat{p} \quad \text{and} \quad u = -s^3\lambda^4\xi^3e^{-2s\alpha}\widehat{p}, \tag{A.18}$$

we use (A.17), and conclude that the pair (w, u) is the weak solution of (A.15). Taking $p = \widehat{p}$ in (A.17) and using (A.16), we get

$$\|\widehat{p}\|_X^2 \leq Cs^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\phi|^2 dxdt. \tag{A.19}$$

Therefore, combining (A.18) and (A.19), we obtain

$$\iint_{Q_T} e^{2s\alpha}|w|^2 dxdt + s^{-3}\lambda^{-4}\iint_{q_T} \xi^{-3}e^{2s\alpha}|u|^2 dxdt \leq Cs^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\phi|^2 dxdt. \tag{A.20}$$

Now, if we take $g = s^3\lambda^4\xi^3e^{-2s\alpha}\phi + u1_\omega$ in (A.14), we get

$$\begin{aligned} s^3\lambda^4 &\iint_{Q_T} e^{-2s\alpha}\xi^3|\phi|^2\, dxdt \\ &= \Re\left(-\iint_{q_T} \phi\bar{u}\, dxdt + \iint_{Q_T} h^0\bar{w}\, dxdt - \iint_{Q_T} h^1\cdot\nabla\bar{w}\, dxdt\right). \end{aligned} \tag{A.21}$$

Let us estimate the right-hand side of (A.21), in order to obtain (A.13).

First, a simple computation shows that,

$$s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}|\nabla w|^2dxdt=\Re\left(s^{-2}\lambda^{-2}(1+ia)\iint_{Q_T}\xi^{-2}e^{2s\alpha}\nabla w\cdot\nabla\bar{w}\,dxdt\right)$$
$$=\Re\left(-s^{-2}\lambda^{-2}(1+ia)\iint_{Q_T}\xi^{-2}e^{2s\alpha}\Delta w\bar{w}dxdt\right.$$
$$\left.-s^{-2}\lambda^{-2}(1+ia)\iint_{Q_T}\left(\nabla(\xi^{-2}e^{2s\alpha})\cdot\nabla w\right)\bar{w}\,dxdt\right). \quad \text{(A.22)}$$

Multiplying the first equation of (A.15) by $s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}\bar{w}$ and integrating on Q_T. we get that

$$\Re\left(-s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}w_t\bar{w}\,dxdt+s^{-2}\lambda^{-2}(1+ia)\iint_{Q_T}\xi^{-2}e^{2s\alpha}\Delta w\bar{w}\,dxdt\right)$$
$$=\Re\left(s\lambda^2\iint_{Q_T}\xi\phi\bar{w}dxdt+s^{-2}\lambda^{-2}\iint_{q_T}\xi^{-2}e^{2s\alpha}u\bar{w}\,dxdt\right). \quad \text{(A.23)}$$

Combining (A.22) and (A.23), it is immediate that

$$s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}|\nabla w|^2\,dxdt$$
$$=-\Re\left(s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}w_t\bar{w}\,dxdt+s\lambda^2\iint_{Q_T}\xi\phi\bar{w}dxdt\right.$$
$$\left.+s^{-2}\lambda^{-2}\iint_{q_T}\xi^{-2}e^{2s\alpha}u\bar{w}dxdt-s^{-2}\lambda^{-2}\iint_{Q_T}\left(\nabla(\xi^{-2}e^{2s\alpha})\cdot\nabla w\right)\bar{w}\,dxdt\right). \quad \text{(A.24)}$$

Let us estimate the first three terms on the right-hand side of (A.24). For the first term, we integrate by parts and use the fact that $w|_{t=0}=w|_{t=T}=0$ to obtain

$$s^{-2}\lambda^{-2}\Re\iint_{Q_T}\xi^{-2}e^{2s\alpha}w_t\bar{w}\,dxdt=s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}\frac{\partial}{\partial t}|w|^2dxdt=$$
$$-s^{-2}\lambda^{-2}\iint_{Q_T}(\xi^{-2}e^{2s\alpha})_t|w|^2dtdx\le CTs^{-1}\lambda^{-2}\iint_{Q_T}e^{2s\alpha}|w|^2dxdt. \quad \text{(A.25)}$$

In the previous inequality, we have used the following estimate $|(\xi^{-2}e^{2s\alpha})_t|\le CT^4se^{2s\alpha}$.

For the next two terms on the right-hand side of (A.24), we use Young's inequality and we get that

$$s\lambda^2\Re\iint_{Q_T}\xi\phi\bar{w}dxdt\le C\left(s^3\lambda^4\iint_{Q_T}\xi^3e^{-2s\alpha}|\phi|^2\,dxdt+s^{-1}\iint_{Q_T}\xi^{-1}e^{2s\alpha}|w|^2\,dxdt\right), \quad \text{(A.26)}$$

and that

$$s^{-2}\lambda^{-2}\Re\iint_{q_T}\xi^{-2}e^{2s\alpha}u\bar{w}\,dxdt$$
$$\le C\left(s^{-3}\lambda^{-4}\iint_{q_T}\xi^{-3}e^{-2s\alpha}|u|^2dxdt+s^{-1}\iint_{Q_T}\xi^{-1}e^{2s\alpha}|w|^2dxdt\right). \quad \text{(A.27)}$$

Finally, for the last term we see that

$$
\begin{aligned}
s^{-2}\lambda^{-2}\Re\iint_{Q_T}\nabla(\xi^{-2}e^{2s\alpha})\cdot\nabla w\bar{w}\,dxdt\\
\leq \frac{1}{2}s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}|\nabla w|^2dxdt + C\iint_{Q_T}e^{2s\alpha}|w|^2\,dxdt. \quad \text{(A.28)}
\end{aligned}
$$

In (A.28), we have used the fact that $|\nabla(\xi^{-2}e^{2s\alpha})| \leq Cs\lambda\xi^{-1}e^{2s\alpha}$.

Using (A.24)-(A.28) we obtain that

$$
\begin{aligned}
s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}|\nabla w|^2\,dxdt \leq C\Big(s^3\lambda^4\iint_{Q_T}\xi^3e^{-2s\alpha}|\phi|^2dxdt\\
+s^{-3}\lambda^{-4}\iint_{q_T}\xi^{-3}e^{2s\alpha}|u|^2dxdt + \iint_{Q_T}e^{2s\alpha}|w|^2\,dxdt\Big). \quad \text{(A.29)}
\end{aligned}
$$

Then, we combine (A.20) and (A.29) and we get that

$$
s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}|\nabla w|^2dxdt \leq Cs^3\lambda^4\iint_{Q_T}\xi^3e^{-2s\alpha}|\phi|^2dxdt. \quad \text{(A.30)}
$$

Now, we use Young's inequality in (A.21), and we obtain that

$$
\begin{aligned}
&s^3\lambda^4\iint_{Q_T}\xi^3e^{-2s\alpha}|\phi|^2\,dxdt\\
&\leq Cs^3\lambda^4\iint_{q_T}\xi^3e^{-2s\alpha}|\phi|^2dxdt + \epsilon s^{-3}\lambda^{-4}\iint_{q_T}\xi^{-3}e^{-2s\alpha}|u|^2\,dxdt\\
&+ C\iint_{Q_T}e^{-2s\alpha}|h^0|^2\,dxdt + \epsilon\iint_{Q_T}e^{2s\alpha}|w|^2\,dxdt\\
&+ Cs^2\lambda^2\iint_{Q_T}\xi^2e^{-2s\alpha}|h^1|^2\,dxdt + \epsilon s^{-2}\lambda^{-2}\iint_{Q_T}\xi^{-2}e^{2s\alpha}|\nabla w|^2\,dxdt, \quad \text{(A.31)}
\end{aligned}
$$

for $\epsilon > 0$ and small.

Combining (A.20), (A.30) and (A.31) we get

$$
\begin{aligned}
s^3\lambda^4\iint_{Q_T}\xi^3e^{-2s\alpha}|\phi|^2\,dxdt \leq C\Big(\iint_{Q_T}e^{-2s\alpha}|h^0|^2\,dxdt + s^2\lambda^2\iint_{Q_T}\xi^2e^{-2s\alpha}|h^1|^2\,dxdt\\
+\iint_{q_T}\xi^3e^{-2s\alpha}|\phi|^2\,dxdt\Big). \quad \text{(A.32)}
\end{aligned}
$$

In what follows, we prove a regularity result for ϕ. More precisely, we show that we can add a weighted gradient term of ϕ on the left-hand side of (A.32). Indeed, integrating by parts we have

$$
\begin{aligned}
s\lambda^2\iint_{Q_T}\xi e^{-2s\alpha}|\nabla\phi|^2dxdt = -s\lambda^2\Re\Big((1-ia)\iint_{Q_T}\xi e^{-2s\alpha}\Delta\phi\bar{\phi}\,dxdt\Big)\\
- s\lambda^2\Re\Big((1-ia)\iint_{Q_T}\nabla\phi\cdot\nabla(\xi e^{-2s\alpha})\overline{\phi}\,dxdt\Big). \quad \text{(A.33)}
\end{aligned}
$$

Proceeding in an analogous way as in (A.28), the last term of (A.33) can be bounded as

$$s\lambda^2\Re\iint_{Q_T}\nabla\phi\cdot\nabla(\xi e^{-2s\alpha})\overline{\phi}\,dxdt \leq \frac{s\lambda^2}{4}\iint_{Q_T}\xi e^{-2s\alpha}|\nabla\phi|^2dxdt + Cs^3\lambda^4\iint_{Q_T}\xi^3e^{-2s\alpha}|\phi|^2dxdt. \quad \text{(A.34)}$$

To estimate the first integral in (A.33), we multiply the first equation of (A.12) by $-s\lambda^2\xi e^{-2s\alpha}\overline{\phi}$ and integrate in Q_T to obtain

$$-s\lambda^2\Re\left((1-ia)\iint_{Q_T}\xi e^{-2s\alpha}\Delta\phi\overline{\phi}\,dxdt\right) = \Re\left(s\lambda^2\iint_{Q_T}\xi e^{-2s\alpha}\phi_t\bar{\phi}\,dxdt - s\lambda^2\iint_{Q_T}\xi e^{-2s\alpha}h^0\bar{\phi}\,dxdt - s\lambda^2\iint_{Q_T}\xi e^{-2s\alpha}\nabla\cdot h^1\bar{\phi}\,dxdt\right). \quad \text{(A.35)}$$

Now, the task is to estimate the three terms on the right-hand side of (A.35). For the first one, we proceed in a similar way as in (A.25) to have

$$s\lambda^2\Re\iint_{Q_T}\xi e^{-2s\alpha}\phi_t\overline{\phi}dxdt \leq CTs^2\lambda^2\Re\iint_{Q_T}\xi^3e^{-2s\alpha}|\phi|^2dxdt. \quad \text{(A.36)}$$

For the second term, we use Young's inequality to obtain

$$s\lambda^2\Re\iint_{Q_T}\xi e^{-2s\alpha}h^0\overline{\phi}dxdt \leq C\iint_{Q_T}e^{-2s\alpha}|h^0|^2dxdt + \epsilon s^2\lambda^4\iint_{Q_T}\xi^2e^{-2s\alpha}|\phi|^2dxdt, \quad \text{(A.37)}$$

for all $\epsilon > 0$, sufficiently small.

For the third term, we integrate by parts and use (A.10) to get

$$s\lambda^2\Re\iint_{Q_T}\xi e^{-2s\alpha}\nabla\cdot h^1\overline{\phi}\,dxdt \leq C\left(s^2\lambda^2\iint_{Q_T}\xi^2e^{-2s\alpha}|h^1|^2\,dxdt + s^2\lambda^4\iint_{Q_T}\xi^2e^{-2s\alpha}|\phi|^2\,dxdt\right) + \epsilon s\lambda^2\iint_{Q_T}\xi e^{-2s\alpha}|\nabla\phi|^2dxdt, \quad \text{(A.38)}$$

for every $\epsilon > 0$, sufficiently small.

Combining (A.32) and (A.33)-(A.38), and taking s and λ sufficiently large, we obtain (A.13). □

Remark A.0.2. By making the change of variable $s = \frac{3}{2}s'$, Carleman inequality (A.13) holds by replacing the weight $e^{-2s\alpha}$ for $e^{-3s'\alpha}$, for every $s' \geq \frac{2}{3}s_2$ (s_2 is the same constant given in Lemma A.6). This particular weight function will be important in forthcoming computations.

Now, we will prove a new Carleman estimate for system (A.7), this will be the main tool to prove Theorem A.3.

Proposition A.7. *There exists a positive constant λ_0 such that, for all $\lambda \geq \lambda_0$, one can find a constant $C > 0$ depending on λ, Ω and ω, such that for all $\psi_0 \in L^2(\Omega)$, $g^0 \in L^2(Q_T)$ and $g^1 \in L^2(0,T;H_0^1(\Omega))$, the solution (φ,ψ) of* (A.7) *satisfies*

$$
\begin{aligned}
s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-3s\alpha}|\varphi|^2 dxdt + s\lambda^2 \iint_{Q_T} \xi e^{-3s\alpha}|\nabla\varphi|^2 dxdt \\
+ s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2 dxdt + s\lambda^2 \iint_{Q_T} \xi e^{-2s\alpha}|\nabla(\Delta\psi)|^2 dxdt \\
\leq C\Big(s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|g^0|^2 dxdt + s^5\lambda^6 \iint_{Q_T} \xi^5 e^{-s\alpha}|\nabla g^1|^2 dxdt \\
+ s^7\lambda^8 \iint_{q_T} \xi^7 e^{-2s\alpha}|\varphi|^2 dxdt\Big). \quad \text{(A.39)}
\end{aligned}
$$

Proof. We will assume that $\psi_0 \in C_0^\infty(\Omega)$. The general case can be proved by using a density argument.

Applying (A.13) with weight function $e^{-3s\alpha}$ instead of $e^{-2s\alpha}$(see Remark A.0.2) for the φ variable, we obtain

$$
\begin{aligned}
s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-3s\alpha}|\varphi|^2 dxdt + s\lambda^2 \iint_{Q_T} \xi e^{-3s\alpha}|\nabla\varphi|^2 dxdt \\
\leq C\Big(\iint_{Q_T} e^{-3s\alpha}|g^0|^2 dxdt + s^2\lambda^2 \iint_{\mathcal{O}_T} \xi^2 e^{-3s\alpha}|\nabla\psi|^2 dxdt \\
+ s^3\lambda^4 \iint_{q_T} \xi^3 e^{-3s\alpha}|\varphi|^2 dxdt\Big), \quad \text{(A.40)}
\end{aligned}
$$

for $\lambda \geq C$ and $s \geq \max\{\frac{2}{3}s_2, CR^{\frac{2}{3}}T^2\}$.

We can bound the second term on the right-hand side of (A.40) as follows

$$
\begin{aligned}
s^2\lambda^2 \iint_{\mathcal{O}_T} \xi^2 e^{-3s\alpha}|\nabla\psi|^2 dxdt &\leq Cs^2\lambda^2 \iint_{Q_T} (\hat{\xi})^2 e^{-3s\hat{\alpha}}|\nabla\psi|^2 \, dxdt \\
&\leq Cs^2\lambda^2 \iint_{Q_T} (\hat{\xi})^2 e^{-3s\hat{\alpha}}|\Delta\psi|^2 \, dxdt.
\end{aligned}
$$

The weights $\hat{\xi}$ and $\hat{\alpha}$ are defined in (A.9).

A simply computation shows that, for λ large enough

$$
3\hat{\alpha} - 2\alpha \geq 0 \quad \text{for every} \quad (x,t) \in Q_T,
$$

in this way we have that

$$
\begin{aligned}
s^2\lambda^2 \iint_{Q_T} (\hat{\xi})^2 e^{-3s\hat{\alpha}}|\Delta\psi|^2 \, dxdt = s^2\lambda^2 \iint_{Q_T} \xi^{-3}\hat{\xi}^2 e^{-(3s\hat{\alpha}-2s\alpha)}\xi^3 e^{-2s\alpha}|\Delta\psi|^2 \, dxdt \\
\leq Cs^2\lambda^2 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2 \, dxdt. \quad \text{(A.41)}
\end{aligned}
$$

Since $g^1 \in L^2(0,T;H_0^1(\Omega))$, the equation satisfied by $\Delta\psi$ is

$$
\begin{cases}
\Delta\psi_t - (1+ia)\Delta^2\psi + R\Delta\psi = \Delta g^1, & \text{in } Q_T, \\
\Delta\psi = 0, & \text{on } \Sigma_T, \\
\Delta\psi|_{t=0} = \Delta\psi^0, & \text{in } \Omega.
\end{cases} \quad \text{(A.42)}
$$

Let $\tilde{\omega} \subset \omega \cap \mathcal{O}$ be an nonempty open subset and define $\tilde{q}_T = \tilde{\omega}\times]0,T[$. Using Carleman estimate (A.13) and the fact that $\Delta g^1 = \nabla \cdot \nabla g^1$, we get

$$s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2 dxdt + s\lambda^2 \iint_{Q_T} \xi e^{-2s\alpha}|\nabla(\Delta\psi)|^2 dxdt \\ \leq C\left(s^2\lambda^2 \iint_{Q_T} \xi^2 e^{-2s\alpha}|\nabla g^1|^2 dxdt + s^3\lambda^4 \iint_{\tilde{q}_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2 dxdt\right), \quad \text{(A.43)}$$

for $\lambda \geq C$ and $s \geq \max\{\frac{2}{3}s_2, CR^{\frac{2}{3}}T^2\}$.

Now, let us estimate the local term of $\Delta\psi$ on the right-hand side of (A.43). By equation (A.7), we see that

$$-\varphi_t - (1-ia)\Delta\varphi + R\varphi = g^0 + \Delta\psi \quad \text{in} \quad \mathcal{O}\times]0,T[. \quad \text{(A.44)}$$

Let $\theta \in C_c^2(\omega)$ a cut-off function such that $\theta \equiv 1$ in $\tilde{\omega}$. Using (A.44) and integrating by parts we obtain

$$s^3\lambda^4 \iint_{\tilde{q}_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2\, dxdt \leq s^3\lambda^4 \iint_{q_T} \theta e^{-2s\alpha}\xi^3|\Delta\psi|^2\, dxdt \\ = s^3\lambda^4\Re \iint_{q_T} \theta\xi^3 e^{-2s\alpha}\overline{\Delta\psi}(-\varphi_t - (1-ia)\Delta\varphi + R\varphi - g^0)\, dxdt \\ = s^3\lambda^4\Re \iint_{q_T} \theta\xi^3 e^{-2s\alpha}\overline{\Delta g^1}\varphi\, dxdt + s^3\lambda^4\Re \iint_{q_T} (\theta\xi^3 e^{-2s\alpha})_t\overline{\Delta\psi}\varphi\, dxdt \\ - s^3\lambda^4\Re \iint_{q_T} \Delta(\theta\xi^3 e^{-2s\alpha})\overline{\Delta\psi}(1-ia)\varphi dxdt \\ - 2s^3\lambda^4\Re \iint_{q_T} [\nabla(\theta\xi^3 e^{-2s\alpha}) \cdot \nabla(\overline{\Delta\psi})](1-ia)\varphi dxdt \\ - s^3\lambda^4\Re \iint_{q_T} \theta\xi^3 e^{-2s\alpha}\overline{\Delta\psi} g^0\, dxdt = \sum_{j=1}^{5} A_j, \quad \text{(A.45)}$$

where A_j represents the j-th term on the right-hand side of (A.45).

