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FLUIDOS MICROPOLARES COM CONVECÇÃO TÉRMICA: existência, unicidade e estimativas de erro do método de Galerkin

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Tese apresentada ao Departamento de Matemática da Universidade Federal de Pernambuco, como parte dos requisitos para obtenção do título de Doutor em Matemática.

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Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

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Para Thaís e Breno

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Charles Braga Amorim

RESUMO

Neste trabalho estudamos aspectos teóricos das equações que modelam o movimento de um fluido assimétrico (micropolar), viscoso, incompressível com convecção térmica em um domínio limitado de $\mathbb{R}^3$ com fronteira suave. De maneira mais objetiva, este estudo foi realizado usando um método iterativo no qual obtivemos aproximação, decaimento da solução e melhoramos a regularidade para o modelo micropolar com convecção térmica, com a vantagem de dispensar argumentos de compacidade que seriam necessários para utilizar o método de Galerkin por exemplo. Aplicamos também ao modelo o método de Galerkin espectral a fim de estimar o erro por potências do inverso dos autovalores $\lambda_{k+1}, \gamma_{k+1}$ e $\widetilde{\gamma}_{k+1}$ dos operadores de Laplace, Stokes e L considerando-se aproximações nos subespaços V_k, H_k e $\widetilde{H}_k$. Estas estimativas de erro para o método de Galerkin são muito importantes, tendo já sido estudadas para outros modelos como Navier-Stokes e Boussinesq por exemplo. As estimativas obtidas tem ampla aplicação em métodos numéricos, como por exemplo o método dos elementos finitos.

Palavras-chave: Equações do tipo Navier-Stokes. Fluidos assimétricos. Método iterativo.

ABSTRACT

In this work we study theorical aspects of the equations that model the motion assymetric fluid (micropolar), viscous and incompressible with thermal convection in a bounded domain of $\mathbb{R}^3$ with smooth boundary, More objectively, this study was carried out using interactive approach we got decay of solution and improve the regularity for model micropolar with termal convection, with the advantage to dispense compactness arguments which are required to use Galerkin method for example. We also apply to the model the Galerkin espectral method in order to estimate the error by inverse powers of the eingvalues $\lambda_{k+1}, \gamma_{k+1}$ and $\widetilde{\gamma}_{k+1}$ of the Stokes, Laplace and L operators considering approximations in subspaces V_k, H_k e $\widetilde{H}_k$. These error estimates for galerkin method are very important, having already been studied for other models such Navier-Stokes and Boussinesq for example. The estimates obtained are widely used in numerical methods, such as the finite element method.

Keywords: Navier-Stokes type equations. assymetric fluids. Interactive approach.

SUMÁRIO

1 INTRODUÇÃO

Fluidos micropolares (assimétricos) são fluidos com microestrutura, ou seja, as partículas imersas no fluido estão sujeitas tanto a rotações, como a translações, o que nos leva de forma natural a considerar a velocidade angular. Exemplos destes fluidos são o sangue e os cristais líquidos. O modelo clássico de Navier-Stokes não descreve de maneira satisfatória a dinâmica destes fluidos, desta forma em 1966 Eringen [6] no trabalho intitulado *Theory of micropolar fluids* introduziu o modelo para estes fluidos. Desde então a teoria se expandiu bastante, veja por exemplo [15], [20], [21] e [22]. Um modelo mais geral do que o proposto por Eringen [6] é o seguinte: seja Ω um domínio limitado de $\mathbb{R}^3$ com fronteira suave $\partial\Omega$, e seja $T > 0$. Vamos considerar o problema que descreve o movimento de um fluído micropolar com convecção térmica, viscoso e incompressível na região $Q_T = \Omega \times (0,T)$:

$$\left\{\begin{aligned} \boldsymbol{u}_t - (\mu+\mu_r)\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= 2\mu_r \operatorname{rot}\boldsymbol{w} + \boldsymbol{f}(\theta), \\ \boldsymbol{w}_t - (c_a+c_d)\Delta\boldsymbol{w} + (u\cdot\nabla)\boldsymbol{w} + 4\mu_r\boldsymbol{w} &= (c_0+c_d-c_a)\nabla\operatorname{div}\boldsymbol{w} + 2\mu_r\operatorname{rot}\boldsymbol{u} + \boldsymbol{g}(\theta), \\ \theta_t + \boldsymbol{u}\cdot\nabla\theta - \kappa\Delta\theta &= \Phi(\boldsymbol{u},\boldsymbol{w}) + h, \\ \operatorname{div}\boldsymbol{u} &= 0, \end{aligned}\right. \tag{1.1}$$

junto com as seguintes condições iniciais e de fronteira

$$\left\{\begin{array}{l} \boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{w} = \boldsymbol{0}, \quad \theta = 0 \text{ em } S_T, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{w}(0) = \boldsymbol{w}_0, \ \theta(0) = \theta_0, \ \text{ em } \Omega, \end{array}\right. \tag{1.2}$$

onde $S_T = \partial\Omega \times (0,T)$. As funções vetoriais $\boldsymbol{u} = (u_1,u_2,u_3)$, $\boldsymbol{w} = (w_1,w_2,w_3)$ e as funções escalares p e θ denotam a velocidade, a velocidade angular do vetor de rotação das partículas, a pressão e a temperatura, respectivamente. As funções $\boldsymbol{f}$, $\boldsymbol{g}$ denotam forças externas do momento linear e angular, e h a fonte de calor externa. As constantes positivas μ, μ_r, c_0, c_a e c_d verificam $c_0 + c_d > c_a$ e a constante positiva κ é a condutividade de calor. A função Φ denota a função dissipação $\Phi = \sum_{i=1}^{5} \Phi_i$, onde

$$
\begin{aligned}
\Phi_1(\boldsymbol{u}) &= \frac{1}{2}\mu \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2, \\
\Phi_2(\boldsymbol{u}, \boldsymbol{w}) &= 4\mu_r \left| \frac{1}{2} \operatorname{rot} \boldsymbol{u} - \boldsymbol{w} \right|^2, \\
\Phi_3(\boldsymbol{w}) &= c_0 (\operatorname{div} \boldsymbol{w})^2, \\
\Phi_4(\boldsymbol{w}) &= (c_a + c_d) \sum_{i,j=1}^{3} \left(\frac{\partial w_i}{\partial x_j} \right)^2, \\
\Phi_5(\boldsymbol{w}) &= (c_d - c_a) \sum_{i,j=1}^{3} \frac{\partial w_i}{\partial x_j} \frac{\partial w_j}{\partial x_i}.
\end{aligned}
$$

Iremos supor que as funções $\boldsymbol{f}, \boldsymbol{g}$ e h verificam

$$
|\boldsymbol{f}(s) - \boldsymbol{f}(t)| \leq M_{\boldsymbol{f}} |t - s|, \quad |\boldsymbol{g}(s) - \boldsymbol{g}(t)| \leq M_{\boldsymbol{g}} |t - s| \tag{1.3}
$$

para $s, t \in \mathbb{R}$ e constantes $M_{\boldsymbol{f}}, M_{\boldsymbol{g}} > 0$, $\boldsymbol{f}(0) = \boldsymbol{g}(0) = 0$ e $h \in L^2(0, T; L^2(\Omega))$.

Os símbolos ∇, Δ, rot e div representam os operadores *gradiente, Laplaciano, rotacional e divergente* respectivamente, $\boldsymbol{u}_t$, $\boldsymbol{w}_t$, e θ_t denotam as derivadas com respeito ao tempo de $\boldsymbol{u}$, $\boldsymbol{w}$ e θ. A j-ésima componente de $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, $(\boldsymbol{u} \cdot \nabla)\boldsymbol{w}$ e $(\boldsymbol{u} \cdot \nabla)\theta$ são dadas, de forma respectiva, por

$$
\begin{aligned}
[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}]_j &= \sum_{i=1}^{3} \boldsymbol{u}_i \frac{\partial \boldsymbol{u}_j}{\partial x_i}, \\
[(\boldsymbol{u} \cdot \nabla)\boldsymbol{w}]_j &= \sum_{i=1}^{3} \boldsymbol{u}_i \frac{\partial \boldsymbol{w}_j}{\partial x_i}, \\
(\boldsymbol{u} \cdot \nabla)\theta &= \sum_{i=1}^{3} \boldsymbol{u}_i \frac{\partial \theta}{\partial x_i}
\end{aligned}
$$

Uma das dificuldades para tratar o problema (1.1) é que a função dissipação Φ é um termo quadrático não linear em termos de $\nabla \boldsymbol{u}$ e $\nabla \boldsymbol{w}$. Assim a equação de balanço de energia de (1.1) fará sentido em $H^{-1}(\Omega)$. Entendemos por solução do problema (1.1) a tripla de funções $(\boldsymbol{u}, \boldsymbol{w}, \theta)$ satisfazendo

$$
\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), && \boldsymbol{u}_t \in L^2(0,T;H),\\
&\boldsymbol{w} \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), && \boldsymbol{w}_t \in L^2(0,T;L^2(\Omega)),\\
&\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), && \theta_t \in L^2(0,T;H^{-1}(\Omega)),
\end{aligned}
$$

as identidades

$$
\begin{aligned}
\int_0^T (\boldsymbol{u}_t - (\mu+\mu_r)\Delta\boldsymbol{u}, \varphi)dt + \int_0^T b(\boldsymbol{u},\boldsymbol{u},\varphi)dt &= \int_0^T (2\mu_r \operatorname{rot}\boldsymbol{w} + \boldsymbol{f}(\theta), \varphi)dt,\\
\int_0^T (\boldsymbol{w}_t + L\boldsymbol{w} + 4\mu_r\boldsymbol{w}, \psi) + \int_0^T b(\boldsymbol{u},\boldsymbol{w},\psi) &= \int_0^T (2\mu_r \operatorname{rot}\boldsymbol{u} + \boldsymbol{g}(\theta), \psi)dt\\
\int_0^T \langle\theta_t, \phi\rangle + \kappa\int_0^T (\nabla\theta, \nabla\phi)dt + \int_0^T (u\cdot\nabla\theta, \phi)dt &= \int_0^T (\Phi(\boldsymbol{u},\boldsymbol{w}) + h, \phi)dt
\end{aligned}
$$

para $\varphi \in L^2(0,T;H), \psi \in L^2(0,T;L^2(\Omega)), \phi \in L^2(0,T;H_0^1(\Omega))$, e as condições iniciais $\boldsymbol{u}(0) = \boldsymbol{u}_0, \boldsymbol{w}(0) = \boldsymbol{w}_0$ e $\theta(0) = \theta_0$.

Como já foi mencionado acima o problema (1.1) inclui, como caso particular, as equações de um fluido micropolar. De fato, basta não considerar a equação de balanço de energia. Desta forma o problema (1.1) se reduz a

$$
\left\{
\begin{aligned}
\boldsymbol{u}_t - (\mu+\mu_r)\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= 2\mu_r \operatorname{rot}\boldsymbol{w} + \boldsymbol{f},\\
\boldsymbol{w}_t - (c_a+c_d)\Delta\boldsymbol{w} + (u\cdot\nabla)\boldsymbol{w} + 4\mu_r\boldsymbol{w} &= (c_0+c_d-c_a)\nabla\operatorname{div}\boldsymbol{w} + 2\mu_r\operatorname{rot}\boldsymbol{u} + \boldsymbol{g},\\
\operatorname{div}\boldsymbol{u} &= 0,
\end{aligned}
\right.
$$

junto com as seguintes condições iniciais e de fronteira

$$
\left\{
\begin{aligned}
&\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{w} = \boldsymbol{0}, \quad \text{em } S_T,\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{w}(0) = \boldsymbol{w}_0, \ \text{ em } \Omega.
\end{aligned}
\right.
$$

Incluímos aqui também as equações de Oberbeck-Boussinesq, para isto basta em (1.1) omitir a equação do momento angular e fazer $\Phi = 0$, desta forma obtemos

$$
\left\{
\begin{aligned}
\boldsymbol{u}_t - (\mu+\mu_r)\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f}(\theta),\\
\theta_t + \boldsymbol{u}\cdot\nabla\theta - \kappa\Delta\theta &= h,\\
\operatorname{div}\boldsymbol{u} &= 0,
\end{aligned}
\right.
$$

junto com as seguintes condições iniciais e de fronteira

$$
\left\{
\begin{aligned}
&\boldsymbol{u} = \boldsymbol{0}, \quad \theta = 0 \text{ on } S_T,\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \theta(0) = \theta_0, \ \text{in } \Omega,
\end{aligned}
\right.
$$

veja [3] e [8].

Por fim ao omitir as equações de energia e do momento angular obtemos as equações clássicas de Navier-Stokes,

$$\begin{cases} \boldsymbol{u}_t - (\mu + \mu_r)\Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \\ \operatorname{div} \boldsymbol{u} &= 0, \end{cases}$$

junto com as seguintes condições iniciais e de fronteira

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0}, \ \text{em } S_T, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \text{ em } \Omega, \end{cases}$$

veja [9], [11] e [23].

O problema estacionário associado a (1.1) foi estudado por Lukaszewicz e Walus em [14]. Eles mostraram a existência e unicidade de uma solução $(\boldsymbol{u}, \boldsymbol{w}, \theta) \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$, supondo $\boldsymbol{f}$ e $\boldsymbol{g}$ limitadas e verificando a condição (1.3), para μ, $c_a + c_d$ suficientemente grandes. Por um argumento de ponto fixo a existência e unicidade local do problema (1.1) foi provado por Kagei e Skowron [12] considerando $\mu_r, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}$ e h suficientemente pequenos. A saber

Teorema 1.1. *Para cada tripla de valores iniciais* $(\boldsymbol{u}_0, \boldsymbol{w}_0, \theta_0) \in V \times H_0^1(\Omega) \times L^2(\Omega)$ *existe um número positivo* T_* *e uma única solução* $(\boldsymbol{u}, \boldsymbol{w}, \theta)$ *para o problema* (1.1) *em* $[0, T_*]$, *onde* T_* *depende apenas de* $\mu, \mu_r, c_a, c_d, c_0, \kappa, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}, \Omega, (\boldsymbol{u}_0, \boldsymbol{w}_0, \theta_0)$ *e* h. *Além disso se* $\mu_r, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}, (\boldsymbol{u}_0, \boldsymbol{w}_0)$ *são suficientemente pequenos, então a solução forte* $(\boldsymbol{u}, \boldsymbol{w}, \theta)$ *existe em todo* $[0, \infty)$.

Nosso trabalho está estruturado da seguinte maneira:

No Capitulo 1 apresentamos conceitos preliminares como, espaços funcionais, desigualdades clássicas e resultados importantes que serão de utilidade no decorrer do texto.

No Capitulo 2 usamos um método iterativo para mostrar existência e unicidade de soluções de (1.1)-(1.2) estes são mostrados no Teorema 3.3 e os resultados obtidos neste teorema são compatíveis com os obtidos por Kagei e Skowron [12] sob a forma do Teorema 1.1. Além disso, no Teorema 3.4 supondo $f, g \in C^1$, obtemos mais regularidade para a solução do problema (1.1)-(1.2), algo que ainda não havia sido abordado. Também determinamos o raio de convergência desta aproximação em varias normas, que é importante do ponto de vista teórico e da análise numérica. Este método foi proposto por Zarubin [24] para encontrar a solução de um problema de valor inicial e de fronteira para as equações de convecção termal da aproximação de Boussinesq. O método foi utilizado nas equações dos fluidos micropolares em [19] por Rojas-Medar e Ortega-Torres. O método consiste em encontrar uma sequência de soluções, $\{(\boldsymbol{u}^n, \boldsymbol{w}^n, \theta^n)\}_{n\geq 1}$, para o problema (1.1)-(1.2) linearizado. Depois de provar algumas estimativas em espaços de Banach adequados é necessário concluir que essa sequência é uma sequência de Cauchy. Assim, o limite $(\boldsymbol{u}, \boldsymbol{w}, \theta)$, quando $n \to \infty$, será a solução desejada do problema (1.1). Este método é mais simples que o utilizado em [12], e mais simples que o método de Galerkin visto que não precisamos de resultados de compacidade.

No Capitulo 3 aplicamos o método de *Galerkin espectral* ao problema (1.1) afim de estimar o erro por potências do inverso dos autovalores $\lambda_{k+1}, \gamma_{k+1}$ e $\widetilde{\gamma}_{k+1}$ dos operadores de Stokes, Laplace e L considerando-se aproximações nos subespaços V_k, H_k e $\widetilde{H}_k$. Estas estimativas de erro para o método de Galerkin são importantes pela ampla aplicação de tais métodos em experimentos numéricos. Em 1980, Rautmann [18] sistematizou as estimativas de erro para o método de Galerkin espectral aplicado às equações de Navier-Stokes clássicas. O caso de fluidos magneto-micropolares foi tratado por Ortega-Torres, Rojas-Medar e Cabrales em [16]. Inspirados nestas ideias obtemos no Teorema 4.4 estimativas na norma de $L^2(\Omega)$ para o erro que se comete ao aproximar $(\boldsymbol{u}, \boldsymbol{w}, \theta)$ por $(\boldsymbol{u}^k, \boldsymbol{w}^k, \theta^k)$ suas respectivas aproximações de Galerkin. No Teorema 4.6, fizemos o mesmo para $\boldsymbol{u}$ e $\boldsymbol{w}$ na norma de $H^1(\Omega)$. Por fim, tratamos de outras normas no Teorema 4.7.

2 PRELIMINARES

Afim de tornar o trabalho autossuficiente apresentaremos aqui conceitos e resultados que serão utilizados nos capítulos seguintes.

2.1 ESPAÇOS DE FUNÇÕES E NOTAÇÕES

No que segue Ω sempre irá denotar um domínio limitado no espaço tridimensional $\mathbb{R}^3$ com fronteira $\partial\Omega$. Denotemos por $L^p(\Omega)$ com $1 \leq p < \infty$, o espaço das funções reais u definidas em Ω, mensuráveis, tais que $|u|^p$, é integrável a Lebesgue em Ω, ou seja

$$L^p(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R}, u \text{ mensurável e } \int_\Omega |u|^p dx < \infty \right\},$$

que é um espaço de Banach se munido da norma usual

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{1/p}.$$

No caso $p = 2$, $L^2(\Omega)$ é um espaço de Hilbert separável munido do produto interno

$$(u, v) = \int_\Omega u(x)v(x)dx.$$

Por simplicidade, neste caso denotaremos $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$. Uma função mensurável u é dita essencialmente limitada em Ω se existe $C \in \mathbb{R}^+$ de modo que $|u(x)| \leq C$ exceto em um conjunto de medida zero de Ω (a.e). Denotaremos por $L^\infty(\Omega)$ o espaço das funções reais u mensuráveis e essencialmente limitadas em Ω. chama-se supremo essencial de u a menor das constantes com tal propriedade, ou seja

$$\sup_{x\in\Omega}\text{ess}|u(x)| := \inf\left\{C \in \mathbb{R}^+; |u(x)| \leq C \text{ a.e em } \Omega\right\}$$

Desta forma, $L^\infty(\Omega)$ é um espaço de Banach munido da norma

$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega}\text{ess}\ |u(x)|.$$

Desde que E seja um espaço de Banach e $0 < T \leq \infty$, $L^q(0,T;E)$ com $1 \leq q \leq \infty$, irá denotar o espaço de Banach das funções com valores em E definidas em $[0,T)$ que são L^q-integráveis a Bochner, com as normas usuais

$$\begin{aligned} \|u\|_{L^q(0,T;E)} &= \left(\int_0^T \|u(t)\|_E^q dt\right)^{1/q}, \ 1 \leq q < \infty, \\ \|u\|_{L^\infty(0,T;E)} &= \sup_{t\in(0,T)}\ \text{ess}\ \|u(x)\|_E. \end{aligned}$$

Para $m \geq 0$ e $1 \leq p \leq \infty$, consideraremos os espaços de Sobolev usuais

$$W^{m,p} = \{f \in L^p(\Omega) : \|D^\alpha f\|_{L^p(\Omega)} < \infty, |\alpha| \leq m\},$$

onde $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ é um multi-índice. D^α denota o operador derivação de ordem α definido da seguinte forma

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$$

onde $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. Para cada $u \in W^{m,p}(\Omega)$, definimos de maneira usual

$$\begin{aligned} \|u\|_{m,p} &= \left(\sum_{|\alpha|\leq m}\int_\Omega |D^\alpha u(x)|^p dx\right)^{1/p}, \ 1 \leq p < \infty \\ \|u\|_{m,\infty} &= \sum_{|\alpha|\leq m}\sup_{x\in\Omega}\text{ess}|D^\alpha u(x)|. \end{aligned}$$

Quando $p = 2$, $W^{m,2}(\Omega)$ é um espaço de Hilbert separável. Denotamos $W^{m,2}(\Omega) = H^m(\Omega)$ e $\|\cdot\|_{m,p} = \|\cdot\|_{H^m}$.

Definimos

$$H_0^m(\Omega) = \text{o fecho de } C_0^\infty(\Omega) \text{ na norma } H^m(\Omega)$$

onde $C_0^\infty(\Omega)$ denota o conjunto de todas as funções $f : \Omega \longrightarrow \mathbb{R}$ com suporte compacto e de classe C^∞. Note que uma "caracterização" para $H_0^1(\Omega)$ é a seguinte

$$H_0^1(\Omega) = \{u \in H^1(\Omega); u|_{\partial\Omega} = 0\},$$

onde $u|_{\partial\Omega}$ é traço de u.(veja [23]).

Como estamos considerando Ω um domínio limitado temos como consequência a desigualdade de Poincaré (veja [1])

$$\|u\| \leq \frac{1}{\lambda_1}\|\nabla u\|; \text{ para todo } u \in H_0^1(\Omega),$$

onde λ_1 é o primeiro autovalor do operador $-\Delta$ em $H_0^1(\Omega)$.

A seguir enunciaremos as desigualdades de Sobolev e o teorema de imersão de Sobolev. Mais detalhes e suas respectivas demonstrações podem ser vistas em [1] (p. 79, p. 97, p. 144).

Teorema 2.1 (Gagliardo-Nirenberg)**.** *Seja $\Omega \in \mathbb{R}^n$ um domínio limitado com fronteira $\partial\Omega$ localmente lipschitz, $m = 0, 1, 2, ...$ e $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$ com $1 \leq r, q \leq \infty$. Se j é um inteiro com $0 \leq j \leq m$ e α um real qualquer no intervalo $\frac{j}{m} \leq \alpha \leq 1$ tal que*

$$\frac{1}{p} = \frac{j}{N} + \alpha(\frac{1}{r} - \frac{m}{N}) + (1 - \alpha)\frac{1}{q},$$

então

$$\|D^j u\|_{L^p} \leq C\|u\|_{m,r}^{\alpha}\|u\|_{L^q}^{(1-\alpha)} \quad se \quad m - j - \frac{N}{r} < 0$$

e

$$\|D^j u\|_{L^p} \leq C\|u\|_{m,r}^{j/m}\|u\|_{L^q}^{(m-j)/m} \quad com \quad \alpha = \frac{j}{m} \; se \; m - j - \frac{N}{r} \geq 0,$$

onde C é uma constante que depende apenas de $\Omega, r, q, m, j, \alpha$.

Note que em particular se $u \in L^r(\Omega) \cap L^q(\Omega)$ com $1 \leq r \leq q \leq \infty$, claramente devemos ter $u \in L^p(\Omega)$ para todo $p \in [r, q]$ e além disso, se α satisfaz $\frac{1}{p} = \frac{\alpha}{r} + \frac{1-\alpha}{q}$ temos a desigualdade de interpolação,

$$\|u\|_{L^p} \leq \|u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}.$$

Teorema 2.2. *Seja $\Omega \in \mathbb{R}^n$ um domínio limitado com fronteira $\partial\Omega$ localmente lipschitz. Sejam m, n números inteiros e p um número real com $0 \leq n \leq m$ e $p \geq 1$. Desta forma temos as injeções contínuas e densas:*

$$W^{m,p}(\Omega) \hookrightarrow W^{n,q}(\Omega),$$

onde

$$\frac{1}{q} = \frac{1}{p} - \frac{m-n}{N} \quad se \quad p(m-n) < N,$$
$$q \in [1, \infty) \quad se \quad p(m-n) = N,$$
$$q = \infty \quad se \quad p(m-n) > N.$$

Além disso, a injeção $W^{n,p}(\Omega) \hookrightarrow W^{n,s}(\Omega)$ *é compacta para cada* $1 \leq s < q$, *com* q *definido como acima. Também se* $mp > N$ *e* k *é o maior número inteiro tal que* $0 \leq k < \dfrac{pm - N}{p}$, *temos a injeção compacta*

$$W^{m,p} \hookrightarrow C^k(\bar{\Omega}).$$

A seguir apresentaremos mais alguns resultados de uso frequente no decorrer do texto.

Teorema 2.3 (Desigualdade de Hölder generalizada)**.** *Sejam* $\Omega \in \mathbb{R}^n$ *um domínio limitado,* $p_i \geq 1$ *e* $f_i \in L^{p_i}(\Omega)$ *para* $i = 1, \cdots, k$, *onde* $\dfrac{1}{p} = \displaystyle\sum_{i=1}^{k} \frac{1}{p_i} \leq 1$. *Então*

$$f = \prod_{i=1}^{k} f_i \in L^p(\Omega) \;\; e \;\; \|f\|_{L^p} \leq \prod_{i=1}^{k} \|f_i\|_{L^{p_i}}.$$

Para demonstração, veja [4], p. 57.

Lema 2.4 (Desigualdade de Young generalizada)**.** *Sejam* a *e* b *números reais não negativos. Se* $p, q \in \mathbb{R}$ *são tais que* $1 \leq p < \infty$ *e* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *então para todo* $\epsilon > 0$, *temos*

$$ab \leq \epsilon a^p + C_\epsilon b^q,$$

onde $C_\epsilon = \dfrac{p-1}{p^q}\epsilon^{1-q}$.

Lema 2.5 (Desigualdade de Gronwall)**.** *Sejam* $a, b : [0, T] \to \mathbb{R}$ *funções. Suponha que* $a(t) \geq 0$ *é absolutamente contínua com* $a'(t) \geq 0$ *e a função* $b(t) \geq 0$ *somável em* $[0, T]$. *Além disso suponha que é satisfeita a seguinte desigualdade integral*

$$\varphi(t) + \int_0^t \varphi^*(\tau) d\tau \leq \frac{a(t)}{\lambda} + \int_0^t b(\tau)\varphi(\tau) d\tau,$$

onde φ *e* φ^* *são funções contínuas e positivas sobre* $[0, T]$ *com* $\lambda > 0$. *Então*

$$\varphi(t) + \int_0^t \varphi^*(\tau) d\tau \leq \frac{1}{\lambda}\left(1 + \int_0^t b(\tau) d\tau\right) a(t) \exp\left(\int_0^t b(\tau) d\tau\right).$$

A seguir, introduzimos alguns espaços funcionais necessários para o desenvolvimento deste trabalho. Considerando

$$\begin{aligned} H(\Omega) &= \text{ fecho de } C^{\infty}_{0,\sigma} \text{ na norma } (L^2(\Omega))^3, \\ V(\Omega) &= \text{ fecho de } C^{\infty}_{0,\sigma} \text{ na norma } (H^1(\Omega))^3, \end{aligned}$$

onde

$$C^{\infty}_{0,\sigma} = \{v \in (C^{\infty}_0(\Omega))^3; \ \text{div } v := \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = 0 \ \text{ em } \ \Omega\},$$

com as respectivas normas e produtos internos

$$\begin{aligned} (u,v)_H &= \sum_{i=1}^{3} \int_{\Omega} u_i(x) v_i(x) dx, \quad \|u\|_H = (u,u)_H^{1/2}, \\ (u,v)_V &= \sum_{i=1}^{3} (u_i, v_i)_{H^1}, \quad \|u\|_V = (u,u)_V^{1/2}. \end{aligned}$$

O espaço $V(\Omega)$ tem uma caracterização muito útil, a saber

$$V(\Omega) = \{v \in (H^1_0(\Omega))^3; \ \text{div } v = 0 \ \text{ em } \ \Omega\}$$

e consequentemente quando $u \in V(\Omega)$ são equivalentes as normas de $H^1(\Omega)$ e $H^1_0(\Omega)$. Ou seja, para $u \in V(\Omega)$ denotamos $\|u\|_V = \|\nabla u\|$.

O complemento ortogonal de $H(\Omega)$ em $(L^2(\Omega))^3$ tem também uma útil caracterização,

$$H^{\perp} = \{\phi \in (L^2(\Omega))^3; \phi = \nabla p \quad \text{para algum } \ p \in H^1(\Omega)\}.$$

Desta forma, o espaço $(L^2(\Omega))^3$ possui a decomposição de Helmholtz $(L^2(\Omega))^3 = H \oplus H^{\perp}$ e podemos então considerar a projeção ortogonal

$$P : (L^2(\Omega))^3 \longrightarrow H.$$

Note que, $P(\nabla g) = 0$, para todo $g \in H^1(\Omega)$.

Considere a aplicação

$$b : H^1_0(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega) \longrightarrow \mathbb{R}$$

definida por $b(u,v,w) = (u \cdot \nabla v, w)$. Esta é uma forma trilinear bem definida cujas propriedades a seguir utilizaremos exaustivamente,

$$b(u,v,v) = 0 \ \text{ e } \ b(u,v,w) = -b(u,w,v), \ \text{ para todo } u \in V, \ v,w \in (H_0^1(\Omega))^3.$$

O operador de Stokes $A = -P\Delta : D(A) = (H^2(\Omega))^3 \cap V(\Omega) \longrightarrow H(\Omega)$ é definido por

$$(Aw, v) = (\nabla w, \nabla v), \ \text{ para todo } \ w \in D(A) \cap V(\Omega), \ \ v \in V(\Omega).$$

Como A é um operador autoadjunto, definido positivo com inversa compacta, resultados de análise funcional garantem para A, uma sequência de autovalores positivos λ_i e uma sequência de autofunções $\{\varphi^i(x)\}_{i=1}^{\infty}$, tal que

$$A\varphi_i = \lambda_i \varphi^i, \quad \varphi^i \in D(A),$$

com $0 < \lambda_1 \leq \cdots \leq \lambda_{i-1} \leq \lambda_i \leq \cdots$ e $\lim\limits_{i \longrightarrow \infty} \lambda_i = \infty$.

