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Bell nonlocality and quantum teleportation under noisy channels in high dimensional systems

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EULISES ALEJANDRO FONSECA PARRA

BELL NONLOCALITY AND QUANTUM TELEPORTATION UNDER NOISY CHANNELS IN HIGH DIMENSIONAL SYSTEMS

Tese apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutor em Física.

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Este trabajo va dedicado a las miles de víctimas que dejó el conflicto armado en Colombia entre los años 1964 y 2017.

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ABSTRACT

The first part of this thesis is devoted to address the problem of Bell nonlocality quantification. For this, it is introduced a measure of nonlocality associated with a quantum state embedded in a specific measurement context dictated by a Bell inequality. Under this approach the amount of nonlocality is closely related to the ability that a quantum state has to reveal non-classical features when submitted to unbiased local random measurements. It is shown that by employing this figure of merit, the most nonlocal states correspond to the maximally entangled ones for the considered scenarios, especially for the case of pairs of entangled d-level systems under two local measurements per party for which the most known measures of nonlocality do not exhibit such an agreement, (Quant. Inf. Comput. 7, 157, 2008). Another interesting observation is that the amount of nonlocality diminishes with the inverse o the local dimension d of the subsystems, and thus revealing the *emergence of classicality* in the limit of large quantum numbers. In the second part it is investigated how to improve the performance of several quantum information protocols when noise affects the system. First it is proved that the replacement of EPR states and measurements by their GHZ counterparts may improve the efficiency in the execution of the teleportation and super-dense coding protocols when the qubits may suffer bit-flip noise in the weak regime. Furthermore, it was performed a characterization of the usual protocol of teleportation for qudits for the case in which the involved parts may be affected by some of the most known instances of noise and/or the channel and measurements are not ideal. The idea of *fighting noise with noise*, presented by Fortes and Rigolin (Phys. Rev. A 92, 012338, 2015), is studied for arbitrary dimension d.

Keywords: Entanglement. Bell nonlocality. Quantum teleportation. Noisy channels.

RESUMO

A primeira parte desta tese é direcionada no problema de quantificação de não-localidade de Bell. Para isto, nos introduzimos uma medida associada a um estado quântico sujeito a um esquema de medidas especifico ditado por uma desigualdade de Bell. Sob esta aproximação, a quantidade de não-localidade esta intimamente relacionada com a habilidade que um estado quântico emaranhado tem para revelar características não clássicas quando submetido a medidas aleatórias locais não enviesadas. É mostrado que empregando esta figura de mérito os estados mais não locais correspondem com os maximamente emaranhados para os cenários considerados, especialmente para o caso de pares de sistemas d-dimensionais emaranhados sujeitos a uma entre duas medições por parte, para os quais a maioria entre as mais conhecidas medidas de não-localidade não exibem dita concordância, (Quant. Inf. Comput. 7, 157, 2008). Outra observação interessante é que a quantidade de não-localidade diminui com o inverso da dimensão local d dos subsistemas, revelando assim a *emergência de classicalidade* no limite de números quânticos grandes. Na segunda parte é investigado como melhorar o desempenho de vários protocolos de informação quântica quando ruido afeta o sistema. Em primeiro lugar é provado que a substituição de pares e medições EPR pelos seus análogos GHZ pode melhorar a eficiência na execução dos protocolos de teletransporte e codificação superdensa quando os qubits podem sofrer ruido do tipo bit-flip no regime de ruido fraco. Adicionalmente foi levada a cabo uma caracterização do protocolo usual de teletransporte de qudits para o caso em que as partes envolvidas podem ser afetadas por alguns dos casos mais conhecidos de ruido e/ou o canal e medições são não-ideais. A ideia de *atacar ruido usando ruido em qubits*, introduzida por Fortes e Rigolin (Phys. Rev. A 92, 012338, 2015) é estudada para sistemas de dimensionalidade arbitrária d.

Palavras-chave: Emaranhamento. Não-Localidade de Bell. Teletransporte quântico. Canais ruidosos.

CONTENTS

1 INTRODUCTION

Along with general relativity, the formulation of quantum theory is probably one of the greatest scientific achievements of the twentieth century, mainly because of two reasons: (i) the induced paradigm shift from the Newtonian view of the world, and (ii) the burst of technological implementations never conceived before its foundation. Although providing very accurate predictions within its validity range, quantum mechanics is not completely understood. Even today there are many open questions and considerable efforts are devoted day by day towards their answer.

One of the most fundamental problems of quantum theory was raised by Einstein himself together with Podolsky and Rosen in 1935 [Einstein, Podolsky e Rosen 1935]. By using a *gedankenexperiment* involving a pair of entangled particles and an elegant set of arguments, they concluded that “the wave function in quantum mechanics does not give a complete description of the physical reality" and therefore there should exist a more general theory in order to not enter into a conflict with relativity. This became known after as the *EPR paradox* and did not receive relevant attention[1] until 1964, when John Bell showed that certain predictions from quantum theory cannot be reproduced by using any *local hidden variables model* [Bell 1964], or in other words he proved the impossibility of the existence of a theory with the features idealized in EPR, conclusion confirmed experimentally some years later by Aspect [Aspect, Grangier e Roger 1982]. Today we have a lot of evidence to believe that in fact nature is able to exhibit such a *spooky action at a distance*, but never violating the principles of relativity i.e. it is not possible to use quantum correlations to send information faster than the speed of light.

[1] Due to accurate predictions of the theory, most of the scientific community was concerned with the development of techniques and models describing the matter, or they simply were not interested in the fundamental details of the formulation, in words of David Mermin “If I were forced to sum up in one sentence what the Copenhagen interpretation says to me, it would be ‘Shut up and calculate!’ ".

Bell's paper represents the starting point of the so called *second quantum revolution* and together with other works (see for instance [Holevo 1973, Ingarden 1976]) contributed to the construction of the theory of quantum information. Subsequent efforts include the development of Shor's algorithm for integer factorization [Shor 1996], and protocols using entanglement as the prime resource, like quantum key distribution [Bennett e Brassard 1984, Ekert 1991], entanglement swapping [Zukowski et al. 1993], super-dense coding [Bennett e Wiesner 1992] and quantum teleportation [Bennett et al. 1993].

Entanglement based protocols offer a great advantage when compared with classical counterparts for several reasons including security in the transmission of information [Macchiavello e Palma 2002], increase in the speed of execution for certain computational tasks[2] [Ekert e Jozsa 1998], among others. Nevertheless, most of the applications require sources capable of producing perfect entanglement (e.g. pairs of maximally entangled states) and it is a well known fact that from the experimentalist point of view, it is a very hard task [Barz et al. 2010, Wagenknecht et al. 2010]. In addition one must also consider the interactions with the environment, usually taking the system to low entangled mixed states. For this, designing techniques heading towards the detection and correction of errors in quantum systems is a very important subject in quantum information theory. The second part of this thesis is devoted to that matter.

Most of the quantum information protocols were initially conceived for systems based on qubits[3], nevertheless recent developments have shown that using high-dimensional systems may lead to a higher performance. For instance, Durt and collaborators showed that the security of quantum key distribution may be enhanced by the usage of entangled qutrits [Durt et al. 2003] (a generalization using qudits is presented in [Durt et al. 2004], see also [Cerf et al. 2002, Acin, Gisin e Scarani 2003]). Qudit systems have also been shown to be a useful resource for quantum computation [Ralph, Resch e Gilchrist 2007, Lanyon et al. 2009, Strauch 2011, Mischuck e Mølmer 2013] and tests of nonlocality [Vértesi, Pironio e Brunner 2010]. Remarkably, Skrzypczyk and Cavalcanti have recently

[2] Note though that entanglement is not the only non-classical resource useful for computation.
[3] Two-dimensional quantum systems. d-dimensional generalizations are usually known in the literature as *qudits*.

shown that through violation of the so called *steering inqualities*, qudits may be used to generate maximal randomness, equal to $\log d$ bits [Skrzypczyk e Cavalcanti 2018]. Moreover, technological limitations have been overcome and nowadays entanglement generation in high dimensions is possible[4] [Martin et al. 2017, Wang et al. 2018].

Motivated by the relevance of quantum high dimensional systems, throughout this thesis we present several results regarding these special kind of states. The first of these is related with the quantification of their nonlocal content, or their ability to reveal non-classical features. There are currently in the literature several ways to assess the amount of nonlocality of a given state. For the case of pure bipartite qubit states most of them agree to state the singlet as the most nonlocal one, however when the dimensionality is increased this feature ceases to happen. This became known as an “anomaly of nonlocality" [Méthot e Scarani 2007]. We have proposed a quantifier of nonlocality in which such an “anomaly" does not appear at least for $d \leq 7$. Our second contribution related to this kind of systems is a characterization of the protocol of qudit teleportation under noisy environments. We extend the ideas introduced in [Fortes e Rigolin 2015] for arbitrary dimension d and present some cases in which addition of noise in the system leads to an increase in the fidelity of teleportation.

This thesis is organized as follows: In the first chapter (Ch. 2) we introduce the main concepts and give the tools necessary to understand the rest of the work. Chapter 3 is devoted to present the probability of violation as a measure of nonlocality [Fonseca e Parisio 2015]. By employing the quantifier proposed in the former chapter, we study the nonlocality of systems in higher dimensions [Fonseca et al. 2018], in chapter 4. In chapter 5 we present a method to partially detect bit-flip noise in entanglement based protocols [Moreno, Fonseca e Cunha 2018]. A characterization of the teleportation protocol for qudits in the presence of noisy environments is given in chapter 6 [Fonseca 2018] (soon in arXiv). At the end we give some of the most relevant conclusions of the thesis.

4 Recently it has been reported an experiment producing entangled states of at least $d = 100$ [Krenn et al. 2014].

2 PRELIMINARIES

Highlights

Quantum entanglement is briefly presented as a resource for informational tasks.

An introduction is given to the different manifestations of quantum nonlocality and proposals to quantify it.

The teleportation protocol is presented along with some extensions.

The last part is devoted to introduce the Kraus operators treatment to quantum noise.

2.1 Quantum States

Before introducing some relevant ideas like entanglement, nonlocality and teleportation, it is important to briefly clarify several concepts often used in quantum theory[1].