For the next computations, the following bounds for the weight functions are needed:

$$|(\xi^3 e^{-2s\alpha})_t| \leq CTse^{-2s\alpha}\xi^5, \\ |\nabla(\xi^3 e^{-2s\alpha})| \leq Cs\lambda e^{-2s\alpha}\xi^4, \\ |\Delta(\xi^3 e^{-2s\alpha})| \leq Cs^2\lambda^2 e^{-2s\alpha}\xi^5. \quad \text{(A.46)}$$

Integrating by parts, using Young's inequality and (A.46) we can bound the A_1 term in (A.45)

$$A_1 = -s^3\lambda^4\Re \iint_{q_T} \nabla(\theta\xi^3 e^{-2s\alpha})\overline{\nabla g^1}\varphi\, dxdt - s^3\lambda^4\Re \iint_{q_T} \theta\xi^3 e^{-2s\alpha}\overline{\nabla g^1}\nabla\varphi\, dxdt \\ \leq \epsilon\left(s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-3s\alpha}|\varphi|^2\, dxdt + s\lambda^2 \iint_{Q_T} \xi e^{-3s\alpha}|\nabla\varphi|^2\, dxdt\right) \\ + Cs^5\lambda^6 \iint_{Q_T} \xi^5 e^{-s\alpha}|\nabla g^1|\, dxdt. \quad \text{(A.47)}$$

For the second term we get

$$A_2 \leq \epsilon s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2\,dxdt + Cs^5\lambda^4 \iint_{q_T} \xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt, \tag{A.48}$$

for all $\epsilon > 0$, sufficiently small. Analogously for the third one we have

$$A_3 \leq \epsilon s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2\,dxdt + Cs^7\lambda^8 \iint_{q_T} \xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt, \tag{A.49}$$

for all $\epsilon > 0$, sufficiently small. For the fourth one has

$$A_4 \leq \epsilon s\lambda^2 \iint_{Q_T} \xi e^{-2s\alpha}|\nabla(\Delta\psi)|^2\,dxdt + Cs^7\lambda^8 \iint_{q_T} \xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt, \tag{A.50}$$

for all $\epsilon > 0$, sufficiently small. Finally, for the last term, we have

$$A_5 \leq \epsilon s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2 dxdt + Cs^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|g^0|^2 dxdt, \tag{A.51}$$

for all $\epsilon > 0$, sufficiently small. In this way, combining (A.45)-(A.51) we obtain

$$\begin{aligned} s^3\lambda^4 \iint_{\tilde{q}_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2\,dxdt &\leq Cs^7\lambda^8 \iint_{q_T} \xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt \\ &+ \epsilon\left(s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|\Delta\psi|^2 dxdt + s\lambda^2 \iint_{Q_T} \xi e^{-2s\alpha}|\nabla(\Delta\psi)|^2\,dxdt\right) \\ &+ C\left(s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\alpha}|g^0|^2\,dxdt + s^5\lambda^6 \iint_{Q_T} \xi^5 e^{-s\alpha}|\nabla g^1|^2\,dxdt\right). \end{aligned} \tag{A.52}$$

Now, taking ϵ sufficiently small, summing (A.40) and (A.43) and using (A.41), (A.52) we obtain (A.39). □

Null Controllability

In this section, we will see how Carleman estimate (A.39) allows us to prove an observability inequality like (A.8). In order to prove this estimate, we first prove a Carleman estimate with weight functions blowing up only at $t = 0$. Here, we also prove Theorem A.4, that is, the null controllability for the linear system (A.5).

Define the new weight functions

$$\begin{aligned} \beta(x,t) &= \frac{e^{2\lambda||\eta||_\infty} - e^{\lambda\eta(x)}}{l(t)}, & \gamma(x,t) &= \frac{e^{\lambda\eta(x)}}{l(t)}, \\ \beta^*(t) &= \max_{x\in\overline{\Omega}} \beta(x,t), & \gamma^*(t) &= \min_{x\in\overline{\Omega}} \zeta(x,t), \\ \widehat{\beta}(t) &= \min_{x\in\overline{\Omega}} \beta(x,t), & \widehat{\gamma}(t) &= \max_{x\in\overline{\Omega}} \zeta(x,t), \end{aligned}$$

where l is given by

$$l(t) = \begin{cases} t(T-t), & 0 \leq t \leq T/2, \\ T^2/4, & T/2 < t \leq T. \end{cases}$$

Notice that $\beta = \alpha$ and $\gamma = \xi$ in $]0, T/2[$.

We have the following result.

Proposition A.8. *There exists a constant $C > 0$ (depending on s, λ), such that the solution (φ, ψ) of* (A.7) *satisfies*

$$\iint_{Q_T} (\gamma^*)^3 e^{-3s\beta^*} |\varphi|^2 dxdt + \iint_{Q_T} \gamma^* e^{-3s\beta^*} |\nabla\varphi|^2 dxdt + \iint_{Q_T} (\gamma^*)^3 e^{-2s\beta^*} |\Delta\psi|^2 dxdt \\ \leq C \left(\iint_{Q_T} \hat{\gamma}^3 e^{-2s\hat{\beta}} |g^0|^2 dxdt + \iint_{Q_T} \hat{\gamma}^5 e^{-s\hat{\beta}} |\nabla g^1|^2 dxdt + \iint_{q_T} \hat{\gamma}^7 e^{-2s\hat{\beta}} |\varphi|^2 dxdt \right). \quad \text{(A.53)}$$

In order to prove Proposition A.8, we are going to use the following technical Lemma.

Lemma A.9. *If (φ, ψ) is a solution of system* (A.7)*, then*

$$\|\varphi\|^2_{L^2(T/2,T;L^2(\Omega))} + \|\nabla\varphi\|^2_{L^2(T/2,T;L^2(\Omega))} \\ \leq C \left(\|(\varphi, \Delta\psi)\|^2_{L^2(T/4,T/2;L^2(\Omega))^2} + \|(g^0, \nabla g^1)\|^2_{L^2(T/2,T;L^2(\Omega))^2} \right), \quad \text{(A.54)}$$

and

$$\|\Delta\psi\|^2_{L^2(T/2,T;L^2(\Omega))} \leq C \left(\|\Delta\psi\|^2_{L^2(T/4,T/2;L^2(\Omega))} + \|\nabla g^1\|^2_{L^2(T/2,T;L^2(\Omega))} \right). \quad \text{(A.55)}$$

Proof. Define $\mathcal{X} \in C^2([0,T])$ such that

$$\chi = \begin{cases} 0, & \text{if} \quad t \in [0, T/4], \\ 1, & \text{if} \quad t \in [T/2, T]. \end{cases}$$

If (φ, ψ) is a solution for (A.7), it is not difficult to see that $(\chi\varphi, \chi\psi)$ satisfies the system

$$\begin{cases} -(\chi\varphi)_t - (1-ia)\Delta(\chi\varphi) + R(\chi\varphi) = \chi g^0 + \nabla\cdot(\nabla(\chi\psi)1_{\mathcal{O}}) - \chi_t\varphi, & \text{in } Q_T, \\ (\chi\psi)_t - (1+ia)\Delta(\chi\psi) + R(\chi\psi) = \chi g^1 + \chi_t\psi, & \text{in } Q_T, \\ \chi\varphi = \chi\psi = 0, & \text{on } \Sigma_T, \\ \chi\psi|_{t=0} = 0, \ \chi\varphi|_{t=T} = 0, & \text{in } \Omega. \end{cases} \quad \text{(A.56)}$$

Since g^1 and ψ belongs to $L^2(0,T;H^1_0(\Omega))$ we have that

$$\frac{d^j\chi}{dt^j} \in L^\infty(\Omega) \quad \text{and} \quad \frac{d^{j-1}\chi}{dt^{j-1}}\psi \in C([0,T];H^1_0(\Omega)) \cap L^2(0,T;H^2(\Omega)), \ j = 1, 2,$$

and the following estimate holds

$$\int_\Omega |\chi(t)\nabla\psi(t)|^2\, dx + \iint_{Q_T} |\chi\Delta\psi|^2\, dxdt \leq C \left(\iint_{Q_T} |\chi\nabla g^1|^2\, dxdt + \iint_{Q_T} |\chi_t\Delta\psi|^2\, dxdt \right). \quad \text{(A.57)}$$

Multiplying the first equation of (A.56) by $\chi\varphi$, integrating on Q_T and taking the real part, we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\chi(t)\varphi(t)|^2\, dx + \int_\Omega |\chi\nabla\varphi|^2\, dx + R\int_\Omega |\chi\varphi|^2\, dx \\ \leq C\left(\int_\Omega |\chi g^0|^2\, dx + \int_\Omega |\chi_t\varphi|^2\, dx + \int_\Omega |\chi\Delta\psi|^2\, dx \right) + \epsilon\int_{\mathcal{O}} |\chi\nabla\varphi|^2\, dx, \quad \text{(A.58)}$$

for all $\epsilon > 0$, sufficiently small. Integrating (A.58) on $[t, T]$ and combining with (A.57), we get

$$\int_\Omega |\chi(t)\varphi(t)|^2\, dx + \iint_{Q_T} |\chi \nabla \varphi|^2\, dxdt \\ \leq C \left(\iint_{Q_T} |\chi|^2 (|g^0|^2 + |\nabla g^1|^2)\, dxdt + \iint_{Q_T} |\chi_t|^2 (|\varphi|^2 + |\Delta \psi|^2)\, dxdt \right), \quad \text{(A.59)}$$

for all $t \in [0, T]$. In this way, using respectively (A.57) and (A.59) we obtain (A.54) and (A.55). □

Now, we use Lemma A.9 to prove the following Carleman estimate where now the new weight functions γ and β are used.

Lemma A.10. *There exists a constant $C > 0$ (depending on s, λ), such that every solution (φ, ψ) of* (A.7) *satisfies*

$$\iint_{Q_T} \gamma^3 e^{-3s\beta} |\varphi|^2 dxdt + \iint_{Q_T} \gamma e^{-3s\beta} |\nabla \varphi|^2 dxdt + \iint_{Q_T} \gamma^3 e^{-2s\beta} |\Delta \psi|^2 dxdt \\ \leq C \left(\iint_{Q_T} \gamma^3 e^{-2s\beta} |g^0|^2 dxdt + \iint_{Q_T} \gamma^5 e^{-s\beta} |\nabla g^1|^2 dxdt \right. \\ \left. + \iint_{q_T} \gamma^7 e^{-2s\beta} |\varphi|^2 dxdt \right). \quad \text{(A.60)}$$

Proof. In one hand, since $\alpha = \beta$ for every $t \in [0, T/2]$, and since l is constant in $[T/2, T]$, we easily see that

$$\iint_{Q_T} \xi^3 e^{-2s\alpha} |g^0|^2 dxdt + \iint_{Q_T} \xi^5 e^{-s\alpha} |\nabla g^1|^2 dxdt + \iint_{q_T} \xi^7 e^{-2s\alpha} |\varphi|^2 dxdt \\ \leq C \left(\iint_{Q_T} \gamma^3 e^{-2s\beta} |g^0|^2 dxdt + \iint_{Q_T} \gamma^5 e^{-s\beta} |\nabla g^1|^2 dxdt \right. \\ \left. + \iint_{q_T} \gamma^7 e^{-2s\beta} |\varphi|^2 dxdt \right). \quad \text{(A.61)}$$

In other hand, in $[0, T/2]$, the weight coincides and then

$$\int_0^{\frac{T}{2}}\int_\Omega \gamma^3 e^{-3s\beta} |\varphi|^2 dxdt + \iint_{Q_T} \gamma e^{-3s\beta} |\nabla \varphi|^2 dxdt + \int_0^{\frac{T}{2}}\int_\Omega \gamma^3 e^{-2s\beta} |\Delta \psi|^2 dxdt \\ = \int_0^{\frac{T}{2}}\int_\Omega \xi^3 e^{-3s\alpha} |\varphi|^2 dxdt + \iint_{Q_T} \xi e^{-3s\alpha} |\nabla \varphi|^2 dxdt + \int_0^{\frac{T}{2}}\int_\Omega \xi^3 e^{-2s\alpha} |\Delta \psi|^2 dxdt. \quad \text{(A.62)}$$

Moreover, in $[T/2,T]$ we use (A.54) and (A.55) to obtain

$$\begin{aligned}
&\int_{\frac{T}{2}}^{T}\!\!\int_\Omega \gamma^3 e^{-3s\beta}|\varphi|^2\,dxdt + \iint_{Q_T} \gamma e^{-3s\beta}|\nabla\varphi|^2 dxdt \\
&\qquad + \int_{\frac{T}{2}}^{T}\!\!\int_\Omega \gamma^3 e^{-2s\beta}|\Delta\psi|^2\,dxdt \le C\int_{\frac{T}{2}}^{T}\!\!\int_\Omega |\varphi|^2 + |\nabla\varphi|^2 + |\Delta\psi|^2\,dxdt \\
&\qquad \le C\left(\int_{\frac{T}{4}}^{\frac{T}{2}}\!\!\int_\Omega |\varphi|^2 + |\Delta\psi|^2\,dxdt + \int_{\frac{T}{2}}^{T}\!\!\int_\Omega |g^0|^2 + |\nabla g^1|^2\,dxdt\right) \\
&\qquad \le C\left(\int_{\frac{T}{4}}^{\frac{T}{2}}\!\!\int_\Omega \xi^3 e^{-3s\alpha}|\varphi|^2\,dxdt + \int_{\frac{T}{4}}^{\frac{T}{2}}\!\!\int_\Omega \xi^3 e^{-2s\alpha}|\Delta\psi|^2\,dxdt + \right. \\
&\qquad\qquad \left. \int_{\frac{T}{2}}^{T}\!\!\int_\Omega \gamma^3 e^{-2s\beta}|g^0|^2\,dxdt + \int_{\frac{T}{2}}^{T}\!\!\int_\Omega \gamma^5 e^{-s\beta}|\nabla g^1|^2\,dxdt\right). \qquad \text{(A.63)}
\end{aligned}$$

Combining (A.39), (A.61),(A.62) and (A.63) we obtain (A.60). □

Now, the proof of Proposition A.8 follows immediately.

Proof of Proposition A.8. We just have to use Lemma A.10 and take the minimum and maximum of the corresponding weight functions. □

Controllability for the Linear Case

In this section, we will see how we can use Carleman inequality (A.53) to prove Theorem A.4. For this, let $\mathcal{L} := \partial_t - (1+ia)\Delta + R$, and $\mathcal{L}^* := -\partial_t - (1-ia)\Delta + R$ and introduce the space

$$\begin{aligned}
\mathcal{C} = \Big\{ &(y,z,v);\ (\hat\gamma)^{-\frac{3}{2}} e^{s\hat\beta} y \in L^2(Q_T),\ (\hat\gamma)^{-\frac{5}{2}} e^{\frac{s\hat\beta}{2}} z \in L^2(0,T;H^{-1}(\Omega)), \\
&(\hat\gamma)^{-\frac{7}{2}} e^{s\hat\beta} v \in L^2(Q_T), (\gamma^*)^{-\frac{3}{2}} e^{\frac{3}{2}s\beta^*}(\mathcal{L}y - v1_\omega) \in L^2(Q_T), \\
&(\gamma^*)^{-\frac{3}{2}} e^{s\beta^*}(\mathcal{L}^* z - \nabla\cdot(\nabla y 1_{\mathcal{O}})) \in L^2(0,T;H^{-1}(\Omega)), \\
&e^{s\hat\beta}(\hat\gamma)^{-\frac{7}{2}} y \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), \\
&e^{\frac{s\hat\beta}{2}}(\hat\gamma)^{-\frac{9}{2}} z \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), z|_{t=T} = 0 \text{ in } \Omega \Big\}.
\end{aligned}$$

We remark that, if a triplet (y,z,v) is in $\mathcal{C}$ then $z|_{t=0} = 0$. This comes from the fact that $e^{\frac{s\hat\beta}{2}}(\hat\gamma)^{-\frac{9}{2}} z$ remains bounded in $L^\infty(0,T;L^2(\Omega))$ and the weight $e^{\frac{s\hat\beta}{2}}(\hat\gamma)^{-\frac{9}{2}}$ blows up at $t=0$.

We have the following result.

Proposition A.11. *Assume that*

$$(\gamma^*)^{-\frac{3}{2}} e^{\frac{3}{2}s\beta^*} f^0 \in L^2(Q_T) \quad \text{and} \quad (\gamma^*)^{-\frac{3}{2}} e^{s\beta^*} f^1 \in L^2(0,T;H^{-1}(\Omega)).$$

There exists a function v such that, if (y,z) is the solution of (A.5) *with control v and right-hand side (f^0,f^1), then the triplet (y,z,v) belongs to $\mathcal{C}$.*

Proof. To start, we define the spaces $X_0 = \{u \in H_0^1(\Omega); \Delta u \in H_0^1(\Omega)\}$,

$Y_0 = C([0,T]; X_0) \cap C^1([0,T]; H_0^1(\Omega))$, and $Y_1 = C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$,

and we set

$$P_0 = \{(\varphi, \psi) \in Y_1 \times Y_0;\ \mathcal{L}^*\varphi - \nabla \cdot (\nabla\psi 1_{\mathcal{O}}) \in L^2(Q_T)\}.$$

From Proposition A.2 and Remark A.2 of the Appendix, it is clear that P_0 is nonempty.

Now, let us consider the bilinear form $a : P_0 \times P_0 \to \mathbb{R}$ given by

$$\begin{aligned} a\left((\hat{\varphi}, \hat{\psi}), (\varphi, \psi)\right) &= \Re \iint_{Q_T} (\hat{\gamma})^3 e^{-2s\hat{\beta}} (\mathcal{L}^*\hat{\varphi} - \nabla \cdot (\nabla\hat{\psi} 1_{\mathcal{O}})) \cdot (\overline{\mathcal{L}^*\varphi - \nabla \cdot (\nabla\psi 1_{\mathcal{O}})})\, dxdt \\ &+ \Re \iint_{Q_T} (\hat{\gamma})^5 e^{-s\hat{\beta}} \nabla(\mathcal{L}\hat{\psi}) \cdot \nabla(\overline{\mathcal{L}\psi})\, dxdt + \Re \iint_{q_T} (\hat{\gamma})^7 e^{-2s\hat{\beta}} \hat{\varphi}\bar{\varphi}\, dxdt, \end{aligned}$$

and the linear form $G : P_0 \to \mathbb{R}$ defined by

$$\langle G, (\varphi, \psi)\rangle = \Re \iint_{Q_T} f^0 \bar{\varphi}\, dxdt + \int_0^T \langle f^1, \psi\rangle_{H^{-1}(\Omega)\, H_0^1(\Omega)}\, dt.$$

It is not difficult to see that $\|(\varphi, \psi)\|_* = a((\varphi, \psi), (\varphi, \psi))^{1/2}$ defines a norm on P_0. Indeed, if we assume that $a((\varphi, \psi), (\varphi, \psi)) = 0$, then we obtain from Proposition A.8 that $\varphi = 0$ and $\Delta\psi = 0$ in Q_T, and combined with the fact that $\psi = 0$ over Σ_T, we get that $\psi = 0$ in Q_T. Therefore, we can consider the space P, the completion of P_0 with the norm $\|\cdot\|_*$. It follows that $a(\cdot,\cdot)$ is continuous and coercive over $P \times P$.

For $(\varphi, \psi) \in P_0$, there exist C such that $\|\nabla\psi(t)\|_{L^2(\Omega)} \le C\|\Delta\psi(t)\|_{L^2(\Omega)}$, for all $t \in [0,T]$. Then, using Proposition A.8, we can estimate G as follows

$$\begin{aligned} \langle G, (\varphi, \psi)\rangle &\le \left(\iint_{Q_T} (\gamma^*)^3 e^{-3s\beta^*} |\varphi|^2\, dxdt + C \iint_{Q_T} (\gamma^*)^3 e^{-2s\beta^*} |\Delta\psi|^2\, dxdt\right)^{\frac{1}{2}} \times \\ &\quad \left(\iint_{Q_T} (\gamma^*)^{-3} e^{3s\beta^*} |f^0|^2\, dxdt + \int_0^T (\gamma^*)^{-3} e^{2s\beta^*} \|f^1\|^2_{H^{-1}(\Omega)}\, dt\right)^{\frac{1}{2}} \\ &\le C\|(\varphi, \psi)\|_* \left(\iint_{Q_T} (\gamma^*)^{-3} e^{3s\beta^*} |f^0|^2\, dxdt + \int_0^T (\gamma^*)^{-3} e^{2s\beta^*} \|f^1\|^2_{H^{-1}(\Omega)}\, dt\right)^{\frac{1}{2}}, \end{aligned}$$

for every $(\varphi, \psi) \in P_0 \times P_0$. By applying the Hahn-Banach extension Theorem, the functional G can be extended to a continuous functional over P. In this way, we apply the Lax-Milgram Theorem and conclude that the variational problem

$$a\left((\hat{\varphi}, \hat{\psi}), (\varphi, \psi)\right) = \langle G, (\varphi, \psi)\rangle,\ \forall(\varphi, \psi) \in P, \tag{A.64}$$

has a unique solution $(\hat{\varphi}, \hat{\psi}) \in P \times P$.