Também serão considerados o operador de Laplace $B = -\Delta$ e o operador fortemente elíptico $L = -(c_a + c_d)\Delta - (c_0 + c_d - c_a)\nabla \text{div}$ onde $c_0 + c_d > c_a$, com condições de contorno de Dirichlet e domínio $D(B) = (H^2(\Omega))^3 \cap (H_0^1(\Omega))^3 = D(L)$. Note que para os operadores B e L são válidas considerações análogas as feitas para o operador A.

Utilizando resultados de Amrouche e Girault (veja [2]), quando Ω é de classe $C^{1,1}$, temos que $\|u\|_{H^2}$ e $\|Au\|$, $\|w\|_{H^2}$ e $\|Bw\|$, $\|\nabla u\|$ e $\|A^{1/2}u\|$, $\|\nabla w\|$ e $\|B^{1/2}w\|$ são normas equivalentes. Além disso, desde que $\|w\|_{H^2}$ e $\|Lw\|$ sejam normas equivalentes, $\|Bw\|$ e $\|Bw\|$, $\|L^{1/2}w\|$ e $\|\nabla w\|$, $\|L^{1/2}w\|$ e $\|B^{1/2}w\|$ também o serão.

O seguinte Lema será útil na hora de estimar a derivada (no tempo) de u_t, veja ([23], p. 260).

Lema 2.6. *Sejam V e H espaços de Hilbert tais que $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ com inclusões contínuas e densas. Se $u \in L^2(0,T;V)$ e $u' := \dfrac{du}{dt} \in L^2(0,T;V^*)$. Então*

$$\frac{d}{dt}\|u\|^2 = 2(u', u)$$

no sentido de distribuição sobre $(0,T)$.

Observação 2.7. *Como já é usual, a partir daqui denotaremos $(L^p(\Omega))^3$, $(H_0^1(\Omega))^3$, $(H^1(\Omega))^3$, $(H^2(\Omega))^3$, etc apenas por $L^p(\Omega)$, $H_0^1(\Omega)$, $H^1(\Omega)$, $H^2(\Omega)$ respectivamente.*

3 FLUIDOS MICROPOLARES COM CONVECÇÃO TÉRMICA

Utilizando um método iterativo proposto por Zarubin [24] nas aproximações de Oberbeck-Boussinesq, obteremos existência, unicidade e regularidade para o problema (1.1). Para as aproximações os devidos raios de convergência em várias normas. A vantagem desta abordagem é que evitamos argumentos de ponto fixo e resultados de compacidade.

3.1 RESULTADOS PRINCIPAIS

Afim de simplificar a notação denotaremos por C uma constante genérica finita positiva, dependendo apenas de Ω e de outros parâmetros fixo do problema. Esta pode assumir diferentes valores em diferentes expressões. Quando necessário, vamos enfatizar que as constantes tem valores diferentes usando a notação C_1, C_2, e assim por diante.

Para $\boldsymbol{u}_0, \boldsymbol{w}_0$ e θ_0, definimos

$$\boldsymbol{u}^1(t) = e^{-t(\mu+\mu_r)A}\boldsymbol{u}_0, \quad \boldsymbol{w}^1(t) = e^{-t(c_a+c_d)B}\boldsymbol{w}_0 \;\text{ e }\; \theta^1(t) = e^{-t\kappa B}\theta_0,$$

onde $e^{-t(\mu+\mu_r)A}$ e $e^{-t(c_a+c_d)B}$ denotam os semigrupos gerados pelos operadores de Stokes e Laplace respectivamente. Consideraremos as sequências $\{\boldsymbol{u}_n\}_{n\geq 1}$, $\{\boldsymbol{w}_n\}_{n\geq 1}$ e $\{\theta_n\}_{n\geq 1}$

dadas como soluções do seguinte problema linearizado:

$$
\begin{aligned}
\boldsymbol{u}_t^{n+1} + (\mu+\mu_r)A\boldsymbol{u}^{n+1} + P(\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^{n+1}) &= 2\mu_r P(\operatorname{rot}\boldsymbol{w}^n) + P\boldsymbol{f}(\theta^n), && (3.1)\\
\boldsymbol{w_t}^{n+1} + L\boldsymbol{w}^{n+1} + \boldsymbol{u}^n\cdot\nabla\boldsymbol{w}^{n+1} + 4\mu_r\boldsymbol{w}^{n+1} &= 2\mu_r\operatorname{rot}\boldsymbol{u}^n + \boldsymbol{g}(\theta^n), && (3.2)\\
\theta_t^{n+1} + (\boldsymbol{u}^n\cdot\nabla)\theta^{n+1} - \kappa\Delta\theta^{n+1} &= \Phi(\boldsymbol{u}^n,\boldsymbol{w}^n) + h && (3.3)\\
\operatorname{div}\boldsymbol{u}^{n+1} &= 0,
\end{aligned}
$$

em $\Omega\times(0,T)$, com fronteira e condições iniciais

$$
\begin{cases}
\boldsymbol{u^{n+1}} = \boldsymbol{w^{n+1}} = \theta^{n+1} = 0 \text{ in } \partial\Omega\times(0,T),\\
\boldsymbol{u^{n+1}}(0) = \boldsymbol{u_0},\ \boldsymbol{w^{n+1}}(0) = \boldsymbol{w_0},\ \theta^{n+1}(0) = \theta_0 \text{ in } \Omega.
\end{cases} \tag{3.4}
$$

Por simplicidade, vamos considerar $\boldsymbol{u_0} = \boldsymbol{w_0} = 0$ e $\theta_0 = 0$. Iremos supor também que as funções $\boldsymbol{f}, \boldsymbol{g}$ verificam

$$
|\boldsymbol{f}(s) - \boldsymbol{f}(t)| \le M_{\boldsymbol{f}}|t-s|,\ \ |\boldsymbol{g}(s) - \boldsymbol{g}(t)| \le M_{\boldsymbol{g}}|t-s|, \tag{3.5}
$$

para $s,t\in\mathbb{R}$ e constantes $M_{\boldsymbol{f}}, M_{\boldsymbol{g}} > 0$, $\boldsymbol{f}(0) = \boldsymbol{g}(0) = 0$. Em nosso primeiro resultado, estabelecemos existência e estimativas de limitação uniforme para estas sequências.

Teorema 3.1. *Sejam $\boldsymbol{f}$, $\boldsymbol{g}$ verificando* (3.5) *e h em $L^2(0,T;L^2(\Omega))$, $\boldsymbol{u}_0 = \boldsymbol{w}_0 = \theta_0 = 0$. Para cada n existe uma única solução $(\boldsymbol{u}^n, \boldsymbol{w}^n, \theta^n)$ do problema (3.1)-(3.4), definida no intervalo $[0,T_1]$, com $0 < T_1 \le T$, tal que*

$$
\begin{aligned}
\boldsymbol{u}^n &\in L^\infty(0,T_1;V)\cap L^2(0,T_1;D(A)),\\
\boldsymbol{w}^n &\in L^\infty(0,T_1;H_0^1(\Omega))\cap L^2(0,T_1;D(B)),\\
\theta^n &\in L^\infty(0,T_1;L^2(\Omega))\cap L^2(0,T_1;H_0^1(\Omega)),\\
\boldsymbol{u}_t^n &\in L^2(0,T_1;H),\ \boldsymbol{w}_t^n \in L^2(0,T_1;L^2(\Omega)), \theta_t^n \in L^2(0,T_1;H^{-1}(\Omega)).
\end{aligned}
$$

Além disso, existe uma constante $M_0 > 0$, independente de n, tal que

$$
\begin{aligned}
\int_0^t \|\nabla\boldsymbol{u}^n(\tau)\|^2 d\tau + \int_0^t \|\nabla\boldsymbol{w}^n(\tau)\|^2 d\tau + \int_0^t \|\nabla\theta^n(\tau)\|^2 d\tau &\le M_0,\\
\sup_{t\in(0,T_1)} \{\|\nabla\boldsymbol{w}^n(t)\|^2 + \|\nabla\boldsymbol{w}^n(t)\|^2 + \|\theta^n(t)\|^2\} &\le M_0,\\
\int_0^t \|A\boldsymbol{u}^n(\tau)\|^2 d\tau + \int_0^t \|B\boldsymbol{w}^n(\tau)\|^2 d\tau &\le M_0,\\
\int_0^t \|\boldsymbol{u}_\tau^n(\tau)\|^2 d\tau + \int_0^t \|\boldsymbol{w}_\tau^n(\tau)\|^2 d\tau + \int_0^t \|\theta_\tau^n(\tau)\|_{H^{-1}}^2 d\tau &\le M_0,
\end{aligned}
$$

para cada $t \in (0, T_1)$.

O valor de T_1 *depende apenas de* $\mu, \mu_r, c_a, c_d, c_0, \kappa, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}, \Omega$ *e* h. *Além disso, se* $\mu_r, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}$ *e* h *são suficientemente pequenos, então é possível escolher* $T_1 = T$.

Observação 3.2. *Para cada* $n \in \mathbb{N}$, *existe* $p^n \in L^2(0, T_1; H^1(\Omega)/\boldsymbol{R})$ *tal que*

$$\boldsymbol{u}_t^{n+1} + \boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^{n+1} - (\mu + \mu_r)\Delta \boldsymbol{u}^{n+1} + \nabla p^n = 2\mu_r \operatorname{rot} \boldsymbol{w}^n + \boldsymbol{f}(\theta^n)$$

em $\Omega \times (0, T_1)$, *e* $\displaystyle\int_0^t \|p^n(\tau)\|^2_{H^1(\Omega)/\boldsymbol{R}} d\tau \leq C_0$, *para cada* $t \in [0, T_1]$ *e alguma constante* $C_0 > 0$. *De fato, de (3.1) nós temos* $(\mu + \mu_r)A\boldsymbol{u}^{n+1} = P(\boldsymbol{F}^{n+1})$, *onde* $\boldsymbol{F}^{n+1} = 2\mu_r \operatorname{rot} \boldsymbol{w}^n + \boldsymbol{f}(\theta^n) - \boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^{n+1} - \boldsymbol{u}_t^{n+1}$. *Assim,*

$$\begin{aligned}
\|\boldsymbol{F}^{n+1}\|^2 &\leq C\|\nabla \boldsymbol{w}^n\|^2 + C\|\theta^n\|^2 + C\|\boldsymbol{u}^n\|_{L^4}\|\nabla \boldsymbol{u}^{n+1}\|_{L^4} + C\|\boldsymbol{u}_t^{n+1}\|^2 \\
&\leq C\|\nabla \boldsymbol{w}^n\|^2 + C\|\theta^n\|^2 + C\|\nabla \boldsymbol{u}^n\|\|A\boldsymbol{u}^{n+1}\| + C\|\boldsymbol{u}_t^{n+1}\|^2.
\end{aligned}$$

Do Teorema 3.1, concluímos que $\displaystyle\int_0^t \|\boldsymbol{F}^{n+1}(\tau)\|^2 d\tau \leq C$, *isto é,* $\boldsymbol{F}^{n+1} \in L^2(0, T; L^2(\Omega))$. *Consequentemente, pelo Teorema 3 de Amrouche e Girault [2], existe uma única função* $p^n \in L^2(0, T; H^1(\Omega)/\boldsymbol{R})$ *tal que* $-(\mu + \mu_r)\Delta \boldsymbol{u}^{n+1} + \nabla p^n = \boldsymbol{F}^{n+1}$ *em* Q_T, *e* $\|p^n\|^2_{H^1(\Omega)/\boldsymbol{R}} \leq C\|\boldsymbol{F}^{n+1}\|^2$.

Estabelecemos agora nosso primeiro resultado.

Teorema 3.3. *Seja* $\boldsymbol{f}$, $\boldsymbol{g}$ *verificando (3.5) e* h *em* $L^2(0, T; L^2(\Omega))$, $\boldsymbol{u}^1 = \boldsymbol{w}^1 = \theta^1 = 0$. *Seja* $(\boldsymbol{u}^n, \boldsymbol{w}^n, \theta^n)$ *a sequência dada pelo Teorema 3.1. Então,* $(\boldsymbol{u}^n, \boldsymbol{w}^n, \theta^n)$ *converge a uma função* $(\boldsymbol{u}, \boldsymbol{w}, \theta)$ *quando* $n \to \infty$. *Além disso,* $(\boldsymbol{u}, \boldsymbol{w}, \theta)$ *é a única solução do problema (1.1), e a convergência satisfaz as seguintes desigualdades:*

$$\begin{aligned}
&\|\nabla \boldsymbol{u}^n(t) - \nabla \boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{w}^n(t) - \nabla \boldsymbol{w}(t)\|^2 + \|\theta^n(t) - \theta(t)\|^2 \leq C\Lambda_{n-1}(t), \\
&\int_0^t \|A\boldsymbol{u}^n(\tau) - A\boldsymbol{u}(\tau)\|^2 d\tau + \int_0^t \|B\boldsymbol{w}^n(\tau) - B\boldsymbol{w}(\tau)\|^2 d\tau + \int_0^t \|\nabla\theta^n(\tau) - \nabla\theta(\tau)\|^2 d\tau \\
&\leq C\Lambda_{n-1}(t), \\
&\int_0^t \|\nabla \boldsymbol{u}^n(\tau) - \nabla \boldsymbol{u}(\tau)\|^2 d\tau + \int_0^t \|\nabla \boldsymbol{w}^n(\tau) - \nabla \boldsymbol{w}(\tau)\|^2 d\tau + \int_0^t \|\theta^n(\tau) - \theta(\tau)\|^2 d\tau \\
&\leq C\Lambda_n(t), \\
&\int_0^t \|\boldsymbol{u}_\tau^n(\tau) - \boldsymbol{u}_t(\tau)\|^2 d\tau + \int_0^t \|\boldsymbol{w}_\tau^n(\tau) - \boldsymbol{w}_t(\tau)\|^2 d\tau + \int_0^t \|\theta_\tau^n(\tau) - \theta_t(\tau)\|^2_{H^{-1}} d\tau \\
&\leq C\sum_{i=0}^{2} \Lambda_{n-i}(t),
\end{aligned}$$

para cada $t \in (0, T_1)$, $\Lambda_n(t) = (t^n/n!)^{1/2}$. *Ainda, se* $\mu_r, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}$, *e* h *são suficientemente pequenos, a solução existe em* $[0, T]$.

Obtemos mais regularidade da solução de (1.1) se supormos $f, g \in C^1$.

Teorema 3.4. *Suponha que* f_t *e* g_t *verificam a condição (3.5), para constantes* $M_{f'}$ *e* $M_{g'}$ *respectivamente. Se* $\boldsymbol{u}_0 = \boldsymbol{w}_0 = \theta_0 = 0$ *e* $(\boldsymbol{u}, \boldsymbol{w}, \theta)$ *é solução do problema (1.1) dada pelo Teorema 3.3. Então, existe* $0 < T_2 \leq T_1$ *de modo que*

$$\boldsymbol{u} \ \in L^\infty(0, T_2; D(A)), \ \boldsymbol{w} \in L^\infty(0, T_2; D(B)), \ \theta \in L^\infty(0, T_2; H_0^1(\Omega)),$$

e temos os seguintes raios de convergência:

$$\|\boldsymbol{u}_t^n(t) - \boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{w}_t^n(t) - \boldsymbol{w}_t(t)\|^2 \leq C \sum_{i=1}^{3} \Lambda_{n-i}(t) + C\Lambda_{n-2}^{1/2}(t),$$

$$\int_0^t \|\nabla \boldsymbol{u}_t^n(\tau) - \nabla \boldsymbol{u}_t(\tau)\|^2 d\tau + \int_0^t \|\nabla \boldsymbol{w}_t^n(\tau) - \nabla \boldsymbol{w}_t(\tau)\|^2 d\tau + \int_0^t \|\theta_t^n(\tau) - \theta_t(\tau)\|_{H^{-1}}^2 d\tau$$
$$\leq C \sum_{i=1}^{3} \Lambda_{n-i}(t) + C\Lambda_{n-2}^{1/2}(t),$$

$$\|A\boldsymbol{u}^n(t) - A\boldsymbol{u}(t)\|^2 + \|B\boldsymbol{w}^n(t) - B\boldsymbol{w}(t)\|^2 + \|\nabla\theta^n(t) - \nabla\theta(t)\|^2$$
$$\leq C \sum_{i=1}^{3} \Lambda_{n-i}(t) + C\Lambda_{n-2}^{1/2}(t),$$

para cada $t \in (0, T_2)$, $\Lambda_n(t) = (t^n/n!)^{1/2}$. *Além disso, se* $\mu_r, M_{\boldsymbol{f}}, M_{\boldsymbol{g}}$, *e* h *são suficientemente pequenos, a solução existe em* $[0, T]$.

3.2 PROVA DOS RESULTADOS

3.2.1 Prova do Teorema 3.1

A existência e unicidade de soluções do problema (3.1)-(3.4) em $(0, T)$ segue dos resultados dos problemas lineares considerados em [12].

Vamos mostrar as estimativas. Algumas delas são obtidas da limitação de θ^n no espaço $L^\infty(0, T; L^2(\Omega))$. Assim, vamos mostrar que existe $T_1 \in (0, T]$ tal que

$$\sup_{t \in (0,T)} \|\theta^n(t)\| \leq 1 \tag{3.6}$$

para todo $n \in \mathbb{N}$, onde T_1 verifica as condições (3.23) e (3.36).

Usaremos o segundo princípio de indução em n. Suponha que (3.6) é válido para $1 \leq j \leq n$. Mostraremos que a estimativa vale para $n+1$.

(i) Estimativas para $\boldsymbol{u}^{j+1}$ e $\boldsymbol{w}^{j+1}$ nos espaços

$$L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V), \ \ e \ L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

respectivamente.

Multiplicando as equações (3.1) e (3.2) (para $n = j$) por $\boldsymbol{u}^{j+1}$ e $\boldsymbol{w}^{j+1}$ respectivamente, obtemos

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^{j+1}\|^2 + (\mu+\mu_r)\|\nabla \boldsymbol{u}^{j+1}\|^2 = 2\mu_r(\operatorname{rot} \boldsymbol{w}^j, \boldsymbol{u}^{j+1}) + (\boldsymbol{f}(\theta^j), \boldsymbol{u}^{j+1}) \tag{3.7}$$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}^{j+1}\|^2 + (c_a+c_d)\|\nabla \boldsymbol{w}^{j+1}\|^2 + (c_0+c_d-c_a)\|\operatorname{div} \boldsymbol{w}^{j+1}\|^2 + 4\mu_r\|\boldsymbol{w}^{j+1}\|^2$$
$$= 2\mu_r(\operatorname{rot} \boldsymbol{u}^j, \boldsymbol{w}^{j+1}) + (\boldsymbol{g}(\theta^j), \boldsymbol{w}^{j+1}) \tag{3.8}$$

Desde que $\|\boldsymbol{u}\|^2 \leq \lambda_1^{-1}\|\nabla \boldsymbol{u}\|^2$ e $\|\operatorname{rot} \boldsymbol{u}\| = \|\nabla \boldsymbol{u}\|$ para tod $\boldsymbol{u} \in V$, usando as desigualdades de Hölder e Young temos

$$|2\mu_r(\operatorname{rot} \boldsymbol{w}^j, \boldsymbol{u}^{j+1})| = |2\mu_r(\boldsymbol{w}^j, \operatorname{rot} \boldsymbol{u}^{j+1})| \leq \frac{4\mu_r^2}{\mu+\mu_r}\|\boldsymbol{w}^j\|^2 + \frac{\mu+\mu_r}{4}\|\nabla \boldsymbol{u}^{j+1}\|^2,$$

$$|(\boldsymbol{f}(\theta^j), \boldsymbol{u}^{j+1})| \leq \frac{\lambda_1^{-1}M_{\boldsymbol{f}}^2}{\mu+\mu_r}\|\theta^j\|^2 + \frac{\mu+\mu_r}{4}\|\nabla \boldsymbol{u}^{j+1}\|^2,$$

$$|2\mu_r(\operatorname{rot} \boldsymbol{u}^j, \boldsymbol{w}^{j+1})| = |2\mu_r(\boldsymbol{u}^j, \operatorname{rot} \boldsymbol{w}^{j+1})| \leq \frac{2\mu_r^2}{c_a+c_d}\|\boldsymbol{u}^j\|^2 + \frac{c_a+c_d}{2}\|\nabla \boldsymbol{w}^{j+1}\|^2,$$

$$|(\boldsymbol{g}(\theta^j), \boldsymbol{w}^{j+1})| \leq \frac{M_{\boldsymbol{g}}^2}{16\mu_r}\|\theta^j\|^2 + 4\mu_r\|\boldsymbol{w}^{j+1}\|^2.$$

Destas estimativas, (3.7) e (3.8) temos

$$\frac{d}{dt}\|\boldsymbol{u}^{j+1}\|^2 + (\mu+\mu_r)\|\nabla \boldsymbol{u}^{j+1}\|^2 \leq \frac{8\mu_r^2}{\mu+\mu_r}\|\boldsymbol{w}^j\|^2 + \frac{2\lambda^{-1}M_{\boldsymbol{f}}^2}{\mu+\mu_r}\|\theta^j\|^2 \tag{3.9}$$

$$\frac{d}{dt}\|\boldsymbol{w}^{j+1}\|^2 + (c_a+c_d)\|\nabla \boldsymbol{w}^{j+1}\|^2 + 2(c_0+c_d-c_a)\|\operatorname{div} \boldsymbol{w}^{j+1}\|^2 \leq \frac{4\mu_r^2}{c_a+c_d}\|\boldsymbol{u}^j\|^2 + \frac{M_{\boldsymbol{g}}^2}{8\mu_r}\|\theta^j\|^2. \tag{3.10}$$

Assim,

$$
\begin{aligned}
&\|\boldsymbol{u}^{j+1}(t)\|^2 + \|\boldsymbol{w}^{j+1}(t)\|^2 + (\mu+\mu_r)\int_0^t \|\nabla \boldsymbol{u}^{j+1}(\tau)\|^2 d\tau \\
&+(c_a+c_d)\int_0^t \|\nabla \boldsymbol{w}^{j+1}(\tau)\|^2 d\tau + 2(c_0+c_d-c_a)\int_0^t \|\operatorname{div}\boldsymbol{w}^{j+1}(\tau)\|^2 d\tau \\
&\leq C_1 + C_2\int_0^t \|\boldsymbol{u}^{j}(\tau)\|^2 + \|\boldsymbol{w}^{j}(\tau)\|^2 d\tau,
\end{aligned}
\tag{3.11}
$$

onde

$$
C_1(T) = \frac{2TM_{\boldsymbol{f}}^2}{\lambda_1(\mu+\mu_r)} + \frac{TM_{\boldsymbol{g}}^2}{8\mu_r} \text{ e } C_2 = \max\left\{\frac{8\mu_r^2}{\mu+\mu_r}, \frac{4\mu_r^2}{c_a+c_d}\right\}.
$$

Agora defina $\phi_k(t) = \|\boldsymbol{u}^k(t\|^2 + \|\boldsymbol{w}^k(t)\|^2$ para $k \geq 1$ e $t \in [0,T]$. Então a estimativa (3.11) implica

$$
\phi_{j+1}(t) \leq C_1 + C_2\int_0^t \phi_j(t_1)dt_1,
$$

para cada $j = 1, ..., n$. Note que

$$
\begin{aligned}
\phi_{j+1}(t) &\leq C_1 + C_2\int_0^t \left[C_1 + C_2\int_0^{t_1}\phi_{j-1}(t_2)dt_2\right]dt_1 \\
&\leq C_1 + C_1C_2t + C_2^2\int_0^t\int_0^{t_1}\left[C_1 + C_2\int_0^{t_2}\phi_{j-2}(t_3)dt_3\right]dt_2dt_1 \\
&\leq C_1 + C_1C_2t + C_1\frac{C_2^2t^2}{2} + C_2^3\int_0^t\int_0^{t_2}\int_0^{t_2}\phi_{j-2}(t_3)dt_3dt_2dt_1 \\
&\leq C_1\sum_{k=0}^{j-1}\frac{(C_2t)^k}{k!} + C_2^j\underbrace{\int_0^t\cdots\int_0^{t_{j-1}}}_{j-vezes}\phi_{j-(j-1)}(t_j)dt_j\cdots dt_1.
\end{aligned}
$$

Desde que $\phi_{j-(j-1)}(t) = \phi_1(t) = 0$ e $\displaystyle\sum_{k=0}^{j-1}\frac{(C_2t)^k}{k!} \leq \sum_{k=0}^{\infty}\frac{(C_2t)^k}{k!} = \exp(C_2t)$, concluímos que $\phi_{j+1}(t) \leq C_1\exp(C_2t)$. Consequentemente,

$$
\sup_{t\in[0,T]}\{\|\boldsymbol{u}^{j+1}(t)\|^2 + \|\boldsymbol{w}^{j+1}(t)\|^2\} \leq C_1\exp(C_2T) \equiv C_3(T),
\tag{3.12}
$$

para $j = 1, ..., n$.

Substituindo a estimativa (3.12) no lado direito da estimativa (3.11) temos

$$
\begin{aligned}
&(\mu+\mu_r)\int_0^t \|\nabla \boldsymbol{u}^{j+1}(\tau)\|^2 d\tau + (c_a+c_d)\int_0^t \|\nabla \boldsymbol{w}^{j+1}(\tau)\|^2 d\tau \\
&\leq C_1 + C_2\int_0^t C_3 d\tau \leq C_1 + C_2C_3T \equiv C_4(T).
\end{aligned}
$$

Assim,

$$\int_0^t \|\boldsymbol{u}^{j+1}(\tau)\|_V^2 d\tau \leq \frac{C_4}{\mu+\mu_r}, \int_0^t \|\boldsymbol{w}^{j+1}(\tau)\|_{H_0^1}^2 d\tau \leq \frac{C_4}{c_a+c_d}, \tag{3.13}$$

for $j = 1, ..., n$, $t \in (0, T)$.

(ii) Estimativas para $\boldsymbol{u}^{j+1}$ e $\boldsymbol{w}^{j+1}$ nos espaços

$$L^\infty(0,T;V) \cap L^2(0,T;D(A)) \text{ e } L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$$

respectivamente.

Multiplicando a equação (3.1)(com $n = j$) por $A\boldsymbol{u}^{j+1}$ temos que

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}^{j+1}\|^2 + (\mu+\mu_r)\|A\boldsymbol{u}^{j+1}\|^2 = \;& 2\mu_r(\operatorname{rot} \boldsymbol{w}^j, A\boldsymbol{u}^{j+1}) + (\boldsymbol{f}(\theta^j), A\boldsymbol{u}^{j+1}) \\ & -(\boldsymbol{u}^j \cdot \nabla \boldsymbol{u}^{j+1}, A\boldsymbol{u}^{j+1}), \end{aligned} \tag{3.14}$$

para $j = 1, ..., n$. Pelas desigualdades de Hölder, Young e pela seguinte desigualdade de Sobolev

$$\|\varphi\|_{L^4} \leq 2^{1/2}\|\varphi\|^{1/4}\|\nabla\varphi\|^{3/4} \tag{3.15}$$

que vale para $\varphi \in H_0^1(\Omega)$, obtemos

$$\begin{aligned} |(\boldsymbol{u}^j \cdot \nabla \boldsymbol{u}^{j+1}, A\boldsymbol{u}^{j+1})| &\leq \|\boldsymbol{u}^j\|_{L^4}\|\nabla \boldsymbol{u}^{j+1}\|_{L^4}\|A\boldsymbol{u}^{j+1}\| \\ &\leq 2^{1/2}\|\boldsymbol{u}^j\|^{1/4}\|\nabla \boldsymbol{u}^j\|^{3/4}\|\nabla \boldsymbol{u}^{j+1}\|_{L^4}\|A\boldsymbol{u}^{j+1}\| \\ &\leq 2^{1/2}\lambda_1^{-1/8}\|\nabla \boldsymbol{u}^j\| \; \|\nabla \boldsymbol{u}^{j+1}\|_{L^4}\|A\boldsymbol{u}^{j+1}\|. \end{aligned} \tag{3.16}$$

Por outro lado, desde que $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$, temos

$$\|\nabla\varphi\|_{L^4} \leq M_1\|\varphi\|_{H^2} \tag{3.17}$$

para todo $\varphi \in H^2(\Omega)$ e alguma constante $M_1 > 0$. Além disso, pela desigualdade de Cattabriga

$$\|\boldsymbol{u}\|_{H^2} \leq M_2\|A\boldsymbol{u}\| \tag{3.18}$$

para $\boldsymbol{u} \in D(A)$, e alguma constante $M_2 > 0$ concluímos que $\|\nabla \boldsymbol{u}\|_{L^4} \leq M_1 M_2\|A\boldsymbol{u}\|$, para todo $\boldsymbol{u} \in D(A)$. Assim de (3.16) obtemos

$$|(\boldsymbol{u}^j \cdot \nabla \boldsymbol{u}^{j+1}, A\boldsymbol{u}^{j+1})| \leq 2^{1/2}\lambda_1^{-1/8}M_1 M_2\|\nabla \boldsymbol{u}^j\| \; \|A\boldsymbol{u}^{j+1}\|^2. \tag{3.19}$$

Para estimar os demais termos do lado direito de (3.14) procederemos da seguinte maneira:

$$|2\mu_r(\operatorname{rot}\boldsymbol{w}^j, A\boldsymbol{u}^{j+1})| \leq \frac{4\mu_r^2}{\mu+\mu_r}\|\nabla\boldsymbol{w}^j\|^2 + \frac{\mu+\mu_r}{4}\|A\boldsymbol{u}^{j+1}\|^2, \tag{3.20}$$

$$|(\boldsymbol{f}(\theta^j), A\boldsymbol{u}^{j+1})| \leq \frac{M_{\boldsymbol{f}}^2}{\mu+\mu_r}\|\theta^j\|^2 + \frac{\mu+\mu_r}{4}\|A\boldsymbol{u}^{j+1}\|^2. \tag{3.21}$$

Substituindo as estimativas (3.19)-(3.21) em (3.14), e integrando de 0 a t, temos

$$\begin{aligned}
&\|\nabla\boldsymbol{u}^{j+1}(t)\|^2 + \int_0^t (\mu+\mu_r - 2^{3/2}\lambda_1^{-1/8}M_1M_2\|\nabla\boldsymbol{u}^j(\tau)\|)\|A\boldsymbol{u}^{j+1}(\tau)\|^2 d\tau \\
&\leq \frac{8\mu_r^2}{\mu+\mu_r}\int_0^t \|\nabla\boldsymbol{w}^j(\tau)\|^2 d\tau + \frac{2M_{\boldsymbol{f}}^2}{\mu+\mu_r}\int_0^t \|\theta^j(\tau)\|^2 d\tau \\
&\leq \frac{8\mu_r^2 C_4}{(\mu+\mu_r)(c_a+c_d)} + \frac{2M_{\boldsymbol{f}}^2 T}{\mu+\mu_r} = \delta^2,
\end{aligned} \tag{3.22}$$

para $j = 1, ..., n$.