Depending on the nature of its preparation, observation and even the level of interaction with the environment, the description of the state of a quantum system may be divided in two classes: *pure* and *mixed*:

A pure state $|\psi\rangle$ describing a N-partite system may be represented as a superposition of kets $|\boldsymbol{j}\rangle$ in the associated Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$:

$$|\psi\rangle = \sum_{\boldsymbol{j}} \alpha_{\boldsymbol{j}} |\boldsymbol{j}\rangle , \tag{2.1}$$

with $\boldsymbol{j} = (j_1, \cdots, j_N)$, $0 \leq j_m \leq d_m - 1$, where d_m is the dimensionality of the m-th subsystem and $\alpha_{j_m} \in \mathbb{C}$, satisfying $\sum_{\boldsymbol{j}} |\alpha_{\boldsymbol{j}}|^2 = 1$. This definition corresponds to the more general kind of pure state, nevertheless throughout this thesis we will be dealing with subsystems of equal dimension d.

[1] For a formal introduction, see [Sakurai e Napolitano 2011, Cohen-Tannoudji, Diu e Laloe 1992, Nielsen e Chuang 2010].

On the other hand, mixed states may be thought as statistical ensembles of pure states[2] $|\psi_k\rangle$, represented by Hermitian, positive semi-definite operators defined on the Hilbert-Schmidt space $\mathcal{HS}$, also known as density matrices[3]:

$$\hat{\rho} = \sum_{k=1}^{M} p_k \, |\psi_k\rangle\langle\psi_k| \,, \tag{2.2}$$

where $p_k \in [0, 1]$ and $\sum_{k=1}^{M} p_k = 1$, in order to satisfy the normalization condition $\mathrm{Tr}(\hat{\rho}) = 1$.

Given a density operator $\hat{\rho}$, it is possible to determine its degree of mixedness by calculating the purity parameter $r = \mathrm{Tr}(\hat{\rho}^2)$, whose extremal values are given by $r = 1$ for a pure state ($\hat{\rho} = |\psi\rangle\langle\psi|$) and $r = 1/D$ for a maximally mixed state ($\hat{\rho} = \hat{1}/D$), where $\hat{1}$ is the identity matrix and D is the dimensionality of the whole Hilbert space, $D = \prod_{j=1}^{N} d_j$.

Next we give a brief introduction to the subject of quantum entanglement. For an exhaustive revision we refer the reader to the Horodecki's review[4] [Horodecki et al. 2009] and Bengtsson & Życzkowski's book [Bengtsson e Zyczkowski 2007], for a more formal construction.

2.2 Entanglement

Although predicted in the early days of quantum theory [Einstein, Podolsky e Rosen 1935, Schrödinger 1935], entanglement only came to play a significant role after the appearance of the Bell's proposal to test quantum mechanics against local hidden variables models [Bell 1964]. Particularly, after the end of the eighties the scientific community became aware of the capacity of this feature to overcome

2 The case mentioned in the text refers to a proper mixture. Nevertheless, a mixed state may also be obtained when we have access to partial information about the whole system i.e. by tracing some part. These are known as improper mixed states.

3 Some textbooks instead of the usual Hilbert-Schmidt space, define the density matrices on $\mathcal{B}(\mathcal{H})$, the space of bounded operators acting on $\mathcal{H}$ [Bengtsson e Zyczkowski 2007].

4 The reader interested in experimental implementations is referred to [Pan et al. 2012]. Moreover, a review focused on entanglement in open systems is given in [Aolita, De Melo e Davidovich 2015] and a personal perspective on the subject by Anton Zelinger can be found in [Zeilinger 2017].

the performance in the execution of certain tasks when compared to the usage of classical resources, giving birth to the field of quantum information.

Quantum entanglement may be seen inherently as a mathematical property of a N-partite[5] quantum state $|\psi\rangle$, namely the impossibility of this to be written as a product of the form

$$|\psi\rangle = \bigotimes_{j=1}^{N} |\phi\rangle_j = |\phi\rangle_1 \otimes \cdots \otimes |\phi\rangle_N \,, \tag{2.3}$$

or more generally, for the case of density operators one can say that the state $\hat{\rho}$ is *entangled* if there is no way to decompose it in a convex sum of *separable* density matrices

$$\hat{\rho} = \sum_k p_k \bigotimes_{j=1}^{N} \hat{\rho}_j^{(k)} = \sum_k p_k \hat{\rho}_1^{(k)} \otimes \cdots \otimes \hat{\rho}_N^{(k)}, \tag{2.4}$$

with coefficients $p_k \in [0,1]$, satisfying $\sum_k p_k = 1$.

In general terms it is possible to produce quantum entanglement between degrees of freedom in continuous variables, e.g. gaussian states [Braunstein e Loock 2005,Adesso e Illuminati 2007], or in discrete variables like spin or number of particles. In this thesis we concentrate our attention on the second type.

2.2.1 Bipartite entanglement

Currently, the most known and widely studied instance of entanglement corresponds to that between systems composed by two parts. A very useful way to represent pure states of this kind is given by the *Schmidt decomposition*[6]:

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5 Note that talking about multipartite entanglement does not necessarily imply that the system as a whole is composed by spatially separated parts inasmuch as we can also have entanglement between local degrees of freedom in a single particle e.g. spin and position variables of an electron subject to a non-uniform magnetic field.

6 The Schmidt decomposition is originally introduced in [Schmidt 1907]. For a more friendly presentation, we refer the reader to [Ekert e Knight 1995].

Schmidt decomposition

In general, a bipartite pure quantum state may be written as:

$$|\psi\rangle = \sum_{mn=0}^{d_A-1,d_B-1} \gamma_{mn} \left|m\right\rangle_A \otimes \left|n\right\rangle_B , \tag{2.5}$$

where $\gamma_{mn} \in \mathbb{C}$. After some calculations it is possible to show that by performing a local change of basis, the state is completely equivalent to:

$$|\psi\rangle = \sum_{j=0}^{d} \sqrt{\lambda_j} \left|\mu_j\right\rangle_A \otimes \left|\nu_j\right\rangle_B , \tag{2.6}$$

or:

$$|\psi\rangle = \sum_{j=0}^{d} \alpha_j \left|jj\right\rangle_{AB} , \tag{2.7}$$

where $d = \min\{d_A, d_B\}$, $\alpha_j \in \mathbb{R}$ and the λ_j's are known as *Schmidt coefficients* which correspond to the eigenvalues of the reduced matrix $\hat{\rho}_A = \mathrm{Tr}_B\{|\psi\rangle\langle\psi|\}$, with $\mathrm{Tr}_B\{\}$ denoting the partial trace on subsystem B[7]. In conclusion, given a bipartite system described by a pure state (eq. 2.5), we can always find a change of basis which describes our state in the Schmidt form (eq. 2.7).

The number of non-null Schmidt coefficients is known as *Schmidt-rank* and it is related to the amount of entanglement of the system as we will see next. For the moment let us analyse the simplest scenario: a state whose Schmidt-rank is equal to one, in this case, by virtue of the definition above (equation 2.3), is separable. Thus, the calculation of the Schmidt coefficients represents a method to witness the presence of entanglement i.e. if from experimental data we can determinate the state coefficients γ_{mn} (eq. 2.5), then we are able to say whether the state is entangled or not.

The Schmidt decomposition is also useful in the quantification of entanglement for bipartite states as we will see in the next part.

[7] It is important to remark that the Schmidt coefficients are also eigenvalues of $\hat{\rho}_B = \mathrm{Tr}_A\{|\psi\rangle\langle\psi|\}$.

Measures of Bipartite Entanglement - Entropy of Entanglement

The superiority of entanglement above classical mechanisms have motivated an operational formulation from the point of view of resource theories. Under this, it is possible to define which rules a *good measure* of entanglement should satisfy and conversely, the set of free operations under which the amount of entanglement of a given state is not increased [Vedral e Plenio 1998, ao e Plenio 2008, Horodecki et al. 2009]. The basic properties an entanglement measure $E(|\psi\rangle)$ has to fulfill may be summarized as: Given a quantum state $|\psi\rangle$, $E(|\psi\rangle) \geq 0$, the equality only holds whenever $|\psi\rangle$ is a separable state, and $E(|\psi\rangle)$ is invariant under *Local Operations and Classical Communication* (hereafter, LOCC). An extensive set of axioms may be further included, for a revision see [Horodecki et al. 2009]. Among the zoo of entanglement measures, due to their operational interpretation we can highlight distillable entanglement, entanglement cost [Bennett et al. 1996] and concurrence [Wootters 1998]. For our case of interest (bipartite pure states) the three examples cited above reduce to the *entanglement entropy* [Bennett et al. 1996], for this reason and due to its simplicity, we employ it to quantify entanglement all through this thesis.

The entropy of entanglement is calculated by using the von Neumann entropy of the reduced density operator of any of the subsystems:

$$E(|\psi\rangle) = S(\hat{\rho}_A) = S(\hat{\rho}_B) = -\operatorname{Tr}\left[\hat{\rho}_A \log(\hat{\rho}_A)\right] = -\operatorname{Tr}\left[\hat{\rho}_B \log(\hat{\rho}_B)\right]. \tag{2.8}$$

For a system composed by two d-dimensional systems (hereafter qudits), described by the state 2.7, the reduced density matrix corresponding to the subsystem A reads:

$$\hat{\rho}_A = \sum_{j=0}^{d-1} \alpha_j^2 |j\rangle\langle j|. \tag{2.9}$$

Using the fact that $\log \hat{\rho}_A = \sum_{j=0}^{d-1} \log \alpha_j^2 |j\rangle\langle j|$, we have:

$$E(|\psi\rangle) = -\sum_{k=0}^{d-1} \langle k| \left(\sum_{l=0}^{d-1} \alpha_l^2 |l\rangle\langle l| \sum_{m=0}^{d-1} \log \alpha_m^2 |m\rangle\langle m| \right) |k\rangle,$$

then:

$$E(|\psi\rangle) = -\sum_{m=0}^{d-1} \alpha_m^2 \log \alpha_m^2. \tag{2.10}$$

From here, it is straightforward to show that the state which maximizes the entropy of entanglement is such that $\alpha_m = 1/\sqrt{d}$. In this way, we can define the maximally entangled state of two qudits as:

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle \, , \tag{2.11}$$

with an entropy of entanglement $\log d$. The states above are not unique, instead we can associate a phase $\exp(i\varphi_j)$ to each coefficient. In particular, let us define the "μ, ν" element of the generalized Bell basis as:

$$\left|\phi^d_{\mu\nu}\right\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega_d^{j\mu} \left|j, j \oplus \nu\right\rangle , \tag{2.12}$$

where $\omega_d = \exp\{\frac{2\pi i}{d}\}$ is the primitive d-th root of unity, the symbol "$\oplus$" denotes sum modulo d and $\mu, \nu = 0, \ldots, d-1$. Note that for $d = 2$ we obtain the usual Bell basis. Moreover, the set of maximally entangled states $\{|\phi_{\mu\nu}\rangle\}$ constitutes a basis for the Hilbert space associated with a pair of qudits, $\mathcal{H}_d \otimes \mathcal{H}_d$.