Define $(\hat{y}, \hat{z}, \hat{v})$ by

$$\begin{cases} \hat{y} = (\hat{\gamma})^3 e^{-2s\hat{\beta}} (\mathcal{L}^*\hat{\varphi} - \nabla \cdot (\nabla\hat{\psi} 1_{\mathcal{O}})), & \text{in } Q_T, \\ \hat{z} = -(\hat{\gamma})^5 e^{-s\hat{\beta}} \Delta(\mathcal{L}\hat{\psi}), & \text{in } Q_T, \\ \hat{v} = -(\hat{\gamma})^7 e^{-2s\hat{\beta}} \hat{\varphi}, & \text{in } Q_T. \end{cases} \tag{A.65}$$

Combining (A.64) and (A.65), we get that

$$\Re \iint_{Q_T} \hat{y}(\overline{\mathcal{L}^*\varphi - \nabla\cdot(\nabla\psi 1_{\mathcal{O}})})\, dxdt + \int_0^T \langle \hat{z}, \mathcal{L}\psi\rangle_{H^{-1}H_0^1}\, dt$$
$$= \Re \iint_{q_T} \hat{v}\bar{\varphi}\, dxdt + \Re \iint_{Q_T} f^0\bar{\varphi}\, dxdt + \int_0^T \langle f^1, \psi\rangle_{H^{-1}(\Omega)\, H_0^1(\Omega)}\, dt, \quad \text{(A.66)}$$

for all $(\varphi, \psi) \in P_0 \subset P$.

For $(g^0, g^1) \in L^2(Q_T) \times L^2(0,T;H_0^1(\Omega))$, we take a sequence $(g_n^0, g_n^1) \in [C_0^\infty(\Omega)]^2$ converging to (g^0, g^1) and (φ_n, ψ_n) the corresponding regular solutions of (A.7) with $\psi_0 = 0$ (see Proposition A.2 and Remark A.2 of the Appendix). From (A.66), we have that

$$\Re \iint_{Q_T} \hat{y} g_n^0\, dxdt + \int_0^T \langle \hat{z}, g_n^1\rangle_{H^{-1}H_0^1}\, dt = \Re \iint_{q_T} \hat{v}\bar{\varphi}_n\, dxdt$$
$$+ \Re \iint_{Q_T} f^0\bar{\varphi}_n\, dxdt + \int_0^T \langle f^1, \psi_n\rangle_{H^{-1}(\Omega)\, H_0^1(\Omega)}\, dt. \quad \text{(A.67)}$$

It can be proved that $(\varphi_n, \psi_n) \rightharpoonup (\varphi, \psi)$ weakly in $[L^2(Q_T)]^2$, where (φ, ψ) is the solution of (A.7) with $\psi_0 = 0$ and (g^0, g^1) on the right-hand side (see Proposition A.1 and Remark A.2). Passing to the limit in (A.67), we obtain that

$$\Re \iint_{Q_T} \hat{y}\bar{g}^0\, dt + \int_0^T \langle \hat{z}, g^1\rangle_{H^{-1}H_0^1}\, dt = \Re \iint_{q_T} \hat{v}\bar{\varphi}\, dxdt$$
$$+ \Re \iint_{Q_T} f^0\bar{\varphi}\, dxdt + \int_0^T \langle f^1, \psi\rangle_{H^{-1}(\Omega)\, H_0^1(\Omega)}\, dt. \quad \text{(A.68)}$$

Let us prove now that $(\hat{y}, \hat{z})$ is solution in the transposition sense for (A.5) (see the Appendix for the definition of transposition solutions). For this, let $(\tilde{y}, \tilde{z}) \in L^2(0,T;H_0^1(\Omega)) \times L^2(Q_T)$ be the transposition solution of (A.5) associated to the control $v = \hat{v}$, source terms f^0 and f^1, and $y_0 = 0$, and let us prove that $\hat{y} = \tilde{y}$ and $\hat{z} = \tilde{z}$. By taking $(g^1, \psi) = (0, 0)$ in (A.68), we get that

$$\Re \iint_{Q_T} \hat{y}\overline{g^0}\, dxdt = \Re \iint_{q_T} \hat{v}\bar{\varphi}\, dxdt + \Re \iint_{Q_T} f^0\overline{\varphi}\, dxdt, \quad \text{(A.69)}$$

for all $g^0 \in L^2(Q_T)$. This implies that $\hat{y} \in L^2(0,T;H_0^1(\Omega))$ and $\hat{y} = \tilde{y}$ (see (A.86)). Since now we have that $\hat{y} \in L^2(0,T;H_0^1(\Omega))$, we use again a density argument to extend (A.68) for all $g^0 \in L^2(0,T;H^{-1}(\Omega))$, and we get that

$$\int_0^T \langle g^0, \hat{y}\rangle_{H^{-1}H_0^1}\, dt + \int_0^T \langle \hat{z}, g^1\rangle_{H^{-1}H_0^1}\, dt = \Re \iint_{q_T} \hat{v}\bar{\varphi}\, dxdt$$
$$+ \Re \iint_{Q_T} f^0\varphi\, dxdt + \int_0^T \langle f^1, \psi\rangle_{H^{-1}(\Omega)\, H_0^1(\Omega)}\, dt, \quad \text{(A.70)}$$

for every $(g^0, g^1) \in L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))$, where (φ, ψ) is the solution of (A.7) with $\psi_0 = 0$ and (g^0, g^1) on the right-hand side.

Now, we assume in (A.67) that $(g^0, \varphi) = (-\nabla \cdot (\nabla \psi 1_{\mathcal{O}}), 0)$, and we obtain that

$$\int_0^T \langle \hat{z}, g^1 \rangle_{H^{-1} H_0^1} \, dt = -\Re \int_0^T\!\!\int_{\mathcal{O}} \nabla \hat{y} \cdot \overline{\nabla \psi} \, dx dt + \int_0^T \langle f^1, \psi \rangle_{H^{-1}(\Omega)\, H_0^1(\Omega)} \, dt, \tag{A.71}$$

for all $g^1 \in L^2(0, T; H_0^1(\Omega))$. Therefore, we have that $\hat{z} \in L^2(Q_T)$ and $\hat{z} = \tilde{z}$, which means that $(\hat{y}, \hat{z})$ is the transposition solution of (A.5) associated to the control $v = \hat{v}$, source terms f^0 and f^1, and $y_0 = 0$.

Let us prove that $(\hat{y}, \hat{z})$ is indeed more regular. We start using (A.65) to see that

$$\|(\hat{\gamma})^{-5/2} e^{s\hat{\beta}/2} \hat{z}\|^2_{L^2(0,T;H^{-1}(\Omega))} =$$

$$\int_0^T (\hat{\gamma})^5 e^{-s\hat{\beta}} \left(\sup_{\|\zeta\|_{H_0^1(\Omega)}=1} (\nabla(\mathcal{L}\hat{\psi}), \nabla \zeta)_{L^2(\Omega)} \right)^2 dt = \iint_{Q_T} (\hat{\gamma})^5 e^{-s\hat{\beta}} |\nabla(\mathcal{L}\hat{\psi})|^2 \, dx dt.$$

Hence, we have that

$$\iint_{Q_T} (\hat{\gamma})^{-3} e^{2s\hat{\beta}} |\hat{y}|^2 \, dx dt + \int_0^T (\hat{\gamma})^{-5} e^{s\hat{\beta}} \|\hat{z}\|^2_{H^{-1}(\Omega)} dt$$
$$+ \iint_{q_T} (\hat{\gamma})^{-7} e^{2s\hat{\beta}} |\hat{v}|^2 \, dx dt = a\left((\hat{\varphi}, \hat{\psi}), (\hat{\varphi}, \hat{\psi}) \right) < \infty.$$

Now, we define

$$\begin{cases} y^* = e^{s\hat{\beta}} (\hat{\gamma})^{-\frac{7}{2}} \hat{y}, \\ z^* = e^{\frac{s\hat{\beta}}{2}} (\hat{\gamma})^{-\frac{9}{2}} \hat{z}, \\ f_*^0 = e^{s\hat{\beta}} (\hat{\gamma})^{-\frac{7}{2}} (f^0 + \hat{v} 1_\omega), \\ f_*^1 = e^{\frac{s\hat{\beta}}{2}} (\hat{\gamma})^{-\frac{9}{2}} f^1, \end{cases}$$

and it is not difficult to see that

$$\begin{cases} \mathcal{L} y^* = f_*^0 + \left(e^{s\hat{\beta}} (\hat{\gamma})^{-\frac{7}{2}} \right)_t \hat{y}, & \text{in } Q_T, \\ \mathcal{L}^* z^* = e^{\frac{-s\hat{\beta}}{2}} (\hat{\gamma})^{-1} \nabla \cdot (\nabla y^* 1_{\mathcal{O}}) + f_*^1 - \left(e^{\frac{s\hat{\beta}}{2}} (\hat{\gamma})^{-\frac{9}{2}} \right)_t \hat{z}, & \text{in } Q_T, \\ y^* = z^* = 0, & \text{on } \Sigma_T, \\ y^*|_{t=0} = 0, z^*|_{t=T} = 0, & \text{in } \Omega. \end{cases}$$

An easy computation shows that

$$\left(e^{s\hat{\beta}} (\hat{\gamma})^{-\frac{7}{2}} \right)_t \leq Cs(\hat{\gamma})^{-\frac{3}{2}} e^{s\hat{\beta}} \quad \text{and} \quad \left(e^{\frac{s\hat{\beta}}{2}} (\hat{\gamma})^{-\frac{9}{2}} \right)_t \leq Cs(\hat{\gamma})^{-\frac{5}{2}} e^{\frac{s\hat{\beta}}{2}}.$$

In this way, $f_*^0 + \left(e^{s\hat{\beta}} (\hat{\gamma})^{-7/2} \right)_t \hat{y} \in L^2(Q)$, $f_*^1 - \left(e^{\frac{s\hat{\beta}}{2}} (\hat{\gamma})^{-9/2} \right)_t \hat{z} \in L^2(0, T; H^{-1}(\Omega))$, and since $e^{-s\hat{\beta}/2} (\hat{\gamma})^{-1}$ is bounded, we have that

$$y^* \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)) \quad \text{and} \quad z^* \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).$$

Therefore, the triplet $(\hat{y}, \hat{z}, \hat{v})$ belongs to $\mathcal{C}$, which completes the proof of Proposition A.11. $\square$

The proof of Theorem A.4, that is, the null controllability for the linear system (A.5), follows immediately from Proposition A.11.

Proof of Theorem A.4. The triplet $(\hat{y}, \hat{z}, \hat{v})$ given in (A.65) solves (A.5) and belongs to $\mathcal{C}$. Since the weight function $e^{s\hat{\beta}/2}$ blows up at $t = 0$ then $z|_{t=0} = 0$, and this ends the proof. □

Controllability for the Nonlinear Case

To deal with the nonlinear case, we are going to use a local inversion mapping theorem (see [30]).

Theorem A.12 (Inverse Mapping Theorem)**.** *Let B_1 and B_2 be Banach spaces, and let $\mathcal{A} : B_1 \to B_2$ with $\mathcal{A} \in C^1(B_1, B_2)$. Given $b_1 \in B_1$ and $b_2 = \mathcal{A}(b_1)$, supose that $\mathcal{A}'(b_1) : B_1 \to B_2$ is surjective. Then there exists $\delta > 0$ such that for all $b' \in B_2$ with $||b' - b_2||_{B_2} < \delta$, one can find $b \in B_1$ such that $\mathcal{A}(b) = b'$.*

We are finally ready to prove Theorem A.3.

Proof of Theorem A.3. We will assume that $N \leq 2$.

Consider $B_1 = \mathcal{C}$ and

$$B_2 = L^2((\hat{\gamma})^{-\frac{21}{2}} e^{3s\hat{\beta}}]0,T[; L^2(\Omega)) \times L^2((\hat{\gamma})^{-\frac{23}{2}} e^{\frac{5s\hat{\beta}}{2}}]0,T[; H^{-1}(\Omega)).$$

We define $\mathcal{A} : B_1 \to B_2$ by expression

$$\mathcal{A}(y,z,v) = (\mathcal{L}y - (1+ib)|y|^2 y - v1_\omega, \mathcal{L}^* z - (1-ib)\overline{y}^2 \overline{z} - 2(1-ib)|y|^2 z - \nabla \cdot (\nabla y 1_{\mathcal{O}})).$$

It is not difficult to see that $\mathcal{A}(0,0,0) = (0,0)$, and since

$$\mathcal{A}'(0,0,0)(\hat{y}, \hat{z}, \hat{v}) = (\mathcal{L}\hat{y} - \hat{v}1_\omega, \mathcal{L}^* \hat{z} - \nabla \cdot (\nabla \hat{y} 1_{\mathcal{O}})),$$

the surjectivity of $\mathcal{A}'(0,0,0)$ follows from Proposition A.11, and the fact that, for λ large enough, the following inclusion holds

$$B_2 \subset L^2((\gamma^*)^{-\frac{3}{2}} e^{\frac{3}{2}s\beta^*}]0,T[; L^2(\Omega)) \times L^2((\gamma^*)^{-\frac{3}{2}} e^{s\beta^*}]0,T[; H^{-1}(\Omega)). \tag{A.72}$$

We remark that, despite the fact that at this point λ is already fixed, we could have taken it even bigger, if necessary, in such a way that (A.72) holds.

Now, in order to apply Theorem A.12, we just have to prove that $\mathcal{A} \in C^1(B_1, B_2)$. To do this, it is sufficient to prove that the maps

$$((y_1, z_1, v_1), (y_2, z_2, v_2), (y_3, z_3, v_3)) \mapsto y_1 y_2 y_3, \quad (y_i, z_i, v_i) \in \mathcal{C},$$

and

$$((y_1, z_1, v_1), (y_2, z_2, v_2), (y_3, z_3, v_3)) \mapsto y_1 y_2 z_1, \quad (y_i, z_i, v_i) \in \mathcal{C},$$

are continuous. Indeed, we have that $e^{s\hat{\beta}}(\hat{\gamma})^{-\frac{7}{2}}y_i \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \subset L^6(Q_T)$. Consequently, we obtain

$$\|(\hat{\gamma})^{-\frac{21}{2}}e^{3s\hat{\beta}}y_1y_2y_3\|_{L^2(Q_T)} = \|(e^{s\hat{\beta}}(\hat{\gamma})^{\frac{-7}{2}}y_1)(e^{s\hat{\beta}}(\hat{\gamma})^{\frac{-7}{2}}y_2)(e^{s\hat{\beta}}(\hat{\gamma})^{\frac{-7}{2}}y_3)\|_{L^2(Q_T)}$$
$$\leq \prod_{i=1}^{3}\|e^{s\hat{\beta}}(\hat{\gamma})^{\frac{-7}{2}}y_i\|_{L^6(Q_T)} \leq \prod_{i=1}^{3}\|(y_i,z_i,v_i)\|_{\mathcal{C}}. \quad \text{(A.73)}$$

Moreover, for $N \leq 2$, $e^{\frac{s\hat{\beta}}{2}}(\hat{\gamma})^{-\frac{9}{2}}z_i \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \subset L^6(Q_T)$, and then

$$\begin{aligned}\|(\hat{\gamma})^{-\frac{23}{2}}e^{\frac{5s\hat{\beta}}{2}}y_1y_2z_1\|_{L^2(Q_T)} &= \|(e^{s\hat{\beta}}(\hat{\gamma})^{\frac{-7}{2}}y_1)(e^{s\hat{\beta}}(\hat{\gamma})^{\frac{-7}{2}}y_2)(e^{\frac{s\hat{\beta}}{2}}(\hat{\gamma})^{\frac{-9}{2}}z_1)\|_{L^2(Q_T)} \\ &\leq \prod_{i=1}^{3}\|(y_i,z_i,v_i)\|_{\mathcal{C}}.\end{aligned}$$

Therefore, we have that $\mathcal{A} \in C^1(B_1,B_2)$, and we can apply Theorem A.12 to guarantee the existence of $\delta > 0$, such that for every (f^0,f^1) satisfying $\|(f^0,f^1)\|_{B_2} \leq \delta$, one can find $(y,z,v) \in B_1$ such that $\mathcal{A}(y,z,v) = (f^0,f^1)$. If we in take in particular $f^0 = f$ in the weighted space $L^2((\hat{\gamma})^{-\frac{21}{2}}e^{3s\hat{\beta}}]0,T[;L^2(\Omega))$ such that $\|(\hat{\gamma})^{-\frac{21}{2}}e^{3s\hat{\beta}}f\| \leq \delta$, and $f^1 = 0$, then the triplet $(y,z,v) = (y,z,v)(f)$ is a solution of system (A.4) with $z|_{t=0} = 0$. □

The next section is dedicated to show some open questions and perspectives related to the problems considered here.

Open Questions

In this section we will indicate several additional problems related to insensitizing control problems for the Ginzburg-Landau equation.

On the Functional (A.2)

In this paper, we have solved the problem of existence of controls which insensitizes the functional J given by (A.2). An interesting and difficult problem that can br considered, is to find controls which turns the state insensitive with respect to the functional

$$J(\tau,v) := \frac{1}{2}\iint_{\mathcal{O}_T}|\Re\left(y(x,t;\tau,v)\right)|^2dxdt.$$

That is, a control which insensitizes only the real part (or imaginary part) of the state. If we follow similar strategies, we are led to prove the existence of controls $v \in L^2(\omega\times]0,T[)$ such that the solution (y,z) of

$$\begin{cases} y_t - (1+ia)\Delta y + Ry = v1_\omega + f^0, & \text{in} \quad Q_T, \\ -z_t - (1-ia)\Delta z + Rz = \Re(y)1_{\mathcal{O}} + f^1, & \text{in} \quad Q_T, \\ y = z = 0, & \text{on} \quad \Sigma_T, \\ y|_{t=0} = y_0, z|_{t=T} = 0, & \text{in} \quad \Omega, \end{cases}$$

satisfy $z|_{t=0} = 0$. The problem here is that the coupling term only depend on the real part of the state, and if we try to prove the observability inequality for the adjoint system using the same strategies of this paper, it turns difficult to eliminate the local term of the imaginary part (see (A.52)). The same question can be formulated for functionals of gradient type.

On the Nonlinear Terms

In this paper, we have applied an inverse mapping theorem to solve the nonlinear control problem, and we have obtained a local result. A natural problem that arises, is to know for which kinds of nonlinearities the problem can still be solved. That is, if $F : \mathbb{C} \to \mathbb{C}$ is nonlinear and continuous differentiable function, and if we consider the system

$$\begin{cases} y_t - (1+ia)\Delta y + F(y) = v1_\omega + f, & \text{in} \quad Q_T, \\ y = 0, & \text{on} \quad \Sigma_T, \\ y|_{t=0} = y_0 + \tau\widehat{y}_0, & \text{in} \quad \Omega, \end{cases} \tag{A.74}$$

can we prove Theorem A.3, where now y is the solution of (A.74)? For instance, if we consider a nonlinearity of the form $f(z) = |z|^4 z$, then in (A.73) we would need that $e^{s\beta^*}(\gamma^*)^{-\frac{7}{2}} y_i \in L^{10}(Q_T)$. But due to the lack of regularity (see Remark A.0.1), this regularity is not possible. A remedy for this is to consider the regularized functional (A.6) instead of (A.2) such as sufficiently regular source terms f. If this is so, it is expected that we can solve the isensitizing control problem considering nonlinearities of the form $g(z) = |z|^{2\alpha} z$.