Desde que $\boldsymbol{u}^1 = 0$, deduzimos de (3.22) que $\|\nabla\boldsymbol{u}^2(t)\| \leq \delta$ para $t \in (0,T)$. Suponha por um momento que

$$\delta < 2^{-3/2}\lambda_1^{1/8}M_1^{-1}M_2^{-1}(\mu+\mu_r), \tag{3.23}$$

então

$$\mu+\mu_r - 2^{3/2}\lambda_1^{-1/8}M_1M_2\|\nabla\boldsymbol{u}^2(t)\|^2 \geq \mu+\mu_r - 2^{3/2}\lambda_1^{-1/8}M_1M_2\delta > 0.$$

Daí, e de (3.22) concluímos que $\|\nabla\boldsymbol{u}^3\| \leq \delta$, e

$$(\mu+\mu_r - 2^{3/2}\lambda_1^{-1/8}M_1M_2\delta)\int_0^t \|A\boldsymbol{u}^3(s)\|^2 ds \leq \delta^2,$$

para $t \in (0,T)$. Repetindo este argumento é possível obter

$$\sup_{t\in[0,T]}\|\nabla\boldsymbol{u}^{j+1}(t)\| \leq \delta, \tag{3.24}$$

e

$$\int_0^T \|A\boldsymbol{u}^{j+1}(s)\|^2 ds \leq \delta^2/(\mu+\mu_r - 2^{3/2}\lambda_1^{-1/8}M_1M_2\delta) = C_5, \tag{3.25}$$

para $j = 1, ..., n$.

Como o operador $L\boldsymbol{w} = (c_a+c_d)B\boldsymbol{w} - (c_0+c_d-c_a)\nabla\operatorname{div}\boldsymbol{w}$, definido para cada $\boldsymbol{w} \in H^2(\Omega)\cap H_0^1(\Omega)$ é um operador fortemente elíptico temos que

$$(L\boldsymbol{w}, B\boldsymbol{w}) \geq (c_a+c_d)\|B\boldsymbol{w}\|^2 - N_0\|\nabla\boldsymbol{w}\|^2, \tag{3.26}$$

onde $N_0 > 0$ é uma constante que depende apenas de $c_a + c_d$, $c_0 + c_d - c_a$ e $\partial\Omega$ (veja [10] p. 70). Assim, da equação (3.2) tem-se

$$\boldsymbol{w_t}^{j+1} + L\boldsymbol{w}^{j+1} + 4\mu_r \boldsymbol{w}^{j+1} = 2\mu_r \operatorname{rot} \boldsymbol{u}^j - \boldsymbol{u}^j \cdot \nabla \boldsymbol{w}^{j+1} + \boldsymbol{g}(\theta^j),$$

para $j = 1, ..., n$. Multiplicando esta equação por $B\boldsymbol{w}^{j+1}$ e usando (3.26) obtemos

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{w}^{j+1}\|^2 + (c_a + c_d)\|B\boldsymbol{w}^{j+1}\|^2 + 4\mu_r\|\nabla \boldsymbol{w}^{j+1}\|^2 \\ &\leq N_0\|\nabla \boldsymbol{w}^{j+1}\|^2 - (\boldsymbol{u}^j \cdot \nabla \boldsymbol{w}^{j+1}, B\boldsymbol{w}^{j+1}) + 2\mu_r(\operatorname{rot} \boldsymbol{u}^j, B\boldsymbol{w}^{j+1}) + (\boldsymbol{g}(\theta^j), B\boldsymbol{w}^{j+1}). \end{aligned} \tag{3.27}$$

Agora estimaremos os termos do lado direito de (3.27). Usando a desigualdade de Poincaré, a estimativa (3.24), a desigualdade

$$\|\nabla \varphi\|_{L^4} \leq M_3 \|\nabla \varphi\|^{1/4} \|B\varphi\|^{3/4},$$

válida para todo $\varphi \in H^2(\Omega)$, alguma constante $M_3 > 0$ e a desigualdade de Young obtemos

$$\begin{aligned} |(\boldsymbol{u}^j \cdot \nabla \boldsymbol{w}^{j+1}, B\boldsymbol{w}^{j+1})| &\leq 2^{1/2}\lambda_1^{-1/8}\|\nabla \boldsymbol{u}^j\| \, \|\nabla \boldsymbol{w}^{j+1}\|_{L^4}\|B\boldsymbol{w}^{j+1}\| \\ &\leq 2\lambda_1^{-1/8}\delta M_3\|\nabla \boldsymbol{w}^{j+1}\|^{1/4}\|B\boldsymbol{w}^{j+1}\|^{7/4} \\ &\leq \alpha\|\nabla \boldsymbol{w}^{j+1}\|^2 + \frac{c_a + c_d}{6}\|B\boldsymbol{w}^{j+1}\|^2, \end{aligned}$$

onde $\alpha = (1/7)\,(8/7)^{-8}\,(2\lambda^{-1/8}\delta M_3)^8\,((c_a + c_d)/6)^{-7}$,

$$|2\mu_r(\operatorname{rot} \boldsymbol{u}^j, B\boldsymbol{w}^{j+1})| \leq \frac{6\mu_r^2}{c_a + c_d}\|\nabla \boldsymbol{u}^j\|^2 + \frac{c_a + c_d}{6}\|B\boldsymbol{w}^{j+1}\|^2,$$

$$|(\boldsymbol{g}(\theta^j), B\boldsymbol{w}^{j+1})| \leq \frac{3M_{\boldsymbol{g}}^2}{2(c_a + c_d)}\|\theta^j\|^2 + \frac{c_a + c_d}{6}\|B\boldsymbol{w}^{j+1}\|^2.$$

Usando estas estimativas em (3.27), integrando de 0 a t , e usando as desigualdades de (3.13) temos

$$\begin{aligned} &\|\nabla \boldsymbol{w}^{j+1}(t)\|^2 + (c_a + c_d)\int_0^t \|B\boldsymbol{w}^{j+1}(\tau)\|^2 d\tau + 8\mu_r \int_0^t \|\nabla \boldsymbol{w}^{j+1}(\tau)\|^2 d\tau \\ &\leq (N_0 + \alpha)\int_0^t \|\nabla \boldsymbol{w}^{j+1}(\tau)\|^2 d\tau + \frac{6\mu_r^2}{c_a + c_d}\int_0^t \|\nabla \boldsymbol{u}^j(\tau)\|^2 d\tau + \frac{3M_{\boldsymbol{g}}^2}{2(c_a + c_d)}\int_0^t \|\theta^j(\tau)\|^2 d\tau \\ &\leq \frac{C_4(N_0 + \alpha)}{c_a + c_d} + \frac{6C_4\mu_r^2}{(c_a + c_b)(\mu + \mu_r)} + \frac{3M_{\boldsymbol{g}}^2 T}{2(c_a + c_d)} = C_6(T), \end{aligned} \tag{3.28}$$

para $j = 1, .., n$.

(iii) Estimativas para θ^{n+1}.

Para obter as estimativas de θ^{n+1} multiplicamos (3.3) por θ^{n+1}. Assim,

$$\frac{1}{2}\frac{d}{dt}\|\theta^{n+1}\|^2 + \kappa\|\nabla\theta^{n+1}\|^2 = (\Phi(\boldsymbol{u}^n, \boldsymbol{w}^n), \theta^{n+1}) + (h, \theta^{n+1}). \tag{3.29}$$

Agora vamos estimar os termos Φ_i, $1 \leq i \leq 5$. Pelas desigualdades de Hölder, Young e seguintes desigualdades

$$\begin{aligned} \|\varphi\|_{L^3} &\leq M_4\|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2}, \text{ for } \varphi \in H_0^1(\Omega), \\ \|\nabla\boldsymbol{u}\|_{L^3} &\leq M_5\|\nabla\boldsymbol{u}\|^{1/2}\|A\boldsymbol{u}\|^{1/2}, \text{ for } u \in D(A), \\ \|\nabla\varphi\|_{L^3} &\leq M_5'\|\nabla\varphi\|^{1/2}\|B\boldsymbol{w}\|^{1/2}, \text{ for } u \in D(B), \end{aligned} \tag{3.30}$$

e

$$\|\varphi\|_{L^6} \leq M_6\|\nabla\varphi\|, \text{ for } \varphi \in H_0^1(\Omega) \tag{3.31}$$

obtemos

$$\begin{aligned} |(\Phi_1(\boldsymbol{u}^n), \theta^{n+1})| &\leq \mu\int|\nabla\boldsymbol{u}^n|^2|\theta^{n+1}|dx \\ &\leq \mu\|\nabla\boldsymbol{u}^n\|\ \|\nabla\boldsymbol{u}^n\|_{L^3}\|\theta^{n+1}\|_{L^6} \\ &\leq \mu M_5M_6\|\nabla\boldsymbol{u}^n\|^{3/2}\|A\boldsymbol{u}^n\|^{1/2}\|\nabla\theta^{n+1}\| \\ &\leq \frac{\kappa}{20}\|\nabla\theta^{n+1}\|^2 + \frac{5(2\mu M_5M_6)^2}{\kappa}\|\nabla\boldsymbol{u}^n\|^3\|A\boldsymbol{u}^n\|, \end{aligned}$$

$$\begin{aligned} |(\Phi_2(\boldsymbol{u}^n, \boldsymbol{w}^n), \theta^{n+1})| &\leq \mu_r\left(\int|\nabla\boldsymbol{u}^n|^2|\theta^{n+1}|dx + \int|\boldsymbol{w}^n|^2|\theta^{n+1}|dx\right) \\ &\leq \mu_rM_5M_6\|\nabla\boldsymbol{u}^n\|^{3/2}\|A\boldsymbol{u}^n\|^{1/2}\|\nabla\theta^{n+1}\|+ \\ &\quad +\mu_r\|\boldsymbol{w}^n\|\|\boldsymbol{w}^n\|_{L^3}\|\theta^{n+1}\|_{L^6} \\ &\leq \mu_rM_5M_6\|\nabla\boldsymbol{u}^n\|^{3/2}\|A\boldsymbol{u}^n\|^{1/2}\|\nabla\theta^{n+1}\|+ \\ &\quad \mu_rM_4M_6\|\boldsymbol{w}^n\|^{3/2}\|\nabla\boldsymbol{w}\|^{1/2}\|\nabla\theta^{n+1}\| \\ &\leq \frac{\kappa}{20}\|\nabla\theta^{n+1}\|^2 + \frac{10(4\mu_rM_5M_6)^2}{\kappa}\|\nabla\boldsymbol{u}^n\|^3\|A\boldsymbol{u}^n\| \\ &+ \frac{10(8\mu_rM_4M_6)^2}{\kappa}\|\boldsymbol{w}^n\|^3\|\nabla\boldsymbol{w}^n\|, \end{aligned}$$

$$
\begin{aligned}
|(\Phi_3(\boldsymbol{w}^n),\theta^{n+1})| &\le c_0\int|\operatorname{div}\boldsymbol{w}^n|^2|\theta^n|dx\\
&\le \frac{c_0}{2}\|\nabla\boldsymbol{w}^n\|\|\nabla\boldsymbol{w}^n\|_{L^3}\|\theta^{n+1}\|_{L^6}\\
&\le \frac{c_0}{2}M_5'M_6\|\nabla\boldsymbol{w}^n\|^{3/2}\|B\boldsymbol{w}^n\|^{1/2}\|\nabla\theta^{n+1}\|\\
&\le \frac{\kappa}{20}\|\nabla\theta^{n+1}\|^2+\frac{5(3c_0M_5'M_6)^2}{2\kappa}\|\nabla\boldsymbol{w}^n\|^3\|B\boldsymbol{w}^n\|\\
|(\Phi_4(\boldsymbol{w}^n),\theta^{n+1})| &\le (c_a+c_d)\int|\nabla\boldsymbol{w}^n|^2|\theta^{n+1}|dx\\
&\le \frac{\kappa}{20}\|\nabla\theta^{n+1}\|^2+\frac{5[(c_a+c_d)M_5'M_6]^2}{2\kappa}\|\nabla\boldsymbol{w}^n\|^3\|B\boldsymbol{w}^n\|,\\
|(\Phi_5(\boldsymbol{w}^n),\theta^{n+1})| &\le (c_a-c_d)\int|\nabla\boldsymbol{w}^n||\theta^{n+1}|dx\\
&\le \frac{\kappa}{20}\|\nabla\theta^{n+1}\|^2+\frac{5[|c_d-c_a|M_5'M_6]^2}{2\kappa}\|\nabla\boldsymbol{w}^n\|^3\|B\boldsymbol{w}^n\|.
\end{aligned}
$$

Combinando estas desigualdades obtemos

$$
\begin{aligned}
&|(\Phi(\boldsymbol{u}^n,\boldsymbol{w}^n),\theta^{n+1})|\\
&\le \frac{\kappa}{4}\|\nabla\theta^{n+1}\|^2+\frac{20}{\kappa}(M_5M_6)^2(\mu^2+2\mu_r^2)\|\nabla\boldsymbol{u}^n\|^3\|A\boldsymbol{u}^n\|+\frac{10}{\kappa}(8\mu_rM_4M_6)^2\|\boldsymbol{w}^n\|^3\|\nabla\boldsymbol{w}^n\|\\
&+\frac{5}{\kappa}(M_5'M_6)^2[c_0^2+(c_a+c_d)^2+(c_d-c_2)^2]\|\nabla\boldsymbol{w}^n\|^3\|B\boldsymbol{w}^n\|.
\end{aligned}
$$

Consequentemente, pela desigualdade de Young, deduzimos que

$$
\begin{aligned}
|(\Phi(\boldsymbol{u}^n,\boldsymbol{w}^n),\theta^{n+1})| \le\ & \frac{\kappa}{4}\|\nabla\theta^{n+1}\|^2+\frac{1}{2}\|A\boldsymbol{u}^n\|^2+\frac{1}{2}\|B\boldsymbol{w}^n\|^2\\
&+\frac{1}{2}\left[\frac{20}{\kappa}(M_5M_6)^2(\mu^2+8\mu_r^2)\right]^2\|\nabla\boldsymbol{u}^n\|^6\\
&+\frac{10}{\kappa}(8\mu_rM_4M_6)^2\|\boldsymbol{w}^n\|^3\|\nabla\boldsymbol{w}^n\|\\
&+\frac{1}{2}\left\{\frac{5}{\kappa}(M_5'M_6)^2[9c_0^2+(c_a+c_d)^2+(c_d-c_2)^2]\right\}^2\|\nabla\boldsymbol{w}^n\|^6
\end{aligned}
\tag{3.32}
$$

Por outro lado, pela desigualdade de Poincaré

$$
|(h,\theta^{n+1})|\le\|h\|\ \|\theta^{n+1}\|\le\lambda_1^{-1}\|h\|\ \|\nabla\theta^{n+1}\|\le\frac{\kappa}{4}\|\nabla\theta^{n+1}\|^2+\frac{1}{\kappa\lambda_1^{-2}}\|h\|^2. \tag{3.33}
$$

De (3.29), usando as estimativas (3.32), (3.33) segue que

$$
\begin{aligned}
&\|\theta^{n+1}(t)\|^2+\kappa\int_0^t\|\nabla\theta^{n+1}(s)\|^2ds\\
&\le C_7T+\int_0^t\|A\boldsymbol{u}^n(s)\|^2ds+\int_0^t\|B\boldsymbol{w}^n(s)\|^2ds+\frac{2}{\kappa\lambda_1^2}\int_0^t\|h(s)\|^2ds,
\end{aligned}
\tag{3.34}
$$

onde

$$\begin{aligned} C_7 &= \left[(M_5M_6)^2(\mu^2+2\mu_r^2)\right]^2\delta^6 + \frac{20}{\kappa}(8\mu_r M_4 M_6)^2 C_3^{3/2}C_6^{1/2} + \\ &\left\{\frac{5}{\kappa}(M_5' M_6)^2[9c_0^2+(c_a+c_d)^2+(c_d-c_a)^2]\right\}^2 C_6^3. \end{aligned}$$

Portanto,

$$\|\theta^{n+1}(t)\|^2 \leq C_7T + C_5 + \frac{C_6}{c_a+c_d} + \frac{2}{\kappa\lambda_1^2}\|h\|^2_{L^2(0,T;L^2)}, \tag{3.35}$$

para $t \in (0,T)$. Logo, se $T_1 \in (0,T)$ verifica a condição (3.23) e

$$C_7T_1 + C_5 + \frac{C_6(T_1)}{c_a+c_d} + \frac{2}{\kappa\lambda_1^2}\|h\|^2_{L^2(0,T_1;L^2)} \leq 1, \tag{3.36}$$

temos que $\sup ess_{t\in(0,T_1)}\|\theta^{n+1}\| \leq 1$. Assim o processo de indução termina. Além disso, de (3.34) concluímos que

$$\int_0^{T_1} \|\nabla\theta^{n+1}(s)\|^2 ds \leq 1/\kappa.$$

Note que se μ_r, $M_{\boldsymbol{f}}$, $M_{\boldsymbol{f}}$ e $\|h\|_{L^2(0,T;L^2(\Omega))}$ são pequenos temos que a condição (3.23) é verificada e as constantes C_5, C_6 e C_7 podem ser escolhidas pequenas de modo que a desigualdade (3.36) seja verificada para $T_1 = T$.

(iv) Estimativas para $\boldsymbol{u}_t^n$, $\boldsymbol{w}_t^n$ e θ_t^n . Multiplicando a equação (3.1) por $\boldsymbol{u}_t^{n+1}$ obtemos

$$\begin{aligned} \|\boldsymbol{u}_t^{n+1}\|^2 &\leq (\mu+\mu_r)\|A\boldsymbol{u}^{n+1}\|\|\boldsymbol{u}_t^{n+1}\| + \|\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^{n+1}\|\|\boldsymbol{u}_t^{n+1}\| \\ &+2\mu_r\|\nabla\boldsymbol{w}^n\|\|\boldsymbol{u}_t^{n+1}\| + \|\boldsymbol{f}(\theta^n)\|\|\boldsymbol{u}_t^{n+1}\|. \end{aligned}$$

Daí, usando (3.31), (3.30) e a desigualdade de Young temos

$$\begin{aligned} \int_0^t \|\boldsymbol{u}_t^{n+1}(\tau)\|^2 d\tau \leq\ & C\int_0^t \|A\boldsymbol{u}^{n+1}(\tau)\|^2 d\tau + C\int_0^t \|\nabla\boldsymbol{u}^n(\tau)\|^{4/3}\|\nabla\boldsymbol{u}^{n+1}(\tau)\|^{2/3} d\tau \\ & +C\int_0^t \|\nabla\boldsymbol{w}^n(\tau)\|^2 d\tau + C\int_0^t \|\theta^n(\tau)\|^2 d\tau \leq C, \end{aligned}$$

a partir das estimativas (3.24), (3.25), (3.28) e (3.6).

Agora, multiplicando a equação (3.2) por $\boldsymbol{w}_t^{n+1}$ temos

$$\begin{aligned} &\|\boldsymbol{w}_t^{n+1}\|^2 + ((c_0+c_d-c_a)/2)\frac{d}{dt}\|\operatorname{div}\boldsymbol{w}^{n+1}\|^2 = -(c_a+c_d)(B\boldsymbol{w}^{n+1},\boldsymbol{w}_t^{n+1}) \\ &-(\boldsymbol{u}^n\cdot\nabla\boldsymbol{w}^{n+1},\boldsymbol{w}_t^{n+1}) - 4\mu_r(\boldsymbol{w}^{n+1},\boldsymbol{w}_t^{n+1}) + 2\mu_r(\operatorname{rot}\boldsymbol{u}^n,\boldsymbol{w}_t^{n+1}) + (\boldsymbol{g}(\theta^n),\boldsymbol{w}_t^{n+1}). \end{aligned} \tag{3.37}$$

Para o lado direito da equação acima temos as seguintes estimativas

$$\begin{aligned} |(c_a+c_d)(B\boldsymbol{w}^{n+1},\boldsymbol{w}_t^{n+1})| &\leq C_\epsilon\|B\boldsymbol{w}^{n+1}\|^2 + \epsilon\|\boldsymbol{w}_t^{n+1}\|^2, \\ |4\mu_r(\boldsymbol{w}^{n+1},\boldsymbol{w}_t^{n+1})| &\leq C_\epsilon\|\boldsymbol{w}^{n+1}\|^2 + \epsilon\|\boldsymbol{w}_t^{n+1}\|^2, \\ |(\boldsymbol{g}(\theta^n),\boldsymbol{w}_t^{n+1})| &\leq C_\epsilon\|\theta^n\|^2 + \epsilon\|\boldsymbol{w}_t^{n+1}\|^2. \end{aligned}$$

Por outro lado, como $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, concluímos usando a estimativa (3.28) e a desigualdade de Cattabriga (3.18) que

$$|(\boldsymbol{u}^n \cdot \nabla \boldsymbol{w}^{n+1}, \boldsymbol{w}_t^{n+1})| \leq C\|\boldsymbol{u}^n\|_{L^\infty}\|\nabla \boldsymbol{w}^{n+1}\|\|\boldsymbol{w}_t^{n+1}\| \leq C_\epsilon\|A\boldsymbol{u}^n\|^2 + \epsilon\|\boldsymbol{w}_t^{n+1}\|^2.$$

Usando as estimativas de (3.37), e integrando de 0 a t temos

$$\begin{aligned}\int_0^t \|\boldsymbol{w}_\tau^{n+1}(\tau)\|^2 d\tau \ & \leq C\int_0^t \|B\boldsymbol{w}^{n+1}(\tau)\|^2 d\tau + C\int_0^t \|\boldsymbol{w}^{n+1}(\tau)\|^2 d\tau \\ & +C\int_0^t \|\nabla \boldsymbol{u}^n(\tau)\|^2 d\tau + C\int_0^t \|A\boldsymbol{u}^n(\tau)\|^2 d\tau + C\int_0^t \|\theta^n(\tau)\|^2 d\tau \leq C,\end{aligned}$$

das estimativas (3.25), (3.28) e (3.35).

Finalmente, de (3.3) temos

$$\|\theta_t^{n+1}\|_{H^{-1}}^2 \leq \|(\boldsymbol{u}^n \cdot \nabla)\theta^{n+1}\|_{H^{-1}}^2 + \kappa\|\Delta\theta^{n+1}\|_{H^{-1}}^2 + \|\Phi(\boldsymbol{u}^n, \boldsymbol{w}^n)\|_{H^{-1}}^2 + \|h\|_{H^{-1}}^2. \quad (3.38)$$

Desde que

$$\begin{aligned}\|\boldsymbol{u}^n \cdot \nabla\theta^{n+1}\|_{H^{-1}} \ & = \sup_{\|\varphi\|_{H_0^1}\leq 1} |(\boldsymbol{u}^n \cdot \nabla\theta^{n+1}, \varphi)| \\ & \leq C \sup_{\|\varphi\|_{H_0^1}\leq 1} \|\boldsymbol{u}^n\|_{L^3}\|\nabla\theta^{n+1}\|\|\nabla\varphi\| \\ & \leq C\|\nabla\theta^{n+1}\|,\end{aligned} \quad (3.39)$$

$$\begin{aligned}\|\Delta\theta^{n+1}\|_{H^{-1}} \ & \leq \sup_{\|\varphi\|_{H_0^1}\leq 1} |(\Delta\theta^{n+1}, \varphi)| \\ & \leq C \sup_{\|\varphi\|_{H_0^1}\leq 1} \|\nabla\theta^{n+1}\|\|\nabla\varphi\| \\ & \leq C\|\nabla\theta^{n+1}\|,\end{aligned} \quad (3.40)$$

$$\begin{aligned}\|\Phi_1(\boldsymbol{u}^n)\|_{H^{-1}} \ & \leq C \sup_{\|\varphi\|_{H_0^1}\leq 1} \|\nabla \boldsymbol{u}^n\|\|\nabla \boldsymbol{u}^n\|_{L^3}\|\varphi\|_{L^6} \\ & \leq C\|A\boldsymbol{u}^n\|,\end{aligned}$$

$$\begin{aligned}\|\Phi_2(\boldsymbol{u}^n, \boldsymbol{w}^n)\|_{H^{-1}} \ & \leq C \sup_{\|\varphi\|_{H_0^1}\leq 1} \|\nabla \boldsymbol{u}^n\|\|\nabla \boldsymbol{u}^n\|_{L^3}\|\varphi\|_{L^6} \\ & \quad +C \sup_{\|\varphi\|_{H_0^1}\leq 1} \|\boldsymbol{w}^n\|\|\boldsymbol{w}^n\|_{L^3}\|\varphi\|_{L^6} \\ & \leq C(\|A\boldsymbol{u}^n\| + \|B\boldsymbol{w}^n\| + 1),\end{aligned} \quad (3.41)$$

$$\|h\|_{H^{-1}}^2 \leq \sup_{\|\varphi\|_{H_0^1}\leq 1} |\langle h, \varphi\rangle| \leq \|h\|,$$

e para $i = 3, 4, 5$ temos

$$\begin{aligned} \|\Phi_i(\boldsymbol{w}^n)\|_{H^{-1}} &\leq C \sup_{\|\varphi\|_{H_0^1} \leq 1} \|\nabla \boldsymbol{w}^n\| \|\nabla \boldsymbol{w}^n\|_{L^3} \|\varphi\|_{L^6} \\ &\leq C\|B\boldsymbol{w}^n\|. \end{aligned}$$

Desta última estimativa, (3.25), (3.28), (3.35) e (3.38) obtemos

$$\int_0^t \|\theta_t^{n+1}(\tau)\|_{H^{-1}}^2 d\tau \leq C \int_0^t (\|\nabla\theta^{n+1}(\tau)\|^2 + \|A\boldsymbol{u}^n(\tau)\|^2 + \|B\boldsymbol{w}^n\|^2 + \|h(\tau)\|^2) d\tau \leq C.$$

3.2.2 Prova do Teorema 3.3

Seja $\boldsymbol{u}^{n,s}(t) = \boldsymbol{u}^{n+s}(t) - \boldsymbol{u}^n(t)$, $\boldsymbol{w}^{n,s}(t) = \boldsymbol{w}^{n+s}(t) - \boldsymbol{w}^n(t)$ e $\theta^{n,s}(t) = \theta^{n+s}(t) - \theta^n(t)$, para cada $n, s \in \mathbb{N}$, $t \in [0, T]$. Então, a partir das equações (3.1)-(3.3) obtemos

$$\begin{aligned} &\boldsymbol{u}_t^{n,s} + (\mu + \mu_r)A\boldsymbol{u}^{n,s} + P(\boldsymbol{u}^{n-1+s} \cdot \nabla \boldsymbol{u}^{n,s}) \\ &= 2\mu_r P(\operatorname{rot} \boldsymbol{w}^{n-1,s}) - P(\boldsymbol{u}^{n-1,s} \cdot \nabla \boldsymbol{u}^n) + P\boldsymbol{f}(\theta^{n-1+s}) - P\boldsymbol{f}(\theta^{n-1}), \end{aligned} \tag{3.42}$$

$$\begin{aligned} &\boldsymbol{w}_t^{n,s} + (c_a + c_d)B\boldsymbol{w}^{n,s} + (\boldsymbol{u}^{n-1+s} \cdot \nabla \boldsymbol{w}^{n,s}) - (c_0 + c_d - c_a)\nabla \operatorname{div} \boldsymbol{w}^{n,s} + 4\mu_r \boldsymbol{w}^{n,s} \\ &= 2\mu_r \operatorname{rot} \boldsymbol{u}^{n-1,s} - (\boldsymbol{u}^{n-1,s} \cdot \nabla \boldsymbol{w}^n) + \boldsymbol{g}(\theta^{n-1+s}) - \boldsymbol{g}(\theta^{n-1}) \end{aligned} \tag{3.43}$$

$$\begin{aligned} &\theta_t^{n,s} - \kappa\Delta\theta^{n,s} + \boldsymbol{u}^{n-1+s} \cdot \nabla\theta^{n,s} \\ &= \Phi(\boldsymbol{u}^{n-1+s}, \boldsymbol{w}^{n-1+s}) - \Phi(\boldsymbol{u}^{n-1}, \boldsymbol{w}^{n-1}) - \boldsymbol{u}^{n-1,s} \cdot \nabla\theta^n. \end{aligned} \tag{3.44}$$

Para obter estimativas dos termos $\boldsymbol{u}^{n,s}$, $\boldsymbol{w}^{n,s}$ e $\theta^{n,s}$ precisaremos do seguinte,

Lema 3.5. *Suponha que para cada $n \in \mathbb{N}$, $\phi_n \in C([0, T])$ e $0 \leq \phi_0(t) \leq M$ para todo $t \in [0, T]$. Se*

$$0 \leq \phi_n(t) \leq C_0 \int_0^t \phi_{n-1}(s) ds,$$

para cada $n \geq 1$ e $t \in [0, T]$, então

$$\phi_n(t) \leq M \frac{(C_0 t)^{n-1}}{(n-1)!} \tag{3.45}$$

para todo $t \in [0, T]$ e $n \geq 1$.