In addition, it is always possible to transform a maximally entangled state into another one by LOCC. In fact we can obtain $|\phi_{00}\rangle$ from any state $|\phi_{\mu\nu}\rangle$ by applying the conjugate of the Weyl operator $\hat{U}_{\mu\nu}$ on the second part:

$$\left|\phi^d_{00}\right\rangle = \hat{1}_A \otimes \hat{U}^*_{\mu\nu} \left|\phi^d_{\mu\nu}\right\rangle , \tag{2.13}$$

where $\hat{U}_{\mu\nu}$ is defined as:

$$\hat{U}_{\mu\nu} = \sum_{j=0}^{d-1} \omega_d^{\mu j} \left|j\right\rangle\!\left\langle j \oplus \nu\right| . \tag{2.14}$$

It is straightforward to show that for $d = 2$, $\hat{U}_{\mu\nu}$ operators are proportional to the Pauli matrices (see figure 4). In the last section we will make use of these objects to define some kinds of noise on qudit systems.

Note that the more non-null Schmidt coefficients, the larger the contributions to the entropy of entanglement, thus it follows that the associated Schmidt rank r_ϕ constitutes a discrete valued measure of entanglement of a given quantum state $|\phi\rangle$.

Now we give a brief description of the phenomenon of entanglement involving more than two parts.

2.2.2 Multipartite entanglement

The bipartite case represents the less complex manifestation of entanglement. Indeed one of the features motivating this assertion is the fact that one can get any two-qudit state from a maximally entangled one by using LOCC [Vidal 1999, Bose, Vedral e Knight 1999]. Moreover, given that we can transform any maximally entangled state (hereafter MES) into any other state by LOCC, then it is possible to state that there is only one kind of *genuine* bipartite entanglement. Nevertheless, this feature is uniquely observed in the two parts scenario. For the next level of complexity, namely three entangled qubits, given an state one can obtain it from one and only one out of two types of entangled states, or in other words, there are two kinds of genuine three partite entanglement [Dür, Vidal e Cirac 2000]. The first class is composed by states which may be obtained by LOCC on the Greenberger-Horne-Zeilinger state (GHZ) [Greenberger, Horne e Zeilinger 1989], defined as:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right). \tag{2.15}$$

Let us define the GHZ basis with elements $|\phi_{\mu\nu\lambda}\rangle$

$$|\phi_{\mu\nu\lambda}\rangle = \frac{1}{\sqrt{2}}\sum_{j=0}^{1}(-1)^{\mu j}\,|j, j\oplus\nu, j\oplus\lambda\rangle\,, \tag{2.16}$$

where $\mu, \nu, \lambda \in \{0, 1\}$. The GHZ state $|GHZ\rangle$ is related to the elements of the basis as:

$$|\phi_{\mu\nu\lambda}\rangle = \left(\hat{\sigma}_z^{\mu}\otimes\hat{\sigma}_x^{\nu}\otimes\hat{\sigma}_x^{\lambda}\right)|GHZ\rangle\,. \tag{2.17}$$

Here, $\hat{\sigma}_k^m$ indicates m applications of the k-Pauli matrix.

The second class of genuine three-partite entanglement is constituted by the so called W states [Dür, Vidal e Cirac 2000, Zeilinger, Horne e Greenberger 1997], defined as:

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right). \tag{2.18}$$

As before, we can build a basis composed by 8 states equivalent to $|W\rangle$ under LOCC, however we do not present it here.

An interesting feature emerges if we increase the number of particles by one. Verstraete and collaborators [Verstraete et al. 2002] have shown that any four qubit pure state is equivalent by SLOCC[8] to one out nine *families* of states. Nevertheless, at least 6 of these depend on a set of continuous parameters, and in this way, different to the $N = 3$ case, it is possible to conclude that there are infinite inequivalent SLOCC classes of four qubit entanglement [Dür, Vidal e Cirac 2000, Eltschka e Siewert 2014, Regula et al. 2014].

More formally, a genuinely entangled state may be defined as a state which cannot be split as a product of two arbitrary partitions i.e. $|\psi\rangle \neq |\phi\rangle \otimes |\varphi\rangle$. This definition may be further extended to mixed states: A genuinely N-partite entangled mixed state $\hat{\varrho}$ is any state indecomposable into arbitrary partitions $\hat{\varrho}_{A_j}$ and $\hat{\varrho}_{B_j}$, as:

$$\hat{\varrho} = \sum_j p_j \hat{\varrho}_{A_j} \otimes \hat{\varrho}_{B_j}. \tag{2.19}$$

Note that the definition above is different from that no separable state because in this, each partition $\hat{\varrho}_{C_j}$ may be composed by several subsystems which in addition can be entangled, if so we can say that $\hat{\varrho}$ is entangled, but not genuinely.

Roughly speaking, the task of determining whether a quantum state is genuinely entangled is very difficult and its complexity increases with the number of parties and dimensions of each subsystem. Recently, Kraft and collaborators [Kraft et al. 2018] have developed a method to address this problem. Under this approach they were able to define the concept of genuinely entangled multilevel states as those which are not decomposable in lower-dimensional subsystems and extend their results to the multipartite case.

A less restrictive though difficult task is the verification of the presence of entanglement in quantum systems. For this there exist a lot of procedures whose effectiveness depends on the specific features of the state in question. The most well known procedure is the Peres-Horodecki criterion which represents a necessary condition for a bipartite system to be separable [Peres 1996, Horodecki, Horodecki e Horodecki 1996], and specifically for the 2×2 and 2×3 cases, it also constitutes a sufficient condition. Its importance lies on the capability of

[8] Stochastic LOCC.

discriminating entanglement in mixed states, on which it is no longer possible to get a Schmidt decomposition. Remarkably, an extension of the Peres-Horodecki criterion for systems in continuous variables has also been proposed [Simon 2000].

The Peres-Horodecki criterion requires a previous complete knowledge of the system, (i.e. the whole elements of the density matrix $\hat{\rho}$), nevertheless most of the time it is by itself a very hard task from the experimentalist point of view. An alternative is to use hermitian operators whose expectation values are always positive or zero for separable states and negative for certain entangled states and are known as *entanglement witnesses* [Horodecki, Horodecki e Horodecki 1996, Terhal 2000]. A closely related method to identify entanglement is the violation of Bell inequalities [Terhal 2000], by virtue of the Gisin's theorem (see next section). For a complete review on the subject, we refer the reader to [Gühne e Tóth 2009].

The following subsection is devoted to present some states relevant in quantum information from the historic and foundational points of view.

2.2.3 Some special families of states

Supose we have a source of entangled qudits in a state $\hat{\rho}$. We send one qudit to Alice and the other to Bob. Then Alice applies an unitary operation $\hat{U}$ chosen at random and informs Bob to carry out either $\hat{U}$ or $\hat{U}^{*}$[9] on his qudit. Regardless the initial state of the system, after many repetitions of the same procedure, the final state shared by Alice and Bob reduces to a Werner or an isotropic state:

Werner states

When Alice and Bob apply local operations $\hat{U}\otimes\hat{U}$, we have [Werner 1989]:

$$\hat{\rho} \to \int \hat{U}\otimes\hat{U}\;\hat{\rho}\;\hat{U}^{\dagger}\otimes\hat{U}^{\dagger}\mathrm{d}U = \hat{\rho}_{W}, \tag{2.20}$$

9 The symbol "*"indicates complex conjugation of the associated matrix elements.

where $\mathrm{d}U$ is the Haar measure of the unitary group $\mathbf{U}(d)$ and $\hat{\rho}_W$ is the Werner state, given by:

$$\hat{\rho}_W = (1-p)\frac{2}{d^2+d}\hat{P}^{(+)} + p\frac{2}{d^2-d}\hat{P}^{(-)}, \tag{2.21}$$

with $\hat{P}^{(\pm)} = \frac{1}{2}\left(\hat{1} \pm \hat{V}\right)$, where $\hat{1}$ is the identity: $\hat{1} = \sum_{jk=0}^{d-1} |jk\rangle\langle jk|$ and $\hat{V}$ is the flip operator: $\hat{V} = \sum_{jk=0}^{d-1} |jk\rangle\langle kj|$. It is important to mention that the Werner state $\hat{\rho}_W$ is invariant under $\hat{U} \otimes \hat{U}$ operations. By using the following relations $\hat{P}^{(+)}\hat{P}^{(-)} = 0$, $\hat{P}^{(\pm)2} = \hat{P}^{(\pm)}$ and $\mathrm{tr}\,\hat{P}^{(-)} = \frac{1}{2}(\mathrm{tr}\,\hat{1} - \mathrm{tr}\,\hat{V}) = \frac{1}{2}(d^2-d)$, it is easy to show that the operation invariant p [10] is equal to $p = \mathrm{tr}\left(\hat{P}^{(-)}\hat{\rho}_W\right)$.

Isotropic states

In the case of local operations $\hat{U} \otimes \hat{U}^*$, the state $\hat{\rho}$ is transformed as [Horodecki e Horodecki 1999]:

$$\hat{\rho} \to \int \hat{U} \otimes \hat{U}^* \; \hat{\rho} \; \left(\hat{U} \otimes \hat{U}^*\right)^\dagger \mathrm{d}U = \hat{\rho}_f, \tag{2.22}$$

where $\hat{\rho}_f$ is the isotropic state, given by:

$$\hat{\rho}_f = \frac{1-f}{d^2-1}\hat{1} + \frac{fd^2-1}{d^2-1}\hat{P}_+, \tag{2.23}$$

with $\hat{P}_+ = |\phi_{00}^d\rangle\langle\phi_{00}^d|$. Analogously to the Werner state, the isotropic state is invariant under $\hat{U} \otimes \hat{U}^*$ operations.

By expanding the identity operator in the generalized Bell basis $\hat{1} = \sum_{\mu\nu=0}^{d-1} |\phi_{\mu\nu}^d\rangle\langle\phi_{\mu\nu}^d|$, the isotropic state takes the form:

$$\hat{\rho}_f = f\,|\phi_{00}^d\rangle\langle\phi_{00}^d| + \frac{1-f}{d^2-1} \sum_{\substack{\mu\nu=0 \\ (\mu,\nu)\neq(0,0)}}^{d-1} |\phi_{\mu\nu}^d\rangle\langle\phi_{\mu\nu}^d|\,. \tag{2.24}$$

In this expression it is possible to see more clearly that the operation invariant f is equal to: $f = \mathrm{tr}\left(|\phi_{00}^d\rangle\langle\phi_{00}^d|\,\hat{\rho}\right) = \mathrm{tr}\left(|\phi_{00}^d\rangle\langle\phi_{00}^d|\,\hat{\rho}_f\right)$.