Another interesting question is the global null controllability for nonlinearities with superlinear growth, when the nonlinear term satisfies the following growth condition

$$\lim_{|z|\to\infty} \frac{f(z)}{z \ln^\alpha |z|} = 0, \quad \alpha > 0.$$

Is it possible to solve the insensitizing control problem considered here for given values of α? Notice that, assuming conditions of this kind, the global null controllability for the heat equation holds (see [31]) when $\alpha < \frac{3}{2}$. The main difficulty here is that, when trying to apply a fixed point theorem in order to obtain global results, the associated linear equation has a potential which depends on space and time, then we cannot obtain an equation for $\Delta\psi$ (see (A.42)).

On the Existence of Solutions

We will show some results about the existence of solutions for system

$$\begin{cases} y_t-(1+ia)\Delta y+Ry=F^0, & \text{in } Q_T,\\ -z_t-(1-ia)\Delta z+Rz=\nabla\cdot(\nabla y 1_{\mathcal{O}})+F^1, & \text{in } Q_T,\\ y=z=0, & \text{on } \Sigma_T,\\ y|_{t=0}=y_0,\, z|_{t=T}=0, & \text{in } \Omega, \end{cases} \tag{A.75}$$

where (F^0,F^1) are given. The proofs in here can be adapted to prove existence of solutions for systems (A.5) and (A.7).

For $a\in\mathbb{R}$, define the linear unbounded operators

$$\begin{cases} D(L_0)=\{u\in H_0^1(\Omega);\Delta u\in H_0^1(\Omega)\},\\ L_0u=(1+ia)\Delta u-Ru\in H_0^1(\Omega), \end{cases} \quad \begin{cases} D(L_1)=H_0^1(\Omega),\\ L_1u=(1-ia)\Delta u-Ru\in H^{-1}(\Omega). \end{cases} \tag{A.76}$$

Both operators are m-dissipative with dense domains, therefore they generates the C_0 semigroups of contractions $\mathcal{T}_0$ and $\mathcal{T}_1$, respectively.

Here, we will need once again the spaces

$$Y_0=C([0,T];D(L_0))\cap C^1([0,T];H_0^1(\Omega))$$
$$\text{and}\quad Y_1=C([0,T];D(L_1))\cap C^1([0,T];H^{-1}(\Omega)). \tag{A.77}$$

We start proving the existence of *mild solutions* for system (A.75).

Proposition A.1 (Mild Solutions). *Assume that $y_0\in H_0^1(\Omega)$,*

$$F^0\in L^2(0,T;H_0^1(\Omega)) \quad \textit{and} \quad F^1\in L^2(0,T;H^{-1}(\Omega)).$$

Then, problem (A.75) possesses a unique mild solution $(y,z)\in C([0,T];H_0^1(\Omega))\times C([0,T];H^{-1}(\Omega))$ in the sense that

$$y(t)=\mathcal{T}_0(t)y_0+\int_0^t \mathcal{T}_0(t-s)F^0(s)\,ds, \tag{A.78}$$

and

$$z(t)=\int_t^T \mathcal{T}_1(s-t)\left(F^1+\nabla\cdot(\nabla y 1_{\mathcal{O}})\right)(s)\,ds. \tag{A.79}$$

Proof. The existence of $y\in C([0,T];H_0^1(\Omega))$ satisfying (A.78) follows directly from the fact that $(L_0,D(L_0))$ is a m-dissipative operator with dense domain (see Lemma 4.1.5 of [32]). Moreover, a simply computation shows that

$$\|\nabla\cdot(\nabla y 1_{\mathcal{O}})(t)-\nabla\cdot(\nabla y 1_{\mathcal{O}})(t_0)\|_{H^{-1}(\Omega)}$$
$$=\sup_{\zeta\in H_0^1(\Omega),\ \|\zeta\|_{H_0^1(\Omega)}=1}\left|\int_{\mathcal{O}}(\nabla y(t)-\nabla y(t_0))\cdot\nabla\zeta\,dx\right|$$
$$\leq\left(\int_\Omega|\nabla y(t)-\nabla y(t_0)|^2\,dx\right)^{\frac12}, \tag{A.80}$$

for every $t, t_0 \in [0,T]$, which means that $\nabla\cdot(\nabla y 1_{\mathcal{O}}) \in C([0,T];H^{-1}(\Omega))$. Now, using that $(L_1, D(L_1))$ is m-dissipative with dense domain and that $F^1+\nabla\cdot(\nabla y 1_{\mathcal{O}}) \in L^2(0,T;H^{-1}(\Omega))$, we have (again by Lemma 4.1.5 of [32]) the existence of $z \in C([0,T];H^{-1}(\Omega))$ satisfying (A.79). □

The next result is dedicated to prove the existence of *regular solutions* for (A.75).

Proposition A.2 (Regular Solutions)**.** *Assume that* $y_0 \in D(L_0)$*,*

$$F^0 \in C([0,T],H_0^1(\Omega)) \cap W^{1,1}(0,T;H_0^1(\Omega)),$$

and

$$F^1 \in C([0,T],H^{-1}(\Omega)) \cap W^{1,1}(0,T;H^{-1}(\Omega)).$$

Then, problem (A.75) *possesses a unique regular solution in the sense that*

$$\begin{cases} (y,z) \in Y_0 \times Y_1, \\ y_t - (1+ia)\Delta y + Ry = F^0, \\ -z_t - (1-ia)\Delta z + Rz = \nabla\cdot(\nabla y 1_{\mathcal{O}}) + F^1, \\ y|_{t=0} = y_0,\ z|_{t=T} = 0. \end{cases} \tag{A.81}$$

Proof. We start by using Proposition 4.1.6. in [32], and we get that the mild solution (A.78) satisfies

$$\begin{cases} y \in Y_0, \\ y_t - (1+ia)\Delta y + Ry = F^0, \\ y|_{t=0} = y_0. \end{cases} \tag{A.82}$$

Now, it is not difficult to see that

$$\int_0^T \|\nabla\cdot\nabla(y_t 1_{\mathcal{O}})(s)\|^2_{H^{-1}(\Omega)}\,ds = \int_0^T \left| \sup_{\zeta\in H_0^1(\Omega),\ \|\zeta\|_{H_0^1(\Omega)}=1} \int_{\mathcal{O}} \nabla y_t \cdot \nabla\zeta\,dx \right|^2 dt$$
$$\leq \iint_{Q_T} |\nabla y_t|^2\,dxdt,$$

and hence $\nabla\cdot(\nabla y 1_{\mathcal{O}}) \in C([0,T],H^{-1}(\Omega)) \cap W^{1,1}(0,T;H^{-1}(\Omega))$. Then, we apply again Proposition 4.1.6. of [32], and we get that the mild solution (A.79) satisfies

$$\begin{cases} z \in Y_1, \\ -z_t - (1-ia)\Delta z + Rz = \nabla\cdot(\nabla y 1_{\mathcal{O}}) + F^1, \\ z|_{t=T} = 0. \end{cases} \tag{A.83}$$

Combining (A.82) and (A.83), we obtain (A.81). □

Remark A.1. If we assume in Proposition A.1 that $y_0 \in L^2(\Omega)$ and that $(F^0, F^1) \in [L^2(0,T;H^{-1}(\Omega))]^2$, we still can prove the existence of mild solutions for (A.75). Indeed, let

$$\begin{cases} D(\tilde{L}_1) = H_0^1(\Omega), \\ \tilde{L}_1 u = (1+ia)\Delta u - Ru \in H^{-1}(\Omega), \end{cases} \tag{A.84}$$

m-dissipative with dense domain, and $\widetilde{\mathcal{T}_1} : H^{-1}(\Omega) \to H^{-1}(\Omega)$ the semigroup generated by it. Then, the mild solution

$$y(t) = \widetilde{\mathcal{T}_1}(t)y_0 + \int_0^t \widetilde{\mathcal{T}_1}(t-s)F^0(s)\,ds,$$

is such that $y \in C([0,T];H^{-1}(\Omega))$. Using a density argument from regular solutions, we can prove that $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, and hence

$$\nabla \cdot (\nabla y 1_{\mathcal{O}}) \in L^2(0,T;H^{-1}(\Omega)).$$

Going back to formula (A.79), we can use again a density argument from regular solutions, and we obtain that $z \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

Remark A.2. The proofs in here can be adapted to prove the existence of regular and mild solutions for system (A.7).

Regular Solutions: if we assume that $\psi_0 \in D(L_0)$, $g^0 \in C([0,T],H^{-1}(\Omega)) \cap W^{1,1}(0,T;H^{-1}(\Omega))$, and that $g^1 \in C([0,T],H_0^1(\Omega)) \cap W^{1,1}(0,T;H_0^1(\Omega))$, we can prove existence of regular solution for (A.7), in the sense that

$$\begin{cases} (\varphi,\psi) \in Y_1 \times Y_0, \\ -\varphi_t - (1-ia)\Delta\varphi + R\varphi = \nabla \cdot (\nabla\psi 1_{\mathcal{O}}) + g^0, \\ \psi_t - (1+ia)\Delta\psi + R\psi = g^1, \\ \psi|_{t=0} = \psi_0, \varphi|_{t=T} = 0. \end{cases} \tag{A.85}$$

Mild Solutions: if we assume that $\psi_0 \in L^2(\Omega)$ and $(g^0, g^1) \in [L^2(0,T;H^{-1}(\Omega))]^2$, there exist a unique mild solution $(\varphi,\psi) \in [C([0,T];H^{-1}(\Omega))]^2$ of (A.7). In the same way as in Remark A.1, we can prove that $(\varphi,\psi) \in [C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))]^2$.

In what follows, we will talk about *transposition solutions*, that are of particular interest for the purposes of this paper.

Let $y_0 \in L^2(\Omega)$ and $(F^0,F^1) \in [L^2(0,T;H^{-1}(\Omega))]^2$. We say that a pair $(y,z) \in L^2(0,T;H_0^1(\Omega)) \times L^2(Q_T)$ is a solution in the transposition sense of (A.75), if it satisfies

$$\begin{cases} \displaystyle\int_0^T \langle G^0, y\rangle_{H^{-1}H_0^1}\,dt = \Re \int_\Omega \varphi(0)\overline{y_0}\,dx + \int_0^T \langle F^0,\varphi\rangle_{H^{-1}H_0^1}\,dt, \\ \displaystyle\Re \iint_{Q_T} G^1 \bar{z}\,dxdt = -\Re \int_0^T\!\!\int_{\mathcal{O}} \nabla y \cdot \overline{\nabla\psi}\,dxdt + \int_0^T \langle F^1,\psi\rangle_{H^{-1}H_0^1}\,dt, \end{cases} \tag{A.86}$$

for every $(G^0, G^1) \in L^2(0,T; H^{-1}(\Omega)) \times L^2(Q_T)$, where (φ, ψ) is the solution of

$$\begin{cases} -\varphi_t - (1-ia)\Delta\varphi + R\varphi = G^0 & \text{in} \quad Q_T, \\ \psi_t - (1+ia)\Delta\psi + R\psi = G^1 & \text{in} \quad Q_T, \\ \varphi = \psi = 0 & \text{on} \quad \Sigma_T, \\ \psi|_{t=0} = 0, \varphi|_{t=T} = 0 & \text{in} \quad \Omega. \end{cases} \tag{A.87}$$

We have the following result about the existence and uniqueness of transposition solutions.

Proposition A.3 (Transposition Solution)**.** *For $(F^0, F^1) \in [L^2(0,T; H^{-1}(\Omega))]^2$ and $y_0 \in L^2(\Omega)$, there exists a unique $(y, z) \in [L^2(0,T; H_0^1(\Omega))]^2$ satisfying* (A.86) *for every $(G^0, G^1) \in L^2(0,T; H^{-1}(\Omega)) \times L^2(Q_T)$, where (φ, ψ) is solution of* (A.87).

Proof. Let $\Phi_1 : L^2(0,T; H_0^1(\Omega)) \to \mathbb{R}$ the operator

$$\Phi_1(h^0) = \Re \int_\Omega \varphi(0)\overline{y_0}\, dx + \int_0^T \langle F^0, \varphi \rangle_{H^{-1}H_0^1}\, dt,$$

where φ satisfies the first equation of (A.87) for $G^0 = -\Delta h^0 \in L^2(0,T; H^{-1}(\Omega))$. From energy estimates, it is easy to see that Φ_1 is continuous. Then, from Lax Milgram theorem, there exists $y \in L^2(0,T; H_0^1(\Omega))$ such that

$$\int_0^T \langle G^0, y \rangle_{H^{-1}H_0^1}\, dt = \Re \iint_{Q_T} \nabla h^0 \cdot \nabla \bar{y}\, dxdt = \Phi_1(h^0),$$

for every $G^0 \in H^{-1}(\Omega)$, where $-\Delta h^0 = G^0$. The existence of $z \in L^2(Q_T)$ satisfying the second equation of (A.86) follows in a completely analogous way, since the linear form

$$\Phi_2(G^1) = -\Re \int_0^T\!\!\int_{\mathcal{O}} \nabla y \cdot \overline{\nabla\psi}\, dxdt + \int_0^T \langle F^1, \psi \rangle_{H^{-1}H_0^1}\, dt,$$

is continuous in $L^2(Q_T)$.

To prove that $z \in L^2(0,T; H_0^1(\Omega))$, we can proceed in the following way. First, we take sequences of regular data such that $y_0^n \to y_0$ in $L^2(\Omega)$ and $(F_n^0, F_n^1) \to (F^0, F^1)$ in $L^2(0,T; H^{-1}(\Omega)) \times L^2(Q_T)$. We show that the regular solutions (y_n, z_n) for (A.75) (whose existence is given in Proposition A.2) with inital data y_0^n and (F_n^0, F_n^1) on the right-hand side, is also a solution in the transposition sense; moreover, it is bounded in $[L^2(0,T; H_0^1(\Omega))]^2$. Hence, in the limit, we obtain that $(y, z) \in [L^2(0,T; H_0^1(\Omega))]^2$.

Now, let us prove that the solution (y, z) is unique. If $(\hat{y}, \hat{z})$ is another solution, then

$$\Re \int_0^T \langle G^0, y - \hat{y} \rangle_{H^{-1}H_0^1}\, dt = 0, \quad \text{for all} \quad G^0 \in L^2(0,T; H^{-1}(\Omega)),$$

and

$$\Re \iint_{Q_T} G^1(z - \hat{z})\, dxdt = -\Re \int_0^T\!\!\int_{\mathcal{O}} (\nabla y - \nabla \hat{y}) \cdot \overline{\nabla\psi}\, dxdt, \quad \text{for all} \quad G^1 \in L^2(Q_T).$$

Hence, $y = \hat{y}$ and $z = \hat{z}$. □

Null condition on the initial data

In this section we justify the condition $y^0 \equiv 0$ in order to obtain the existence of insensitizing controls. We now prove that one cannot expect to obtain insensitizing controls for all given data $y^0 \in L^2(\Omega)$, even in the simplest case, dealing with the following linear version of (A.1),

$$\begin{cases} y_t - (1+ia)\Delta y = v1_\omega + f, & \text{in} \quad Q_T, \\ y = 0, & \text{on} \quad \Sigma_T, \\ y(0) = y_0 + \tau\widehat{y}_0, & \text{in} \quad \Omega. \end{cases} \tag{A.88}$$

More precisely, we are going to prove the following result

Theorem A.13. *Assume that $\Omega\backslash\overline{\omega} \neq \emptyset$. There exists a data $y^0 \in L^2(\Omega)$ such that for all control $v \in L^2(Q)$, we have $z(0) \neq 0$, which means that the functional in (A.3) cannot be insensitized.*

The main ideia of the proof is to construct a concrete example by means of the fundamental solution and prove that it does not satisfies the null controlability that we stated before.

Proof. Supose that for all $y^0 \in L^2(\Omega)$ there exists $v \in L^2(Q)$ and $C > 0$ satisfying

$$||v||_{L^2(Q)} \leq C||y^0||_{L^2(\Omega)}, \tag{A.89}$$

and such that the solutions (w, z) of the following systems

$$\begin{cases} w_t - (1+ia)\Delta w = v1_\omega, & \text{in} \quad Q_T, \\ w = 0, & \text{on} \quad \Sigma_T, \\ w(0) = y_0, & \text{in} \quad \Omega. \end{cases} \tag{A.90}$$

$$\begin{cases} -z_t - (1-ia)\Delta z = \nabla \cdot (\nabla w 1_{\mathcal{O}}), & \text{in} \quad Q_T, \\ z = 0, & \text{on} \quad \Sigma_T, \\ z(T) = 0, & \text{in} \quad \Omega. \end{cases} \tag{A.91}$$

satisfies

$$z(0) = 0 \text{ in } \Omega. \tag{A.92}$$

Now, consider (ψ, φ) solutions of the respective adjoint systems,

$$\begin{cases} \psi_t - (1+ia)\Delta \psi = 0, & \text{in} \quad Q_T, \\ \psi = 0, & \text{on} \quad \Sigma_T, \\ \psi(0) = \psi_0, & \text{in} \quad \Omega. \end{cases} \tag{A.93}$$

$$\begin{cases} -\varphi_t - (1-ia)\Delta \varphi = \nabla \cdot (\nabla \psi 1_{\mathcal{O}}), & \text{in} \quad Q_T, \\ \varphi = 0, & \text{on} \quad \Sigma_T, \\ \varphi(T) = 0, & \text{in} \quad \Omega. \end{cases} \tag{A.94}$$

Multiplying $(A.89)_1$ by $\overline{\varphi}$ and $(A.90)_1$ by $\overline{\psi}$, integrating in Q and using (A.88), one gets the following inequality

$$\int_\Omega |\varphi(0)|^2 dx \le C \iint_{q_T} |\varphi|^2 dxdt. \tag{A.95}$$

Conversely, we can prove that if the observability inequality (A.94) holds, then we can obtain $z(0) = 0$ in Ω with the control v satisfying (A.88), this means that condition (A.94) is equivalent to the null controllability for (A.88) - (A.89) when $y^0 \in L^2(\Omega)$. We now prove that the condition (A.94) does not holds even in the simplest case when $\mathcal{O} = \Omega$ and $\Omega \backslash \overline{\omega} \neq \emptyset$. This will be done is three steps.