Demonstração. Vamos utilizar o primeiro principio de indução. Para $n = 2$, é claro que (3.45) é satisfeita. Suponha que (3.45) se cumpre. Então

$$\phi_{n+1}(t) \leq C_0 \int_0^t \phi_n(s)ds \leq M\frac{C_0^n}{(n-1)!}\int_0^t s^{n-1}ds \leq M\frac{(C_0t)^n}{(n)!}.$$

Portanto, a estimativa (3.45) vale para $n+1$. □

Lema 3.6. *Existe uma constante $C > 0$,independente de n e s, tal que*

$$\begin{aligned}&\|\nabla \boldsymbol{u}^{n,s}(t)\|^2 + \|\nabla \boldsymbol{w}^{n,s}(t)\|^2 + (\mu+\mu_r)\int_0^t \|A\boldsymbol{u}^{n,s}(\tau)\|^2 d\tau + (c_a+c_d)\int_0^t \|B\boldsymbol{w}^{n,s}(\tau)\|^2 d\tau \\ &\leq C_8 \int_0^t (\|A\boldsymbol{u}^n(\tau)\| + \|B\boldsymbol{w}^n(\tau)\| + 1)(\|\nabla \boldsymbol{u}^{n-1,s}(\tau)\|^2 + \|\nabla \boldsymbol{w}^{n-1,s}(\tau)\|^2 + \|\theta^{n-1,s}(\tau)\|^2)d\tau.\end{aligned} \tag{3.46}$$

Demonstração. Multiplicando a equação (3.42) por $A\boldsymbol{u}^{n,s}$ e integrando em Ω obtemos

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}^{n,s}\|^2 + (\mu+\mu_r)\|A\boldsymbol{u}^{n,s}\|^2 \\ &= -(\boldsymbol{u}^{n-1+s}\cdot\nabla \boldsymbol{u}^{n,s}, A\boldsymbol{u}^{n,s}) + 2\mu_r(\operatorname{rot}\boldsymbol{w}^{n-1,s}, A\boldsymbol{u}^{n,s}) \\ &\quad +(\boldsymbol{u}^{n-1,s}\cdot\nabla \boldsymbol{u}^{n}, A\boldsymbol{u}^{n,s}) + (\boldsymbol{f}(\theta^{n-1+s}) - \boldsymbol{f}(\theta^{n-1}), A\boldsymbol{u}^{n,s}).\end{aligned} \tag{3.47}$$

Usando as desigualdades de Hölder, Young, (3.31) e (3.30), temos

$$\begin{aligned}
|(\boldsymbol{u}^{n-1+s}\cdot\nabla \boldsymbol{u}^{n,s}, A\boldsymbol{u}^{n,s})| &\leq \|\boldsymbol{u}^{n-1+s}\|_{L^6}\|\nabla \boldsymbol{u}^{n,s}\|_{L^3}\|A\boldsymbol{u}^{n,s}\| \\
&\leq C\|\nabla \boldsymbol{u}^{n-1+s}\|\|\nabla \boldsymbol{u}^{n,s}\|^{1/2}\|A\boldsymbol{u}^{n,s}\|^{3/2} \\
&\leq C\|\nabla \boldsymbol{u}^{n,s}\|^2 + \frac{\mu+\mu_r}{8}\|A\boldsymbol{u}^{n,s}\|^2 \\
|2\mu_r(\operatorname{rot}\boldsymbol{w}^{n-1,s}, A\boldsymbol{u}^{n,s})| &\leq C\|\nabla \boldsymbol{w}^{n-1,s}\|^2 + \frac{\mu+\mu_r}{8}\|A\boldsymbol{u}^{n,s}\|^2 \\
|(\boldsymbol{u}^{n-1,s}\cdot\nabla \boldsymbol{u}^{n}, A\boldsymbol{u}^{n,s})| &\leq \|\boldsymbol{u}^{n-1,s}\|_{L^6}\|\nabla \boldsymbol{u}^{n}\|_{L^3}\|A\boldsymbol{u}^{n,s}\| \\
&\leq C\|\nabla \boldsymbol{u}^{n-1,s}\|\|\nabla \boldsymbol{u}^{n}\|^{1/2}\|A\boldsymbol{u}^{n}\|^{1/2}\|A\boldsymbol{u}^{n,s}\| \\
&\leq C\|A\boldsymbol{u}^{n}\|\|\nabla \boldsymbol{u}^{n-1,s}\|^2 + \frac{\mu+\mu_r}{8}\|A\boldsymbol{u}^{n,s}\|^2 \\
|(\boldsymbol{f}(\theta^{n-1+s}) - \boldsymbol{f}(\theta^{n-1}), A\boldsymbol{u}^{n,s})| &\leq \|\boldsymbol{f}(\theta^{n-1+s}) - \boldsymbol{f}(\theta^{n-1})\|\|A\boldsymbol{u}^{n,s}\| \\
&\leq C\|\theta^{n,s}\|\|A\boldsymbol{u}^{n,s}\| \\
&\leq C\|\theta^{n-1,s}\|^2 + \frac{\mu+\mu_r}{8}\|A\boldsymbol{u}^{n,s}\|^2.
\end{aligned}$$

Destas estimativas e de (3.47) obtêm-se

$$\begin{aligned}&\|\nabla \boldsymbol{u}^{n,s}(t)\|^2 + (\mu+\mu_r)\int_0^t \|A\boldsymbol{u}^{n,s}(\tau)\|^2 d\tau\\ &\leq C\int_0^t (\|\nabla \boldsymbol{u}^{n,s}(\tau)\|^2 + \|\nabla \boldsymbol{w}^{n-1,s}(\tau)\|^2 + \|\theta^{n-1,s}(\tau)\|)d\tau\\ &+C\int_0^t \|\nabla \boldsymbol{u}^{n-1,s}(\tau)\|^2\|A\boldsymbol{u}^{n}(\tau)\|d\tau.\end{aligned} \tag{3.48}$$

De modo similar, multiplicando (3.43) por $B\boldsymbol{w}^{n,s}$ e usando (3.26), temos

$$\begin{aligned}&\|\nabla \boldsymbol{w}^{n,s}(t)\|^2 + (c_a+C_d)\int_0^t \|B\boldsymbol{w}^{n,s}(\tau)\|^2 d\tau\\ &\leq C\int_0^t (\|\nabla \boldsymbol{w}^{n,s}(\tau)\|^2 + \|\nabla \boldsymbol{u}^{n-1,s}(\tau)\|^2 + \|\theta^{n-1,s}(\tau)\|^2)d\tau\\ &+C\int_0^t \|\nabla \boldsymbol{u}^{n-1,s}(\tau)\|^2\|B\boldsymbol{w}^{n}(\tau)\|d\tau.\end{aligned} \tag{3.49}$$

Somando as desigualdades (3.48), (3.49), e usando a desigualdade de Gronwall o resultado segue. $\square$

Lema 3.7. *Existe uma constante $C_9 > 0$, que independe de n e s, tal que*

$$\begin{aligned}&\int_0^t \|\boldsymbol{u}_t^{n,s}(\tau)\|^2 d\tau + \int_0^t \|\boldsymbol{w}_t^{n,s}(\tau)\|^2 d\tau\\ &\leq C_9\int_0^t \|\nabla \boldsymbol{w}^{n,s}(\tau)\|^2 d\tau\\ &+C_9\int_0^t (\|A\boldsymbol{u}^{n}(\tau)\| + \|B\boldsymbol{w}^{n}(\tau)\|+1)(\|\nabla \boldsymbol{u}^{n-1,s}(\tau)\|^2 + \|\nabla \boldsymbol{w}^{n-1,s}(\tau)\|^2 + \|\theta^{n-1,s}(\tau)\|^2)d\tau.\end{aligned}$$

Demonstração. Multiplicando (3.42) por $\boldsymbol{u}_t^{n,s}$, e integrando em Ω temos

$$\begin{aligned}\|\boldsymbol{u}_t^{n,s}\|^2 &= -(\mu+\mu_r)(A\boldsymbol{u}^{n,s},\boldsymbol{u}_t^{n,s}) - (\boldsymbol{u}^{n-1+s}\cdot\nabla \boldsymbol{u}^{n,s},\boldsymbol{u}_t^{n,s}) + 2\mu_r(\operatorname{rot}\boldsymbol{w}^{n-1,s},\boldsymbol{u}_t^{n,s})\\ &\quad -(\boldsymbol{u}^{n-1,s}\cdot\nabla\boldsymbol{u}^{n},\boldsymbol{u}_t^{n,s}) + (\boldsymbol{f}(\theta^{n-1+s}) - \boldsymbol{f}(\theta^{n-1}),\boldsymbol{u}_t^{n,s})\\ &\leq C\|A\boldsymbol{u}^{n,s}\|\|\boldsymbol{u}_t^{n,s}\| + C\|\boldsymbol{u}^{n-1+s}\|_{L^4}\|\nabla\boldsymbol{u}^{n,s}\|_{L^4}\|\boldsymbol{u}_t^{n,s}\| + C\|\nabla\boldsymbol{w}^{n-1,s}\|\|\boldsymbol{u}_t^{n,s}\|\\ &\quad +C\|\boldsymbol{u}^{n-1,s}\|_{L^6}\|\nabla\boldsymbol{u}^{n}\|_{L^3}\|\boldsymbol{u}_t^{n,s}\| + C\|\theta^{n-1,s}\|\|\boldsymbol{u}_t^{n,s}\|\\ &\leq C_\epsilon\|A\boldsymbol{u}^{n,s}\|^2 + C_\epsilon\|\nabla\boldsymbol{w}^{n-1,s}\|^2 + C_\epsilon\|\nabla\boldsymbol{u}^{n-1,s}\|^2\|A\boldsymbol{u}^{n}\|\\ &\quad +C_\epsilon\|\theta^{n-1,s}\|^2 + \epsilon\|\boldsymbol{u}_t^{n,s}\|^2.\end{aligned} \tag{3.50}$$

Analogamente, obtemos

$$
\begin{aligned}
&\|\boldsymbol{w}_t^{n,s}\|^2 + \frac{1}{2}(c_0 + c_d - c_a)\frac{d}{dt}\|\operatorname{div}\boldsymbol{w}^{n,s}\|^2 \\
&= -(c_a + c_d)(B\boldsymbol{w}^{n,s}, \boldsymbol{w}_t^{n,s}) - (\boldsymbol{u}^{n-1+s}\cdot\nabla\boldsymbol{w}^{n,s}, \boldsymbol{w}_t^{n,s}) - 4\mu_r(\boldsymbol{w}^{n,s}, \boldsymbol{w}_t^{n,s}) \\
&\quad +2\mu_r(\operatorname{rot}\boldsymbol{u}^{n-1,s}, \boldsymbol{w}_t^{n,s}) - (\boldsymbol{u}^{n-1,s}\cdot\nabla\boldsymbol{w}^{n}, \boldsymbol{w}_t^{n,s}) + (\boldsymbol{g}(\theta^{n-1+s}) - \boldsymbol{g}(\theta^{n-1}), \boldsymbol{w}_t^{n,s}) \\
&\le C\|B\boldsymbol{w}^{n,s}\|\|\boldsymbol{w}_t^{n,s}\| + C\|\boldsymbol{u}^{n-1+s}\|_{L^4}\|\nabla\boldsymbol{w}^{n,s}\|_{L^4}\|\boldsymbol{w}_t^{n,s}\| + C\|\boldsymbol{w}^{n,s}\|\|\boldsymbol{w}_t^{n,s}\| \\
&\quad +C\|\nabla\boldsymbol{u}^{n-1,s}\|\|\boldsymbol{w}_t^{n,s}\| + \|\boldsymbol{u}^{n-1,s}\|_{L^6}\|\nabla\boldsymbol{w}^{n}\|_{L^3}\|\boldsymbol{w}_t^{n,s}\| + C\|\theta^{n-1,s}\|\|\boldsymbol{w}_t^{n,s}\| \\
&\le C_\eta\|B\boldsymbol{w}^{n,s}\|^2 + C_\eta\|\nabla\boldsymbol{w}^{n,s}\|^2 + C_\eta\|\nabla\boldsymbol{u}^{n-1,s}\|^2 + C_\eta\|\nabla\boldsymbol{u}^{n-1,s}\|^2\|B\boldsymbol{w}^{n}\| \\
&\quad +C_\epsilon\|\theta^{n-1,s}\|^2 + \eta\|\boldsymbol{w}_t^{n,s}\|^2.
\end{aligned}
\tag{3.51}
$$

Tomando $\epsilon > 0$ e $\eta > 0$ de maneira apropriada, somando (3.50) - (3.51) temos

$$
\begin{aligned}
&\int_0^t \|\boldsymbol{u}_t^{n,s}(\tau)\|^2 d\tau + \int_0^t \|\boldsymbol{w}_t^{n,s}(\tau)\| d\tau \\
&\le C\int_0^t (\|A\boldsymbol{u}^{n,s}\|^2 + \|B\boldsymbol{w}^{n,s}\|^2) d\tau \\
&+C\int_0^t (\|\nabla\boldsymbol{u}^{n-1,s}(\tau)\|^2 + \|\nabla\boldsymbol{w}^{n-1,s}(\tau)\|^2 + \|\theta^{n-1,s}(\tau)\|^2) d\tau \\
&+C\int_0^t \|\nabla\boldsymbol{u}^{n-1,s}(\tau)\|^2(\|A\boldsymbol{u}^{n}(\tau)\| + \|B\boldsymbol{w}^{n}(\tau)\|) d\tau + \int_0^t \|\nabla\boldsymbol{w}^{n,s}(\tau)\|^2 d\tau,
\end{aligned}
$$

onde $C > 0$ não depnde de n e s. A conclusão segue do Lema 3.6. □

Prova do Teorema 3.3 Multiplicando (3.44) por $\theta^{n,s}$ e integrando em Ω temos

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta^{n,s}\|^2 + \kappa\|\nabla\theta^{n,s}\|^2 \\
&= (\Phi(\boldsymbol{u}^{n-1+s}, \boldsymbol{w}^{n-1+s}) - \Phi(\boldsymbol{u}^{n-1}, \boldsymbol{w}^{n-1}), \theta^{n,s}) - (\boldsymbol{u}^{n-1,s}\cdot\nabla\theta^{n}, \theta^{n,s}).
\end{aligned}
$$

Das desigualdades (3.30), (3.31) e das estimativas do Teorema 3.1 nós temos que

$$
\begin{aligned}
|(\boldsymbol{u}^{n-1,s}\cdot\nabla\theta^{n}, \theta^{n,s})| &= |(\boldsymbol{u}^{n-1,s}\theta^{n}, \nabla\theta^{n,s})| \\
&\le \|\theta^{n}\|_{L^3}\|\boldsymbol{u}^{n-1,s}\|_{L^6}\|\nabla\theta^{n,s}\| \\
&\le M_0^{1/2}M_6\|\nabla\theta^{n}\|^{1/2}\|\nabla\boldsymbol{u}^{n-1,s}\|\|\theta^{n,s}\| \\
&\le \frac{3}{\kappa}M_0M_6^2\|\nabla\boldsymbol{u}^{n-1,s}\|^2\|\nabla\theta^{n}\| + \frac{\kappa}{12}\|\nabla\theta^{n,s}\|^2,
\end{aligned}
$$

Para estimar os termos da função Φ usaremos as seguintes desigualdades:

$$\left\{\begin{aligned} |\Phi_1(\boldsymbol{u}_1)-\Phi_1(\boldsymbol{u}_2)| &\leq 18\mu(|\nabla \boldsymbol{u}_1|+|\nabla \boldsymbol{u}_2|)|\nabla(\boldsymbol{u_1}-\boldsymbol{u_1})|,\\ |\Phi_2(\boldsymbol{u}_1,\boldsymbol{w}_1)-\Phi_2(\boldsymbol{u}_2,\boldsymbol{w}_2)| &\leq 4\mu_r(|\nabla \boldsymbol{u}_1|+|\nabla \boldsymbol{u}_2|+|\boldsymbol{w}_1|+|\boldsymbol{w}_2|))\\ &.(|\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)|+|\boldsymbol{w}_1-\boldsymbol{w}_2|),\\ |\Phi_3(\boldsymbol{w}_1)-\Phi_3(\boldsymbol{w}_2)| &\leq 9c_0|\nabla(\boldsymbol{w}_1+\boldsymbol{w}_2)||\nabla(\boldsymbol{w}_1-\boldsymbol{w}_2)|,\\ |\Phi_4(\boldsymbol{w}_1)-\Phi_3(\boldsymbol{w}_2)| &\leq 9(c_a+c_d)(|\nabla \boldsymbol{w}_1|+|\nabla \boldsymbol{w}_2|)|\nabla(\boldsymbol{w}_1-\boldsymbol{w}_2)|,\\ |\Phi_5(\boldsymbol{w}_1)-\Phi_3(\boldsymbol{w}_2)| &\leq 18|c_d-c_a|(|\nabla \boldsymbol{w}_1|+|\nabla \boldsymbol{w}_2|)|\nabla(\boldsymbol{w}_1-\boldsymbol{w}_2)|. \end{aligned}\right. \tag{3.52}$$

Usando estas estimativas temos

$$\begin{aligned} &|(\Phi_1(\boldsymbol{u}^{n-1+s})-\Phi_1(\boldsymbol{u}^{n-1}),\theta^{n,s})|\\ &\leq 18\mu(\|\nabla \boldsymbol{u}^{n-1+s}\|_{L^3}+\|\nabla \boldsymbol{u}^{n-1}\|_{L^3})\|\nabla \boldsymbol{u}^{n-1,s}\|\|\theta^{n,s}\|_{L^6}\\ &\leq 18\mu M_5M_6M_0^{1/2}(\|A\boldsymbol{u}^{n-1+s}\|^{1/2}+\|A\boldsymbol{u}^{n-1}\|^{1/2})\|\nabla \boldsymbol{u}^{n-1,s}\|\|\nabla\theta^{n,s}\|\\ &\leq \frac{6}{\kappa}M_0(18\mu M_5M_6)^2(\|A\boldsymbol{u}^{n-1+s}\|+\|A\boldsymbol{u}^{n-1}\|)\|\nabla \boldsymbol{u}^{n-1,s}\|^2+\frac{\kappa}{12}\|\nabla\theta^{n,s}\|^2, \end{aligned}$$

$$\begin{aligned} &|(\Phi_2(\boldsymbol{u}^{n-1+s},\boldsymbol{w}^{n-1+s})-\Phi_2(\boldsymbol{u}^{n-1},\boldsymbol{w}^{n-1}),\theta^{n,s})|\\ &\leq 4\mu_r(\|\nabla \boldsymbol{u}^{n-1+s}\|_{L^3}+\|\nabla \boldsymbol{u}^{n-1}\|_{L^3}+\|\boldsymbol{w}^{n-1+s}\|_{L^3}+\|\boldsymbol{w}^{n-1}\|_{L^3})\\ &.(\|\nabla \boldsymbol{u}^{n-1,s}\|+\|\boldsymbol{w}^{n-1,s}\|)\|\theta^{n,s}\|_{L^6}\\ &\leq 4\mu_rM_5M_6M_0^{1/2}(\|A\boldsymbol{u}^{n-1+s}\|+\|A\boldsymbol{u}^{n-1}\|+\|\nabla \boldsymbol{w}^{n-1+s}\|+\|\nabla \boldsymbol{w}^{n-1}\|)\cdot\\ &\quad(\|\nabla \boldsymbol{u}^{n-1,s}\|^2+\|\nabla \boldsymbol{w}^{n-1,s}\|^2)+\frac{\kappa}{12}\|\nabla\theta^{n,s}\|\\ &\leq \frac{18}{\kappa}(4\mu_rM_5M_6M_0^{1/2})^2(1+2M_0^{1/2})^2(\|A\mathbf{u}^{n-1+s}\|+\|A\mathbf{u}^{n-1}\|+1\|)\\ &.(\|\nabla \mathbf{u}^{n-1,s}\|^2+\|\nabla \mathbf{w}^{n-1,s}\|^2)+\frac{\kappa}{12}\|\nabla\theta^{n,s}\|, \end{aligned}$$

e para $i=3,4,5$

$$\begin{aligned} (\Phi_i(\boldsymbol{w}^{n-1+s})-\Phi_i(\boldsymbol{w}^{n-1}),\theta^{n,s})| &\leq L_i(\|\nabla \boldsymbol{w}^{n-1+s}\|_{L^3}+\|\nabla \boldsymbol{w}^{n-1}\|_{L^3})\|\nabla \boldsymbol{w}^{n-1,s}\|\|\theta^{n,s}\|_{L^6}\\ &\leq M_5M_6M_0^{1/2}L_i(\|B\boldsymbol{w}^{n-1+s}\|^{1/2}\\ &+ \|B\boldsymbol{w}^{n-1}\|^{1/2})\|\nabla \boldsymbol{w}^{n-1,s}\|\|\nabla\theta^{n,s}\|\\ &\leq C_\epsilon(\|B\boldsymbol{w}^{n-1+s}\|+\|B\boldsymbol{w}^{n-1}\|)\|\nabla \boldsymbol{w}^{n-1,s}\|^2\\ &+ \frac{\kappa}{12}\|\nabla\theta^{n,s}\|^2, \end{aligned}$$

onde $L_3=9c_0, L_4=9(c_a+c_d)$ e $L_5=18|c_d-c_a|$.

Para $n,s\in\mathbb{N}$, $t\in[0,T_1]$ seja

$$M_{n,s}(t)=\|A\boldsymbol{u}^{n-1+s}\|+\|A\boldsymbol{u}^{n-1}\|+\|B\boldsymbol{w}^{n-1+s}\|+\|B\boldsymbol{w}^{n-1}\|+\|\theta^n\|+1.$$

Das estimativas anteriores concluímos que

$$\begin{aligned}
&\|\theta^{n,s}(t)\|^2 + \kappa \int_0^t \|\nabla\theta^{n,s}(\tau)\|^2 d\tau \\
&\leq L_6 \int_0^t M_{n,s}(\tau)(\|\nabla \boldsymbol{u}^{n-1,s}(\tau)\|^2 + \|\nabla \boldsymbol{w}^{n-1,s}(\tau)\|^2) d\tau,
\end{aligned} \tag{3.53}$$

onde

$$\begin{aligned}
L_6 &= \frac{6}{\kappa} M_0 (M_4 M_6)^2 + \frac{12}{\kappa} M_0 (18\mu M_5 M_6)^2 \\
&+ \frac{36 M_0}{\kappa} (4\mu_r M_5 M_6)^2 (1 + 2M_0^{1/2})^2 + \frac{12}{\kappa} M_0 \sum_{i=1}^{3} (M_5 M_6 L_i)^2.
\end{aligned}$$

Somando as desigualdades (3.46), (3.53), e definindo

$$\phi_{n,s}(t) \;=\; \|\nabla \boldsymbol{u}^{n,s}(t)\|^2 + \|\nabla \boldsymbol{w}^{n,s}(t)\|^2 + \|\theta^{n,s}(t)\|^2,$$

obtemos

$$\begin{aligned}
&\phi_{n,s}(t) + (\mu + \mu_r) \int_0^t \|A\boldsymbol{u}^{n,s}(\tau)\|^2 d\tau + (c_a + c_d) \int_0^t \|B\boldsymbol{w}^{n,s}(\tau)\|^2 d\tau + \kappa \int_0^t \|\nabla\theta^{n,s}(\tau)\|^2 d\tau \\
&\leq C_{10} \int_0^t \widetilde{M}_{n,s}(\tau) \phi_{n-1,s}(\tau) d\tau,
\end{aligned} \tag{3.54}$$

onde $\widetilde{M}_{n,s} = M_{n,s}(\tau) + \|A\boldsymbol{u}^n\| + \|B\boldsymbol{u}^n\| + 1$ e $C_{10} = C_8 + L_6$. Pelo Teorema 3.1, $\widetilde{M}_{n,s} \in L^2(0,T)$. Assim pela desigualdade de Hölder obtemos

$$\phi_{n,s}(t) \leq C_{10} \left(\int_0^t \widetilde{M}^2_{n,s}(\tau) d\tau \right)^{1/2} \left(\int_0^t \phi^2_{n-1,s}(\tau) d\tau \right)^{1/2}.$$

Assim,

$$\phi^2_{n,s}(t) \leq C_{11} \int_0^t \phi^2_{n-1,s}(\tau) d\tau,$$

onde $C_{11} = C_{10}^2 \int_0^{T_1} \widetilde{M}^2_{n,s}(\tau) d\tau < \infty$.

Do Teorema 3.1, $\phi_{0,s}(t) = \|\nabla \boldsymbol{u}^s(t)\|^2 + \|\nabla \boldsymbol{w}^s(t)\|^2 + \|\theta^s(t)\|^2 \leq M_0$ para todo $t \in [0, T_1]$. Portanto, pelo Lema 3.5 concluímos que

$$\|\nabla \boldsymbol{u}^{n,s}(t)\|^2 + \|\nabla \boldsymbol{w}^{n,s}(t)\|^2 + \|\theta^{n,s}(t)\|^2 = \phi_{n,s} \leq M_0^{1/2} \left[\frac{(C_{11} t)^{n-1}}{(n-1)!} \right]^{1/2} = M_0^{1/2} \Lambda_{n-1}(t), \tag{3.55}$$

para todo $t \in [0, T_1]$, $n \geq 2$. Substituindo esta estimativa em (3.54) segue que

$$
\begin{aligned}
&(\mu + \mu_r) \int_0^t \|A\boldsymbol{u}^{n,s}(\tau)\|^2 d\tau + (c_a + c_d) \int_0^t \|B\boldsymbol{w}^{n,s}(\tau)\|^2 d\tau + \kappa \int_0^t \|\nabla\theta^{n,s}(\tau)\|^2 d\tau \\
&\leq C_{11}^{1/2} \left(\int_0^t \phi_{n-1,s}^2 d\tau \right)^{1/2} \\
&\leq M_0^{1/2} \left[\frac{(C_{11}t)^{n-1}}{(n-1)!} \right]^{1/2} = M_0^{1/2} \Lambda_{n-1}(t).
\end{aligned}
\tag{3.56}
$$

Integrando (3.55) e usando a desigualdade de Hölder, obtemos

$$
\begin{aligned}
&\int_0^t \|\nabla\boldsymbol{u}^{n,s}(\tau)\|^2 d\tau + \int_0^t \|\nabla\boldsymbol{w}^{n,s}(\tau)\|^2 d\tau + \int_0^t \|\theta^{n,s}(\tau)\|^2 d\tau \\
&\leq M_0^{1/2} \int_0^t \left[\frac{(C_{11}\tau)^{n-1}}{(n-1)!} \right]^{1/2} \\
&\leq C_{12} \left[\frac{(C_{11}t)^{n}}{n!} \right]^{1/2} = C_{12}\Lambda_n(t)
\end{aligned}
\tag{3.57}
$$

onde $C_{12} = (M_0 T_1)^{1/2}$. Em consequência do Lema 3.7, Teorema 3.1 e (3.55)

$$
\begin{aligned}
&\int_0^t \|\boldsymbol{u}_t^{n,s}(\tau)\|^2 d\tau + \int_0^t \|\boldsymbol{w}_t^{n,s}(\tau)\|^2 d\tau \\
&\leq C_9 \int_0^t \|\nabla\boldsymbol{w}^{n,s}(\tau)\|^2 d\tau + C_9 \int_0^t \phi_{n-1,s}(\tau)(\|A\boldsymbol{u}^n\| + \|B\boldsymbol{w}^n\| + 1) d\tau \\
&\leq C_9 C_{12} \left[\frac{(C_{11}t^n)}{n!} \right]^{1/2} + \frac{C_9}{4} \left(\int_0^t \phi_{n-1,s}^2(\tau) d\tau \right)^{1/2} \left[\int_0^t (\|A\boldsymbol{u}^n\|^2 + \|B\boldsymbol{w}^n\|^2 + 1) d\tau \right]^{1/2} \\
&\leq C_{13} \left[\frac{(C_{11}t)^n}{n!} \right]^{1/2} + C_{14} \left[\frac{(C_{11}t)^{n-1}}{(n-1)!} \right]^{1/2} \\
&\leq C_{13}\Lambda_n(t) + C_{14}\Lambda_{n-1}(t)
\end{aligned}
\tag{3.58}
$$

onde $C_{13} = C_9 C_{12}$ e $C_{14} = \dfrac{C_9(M_0 + T_1)^{1/2} M_0}{4C_{11}}$.