[10] Particularly in the case $d = 2$, p denotes the singlet fraction or in other words the degree of similarity of the state before ($\hat{\rho}$) and after ($\hat{W}_p$) twirling operations, to the singlet state $|\phi_{11}^2\rangle$, i.e. $p = \mathrm{tr}\left(|\phi_{11}^2\rangle\langle\phi_{11}^2|\,\hat{\rho}\right) = \mathrm{tr}\left(|\phi_{11}^2\rangle\langle\phi_{11}^2|\,\hat{W}_p\right)$.

2.2.4 Applications of entanglement

The advent of the so called *second quantum revolution* has been possible due to key theoretical developments and applications of entanglement in quantum systems, let us mention some of the most relevant:

In 1984 Charles Bennet and Guilles Brassard presented a quantum key distribution protocol (BB84) [Bennett e Brassard 1984], which under ideal conditions is able to exhibit an improved performance when compared to the case in which only classical resources are employed. Moreover, in this it is possible to detect an eavesdrop attack due to the fundamental 5fact that once a quantum system is measured its state is modified.

Another interesting application using quantum resources is known as *super-dense coding* [Bennett e Wiesner 1992]. Consider Alice and Bob sharing a maximally entangled state $|\phi^2_{00}\rangle$, she performs one out of the following operations on her qubit $\{\hat{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ and sends it to Bob, who carries out a Bell measurement on the pair. It is easy to show that there is a one to one correspondence between the measurement outcomes obtained by Bob and the operations performed by Alice (see equation 2.13). In this way Alice by sending only one quantum bit can transmit 2 classical bits, attaining a compression factor of two per message. We devote one chapter of this thesis to present results of noise detection in protocols like this when it is made use of GHZ states and measurements instead of bipartite counterparts (Bell).

One of the most known applications of entanglement is the so called *quantum teleportation* [Bennett et al. 1993]. In this case Alice and Bob share an entangled state, additionally Alice possesses a qubit in a state that may be unknown. By carrying out a Bell measurement on her particles and using a classical channel to send the information about her outcome, Bob's qubit can recover the state of Alice's particle. In other words Alice and Bob employed a shared entangled state and LOCC to transmit the quantum state of a particle without the need to transmit it physically. We extend this idea in a subsequent section and present several results in the last two chapters of this thesis.

Another related protocol is the *entanglement swapping* [Zukowski et al.

1993]. In this it is possible to entangle two qubits which have never interacted but share entanglement with a third party that performs Bell measurements on his qubits and sends classically the information about the outcomes to one of the parties.

The next section is focused on a phenomenon closely related to entanglement, i.e. *quantum nonlocality*.

2.3 Bell Nonlocality

To date there exist several compilations in the area from diverse perspectives. Remarkably, in [Reid et al. 2009] it is given a focus mainly on experimental implementations. Buhrman and collaborators review the relationship between quantum information science and fundamental concepts in quantum mechanics like nonlocality in [Buhrman et al. 2010]. The most recent (and probably the most known) is the "Bell Nonlocality" review by Nicolas Brunner and co-workers [Brunner et al. 2014]. Finally, in Gláucia Murta master's dissertation it is presented a series of mathematical tools essential to understand the geometrical structure behind quantum nonlocality [Murta 2012].

Although the study of Bell nonlocality was initially motivated by the Einstein-Podolsky-Rosen (EPR) paradox [Einstein, Podolsky e Rosen 1935,Bohm e Aharonov 1957] and posteriorly by the work of Bell [Bell 1964], here we adopt an alternative route to introduce the concept and after we review the most important ideas behind these seminal papers. Let us start the discussion evoking one of the most intuitive ways to see nonlocality in action: The CHSH[11] game:

2.3.1 The CHSH game

Imagine two players, Alice and Bob as usual, who are cooperating to win a game against an external referee. In each round the referee sends a pair of

[11] CHSH, after John Clauser, Michael Horne, Abner Shimony and Richard Holt, responsible for the derivation of the unique tight Bell inequality for the scenario $2, 2, 2$ (Two parts, two inputs each with two possible outcomes) [Clauser et al. 1969]. In the forthcoming paragraphs, some technical details above mentioned will become more clear.

classical bits (x, y) (questions) separately to Alice and Bob, and immediately after, each of them sends back a bit a and b respectively (answers). They receive a point whenever the relation $a \oplus b = x \cdot y$ is satisfied, where the symbol "$\oplus$"indicates sum modulo 2. The only constraint in the game is that as soon as it begins, both players are forbidden to interchange any kind of information (including the questions sent by the referee). Nevertheless, they can previously agree on an strategy such as a set of random shared bits, or simply as to ignore the questions and both always provide the same answer i.e. $a = b = 0$. In fact, it is easy to see that if the referee chooses the questions from a random uniformly distributed sample, the last mentioned strategy is optimal, given that they can win three out of four times and there is no way to overcome the limit $p_{win} = 75\%$ by only using classical strategies. Moreover, Alice and Bob may adopt another strategy to win the game with a higher probability (even keeping the *no-signalling* restriction, defined in the next section), that is, by using a quantum resource: pairs of maximally entangled qubits and local incompatible measurements. In particular, consider the players sharing a state:

$$\left|\phi_{00}^{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right). \tag{2.25}$$

The choice of the measurement basis is conditioned to the question sent by the referee. It is not difficult to show that the optimal set of local measurement basis are given by:

$$|a=0\rangle = |0\rangle \quad \text{and} \quad |a=1\rangle = -\,|1\rangle\,,$$
$$|a=0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad \text{and} \quad |a=1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right),$$

for questions $x = 0$ and $x = 1$ in Alice respectively. For Bob we have:

$$|b=0\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle \text{ and } |b=1\rangle = \sin\left(\frac{\pi}{8}\right)|0\rangle - \cos\left(\frac{\pi}{8}\right)|1\rangle\,,$$

$$|b=0\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle - \sin\left(\frac{\pi}{8}\right)|1\rangle \text{ and } |b=1\rangle = -\sin\left(\frac{\pi}{8}\right)|0\rangle - \cos\left(\frac{\pi}{8}\right)|1\rangle\,,$$

for measurement choices $y = 0$ and $y = 1$, subsequently.

After some calculations, it is possible to show that the probability of winning the game is $p_{win}^{Q} = \cos^2(\frac{\pi}{8}) = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.853$, which is significantly greater than the larger value attainable by only using classical means, 75%!

In this simple example we have shown that nature is capable of providing correlations which are in fact stronger than classical ones by the mere application of the laws of quantum mechanics. Now we turn our attention to a presentation of nonlocality based on the statistics of the outcomes rather than the physical mechanisms responsible of producing them: the *device independent* approach (the interested reader may find very useful the Valerio Scarani's work on the subject [Scarani 2012]).

2.3.2 Device independent approach to nonlocality

The Device Independent (DI) certification program has a huge importance because it only relies on the statistics of a given experiment and it is not necessary to make any extra assumption on the system to be tested[12]. A system under the DI approach is treated as an assortment of black-boxes equipped with buttons (or simply inputs) which after being pushed produce one out of an array of outcomes, in general different in each run of the experiment. At the end of the day, from the set of frequencies it is possible to infer underlying properties of the whole system.

For simplicity and due to the fact that this thesis is mainly devoted to bipartite systems, we present the main features of the device independent scenario restricted to the $N = 2$ case. Nevertheless, the N-partite generalization is straightforward.

Consider a system composed by two parts, Alice and Bob each possessing a black-box as in figure 1. At each run of the experiment, Alice chooses an input x out of m_A options and likewise, Bob selects his input y from m_B alternatives. Subsequently their boxes give back outputs a and b respectively. From the frequencies and after many repetitions Alice and Bob can build good estimates of the set of joint probabilities $\{p(ab|xy)\}$. Here we adopt the standard convention for the outputs labels in which they can attain discrete values between 0 and $d_k - 1$, where k stands for the part of the system, say Alice or Bob. Although, in this work we do not consider subsystems with different number of outputs e.g.

[12] The only physically motivated restriction imposed on the system is that communication among parts is forbidden, thus in agreement with relativity. This is also known as the *non-signalling* condition, introduced more formally later.

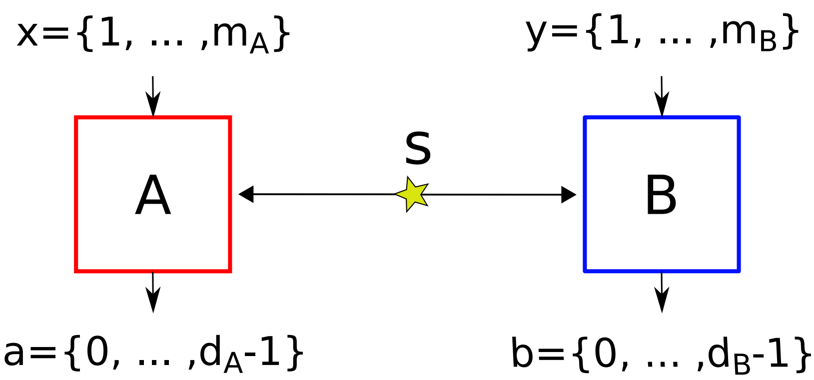

Figure 1 – Usual Bell nonlocality bipartite scenario. Alice's (Bob's) box has m_A (m_B) possible inputs and d_A (d_B) potential outcomes. After many rounds of the experiment they can construct the complete set of joint probabilities $\{p(ab|xy)\}$ from which it is possible to infer the presence of a shared resource S and even to build a characterization of this.

$d_A = d_B = d$, we tend to maintain the generality in the introductory part. The vector $\mathbf{p}(ab|xy)$ with elements $\{p(ab|xy)\}$ is often called a *behaviour* and the set of all possible behaviours define the probabilities space $\mathcal{P}$, which may be seen as the "positive hyperoctant" (simplex) in a $m_A m_B d_A d_B$-dimensional flat space. The first an most natural constraint on $\mathcal{P}$ is the normalization:

$$\sum_{a,b=0}^{d_A-1,d_B-1} p(ab|xy) = 1 \ \ \forall\ x, y. \tag{2.26}$$

Given that we have in total $m_A m_B$ equations, then the effective dimensionality is reduced to $m_A m_B (d_A d_B - 1)$.