Step 1. We study the following auxiliary cascade system in $\mathbb{R}^n$,

$$\begin{cases} -\theta_t - (1 - ia)\Delta\theta = \Delta\sigma, & \text{in} \quad Q_T, \\ \theta(T) = 0, & \text{in} \quad \Omega. \end{cases} \tag{A.96}$$

$$\begin{cases} \sigma_t - (1 + ia)\Delta\sigma = 0, & \text{in} \quad Q_T, \\ \sigma(0) = \Delta\delta_0, & \text{in} \quad \Omega. \end{cases} \tag{A.97}$$

Here, δ_0 denotes the Dirac measure at zero, i.e., for all $f \in C_0^\infty(\mathbb{R}^n)$, $\delta_0(f) = f(0)$ and $\Delta\delta_0(f) = \Delta f(0)$. Let $G(x,t)$ be the fundamental solution for the homogenous and linear ginzburg-landau equation, i.e.,

$$G(x,t) = [4(1+ia)\pi t]^{-n/2} e^{-\frac{|x|^2}{4(1+ia)t}}, \tag{A.98}$$

By properties of the fundamental solution and convolution, since $\sigma(0) = \Delta\delta_0$, we get that $\sigma(x,t) = \Delta G(x,t)$ since one can obtain solution σ by means of a convolution of G by $\Delta\delta_0$. Now, variation of parameters formula allow us to write

$$\theta(t) = \int_t^T G(s-t) * \Delta^2 G(s) ds = \Delta^2 \int_t^T G(s-t) * G(s) ds, \tag{A.99}$$

We now compute the convolution apearing in the right-hand side of (118).

$$\begin{aligned} G(s-t) * G(s) &= \int_{\mathbb{R}^n} G(s-t, x-y) G(s,y) dy \\ &= \int_{\mathbb{R}^n} [4(1+ia)\pi(s-t)]^{-n/2} e^{-\frac{|x-y|^2}{4(1+ia)(s-t)}} \cdot [4(1+ia)\pi s]^{-n/2} e^{-\frac{|y|^2}{4(1+ia)s}} dy \\ &= [4\pi(1+ia)(s-t)4\pi(1+ia)s]^{-n/2} e^{-\frac{|x|^2}{4(1+ia)(s-t)}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2 - 2xy}{4(1+ia)(s-t)} - \frac{|y|^2}{4(1+ia)s}} dy. \end{aligned} \tag{A.100}$$

We now deal with the last integral in (119). The case with $n = 1$ will be enough, since the $\mathbb{R}^n$ case is an imediate consequence.

$$\begin{aligned}
\int_{\mathbb{R}} e^{-\frac{|y|^2-2xy}{4(1+ia)(s-t)}-\frac{|y|^2}{4(1+ia)s}}dy &= \int_{\mathbb{R}} e^{\frac{s(-|y|^2+2xy)+(s-t)(-|y|^2)}{4(1+ia)(s-t)s}}dy = \int_{\mathbb{R}} e^{\frac{-s|y|^2+2sxy-s|y|^2+t|y|^2}{4(1+ia)(s-t)s}}dy \\
&= \int_{\mathbb{R}} e^{\frac{(t-2s)|y|^2+2sxy}{4(1+ia)(s-t)s}}dy = \int_{\mathbb{R}} e^{\left(\frac{t-2s}{4(1+ia)(s-t)s}\right)\left(y^2+2\cdot y\cdot\frac{sx}{t-2s}+\left(\frac{sx}{t-2s}\right)^2-\left(\frac{sx}{t-2s}\right)^2\right)}dy \\
&= e^{\frac{2s-t}{4(1+ia)(s-t)s}\left(\frac{sx}{t-2s}\right)^2}\int_{\mathbb{R}} e^{\left(\frac{t-2s}{4(1+ia)(s-t)s}\right)\left(y+\frac{sx}{t-2s}\right)^2}dy \\
&= e^{-\frac{s|x|^2}{4(1+ia)(s-t)(t-2s)}}\int_{\mathbb{R}} e^{\left(\frac{t-2s}{4(1+ia)(s-t)s}\right)\left(y+\frac{sx}{t-2s}\right)^2}dy \\
&= e^{-\frac{s|x|^2}{4(1+ia)(s-t)(t-2s)}}\int_{\mathbb{R}} e^{\left(\frac{t-2s}{4(1+ia)(s-t)s}\right)\alpha^2}d\alpha \\
&= e^{-\frac{s|x|^2}{4(1+ia)(s-t)(t-2s)}}\int_{\mathbb{R}}\left(\frac{2s-t}{4(1+ia)(s-t)s}\right)^{-1/2}e^{-\beta^2}d\beta \\
&= \left(\frac{2s-t}{4(1+ia)\pi(s-t)s}\right)^{-1/2}e^{-\frac{s|x|^2}{4(1+ia)(s-t)(t-2s)}}. \qquad (A.101)
\end{aligned}$$

Combining (A.99) and (A.100), we get that

$$\begin{aligned}
&\int_{\mathbb{R}^n} G(s-t,x-y)G(s,y)dy \\
&= \left(\frac{4\pi(1+ia)(s-t)4\pi(1+ia)s(2s-t)}{4(1+ia)\pi(s-t)s}\right)^{-n/2}e^{-\frac{|x|^2}{4(1+ia)(s-t)}-\frac{\sigma|x|^2}{4(1+ia)(s-t)(t-2s)}} \\
&= [4(1+ia)\pi(2s-t)]^{-n/2}e^{-\frac{|x|^2}{4(1+ia)(2s-t)}} = G(2s-t,x). \qquad (A.102)
\end{aligned}$$

Combining (A.101) with (A.90) we get that $\theta(t) = \Delta^2\int_t^T G(2s-t)ds$. We now prove that

1. $\theta(0) \notin L^2_{loc}(\mathbb{R}^n)$,
2. $\theta \in L^2(\omega\times(0,T))$ if $0 \notin \omega$,

which means that the inequality

$$\int_0^T |\theta|^2 dxdt \geq C||\theta(0)||^2_{L^2(\mathbb{R}^n)}, \qquad (A.103)$$

is false. Moreover, we prove that $D^m\theta \in L^2(\omega\times(0,T))$ if $0\notin\omega$, for all $m\geq 1$, which means that the inequality

$$\iint_{q_T} |D^m\theta|^2 dxdt \geq C||\theta(0)||^2_{L^2(\mathbb{R}^n)}, \qquad (A.104)$$

is also false. Let us prove 1. Note that

$$\theta(0) = \int_0^T \Delta^2 G(2s)ds = \int_0^T h(s)ds, \qquad (A.105)$$

with $h(s) = \Delta^2 G(2s)$ solution for

$$h_s = 4\Delta^2 h. \qquad (A.106)$$

Note that $\theta(0) = \int_0^T hds$ satisfies the following elipitic equation

$$4\Delta^2\theta(0) = 4\Delta^2 \int_0^T hds = \int_0^T 4\Delta^2 h(s)ds = \int_0^T h_s ds = h(T) - h(0)$$
$$= \Delta^2(G(2T) - \delta_0) \quad \text{(A.107)}$$

Therefore,

$$\theta(0) = \frac{G(2T) - \delta_0}{4} \quad \text{(A.108)}$$

This proves that $\theta(0) \notin L^2_{loc}(\mathbb{R}^n)$, since G is smooth but $\delta_0 \notin L^2_{loc}(\mathbb{R}^n)$. Now, let us prove 2. Note that

$$\begin{aligned}
\theta(t,x) &= C\int_T^{2T-t} \Delta^2\left(s^{-n/2}e^{-\frac{-|x|^2}{4(1+ia)s}}\right)ds \\
&= C\int_T^{2T-t} \Delta\left(\frac{4s^{-n/2}e^{-\frac{|x|^2}{4(1+ia)s}}|x|^2}{[4(1+ia)s]^2} - \frac{2s^{-n/2}e^{-\frac{|x|^2}{4(1+ia)s}}}{4(1+ia)s}\right)ds \\
&= C\int_T^{2T-t} \Delta\left(\frac{4s^{-n/2}e^{-\frac{|x|^2}{4(1+ia)s}}|x|^2}{[4(1+ia)s]^2}\right)ds - C\int_T^{2T} \Delta\left(\frac{2s^{-n/2}e^{-\frac{|x|^2}{4(1+ia)s}}}{4(1+ia)s}\right)ds \\
&= C\int_T^{2T-t} \frac{4s^{-n/2}}{[4(1+ia)s]^2}\Bigg[\Delta\left(e^{-\frac{|x|^2}{4(1+ia)s}}\right)|x|^2 + 2\nabla\left(e^{-\frac{|x|^2}{4(1+ia)s}}\right)\cdot\nabla(|x|^2) \\
&\qquad + e^{-\frac{|x|^2}{4(1+ia)s}}\Delta(|x|^2)\Bigg]ds \\
&\qquad - C\int_T^{2T-t} \frac{2}{4(1+ia)s}\Delta\left(s^{-n/2}e^{-\frac{|x|^2}{4(1+ia)s}}\right)ds \\
&= C\int_T^{2T-t} \frac{4s^{-n/2}}{[4(1+ia)s]^2}e^{-\frac{|x|^2}{4(1+ia)s}}\left(\frac{4|x|^2}{[4(1+ia)s]^2} - \frac{12|x|^2}{4(1+ia)s} + 2n + 1\right)ds. \quad \text{(A.109)}
\end{aligned}$$

Since $-\frac{|x|^2}{4(1+ia)s} = -\frac{|x|^2}{4(1+a^2)s} + \frac{a|x|^2}{4(1+a^2)s}i$, we obtain the following estimate

$$|\theta(x,t)| \le C\int_T^{2T-t} s^{-(n+8)/2}e^{-\frac{|x|^2}{4(1+a^2)s}}|x|^4 ds + \int_T^{2T-t} s^{-(n+6)/2}e^{-\frac{|x|^2}{4(1+a^2)s}}ds$$
$$+ \int_T^{2T-t} s^{-(n+4)/2}e^{-\frac{|x|^2}{4(1+a^2)s}}(2n+1)ds. \quad \text{(A.110)}$$

Using diferential calculus methods, we obtain that

$$|\theta(x,t)| \le C|x|^{-(n+4)}(T-t). \quad \text{(A.111)}$$

Since $0 \notin \omega$, there exists $r > 0$ such that $w \subset \mathbb{R}^n \backslash B_r$ where B_r denotes the r radius ball with center in 0. This give us

$$\iint_{q_T} |\theta(x,t)|^2 dxdt \le C\int_0^T \int_{\mathbb{R}^n\backslash B_r} |x|^{-2(n+4)}(T-t)^2 dxdt \le Cr^{-2(n+4)}. \quad \text{(A.112)}$$

one can prove (A.103) as proceeding as before.

Step 2. We now deal with the case where Ω is a bounded domain. Without generality loss, we can supose that $0 \in \Omega$ but $0 \notin \omega$. We now considere the following system

$$\begin{cases} \psi_t - (1+ia)\Delta\psi = 0, & \text{in} \quad Q_T, \\ \psi = 0, & \text{on} \quad \Sigma_T, \\ \psi(0) = \Delta\delta_0, & \text{in} \quad \Omega. \end{cases} \tag{A.113}$$

$$\begin{cases} -\varphi_t - (1-ia)\Delta\varphi = \nabla\cdot(\nabla\psi 1_{\mathcal{O}}), & \text{in} \quad Q_T, \\ \varphi = 0, & \text{on} \quad \Sigma_T, \\ \varphi(T) = 0, & \text{in} \quad \Omega. \end{cases} \tag{A.114}$$

By putting $\phi = \sigma - \psi$ and $\zeta = \theta - \varphi$, where σ, ψ, θ and φ are solutions for (A.96), (A.92) and (A.93) respectively. Note now that the new variables ϕ and ζ satisfies the following system

$$\begin{cases} \phi_t - (1+ia)\Delta\phi = 0, & \text{in} \quad Q_T, \\ \phi = \sigma, & \text{on} \quad \Sigma_T, \\ \phi(0) = 0, & \text{in} \quad \Omega. \end{cases} \tag{A.115}$$

$$\begin{cases} -\zeta_t - (1-ia)\Delta\zeta = \Delta\phi, & \text{in} \quad Q_T, \\ \zeta = \theta, & \text{on} \quad \Sigma_T, \\ \zeta(T) = 0, & \text{in} \quad \Omega. \end{cases} \tag{A.116}$$

Note that $\sigma|_\Sigma = \Delta G|_\Sigma$ and G is smooth, therefore ϕ is smooth. Since $\theta|_\Sigma$ is also smooth, we get that ζ is also smooth. Therefore φ satisfies conditions 1. and 2., since $\varphi = \theta - \zeta$.

Step 3. Define $\psi_\epsilon(0) = \psi(\epsilon)$ and let $(\psi_\epsilon, \varphi_\epsilon)$ be the corresponding solutions for system (A.112) and (A.113). Proceeding as steps 1 and 2 before, it is straightforward to see that

$$\iint_{q_T} |\varphi_\epsilon|^2 dxdt \leq C, \tag{A.117}$$

with C independent of ϵ. By the other hand,

$$\lim_{\epsilon\to 0^+} \frac{\int_\Omega |\varphi_\epsilon(0)|^2}{\iint_{q_T} |\varphi_\epsilon|^2 dxdt} = \infty, \tag{A.118}$$

since, $\varphi_\epsilon(0) = \varphi(\epsilon) = \theta(\epsilon) - \zeta(\epsilon)$. This proves that (A.94) does not holds. □

Conclusion of Appendix A

In this paper, we have been able to prove the existence of controls which insensitize a functional depending on the gradient of solutions for a nonlinear Ginzburg Landau equation with partially unknown data. This problem was solved by first writing its equivalence with a partial null controllability problem for a nonlinear cascade system with second order coupling coefficients. To solve this problem we first studied a linearized system around zero and to deal with the coupling terms, we proved a new Carleman estimate for the adjoint state. Then, we have applied successfully an inverse mapping theorem and obtained a local partial null controllability result for the optimality system, and this result is completely equivalent to the existence of insensitizing controls. We have also formulated some open questions that arise naturally. They remain open at present and will be considered in a forthcoming paper. We also indicate that the techniques of this paper can be of help to other kinds of problems, like for multiobjective control problems or for the controllability of systems of Schrödinger type.

Acknowledgment: The authors would like to thank professor Diego A. Souza, from Universidade Federal de Pernambuco (Brazil), to have participated in many discussions concerning this problem, the authors are in debt for his support.

APÊNDICE B – UNIFORM NULL CONTROLLABILITY FOR FAST DIFFUSION PARABOLIC SYSTEMS WITH FIRST ORDER COUPLING TERMS

Abstract

In this paper, we prove a uniform null controllability result for a family of coupled linear parabolic systems which one of the equations degenerates into an elliptic one. More precisely, we extend the results of *Chaves-Silva et al.* by considering parabolic systems with first order coupling terms. The strategy is to prove a suitable observability inequality with observability constant uniformly bounded with resperct to the degenerating parameter. As a consequence, when the parameter goes to zero, we obtain the null controllability for the limiting system. Also, our proof simplifies the one given in *Chaves-Silva et al.* for the case of zero order coupling terms.

Keywords: Uniform controllability, Null controllability, Carleman estimates, parabolic systems.

In many applications, it turns out to be very difficult, or even impossible, to analyze and implement complete systems. Thus, it becomes important to seek for simplified or approximated models that keep the main features of the problem under consideration. From the control point of view, this leads to the so-called *uniform controllability problem*, in which one aims to understand what happens with the classical control properties, such as controllability and observability, when a system is approximated by a simplified one.

This paper deals with with the uniform null controllability problem for a family of parabolic systems with distributed controls. More precisely, we consider systems of two parabolic equations with first order coupling terms which one of the equations is degenerating into an elliptic one, and we show that it is possible to drive each solution to zero by the action of controls which are uniformly bounded with respect to the degenerating parameter. In particular, our result extends those in [27] where the authors solved an uniform controllability problem for coupled parabolic systems with zero order coupling terms. Moreover, we simplify the proofs given therein. Our main strategy is the same as that in [27], which consists in obtaining an uniform observability inequality for the associated adjoint system, that is, the cost of observability remains bounded when the degenerative parameter goes to zero. This will follow from suitable global Carleman estimates combined with energy inequalities.

As far as we know, the notion of uniform controllability emerged with Kalman in [7] in the context of time varying systems of finite dimension. Since then, there have been great interest in uniform controllability problems for systems of PDEs. For instance, in [25] the authors consider the case of the heat equation being approximated by a family of hyperbolic equations. In [21], a transport equation is approximated by a family of parabolic equations. In [33], the authors study the controllability of the Keller-Segel parabolic system by approximating its parabolic-elliptic counterpart. In [34], the authors consider a family of KdV equations degenerating into a transport equation and obtain a boundary uniform controllability result.

This work is divided as follows: In Section B, we present in more details the problem and its difficulties. In Section B, we discuss some strategies on how to deal with first order terms. In Section B, by energy estimates, we prove the uniform null controllability result for the family of systems. In Section B, we give some related open questions. Finally, in Section B, we make some conclusions.

Statement of the problem

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded connected open set with regular boundary $\partial\Omega$. Let $T > 0$ and let ω_1 and ω_2 be two nonempty subsets of Ω called *control domains*. We will use the notation $Q_T = \Omega\times]0,T[$, $\Sigma_T = \partial\Omega\times]0,T[$ and for every $\omega \subset \Omega$ we define $\omega_T := \omega\times]0,T[$.

This paper is dedicated to the study of a uniform null controllability problem for linear systems of parabolic equations with first order coupling terms, in which one of the equations is degenerating into an elliptic one as a given parameter ϵ, a diffusion parameter, goes to zero.

For each $\epsilon > 0$, consider the following system of parabolic equations

$$\begin{cases} y_t^\epsilon - \Delta y^\epsilon = a_{1,1}y^\epsilon + A_{1,1}\cdot\nabla y^\epsilon + a_{1,2}z^\epsilon + A_{1,2}\cdot\nabla z^\epsilon + f_1^\epsilon 1_{\omega_1}, & \text{in } Q_T,\\ \epsilon z_t^\epsilon - \Delta z^\epsilon = a_{2,1}z^\epsilon + A_{2,1}\cdot\nabla z^\epsilon + a_{2,2}y^\epsilon + A_{2,2}\cdot\nabla y^\epsilon + f_2^\epsilon 1_{\omega_2}, & \text{in } Q_T,\\ y^\epsilon = z^\epsilon = 0, & \text{on } \Sigma_T,\\ y^\epsilon|_{t=0} = y_0, z^\epsilon|_{t=0} = z_0, & \text{in } \Omega, \end{cases} \tag{B.1}$$

with $a_{i,j}$ in $L^\infty(Q_T;\mathbb{R})$ and $A_{i,j}$ in $L^\infty(Q_T;\mathbb{R}^N)$, for $i,j = 1,2$. It is well known that, for each $\epsilon > 0$ fixed and (y_0, z_0) in $[L^2(\Omega)]^2$, system (B.1) is null controllable in the sense that there exists functions $(f_1^\epsilon, f_2^\epsilon) \in [L^2(Q_T)]^2$ such that $y^\epsilon(\cdot,T) = z^\epsilon(\cdot,T) = 0$ (see [29, 35]). Here, we want to know whether it is possible to find controls $(f_1^\epsilon, f_2^\epsilon)$ which are uniform with respect to ϵ in the sense that its costs is uniformly bounded with respect to this parameter. It is important to mention that, in [27], *Chaves-Silva et al.* already proved the existence of uniform controls for systems like (B.1) but for zero order couplings, that is, assuming that $A_{1,1} = A_{1,2} = A_{2,1} = A_{2,2} = 0$. In this work we extend their results for the case of first order couplings. Moreover, the proofs contained here are much simpler than the ones in [27], which can be of use when dealing with more complex problems.

The reason to study controllability properties for system (B.1) is that, in many applications, the it can be seen as an approximated version of the following parabolic-elliptic system

$$\begin{cases} y_t - \Delta y = a_{1,1}y + A_{1,1}\cdot\nabla y + a_{1,2}z + A_{1,2}\cdot\nabla z + f_1 1_{\omega_1}, & \text{in } Q_T \\ -\Delta z = a_{2,1}z + A_{2,1}\cdot\nabla z + a_{2,2}y + A_{2,2}\cdot\nabla y + f_2 1_{\omega_2}, & \text{in } Q_T, \\ y = z = 0, & \text{on } \Sigma_T, \\ y|_{t=0} = y_0, & \text{in } \Omega, \end{cases} \tag{B.2}$$

which can be seen, for instance, as a model for aggregation phenomena or also for chemical reactions of substances with different concentrations (see [36, 37]).

The strategy of controlling a system by approximating it by a family of systems having different properties has already been considered in past years. For instance, one can see [25] where the authors studied the case of a damped wave equation degenerating into a heat equation, but with controls not uniform with respect to the parameter. In [21], the authors studied the uniform null controllability for a transport-diffusion equation degenerating into a transport equation, with distributed and boundary controls, assuming some geometrical conditions and sufficiently large time. More connected to the problem considered on this paper, we have the results of *Chaves-Silva et al.* in [27], where they proved the uniform null controllability for system (B.1) when only zero order coupling coefficients are considered. See also [33], where the authors consider the case of a nonlinear parabolic-elliptic system as a simplification of a coupled parabolic system.