Por outro lado, de (3.44) temos

$$
\begin{aligned}
\|\theta_t^{n,s}\|_{H^{-1}}^2 &\leq \kappa \|\Delta\theta^{n,s}\|_{H^{-1}}^2 + \|\boldsymbol{u}^{n-1+s} \cdot \nabla\theta^{n,s}\|_{H^{-1}}^2 + \|\boldsymbol{u}^{n-1,s} \cdot \nabla\theta^{n}\|_{H^{-1}}^2 \\
&\quad + \|\Phi(\boldsymbol{u}^{n-1+s}, \boldsymbol{w}^{n-1+s}) - \Phi(\boldsymbol{u}^{n-1}, \boldsymbol{w}^{n-1})\|_{H^{-1}}^2.
\end{aligned}
\tag{3.59}
$$

Agora, vamos estimar os termos do lado direito de (3.59):

$$
\begin{aligned}
\kappa \|\Delta\theta^{n,s}\|_{H^{-1}} &\leq \kappa \sup_{\|\varphi\|_{H_0^1} \leq 1} |(\nabla\theta^{n,s}, \nabla\varphi)| \\
&\leq \kappa \|\nabla\theta^{n,s}\|.
\end{aligned}
$$

Pela desigualdade de Poincaré, (3.31) e pelo Teorema 3.1 temos

$$\begin{aligned}\|\boldsymbol{u}^{n-1+s}\cdot\nabla\theta^{n,s}\|_{H^{-1}} &\le \sup_{\|\varphi\|_{H_0^1}\le 1}|(\boldsymbol{u}^{n-1+s}\cdot\nabla\theta^{n,s},\varphi)| \\ &\le \sup_{\|\varphi\|_{H_0^1}\le 1}\|\boldsymbol{u}^{n-1+s}\|_{L^3}\|\nabla\theta^{n,s}\|\|\varphi\|_{L^6} \\ &\le M_0M_6/\lambda_1\|\nabla\theta^{n,s}\|.\end{aligned}$$

Usando a imersão de Sobolev $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, isto é, $\|\psi\|_{L^\infty}\le M_8\|\psi\|_{H^2}$ para todo $\psi\in H^2(\Omega)$, e por (3.18) temos

$$\begin{aligned}\|\boldsymbol{u}^{n-1,s}\cdot\nabla\theta^{n}\|_{H^{-1}} &\le \sup_{\|\varphi\|\le 1}|(\boldsymbol{u}^{n-1,s}\cdot\nabla\theta^{n},\varphi)| \\ &\le \sup_{\|\varphi\|_{H_0^1}\le 1}\|\boldsymbol{u}^{n-1,s}\|_\infty\|\theta^n\|\|\|\nabla\varphi\| \\ &\le M_2M_8\|A\boldsymbol{u}^{n-1,s}\|.\end{aligned}$$

De forma similar, usando (3.52) temos

$$\begin{aligned}\|\Phi_1(\boldsymbol{u}^{n-1+s})-\Phi_1(\boldsymbol{u}^{n-1})\|_{H^{-1}} &\le 18\mu\sup_{\|\varphi\|_{H_0^1}\le 1}(\|\nabla\boldsymbol{u}^{n-1+s}\|+\|\nabla\boldsymbol{u}^{n-1}\|)\|\nabla\boldsymbol{u}^{n,s}\|_{L^3}\|\varphi\|_{L^6} \\ &\le 18\mu M_2M_5'(\|\nabla\boldsymbol{u}^{n-1+s}\|+\|\nabla\boldsymbol{u}^{n-1}\|)\|A\boldsymbol{u}^{n-1,s}\| \\ &\le 36\mu M_0M_2M_5'\|A\boldsymbol{u}^{n-1,s}\|,\end{aligned}$$

$$\begin{aligned}&\|\Phi_2(\boldsymbol{u}^{n-1+s},\boldsymbol{w}^{n-1+s})-\Phi_2(\boldsymbol{u}^{n-1},\boldsymbol{w}^{n-1+s})\|_{H^{-1}} \\ &\le 4\mu_rM_6(\|\nabla\boldsymbol{u}^{n-1+s}\|+\|\nabla\boldsymbol{u}^{n-1}\|+\|\boldsymbol{w}^{n-1+s}\|+\|\boldsymbol{w}^{n-1}\|)(\|\nabla\boldsymbol{u}^{n-1,s}\|_{L^3}+\|\boldsymbol{w}^{n-1,s}\|_{L^3}) \\ &\le C_{15}(\|A\boldsymbol{u}^{n-1,s}\|+\|\nabla\boldsymbol{w}^{n-1,s}\|),\end{aligned}$$

onde $C_{15}=16\mu_rM_0M_6(M_2M_5'+M_4\lambda_1^{-1/2})$. Além disso, para $i=3,4,5$

$$\begin{aligned}\|\Phi_i(\boldsymbol{u}^{n-1+s})-\Phi_i(\boldsymbol{u}^{n-1})\|_{H^{-1}} &\le C_{16}M_6(\|\nabla\boldsymbol{w}^{n-1+s}\|+\|\nabla\boldsymbol{w}^{n-1}\|)\|\nabla\boldsymbol{w}^{n-1,s}\|_{L^3} \\ &\le 2C_{16}M_0M_6M_4\lambda_1^{-1/2}\|\nabla\boldsymbol{w}^{n-1,s}\|,\end{aligned}$$

onde $C_{16}=\max\{c_a+c_d,9c_0,18|c_d-c_a|\}$.

Portanto, de (3.56) e (3.57)

$$\begin{aligned}\int_0^t\|\theta_t^{n,s}(\tau)\|_{H^{-1}}^2d\tau &\le C_{17}\int_0^t(|\nabla\theta^{n,s}(\tau)\|^2+\|A\boldsymbol{u}^{n-1,s}(\tau)\|^2+\|\nabla\boldsymbol{w}^{n-1,s}(\tau)\|^2)dt \\ &\le C_{18}[\Lambda_n(t)+\Lambda_{n-2}(t)+\Lambda_{n-1}(t)]\end{aligned} \tag{3.60}$$

onde
$C_{17} = \max\{\kappa + M_0M_6/\lambda_1, M_2M_8 + 36\mu M_0M_2M_5' + C_{15}/2, C_{15}/2 + 6C_{16}M_0M_6M_4\lambda_1^{-1/2}\}$, e
$C_{18} = C_{17}\max\{M_0^{1/2}/\kappa, M_0^{1/2}/(\mu+\mu_r), C_{12}\}$.

Como os espaços $L^\infty(0,T;V)$, $L^\infty(0,T;H_0^1(\Omega))$, $L^\infty(0,T;L^2(\Omega))$, $L^2(0,T;D(A))$, $L^2(0,T;D(B))$, $L^2(0,T;H_0^1(\Omega))$, $L^2(0,T;H)$, $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$ são espaços de Banach, das estimativas (3.55)-(3.58) e (3.60) concluímos que existem funções $\boldsymbol{u}$, $\boldsymbol{w}$ e θ tais que

$$\begin{aligned}
\boldsymbol{u}^n &\longrightarrow \boldsymbol{u} \quad \text{fortemente em} \quad L^\infty(0,T;V)\cap L^2(0,T;D(A)),\\
\boldsymbol{u}_t^n &\longrightarrow \boldsymbol{u}_t \quad \text{fortemente em} \quad L^2(0,T;H),\\
\boldsymbol{w}^n &\longrightarrow \boldsymbol{w} \quad \text{fortemente em} \quad L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;D(B)),\\
\boldsymbol{w}_t^n &\longrightarrow \boldsymbol{w}_t \quad \text{fortemente em} \quad L^2(0,T;L^2(\Omega)),\\
\theta^n &\longrightarrow \theta \quad \text{fortemente em} \quad L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)),\\
\theta_t^n &\longrightarrow \theta_t \quad \text{fortemente em} \quad L^2(0,T;H^{-1}(\Omega)),
\end{aligned}$$

quando $n \to \infty$. Desde que $(\boldsymbol{u}^n, \boldsymbol{w}^n, \theta^n)$ verifique o problema (3.4)-(3.2) é claro que o limite $(\boldsymbol{u}, \boldsymbol{w}, \theta)$ é solução do problema (1.1)-(1.2).

Agora iremos mostrar a unicidade. Sejam $(\boldsymbol{u}_1, \boldsymbol{w}_1, \theta_1)$ e $(\boldsymbol{u}_2, \boldsymbol{w}_2, \theta_2)$ duas soluções do problema (1.1). Seja $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{w} = \boldsymbol{w}_1 - \boldsymbol{w}_2$ e $\theta = \theta_1 - \theta_2$. Então, temos

$$\begin{aligned}
(\boldsymbol{u}_t - (\mu+\mu_r)A\boldsymbol{u}, \varphi) + b(\boldsymbol{u}, \boldsymbol{u}_1, \varphi) + b(u_2, \boldsymbol{u}, \varphi) &= (2\mu_r \operatorname{rot} \boldsymbol{w} + f(\theta_1) - f(\theta_2), \varphi),\\
(\boldsymbol{w}_t + L\boldsymbol{w} + 4\mu_r \boldsymbol{w}, \psi) + b(\boldsymbol{u}_1, w, \psi) + b(\boldsymbol{u}, \boldsymbol{w}_2, \psi) &= (2\mu_r \operatorname{rot} \boldsymbol{u} + g(\theta_1) - g(\theta_2), \psi),
\end{aligned}$$

e

$$\langle \theta_t, \phi\rangle + \kappa(\nabla\theta, \nabla\phi) + (\boldsymbol{u}_1\cdot\nabla\theta + \boldsymbol{u}\cdot\nabla\theta_2, \phi) = (\Phi(\boldsymbol{u}_1, \boldsymbol{w}_1) - \Phi(\boldsymbol{u}_2, \boldsymbol{w}_2), \phi), \tag{3.61}$$

para $\varphi \in H$, $\psi \in L^2(\Omega)$ e $\phi \in H_0^1(\Omega)$. Defina $\varphi = A\boldsymbol{u}$ podemos argumentar como em (3.19)-(3.21) para concluir

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|^2 + (\mu+\mu_r)\|A\boldsymbol{u}(t)\|^2 &= -(\boldsymbol{u}\cdot\nabla\boldsymbol{u}_1, A\boldsymbol{u})\\
&\quad +(\operatorname{rot} \boldsymbol{w}, A\boldsymbol{u}) + (\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2), A\boldsymbol{u}),
\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{u}\cdot\nabla\boldsymbol{u}_1, A\boldsymbol{u})| &\le \|\boldsymbol{u}\|_{L^4}\|\nabla\boldsymbol{u}_1\|_{L^4}\|A\boldsymbol{u}\|\\
&\le C\|\nabla\boldsymbol{u}\|\|A\boldsymbol{u}_1\|\|A\boldsymbol{u}\|\\
&\le C\|\nabla\boldsymbol{u}\|^2\|A\boldsymbol{u}_1\|^2 + \epsilon\|A\boldsymbol{u}\|^2,
\end{aligned}$$

$$
\begin{aligned}
|(\operatorname{rot}\boldsymbol{w}, A\boldsymbol{u})| &\le \|\nabla\boldsymbol{w}\|\|A\boldsymbol{u}\| \le C\|\nabla\boldsymbol{w}\|^2 + \epsilon\|A\boldsymbol{u}\|^2,\\
|(\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2), A\boldsymbol{u})| &\le C\|\theta\|\|A\boldsymbol{u}\| \le C\|\theta\|^2 + \epsilon\|A\boldsymbol{u}\|^2.
\end{aligned}
$$

Escolhendo $\epsilon > 0$ apropriadamente temos

$$
\frac{d}{dt}\|\nabla\boldsymbol{u}(t)\|^2 + (\mu + \mu_r)\|A\boldsymbol{u}(t)\|^2 \le C(\|A\boldsymbol{u}_1\|^2\|\nabla\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{w}\|^2 + \|\theta\|^2). \tag{3.62}
$$

De modo similar, usando (3.26) podemos concluir que

$$
\frac{d}{dt}\|\nabla\boldsymbol{w}\|^2 + (c_a + c_d)\|B\boldsymbol{w}\|^2 \le C[(\|B\boldsymbol{w}_2\|^2 + 1)\|\nabla\boldsymbol{u}\|^2 + \|\nabla\boldsymbol{w}\|^2 + \|\theta\|^2]. \tag{3.63}
$$

Seja $\phi = \theta$ em (3.61) desta forma

$$
\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \kappa\|\nabla\theta\|^2 = -(\boldsymbol{u}\cdot\nabla\theta_2, \theta) + (\Phi(\boldsymbol{u}_1, \boldsymbol{w}_1) - \Phi(\boldsymbol{u}_2, \boldsymbol{w}_2), \theta). \tag{3.64}
$$

Pela imersão $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, (3.18), (3.15), (3.17) e a desigualdade de Young temos

$$
\begin{aligned}
|(\boldsymbol{u}\cdot\nabla\theta_2, \theta)| &\le \|\boldsymbol{u}\|_{L^\infty}\|\nabla\theta_2\|\|\theta\|\\
&\le C\|A\boldsymbol{u}\|\|\nabla\theta_2\|\|\theta\|\\
&\le C\|\nabla\theta_2\|^2\|\theta\|^2 + \epsilon\|A\boldsymbol{u}\|^2.
\end{aligned}
$$

De maneira similar, por (3.18), (3.15), (3.17) e (3.52)

$$
\begin{aligned}
|(\Phi_1(\boldsymbol{u}_1) - \Phi_1(\boldsymbol{u}_2), \theta)| &\le C(\|\nabla\boldsymbol{u}_1\|_{L^4} + \|\nabla\boldsymbol{u}_2\|_{L^4})\|\boldsymbol{u}\|_{L^4}\|\theta\|\\
&\le C(\|A\boldsymbol{u}_1\| + \|A\boldsymbol{u}_2\|)\|A\boldsymbol{u}\|\|\theta\|\\
&\le C(\|A\boldsymbol{u}_1\|^2 + \|A\boldsymbol{u}_2\|^2)\|\theta\|^2 + \epsilon\|A\boldsymbol{u}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&|(\Phi_2(\boldsymbol{u}_1, \boldsymbol{w}_1) - \Phi_2(\boldsymbol{u}_2, \boldsymbol{w}_2), \theta)|\\
&\le C(\|\nabla\boldsymbol{u}_1\|_{L^4} + \|\nabla\boldsymbol{u}_2\|_{L^4} + \|\boldsymbol{w}_1\|_{L^4} + \|\nabla\boldsymbol{w}_2\|_{L^4})(\|\nabla\boldsymbol{u}\|_{L^4} + \|\boldsymbol{w}\|_{L^4})\|\theta\|\\
&\le C(\|A\boldsymbol{u}_1\|^2 + \|A\boldsymbol{u}_2\|^2 + 1)\|\theta\|^2 + \epsilon(\|A\boldsymbol{u}\|^2 + \|B\boldsymbol{w}\|^2)
\end{aligned}
$$

$$
\begin{aligned}
|(\Phi_i(\boldsymbol{w}_1) - \Phi_i(\boldsymbol{w}_2), \theta)| &\le C(\|\nabla\boldsymbol{w}_1\|_{L^4} + \|\nabla\boldsymbol{w}_2\|_{L^4})\|\nabla\boldsymbol{w}\|_{L^4}\|\theta\|\\
&\le C(\|B\boldsymbol{w}_1\|^2 + \|B\boldsymbol{w}_2\|^2)\|\theta\|^2 + \epsilon\|B\boldsymbol{w}\|^2,
\end{aligned}
$$

para $i = 3, 4, 5$.

Escolhendo $\epsilon > 0$, das estimativas acima e de (3.63)-(3.64) se obtêm

$$
\Psi'(t) \le C(\|A\boldsymbol{u}_1\|^2 + \|A\boldsymbol{u}_2\|^2 + \|B\boldsymbol{u}_1\|^2 + \|B\boldsymbol{u}_2\|^2 + \|\nabla\theta_2\|^2 + 1)\Psi,
$$

onde $\Psi(t) = \|\nabla\boldsymbol{u}(t)\|^2 + \|\nabla\boldsymbol{w}\|^2 + \|\theta\|^2$. Assim, pela desigualdade de Gronwall, segue que $\Psi = 0$. Portanto, $\boldsymbol{u}_1 = \boldsymbol{u}_2, \boldsymbol{w}_1 = \boldsymbol{w}_2$, e $\theta_1 = \theta_2$ em $[0, T_1]$.

3.2.3 Prova do Teorema 3.4

Para cada $n \in \mathbb{N}$, a regularidade da solução aproximada (3.1)-(3.3) pode ser provado usando o método espectral de Galerkin.

Para estabelecer as estimativas, vamos provar primeiro que existe $T_2 > 0$ tal que

$$\int_0^{T_2} \|\theta_t^n(\tau)\|^2 d\tau \leq 1, \tag{3.65}$$

para todo $n \in \mathbb{N}$. Usaremos o segundo principio de indução em n, isto é, vamos supor que (3.65) vale para $1 \leq j \leq n$.

Diferenciando com respeito a t a equação (3.1) temos

$$\boldsymbol{u}_{tt}^{j+1} + (\mu+\mu_r)A\boldsymbol{u}_t^{j+1} + P(\boldsymbol{u}_t^j \cdot \nabla \boldsymbol{u}^{j+1}) + P(\boldsymbol{u}^j \cdot \nabla \boldsymbol{u}_t^{j+1}) = 2\mu_r P(\operatorname{rot} \boldsymbol{w}_t^j) + Pf'(\theta^j)\theta_t^j.$$

Multiplicando por u_t^{j+1}, e integrando em Ω, obtemos

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t^{j+1}\|^2 + (\mu+\mu_r)\|\nabla \boldsymbol{u}_t^{j+1}\|^2 = -(\boldsymbol{u}_t^j \cdot \nabla \boldsymbol{u}^{j+1}, \boldsymbol{u}_t^{j+1}) + 2\mu_r(\operatorname{rot} \boldsymbol{w}_t^j, \boldsymbol{u}_t^{j+1}) + (f'(\theta^j)\theta_t^j, \boldsymbol{u}_t^{j+1}). \tag{3.66}$$

Agora vamos estimar os termos do lado direito de (3.66).

$$\begin{aligned}
2\mu_r|(\operatorname{rot} \boldsymbol{w}_t^j, \boldsymbol{u}_t^{j+1})| &\leq C\|\boldsymbol{w}_t^j\|^2 + \frac{\mu+\mu_r}{6}\|\nabla \boldsymbol{u}_t^{j+1}\|^2 \\
|(f'(\theta^j)\theta_t^j, \boldsymbol{u}_t^{j+1})| &\leq C|(\theta_t^j, \boldsymbol{u}_t^{j+1})| \leq C\|\theta_t^j\|^2 + \frac{\mu+\mu_r}{6}\|\nabla \boldsymbol{u}_t^{j+1}\|^2 \\
|(\boldsymbol{u}_t^j \cdot \nabla \boldsymbol{u}^{j+1}, \boldsymbol{u}_t^{j+1})| &\leq C\|\boldsymbol{u}_t^j\|\|A\boldsymbol{u}^{j+1}\|\|\nabla \boldsymbol{u}_t^{j+1}\| \\
&\leq C\|\boldsymbol{u}_t^j\|^2\|A\boldsymbol{u}^{j+1}\|^2 + \frac{\mu+\mu_r}{6}\|\nabla \boldsymbol{u}_t^{j+1}\|.
\end{aligned}$$

Usando estas estimativas em (3.66) temos

$$\frac{d}{dt}\|\boldsymbol{u}_t^{j+1}\|^2 + (\mu+\mu_r)\|\nabla \boldsymbol{u}_t^{j+1}\|^2 \leq C\|\boldsymbol{w}_t^j\|^2 + C\|\theta_t^j\|^2 + C\|\boldsymbol{u}_t^j\|^2\|A\boldsymbol{u}^{j+1}\|^2, \tag{3.67}$$

para $1 \leq j \leq n$.

Para estimar $\|A\boldsymbol{u}^{j+1}\|$ multiplique (3.1)(para $n = j$) por $A\boldsymbol{u}^{j+1}$. Desta forma

$$\begin{aligned}
(\mu+\mu_r)\|A\boldsymbol{u}^{j+1}\|^2 &= -(\boldsymbol{u}_t^{j+1}, A\boldsymbol{u}^{j+1}) - (\boldsymbol{u}^j \cdot \nabla u^{j+1}, A\boldsymbol{u}^{j+1}) + (2\mu_r \operatorname{rot} \boldsymbol{w}^j, A\boldsymbol{u}^{j+1}) \\
&+ (f(\theta^j), A\boldsymbol{u}^{j+1}).
\end{aligned} \tag{3.68}$$

Agora vamos estimar os termos do lado direito de (3.68)

$$
\begin{aligned}
|(\boldsymbol{u}_t^{j+1}, A\boldsymbol{u}^{j+1})| &\leq C\|\boldsymbol{u}_t^{j+1}\|^2 + \frac{\mu+\mu_r}{8}\|A\boldsymbol{u}^{j+1}\|^2,\\
|(\boldsymbol{u}^j \cdot \nabla \boldsymbol{u}^{j+1}, A\boldsymbol{u}^{j+1})| &\leq \|\boldsymbol{u}^j\|_{L^6}\|\nabla \boldsymbol{u}^{j+1}\|_{L^3}\|A\boldsymbol{u}^{j+1}\|\\
&\leq C\|\nabla \boldsymbol{u}^j\|^4\|\nabla \boldsymbol{u}^{j+1}\|^2 + \frac{\mu+\mu_r}{8}\|A\boldsymbol{u}^{j+1}\|^2,\\
|(2\mu_r \operatorname{rot} \boldsymbol{w}^j, A\boldsymbol{u}^{j+1})| &\leq C\|\nabla \boldsymbol{w}^j\|^2 + \frac{\mu+\mu_r}{8}\|A\boldsymbol{u}^{j+1}\|^2,\\
|(f(\theta^j), A\boldsymbol{u}^{j+1})| &\leq C\|\theta^j\|^2 + \frac{\mu+\mu_r}{8}\|A\boldsymbol{u}^{j+1}\|^2.
\end{aligned}
$$

Disto e das estimativas dadas pelo Teorema (3.1) se pode concluir

$$\|A\boldsymbol{u}^{j+1}\|^2 \leq C(1 + \|\boldsymbol{u}_t^{j+1}\|^2). \tag{3.69}$$

De (3.69) e (3.67) temos

$$\frac{d}{dt}\|\boldsymbol{u}_t^{j+1}\|^2 + (\mu+\mu_r)\|\nabla \boldsymbol{u}_t^{j+1}\|^2 \leq C\|\boldsymbol{w}_t^j\|^2 + C\|\theta_t^j\|^2 + C\|\boldsymbol{u}_t^j\|^2(1 + \|\boldsymbol{u}_t^{j+1}\|^2).$$

Por (3.65) e das estimativas do Teorema 3.1 temos

$$\|\boldsymbol{u}_t^{j+1}\|^2 + (\mu+\mu_r)\int_0^t \|\nabla \boldsymbol{u}_t^{j+1}(\sigma)\|^2 d\sigma \leq C + C\int_0^t \|\boldsymbol{u}_t^j(\sigma)\|^2\|\boldsymbol{u}_t^{j+1}(\sigma)\|^2 d\sigma.$$

Aplicando a desigualdade de Gronwall concluímos

$$\|\boldsymbol{u}_t^{j+1}\|^2 + (\mu+\mu_r)\int_0^t \|\nabla \boldsymbol{u}_t^{j+1}(\sigma)\|^2 d\sigma \leq C\exp\left(C\int_0^t \|\boldsymbol{u}_t^j(\sigma)\|^2 d\sigma\right) \leq C \tag{3.70}$$

para $1 \leq j \leq n$ e $t \in (0, T_2)$. Assim, de (3.69) temos

$$\|A\boldsymbol{u}^{j+1}\|^2 \leq C, \tag{3.71}$$

para $1 \leq j \leq n$ e $t \in (0, T_2)$.

Analogamente, é possível concluir que

$$\|B\boldsymbol{w}^{j+1}\|^2 \leq C, \tag{3.72}$$

$$\|\boldsymbol{w}_t^{j+1}\|^2 + (c_a + c_d)\int_0^t \|\nabla \boldsymbol{w}_t^{j+1}(\sigma)\|^2 d\sigma \leq C, \tag{3.73}$$

for $1 \leq j \leq n$ and $t \in (0, T_2)$.

Finalmente, Multiplique a equação (3.3) por θ_t^{n+1}. Assim,

$$\|\theta_t^{n+1}\|^2 + \frac{\kappa}{2}\frac{d}{dt}\|\nabla \theta^{n+1}\|^2 = -(\boldsymbol{u}^n \nabla \theta^{n+1}, \theta_t^{n+1}) + (\Phi(\boldsymbol{u}^n, \boldsymbol{w}^n), \theta_t^{n+1}). \tag{3.74}$$

Usando as estimativas do Teorema (3.1) e (3.69)-(3.73) temos

$$\begin{aligned}
|(\boldsymbol{u}^n\nabla\theta^{n+1},\theta_t^{n+1})| &\leq \|\boldsymbol{u}^n\||_{L^\infty}\|\nabla\theta^{n+1}\|\|\theta_t^{n+1}\| \\
&\leq C\|A\boldsymbol{u}^n\|\|\nabla\theta^{n+1}\|\|\theta_t^{n+1}\| \\
&\leq C\|\nabla\theta^{n+1}\|^2+\frac{1}{12}\|\theta_t^{n+1}\| \\
|(\Phi_1(\boldsymbol{u}^n),\theta_t^{n+1})| &\leq C(|\nabla\boldsymbol{u}^n|^2,\theta_t^{n+1}) \\
&\leq C\|\nabla\boldsymbol{u}^n\|_{L^4}^2\|\theta_t^{n+1}\| \\
&\leq C\|A\boldsymbol{u}^n\|^4+\frac{1}{12}\|\theta_t^{n+1}\|^2 \\
|(\Phi_2(\boldsymbol{u}^n,\boldsymbol{w}^n))| &\leq C(|\nabla\boldsymbol{u}^n|^2+|\boldsymbol{w}^n|^2,\theta_t^{n+1})| \\
&\leq C(\|\nabla\boldsymbol{u}^n\|_{L^4}^2+\|\boldsymbol{w}\|_{L^4}^2)\|\theta_t^{n+1}\| \\
&\leq C(\|A\boldsymbol{u}^n\|^4+\|B\boldsymbol{w}^n\|^4)+\frac{1}{12}\|\theta_t^{n+1}\|^2 \\
|(\Phi_i(\boldsymbol{w}^n),\theta_t^{n+1})| &\leq C(|\nabla\boldsymbol{w}^n|^2,\theta_t^{n+1}) \\
&\leq C\|\nabla\boldsymbol{w}^n\|_{L^4}^2\|\theta_t^{n+1}\| \\
&\leq C\|B\boldsymbol{w}^n\|^4+\frac{1}{12}\|\theta_t^{n+1}\|^2,
\end{aligned}$$

$i=3,4,5$. Portanto, de (3.74), (3.71), (3.72) e das estimativas acima, temos

$$\|\theta_t^{n+1}\|^2+\kappa\frac{d}{dt}\|\nabla\theta^{n+1}\|^2\leq C+C\|\nabla\theta^{n+1}\|^2.$$

Integrando, e usando a desigualdade de Gronwall nós temos

$$\kappa\|\nabla\theta^{n+1}\|^2+\int_0^t\|\theta_t^{n+1}(\sigma)\|^2d\sigma\leq CT_2\leq 1, \tag{3.75}$$

se $T_2\leq T_1$é suficientemente pequeno. Portanto, (3.65) vale para $n+1$ e o argumento de indução está concluído.