From Einstein's theory of relativity we know that instantaneous transmission of information is fundamentally forbidden, thus in order to take into account all kind of possibilities, even those events in which Alice and Bob give their inputs to their boxes are separated by space-like intervals, then the outputs of a given part cannot depend of the choice on the other. It lead us to the so called *no-signalling* conditions: Alice's outcomes do not depend on the choice made by Bob:

$$\sum_{b=0}^{d_B-1} p(ab|xy) = \sum_{b=0}^{d_B-1} p(ab|xy') = p(a|x), \quad \forall\ a, x, y, y'. \tag{2.27}$$

And conversely for Bob, we have:

$$\sum_{a=0}^{d_A-1} p(ab|xy) = \sum_{a=0}^{d_A-1} p(ab|x'y) = p(b|y), \quad \forall\, b, x, x', y. \tag{2.28}$$

These restrictions lead also to a decrease in the effective dimensionality of $\mathcal{P}$.

Given two arbitrary vectors $\mathbf{p}_1(ab|xy), \mathbf{p}_2(ab|xy) \in \mathcal{P}$, satisfying normalization and no-signalling conditions, any convex sum of them is also a vector in $\mathcal{P}$ and fulfils the same requirements above, then, the set of no-signalling correlations constitutes a convex set. Additionally, it is possible to show that normalization and no-signalling constraints induce the existence of a *finite set of vertices* or extremal points[13], therefore the no-signalling constraint defines a *convex polytope*. The no-signalling polytope $\mathcal{NS}$ has a huge relevance because it imposes an *outer bound* on models trying to explain nature.

An additional constraint, inspired by our newtonian-like perception of nature is given by the *local realism* constraint. Locality is a natural restriction which arises from the notion that nothing can alter the properties of something that is separated enough and the realism prescription suggests that the outcomes of a given measurement exists even prior to the measurement itself. According to these ideas, the most basic probability distribution following the local realism condition may be written as: $p_L(ab|xy) = \delta_{a,f(x)}\delta_{b,g(y)}$, where f and g are auxiliary functions which account for the determinism in the outputs. However, we can generalize this condition and instead of fixed values, a and b are determined now by probability distributions. Furthermore, apart from the inputs we can consider a set of factors λ influencing the final outcomes, in general unknown to us and for this reason these are often called *hidden variables*. Thus, if a joint probability can be constructed from a *Local Hidden Variables Model* (LHVM), we can write it as:

$$p_L(ab|xy) = \int_\Lambda q(\lambda)p(a|x,\lambda)p(b|y,\lambda)d\lambda, \tag{2.29}$$

where $q(\lambda)$ is the probability distribution of the hidden variables and satisfies normalization as usual $\int_\Lambda q(\lambda)d\lambda = 1$. An important element here is that we

[13] Extremal points in a convex set are all points which cannot be written as a convex combination of other points inside the set (p. 30, [Murta 2012]).

assume *free will* in the choice of the inputs, given that they enter in the LHVM as free parameters and do not depend on particular values of λ, for instance. It can be shown that probability distributions satisfying 2.29 form a convex set and furthermore the set of deterministic strategies describe extremal points on it, then as in the case of the no-signalling correlations[14], the set of probability distributions admitting a LHVM constitute a polytope, called the *local polytope* $\mathcal{L}$. In order to determine whether a given behaviour $p(ab|xy)$ admits a LHVM description ($p(ab|xy) \in \mathcal{L}$), one can find the related hyperplane equations corresponding to the facets of the local polytope and test if the given probability is inside or not. In general terms, those constraints may be written as [Brunner et al. 2014]:

$$\mathbf{s} \cdot \mathbf{p} = \sum_{abxy} s_{xy}^{ab} p(ab|xy) \leq S_L. \tag{2.30}$$

Expressions like this are known as *Bell inequalities*, where **s** is a vector with elements $s_{xy}^{ab} \in \mathbb{R}$ and $S_L = \max_{\mathbf{p}\in\mathcal{L}} \mathbf{s} \cdot \mathbf{p}$, is the locality bound. The inequality is satisfied whenever the probability distribution can be decomposed as eq. 2.29, i.e. $p(ab|xy)$ allows a local hidden variable model description. The following section is devoted to present several examples of Bell inequalities.

Now we proceed to describe the set formed by behaviours resulting from predictions of quantum mechanics: the *quantum set* $\mathcal{Q}$, which comprises probability distributions that may written as:

$$p_Q(ab|xy) = \mathrm{tr}\Big(\hat{M}_{a|x} \otimes \hat{M}_{b|y}\, \hat{\rho}_{AB}\Big), \tag{2.31}$$

where $\hat{\rho}_{AB} \in \mathcal{HS}_A \otimes \mathcal{HS}_B$ is the density operator of the system shared between Alice and Bob, and $\{\hat{M}_{a|x}\}$ is the positive operator valued measurement (POVM) associated with the input choice x for Alice's subsystem, with $\hat{M}_{a|x}$ a positive semidefinite operator acting on $\mathcal{H}_A$, satisfying $\sum_a \hat{M}_{a|x} = \hat{1}_{d_A}$ for all x (analogous relations hold also for Bob's POVM). It is important to note that the amount of operators in the POVM is not necessarily equal to the dimensionality of the system. However, in the case of conventional von-Neumann measurements in which the measurement operators represent rank-1 projectors $\hat{M}_{a|x}^{(p)} = |a\rangle\langle a|_x$, these two

[14] More precisely the set of vertices in $\mathcal{NS}$ is constituted by the local deterministic strategies (extremal points of $\mathcal{L}$) and the so called PR boxes which we will explain latter.

numbers must coincide in order to ensure that the total probability is always equal to one. For this the joint conditional probability reads:

$$p_Q(ab|xy) = \mathrm{tr}\Big(|a\rangle\langle a|_x \otimes |b\rangle\langle b|_y \ \hat{\rho}_{AB}\Big). \tag{2.32}$$

The work presented in this thesis makes use of projective measurements only, then the expression above will be very useful in several contexts.

It can be proven that given two arbitrary behaviours inside the quantum set, a convex sum of them is also in $\mathcal{Q}$, thus we can conclude that the quantum set is convex. Moreover, differently from its local and no-signalling counterparts, the quantum set possesses an infinite number of extremal points, so it is not a convex polytope and for this reason its characterization is more complex. Several efforts have been made in this direction. From a numerical perspective, Navascués, Pironio and Acín (NPA) introduced a method based on semi-definite programming able to find the outer boundary of the quantum set [Navascués, Pironio e Acín 2007], for recent developments and extensions see [Navascués, Torre e Vértesi 2014, Navascués e Vértesi 2015, Navascués et al. 2015]. From an analytical point of view, de Vicente derived a set of conditions that a given behaviour inside the quantum set must satisfy [De Vicente 2015]. Recently, by using tools of convex geometry it has been presented a characterization of $\mathcal{Q}$ for the $(2, 2, 2)$ scenario (two parts, two inputs and two outcomes per part), in which the authors found several interesting features like flat boundaries and non-exposed extremal points, in addition they show that more complex scenarios may lead to other previously unexpected properties. For details, we refer the interested reader to [Goh et al. 2018].

After introducing $\mathcal{NS}$, $\mathcal{L}$ and $\mathcal{Q}$, a natural question that arises is how they are distributed in $\mathcal{P}$ and whether exists a kind of hierarchy between them, in figure 2 we show a section of the space of behaviours and corresponding constraints for the scenario $(2, 2, 2)$. It can be shown [Murta 2012] and as we can see in figure 2, the three sets are related as $\mathcal{L} \subsetneq \mathcal{Q} \subsetneq \mathcal{NS}$. It is important to note that due to the existence of particular kinds of behaviours the inclusion is strict. For instance the condition $\mathcal{L} \subsetneq \mathcal{NS}$ is satisfied because of the presence of the so called Popescu-Rohlrich behaviours. Such probability distributions can be generated by PR-Boxes,

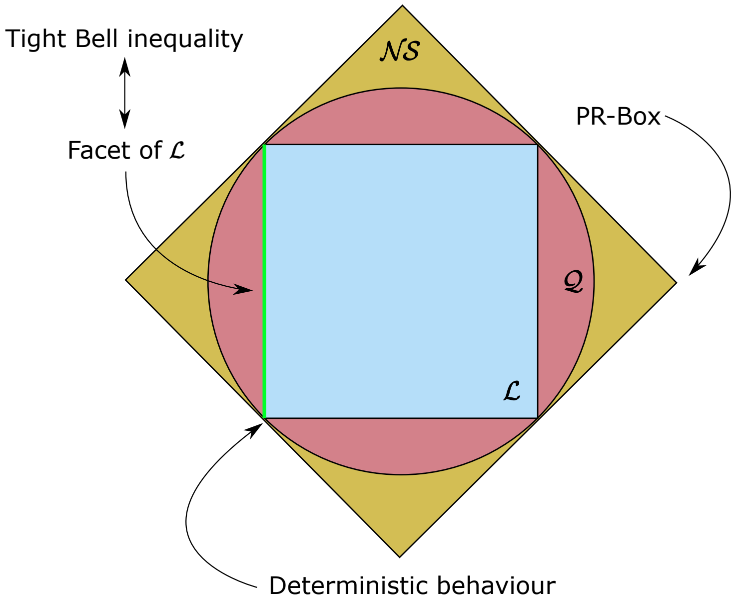

Figure 2 – Pictorial representation of the space of behavious and constraints on it. $\mathcal{Q}$, $\mathcal{L}$ and $\mathcal{NS}$ represent the quantum set, local and no-signalling polytopes. The green line describes a facet of the local polytope for which we can always associate a Bell inequality. The extremal points of the local polytope are local deterministic strategies and those corresponding to the no-signalling part outside the local region are known as Popescu-Rohrlich-Boxes (PR-boxes). Note that the relation $\mathcal{L} \subsetneq \mathcal{Q} \subsetneq \mathcal{NS}$ is satisfied.

hypothetical objects capable of generating outputs correlated in such a way that violate a determinate Bell inequality up to its maximal algebraic limit. Although proposed by Rastall, Khalfin and Tsirelson [Khalfin e Tsirelson 1985,Rastall 1985], the PR-boxes receive their name after Popescu and Rohlrich's pioneering attempt to formalize quantum theory from first principles instead of the usual postulates [Popescu e Rohrlich 1994]. On the other hand, it is straightforward to show that any behaviour inside the local polytope may be "simulated by using quantum resources" (a separable state plus local measurements), i.e. for any $p_L(ab|xy)$, it is always possible to find a decomposition leading to $p_L(ab|xy) = p_Q(ab|xy)$. Nevertheless the converse is not true, and actually there are quantum behaviours generated by the use of *entangled states* and properly chosen local *non-commuting measurement operators* leading to violation of Bell inequalities (eq. 2.30), and in this way outside the local polytope. From this we can conclude that $\mathcal{L} \subsetneq \mathcal{Q}$.