It is important to remark that the controllability of a family of systems does not imply the controllabiity for limiting system, see [38] for some examples. We also remark that even non uniform controllability results for systems of parabolic equations are difficult to obtain. For instance, in [39] the authors analyse the null and approximate controllability for linear coupled parabolic equations with zero and first order coupling terms, and many conditions for the coefficients are imposed. Moreover, it is not clear that the result in [39], applied for system (B.1), produces uniform controls.

In this paper, we study the case where only one control force is active, that is, in system (B.1) we have $f_1^\epsilon \equiv 0$ or $f_2^\epsilon \equiv 0$, not simultaneously. In the particular case where $f_2^\epsilon \equiv 0$, it is standard that the null controllability of system (B.1) is equivalent to the following observability inequality

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \leq C_* \int_0^T\!\!\int_{\omega_1} |\varphi|^2 dxdt, \tag{B.3}$$

for every (φ, ξ) solution of

$$\begin{cases} -\varphi_t - \Delta\varphi = (a_{1,1} - \operatorname{div} A_{1,1})\varphi + (a_{2,2} - \operatorname{div} A_{2,2})\xi - A_{1,1} \cdot \nabla\varphi - A_{2,2} \cdot \nabla\xi, & \text{in } Q_T, \\ -\epsilon\xi_t - \Delta\xi = (a_{1,2} - \operatorname{div} A_{1,2})\varphi + (a_{2,1} - \operatorname{div} A_{2,1})\xi - A_{1,2} \cdot \nabla\varphi - A_{2,1} \cdot \nabla\xi, & \text{in } Q_T, \\ \varphi = \xi = 0, & \text{on } \Sigma_T, \\ \varphi|_{t=T} = \varphi_T, \xi|_{t=T} = \xi_T, & \text{in } \Omega. \end{cases} \tag{B.4}$$

In this way, in order to prove an uniform null controllability result, we have to prove (B.3) in such a way that C_* does not depend on ϵ.

Analogously, if the control active is on the second equation ($f_1^\epsilon \equiv 0$), then the observability inequality we need to prove assumes the form

$$||\varphi(0)||^2_{L^2(\Omega)} + \epsilon||\xi(0)||^2_{L^2(\Omega)} \le C_{**} \int_0^T\int_{\omega_2} |\xi|^2 dxdt, \tag{B.5}$$

for every (φ, ψ) solution of (B.4), where C_{**} does not depend on ϵ.

For each $\epsilon > 0$, one can find several works in the literature which prove estimates (B.3) and (B.5), see for instance [22, 39, 40]. Nevertheless, if one follows [22, 40] it is not difficult to see that C_* is of order ϵ^{-1} while C_{**} does not depend on ϵ. On the other hand, it is not clear if the arguments of [39] can be used to obtain an uniform bound to C_*. Thus, in this paper, we consider only the case of a control in the first equation.

The main result of this paper is the following:

Theorem B.1. *Let $(y_0, z_0) \in [L^2(\Omega)]^2$, $\omega_1 \subset \Omega$ an open subset and let μ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and $\epsilon > 0$ sufficiently small. For $i, j = 1, 2$, let $a_{i,j} \in C^2(\bar{Q}_T; \mathbb{R})$ and $A_{i,j} \in C^2(\bar{Q}_T; \mathbb{R}^N)$, such that*

$$a_{2,1} - \frac{\operatorname{div} A_{2,1}}{2} < \mu_1. \tag{B.6}$$

Moreover, suppose that there is a $C > 0$ such that one of the conditions hold:

(A) $(\operatorname{supp} A_{2,2}) \cap \omega_1 = \emptyset$ *and* $a_{2,2} \ge C > 0$ *in* $\overline{\omega_1} \times [0, T]$;

(B) $N = 1$, $\omega_1 \cap \partial\Omega \neq \emptyset$, $\operatorname{supp}(a_{2,2} - \partial_x A_{2,2}) \cap \omega_1 = \emptyset$ *and* $A_{2,2} \ge C > 0$ *in* $\overline{\omega_1} \times [0, T]$.

Then, there exists a control f_1^ϵ such that the solution (y^ϵ, z^ϵ) of (B.1) *satisfies $y^\epsilon(\cdot, T) = z^\epsilon(\cdot, T) = 0$ and having the following estimate*

$$||f_1^\epsilon 1_{\omega_1}||^2_{L^2(Q_T)} \le C \left(||y_0||^2_{L^2(\Omega)} + \epsilon||z_0||^2_{L^2(\Omega)}\right), \tag{B.7}$$

where the constant C does not depends on ϵ.

An imediate consequence of Theorem B.1 is the null controllability of the system (B.2) with only one control f_1 in the first equation. Indeed, since f_1^ϵ are uniformly bounded with respect to ϵ, it weakly converges in $L^2(\omega_1 \times]0, T[)$ to a function f_1. The corresponding solutions (y^ϵ, z^ϵ) of (B.1) are bounded in $[L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))]^2$ and converges to a solution (y, z) of (B.2), with control f_1 and such that $y(T) = z(T) = 0$.

Carleman Estimates

In this section, we are going to prove some Carleman estimates for the adjoint system (B.4), which will be the main tool to prove inequality (B.3). It states that there exists weight functions ρ and ρ_1, both converging to zero when $t \to 0^+$ or $t \to T^-$, and a constant C_* such that

$$\iint_{Q_T} \rho^2(|\varphi|^2 + |\xi|^2)\, dxdt \leq C_* \int_0^T\!\!\int_{\omega_1} \rho_1^2 |\varphi|^2\, dxdt,$$

for every (φ, ξ) solution of (B.4). The difficulty here is that we obtain this inequality in such a way that C_* does not depend on ϵ. In Section B, we will see that combining this Carleman estimate with energy estimates for system (B.2) we are able to prove the observability inequality (B.3).

As usual in this kind of approach, we define some weight functions whose existence can be found in [29]. Let $\omega_0 \subset \omega_1 \subset \Omega$ an open subset and $\psi \in C^2(\overline{\Omega})$ such that

$$\psi(x) > 0 \text{ in } \Omega,\ \psi \equiv 0 \text{ on } \partial\Omega \text{ and } |\nabla\psi| \geq C > 0 \text{ for all } x \in \overline{\Omega \backslash \omega_0}. \tag{B.8}$$

Given a positive real number λ, we define

$$\begin{aligned} \phi(x,t) &= \frac{e^{\lambda\psi(x)}}{t(T-t)}; & \alpha(x,t) &= \frac{e^{\lambda\psi(x)} - e^{2\lambda||\psi||_\infty}}{t(T-t)}; \\ \widehat{\phi}(t) &= \min_{x\in\overline{\Omega}} \phi(x,t); & \widehat{\alpha}(t) &= \min_{x\in\overline{\Omega}} \alpha(x,t); \\ \phi^*(t) &= \max_{x\in\overline{\Omega}} \phi(x,t); & \alpha^*(t) &= \max_{x\in\overline{\Omega}} \alpha(x,t). \end{aligned} \tag{B.9}$$

For $i, j = 1, ..., N$, the following estimates for the weight functions in (B.9) holds in Q_T:

$$\begin{aligned} |\alpha_t| + |\xi_t| &\leq CT\xi^2, \\ |\partial_{x_i}\alpha| + |\partial_{x_i}\xi| &\leq C\lambda\xi, \\ |\partial^2_{x_i x_j}\alpha| + |\partial^2_{x_i x_j}\xi| &\leq C\lambda^2\xi, \end{aligned} \tag{B.10}$$

these estimations will be important in the proof of the next theorem.

Let s, β, σ be real parameters and $\rho = \rho(x,t)$ a function. Along this paper we will use the notation

$$\begin{aligned} I^m_\beta(s,\lambda,\sigma;\rho) = \ & s^{\beta+3}\lambda^{\beta+4} \iint_{Q_T} \phi^{\beta+3} e^{ms\alpha} |\rho|^2 dxdt + s^{\beta+1}\lambda^{\beta+2} \iint_{Q_T} \phi^{\beta+1} e^{ms\alpha} |\nabla\rho|^2 dxdt \\ & + s^{\beta-1}\lambda^{\beta} \iint_{Q_T} \phi^{\beta-1} e^{ms\alpha} \left(\sigma^2|\rho_t|^2 + |\Delta\rho|^2\right) dxdt. \end{aligned} \tag{B.11}$$

Now we present a classical Carleman estimate for parabolic equations. For a complete proof of it see [29].

Lemma B.2. *Let $\beta \in \mathbb{R}$, $0 < \sigma \leq 1$ and let ω an open subset of Ω. There exists a constant $\lambda_0 = \lambda_0(\Omega, \omega)$ such that for every $\lambda \geq \lambda_0$, there exists $s_0 = s_0(\Omega, \omega, \lambda)$ and $C = C(\Omega, \omega, \lambda)$ such that, for all $s \geq s_0(T + T^2)$, the following inequality holds*

$$\begin{aligned} &I_\beta^m(s, \lambda, \sigma; q) \\ &\quad \leq C\left(s^\beta \lambda^\beta \iint_{Q_T} \phi^\beta e^{ms\alpha} |\sigma q_t + \Delta q|^2 dxdt + s^{\beta+3}\lambda^{\beta+4} \int_0^T\!\!\int_\omega \phi^{\beta+3} e^{ms\alpha} |q|^2 dxdt\right), \end{aligned} \quad \text{(B.12)}$$

for all $q \in C^2(Q_T)$ with $q = 0$ on Σ_T.

By using Lemma B.2, we prove a new Carleman estimate for the solutions of (B.4). This will be the main tool to prove observability inequality (B.3). This estimate is given in the next theorem.

Theorem B.3. *Let $\omega_1 \subset \Omega$ be an open subset. For $i, j = 1, 2$, let $a_{i,j} \in C^2(Q_T; \mathbb{R})$ and $A_{i,j} \in C^2(Q_T; \mathbb{R}^N)$ satisfying* (B.6) *and one of the assumptions (A) or (B). There exists a constant $\lambda_0 = \lambda_0(\Omega, \omega_1) \geq 1$ such that for all $\lambda \geq \lambda_0$, there exists a constant $s_0 = s_0(\Omega, \omega_1, \lambda_0)$ such that for every $s > s_0(T + T^2)$, the solution (φ, ξ) of system* (B.4) *satisfies*

$$s^4 \lambda^5 \iint_{Q_T} \phi^4 e^{2s\alpha} \left(|\varphi|^2 + |\xi|^2\right) dxdt \leq C \int_0^T\!\!\int_{\omega_1} \rho_1^2(t) |\varphi|^2 dxdt, \quad \text{(B.13)}$$

where $\rho_1(t) = s^{\frac{11}{2}} \lambda^{\frac{13}{2}} (\phi^*)^{\frac{11}{2}} e^{2s\alpha^* - \frac{3}{2}s\hat{\alpha}}$ if (A) holds or $\rho_1(t) = s^6 \lambda^{\frac{13}{2}} (\phi^*)^6 e^{2s\alpha^* - \frac{3}{2}s\hat{\alpha}}$ if (B) holds and C is a constant only depending on Ω, ω_1 and λ.

Remark B.4. For s and λ large enough, we have that $4s\alpha^* - 3s\widehat{\alpha} < 0$, which implies that both $(\phi^*)^{11} e^{4s\alpha^* - 3s\widehat{\alpha}}$ and $(\phi^*)^{12} e^{4s\alpha^* - 3s\widehat{\alpha}}$ are bounded.

Proof of Theorem 3.1. For a better understanding, we will divide the proof into three steps. In the first step we use Lemma B.2 for φ and ξ, solutions of (B.4), combine both estimates and absorb some lower order terms in the right-hand side. In this step we obtain an estimate of the form

$$\iint_{Q_T} \rho^{-2} (|\varphi|^2 + |\xi|^2)\, dxdt \leq C \int_0^T\!\!\int_{\omega_1} \rho_1^{-2} (|\varphi|^2 + |\xi|^2)\, dxdt. \quad \text{(B.14)}$$

In the second step, we use energy estimates and add a global weighted norm of $\nabla \xi_t$ in the left hand side of (B.14). Finally, in the third step, we combine steps one and two and estimate the local term of ξ by a local term of φ and other terms of small order. Only in this last step, conditions (A) or (B) are needed.

Let ω' be a nonempty open subset of Ω such that $\omega_0 \subset\subset \omega' \subset\subset \omega_1$ (ω_0 appears on the definition of the weight function ψ, see (B.8)) and let (φ, ξ) a solution of (B.4). Applying Lemma B.2 for the first equation of (B.4) with $m = 2$, $\beta = 1$, $\sigma = 1$ and for the second equation of $(B.4)_2$ with $m = 2$, $\beta = 1$, $\sigma = \epsilon$, and taking $\omega = \omega'$, we obtain

$$I_1^2(s,\lambda,1;\varphi) \le C \left(s^4\lambda^5 \int_0^T\!\!\int_{\omega'} \phi^4 e^{2s\alpha}|\varphi|^2 dxdt \right. \\ +s\lambda \iint_{Q_T} \phi e^{2s\alpha}|(a_{1,1} - \operatorname{div} A_{1,1})\varphi + (a_{2,2} - \operatorname{div} A_{2,2})\xi \\ \left. -A_{1,1}\cdot\nabla\varphi - A_{2,2}\cdot\nabla\xi|^2 dxdt \right), \quad (B.15)$$

and

$$I_1^2(s,\lambda,\epsilon;\xi) \le C \left(s^4\lambda^5 \int_0^T\!\!\int_{\omega'} \phi^4 e^{2s\alpha}|\xi|^2 dxdt \right. \\ +s\lambda \iint_{Q_T} \phi e^{2s\alpha}|(a_{1,2} - \operatorname{div} A_{1,2})\varphi + (a_{2,1} - \operatorname{div} A_{2,1})\xi \\ \left. -A_{1,2}\cdot\nabla\varphi - A_{2,1}\cdot\nabla\xi|^2 dxdt \right). \quad (B.16)$$

By (B.15) and (B.16) we have,

$$I_1^2(s,\lambda,1;\varphi) + I_1^2(s,\lambda,\epsilon;\xi) \le C \left(s\lambda \iint_{Q_T} \phi e^{2s\alpha}(|\varphi|^2 + |\xi|^2 + |\nabla\varphi|^2 + |\nabla\xi|^2) dxdt \right. \\ \left. +s^4\lambda^5 \int_0^T\!\!\int_{\omega'} \phi^4 e^{2s\alpha}|\varphi|^2 dxdt + s^4\lambda^5 \int_0^T\!\!\int_{\omega'} \phi^4 e^{2s\alpha}|\xi|^2 dxdt \right). \quad (B.17)$$

Hence, for s and λ large enough we get

$$I_1^2(s,\lambda,1;\varphi) + I_1^2(s,\lambda,\epsilon;\xi) \\ \le C \left(s^4\lambda^5 \int_0^T\!\!\int_{\omega'} \phi^4 e^{2s\alpha}|\varphi|^2 dxdt + s^4\lambda^5 \int_0^T\!\!\int_{\omega'} \phi^4 e^{2s\alpha}|\xi|^2 dxdt \right). \quad (B.18)$$

Step 2: To absorb the local term of ξ in the right-hand side of (B.18), we will include a global term of $\nabla\xi_t$ in the left-hand side of (B.18). Applying the time derivative on the second equation of (B.4), we get

$$-\epsilon(\xi_t)_t - \Delta(\xi_t) = \partial_t[(a_{1,2} - \operatorname{div} A_{1,2})\varphi + (a_{2,1} - \operatorname{div} A_{2,1})\xi - A_{1,2}\cdot\nabla\varphi - A_{2,1}\cdot\nabla\xi].$$

Then, we get

$$s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\nabla\xi_t|^2 dxdt = -s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} \Delta\xi_t \xi_t dxdt \\ = s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} \xi_t[\epsilon\xi_{tt} + \partial_t(a_{1,2} - \operatorname{div} A_{1,2})\varphi + (a_{1,2} - \operatorname{div} A_{1,2})\varphi_t \\ + \partial_t(a_{2,1} - \operatorname{div} A_{2,1})\xi + (a_{2,1} - \operatorname{div} A_{2,1})\xi_t - \partial_t A_{1,2}\cdot\nabla\varphi - A_{1,2}\cdot\nabla\varphi_t \\ - \partial_t A_{2,1}\cdot\nabla\xi - A_{2,1}\cdot\nabla\xi_t] dxdt = \sum_{k=1}^{9} P_k. \quad (B.19)$$

Let us estimate each of the nine terms above. For the first one we have

$$\begin{aligned} P_1 &= s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}(\epsilon\xi_{tt})\xi_t dxdt = \frac{\epsilon s^{-3}\lambda^{-3}}{2}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}\frac{d}{dt}|\xi_t|^2 dxdt \\ &= -\frac{\epsilon s^{-3}\lambda^{-3}}{2}\iint_{Q_T}\partial_t((\hat{\phi})^{-3}e^{3s\hat{\alpha}})|\xi_t|^2 dxdt \le C\epsilon s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}|\xi_t|^2 dxdt. \end{aligned} \tag{B.20}$$

From now on δ represents a sufficiently small and positive real number. Using Young's inequality we get

$$\begin{aligned} P_2 &= s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}\partial_t(a_{1,2} - \operatorname{div} A_{1,2})\varphi\xi_t dxdt \\ &\le ||\partial_t(a_{1,2} - \operatorname{div} A_{1,2})||_\infty s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\varphi||\xi_t| dxdt \\ &\le Cs^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\varphi|^2 dxdt + \delta s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\xi_t|^2 dxdt \\ &\le Cs^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\varphi|^2 dxdt + \frac{\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2 dxdt, \end{aligned} \tag{B.21}$$

and

$$\begin{aligned} P_3 &= s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}(a_{1,2} - \operatorname{div} A_{1,2})\varphi_t\xi_t dxdt \\ &\le ||a_{1,2} - \operatorname{div} A_{1,2}||_\infty s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\varphi_t||\xi_t| dxdt \\ &\le Cs^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\varphi_t|^2 dxdt + \frac{\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2 dxdt. \end{aligned} \tag{B.22}$$

Analogously we have

$$\begin{aligned} P_4 &= s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}\partial_t(a_{2,1} - \operatorname{div} A_{2,1})\xi\xi_t dxdt \\ &\le ||\partial_t(a_{2,1} - \operatorname{div} A_{2,1})||_\infty s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\xi||\xi_t| dxdt \\ &\le Cs^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\xi|^2 dxdt + \frac{\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2 dxdt. \end{aligned} \tag{B.23}$$

For the fifth term, it is easy to see that

$$P_5 = s^{-3}\lambda\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}(a_{2,1} - \operatorname{div} A_{2,1})|\xi_t|^2 dxdt. \tag{B.24}$$

Using again Young's inequality we get

$$\begin{aligned} P_6 &= s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}(-\partial_t A_{1,2})\cdot\nabla\varphi\xi_t dxdt \\ &\le ||\partial_t A_{1,2}||_\infty s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\varphi||\xi_t| dxdt \\ &\le Cs^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\varphi|^2 dxdt + \frac{\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2 dxdt, \end{aligned} \tag{B.25}$$

and

$$
\begin{aligned}
P_7 = s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} (-A_{1,2}) \cdot \nabla\varphi_t \xi_t dxdt \\
= s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} \mathrm{div}\, A_{1,2} \varphi_t \xi_t dxdt \\
+ s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} A_{1,2} \cdot \nabla\xi_t \varphi_t dxdt \\
\leq \|\mathrm{div}\, A_{1,2}\|_\infty s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\varphi_t||\xi_t| dxdt \\
+ \|A_{1,2}\|_\infty s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\nabla\xi_t||\varphi_t| dxdt \\
\leq C s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\varphi_t|^2 dxdt + \delta s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\nabla\xi_t|^2 dxdt. \quad \text{(B.26)}
\end{aligned}
$$