Vamos obter agora as estimativas para $\boldsymbol{u}^{n,s}$, $\boldsymbol{w}^{n,s}$ e $\theta^{n,s}$ definidos por (3.42)-(3.44). Para tal, diferenciamos a equação (3.42), com respeito a t, e multiplicar por $\boldsymbol{u}_t^{n,s}$. Assim,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t^{n,s}\|^2+(\mu+\mu_r)\|\nabla\boldsymbol{u}_t^{n,s}\|^2 &= -(\boldsymbol{u}_t^{n-1+s}\nabla\boldsymbol{u}^{n,s},\boldsymbol{u}_t^{n,s})+2\mu_r(\operatorname{rot}\boldsymbol{w}_t^{n-1,s},\boldsymbol{u}_t^{n,s}) \\
&\quad -(\boldsymbol{u}_t^{n-1,s}\cdot\nabla\boldsymbol{u}^n,\boldsymbol{u}_t^{n,s})-(\boldsymbol{u}^{n-1,s}\nabla\boldsymbol{u}_t^n,\boldsymbol{u}_t^{n,s}) \\
&\quad +(f'(\theta^{n-1+s})\theta_t^{n-1,s}+[f'(\theta^{n-1+s})-f'(\theta^{n-1})]\theta_t^{n-1},\boldsymbol{u}_t^{n,s}).
\end{aligned}$$

Estimando o lado direito obtemos

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{u}_t^{n,s}\|^2+(\mu+\mu_r)\|\nabla\boldsymbol{u}_t^{n,s}\|^2 &\leq C\|\boldsymbol{u}_t^{n-1,s}\|^2+C\|A\boldsymbol{u}^{n,s}\|^2+C\|\boldsymbol{w}_t^{n-1,s}\|^2 \\
&\quad +C\|\nabla\boldsymbol{u}^{n-1,s}\|^2\|\nabla\boldsymbol{u}_t^n\|^2+C\|\theta_t^{n-1,s}\|_{H^{-1}}^2 \\
&\quad +C\|\theta_t^{n-1}\|^2\|\nabla\theta^{n-1,s}\|.
\end{aligned}$$

Onde, o último termo foi estimado usando (3.75)

$$\begin{aligned}
|([f'(\theta^{n-1+s})-f'(\theta^{n-1})]\theta_t^{n-1},\boldsymbol{u}_t^{n,s})| &\le L_{f'}|(\theta^{n-1,s}\theta_t^{n-1},\boldsymbol{u}_t^{n,s})|\\
&\le L_{f'}\|\theta^{n-1,s}\|_{L^3}\|\theta_t^{n-1}\|\|\boldsymbol{u}_t^{n,s}\|_{L^6}\\
&\le C\|\theta^{n-1,s}\|^{1/2}\|\nabla\theta^{n-1,s}\|^{1/2}\|\theta_t^{n-1}\|\|\nabla\boldsymbol{u}_t^{n,s}\|\\
&\le C\|\theta^{n-1,s}\|^{1/2}(\|\nabla\theta^{n+s-1}\|\\
&+\|\nabla\theta^{n-1}\|)^{1/2}\|\theta_t^{n-1}\|\|\nabla\boldsymbol{u}_t^{n,s}\|\\
&\le C\|\theta^{n-1,s}\|\|\theta_t^{n-1}\|^2+\frac{\mu+\mu_r}{12}\|\nabla\boldsymbol{u}_t^{n,s}\|
\end{aligned}$$

De modo similar, temos

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{w}_t^{n,s}\|^2+(c_a+c_d)\|\nabla\boldsymbol{w}_t^{n,s}\|^2 &\le C\|\boldsymbol{w}_t^{n-1,s}\|^2+C\|B\boldsymbol{w}^{n,s}\|^2+C\|\boldsymbol{u}_t^{n-1,s}\|^2\\
&+C\|\nabla\boldsymbol{u}^{n-1,s}\|^2\|\nabla\boldsymbol{w}_t^{n}\|^2+C\|\theta_t^{n-1,s}\|_{H^{-1}}^2\\
&+C\|\theta_t^{n-1}\|^2\|\nabla\theta^{n-1,s}\|.
\end{aligned}$$

Somando as últimas duas desigualdades, integrando e usando (3.55), (3.70), (3.73) se obtêm

$$\begin{aligned}
&\|\boldsymbol{u}_t^{n,s}\|^2+\|\boldsymbol{w}_t^{n,s}\|^2+(\mu+\mu_r)\int_0^t\|\nabla\boldsymbol{u}_t^{n,s}(\tau)\|^2d\tau+(c_a+c_d)\int_0^t\|\nabla\boldsymbol{w}_t^{n,s}(\tau)\|^2d\tau\\
&\le C\int_0^t\left(\|\boldsymbol{u}_t^{n-1,s}(\tau)\|^2+C\|\boldsymbol{w}_t^{n-1,s}(\tau)\|^2\right)d\tau+C\int_0^t\left(\|A\boldsymbol{u}^{n,s}(\tau)\|^2+\|B\boldsymbol{w}^{n,s}(\tau)\|^2\right)d\tau\\
&+C\int_0^t\|\nabla\boldsymbol{u}^{n-1,s}(\tau)\|^2\left(\|\nabla\boldsymbol{u}_t^{n}(\tau)\|^2+\|\nabla\boldsymbol{w}_t^{n}(\tau)\|^2\right)d\tau\\
&+C\int_0^t\|\theta_t^{n-1,s}(\tau)\|_{H^{-1}}^2d\tau+C\int_0^t\|\theta_t^{n-1}(\tau)\|^2\|\theta^{n-1,s}(\tau)\|d\tau\\
&\le C[\Lambda_{n-1}(t)+\Lambda_{n-2}(t)+\Lambda_{n-3}(t)+\Lambda_{n-2}^{1/2}(t)].
\end{aligned} \tag{3.76}$$

Assim, usando (3.42), (3.43) e (3.55) é fácil mostrar que

$$\begin{aligned}
\|A\boldsymbol{u}^{n,s}\|^2+\|B\boldsymbol{w}^{n,s}\|^2 &\le C(\|\nabla\boldsymbol{u}^{n,s}\|^2+\|\nabla\boldsymbol{w}^{n,s}\|^2)+C(\|\boldsymbol{u}_t^{n,s}\|^2+\|\boldsymbol{w}_t^{n,s}\|^2)\\
&+C(\|\nabla\boldsymbol{u}^{n-1,s}\|^2+\|\nabla\boldsymbol{w}^{n-1,s}\|^2)+C\|\theta^{n-1,s}\|^2\\
&\le C[\Lambda_{n-1}(t)+\Lambda_{n-2}+\Lambda_{n-3}(t)+\Lambda_{n-2}^{1/2}(t)]
\end{aligned} \tag{3.77}$$

Multiplicando (3.44) por $\theta_t^{n,s}$ temos

$$\begin{aligned}
\|\theta_t^{n,s}\|^2+\frac{\kappa}{2}\frac{d}{dt}\|\nabla\theta^{n,s}\|^2 &= -(\boldsymbol{u}^{n-1+s}\nabla\theta^{n,s},\theta_t^{n,s})-(\boldsymbol{u}^{n-1,s}\nabla\theta^{n},\theta_t^{n,s})\\
&+(\Phi(\boldsymbol{u}^{n-1+s},\boldsymbol{w}^{n-1+s})-\Phi(\boldsymbol{u}^{n-1},\boldsymbol{w}^{n-1}),\theta_t^{n,s})
\end{aligned}$$

Vamos estimar os termos do lado direito usando (3.70), (3.73)

$$
\begin{aligned}
|(\boldsymbol{u}^{n-1+s}\nabla\theta^{n,s},\theta_t^{n,s})| &\le \|\boldsymbol{u}^{n-1+s}\|_{L^\infty}\|\nabla\theta^{n,s}\|\theta_t^{n,s}\| \\
&\le C\|A\boldsymbol{u}^{n-1+s}\|\|\nabla\theta^{n,s}\|\|\theta_t^{n,s}\| \\
&\le C\|\nabla\theta^{n,s}\|^2+\frac{\kappa}{14}\|\theta_t^{n,s}\|^2 \\
|(\boldsymbol{u}^{n-1,s}\nabla\theta^{n},\theta_t^{n,s})| &\le \|\boldsymbol{u}^{n-1,s}\|_{L^\infty}\|\nabla\theta^{n}\|\|\theta_t^{n,s}\| \\
&\le C\|A\boldsymbol{u}^{n-1,s}\|_{L^\infty}\|\nabla\theta^{n}\|\|\theta_t^{n,s}\| \\
&\le C\|A\boldsymbol{u}^{n-1,s}\|^2+\frac{\kappa}{14}\|\theta_t^{n,s}\|^2 \\
|(\Phi_1(\boldsymbol{u}^{n-1+s})-\Phi_1(\boldsymbol{u}^{n-1}),\theta_t^{n,s})| &\le C(\|\nabla\boldsymbol{u}^{n-1+s}\|_{L^4} \\
&+\|\nabla\boldsymbol{u}^{n-1}\|_{L^4})\|\nabla\boldsymbol{u}^{n-1,s}\|_{L^4}\|\theta_t^{n,s}\| \\
&\le C(\|A\boldsymbol{u}^{n-1+s}\|+\|A\boldsymbol{u}^{n-1}\|)\|A\boldsymbol{u}^{n-1,s}\|\|\theta_t^{n,s}\| \\
&\le C\|A\boldsymbol{u}^{n-1,s}\|^2+\frac{\kappa}{14}\|\theta_t^{n,s}\|^2 \\
|(\Phi_2(\boldsymbol{u}^{n-1+s},\boldsymbol{w}^{n-1+s})-\Phi_2(\boldsymbol{u}^{n-1},\boldsymbol{w}^{n-1}),\theta_t^{n,s})| &\le C(\|\nabla\boldsymbol{u}^{n-1+s}\|_{L^4} \\
&+\|\boldsymbol{w}^{n-1+s}\|_{L^4})\|\nabla\boldsymbol{u}^{n-1,s}\|_{L^4}\|\theta_t^{n,s}\| \\
&+(\|\nabla\boldsymbol{u}^{n-1}\|_{L^4}+\|\boldsymbol{w}^{n-1}\|_{L^4})\|\nabla\boldsymbol{u}^{n-1,s}\|_{L^4}\|\theta_t^{n,s}\| \\
&\le C(\|A\boldsymbol{u}^{n-1+s}\|+\|A\boldsymbol{u}^{n-1}\|)\|A\boldsymbol{u}^{n-1,s}\|\|\theta_t^{n,s}\| \\
&+(\|B\boldsymbol{w}^{n-1+s}\|+\|B\boldsymbol{w}^{n-1}\|)\|A\boldsymbol{u}^{n-1,s}\|\|\theta_t^{n,s}\| \\
&\le C\|A\boldsymbol{u}^{n-1,s}\|^2+\frac{\kappa}{14}\|\theta_t^{n,s}\|^2 \\
|(\Phi_i(w^{n-1+s})-\Phi_i(w^{n-1}),\theta_t^{n,s})| &\le (\|\nabla\boldsymbol{w}^{n-1+s}\|_{L^4} \\
&+\|\nabla\boldsymbol{w}^{n-1}\|_{L^4})\|\nabla\boldsymbol{w}^{n-1,s}\|_{L^4}\|\theta_t^{n,s}\| \\
&\le (\|B\boldsymbol{w}^{n-1+s}\|+\|B\boldsymbol{w}^{n-1}\|\|B\boldsymbol{w}^{n-1,s}\|\|\theta_t^{n,s}\| \\
&\le C\|B\boldsymbol{w}^{n-1,s}\|^2+\frac{\kappa}{14}\|\theta_t^{n,s}\|^2,
\end{aligned}
$$

$i=3,4,5$. Assim, temos

$$
\begin{aligned}
\|\nabla\theta^{n,s}\|^2+\int_0^t\|\theta_t^{n,s}(\tau)\|^2d\tau &\le C\int_0^t\|\nabla\theta^{n,s}(\tau)\|^2d\tau \\
&+ C\int_0^t(\|A\boldsymbol{u}^{n-1,s}(\tau)\|^2+\|B\boldsymbol{w}^{n-1,s}(\tau)\|^2)d\tau.
\end{aligned}
$$

Pela desigualdade de Gronwall, (3.77) e (3.56) concluímos

$$\begin{aligned} \|\nabla\theta^{n,s}\|^2 + \int_0^t \|\theta_t^{n,s}(\tau)\|^2 d\tau &\leq C\int_0^{T_1} (\|A\boldsymbol{u}^{n-1,s}(\tau)\|^2 + \|B\boldsymbol{w}^{n-1,s}(\tau)\|^2) d\tau \\ &\leq C\Lambda_{n-2}(t). \end{aligned} \tag{3.78}$$

Novamente como os espaços $L^2(0,T;V)$, $L^\infty(0,T;H)$, $L^\infty(0,T;H_0^1(\Omega))$, $L^\infty(0,T;L^2(\Omega))$, $L^\infty(0,T;D(A))$, $L^\infty(0,T;D(B))$, $L^2(0,T;H_0^1(\Omega))$, $L^\infty(0,T;H_0^1(\Omega))$, $L^2(0,T;H)$, $L^\infty(0,T;L^2(\Omega))$ e $L^2(0,T;H^{-1}(\Omega))$ são espaços de Banach, das estimativas (3.76)-(3.78) podemos concluir que existem funções $\boldsymbol{u}$, $\boldsymbol{w}$ e θ tal que

$$\begin{aligned} \boldsymbol{u}^n &\longrightarrow \boldsymbol{u} && \text{fortemente em} \quad L^\infty(0,T;D(A)), \\ \boldsymbol{u}_t^n &\longrightarrow \boldsymbol{u}_t && \text{fortemente em} \quad L^\infty(0,T;H)\cap L^2(0,T;V), \\ \boldsymbol{w}^n &\longrightarrow \boldsymbol{w} && \text{fortemente em} \quad L^\infty(0,T;D(B)), \\ \boldsymbol{w}_t^n &\longrightarrow \boldsymbol{w}_t && \text{fortemente em} \quad L^\infty(0,T;L^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega)), \\ \theta^n &\longrightarrow \theta && \text{fortemente em} \quad L^\infty(0,T;H_0^1(\Omega)), \\ \theta_t^n &\longrightarrow \theta_t && \text{fortemente em} \quad L^2(0,T;H^{-1}(\Omega)), \end{aligned}$$

quando $n \to \infty$.

4 ESTIMATIVAS DE ERRO NO TEMPO

O sistema considerado é o mesmo do capítulo anterior, dado pelas equações de (1.1). Nosso interesse aqui são nas estimativas de erro uniforme no tempo das aproximações de Galerkin para as equações de um fluído micropolar com convecção térmica. É muito importante obter estimativas de erro para métodos de Galerkin, visto sua aplicação em métodos numéricos, como por exemplo o método dos elementos finitos.

4.1 PRELIMINARES E RESULTADOS

Ao longo deste capítulo C, C_0, M_0 denotarão constantes positivas genéricas que dependem apenas de Ω e de outros parâmetros do problema que podem variar de acordo com a fórmula. Seja P a projeção de $L^2(\Omega)$ em H, $A = -P\Delta$ com $D(A) = H^2(\Omega) \cap V$ o operador usual de Stokes. Suas autofunções e seus autovalores são denotados por $\boldsymbol{\varphi}^k(x)$ e λ^k respectivamente. O operador $B = -\Delta$ denota o operador de Laplace com condições de fronteira de Dirichlet e $D(B) = H^2(\Omega) \cup H_0^1(\Omega)$. Denotamos suas autofunções por ϕ^k e seus autovalores por γ^k. Além disso, vamos considerar o operador fortemente elíptico L $(c_0 + c_d > c_a)$;

$$Lw = -(c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla \operatorname{div} w, \quad \text{para} \;\; w \in H^2 \cap H_0^1.$$

Sabemos que a seguinte estimativa é válida,

$$\|L^{1/2}w\|^2 = (Lw, w) \geq \bar{c}\|\nabla w\|^2$$

onde $\bar{c} = \min\{c_a + c_d, c_0 + 2c_d\}$. Denotaremos por $\boldsymbol{\psi}^k(x)$ e $\widetilde{\gamma}^k$ as autofunções e os autovalores de L, respectivamente.

Para cada $k \in \mathbb{N}$, denotamos por $\boldsymbol{P}_k$, $\boldsymbol{R}_k$ e $\widetilde{\boldsymbol{R}}_k$ as projeções ortogonais de $L^2(\Omega)$ em $V_k = \text{span}\{\boldsymbol{\varphi}^1(x), \ldots, \boldsymbol{\varphi}^k(x)\}$, $H_k = \text{span}\{\boldsymbol{\phi}^1(x), \ldots, \boldsymbol{\phi}^k(x)\}$, e $\widetilde{H}_k = \text{span}\{\boldsymbol{\psi}^1(x), \ldots, \boldsymbol{\psi}^k(x)\}$ respectivamente. Para todo $\boldsymbol{z}_1, \boldsymbol{z}_2 \in L^2(\Omega)$ e $k, m \in \mathbb{N}$, segue

(i) $(\boldsymbol{P}_k \boldsymbol{z}_1, \boldsymbol{z}_2) = (\boldsymbol{z}_1, \boldsymbol{P}_k \boldsymbol{z}_2)$,

(ii) $(\boldsymbol{P} \boldsymbol{z}_1, \boldsymbol{z}_2) = (\boldsymbol{z}_2, \boldsymbol{P} \boldsymbol{z}_2)$,

(iii) $((\boldsymbol{P}_m - \boldsymbol{P}_k)\boldsymbol{z}_1, \boldsymbol{z}_2) = (\boldsymbol{z}_1, (\boldsymbol{P}_m - \boldsymbol{P}_k)\boldsymbol{z}_2)$,

(iv) $((\boldsymbol{P} - \boldsymbol{P}_k)\boldsymbol{z}_1, \boldsymbol{z}_2) = (\boldsymbol{z}_1, (\boldsymbol{P} - \boldsymbol{P}_k)\boldsymbol{z}_2)$

Com as notações acima podemos escrever o problema (1.1) da seguinte maneira

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\mu + \mu_r) A \boldsymbol{u} + \boldsymbol{P}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = 2\mu_r \boldsymbol{P}(\text{rot}\, \boldsymbol{w}) + \boldsymbol{f}(\theta), \\ \dfrac{\partial \boldsymbol{w}}{\partial t} + L\boldsymbol{w} + (u \cdot \nabla)\boldsymbol{w} + 4\mu_r \boldsymbol{w} = 2\mu_r \,\text{rot}\, \boldsymbol{u} + \boldsymbol{g}(\theta), \\ \dfrac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta - \kappa \Delta \theta = \Phi(\boldsymbol{u}, \boldsymbol{w}) + h, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x),\ \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x),\ \theta(x,0) = \theta_0 \end{cases} \tag{4.1}$$

ou a formulação equivalente

$$\begin{cases} (\boldsymbol{u}_t, \boldsymbol{\varphi}) + (\mu + \mu_r)(A\boldsymbol{u}, \boldsymbol{\varphi}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) = 2\mu_r(\text{rot}\, \boldsymbol{w}, \boldsymbol{\varphi}) + (\boldsymbol{f}(\theta), \boldsymbol{\varphi}) \\ (\boldsymbol{w}_t, \boldsymbol{\phi}) + (L\boldsymbol{w}, \boldsymbol{\phi}) + (u \cdot \nabla \boldsymbol{w}, \boldsymbol{\phi}) + 4\mu_r(\boldsymbol{w}, \boldsymbol{\phi}) = 2\mu_r(\text{rot}\, \boldsymbol{u}, \boldsymbol{\phi}) + (\boldsymbol{g}(\theta), \boldsymbol{\phi}), \\ \langle \theta_t, \boldsymbol{\psi} \rangle + (\boldsymbol{u} \cdot \nabla \theta, \boldsymbol{\psi}) - \kappa \langle \Delta \theta, \boldsymbol{\psi} \rangle = (\Phi(\boldsymbol{u}, \boldsymbol{w}), \boldsymbol{\psi}) + (h, \boldsymbol{\psi}), \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x),\ \boldsymbol{w}(x,0) = \boldsymbol{w}_0(x),\ \theta(x,0) = \theta_0 \end{cases} \tag{4.2}$$

para todo $\boldsymbol{\varphi} \in \boldsymbol{V}$ e para todo $\boldsymbol{\phi}, \boldsymbol{\psi} \in H_0^1(\Omega)$. Quanto a existências de soluções para (4.1), pode-se proceder como em [12] ou ainda usando o método de Galerkin.

Consideramos a aproximação de Galerkin definida por

$$\boldsymbol{u}^k(x,t) = \sum_{i=1}^{k} c_{ik}(t)\boldsymbol{\varphi}^i(x),\ \boldsymbol{w}^k(x,t) = \sum_{i=1}^{k} d_{ik}(t)\boldsymbol{\phi}^i(x),\ \theta^k(x,t) = \sum_{i=1}^{k} e_{ik}\boldsymbol{\psi}^i(x),$$

verificando as seguintes equações

$$
\begin{aligned}
\boldsymbol{u}_t^k + (\mu+\mu_r)A\boldsymbol{u}^k + \boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k &= 2\mu_r \operatorname{rot}\boldsymbol{w}^k + \boldsymbol{P}_k\boldsymbol{f}(\theta^k)\\
\boldsymbol{w}_t^k + L\boldsymbol{w}^k + \boldsymbol{u}^k\cdot\nabla\boldsymbol{w}^k &= 2\mu_r \operatorname{rot}\boldsymbol{u}^k + \boldsymbol{R}_k\boldsymbol{g}(\theta^k)\\
\theta_t^k - \kappa\Delta\theta^k + \boldsymbol{u}^k\cdot\nabla\theta^k &= \Phi(\boldsymbol{u}^k,\boldsymbol{w}^k) + \widetilde{\boldsymbol{R}}_k h\\
\boldsymbol{u}^k(0) = \boldsymbol{P}_k u_0,\ \boldsymbol{w}^k(0) = \boldsymbol{R}_k\boldsymbol{w}_0,\ & \theta^k(0) = \widetilde{\boldsymbol{R}}_k\theta_0
\end{aligned}
$$

com equivalente formulação

$$
\begin{cases}
(\boldsymbol{u}_t^k,\boldsymbol{\varphi}) + (\mu+\mu_r)(A\boldsymbol{u}^k,\boldsymbol{\varphi}) + (\boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k,\boldsymbol{\varphi}) = 2\mu_r(\operatorname{rot}\boldsymbol{w}^k,\boldsymbol{\varphi}) + (\boldsymbol{f}(\theta^k),\boldsymbol{\varphi})\\
(\boldsymbol{w}_t^k,\boldsymbol{\phi}) + (L\boldsymbol{w}^k,\boldsymbol{\phi}) + (\boldsymbol{u}^k\cdot\nabla\boldsymbol{w}^k,\boldsymbol{\phi}) + 4\mu_r(\boldsymbol{w}^k,\boldsymbol{\phi}) = 2\mu_r(\operatorname{rot}\boldsymbol{u}^k,\boldsymbol{\phi}) + (\boldsymbol{g}(\theta^k),\boldsymbol{\phi}),\\
(\theta_t^k,\boldsymbol{\psi}) + (\boldsymbol{u}^k\cdot\nabla\theta^k,\boldsymbol{\psi}) - \kappa(\Delta\theta^k,\boldsymbol{\psi}) = (\Phi(\boldsymbol{u}^k,\boldsymbol{w}^k),\boldsymbol{\psi}) + (h,\boldsymbol{\psi}),\\
\boldsymbol{u}^k(0) = \boldsymbol{P}_k\boldsymbol{u}_0,\ \boldsymbol{w}^k(0) = \boldsymbol{R}_k\boldsymbol{w}_0,\ \theta^k(0) = \widetilde{\boldsymbol{R}}_k\theta_0
\end{cases}
\tag{4.3}
$$

para $\varphi \in V_k$, $\phi \in H_k$ e $\psi \in \widetilde{H}_k$.

As condições a seguir dadas em forma de proposição foram provadas no capítulo anterior e garantem existência, unicidade e regularidade de solução forte para o problema (1.1).

Proposição 4.1. *Suponha que f, g, f_t e g_t verificam a condição (3.5), para constantes M_f, M_g, M_{f_t} e M_{g_t} respectivamente. Existe $T_2 > 0$ e uma única solução do problema (1.1) no intervalo $[0,T_2]$. Além disso,*

$$\boldsymbol{u} \in L^\infty(0,T_2;D(A)),\ \boldsymbol{w} \in L^\infty(0,T_2;D(B)),\ \theta \in L^\infty(0,T_2;H_0^1(\Omega)).$$

O mesmo resultado vale para a solução $(\boldsymbol{u}^k,\boldsymbol{w}^k,\theta^k)$ do problema (4.3).

Para nossas estimativas será útil o seguinte resultado. As estimativas para o operador $\boldsymbol{P}_k$ podem ser encontradas em [18] e são devidas a Rautmann. Para os operadoradores L e $\boldsymbol{R}_k$ apresentaremos na próxima seção uma prova mais detalhada da encontrada em Ortega-Torres, Rojas-Medar e Cabrales [16].

Lema 4.2. *Se $\boldsymbol{u} \in \boldsymbol{V}$ e $\boldsymbol{w} \in H_0^1(\Omega)$ então*

$$\|\boldsymbol{u} - \boldsymbol{P}_k\boldsymbol{u}\|^2 \le \frac{1}{\lambda_{k+1}}\|\nabla\boldsymbol{u}\|^2,\ \ \|\boldsymbol{w} - \boldsymbol{R}_k\boldsymbol{w}\|^2 \le \frac{1}{\gamma_{k+1}}\|L^{1/2}\boldsymbol{w}\|^2.$$

Além disso, se $\boldsymbol{u} \in \boldsymbol{V} \cap H^2(\Omega)$ *e* $\boldsymbol{w} \in H_0^1(\Omega) \cap H^2(\Omega)$ *temos*

$$\|\boldsymbol{u} - \boldsymbol{P}_k\boldsymbol{u}\|^2 \leq \frac{1}{\lambda_{k+1}^2}\|A\boldsymbol{u}\|^2, \quad \|\boldsymbol{w} - \boldsymbol{R}_k\boldsymbol{w}\|^2 \leq \frac{1}{\gamma_{k+1}^2}\|L\boldsymbol{w}\|^2,$$

$$\|\nabla\boldsymbol{u} - \nabla\boldsymbol{P}_k\boldsymbol{u}\|^2 \leq \frac{1}{\lambda_{k+1}}\|A\boldsymbol{u}\|^2, \quad \|L^{1/2}(\boldsymbol{w} - \boldsymbol{R}_k\boldsymbol{w})\|^2 \leq \frac{1}{\gamma_{k+1}}\|L\boldsymbol{w}\|^2.$$

Para cada $k \in \mathbb{N}$, defina

$$\boldsymbol{v}^k = \boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{u}^k, \;\; \boldsymbol{z}^k = \boldsymbol{R}_k\boldsymbol{w} - \boldsymbol{w}^k, \;\; \boldsymbol{h}^k = \tilde{\boldsymbol{R}}_k\theta - \theta^k,$$

$$\boldsymbol{\eta}^k = \boldsymbol{u} - \boldsymbol{P}_k\boldsymbol{u}, \;\; \boldsymbol{\zeta}^k = \boldsymbol{w} - \boldsymbol{R}_k\boldsymbol{w}, \;\; \boldsymbol{\xi}^k = \theta - \tilde{\boldsymbol{R}}_k\theta.$$

Note que $\boldsymbol{v}^{\boldsymbol{k}}, \boldsymbol{z}^k, \boldsymbol{h}^k$ satisfazem o seguinte sistema de equações:

$$\left\{\begin{aligned}
(\boldsymbol{v}_t^k, \boldsymbol{\varphi}) + (\mu + \mu_r)(A\boldsymbol{v}^k, \boldsymbol{\varphi}) &= 2\mu_r(\operatorname{rot}\boldsymbol{\zeta}^{\boldsymbol{k}}, \boldsymbol{\varphi}) + 2\mu_r(\operatorname{rot}\boldsymbol{z}^{\boldsymbol{k}}, \boldsymbol{\varphi}) - (\boldsymbol{u}.\nabla\boldsymbol{\eta}^k, \boldsymbol{\varphi}) \\
&\quad -(\boldsymbol{u}.\nabla\boldsymbol{v}^k, \boldsymbol{\varphi}) - (\boldsymbol{\eta}^k.\nabla\boldsymbol{u}^k, \boldsymbol{\varphi}) - (\boldsymbol{v}^k.\nabla\boldsymbol{u}^k, \boldsymbol{\varphi}) \\
&\quad +(\boldsymbol{f}(\theta) - \boldsymbol{f}(\theta^k), \boldsymbol{\varphi}) \\
(\boldsymbol{z}_t^k, \boldsymbol{\phi}) + (L\boldsymbol{z}^k, \boldsymbol{\phi}) + 4\mu_r(\boldsymbol{z}^k, \boldsymbol{\phi}) &= 2\mu_r(\operatorname{rot}\boldsymbol{\eta}^k, \boldsymbol{\phi}) + 2\mu_r(\operatorname{rot}\boldsymbol{v}^k, \boldsymbol{\phi}) - (\boldsymbol{\eta}^k.\nabla\boldsymbol{w}, \phi) \\
&\quad -(\boldsymbol{u}^k.\nabla\boldsymbol{z}^k, \phi) - (\boldsymbol{v}^k.\nabla\boldsymbol{w}, \phi) - (\boldsymbol{u}^k.\nabla\boldsymbol{\zeta}^k, \phi) \\
&\quad +(\boldsymbol{g}(\theta) - \boldsymbol{g}(\theta^k), \boldsymbol{\phi}) \\
(\boldsymbol{h}_t^k, \boldsymbol{\psi}) - \kappa(\Delta\boldsymbol{h}^k, \boldsymbol{\psi}) &= -(\boldsymbol{\eta}^k.\nabla\theta, \boldsymbol{\psi}) - (\boldsymbol{u}^k.\nabla\boldsymbol{h}^k, \boldsymbol{\psi}) - (\boldsymbol{v}^k.\nabla\theta, \boldsymbol{\psi}) \\
&\quad -(\boldsymbol{u}^k.\nabla\boldsymbol{\xi}^k, \boldsymbol{\psi}) + (\Phi(\boldsymbol{u}, \boldsymbol{w}) - \Phi(\boldsymbol{u}^k, \boldsymbol{w}^k), \boldsymbol{\psi}), \\
\boldsymbol{u}(0) = \boldsymbol{u}_0, \;\; \boldsymbol{w}(0) &= \boldsymbol{w}_0, \;\; \theta(0) = \theta_0,
\end{aligned}\right. \tag{4.4}$$

para todo $\varphi \in V_k$, $\phi \in H_k$ e $\psi \in \widetilde{H}_k$.

Lema 4.3. *Sob as hipóteses da Proposição 4.1, existe* $C > 0$ *tal que para todo* $k \in \mathbb{N}$ *e todo* $t \geq 0$, *temos*

$$\int_0^t \|\boldsymbol{\eta}^k(\tau)\|^2 d\tau \leq \frac{C}{\lambda_{k+1}^2}, \quad \int_0^t \|\boldsymbol{\zeta}^k(\tau)\|^2 d\tau \leq \frac{C}{\gamma_{k+1}^2}, \quad \int_0^t \|\boldsymbol{\xi}^k(\tau)\|^2 d\tau \leq \frac{C}{\widetilde{\gamma}_{k+1}^2},$$

$$\int_0^t \|\nabla\boldsymbol{\eta}^k(\tau)\|^2 d\tau \leq \frac{C}{\lambda_{k+1}}, \quad \int_0^t \|L^{1/2}\boldsymbol{\zeta}^k(\tau)\|^2 d\tau \leq \frac{C}{\gamma_{k+1}},$$

$$\|\boldsymbol{\eta}^k\|^2 \leq \frac{C}{\lambda_{k+1}^2}, \quad \|\boldsymbol{\zeta}^k\|^2 \leq \frac{C}{\gamma_{k+1}^2}, \quad \|\boldsymbol{\xi}^k\|^2 \leq \frac{C}{\widetilde{\gamma}_{k+1}},$$

$$\|\nabla\boldsymbol{\eta}^k\|^2 \leq \frac{C}{\lambda_{k+1}}, \quad \|L^{1/2}\boldsymbol{\zeta}^k\|^2 \leq \frac{C}{\gamma_{k+1}}.$$

Em nosso primeiro resultado estabelecemos a estimativa na norma $L^2(\Omega)$ do erro da aproximação de Galerkin.

Teorema 4.4. *Sob as hipóteses da Proposição 4.1 as aproximações* $(\boldsymbol{u}^k, \boldsymbol{w}^k, \theta^k)$ *satisfazem*

$$\|\boldsymbol{u}(t)-\boldsymbol{u}^k(t)\|^2+\|\boldsymbol{w}(t)-\boldsymbol{w}^k(t)\|^2+\|\theta(t)-\theta^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}^2}+\frac{C}{\gamma_{k+1}^2}+\frac{C}{\widetilde{\gamma}_{k+1}^2}+\frac{C}{\lambda_{k+1}}+\frac{C}{\gamma_{k+1}} \quad (4.5)$$

para todo $t \geq 0$ *e* $C > 0$ *uma constante genérica que não depende de* $k \in \mathbb{N}$.

Observação 4.5. *A aproximação de Galerkin do sistema de fluídos micropolares é da ordem* $\frac{1}{\lambda_{k+1}^2} + \frac{1}{\gamma_{k+1}^2}$. *Esperavamos que a estimativa (4.4) fosse da ordem* $\frac{1}{\lambda_{k+1}^2} + \frac{1}{\gamma_{k+1}^2} + \frac{1}{\widetilde{\gamma_{k+1}}^2}$, *mas isto não ocorre. A presença do termo não linear* Φ *impede o resultado esperado.*

De maneira análoga estabelecemos para $\boldsymbol{v}$ e $\boldsymbol{z}$ na norma $H^1(\Omega)$ o seguinte

Teorema 4.6. *Sob as hipóteses da Proposição 4.1 as aproximações* $(\boldsymbol{u}^k, \boldsymbol{w}^k)$ *satisfazem*

$$\|\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}^k(t)\|^2 + \|L^{1/2}\boldsymbol{w}(t) - L^{1/2}\boldsymbol{w}^k(t)\|^2 \leq \frac{C}{\gamma_{k+1}} + \frac{C}{\lambda_{k+1}} + \frac{C}{\tilde{\gamma}_{k+1}}$$

para todo $t \geq 0$ *e* $C > 0$ *uma constante que não depende de* k.