Behaviours violating Bell inequalities are usually termed as *nonlocal*[15] due to the impossibility of finding a model based on local hidden variables to explain them[16].

The EPR Argument and Bell's Theorem

The result above constitutes the basis of Bell's theorem which establishes that there is no physical theory based on local hidden variables capable of reproducing all the predictions of quantum mechanics[17] [Bell 1964]. Bell's conclusion came after almost thirty years since the publication of the seminal paper by Einstein, Podolsky and Rosen[18] [Einstein, Podolsky e Rosen 1935], today known as the EPR argument. In their work, EPR conceived an experiment in which Alice and Bob share an ensemble of entangled pairs of qubits, and each of them is able to perform local measurements. Under this scenario, the measurement events are separated by space-like intervals. At each instant Alice may choose one out of two incompatible observables $\hat{A}_1$ or $\hat{A}_2$ (analogously for Bob, $\hat{B}_1$ or $\hat{B}_2$). Assuming that any local action on each particle cannot influence its counterpart (locality), and that measurement results pre-exist for any observable independent of the choice (realism)[19], they were able to show that under special cases concerning systems with a high degree of symmetry, two local measurements (one in Alice and the other in Bob's location) allow for the determination of the values associated to the four involved observables, and thus in contradiction with Heisenberg's uncertainty principle. EPR concluded that there is no way in which QM satisfy the local realism assumption, and then there should exist a more general theory possibly described by a set of hidden variables (not available for the experimenter), in analogy to the relation between thermodynamics and statistical mechanics, in

[15] Some authors rather use the term *nonclassical* instead of *nonlocal* correlations.

[16] In fact in order to reproduce an arbitrary quantum correlation it is necessary to use a model based on *nonlocal* hidden variables only [Colbeck e Renner 2008].

[17] A very interesting extension of Bell's statements is given by the so called Fine's Theorem [Fine 1982, Fine 1982].

[18] An alternative and more friendly formulation of the EPR argument was given by Bohm and Aharonov in 1957 [Bohm e Aharonov 1957] in terms of discrete degrees of freedom (spin), instead of continuous variables as originally proposed. Moreover Bell's deduction is built following Bohm and Aharonov's presentation.

[19] The junction of both premises is known as *local realism* assumption.

which the position of particles in the phase space play the role of hidden variables. Inspired by this, Bell derived a set of conditions (Bell inequalities) satisfied by predictions from any theory based on LHV, which as mentioned before, quantum mechanics violates under certain scenarios. Since then many efforts have been concentrated to experimentally test QT against local realism hypothesis, with a vast majority in favor of the first one. In an upcoming section we briefly discuss some of these experiments along with the so called “loopholes" closed up to date.

The conclusion we can take from Bell’s theorem is that by the usage of systems that even spatially separated by large distances behave like a single one, we can observe how nature is able to exhibit correlations which are stronger than those obtained when classical resources are used, and most important, those correlations do not violate any physical principle and in this way, features like faster than light communication are not allowed.

Back to the quantum set, it is not hard to show that quantum correlations satisfy the no-signalling constraints (equations 2.27 and 2.28). In addition Tsirelson showed that joint probabilities from quantum mechanics cannot exceed the value $\max_{\mathcal{Q}} I_{CHSH} = 2\sqrt{2}$ (*Tsirelson limit*), for the Bell-CHSH inequality [Clauser et al. 1969], instead of its maximal algebraic value, $\max_{\mathcal{NS}} I_{CHSH} = 4$ [Cirel’son 1980]. Then we can safely ensure that the relation $\mathcal{Q} \subsetneq \mathcal{NS}$ is satisfied. This result has inspired many efforts in order to understand what physical principles are behind the Tsirelson bound and why nature does not exhibit violation of Bell inequalities up to their algebraic limit, we refer the interested reader to references [Pawłowski et al. 2009, Masanes e Müller 2011, Popescu e Rohrlich 1994, Navascués e Wunderlich 2010].

2.3.3 *Bell inequalities*

From a geometrical point of view, tight Bell inequalities are known to be hyperplane equations describing the boundary of the local polytope (facets), then we can use them to discriminate local from nonlocal behaviours[20]. In this section

[20] Note that not every Bell inequality is tight i.e. doesn’t represent a facet of $\mathcal{L}$, and though helpful in the task of nonlocality detection those are not optimal i.e. by employing non-tight inequalities it is possible that a fraction of the nonlocal correlations remain unrevealed. The

we present the Bell inequalities that we will employ throughout this thesis. A compilation of Bell inequalities for diverse scenarios may be found in [Brunner et al. 2014].

CHSH inequality

Let us start by the simplest scenario: two parts, two inputs and two outcomes $(2, 2, 2)$. In this case, John Clauser, Michael Horne, Abner Shimony and Richard Holt (CHSH) derived an inequality for a system composed by two spin-$1/2$ entangled particles under Pauli measurements [Clauser et al. 1969], which is equivalent to:

$$-2 \leq E(\mathbf{m}, \mathbf{n}) + E(\mathbf{m}, \mathbf{n}') + E(\mathbf{m}', \mathbf{n}') - E(\mathbf{m}', \mathbf{n}) \leq 2, \tag{2.33}$$

where $E(\mathbf{m}, \mathbf{n})$ is the correlation funtion between $\hat{\mathbf{S}}_A \cdot \mathbf{m}$ and $\hat{\mathbf{S}}_B \cdot \mathbf{n}$ measurements performed along directions $\mathbf{m}$ and $\mathbf{n}$ on Alice and Bob's particles respectively, and $\hat{\mathbf{S}}_j = \frac{\hbar}{2}\left(\hat{\sigma}_x^{(j)}, \hat{\sigma}_y^{(j)}, \hat{\sigma}_z^{(j)}\right)$ denotes a vector whose components are the cartesian projections of the spin operator associated to the j-th part. For convenience we relabel the outcomes as $\frac{1}{2} \to 0$ and $-\frac{1}{2} \to 1$, in this way we can write the correlation function $E(\mathbf{m}, \mathbf{n})$ in terms of joint probabilities as

$$E(\mathbf{m}, \mathbf{n}) = \sum_{a,b=0}^{1} (-1)^{a \oplus b} p(a, b|\mathbf{m}, \mathbf{n}). \tag{2.34}$$

Note furthermore that when Alice and Bob possess one of their measurement directions alligned (e.g. $\mathbf{m}' = \mathbf{n}'$) and if we allow perfect anti-correlations, i.e. $E(\mathbf{m}', \mathbf{m}') = -1$, then the CHSH inequality reduces to the first Bell inequality [Bell 1964].

Several expressions equivalent to the CHSH inequality (eq. 2.33) are currently known in the literature. Let us point out the version derived by Clauser and Horne [Clauser e Horne 1974], which is written in function of joint probabilities instead of correlation functions:

$$-1 \leq p(A_x, B_y) - p(A_x, B_{y'}) + p(A_{x'}, B_y) + p(A_{x'}, B_{y'}) - p(A_{x'}) - p(B_y) \leq 0, \tag{2.35}$$

most known instance is the first inequality derived by Bell [Bell 1964].

where $p(A_x, B_y) \equiv p(0, 0|A_x, B_y)$, is the joint probability of Alice and Bob obtain the outcomes labeled by "0" each, given the observables choices A_x and B_y as inputs for both parts respectively.

It is worth mentioning that its maximal violation may be attained by using a maximally entangled state of two qubits subjected to properly chosen local projective measurements [Cirel'son 1980].

I_{mm22} inequalities

The family of I_{mm22} tight Bell inequalities, first proposed by Froissart in 1981 [Froissart 1981] and then explored in [Collins e Gisin 2004, Śliwa 2003], constitutes the generalization of the CHSH inequality ($m = 2$) for the case in which Alice and Bob may choose among m inputs each. In terms of joint probabilities, the I_{mm22} inequality may be written as [Acín, Gisin e Toner 2006]:

$$I_{mm22} = -p(A_1) - \sum_{j=1}^{m-1}(m-j)p(B_j) + \sum_{\substack{j=1 \\ k=1}}^{\substack{m-j+1 \\ m}} p(A_k, B_j) - \sum_{j=2}^{m} p(A_{m-j+2}, B_j) \leq 0. \tag{2.36}$$

The importance of this family relies on its "ability" to reveal several non-classical features impossible to detect if we restrict the scenario to two inputs only. For instance, Collins and Gisin found a family of mixed entangled states which admits a LHVM under a CHSH scenario and projective measurements, but violates the I_{3322} inequality [Collins e Gisin 2004]. Furthermore in [Brunner, Gisin e Scarani 2005] the authors show that for some non-maximally entangled states it is impossible to simulate correlations under a $(2, 3, 2)$ scenario with local operations and a single use of the so called non-local machine (a hypothetical apparatus capable of generating PR-box correlations), even though Cerf and collaborators [Cerf et al. 2005] had previously proven that this kind of task is always attainable when restricted to a CHSH scheme. In other words, the simulation of $(2, 3, 2)$ correlations requires a higher amount of nonlocal resources.

Another interesting feature of the I_{3322} inequality is that as conjectured by Pál and Vértesi [Pál e Vértesi 2010], its maximal violation is only attained by using infinite-dimesional systems as opposed to the CHSH case, in which two-dimensional subsystems are sufficient, as mentioned above.

The complexity associated to the task of finding the corresponding facets of the local polytope increases very quickly with the number of parts, settings and outcomes. Nevertheless, note that for some of the simplest scenarios (in addition to the CHSH), only one more inequality suffices to fully describe it: the I_{3322} for $(2,3,2)$ and the CGLMP inequality (introduced below) for $(2,2,3)$.

CGLMP inequality

The Collins, Gisin, Linden, Massar, Popescu (CGLMP) inequality [Collins et al. 2002] represents a facet of the associated local polytope in scenarios involving d outcomes per part $(2,2,d)$ [Masanes 2003] and may be written as:

$$I_d = \sum_{k=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1}\right) \left\{\mathcal{B}_k - \mathcal{B}_{-(k+1)}\right\} \leq 2, \tag{2.37}$$

where $[x]$ indicates the integer part of x, $\mathcal{B}_k = P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k)$ and $P(A_x = B_y + k)$ is the joint probability that the outcomes corresponding to the observables A_x in Alice and B_y in Bob respectively differ by k, modulo d:

$$P(A_x = B_y + k) \equiv \sum_{j=0}^{d-1} P(A_x = j \oplus k, B_y = j) = \sum_{j=0}^{d-1} P(j \oplus k, j|x,y).$$

Given that entanglement is one of the necessary resources to reveal nonlocality, one could in principle argue that a maximally entangled state leads to maximal violation of this inequality for arbitrary dimension d, however Acín and collaborators were able to show that this is not the case for $d > 2$ [Acín et al. 2002] (It is important to mention that the same results were obtained later by employing a different approach [Navascués, Pironio e Acín 2007]). Such a discrepancy among maximally entangled states and states that maximally violate the CGLMP inequality together with other results regarding quantification of nonlocality (wich we will explain later) became known in the literature as an anomaly of nonlocality [Méthot e Scarani 2007]. The next two chapters are devoted to present a proposal to quantify nonlocality in which this divergence does not appear anymore.