For the eighth term we get

$$
\begin{aligned}
P_8 = s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} (-\partial_t A_{2,1}) \cdot \nabla\xi \xi_t dxdt \\
\leq \|\partial_t A_{2,1}\|_\infty s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\nabla\xi||\xi_t| dxdt \\
\leq C s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\nabla\xi|^2 dxdt + \frac{\delta}{\mu_1} s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\nabla\xi_t|^2 dxdt. \quad \text{(B.27)}
\end{aligned}
$$

For the last term we have

$$
\begin{aligned}
P_9 = s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} (-A_{2,1}) \cdot \nabla\xi_t \xi_t dxdt \\
= s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} \left(\frac{\mathrm{div}\, A_{2,1}}{2} \right) |\xi_t|^2 dxdt. \quad \text{(B.28)}
\end{aligned}
$$

Using (B.20)-(B.28) in (B.19), we get

$$
\begin{aligned}
s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\nabla\xi_t|^2 dxdt \leq C \Big(\epsilon s^{-1}\lambda^{-1} \iint_{Q_T} (\hat{\phi})^{-1} e^{3s\hat{\alpha}} |\xi_t|^2 dxdt \\
+ s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} (|\varphi|^2 + |\xi|^2 + |\nabla\varphi|^2 + |\nabla\xi|^2 + |\varphi_t|^2 \Big) dxdt \\
+ \delta \left(1 + \frac{5}{\mu_1}\right) s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} |\nabla\xi_t|^2 dxdt \\
+ s^{-3}\lambda^{-3} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}} \left(a_{2,1} - \frac{\mathrm{div}\, A_{2,1}}{2} \right) |\xi_t|^2 dxdt. \quad \text{(B.29)}
\end{aligned}
$$

Let us estimate the first term on the right hand side of (B.29). By the second equation of (B.4), we can write

$$
\begin{aligned}
\epsilon s^{-1}\lambda^{-1} \iint_{Q_T} (\hat{\phi})^{-1} e^{3s\hat{\alpha}} |\xi_t|^2 dxdt = s^{-1}\lambda^{-1} \iint_{Q_T} (\hat{\phi})^{-1} e^{3s\hat{\alpha}} \xi_t (\epsilon \xi_t) dxdt \\
= s^{-1}\lambda^{-1} \iint_{Q_T} (\hat{\phi})^{-1} e^{3s\hat{\alpha}} \xi_t [-\Delta\xi - (a_{1,2} - \mathrm{div}\, A_{1,2})\varphi - (a_{2,1} + \mathrm{div}\, A_{2,1})\xi \\
+ A_{1,2}\varphi_x + A_{2,1} \cdot \nabla\xi] dxdt = \sum_{k=1}^{5} P_{1,k}. \quad \text{(B.30)}
\end{aligned}
$$

Let us estimate each of the five terms above. For the first one we have

$$
\begin{aligned}
P_{1,1} = s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}\xi_t(-\Delta\xi)dxdt \\
= s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}\nabla\xi_t\cdot\nabla\xi dxdt \\
= \frac{s^{-1}\lambda^{-1}}{2}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}\frac{d}{dt}|\nabla\xi|^2dxdt \\
= -\frac{s^{-1}\lambda^{-3+2}}{2}\iint_{Q_T}\left((\hat{\phi})^{-1}e^{3s\hat{\alpha}}\right)_t|\nabla\xi|^2dxdt \\
\leq Cs^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}|\nabla\xi|^2dxdt. \quad \text{(B.31)}
\end{aligned}
$$

For the second and third terms we obtain

$$
\begin{aligned}
P_{1,2} = s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}\xi_t[-(a_{1,2}-\operatorname{div}A_{1,2})\varphi]dxdt \\
\leq ||a_{1,2}-\operatorname{div}A_{1,2}||_\infty s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}|\xi_t||\varphi|dxdt \\
\leq Cs^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}|\varphi|^2dxdt + \frac{\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt, \quad \text{(B.32)}
\end{aligned}
$$

and

$$
\begin{aligned}
P_{1,3} = s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}\xi_t[-(a_{2,1}-\operatorname{div}A_{2,1})]\xi dxdt \\
\leq ||a_{2,1}-\operatorname{div}A_{2,1}||_\infty s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-3+2}e^{3s\hat{\alpha}}|\xi_t||\xi|dxdt \\
\leq Cs\lambda\iint_{Q_T}(\hat{\phi})e^{3s\hat{\alpha}}|\xi|^2dxdt + \frac{\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt. \quad \text{(B.33)}
\end{aligned}
$$

For the fourth one we get

$$
\begin{aligned}
P_{1,4} = s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}\xi_t(A_{1,2}\cdot\nabla\varphi)dxdt \\
\leq ||A_{1,2}||_\infty s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}|\xi_t||\nabla\varphi|dxdt \\
\leq Cs\lambda\iint_{Q_T}(\hat{\phi})e^{3s\hat{\alpha}}|\nabla\varphi|^2dxdt + \frac{\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt. \quad \text{(B.34)}
\end{aligned}
$$

Finally, for the last term

$$
\begin{aligned}
P_{1,5} = s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}\xi_t(A_{2,1}\cdot\nabla\xi)dxdt \\
\leq ||A_{2,1}||_\infty s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}\xi_t|\nabla\xi|dxdt \\
\leq Cs^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}|\nabla\xi|^2dxdt + \frac{\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt. \quad \text{(B.35)}
\end{aligned}
$$

Combining (B.31)-(B.35) and (B.30), we obtain,

$$
\begin{aligned}
\epsilon s^{-1}\lambda^{-1}\iint_{Q_T}(\hat{\phi})^{-1}e^{3s\hat{\alpha}}|\xi_t|^2dxdt \leq \frac{4\delta}{\mu_1}s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt \\
+ Cs\lambda\iint_{Q_T}(\hat{\phi})e^{3s\hat{\alpha}}(|\varphi|^2+|\xi|^2+|\nabla\varphi|^2+|\nabla\xi|^2)^2dxdt. \quad \text{(B.36)}
\end{aligned}
$$

Now, putting (B.36) in (B.29), we get

$$\begin{aligned} s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt \\ \leq Cs\lambda\iint_{Q_T}(\hat{\phi})e^{3s\hat{\alpha}}(|\varphi|^2+|\xi|^2+|\nabla\varphi|^2+|\nabla\xi|^2+|\varphi_t|^2)dxdt \\ +\delta\left(1+\frac{9\delta}{\mu_1}\right)s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt \\ +s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}\left(a_{2,1}-\frac{\operatorname{div}A_{2,1}}{2}\right)|\xi_t|^2dxdt. \end{aligned} \tag{B.37}$$

Combining (B.37) and (B.18), we get

$$\begin{aligned} I_1^2(s,\lambda,1;\varphi)+I_1^2(s,\lambda,\epsilon;\xi)+s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt \\ \leq C\left(s^4\lambda^5\int_0^T\!\!\int_{\omega'}\phi^4e^{2s\alpha}|\varphi|^2dxdt+s^4\lambda^5\int_0^T\!\!\int_{\omega'}\phi^4e^{2s\alpha}|\xi|^2dxdt\right. \\ \left.+s\lambda\iint_{Q_T}(\hat{\phi})e^{3s\hat{\alpha}}(|\varphi|^2+|\xi|^2+|\nabla\varphi|^2+|\nabla\xi|^2+|\varphi_t|^2)dxdt\right) \\ +\delta\left(1+\frac{9\delta}{\mu_1}\right)s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt \\ +s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}\left(a_{2,1}-\frac{\operatorname{div}A_{2,1}}{2}\right)|\xi_t|^2dxdt. \end{aligned} \tag{B.38}$$

Using (B.6), taking s,λ sufficiently large and δ sufficiently small, we get

$$\begin{aligned} I_1^2(s,\lambda,1;\varphi)+I_1^2(s,\lambda,\epsilon;\xi)+s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\nabla\xi_t|^2dxdt \\ \leq C\left(s^4\lambda^5\int_0^T\!\!\int_{\omega'}\phi^4e^{2s\alpha}|\varphi|^2dxdt+s^4\lambda^5\int_0^T\!\!\int_{\omega'}\phi^4e^{2s\alpha}|\xi|^2dxdt\right). \end{aligned} \tag{B.39}$$

Step 3: In order to bound the local term of ξ, we are going to use one of the conditions (A) or (B). In this way, this step will be divided into two steps.

Step 3.1: Assume that (A) is valid. Let ω'' be an open subset, such that $\omega'\subset\subset\omega''\subset\subset\omega_1$ and $\theta\in C_0^\infty(\omega'')$ a cut-off function, such that $0\leq\theta\leq 1$ and $\theta\equiv 1$ in ω'. Since $(\operatorname{supp}A_{2,2})\cap\omega=\emptyset$ and $a_{2,2}\geq C>0$ in $\overline{\omega_T}$, we get

$$\begin{aligned} s^4\lambda^5\int_0^T\!\!\int_{\omega'}\theta(\phi^*)^4e^{2s\alpha^*}|\xi|^2dxdt \leq Cs^4\lambda^5\int_0^T\!\!\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}\xi(a_{2,2}\xi)dxdt \\ = Cs^4\lambda^5\int_0^T\!\!\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}\xi[-\varphi_t-\Delta\varphi-(a_{1,1}-\operatorname{div}A_{1,1})\varphi+A_{1,1}\cdot\nabla\varphi]dxdt \\ =\sum_{k=1}^4 Q_k. \end{aligned} \tag{B.40}$$

Now, let us estimate these four terms. For the first one, we integrate by parts in time and use Young and Poincaré inequalities to obtain

$$
\begin{aligned}
Q_1 &= s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}\xi(-\varphi_t)dxdt \\
&= s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \left(\theta(\phi^*)^4 e^{2s\alpha^*}\right)_t \xi\varphi dxdt + s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}\xi_t\varphi dxdt \\
&\le Cs^6\lambda^6 \int_0^T\!\!\int_{\omega''} (\phi^*)^6 e^{2s\alpha^*}\xi\varphi dxdt + Cs^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\phi^*)^4 e^{2s\alpha^*}\xi_t\varphi dxdt \\
&\le \delta\left(s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\hat{\phi})^4 e^{2s\hat{\alpha}}|\xi|^2 dxdt + s^{-3}\lambda^{-3} \int_0^T\!\!\int_{\omega''} (\hat{\phi})^{-3} e^{3s\hat{\alpha}}|\xi_t|^2 dxdt\right) \\
&\quad + C\left(s^8\lambda^8 \int_0^T\!\!\int_{\omega''} (\phi^*)^{12}(\hat{\phi})^{-4} e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt\right. \\
&\quad \left. + s^{11}\lambda^{13} \int_0^T\!\!\int_{\omega''} (\phi^*)^8(\hat{\phi})^3 e^{4s\alpha^*-3s\hat{\alpha}}|\varphi|^2 dxdt\right) \\
&\le \delta\left(s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\hat{\phi})^4 e^{2s\hat{\alpha}}|\xi|^2 dxdt + \frac{s^{-3}\lambda^{-3}}{\mu_1} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}}|\nabla\xi_t|^2 dxdt\right) \\
&\quad + Cs^{11}\lambda^{13} \int_0^T\!\!\int_{\omega''} (\phi^*)^{11} e^{4s\alpha^*-3s\hat{\alpha}}|\varphi|^2 dxdt. \qquad \text{(B.41)}
\end{aligned}
$$

For the second one we have

$$
\begin{aligned}
Q_2 &= s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}\xi(-\Delta\varphi)dxdt = -s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\phi^*)^4 e^{2s\alpha^*}\Delta(\theta\xi)\varphi dxdt \\
&= -s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\phi^*)^4 e^{2s\alpha^*}\left(\Delta\theta\xi + 2\nabla\theta\cdot\nabla\xi + \theta\Delta\xi\right)\varphi dxdt \\
&\le Cs^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\phi^*)^4 e^{2s\alpha^*}(|\xi| + |\nabla\xi| + |\Delta\xi|)|\varphi| dxdt \\
&\le \delta\left(s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\hat{\phi})^4 e^{2s\hat{\alpha}}|\xi|^2 dxdt + s^2\lambda^3 \int_0^T\!\!\int_{\omega''} (\hat{\phi})^2 e^{2s\hat{\alpha}}|\nabla\xi|^2 dxdt\right. \\
&\quad \left. + \lambda \int_0^T\!\!\int_{\omega''} e^{2s\hat{\alpha}}|\Delta\xi|^2 dxdt\right) + C\left(s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\phi^*)^8(\hat{\phi})^{-4} e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt\right. \\
&\quad \left. + s^6\lambda^7 \int_0^T\!\!\int_{\omega''} (\phi^*)^8(\hat{\phi})^{-2} e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt + s^8\lambda^9 \int_0^T\!\!\int_{\omega''} (\phi^*)^8 e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt\right). \qquad \text{(B.42)}
\end{aligned}
$$

For the third one, a simply computation shows

$$
\begin{aligned}
Q_3 &= s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}\xi[-(a_{1,1} - \operatorname{div} A_{1,1})\varphi]dxdt \\
&\le ||a_{1,1} - \operatorname{div} A_{1,1}||_\infty s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}\xi\varphi dxdt \\
&\le \delta s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\hat{\phi})^4 e^{2s\hat{\alpha}}|\xi|^2 dxdt + Cs^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\phi^*)^8(\hat{\phi})^{-4} e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt. \qquad \text{(B.43)}
\end{aligned}
$$

Finally, for the fourth one we have

$$
\begin{aligned}
Q_4 = s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}\xi(A_{1,1}\cdot\nabla\varphi)dxdt \\
\leq ||A_{1,1}||_\infty s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}|\xi||\nabla\varphi|dxdt \\
\leq \delta s^2\lambda^3 \int_0^T\!\!\int_{\omega''} (\hat{\phi})^2 e^{2s\hat{\alpha}}|\nabla\xi|^2 dxdt + Cs^6\lambda^7 \int_0^T\!\!\int_{\omega''} (\phi^*)^8(\hat{\phi})^{-2} e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt. \quad \text{(B.44)}
\end{aligned}
$$

Putting (B.41)-(B.44) together with (B.40), we get

$$
\begin{aligned}
s^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}|\xi|^2 dxdt \\
\leq \delta \Bigg(s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\hat{\phi})^4 e^{2s\hat{\alpha}}|\xi|^2 dxdt + s^2\lambda^3 \int_0^T\!\!\int_{\omega''} (\hat{\phi})^2 e^{2s\hat{\alpha}}|\nabla\xi|^2 dxdt \\
+\lambda \int_0^T\!\!\int_{\omega''} e^{2s\hat{\alpha}}|\Delta\xi|^2 dxdt + \frac{s^{-3}\lambda^{-3}}{\mu_1} \iint_{Q_T} (\hat{\phi})^{-3} e^{3s\hat{\alpha}}|\nabla\xi_t|^2 dxdt \Bigg) \\
+ C \Bigg(s^4\lambda^5 \int_0^T\!\!\int_{\omega''} (\phi^*)^8(\hat{\phi})^{-4} e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt \\
+s^8\lambda^8 \int_0^T\!\!\int_{\omega''} (\phi^*)^{12}(\hat{\phi})^{-4} e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt \\
+s^{11}\lambda^{13} \int_0^T\!\!\int_{\omega''} (\phi^*)^{11} e^{4s\alpha^*-3s\hat{\alpha}}|\varphi|^2 dxdt \\
+s^6\lambda^7 \int_0^T\!\!\int_{\omega''} (\phi^*)^8(\hat{\phi})^{-2} e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt + s^8\lambda^9 \int_0^T\!\!\int_{\omega''} (\phi^*)^8 e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2 dxdt \Bigg). \quad \text{(B.45)}
\end{aligned}
$$

Again, if s and λ are large enough and δ is small enough, we can absorb all terms of ξ and its derivatives with (B.39). We conclude that

$$
\begin{aligned}
I_1^2(s,\lambda,1;\varphi) + I_1^2(s,\lambda,\epsilon;\xi) + s^{-3}\lambda^{-3} \iint_Q (\hat{\phi})^{-3} e^{3s\hat{\alpha}}|\nabla\xi_t|^2 dxdt \\
\leq Cs^{11}\lambda^{13} \int_0^T\!\!\int_{\omega''} (\phi^*)^{11} e^{4s\alpha^*-3s\hat{\alpha}}|\varphi|^2 dxdt, \quad \text{(B.46)}
\end{aligned}
$$

which implies in (B.13).

Step 3.2: Assume now that (B) holds, we remind that in this case $N = 1$. We take again ω' and ω'' open subsets such that $\omega' \subset\subset \omega'' \subset\subset \omega_1$, and since $\omega_1 \cap \partial\Omega \neq \emptyset$, we can assume that $\omega' \cap \partial\Omega \neq \emptyset$. Now, we define a cut-off function $\theta \in C_0^\infty(\omega'')$ such that $0 \leq \theta \leq 1$ and $\theta \equiv 1$ in ω'. By Poincaré's inequality we get

$$
\begin{aligned}
s^4\lambda^5 \int_0^T\!\!\int_{\omega'} \phi^4 e^{2s\alpha}|\xi|^2 dxdt \leq s^4\lambda^5 \int_0^T\!\!\int_{\omega'} (\phi^*)^4 e^{2s\alpha^*}|\xi|^2 dxdt \\
\leq Cs^4\lambda^5 \int_0^T\!\!\int_{\omega'} (\phi^*)^4 e^{2s\alpha^*}|\xi_x|^2 dxdt \leq Cs^4\lambda^5 \int_0^T\!\!\int_{\omega''} \theta(\phi^*)^4 e^{2s\alpha^*}|\xi_x|^2 dxdt. \quad \text{(B.47)}
\end{aligned}
$$

Therefore, given the fact that $\text{supp}(a_{2,2}-\partial_x A_{2,2})\cap\omega=\emptyset$ and $A_{2,2}\geq C>0$ in $\overline{\omega_1}\times(0,T)$, it follows that

$$\begin{aligned} s^4\lambda^5\int_0^T\!\!\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}|\xi_x|^2dxdt &\leq Cs^4\lambda^5\int_0^T\!\!\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}\xi_x(A_{2,2}\xi_x)dxdt \\ &= Cs^4\lambda^5\int_0^T\!\!\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}\xi_x[-\varphi_t-\varphi_{xx}-(a_{1,1}-\partial_x A_{1,1})\varphi+A_{1,1}\varphi_x]dxdt \\ &= \sum_{k=1}^4 R_k. \end{aligned} \tag{B.48}$$

In this way, we need to estimate these four terms above. For the first one we have

$$\begin{aligned} R_1 &= s^4\lambda^5\int_0^T\!\!\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}\xi_x(-\varphi_t)dxdt \\ &= s^4\lambda^5\int_0^T\!\!\int_{\omega''}\left(\theta(\phi^*)^4e^{2s\alpha^*}\right)_t\xi_x\varphi dxdt + s^4\lambda^5\int_0^T\!\!\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}\xi_{tx}\varphi dxdt \\ &\leq Cs^6\lambda^6\int_0^T\!\!\int_{\omega''}(\phi^*)^6e^{2s\alpha^*}\xi_x\varphi dxdt + Cs^4\lambda^5\int_0^T\!\!\int_{\omega''}(\phi^*)^4e^{2s\alpha^*}\xi_{tx}\varphi dxdt \\ &\leq \delta\left(s^2\lambda^3\int_0^T\!\!\int_{\omega''}(\hat\phi)^2e^{2s\hat\alpha}|\xi_x|^2dxdt + s^{-3}\lambda^{-3}\int_0^T\!\!\int_{\omega''}(\hat\phi)^{-3}e^{3s\hat\alpha}|\xi_{tx}|^2dxdt\right) \\ &\quad + C\left(s^{10}\lambda^{10}\int_0^T\!\!\int_{\omega''}(\phi^*)^{12}(\hat\phi)^{-2}e^{4s\alpha^*-2s\hat\alpha}|\varphi|^2dxdt\right. \\ &\quad \left. + s^{11}\lambda^{13}\int_0^T\!\!\int_{\omega''}(\phi^*)^8(\hat\phi)^3e^{4s\alpha^*-3s\hat\alpha}|\varphi|^2dxdt\right). \end{aligned} \tag{B.49}$$