Obtemos também

Teorema 4.7. *Sob as hipóteses da proposição 4.1 as aproximações* $(\boldsymbol{u}^k, \boldsymbol{w}^k, \theta^k)$ *satisfazem*

$$\|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^k(t)\|_{V^*}^2 + \|\boldsymbol{w}_t(t) - \boldsymbol{w}_t^k(t)\|_{H^{-1}}^2 + \int_0^t \|\theta_t(\tau) - \theta_t^k(\tau)\|_{H^{-1}}^2 d\tau \leq \frac{C}{\gamma_{k+1}} + \frac{C}{\lambda_{k+1}} + \frac{C}{\tilde{\gamma}_{k+1}}$$

para todo $t \geq 0$ *e* $C > 0$ *uma constante que não depende de* k.

4.2 ESTIMATIVAS PARA A NORMA $L^2(\Omega)$

Demonstração do Lema 4.2. Seja $k \in \mathbb{N}$, $\boldsymbol{w} \in H_0^1(\Omega)$ e $\gamma_i \geq \gamma_{k+1}$ para $i \geq k+1$. Desta forma temos,

$$\|\boldsymbol{w} - \boldsymbol{R}_k\boldsymbol{w}\|^2 = \|\sum_{i=k+1}^{\infty} (\boldsymbol{w}, \boldsymbol{\phi}^i)\boldsymbol{\phi}^i\|^2 = \sum_{i=k+1}^{\infty} |(\boldsymbol{w}, \boldsymbol{\phi}^i)|^2 = \sum_{i=k+1}^{\infty} \left(\int_\Omega \boldsymbol{w}\boldsymbol{\phi}^i\right)^2. \quad (4.6)$$

Tais igualdades ocorrem pois $\boldsymbol{w} = \sum_{i=1}^{\infty}(\boldsymbol{w}, \boldsymbol{\phi}^i)\boldsymbol{\phi}^i$, $\boldsymbol{R}_k\boldsymbol{w} = \sum_{i=1}^{k}(\boldsymbol{w}, \boldsymbol{\phi}^i)\boldsymbol{\phi}^i$ e $\{\boldsymbol{\phi}^i\}$ é um sistema ortonormal em $L^2(\Omega)$. Agora estimaremos $\left(\int_\Omega \boldsymbol{w}\boldsymbol{\phi}^i\right)^2$,

$$\begin{aligned} \left(\int_\Omega \boldsymbol{w}\boldsymbol{\phi}^i\right)^2 &= \frac{1}{\gamma_i^2}\left(\int_\Omega (L\boldsymbol{\phi}^i)\boldsymbol{w}\right)^2 \\ &= \frac{1}{\gamma_i^2}\left(\int_\Omega L^{1/2}\boldsymbol{\phi}^i L^{1/2}\boldsymbol{w}\right)^2 \\ &\leq \frac{1}{\gamma_{k+1}}\left(\int_\Omega L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma_i}}\right) L^{1/2}\boldsymbol{w}\right)^2. \end{aligned} \tag{4.7}$$

Combinando (4.6) e (4.7) temos,

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{R}_k\boldsymbol{w}\|^2 &= \frac{1}{\gamma_{k+1}}\sum_{i=k+1}^{\infty}\left(\int_\Omega L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma_i}}\right) L^{1/2}\boldsymbol{w}\right)^2 \\ &\leq \frac{1}{\gamma_{k+1}}\sum_{i=1}^{\infty}\left(\int_\Omega L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma_i}}\right) L^{1/2}\boldsymbol{w}\right)^2 \\ &\leq \frac{1}{\gamma_{k+1}}\|L^{1/2}\boldsymbol{w}\|^2, \end{aligned} \tag{4.8}$$

onde a última desigualdade decorre do fato que $\left\{\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right\}$ é um conjunto ortonormal em $H_0^1(\Omega)$, com respeito ao produto interno $\int_\Omega L^{1/2}\boldsymbol{f}L^{1/2}\boldsymbol{g}$ e da desigualdade de Bessel. Para provar as desigualdades restantes, consideremos, $\boldsymbol{w} \in H_0^1(\Omega) \cap H^2(\Omega)$. Deste modo como L é simétrico temos

$$\begin{aligned} \left(\int_\Omega \boldsymbol{w}\boldsymbol{\phi}^i\right)^2 &= \frac{1}{\gamma_i^2}\left(\int_\Omega (L\boldsymbol{\phi}^i)\boldsymbol{w}\right)^2 \\ &= \frac{1}{\gamma_i^2}\left(\int_\Omega \boldsymbol{\phi}^i(L\boldsymbol{w})\right)^2 \\ &\leq \frac{1}{\gamma_{k+1}^2}\left(\int_\Omega \boldsymbol{\phi}^i(L\boldsymbol{w})\right)^2. \end{aligned} \tag{4.9}$$

Combinando (4.6), (4.9) e usando novamente a desigualdade de Bessel, temos

$$
\begin{aligned}
\|\boldsymbol{w}-\boldsymbol{R}_k\boldsymbol{w}\|^2 &= \sum_{i=k+1}^{\infty}\left(\int_\Omega \boldsymbol{w}\boldsymbol{\phi}^i\right)^2 \\
&\le \sum_{i=1}^{\infty}\left(\int_\Omega \boldsymbol{w}\boldsymbol{\phi}^i\right)^2 \\
&\le \frac{1}{\gamma_{k+1}^2}\sum_{i=1}^{\infty}|(L\boldsymbol{\omega},\boldsymbol{\phi}^i)| \\
&\le \frac{1}{\gamma_{k+1}^2}\|L\boldsymbol{w}\|^2. \qquad (4.10)
\end{aligned}
$$

Para a última desigualdade, como $\boldsymbol{w}-\boldsymbol{R}_k\boldsymbol{w}$ converge em $H_0^1(\Omega)$

$$
\begin{aligned}
L^{1/2}(\boldsymbol{w}-\boldsymbol{R}_k\boldsymbol{w}) &= L^{1/2}\left(\sum_{i=k+1}^{\infty}\left(\int_\Omega \boldsymbol{w}\boldsymbol{\phi}^i\right)\boldsymbol{\phi}^i\right) \\
&= \sum_{i=k+1}^{\infty}\left(\int_\Omega \boldsymbol{w}\boldsymbol{\phi}^i\right)L^{1/2}\boldsymbol{\phi}^i \\
&= \sum_{i=k+1}^{\infty}L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right)\left(\int_\Omega L^{1/2}\boldsymbol{w}L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right)\right), \qquad (4.11)
\end{aligned}
$$

onde a última igualdade decorre de um raciocínio análogo ao empregado em (4.7). Note agora que $\boldsymbol{w}-\boldsymbol{R}_k\boldsymbol{w}$ é ortogonal a $\boldsymbol{\phi}^i$ para $1\le i\le k$ em $H_0^1(\Omega)$. De fato, para $j=1,\cdots,k$

$$
\begin{aligned}
(L^{1/2}(\boldsymbol{w}-\boldsymbol{R}_k\boldsymbol{w}),L^{1/2}\boldsymbol{\phi}^j) &= \left(\sum_{i=k+1}^{\infty}L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right)\left(\int_\Omega L^{1/2}\boldsymbol{w}L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right)\right),L^{1/2}\boldsymbol{\phi}^j\right) \\
&= \sum_{i=k+1}^{\infty}\left(\int_\Omega L^{1/2}\boldsymbol{w}L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right)\right)\left(L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right),L^{1/2}\boldsymbol{\phi}^j\right) \\
&= \sum_{i=k+1}^{\infty}\left(\int_\Omega L^{1/2}\boldsymbol{w}L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right)\right) \\
&\quad \cdot\left(L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right),L^{1/2}\left(\frac{\boldsymbol{\phi}^j}{\sqrt{\gamma^j}}\right)\right) \qquad (4.12) \\
&= 0.
\end{aligned}
$$

Por fim, usando a desigualdade de Bessel e a identidade de Parseval,

$$\begin{aligned}
\|L^{1/2}(\boldsymbol{w}-\boldsymbol{R}_k\boldsymbol{w})\|^2 &= \sum_{i=1}^{\infty}\left|\left(L^{1/2}(\boldsymbol{w}-\boldsymbol{R}_k\boldsymbol{w}), L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right)\right)\right|^2 \\
&= \sum_{i=k+1}^{\infty}\left|\left(L^{1/2}\boldsymbol{w}, L^{1/2}\left(\frac{\boldsymbol{\phi}^i}{\sqrt{\gamma^i}}\right)\right)\right|^2 \\
&\leq \frac{1}{\gamma_{k+1}}\sum_{i=k+1}^{\infty}|(L\boldsymbol{w},\boldsymbol{\phi}^i)|^2 \leq \frac{1}{\gamma_{k+1}}\|L\boldsymbol{w}\|^2
\end{aligned}$$

□

Lema 4.8. *Sob as hipóteses da Proposição 4.1 existe uma constante $C>0$, independente de $t\geq 0$ e $k\in\mathbb{N}$, tal que*

$$\|\boldsymbol{v}^k\|^2+\|\boldsymbol{z}^k\|^2+\|\boldsymbol{h}^k\|^2+\int_0^t \|\nabla \boldsymbol{v}(\tau)\|^2+\|L^{1/2}\boldsymbol{z}^k(\tau)\|^2+\|\nabla\boldsymbol{h}^k(\tau)\|^2 d\tau \leq \frac{C}{\lambda_{k+1}^2}+\frac{C}{\gamma_{k+1}^2}+\frac{C}{\tilde{\gamma}_{k+1}^2}+\frac{C}{\lambda_{k+1}}+\frac{C}{\gamma_{k+1}} \tag{4.13}$$

Demonstração. Considerando $\boldsymbol{\varphi}=\boldsymbol{v}^k$, $\boldsymbol{\phi}=\boldsymbol{z}^k$ e $\boldsymbol{\psi}=\boldsymbol{h}^k$ em (4.4) obtemos,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^k\|^2+(\mu+\mu_r)\|\nabla\boldsymbol{v}^k\|^2 &= 2\mu_r(\operatorname{rot}\boldsymbol{\zeta^k},\boldsymbol{v}^k)+2\mu_r(\operatorname{rot}\boldsymbol{z^k},\boldsymbol{v}^k) \\
&- (\boldsymbol{u}.\nabla\boldsymbol{\eta}^k,\boldsymbol{v}^k)-(\boldsymbol{u}.\nabla\boldsymbol{v}^k,\boldsymbol{v}^k)-(\boldsymbol{\eta}^k.\nabla\boldsymbol{u}^k,\boldsymbol{v}^k) \\
&- (\boldsymbol{v}^k.\nabla\boldsymbol{u}^k,\boldsymbol{v}^k)+(\boldsymbol{f}(\theta)-\boldsymbol{f}(\theta^k),\boldsymbol{v}^k),
\end{aligned}\tag{4.14}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{z}^k\|^2+\|L^{1/2}\boldsymbol{z}^k\|^2+4\mu_r\|\boldsymbol{z}^k\|^2 &= 2\mu_r(\operatorname{rot}\boldsymbol{\eta}^k,\boldsymbol{z}^k)+2\mu_r(\operatorname{rot}\boldsymbol{v}^k,\boldsymbol{z}^k) \\
&- (\boldsymbol{\eta}^k.\nabla\boldsymbol{w},\boldsymbol{z}^k)-(\boldsymbol{v}^k.\nabla\boldsymbol{w},\boldsymbol{z}^k) \\
&- (\boldsymbol{u}^k.\nabla\boldsymbol{\zeta}^k,\boldsymbol{z}^k)+(\boldsymbol{g}(\theta)-\boldsymbol{g}(\theta^k),\boldsymbol{z}^k),
\end{aligned}\tag{4.15}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{h}^k\|^2+\kappa\|\nabla\boldsymbol{h}^k\|^2 &= -(\boldsymbol{\eta}^k.\nabla\theta,\boldsymbol{h}^k)-(\boldsymbol{v}^k.\nabla\theta,\boldsymbol{h}^k)-(\boldsymbol{u}^k.\nabla\boldsymbol{\xi}^k,\boldsymbol{h}^k) \\
&+ (\Phi(\boldsymbol{u},\boldsymbol{w})-\Phi(\boldsymbol{u}^k,\boldsymbol{w}^k),\boldsymbol{h}^k).
\end{aligned}\tag{4.16}$$

Estimaremos o lado direito das equações (4.14)-(4.16). Começamos pelo lado direito de (4.14). Para isto basta utilizar as desigualdades de Hölder, Young e desigualdades do tipo

Sobolev $\|u\|_4 \leq 2^{1/2}\|u\|^{1/4}\|\nabla u\|^{3/4}$ para $u \in H_0^1(\Omega)$ e $\|\nabla u\|_4 \leq C\|Au\|$ para $u \in D(A)$, da seguinte maneira

$$\begin{aligned}
|2\mu_r(\operatorname{rot}\boldsymbol{\zeta^k}, \boldsymbol{v}^k)| + |2\mu_r(\operatorname{rot}\boldsymbol{z^k}, \boldsymbol{v}^k)| &= |2\mu_r(\boldsymbol{\zeta^k}, \operatorname{rot}\boldsymbol{v}^k)| + |2\mu_r(\boldsymbol{z^k}, \operatorname{rot}\boldsymbol{v}^k)| \\
&\leq 2\mu_r\|\boldsymbol{\zeta}^k\|\|\nabla\boldsymbol{v}^k\| + 2\mu_r\|\boldsymbol{z}^k\|\|\nabla\boldsymbol{v}^k\| \\
&\leq C_\epsilon\|\boldsymbol{\zeta}^k\|^2 + \epsilon\|\nabla\boldsymbol{v}^k\|^2 + 2\mu_r\|\boldsymbol{z}^k\|^2 + \frac{\mu_r}{2}\|\nabla\boldsymbol{v}^k\|^2
\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{u}.\nabla\boldsymbol{\eta}^k, \boldsymbol{v}^k)| &= |(\boldsymbol{u}.\nabla\boldsymbol{v}^k, \boldsymbol{\eta}^k)| \\
&\leq \|\boldsymbol{u}\|_\infty\|\nabla\boldsymbol{v}^k\|\|\boldsymbol{\eta}^k\| \\
&\leq C_\epsilon\|A\boldsymbol{u}\|^2\|\boldsymbol{\eta}^k\|^2 + \epsilon\|\nabla\boldsymbol{v}^k\|^2 \\
|(\boldsymbol{\eta}^k.\nabla\boldsymbol{u}^k, \boldsymbol{v}^k)| &\leq \|\boldsymbol{\eta}^k\|\|\nabla\boldsymbol{u}^k\|_4\|\boldsymbol{v}^k\|_4 \\
&\leq C\|\boldsymbol{\eta}^k\|\|\nabla\boldsymbol{u}^k\|^{1/4}\|A\boldsymbol{u}^k\|^{3/4}\|\boldsymbol{v}^k\|^{1/4}\|\nabla\boldsymbol{v}^k\|^{3/4} \\
&\leq C_\epsilon\|\boldsymbol{\eta}^k\|^2\|A\boldsymbol{u}^k\|^2 + \epsilon\|\nabla\boldsymbol{v}^k\|^2
\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{v}^k.\nabla\boldsymbol{u}^k, \boldsymbol{v}^k)| &\leq \|\boldsymbol{v}^k\|_4^2\|\nabla\boldsymbol{u}^k\| \\
&\leq C\|\boldsymbol{v}^k\|^{1/2}\|\nabla\boldsymbol{v}^k\|^{3/2}\|\nabla\boldsymbol{u}\| \\
&\leq C_\epsilon\|\boldsymbol{v}^k\|^2\|\nabla\boldsymbol{u}^k\|^4 + \epsilon\|\nabla\boldsymbol{v}^k\|^2 \\
|(\boldsymbol{f}(\theta) - \boldsymbol{f}(\theta^k), \boldsymbol{v}^k)| &\leq \|\boldsymbol{f}(\theta) - \boldsymbol{f}(\theta^k)\|\|\boldsymbol{v}^k\| \\
&\leq M_{\boldsymbol{f}}\|\theta - \theta^k\|\|\boldsymbol{v}^k\| \\
&= M_{\boldsymbol{f}}\|\boldsymbol{\xi}^k + \boldsymbol{h}^k\|\|\boldsymbol{v}^k\| \\
&\leq M_{\boldsymbol{f}}\|\boldsymbol{\xi}^k\|\|\boldsymbol{v}^k\| + M_{\boldsymbol{f}}\|\boldsymbol{h}^k\|\|\boldsymbol{v}^k\| \\
&\leq \frac{M_{\boldsymbol{f}}^2\lambda_1^{-1}}{2}\|\boldsymbol{\xi}^k\|^2 + \frac{M_{\boldsymbol{f}}^2\lambda_1^{-1}}{2}\|\nabla\boldsymbol{h}^k\|^2 + \frac{1+\lambda_1^{-1}}{2}\|\nabla\boldsymbol{v}^k\|^2.
\end{aligned}$$

Desta forma

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^k\|^2 + (\mu+\mu_r)\|\nabla\boldsymbol{v}^k\|^2 &\leq C_\epsilon\|\boldsymbol{\zeta}^k\|^2 + 2\mu_r\|\boldsymbol{z}^k\|^2 + C_\epsilon(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{u}^k\|^2)\|\boldsymbol{\eta}^k\|^2 \\
&+ C_\epsilon\|\boldsymbol{v}^k\|^2\|\nabla\boldsymbol{u}^k\|^4 + [4\epsilon + (1+\mu_r+\lambda_1^{-1})/2]\|\nabla\boldsymbol{v}^k\|^2 \\
&+ \frac{M_{\boldsymbol{f}}^2\lambda_1^{-1}}{2}\|\boldsymbol{\xi}^k\|^2 + \frac{M_{\boldsymbol{f}}^2\lambda_1^{-1}}{2}\|\nabla\boldsymbol{h}^k\|^2. \qquad (4.17)
\end{aligned}$$

Procedendo como acima estimaremos o lado direito da equação (4.15),

$$
\begin{aligned}
|2\mu_r(\operatorname{rot}\boldsymbol{\eta}^k,\boldsymbol{z}^k)|+|2\mu_r(\operatorname{rot}\boldsymbol{v}^k,\boldsymbol{z}^k)| &= |2\mu_r(\boldsymbol{\eta}^k,\operatorname{rot}\boldsymbol{z}^k)|+|2\mu_r(\boldsymbol{v}^k,\operatorname{rot}\boldsymbol{z}^k)|\\
&\le 2\mu_r C\|\boldsymbol{\eta}^k\|\|L^{1/2}\boldsymbol{z}^k\|+2\mu_r C\|\boldsymbol{v}^k\|\|L^{1/2}\boldsymbol{z}^k\|\\
&\le C_\delta\|\boldsymbol{\eta}^k\|^2+\delta\|L^{1/2}\boldsymbol{z}^k\|^2+2\mu_r\|\boldsymbol{v}^k\|^2\\
&+ \frac{\mu_r}{2}\|L^{1/2}\boldsymbol{z}^k\|^2\\
|(\boldsymbol{u}^k.\nabla\boldsymbol{\zeta}^k,\boldsymbol{z}^k)| &= |(\boldsymbol{u}^k.\nabla\boldsymbol{z}^k,\boldsymbol{\zeta}^k)|\\
&\le \|\boldsymbol{u}^k\|_\infty\|L^{1/2}\boldsymbol{z}^k\|\|\boldsymbol{\zeta}^k\|\\
&\le C_\delta\|A\boldsymbol{u}^k\|^2\|\boldsymbol{\zeta}^k\|^2+\delta\|L^{1/2}\boldsymbol{z}^k\|^2\\
|(\boldsymbol{\eta}^k.\nabla\boldsymbol{w},\boldsymbol{z}^k)| &\le C\|\boldsymbol{\eta}^k\|\|\nabla\boldsymbol{w}\|_4\|\boldsymbol{z}^k\|_4\\
&\le C\|\boldsymbol{\eta}^k\|\|\nabla\boldsymbol{w}\|^{1/4}\|L\boldsymbol{w}\|^{3/4}\|\boldsymbol{z}^k\|^{1/4}\|L^{1/2}\boldsymbol{z}^k\|^{3/4}\\
&\le C_\delta\|\boldsymbol{\eta}^k\|^2\|L\boldsymbol{w}^k\|^2+\delta\|L^{1/2}\boldsymbol{z}^k\|^2\\
|(\boldsymbol{v}^k.\nabla\boldsymbol{w},\boldsymbol{z}^k)| &\le \|\boldsymbol{v}^k\|_6\|\nabla\boldsymbol{w}\|_3\|\boldsymbol{z}^k\|\\
&\le C_\epsilon\|L\boldsymbol{w}\|^2\|\boldsymbol{z}^k\|^2+\epsilon\|\nabla\boldsymbol{v}^k\|^2
\end{aligned}
$$

$$
\begin{aligned}
|(\boldsymbol{g}(\theta)-\boldsymbol{g}(\theta^k),\boldsymbol{z}^k)| &\le \|\boldsymbol{g}(\theta)-\boldsymbol{g}(\theta^k)\|\|\boldsymbol{z}^k\|\\
&\le M_{\boldsymbol{g}}\|\theta-\theta^k\|\|\boldsymbol{z}^k\|\\
&= M_{\boldsymbol{g}}\|\boldsymbol{\xi}^k+\boldsymbol{h}^k\|\|\boldsymbol{z}^k\|\\
&\le M_{\boldsymbol{g}}\|\boldsymbol{\xi}^k\|\|\boldsymbol{z}^k\|+M_{\boldsymbol{g}}\|\boldsymbol{h}^k\|\|\boldsymbol{z}^k\|\\
&\le \frac{M_{\boldsymbol{g}}^2\lambda_1^{-1}}{2}\|\boldsymbol{\xi}^k\|^2+\frac{M_{\boldsymbol{g}}^2\lambda_1^{-1}}{2}\|\nabla\boldsymbol{h}^k\|^2+\frac{1+\lambda_1^{-1}}{2}\|L^{1/2}\boldsymbol{z}^k\|^2,
\end{aligned}
$$

onde novamente λ_1 é o primeiro autovalor do operador Laplaciano, desta forma

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{z}^k\|^2+\|L^{1/2}\boldsymbol{z}^k\|^2+4\mu_r\|\boldsymbol{z}^k\|^2 &\le C_\delta(1+\|L\boldsymbol{w}\|^2)\|\boldsymbol{\eta}^k\|^2+\epsilon\|\nabla\boldsymbol{v}^k\|^2\\
&+ C_\epsilon\|L\boldsymbol{w}\|^2\|\boldsymbol{z}^k\|^2+2\mu_r\|\boldsymbol{v}^k\|^2+C_\delta\|A\boldsymbol{u}^k\|^2\|\boldsymbol{\zeta}^k\|^2\\
&+ [3\delta+(1+\mu_r+\lambda_1^{-1})/2]\|L^{1/2}\boldsymbol{z}^k\|^2\\
&+ \frac{M_{\boldsymbol{g}}^2\lambda_1^{-1}}{2}\|\boldsymbol{\xi}^k\|^2+\frac{M_{\boldsymbol{g}}^2\lambda_1^{-1}}{2}\|\nabla\boldsymbol{h}^k\|^2. \qquad (4.18)
\end{aligned}
$$

Para o lado direito de (4.16) temos,

$$\begin{aligned}
|(\boldsymbol{\eta}.\nabla\theta, \boldsymbol{h}^k)| &\leq \|\nabla\theta\|\|\boldsymbol{\eta}^k\|_4\|\boldsymbol{h}^k\|_4 \\
&\leq C\|\nabla\theta\|\|\boldsymbol{\eta}^k\|^{1/4}\|\|\nabla\boldsymbol{\eta}^k\|^{3/4}\|\nabla\boldsymbol{h}^k\| \\
&\leq C_{\sigma\epsilon}\|\nabla\theta\|^8\|\boldsymbol{\eta}^k\|^2 + \epsilon\|\nabla\boldsymbol{\eta}^k\|^2 + \sigma\|\boldsymbol{\nabla} h^k\|^2 \\
|(\boldsymbol{v}^k.\nabla\theta, \boldsymbol{h}^k)| &\leq \|\nabla\theta\|\|\boldsymbol{h}^k\|_4\|\boldsymbol{v}^k\|_4 \\
&\leq C_{\sigma\epsilon}\|\nabla\theta\|^8\|\boldsymbol{v}^k\|^2 + \epsilon\|\nabla\boldsymbol{v}^k\|^2 + \sigma\|\boldsymbol{\nabla} h^k\|^2 \\
|(\boldsymbol{u}^k.\nabla\boldsymbol{\xi}^k, \boldsymbol{h}^k)| &= |(\boldsymbol{u}^k.\nabla\boldsymbol{h}^k, \boldsymbol{\xi}^k)| \\
&\leq C\|\boldsymbol{u}^k\|_\infty\|\nabla\boldsymbol{h}^k\|\|\boldsymbol{\xi}^k\| \\
&\leq C_\sigma\|A\boldsymbol{u}^k\|^2\|\boldsymbol{\xi}^k\|^2 + \sigma\|\nabla\boldsymbol{h}^k\|^2.
\end{aligned}$$

Para função $\boldsymbol{\Phi}$ temos,

$$|(\boldsymbol{\Phi}_1(\boldsymbol{u},\boldsymbol{w}) - \boldsymbol{\Phi}_1(\boldsymbol{u}^k,\boldsymbol{w}^k), \boldsymbol{h}^k)| \leq \frac{1}{2}(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{u}^k\|^2)(\|\nabla\boldsymbol{\eta}^k\|^2 + \|\nabla\boldsymbol{v}^k\|^2) + \frac{1}{2}\|\nabla\boldsymbol{h}^k\|^2$$

$$\begin{aligned}
|(\boldsymbol{\Phi}_2(\boldsymbol{u},\boldsymbol{w}) - \boldsymbol{\Phi}_2(\boldsymbol{u}^k,\boldsymbol{w}^k), \boldsymbol{h}^k)| &\leq \frac{1}{2}(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{u}^k\|^2 + \|L^{1/2}\boldsymbol{w}\|^2 + \|L^{1/2}\boldsymbol{w}^k\|^2) \\
&\quad . \quad (\|\nabla\boldsymbol{\eta}^k\|^2 + \|\nabla\boldsymbol{v}^k\|^2 + \|\boldsymbol{\zeta}^k\|^2 + \|\boldsymbol{z}^k\|^2) + \frac{1}{2}\|\nabla\boldsymbol{h}^k\|^2.
\end{aligned}$$

Para $\boldsymbol{\Phi}_i = 3, 4, 5$ temos,

$$|(\boldsymbol{\Phi}_i(\boldsymbol{u},\boldsymbol{w}) - \boldsymbol{\Phi}_i(\boldsymbol{u}^k,\boldsymbol{w}^k), \boldsymbol{h}^k)| \leq \frac{1}{2}(\|L\boldsymbol{w}\|^2 + \|L\boldsymbol{w}^k\|^2)(\|\nabla\boldsymbol{\zeta}^k\|^2 + \|\nabla\boldsymbol{z}^k\|^2) + \frac{1}{2}\|\nabla\boldsymbol{h}^k\|^2$$

. Portanto usando as Proposições 3.3 e 3.4

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{h}^k\|^2 + \kappa\|\nabla\boldsymbol{h}^k\|^2 &\leq C_{\sigma\epsilon}\|\nabla\theta\|^8(\|\boldsymbol{\eta}^k\|^2 + \|\boldsymbol{v}^k\|^2) \\
&+ (c+\epsilon)\|\nabla\boldsymbol{\eta}^k\|^2 + (c+\epsilon)\|\nabla\boldsymbol{v}^k\|^2 + c\|\boldsymbol{z}^k\|^2 \\
&+ C_\sigma\|A\boldsymbol{u}^k\|^2\|\boldsymbol{\xi}^k\|^2 + \frac{3c}{2}\|L^{1/2}\boldsymbol{\zeta}^k\|^2 + \frac{3c}{2}\|L^{1/2}\boldsymbol{z}^k\|^2 \\
&+ \frac{c}{2}\|\boldsymbol{\zeta}^k\|^2 + \frac{c}{2}\|\boldsymbol{z}^k\|^2 \\
&+ (3\sigma + 5/2)\|\nabla\boldsymbol{h}^k\|^2 \qquad (4.19)
\end{aligned}$$

onde $c > 0$ é uma constante genérica que independe de k. Tomando ϵ, δ e σ de maneira adequada, somando (4.17), (4.18) e (4.19) e utilizando novamente as Proposições 3.3 e 3.4

temos,

$$\frac{d}{dt}(\|\boldsymbol{v}^k\|^2 + \|\boldsymbol{z}^k\|^2 + \|\boldsymbol{h}^k\|^2) + (\mu + \mu_r)\|\nabla \boldsymbol{v}^k\|^2 + \|L^{1/2}\boldsymbol{z}^k\|^2 + \kappa\|\nabla \boldsymbol{h}^k\|^2 \leq$$
$$c(\|\boldsymbol{\eta}^k\|^2 + \|\boldsymbol{\zeta}^k\|^2 + \|\boldsymbol{\xi}^k\|^2 + \|\nabla\boldsymbol{\eta}^k\|^2 + \|L^{1/2}\boldsymbol{\zeta}^k\|^2)$$
$$+c(\|\nabla \boldsymbol{u}^k\|^2 + \|\nabla\theta\|^2 + \|L\boldsymbol{\omega}\|^2)(\|\boldsymbol{v}^k\|^2 + \|\boldsymbol{z}^k\|^2 + \|\boldsymbol{h}^k\|^2).$$

Integrando de 0 a t, considerando o mínimo entre $\{1, \mu + \mu_r, \kappa\}$ e usando o Lema 4.2 temos

$$\|\boldsymbol{v}^k\|^2 + \|\boldsymbol{z}^k\|^2 + \|\boldsymbol{h}^k\|^2 + \int_0^t \|\nabla \boldsymbol{v}(\tau)\|^2 + \|L^{1/2}\boldsymbol{\omega}(\tau)\|^2 + \|\nabla \boldsymbol{h}^k(\tau)\|^2 d\tau \leq$$
$$\frac{C}{\lambda_{k+1}^2} + \frac{C}{\gamma_{k+1}^2} + \frac{C}{\widetilde{\gamma}_{k+1}^2} + \frac{C}{\lambda_{k+1}} + \frac{C}{\gamma_{k+1}} + C\int_0^t \beta(\tau)(\|\boldsymbol{v}^k(\tau)\|^2 + \|\boldsymbol{z}^k(\tau)\|^2 + \|\boldsymbol{h}^k(\tau)\|^2)d\tau$$

onde $\beta(t) = \|\nabla \boldsymbol{u}^k(t)\|^2 + \|\nabla\theta(t)\|^2 + \|L\boldsymbol{\omega}(t)\|^2$. Aplicando a desigualdade de Gronwall e utilizando as estimativas das Proposições 3.3 e 3.4 o resultado segue. $\square$

4.2.1 Prova do Teorema 4.4

Note que

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^k(t)\|^2 \leq 2(\|\boldsymbol{u}(t) - \boldsymbol{P}_k\boldsymbol{u}(t)\|^2 + \|\boldsymbol{P}_k\boldsymbol{u}(t) - \boldsymbol{u}^k(t)\|^2) = 2(\|\boldsymbol{\eta}^k(t)\|^2 + \|\boldsymbol{v}^k(t)\|^2)$$

e de forma análoga,

$$\|\boldsymbol{w}(t) - \boldsymbol{w}^k(t)\|^2 \leq 2(\|\boldsymbol{\zeta}^k(t)\|^2 + \|\boldsymbol{z}^k(t)\|^2)$$

$$\|\theta(t) - \theta^k(t)\|^2 \leq 2(\|\boldsymbol{\xi}^k(t)\|^2 + \|\boldsymbol{h}^k(t)\|^2).$$

Portanto

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^k(t)\|^2 + \|\boldsymbol{w}(t) - \boldsymbol{w}^k(t)\|^2 + \|\theta(t) - \theta^k(t)\|^2 \leq$$
$$2(\|\boldsymbol{\eta}^k(t)\|^2 + \|\boldsymbol{\zeta}^k(t)\|^2 + \|\boldsymbol{\xi}^k(t)\|^2) + 2(\|\boldsymbol{v}^k(t)\|^2 + \|\boldsymbol{z}^k(t)\|^2 + \|\boldsymbol{h}^k(t)\|^2).$$

Usando o Lema 4.3 e o Lema 4.8 temos o resultado.