It is important to note that both CGLMP (for $d = 2$) and the I_{2222} reduce to the CHSH inequality, as expected[21].

Currently there are several Bell inequalities for the multipartite scenario. For instance in [Laskowski et al. 2004] the authors present inequalities for arbitrary number of measurement settings. More interestingly, to date (to the knowledge of the author) there is no Bell inequality for the most general case (i.e. arbitrary parts, inputs and outcomes). A complete list of known Bell inequalities may be found in Bell nonlocality's review [Brunner et al. 2014] and the "*Zoo of Bell inequalities*" section in Jeong-Cherng Liang's thesis [Liang 2008].

2.3.4 Quantification of nonlocality

As we have already mentioned, there is plenty of tasks whose performance may be enhanced by the usage of quantum resources like entanglement. Nevertheless, in several cases the mere presence of this feature is not enough, it is needed to be able to generate nonlocal correlations as well. In addition, it has been proved from several perspectives that entanglement and nonlocality represent strictly different resources [Brunner, Gisin e Scarani 2005, Quintino et al. 2015, Augusiak et al. 2015], in fact the former is an ingredient of the later, but its presence does not guarantee the observation of nonclassical correlations, given that it is necessary to be able to perform non-commuting local measurements on the involved parties. For this reason it is very important to have a way to assess how nonlocal a given system is. A possible approach consists in treating the problem from the point of view of resource theories [Brandão e Gour 2015]. Under this program the objects which possess the property are defined, together with the rules a proper measure must satisfy and the free operations under which the resource is not increased. The resource theory of entanglement is the most known and well established of this kind [Vedral e Plenio 1998, ao e Plenio 2008] and has served as a reference to develop a wide variety of analogous constructions for quantities

21 However note that in order ensure agreement between $mm22$ and CHSH expressions, we have to take into account the lower bound in the latter, namely $S_{L<} = -1$.

like coherence [Baumgratz, Cramer e Plenio 2014], quantum steering [Gallego e Aolita 2015], contextuality [Amaral et al. 2018], among others. Regarding Bell nonlocality, several efforts have been put towards this formulation, for further details we refer the reader to [Gallego et al. 2012, Vicente 2014, Gallego e Aolita 2017].

Let us explore several quantifiers of nonlocality:

Maximal algebraic value of the Bell inequality - Robustness against noise

Due to its simplicity, it is quite usual to find works in the literature relating the maximal violation of a Bell inequality to the amount of nonlocality. However as pointed out in [Brunner et al. 2014], care must be taken in order to ensure that such a figure of merit in fact have a physical meaning (a possible solution consists in carrying out appropriate rescaling and normalization).

Furthermore, it is possible to show that the maximal violation of a Bell inequality is intimately linked to the robustness of nonlocal correlations when submitted to quantum noise. For instance assume the system is initially prepared in a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and then it is subjected to white noise. The state of the system becomes $\hat{\rho} = (1-\lambda)\,|\psi\rangle\langle\psi| + \hat{1}_D\;\lambda/D$, where λ is the noise fraction, $D = \dim \mathcal{H}_A \otimes \mathcal{H}_B$ and $\hat{1}_D$ is the $D \times D$ unity matrix. It is straightforward to prove that the second part of $\hat{\rho}$ admits a local hidden variables model description and in this way has a null contribution to the Bell function. On the other hand if we define S_ψ as the maximal value of a given Bell inequality when the pure state $|\psi\rangle$ is used, then it is not difficult to show that above a noise threshold given by

$$\lambda^* = \frac{S_\psi - S_L}{S_\psi}. \tag{2.38}$$

The arrangement in question is not able to exhibit Bell nonlocality anymore, where S_L denotes the local limit associated with the Bell inequality. This criteria has been applied to test the nonlocality of multidimensional systems [Kaszlikowski et al. 2000, Durt, Kaszlikowski e Żukowski 2001, Acín et al. 2002], results are discussed in the next chapter.

Other measures of nonlocality

In addition to the noise robustness and maximal violation of a Bell inequality, there exist several interesting approaches to quantify the amount of nonlocality of a given setup. Let us explore briefly some of them.

Recently it has been proposed a nonlocality quantifier based on the minimum trace distance between a given behaviour and the local polytope [Brito, Amaral e Chaves 2018], which satisfies the whole requirements to be an operational quantifier [Vicente 2014]. Particularly for the $(2, 2, d)$ scenario the authors found that this measure is proportional to the maximal violation of a scaled-normalized version of the CGLMP inequality, thus results concerning the optimal state and measurements are the same as in the former case.

Also known as classical relative entropy, the *Kullback–Leibler* (hereafter KL) divergence measures "how separated" are two probability distributions in terms of information. In a statistical test it can be interpreted as the average support of a hypothesis against another. In [Dam, Gill e Grünwald 2005] (see also [Acín, Gill e Gisin 2005]), roughly speaking the authors use the KL divergence to test the nonlocality of a given probability distribution produced in a Bell experiment, testing the hypothesis $\mathcal{Q}$ against $\mathcal{L}$. Interestingly, they obtain the same set of optimal measurements as in the previous measure and it was observed an increase in the amount of nonlocality with the local dimension of the subsystems.

A more complete list of nonlocality quantifiers may be found in [Vicente 2014]. See also [Bernhard et al. 2014] for a experimental-theoretical comparison of some proposed measures of nonlocality mentioned in this thesis for pairs of entangled qutrits.

2.3.5 Entanglement and Nonlocality

Although together with measurements incompatibility[22], entanglement is one of the main ingredients needed to generate nonlocal correlations, the relation

[22] Note though that even possessing an arbitrary entangled state, measurement incompatibility does not necessarily imply violation of a Bell inequality [Quintino et al. 2016,Hirsch, Quintino e Brunner 2018, Bene e Vértesi 2018].

between these two quantities is very subtle and more complex than first thought.

Not every entangled state is useful to reveal nonlocality

The first and more clear observation is that the presence of entanglement is necessary for the existence of nonlocal correlations, for it can be easily shown that if we use a separable state $\hat{\rho}_{AB} = \sum_j \hat{\rho}_j^A \otimes \hat{\rho}_j^B$, then it will be possible to describe any quantum correlation (eq. 2.31) in terms of a LHVM (eq 2.29). Nevertheless the converse is not always true i.e. not all entangled states are able to reveal nonlocality[23]. Regarding the simplest case: pairs of entangled qubits in a pure state $|\psi\rangle = \alpha^2 |00\rangle + (1-\alpha^2)|11\rangle$, Gisin proved that any state of this kind may violate the CHSH inequality whenever $\alpha \neq 0, 1$ (non vanishing entanglement) [Gisin 1991]. This result is currently known as *Gisin's theorem* and have been generalized to the case of two [Chen, Deng e Hu 2008], and arbitrary number of qudits [Li e Fei 2010] in pure states. In conclusion, any pure entangled state under proper measurements is capable of generating nonlocal correlations.

The situation turns out to be more complex when we deal with mixed states. For instance, by using the Horodecki's criterion for violation of the CHSH inequality[24] [Horodecki, Horodecki e Horodecki 1995], it can be shown that a two-qubit Werner state (see section 2.2.3) $\hat{\rho}_W(p) = p\,|\phi_{11}^2\rangle\langle\phi_{11}^2| + (1-p)\frac{\hat{1}}{4}$ subjected to local projective measurements is nonlocal whenever $p > 1/\sqrt{2}$ [Werner 1989]. Nonetheless this state is entangled for $p > 1/3$, then there in an interval for p in which there is entanglement but no violation of the CHSH inequality. Subsequent efforts include the construction of a class of LHVM using generalized measurements [Barrett 2002], derivation of Bell inequalities more efficient than the CHSH one revealing nonlocality [Vértesi 2008], and more general classes of LHVM [Acín, Gisin e Toner 2006, Hirsch et al. 2017] closely

23 This assertion is valid at least under the usual Bell scenario where the inputs and outputs may be conceived as classical bits. For the case in which the inputs are quantum, Buscemi showed that any entangled state is able to reveal nonlocality [Buscemi 2012]. Note also that some local states are capable of violating a Bell inequality after certain processes (like filtering), and in this way revealing some kind of hidden nonlocality [Popescu 1995, Gisin 1996].

24 Do not confuse with the Peres-Horodecki separability criterion [Horodecki, Horodecki e Horodecki 1996].

related to a mathematical problem: The determination of the value for the Grothendieck's constant. The complete characterization of Werner and isotropic entangled states is still an open problem given that for certain parameter regions it is not known how to build a LHVM explaining the correlations produced nor violate a Bell inequality. For a revision on the subject see [Augusiak, Demianowicz e Acín 2014].

An anomaly of nonlocality

As we have seen, due to Gisin's theorem we can guarantee that any entangled pure state is capable of producing non-classical correlations, and given that entanglement is one of the resources of nonlocality, then one could in principle argue that states possessing a larger amount of entanglement lead to stronger correlations. This statement is valid for systems composed by two qubits when we employ any of the operational measures of nonlocality mentioned in a previous section i.e. the most nonlocal state corresponds to the maximally entangled one, say any of the Bell states. When we increase the local dimension of the subsystems it is observed that the coincidence between maximal entanglement and quantum nonlocality ceases to happen. The first observation of this feature was done by taking into account the maximal violation of the CGLMP inequality (or equivalently, the noise robustness) [Acín et al. 2002] and this value is also proportional to the recently proposed *trace distance* to the set of classical correlations as a nonlocality quantifier [Brito, Amaral e Chaves 2018], for the specific case of two entangled qudits. Remarkably, by using the KL divergence as figure of merit [Acín, Gill e Gisin 2005], the more nonlocal state is neither the maximally entangled nor that leading to the maximal violation of the CGLMP inequality but another one, suggesting that the task of assessing how nonlocal a state depends strongly on the potential applicability from the point of view of the measure employed. Note also that there are several proposed measures of nonlocality leading to the same results mentioned above that we do not mention in this thesis for the sake of simplicity. Such a disagreement between maximal entanglement and nonlocality has been known since then in the literature as an *anomaly of nonlocality* [Méthot e Scarani 2007]. A very interesting characteristic not mentioned until now is that under the quantifiers cited above, the amount of nonlocality increases with the dimension of the involved subsystems.