Now, for the second one

$$\begin{aligned} R_2 &= s^4\lambda^5\int_0^T\!\!\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}\xi_x(-\varphi_{xx})dxdt = s^4\lambda^5\int_0^T\!\!\int_{\omega''}(\phi^*)^4e^{2s\alpha^*}\left(\theta\xi_x\right)_x\varphi_x dxdt \\ &\leq \delta\left(s^2\lambda^3\int_0^T\!\!\int_{\omega''}(\hat\phi)^2e^{2s\hat\alpha}|\xi_x|dxdt + \lambda\int_0^T\!\!\int_{\omega''}e^{2s\hat\alpha}|\xi_{xx}|dxdt\right) \\ &\quad + C\left(s^6\lambda^7\int_0^T\!\!\int_{\omega''}(\phi^*)^8(\hat\phi)^{-2}e^{4s\alpha^*-2s\hat\alpha}|\varphi_x|^2dxdt\right. \\ &\quad \left. + s^8\lambda^{10}\int_0^T\!\!\int_{\omega''}(\phi^*)^8e^{4s\alpha^*-2s\hat\alpha}|\varphi_x|^2dxdt\right). \end{aligned} \tag{B.50}$$

Before proceeding to R_3, we will estimate in (B.50) the local term of φ_x by a local term of φ. Let $\tilde\theta\in C_0^\infty(\omega''')$, where $\omega''\subset\subset\omega'''\subset\subset\omega_1$, a function satisfying $0\leq\tilde\theta\leq 1$ and $\tilde\theta=1$ in ω'', then

$$\begin{aligned} & s^6\lambda^7\int_0^T\!\!\int_{\omega'''}\tilde\theta(\phi^*)^6e^{4s\alpha^*-2s\widehat{\alpha}}|\varphi_x|^2dxdt \\ &\qquad \leq \delta\lambda\int_0^T\!\!\int_{\omega'''}e^{2s\hat\alpha}|\varphi_{xx}|^2dxdt + Cs^{12}\lambda^{13}\int_0^T\!\!\int_{\omega'''}(\phi^*)^{12}e^{8s\alpha^*-6s\hat\alpha}|\varphi|^2dxdt, \end{aligned} \tag{B.51}$$

hence,

$$\begin{aligned} R_2 &\leq \delta\left(s^4\lambda^5\int_0^T\!\!\int_{\omega''}(\hat\phi)^4e^{2s\hat\alpha}|\xi|dxdt + s^2\lambda^3\int_0^T\!\!\int_{\omega''}(\hat\phi)^2e^{2s\hat\alpha}|\xi_x|dxdt\right. \\ &\quad \left. + \lambda\int_0^T\!\!\int_{\omega'''}e^{2s\hat\alpha}|\varphi_{xx}|^2dxdt\right) + Cs^{12}\lambda^{13}\int_0^T\!\!\int_{\omega'''}(\phi^*)^{12}e^{8s\alpha^*-6s\hat\alpha}|\varphi|^2dxdt. \end{aligned} \tag{B.52}$$

Now, for R_3 we get

$$\begin{aligned} R_3 &= s^4\lambda^5\int_0^T\int_{\omega''}\tilde{\theta}(\phi^*)^4e^{2s\alpha^*}\xi_x[-(a_{1,1}-\partial_xA_{1,1})\varphi]dxdt \\ &\le ||a_{1,1}-\partial_xA_{1,1}||_\infty s^4\lambda^5\int_0^T\int_{\omega''}\tilde{\theta}(\phi^*)^4e^{2s\alpha^*}|\xi_x||\varphi|dxdt \\ &\le \delta s^2\lambda^3\int_0^T\int_{\omega'''}(\hat{\phi})^2e^{2s\hat{\alpha}}|\xi_x|^2dxdt + Cs^6\lambda^7\int_0^T\int_{\omega'''}(\phi^*)^8(\hat{\phi})^{-2}e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2dxdt. \quad \text{(B.53)} \end{aligned}$$

Finally

$$\begin{aligned} R_4 &= s^4\lambda^5\int_0^T\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}\xi_x(A_{1,1}\varphi_x)dxdt \\ &\le C\Bigg(||\partial_xA_{1,1}||_\infty s^4\lambda^5\int_0^T\int_{\omega''}(\phi^*)^4e^{2s\alpha^*}|\xi_x||\varphi|dxdt \\ &\quad +||A_{1,1}||_\infty s^4\lambda^5\int_0^T\int_{\omega''}(\phi^*)^4e^{2s\alpha^*}|\xi_{xx}||\varphi|dxdt\Bigg) \\ &\le \delta\left(s^2\lambda^3\int_0^T\int_{\omega''}(\phi^*)^2e^{2s\widehat{\alpha}}|\xi_x|^2dxdt + \lambda\int_0^T\int_{\omega'''}e^{2s\hat{\alpha}}|\xi_{xx}|^2dxdt\right) \\ &\quad + Cs^8\lambda^9\int_0^T\int_{\omega'''}(\phi^*)^8e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2dxdt. \quad \text{(B.54)} \end{aligned}$$

Combining (B.49)-(B.54) and (B.48), we obtain

$$\begin{aligned} &s^4\lambda^5\int_0^T\int_{\omega''}\theta(\phi^*)^4e^{2s\alpha^*}|\xi|^2dxdt \\ &\le \delta\Bigg(s^4\lambda^5\int_0^T\int_{\omega''}(\hat{\phi})^4e^{2s\hat{\alpha}}|\xi|^2dxdt + s^2\lambda^3\int_0^T\int_{\omega''}(\hat{\phi})^2e^{2s\hat{\alpha}}|\xi_x|^2dxdt \\ &\quad +\lambda\int_0^T\int_{\omega''}e^{2s\hat{\alpha}}|\xi_{xx}|^2dxdt \\ &\quad +\lambda\int_0^T\int_{\omega''}e^{2s\hat{\alpha}}|\varphi_{xx}|dxdt + s^{-3}\lambda^{-3}\int_0^T\int_{\omega''}(\hat{\phi})^{-3}e^{3s\hat{\alpha}}|\xi_{tx}|^2dxdt\Bigg) \\ &\quad + C\Bigg(s^{10}\lambda^{10}\int_0^T\int_{\omega''}(\phi^*)^{12}(\hat{\phi})^{-2}e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2dxdt \\ &\quad + s^{11}\lambda^{13}\iint_{\omega''\times(0,T)}(\phi^*)^8(\hat{\phi})^3e^{4s\alpha^*-3s\hat{\alpha}}|\varphi|^2dxdt \\ &\quad + s^{12}\lambda^{13}\int_0^T\int_{\omega'''}(\phi^*)^{12}e^{8s\alpha^*-6s\hat{\alpha}}|\varphi|^2dxdt + s^6\lambda^7\int_0^T\int_{\omega''}(\phi^*)^8(\hat{\phi})^{-2}e^{4s\alpha^*-2s\hat{\alpha}}|\varphi|^2dxdt \\ &\quad s^8\lambda^9\int_0^T\int_{\omega''}(\phi^*)^8e^{8s\alpha^*-6s\hat{\alpha}}|\varphi|^2dxdt\Bigg). \quad \text{(B.55)} \end{aligned}$$

Combining (B.55) with (B.39) and taking s and λ large enough with δ small enough, we obtain

$$\begin{aligned} &I_1^2(s,\lambda,1;\varphi) + I_1^2(s,\lambda,\epsilon;\xi) + s^{-3}\lambda^{-3}\iint_{Q_T}(\hat{\phi})^\gamma e^{3s\hat{\alpha}}|\xi_{tx}|^2dxdt \\ &\le Cs^{12}\lambda^{13}\int_0^T\int_{\omega'''}(\phi^*)^{12}e^{4s\alpha^*-3s\hat{\alpha}}|\varphi|^2dxdt. \quad \text{(B.56)} \end{aligned}$$

□

Observability Inequality

In this section we will see how to we can prove the observability inequality (B.3). For simplicity, we make the change of variable $t \to T-t$ in system (B.4) obtaining the following system

$$\begin{cases} \varphi_t - \Delta\varphi = (a_{1,1} - \operatorname{div} A_{1,1})\varphi + (a_{2,2} - \operatorname{div} A_{2,2})\xi - A_{1,1}\cdot\nabla\varphi - A_{2,2}\cdot\nabla\xi, & \text{in } Q_T, \\ \epsilon\xi_t - \Delta\xi = (a_{1,2} - \operatorname{div} A_{1,2})\varphi + (a_{2,1} - \operatorname{div} A_{2,1})\xi - A_{1,2}\cdot\nabla\varphi - A_{2,1}\cdot\nabla\xi, & \text{in } Q_T, \\ \varphi = \xi = 0, & \text{on } \Sigma_T, \\ \varphi|_{t=0} = \varphi_T; \xi|_{t=0} = \xi_T, & \text{in } \Omega. \end{cases} \tag{B.57}$$

Then, it is sufficient to prove that

$$\|\varphi(T)\|^2_{L^2(\Omega)} + \epsilon\|\xi(T)\|^2_{L^2(\Omega)} \leq C\int_0^T\int_{\omega_1}|\varphi|^2dxdt, \tag{B.58}$$

for every (φ,ξ) solution of (B.57), where C does not depends on ϵ. To do this, we multiply the first equation of (B.57) by φ and the second equation of (B.57) by ξ and integrate over Ω obtaining

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\varphi(t)||^2_{L^2(\Omega)} + ||\nabla\varphi(t)||^2_{L^2(\Omega)} \\ = \int_\Omega \Big[(a_{1,1} - \operatorname{div} A_{1,1})|\varphi(t)|^2 - \frac{1}{2}A_{1,1}\cdot\nabla(|\varphi(t)|^2) \\ +\varphi(t)[(a_{2,2} - \operatorname{div} A_{2,2})\xi(t) - A_{2,1}\cdot\nabla\xi]\Big]\,dx, \end{aligned} \tag{B.59}$$

and

$$\begin{aligned} \frac{\epsilon}{2}\frac{d}{dt}||\xi(t)||^2_{L^2(\Omega)} + ||\nabla\xi(t)||^2_{L^2(\Omega)} \\ = \int_\Omega \Big[(a_{2,1} - \operatorname{div} A_{2,1})|\xi(t)|^2 - \frac{A_{2,1}}{2}\nabla(|\xi(t)|^2) \\ +\xi(t)[(a_{1,2} - \operatorname{div} A_{1,2})\varphi(t) - A_{1,2}\cdot\nabla\varphi(t)]\Big]\,dx. \end{aligned} \tag{B.60}$$

Using Young and Poincaré inequalities, we get that

$$\frac{1}{2}\frac{d}{dt}||\varphi(t)||^2_{L^2(\Omega)} + \frac{\epsilon}{2}\frac{d}{dt}||\xi(t)||^2_{L^2(\Omega)} \leq C\|\varphi(t)\|^2_{L^2(\Omega)}, \tag{B.61}$$

where C does not depends on ϵ. Now, by Gronwall Lemma

$$\frac{1}{2}||\varphi(T)||^2_{L^2(\Omega)} + \frac{\epsilon}{2}||\xi(T)||^2_{L^2(\Omega)} \leq C\left(||\varphi(t)||^2_{L^2(\Omega)} + \epsilon||\xi(t)||^2_{L^2(\Omega)}\right), \tag{B.62}$$

and again C does not depends on ϵ. Integrating (B.62) from $t = T/4$ to $t = 3T/4$ we get

$$\frac{1}{2}||\varphi(T)||^2_{L^2(\Omega)} + \frac{\epsilon}{2}||\xi(T)||^2_{L^2(\Omega)} \leq C\int_{T/4}^{3T/4}\int_\Omega\left(|\varphi(t)|^2 + \epsilon|\xi(t)|^2\right)dxdt, \tag{B.63}$$

where C also does not depends on ϵ. Finally, using (B.13) and the fact that $e^{4s\alpha^* - 3s\widehat{\alpha}}(\phi^*)^{11}$ is bounded from bellow in $\Omega\times(T/4, 3T/4)$, we obtain the observability inequality (B.3).

By the Hilbert Uniqueness Method, this inequality is equivalent to the null controllability of system (B.1) with controls $\{f_1^\epsilon\}_{\epsilon>0}$ uniformly bounded with respect to ϵ in the $L^2(\omega_1\times]0,T[)$. Then, if we take f_1 to be the weak limit of $\{f_1^\epsilon\}$ in $L^2(\omega_1\times]0,T[)$ as $\epsilon\to 0$, it is not difficult to see that f_1 is a control for system (B.2), therefore the uniform null control problem given by Theorem B.1 is proved.

□

Comments and Open problems

In this section we make some comments and give some open questions related to this problem.

The case where only f_2 is active

The problem of controlling system (B.1) with controls only on the second equation, that is $f_1^\epsilon \equiv 0$ and f_2^ϵ uniformly bounded with respect to ϵ, is simpler to solve. Indeed, in the proof of Theorem B.3, specifically in estimate (B.41), we made use of Poicaré inequality combined with energy estimates in order to absorb an integral of ξ_t. The reason for adopting this strategy is that in the Carleman estimate for ξ (see (B.17)) we have, in the left-hand side, a global weighted norm of ξ_t multiplied by ϵ^2, meaning that we could not absorb the ξ_t appearing in (B.41) directly. Now, if $f_1\equiv 0$ and f_2 is active, the observability inequality we have to prove is (B.5), then we need a local norm of ξ in the right-hand side. When proceeding in a similar way as in (B.41), we would obtain an local integral of φ_t multiplied by ϵ^2 on the right hand-side but in the Carleman estimate for φ (see again (B.17)) we have no ϵ^2 multiplying the global integral of φ_t, then we can absorb this term directly by taking ϵ sufficiently small.

On the conditions (A) and (B)

Conditions (A) and (B) are only sufficient. The problem of controlling systems of parabolic equations with less control forces than equations is far from being completely understood (see [39, 40]). An interesting problem is to find new conditions, different from those in (A) or (B) such that Theorem B.1 still holds.

Second order couplings

Inspired on the results of [22], we can formulate a problem of uniform null controllability for systems of parabolic equations of the form

$$\begin{cases} y_t^\epsilon - \Delta y^\epsilon - a_{1,1}y^\epsilon - A_{1,1}\cdot\nabla y^\epsilon = P_1(z^\epsilon\theta_1) + f_1^\epsilon 1_{\omega_1}, & \text{in } Q_T, \\ \epsilon z_t^\epsilon - \Delta z^\epsilon - a_{2,1}z^\epsilon - A_{2,1}\cdot\nabla z^\epsilon = P_2(y^\epsilon\theta_2), & \text{in } Q_T, \\ y^\epsilon = z^\epsilon = 0, & \text{on } \Sigma_T, \\ y^\epsilon|_{t=0} = y_0, z|_{t=0} = z_0, & \text{in } \Omega, \end{cases} \tag{B.64}$$

where $a_{i,j}$ are constants and θ_i are smooth enough with $|\theta_2| \geq C > 0$ in $\omega' \subset \omega_1$ and P_j are differential operator of order j in the space variable. The problem here is to prove Theorem B.1 by replacing system (B.1) by (B.64).

Insensitizing control problem

Another interesting question is concerned to the uniform insensitizing control problem for the solutions of

$$\begin{cases} y_t^\epsilon - \Delta y^\epsilon = az^\epsilon + \eta_1 + f_1^\epsilon 1_{\omega_1}, & \text{in } Q_T, \\ \epsilon z_t^\epsilon - \Delta z^\epsilon = by^\epsilon + \eta_2 + f_2^\epsilon 1_{\omega_2}, & \text{in } Q_T, \\ y^\epsilon = z^\epsilon = 0, & \text{on } \Sigma_T, \\ y^\epsilon|_{t=0} = y_0 + \tau_1\hat{y}_0, z|_{t=0} = z_0 + \tau_2\hat{z}_0, & \text{in } \Omega. \end{cases} \tag{B.65}$$

Here a and b are constants, $\eta_j \in L^2(Q_T)$, $y_0, z_0 \in L^2(\Omega)$ and τ_1, τ_2 are small real numbers and $\hat{y}_0$ and $\hat{z}_0$ are unknown function such that $||\hat{y}_0||_{L^2(\Omega)} = ||\hat{z}_0||_{L^2(\Omega)} = 1$. For (y,z) solution of (B.65) and $\mathcal{O}_1$, $\mathcal{O}_2$ open subsets of Ω, consider the *sentinel* functional

$$J(y^\epsilon, z^\epsilon; \tau_1, \tau_2) = \frac{\alpha}{2}\int_0^T\int_{\mathcal{O}_1} |y^\epsilon|^2 dxdt + \frac{\beta}{2}\int_0^T\int_{\mathcal{O}_2} |z^\epsilon|^2 dxdt.$$

These integrals can be viewed as the kinetic or also potential energy for variables y^ϵ or z^ϵ.

The problem consists in finding controls $(f_1^\epsilon, f_2^\epsilon)$, uniformly bounded with respect to ϵ, such that

$$\frac{\partial}{\partial\tau_1} J(y,z;\tau_1,\tau_2)\bigg|_{\tau_1=0} = 0, \tag{B.66}$$

or

$$\frac{\partial}{\partial\tau_2} J(y,z;\tau_1,\tau_2)\bigg|_{\tau_2=0} = 0. \tag{B.67}$$

It is standard in this kind of problem that conditions (B.66) and (B.67) are equivalent to the partial null controllability for the following system

$$\begin{cases} p_t^\epsilon - \Delta p^\epsilon = aq^\epsilon + \eta_1 + f_1^\epsilon 1_{\omega_1}, & \text{in } Q_T, \\ -u_t^\epsilon - \Delta u^\epsilon = av^\epsilon + \alpha p^\epsilon 1_{\omega_1}, & \text{in } Q_T, \\ \epsilon q_t^\epsilon - \Delta q^\epsilon = bp^\epsilon + \eta_2 + f_2^\epsilon 1_{\omega_2}, & \text{in } Q_T, \\ -\epsilon v_t^\epsilon - \Delta v^\epsilon = bu^\epsilon + \beta q^\epsilon 1_{\omega_2}, & \text{in } Q_T, \\ p^\epsilon = q^\epsilon = u^\epsilon = v^\epsilon = 0, & \text{on } \Sigma_T, \\ p^\epsilon|_{t=0} = y_0^\epsilon, u^\epsilon|_{t=T} = 0, q^\epsilon|_{t=0} = z_0^\epsilon, v|_{t=T} = 0, & \text{in } \Omega_T. \end{cases} \tag{B.68}$$

More precisely, the problem is reduced to finding controls $(f_1^\epsilon, f_2^\epsilon)$ uniformly bounded with respect to ϵ, such that $u^\epsilon(0) = v^\epsilon(0) = 0$ in Ω.

Conclusion of Appendix B

In this paper we proved an uniform null controllability result for a family of parabolic-parabolic systems with first order coupling terms, degenerating into a parabolic-elliptic system when a diffusion parameter goes to zero. By proving an appropriated Carleman inequality and combining it with energy estimates we were able to prove an uniform observability inequality, in the sense that the observability constant is uniformly bounded with respect to the diffusion parameter. The difficulty here is that, in every computation we had to be careful about this parameter in order to have an uniform result. A natural strategy to solve this problem is first to follow the arguments of *Chaves-Silva et al* in [27]. By doing so, we found some difficulties in absorbing some global weighted norms, and we could not prove the result in this way due to the first order coupling terms. Then, looking carefully to the proof of [27], we realized that the proof in there could be simplified and by doing so we could apply to case considered here. We expect that the strategy adopted here can be of use in many other situation such as the ones presented in Section B.