4.3 ESTIMATIVAS PARA NORMA $H^1(\Omega)$

Lema 4.9. *Sob as hipóteses da Proposição 4.1 existe uma constante $C > 0$, independente de $t \geq 0$ e $k \in \mathbb{N}$, tal que*

$$\|\nabla \boldsymbol{v}^k\|^2 + \|L^{1/2}\boldsymbol{z}^k\|^2 + \int_0^t \left(\|\boldsymbol{v}_t^k(\tau)\|^2 + \|\boldsymbol{z}_t^k(\tau)\|^2\right) d\tau \leq \frac{C}{\gamma_{k+1}} + \frac{C}{\lambda_{k+1}} + \frac{C}{\widetilde{\gamma}_{k+1}} \tag{4.20}$$

Demonstração. Novamente considerando $\boldsymbol{\varphi} = \boldsymbol{v}_t^k$ e $\boldsymbol{\phi} = \boldsymbol{z}_t^k$ em (4.4) obtemos,

$$\begin{aligned} \|\boldsymbol{v}_t^k\|^2 + \frac{(\mu+\mu_r)}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}^k\|^2 &= 2\mu_r(\operatorname{rot}\boldsymbol{\zeta}^k, \boldsymbol{v}_t^k) + 2\mu_r(\operatorname{rot}\boldsymbol{z}^k, \boldsymbol{v}_t^k) - (\boldsymbol{u}\cdot\nabla\boldsymbol{\eta}^k, \boldsymbol{v}_t^k) \\ &- (\boldsymbol{u}\cdot\nabla\boldsymbol{v}^k, \boldsymbol{v}_t^k) - (\boldsymbol{\eta}^k\cdot\nabla\boldsymbol{u}^k, \boldsymbol{v}_t^k) \\ &- (\boldsymbol{v}^k\cdot\nabla\boldsymbol{u}^k, \boldsymbol{v}_t^k) + (f(\theta) - f(\theta^k), \boldsymbol{v}_t^k) \end{aligned} \tag{4.21}$$

e

$$\begin{aligned} \|\boldsymbol{z}_t^k\|^2 + \frac{1}{2}\frac{d}{dt}\|L^{1/2}\boldsymbol{z}^k\|^2 + 4\mu_r\frac{d}{dt}\|\boldsymbol{z}^k\|^2 &= 2\mu_r(\operatorname{rot}\boldsymbol{\eta}^k, \boldsymbol{z}_t^k) + 2\mu_r(\operatorname{rot}\boldsymbol{v}^k, \boldsymbol{z}_t^k) - (\boldsymbol{\eta}^k\cdot\nabla\boldsymbol{w}, \boldsymbol{z}_t^k) \\ &- (\boldsymbol{v}^k\cdot\nabla\boldsymbol{w}, \boldsymbol{z}_t^k) - (\boldsymbol{u}^k\cdot\nabla\boldsymbol{\zeta}^k, \boldsymbol{z}_t^k) - (\boldsymbol{u}^k\cdot\nabla\boldsymbol{z}^k, \boldsymbol{z}_t^k) \\ &+ (g(\theta) - g(\theta^k), \boldsymbol{z}_t^k) \end{aligned} \tag{4.22}$$

Vamos estimar (4.21) e (4.22) de forma análoga a que fizemos no Lema anterior. Para os termos do lado direito de (4.21) temos,

$$
\begin{aligned}
|2\mu_r(\operatorname{rot}\boldsymbol{\zeta}^k,\boldsymbol{v}_t^k)|+|2\mu_r(\operatorname{rot}\boldsymbol{z}^k,\boldsymbol{v}_t^k)| &\leq \|L^{1/2}\boldsymbol{\zeta}^k\|\|\boldsymbol{v}_t^k\|+\|L^{1/2}\boldsymbol{z}^k\|\|\boldsymbol{v}_t^k\| \\
&\leq C_\epsilon\|L^{1/2}\boldsymbol{\zeta}^k\|^2+C_\epsilon\|L^{1/2}\boldsymbol{z}^k\|^2+2\epsilon\|\boldsymbol{v}_t^k\|^2 \\
|(\boldsymbol{u}\cdot\nabla\boldsymbol{\eta}^k,\boldsymbol{v}_t^k)| &\leq \|\boldsymbol{u}\|_\infty\|\nabla\boldsymbol{\eta}^k\|\|\boldsymbol{v}_t^k\| \\
&\leq C_\epsilon\|A\boldsymbol{u}\|^2\|\nabla\boldsymbol{\eta}^k\|^2+\epsilon\|\boldsymbol{v}_t^k\|^2 \\
|(\boldsymbol{u}\cdot\nabla\boldsymbol{v}^k,\boldsymbol{v}_t^k)| &\leq \|\boldsymbol{u}\|_\infty\|\nabla\boldsymbol{v}^k\|\|\boldsymbol{v}_t^k\| \\
&\leq C_\epsilon\|A\boldsymbol{u}\|^2\|\nabla\boldsymbol{v}^k\|^2+\epsilon\|\boldsymbol{v}_t^k\|^2 \\
|\boldsymbol{\eta}^k\cdot\nabla\boldsymbol{u}^k,\boldsymbol{v}_t^k)| &\leq \|\boldsymbol{\eta}^k\|_3\|\nabla\boldsymbol{u}^k\|_6\|\boldsymbol{v}_t^k\| \\
&\leq C_\epsilon\|\nabla\boldsymbol{\eta}^k\|^2\|A\boldsymbol{u}^k\|^2+\epsilon\|\boldsymbol{v}_t^k\|^2 \\
(\boldsymbol{f}(\theta)-\boldsymbol{f}(\theta^k),\boldsymbol{v}_t^k)| &\leq \|\boldsymbol{f}(\theta)-\boldsymbol{f}(\theta^k)\|\|\boldsymbol{v}_t^k\| \\
&\leq M_{\boldsymbol{f}}\|\theta-\theta^k\|\|\boldsymbol{v}_t^k\| \\
&= M_{\boldsymbol{f}}\|\boldsymbol{\xi}^k+\boldsymbol{h}^k\|\|\boldsymbol{v}_t^k\| \\
&\leq M_{\boldsymbol{f}}\|\boldsymbol{\xi}^k\|\|\boldsymbol{v}_t^k\|+M_{\boldsymbol{f}}\|\boldsymbol{h}^k\|\|\boldsymbol{v}_t^k\| \\
&\leq C_\epsilon\|\boldsymbol{\xi}^k\|^2+C_\epsilon\|\nabla\boldsymbol{h}^k\|^2+\epsilon\|\boldsymbol{v}_t^k\|^2.
\end{aligned}
$$

Daí

$$
\begin{aligned}
\|\boldsymbol{v}_t^k\|^2+\frac{(\mu+\mu_r)}{2}\frac{d}{dt}\|\nabla\boldsymbol{v}^k\|^2 &\leq C_\epsilon\|L^{1/2}\boldsymbol{\zeta}^k\|^2+C_\epsilon\|L^{1/2}\boldsymbol{z}^k\|^2 \\
&+ C_\epsilon\|A\boldsymbol{u}\|^2\|\nabla\boldsymbol{\eta}^k\|^2+C_\epsilon\|A\boldsymbol{u}\|^2\|\nabla\boldsymbol{v}^k\|^2 \\
&+ C_\epsilon\|\nabla\boldsymbol{\eta}^k\|^2\|A\boldsymbol{u}^k\|^2+C_\epsilon\|\nabla\boldsymbol{v}^k\|^2\|A\boldsymbol{u}^k\|^2 \\
&+ C_\epsilon\|\boldsymbol{\xi}^k\|^2+C_\epsilon\|\nabla h^k\|^2+7\epsilon\|\boldsymbol{v}_t^k\|^2,
\end{aligned}
\tag{4.23}
$$

agora estimando os termos do lado direito de (4.22) temos

$$
\begin{aligned}
|2\mu_r(\operatorname{rot}\boldsymbol{\eta}^k,\boldsymbol{z}_t^k)|+|2\mu_r(\operatorname{rot}\boldsymbol{v}^k,\boldsymbol{z}_t^k)| &\leq \|\nabla\boldsymbol{\eta}^k\|\|\boldsymbol{z}_t^k\|+\|\nabla\boldsymbol{v}^k\|\|\boldsymbol{z}_t^k\| \\
&\leq C_\delta\|\nabla\boldsymbol{\eta}^k\|^2+C_\delta\|\nabla\boldsymbol{v}^k\|^2+2\delta\|\boldsymbol{z}_t^k\|^2 \\
|(\boldsymbol{\eta}^k\cdot\nabla\boldsymbol{w},\boldsymbol{z}_t^k)| &\leq \|\boldsymbol{\eta}^k\|_3\|\nabla\boldsymbol{w}\|_6\|\boldsymbol{z}_t^k\| \\
&\leq C_\delta\|\nabla\boldsymbol{\eta}^k\|^2\|L\boldsymbol{w}\|^2+\delta\|\boldsymbol{z}_t^k\|^2 \\
|(\boldsymbol{v}^k\cdot\nabla\boldsymbol{w},\boldsymbol{z}_t^k)| &\leq \|\boldsymbol{v}^k\|_3\|\nabla\boldsymbol{w}\|_6\|\boldsymbol{z}_t^k\| \\
&\leq C_\delta\|\nabla\boldsymbol{v}^k\|^2\|L\boldsymbol{w}\|^2+\delta\|\boldsymbol{z}_t^k\|^2
\end{aligned}
$$

$$
\begin{aligned}
|(\boldsymbol{u}^k \cdot \nabla \boldsymbol{\zeta}^k, \boldsymbol{z}_t^k)| &\leq \|\boldsymbol{u}^k\|_\infty \|\nabla \boldsymbol{\zeta}^k\| \|\boldsymbol{z}_t^k\| \\
&\leq C_\delta \|A\boldsymbol{u}^k\|^2 \|\nabla \boldsymbol{\zeta}^k\|^2 + \delta \|\boldsymbol{z}_t^k\| \\
|(\boldsymbol{u}^k \cdot \nabla \boldsymbol{z}^k, \boldsymbol{z}_t^k)| &\leq \|\boldsymbol{u}^k\|_\infty \|\nabla \boldsymbol{z}^k\| \|\boldsymbol{z}_t^k\| \\
&\leq C_\delta \|A\boldsymbol{u}^k\|^2 \|\nabla \boldsymbol{z}^k\|^2 + \delta \|\boldsymbol{z}_t^k\| \\
(\boldsymbol{g}(\theta) - \boldsymbol{g}(\theta^k), \boldsymbol{z}_t^k)| &\leq \|\boldsymbol{g}(\theta) - \boldsymbol{g}(\theta^k)\| \|\boldsymbol{z}_t^k\| \\
&\leq M_{\boldsymbol{g}} \|\theta - \theta^k\| \|\boldsymbol{z}_t^k\| \\
&= M_{\boldsymbol{g}} \|\boldsymbol{\xi}^k + \boldsymbol{h}^k\| \|\boldsymbol{z}_t^k\| \\
&\leq M_{\boldsymbol{g}} \|\boldsymbol{\xi}^k\| \|\boldsymbol{z}_t^k\| + M_{\boldsymbol{g}} \|\boldsymbol{h}^k\| \|\boldsymbol{z}_t^k\| \\
&\leq C_\delta \|\boldsymbol{\xi}^k\|^2 + C_\delta \|\nabla \boldsymbol{h}^k\|^2 + \delta \|\boldsymbol{z}_t^k\|^2,
\end{aligned}
$$

desta forma

$$
\begin{aligned}
\|\boldsymbol{z}_t^k\|^2 + \frac{1}{2}\frac{d}{dt}\|L^{1/2}\boldsymbol{z}^k\|^2 + 4\mu_r \frac{d}{dt}\|\boldsymbol{z}^k\|^2 &\leq C_\delta \|\nabla \boldsymbol{\eta}^k\|^2 + C_\delta \|\nabla \boldsymbol{v}^k\|^2 + C_\delta \|L\boldsymbol{w}\|^2 \|\nabla \boldsymbol{\eta}^k\|^2 \\
&+ C_\delta \|L\boldsymbol{w}\|^2 \|\nabla \boldsymbol{v}^k\|^2 + C_\delta \|A\boldsymbol{u}^k\|^2 \|L^{1/2}\boldsymbol{\zeta}^k\|^2 \\
&+ C_\delta \|A\boldsymbol{u}^k\|^2 \|L^{1/2}\boldsymbol{z}^k\|^2 + C_\delta \|\boldsymbol{\xi}^k\|^2 \\
&+ C_\delta \|\nabla h^k\|^2 + 7\delta \|\boldsymbol{z}_t^k\|^2 \qquad (4.24)
\end{aligned}
$$

Somando as desigualdades (4.23), (4.24) e tomando $\epsilon = \delta = 1/14$ temos,

$$
\begin{aligned}
\|\boldsymbol{v}_t^k\|^2 + \|\boldsymbol{z}_t^k\|^2 + (\mu + \mu_r)\frac{d}{dt}\|\nabla \boldsymbol{v}^k\|^2 + \frac{d}{dt}\|L^{1/2}\boldsymbol{z}^k\|^2 + 8\mu_r \frac{d}{dt}\|\boldsymbol{z}^k\|^2 \quad &\leq \\
C(1 + \|A\boldsymbol{u}^k\|^2)\|\nabla \boldsymbol{\zeta}^k\|^2 + C(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{u}^k\| + \|L\boldsymbol{w}\|^2 + 1)\|\nabla \boldsymbol{\eta}^k\|^2 & \\
+C(1 + \|A\boldsymbol{u}^k\|)\|L^{1/2}\boldsymbol{z}^k\|^2 + C(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{u}^k\| & \\
+\|L\boldsymbol{w}\|^2 + 1)\|\nabla \boldsymbol{v}^k\|^2 + C\|\boldsymbol{\xi}^k\|^2 + C\|\nabla h^k\|^2. &
\end{aligned}
$$

Portanto, novamente pela Proposição 4.1 temos

$$
\begin{aligned}
\|\boldsymbol{v}_t^k\|^2 + \|\boldsymbol{z}_t^k\|^2 + (\mu + \mu_r)\frac{d}{dt}\|\nabla \boldsymbol{v}^k\|^2 + \frac{d}{dt}\|L^{1/2}\boldsymbol{z}^k\|^2 + 8\mu_r \frac{d}{dt}\|\boldsymbol{z}^k\|^2 \quad &\leq \\
C\|\nabla \boldsymbol{\zeta}^k\|^2 + C\|\nabla \boldsymbol{\eta}^k\|^2 + C\|L^{1/2}\boldsymbol{z}^k\|^2 + C\|\nabla \boldsymbol{v}^k\|^2 + C\|\boldsymbol{\xi}^k\|^2 + C\|\nabla h^k\|^2. &
\end{aligned}
$$

Daí, integrando de 0 a t, considerando o $\min\{1, \mu+\mu_r\}$ e os Lemas 4.3 e 4.8 temos

$$\begin{aligned}
\int_0^t (\|\boldsymbol{v}_t^k(\tau)\|^2 + \|\boldsymbol{z}_t^k(\tau)\|^2)d\tau + \|\nabla \boldsymbol{v}^k\|^2 + \|L^{1/2}\boldsymbol{z}^k\|^2 &\leq \\
C\int_0^t \|L^{1/2}\boldsymbol{\zeta}^k(\tau)\|^2 d\tau + C\int_0^t \|\nabla \boldsymbol{\eta}^k(\tau)\|^2 d\tau + C\int_0^t \|\boldsymbol{\xi}(\tau)\|^2 d\tau & \\
+C\int_0^t \|\nabla \boldsymbol{v}^k(\tau)\|^2 d\tau + C\int_0^t \|L^{1/2}\boldsymbol{z}^k(\tau)\|^2 d\tau + C\int_0^t \|\nabla h^k\|^2 d\tau & \\
\leq \frac{C}{\gamma_{k+1}} + \frac{C}{\lambda_{k+1}} + \frac{C}{\widetilde{\gamma}_{k+1}^2} + \frac{C}{\widetilde{\gamma}_{k+1}} + \frac{C}{\gamma_{k+1}^2} + \frac{C}{\lambda_{k+1}^2} & \\
\leq \frac{C}{\gamma_{k+1}} + \frac{C}{\lambda_{k+1}} + \frac{C}{\widetilde{\gamma}_{k+1}}. &
\end{aligned}$$

□

4.3.1 Prova do Teorema 4.6

Note que

$$\begin{aligned}
\|\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}^k(t)\|^2 &\leq 2(\|\nabla \boldsymbol{u}(t) - \boldsymbol{P}_k \nabla \boldsymbol{u}(t)\|^2 + \|\boldsymbol{P}_k \nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}^k(t)\|^2) \\
&= 2(\|\nabla \eta^k(t)\|^2 + \|\nabla \boldsymbol{v}^k(t)\|^2)
\end{aligned}$$

e de forma análoga,

$$\|L^{1/2}\boldsymbol{w}(t) - L^{1/2}\boldsymbol{w}^k(t)\|^2 \leq 2(\|L^{1/2}\boldsymbol{\zeta}^k(t)\|^2 + \|L^{1/2}\boldsymbol{z}^k(t)\|^2).$$

Portanto

$$\begin{aligned}
&\|\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}^k(t)\|^2 + \|L^{1/2}\boldsymbol{w}(t) - L^{1/2}\boldsymbol{w}^k(t)\|^2 \\
2(\|\nabla \boldsymbol{\eta}^k(t)\|^2 + \|L^{1/2}\boldsymbol{\zeta}^k(t)\|^2 &+ 2(\|\nabla \boldsymbol{v}^k(t)\|^2 + \|L^{1/2}\boldsymbol{z}^k(t)\|^2)
\end{aligned}$$

usando o Lema 4.3 e o Lema 4.9 o resultado segue.

4.3.2 Prova do Teorema 4.7

Note que

$$\begin{aligned}
\boldsymbol{u}_t - \boldsymbol{u}_t^k = -(\mu+\mu_r)A(\boldsymbol{u} - \boldsymbol{u}^k) - (P(\boldsymbol{u}\cdot\nabla \boldsymbol{u}) - P_k(\boldsymbol{u}^k \cdot \nabla \boldsymbol{u}^k)) \\
+2\mu_r(P \operatorname{rot} \boldsymbol{w} - P_k \operatorname{rot} \boldsymbol{w}^k) + Pf(\theta) - P_k f(\theta^k).
\end{aligned}$$

Desta forma

$$\begin{aligned}\|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^k(t)\|_{V^*} &= \sup_{\|v\|_V \le 1} |(-(\mu+\mu_r)A(\boldsymbol{u}-\boldsymbol{u}^k) - (P(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) - P_k(\boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k)) \\ &+ 2\mu_r(P\operatorname{rot}\boldsymbol{w} - P_k\operatorname{rot}\boldsymbol{w}^k) + Pf(\theta) - P_kf(\theta^k), v)|. \end{aligned} \tag{4.25}$$

Estimando temos,

$$|(-(\mu+\mu_r)A(\boldsymbol{u}-\boldsymbol{u}^k),\boldsymbol{v})| \le \|\nabla(\boldsymbol{u}-\boldsymbol{u}^k)\|\|\nabla\boldsymbol{v}\| \le \left(\frac{C}{\gamma_{k+1}} + \frac{C}{\lambda_{k+1}} + \frac{C}{\tilde{\gamma}_{k+1}}\right)^{\frac{1}{2}}\|\nabla\boldsymbol{v}\|$$

$$\begin{aligned}|(\boldsymbol{P}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) - \boldsymbol{P}_k(\boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k),\boldsymbol{v})| &= |((\boldsymbol{P}-\boldsymbol{P}_k)(\boldsymbol{u}\cdot\nabla\boldsymbol{u}),\boldsymbol{v})| \\ &+ |(\boldsymbol{P}_k(\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k),\boldsymbol{v}) \\ &\le |(\boldsymbol{u}\cdot\nabla\boldsymbol{u},(\boldsymbol{P}-\boldsymbol{P}_k)\boldsymbol{v})| + |((\boldsymbol{u}-\boldsymbol{u}^k)\cdot\nabla\boldsymbol{u},\boldsymbol{P}_k\boldsymbol{v}) \\ &+ |(\boldsymbol{u}^k\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}^k),\boldsymbol{P}_k\boldsymbol{v})| \\ &\le \frac{C}{\lambda_{k+1}^{\frac{1}{2}}}\|\nabla\boldsymbol{v}\|\end{aligned}$$

$$\begin{aligned}|2\mu_r(\boldsymbol{P}\operatorname{rot}\boldsymbol{w} - \boldsymbol{P}_k\operatorname{rot}\boldsymbol{w}^k)| &= |2\mu_r((\boldsymbol{P}-\boldsymbol{P}_k)\operatorname{rot}\boldsymbol{w},\boldsymbol{v})| + |2\mu_r(\boldsymbol{P}_k\operatorname{rot}(\boldsymbol{w}-\boldsymbol{w}^k),\boldsymbol{v})| \\ &= |2\mu_r(\operatorname{rot}\boldsymbol{w},(\boldsymbol{P}-\boldsymbol{P}_k)\boldsymbol{v})| + |2\mu_r(\operatorname{rot}(\boldsymbol{w}-\boldsymbol{w}^k),\boldsymbol{P}_k\boldsymbol{v})| \\ &\le \frac{C}{\lambda_{k+1}^{\frac{1}{2}}}\|\nabla\boldsymbol{v}\|\end{aligned}$$

$$\begin{aligned}|(\boldsymbol{P}f(\theta) - \boldsymbol{P}_kf(\theta^k),\boldsymbol{v})| &= |((\boldsymbol{P}-\boldsymbol{P}_k)f(\theta),\boldsymbol{v})| + |\boldsymbol{P}_k(f(\theta)-f(\theta^k)),\boldsymbol{v})| \\ &= |(f(\theta),(\boldsymbol{P}-\boldsymbol{P}_k)\boldsymbol{v})| + |(f(\theta)-f(\theta^k),\boldsymbol{P}_k\boldsymbol{v})| \\ &\le \frac{C}{\lambda_{k+1}^{\frac{1}{2}}}\|\nabla\boldsymbol{v}\|.\end{aligned}$$

Consequentemente utilizando as estimativas acima em (4.25), obtemos

$$\|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^k(t)\|_{V^*} \le \frac{C}{\gamma_{k+1}} + \frac{C}{\lambda_{k+1}} + \frac{C}{\tilde{\gamma}_{k+1}}.$$

As estimativas para w são análogas. Para a equação de balanço de energia temos,

$$\begin{aligned}\|\theta_t - \theta_t^k\|_{H^{-1}}^2 &\le \kappa\|\Delta\theta - \Delta\theta^k\|_{H^{-1}}^2 + \|u^k\cdot\nabla(\theta-\theta^k)\|_{H^{-1}}^2 + \|(u-u^k)\cdot\nabla\theta\|_{H^{-1}}^2 \\ &+ \|\Phi(\boldsymbol{u},\boldsymbol{w}) - \Phi(\boldsymbol{u}^k,\boldsymbol{w}^k)\|_{H^{-1}}^2\end{aligned}$$

estimando

$$\begin{aligned}\|\Delta(\theta-\theta^k)\|_{H^{-1}} &= \sup_{\|\varphi\|_{H_0^1}\le 1} |(\Delta(\theta-\theta^k),\varphi)| \\ &\le \sup_{\|\varphi\|_{H_0^1}\le 1} \|\nabla(\theta-\theta^k)\|\|\nabla\varphi\| \\ &\le C\|\nabla(\theta-\theta^k)\|,\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{u}^k\cdot\nabla(\theta-\theta^k)\|_{H^{-1}} &= \sup_{\|\varphi\|_{H_0^1}\le 1} |(\boldsymbol{u}^k\cdot\nabla(\theta-\theta^k),\varphi)| \\ &\le \sup_{\|\varphi\|_{H_0^1}\le 1} \|\boldsymbol{u}^k\|_{L^3}\|\nabla(\theta-\theta^k)\|\|\nabla\varphi\| \\ &\le C\|\nabla(\theta-\theta^k)\|\end{aligned}$$

$$\begin{aligned}\|(\boldsymbol{u}-\boldsymbol{u}^k)\cdot\nabla\theta\|_{H^{-1}} &= \sup_{\|\varphi\|_{H_0^1}\le 1} |((\boldsymbol{u}-\boldsymbol{u}^k)\cdot\nabla\theta,\varphi)| \\ &\le \sup_{\|\varphi\|_{H_0^1}\le 1} \|\boldsymbol{u}-\boldsymbol{u}^k\|_{L^3}\|\nabla(\theta-\theta^k)\|\|\nabla\varphi\| \\ &\le C\|\nabla(\boldsymbol{u}-\boldsymbol{u}^k)\|,\end{aligned}$$

$$\begin{aligned}\|\Phi_1(\boldsymbol{u})-\Phi_1(\boldsymbol{u}^k)\|_{H^{-1}} &\le C\sup_{\|\varphi\|_{H_0^1}\le 1} (\|\nabla\boldsymbol{u}\|_{L^3}+\|\nabla\boldsymbol{u}^k\|_{L^3})\|\nabla(\boldsymbol{u}-\boldsymbol{u}^k)\|\|\nabla\varphi\| \\ &\le C\|\nabla(\boldsymbol{u}-\boldsymbol{u}^k)\|,\end{aligned}$$

$$\begin{aligned}\|\Phi_2(\boldsymbol{u},\boldsymbol{w})-\Phi_2(\boldsymbol{u}^k,\boldsymbol{w}^k)\|_{H^{-1}} &\le \sup_{\|\varphi\|_{H_0^1}\le 1} C(\|\nabla\boldsymbol{u}\|_{L^3}+\|\nabla\boldsymbol{u}^k\|_{L^3}+\|\boldsymbol{w}\|_{L^3}+\|\boldsymbol{w}^k\|_{L^3}) \\ &\quad (\|\nabla(\boldsymbol{u}-\boldsymbol{u}^k)\|+\|\nabla(\boldsymbol{w}-\boldsymbol{w}^k)\|)\|\nabla\varphi\| \\ &\le C(\|\nabla(\boldsymbol{u}-\boldsymbol{u}^k)\|+\|\nabla(\boldsymbol{w}-\boldsymbol{w}^k)\|),\end{aligned}$$

e para $i=3,4,5$

$$\begin{aligned}\|\Phi_i(\boldsymbol{u},\boldsymbol{w})-\Phi_i(\boldsymbol{u}^k,\boldsymbol{w}^k)\| &\le C\sup_{\|\varphi\|_{H_0^1}\le 1} (\|\nabla\boldsymbol{w}\|_{L^3}+\|\nabla\boldsymbol{w}^k\|_{L^3})\|\nabla(\boldsymbol{w}-\boldsymbol{w}^k)\|\|\nabla\varphi\| \\ &\le C\|\nabla(\boldsymbol{w}-\boldsymbol{w}^k)\|.\end{aligned}$$

Assim das estimativas acima obtemos,

$$\begin{aligned}\int_0^t \|\theta_t(\tau)-\theta_t^k(\tau)\|_{H^{-1}}^2 d\tau &\leq \int_0^t \|\nabla(\boldsymbol{u}-\boldsymbol{u}^k)(\tau)\|^2 d\tau \\ &+ \int_0^t \|\nabla(\boldsymbol{w}-\boldsymbol{w}^k)(\tau)\|^2 + \|\nabla(\theta-\theta^k)(\tau)\|^2 d\tau \\ &\leq \frac{C}{\lambda_{k+1}^2}+\frac{C}{\gamma_{k+1}^2}+\frac{C}{\tilde{\gamma}_{k+1}}+\frac{C}{\lambda_{k+1}} \\ &\leq \frac{C}{\gamma_{k+1}^2}+\frac{C}{\tilde{\gamma}_{k+1}}+\frac{C}{\lambda_{k+1}}.\end{aligned}$$

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