From an experimental point of view it has also been shown that Bell experiments carried out using imperfect detectors require weaker entangled states to violate optimally the CHSH inequality [Eberhard 1993]. For recent developments see [Dilley e Chitambar 2018] and references therein.

Another very interesting manifestation of nonclassicality was given in 1993 by Hardy who derived a set of relations satisfied by joint probabilities respecting local determinism [Hardy 1992, Hardy 1993]. Hardy's argument constitutes the simplest proof of the impossibility to explain quantum predictions in terms of a local hidden variables model. A very short and intuitive derivation may be found in [Goldstein 1994]. The most impressive result is that nonmaximally entangled states lead to the larger deviation from predictions of local deterministic models and remarkably when a maximally entangled state is employed, no deviations from classical predictions are observed. Extensions have been proposed for high dimensional systems [Chen et al. 2013], three partite states [Cabello 2002] (sometimes refered to as Cabello's nonlocality), arbitrary number of parties [Jiang et al. 2018], and remarkably a device-independent derivation of the bounds for the Hardy's experiment [Rabelo, Zhi e Scarani 2012].

Besides the Hardy's argument, there is another proof of Bell's theorem which does not make use of inequalities, namely the *GHZ paradox* [Greenberger, Horne e Zeilinger 1989] in which the authors were able to show that if results of local measurements on a three partite system prepared in a GHZ state are previously defined, independent of the measurements and outcomes of the other particles, one would get a mathematical contradiction $(1 = -1\ !)$, once again showing the nonclassical nature of quantum correlations.

It is worth to remark that the knowledge about the relation between entanglement and nonlocality in the multipartite scenario is even more obscure because to date it has not been possible to define a proper way to quantify the amount of entanglement of a given general state. Nevertheless, in [Augusiak et al. 2015] the authors construct families of genuine entangled states which are not able to show genuine nonlocality [Brunner et al. 2014], proving in this way the inequivalence between entanglement and nonlocality for a system composed by several parts.

2.3.6 Experimental tests of nonlocality

After the publication of Bell's work, several attempts were carried out in order to test the validity of the predicted correlations by quantum mechanics[25], but it was only in 1982 that Alain Aspect and collaborators were able to perform an experiment under reasonable assumptions [Aspect, Grangier e Roger 1982]. Although the observations of Aspect's experiment favoured quantum predictions, due to the possibility of alternative explanations for the results (loopholes), it is not considered as a conclusive test. There exist several loopholes, let us mention the two most relevant: The *locality loophole* emerges due to the possibility of communication among parts during the intervals choice - outcome detection. In order to avoid it, the experimenter must ensure that it is fundamentally impossible to transmit any kind of information about the measurement choice and outcome from Alice to Bob and vice versa. This kind of problem is usually attacked by using ultra-fast detectors and disposing Alice as far as possible from Bob. Moreover, note that one of the challenges in this kind of experiments is dealing with the unavoidable interaction between particles and environment. This is related to the so called *detection loophole*, which relies on a series of factors like non-ideality of detectors and sources of entangled pairs. Recent years have experienced a technological burst and currently it is possible to overcome the problems mentioned above. Up to the year 2015 it had been possible to perform tests avoiding either the locality or the detection loophole, but not both at the same time. It was in the same year that emerged three independent papers reporting results of loophole-free Bell tests [Hensen et al. 2015, Shalm et al. 2015, Giustina et al. 2015], all of them revealing results in favour of quantum mechanics. Another problem to be overcome during the execution of a Bell test is related to the randomness in the choice of the inputs for Alice and Bob. This

[25] For a historical synthesis of Bell experiments up to the year 2000, we refer the reader to Alain Aspect's review [Aspect 1999].

problem has been recently tackled from two different perspectives, one of them using cosmic photons from opposite sources, thus guaranteeing independence between the measurement choices in each side of the experiment [Handsteiner et al. 2017]. The second approach is based on a random selection made by thousands of people around the world by means of a video game [Abellán et al. 2018]. Bell tests have also been carried out using high dimensional systems, for further details see [Vaziri, Weihs e Zeilinger 2002, Howell, Lamas-Linares e Bouwmeester 2002] and references therein.

2.3.7 Related concepts

Besides Bell nonlocality there are several manifestations of the non-classical nature of quantum correlations. Let us mention briefly two very well known examples:

Steering

The term *steering* was initially introduced by Shrödinger [Schrödinger 1935] in reference to the ability of Alice to induce a change in the state of Bob's particle by carrying out local measurements when both share an entangled state. Even introduced as early as entanglement itself, the concept of quantum steering only came to be formalized in 2007 by Wiseman, Jones and Doherty [Wiseman, Jones e Doherty 2007, Cavalcanti et al. 2009]. It can be used as a semi-device independent method to test the presence of entanglement in which we may trust the results coming from quantum operations in one of the parties. Steering is also an intermediate phenomenon between entanglement and nonlocality, moreover it has been recently shown that there is an inclusion relation between these quantities, with nonlocality being the strongest one [Quintino et al. 2015]. Remarkably, not long ago Gallego and Aolita have formulated a resource theory for steering [Gallego e Aolita 2015]. For a review on the subject with focus on semidefinite programming see [Cavalcanti e Skrzypczyk 2017].

Contextuality

Contextuality refers to the fundamental impossibility of assigning determinate values to a set of observables prior to the measurement act, independent of the order those observables are applied [Bell 1966, Kochen e Specker 1967]. It may be thought as a generalization of nonlocality in which the space-like separation between measurement events on different parts is removed. The analogous version of Bell's theorem for contextuality is given by the *Kochen-Specker theorem* (KS) [Kochen e Specker 1967]. A simple presentation and derivation of the KS theorem may be found in [Cabello 1994]. Regarding the connection among contextuality and nonlocality, Kurzynski and collaborators were able to show that there is a monogamy relation between these two quantities, then it is either observed contextuality or nonlocality, but not both at the same time for the particular scenario in which Alice may perform one out of five dichotomic observables and Bob one out of two [ński, Cabello e Kaszlikowski 2014]. It is important to point that it has been very recently conceived a resource theory of contextuality together with the free wirings any operational measure must satisfy [Amaral et al. 2018]. The interested reader may find useful an unpublished review by Thompson and collaborators [Thompson et al. 2013].

At this point we finish the introduction to Bell nonlocality and turn our attention to quantum teleportation, matter of the last part of this thesis.

2.4 Qudit Teleportation

As mentioned in a previous section, qubit teleportation [Bennett et al. 1993] represents one of the most known applications of quantum entanglement. Let us introduce a more general version of the teleportation protocol for d-dimensional systems (qudits) using non-maximally entangled measurement basis. This protocol involves two parts, Alice and Bob as usual, sharing a pair of entangled qudits described by the density operator $\hat{\rho}_{ch}$[26]. Alice is intended to send an arbitrary unknown qudit state $|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle$ to Bob. For this, she carries out a d^2-outcomes joint projective measurement on her pair of qudits in a Bell-like

[26] The sub-index "ch" stands for channel

basis $\{|\phi^d_{mn}\rangle\}$, with elements given by:

$$\left|\phi^d_{mn}\right\rangle = \sum_{k=0}^{d-1} \omega_d^{k\cdot m} \beta_k \left|k, k\oplus n\right\rangle , \tag{2.39}$$

where the amount of entanglement is controlled by the β coefficients, in this way we recover usual maximally entangled joint measurements whenever $\beta_k = 1/\sqrt{d}$.

By using classical communication, Alice sends the information about her outcome (m, n) to Bob who applies a local unitary operation on his qudit, given by one out of the d^2 Weyl operators $\hat{U}_{mn}$[27]. A schematic representation of the teleportation protocol is given in figure 3. After each run of the experiment, the state of the Bob's part of the system (up to normalization) reads:

$$\hat{\rho}_{mn} = \hat{U}_{mn}\ \mathrm{Tr}_A \left\{ \left(|\phi^d_{mn}\rangle\langle\phi^d_{mn}| \otimes \hat{1}_B \right) |\psi\rangle\langle\psi| \otimes \hat{\rho}_{ch} \right\} \hat{U}^\dagger_{mn}, \tag{2.40}$$

where Tr_A denotes the partial trace on the Alice's part of the system.

The reliability of the protocol is usually assesed by calculating the fidelity of teleportation i.e. how close is the state in Bob after the process to that initially possessed by Alice:

$$F_{mn} = \frac{\mathrm{Tr}\left(\hat{\rho}_{mn} |\psi\rangle\langle\psi|\right)}{\mathrm{Tr}\left(\hat{\rho}_{mn}\right)}. \tag{2.41}$$

This expression gives the fidelity related to the cases in which the measurement outcome (m, n) is obtained. In order to estimate reliability of the whole protocol, we have to take into account the whole possibilities. It is not hard to show that the probability of occurrence of a determinate output (m, n) is given by $\mathrm{Tr}\left(\hat{\rho}_{mn}\right)$, then the mean value of the fidelity of teleportation reduces to:

$$F = \sum_{\mu\nu} \mathrm{Tr}\left\{|\psi\rangle\langle\psi|\, \hat{\rho}_{\mu\nu}\right\}. \tag{2.42}$$

This expression typically depends on the state coefficients α_j[28], for this reason it

[27] In the 2×2 case the corresponding Weyl operators are: $\{\hat{U}_{00}, \hat{U}_{01}, \hat{U}_{10}, \hat{U}_{11}\} = \{\hat{1}, \hat{\sigma}_x, \hat{\sigma}_z, \hat{\sigma}_z\hat{\sigma}_x\}$, in agreement with the standard teleportation protocol for qubits.

[28] Actually, the fidelity F does not depend on the state coefficients α_j only when we deal with an ideal channel and measurements.

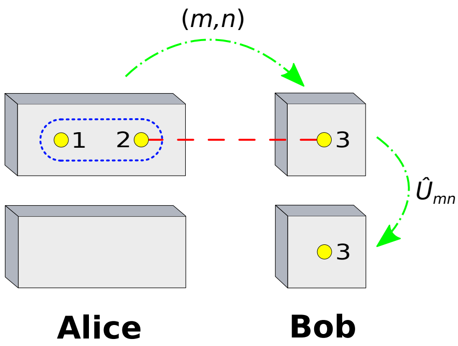

Figure 3 – Teleportation scheme: Alice and Bob share a channel composed by two entangled qudits (red dashed), Alice performs a joint measurement on qudits 1 and 2 (blue dotted). After a LOCC (green dot dashed) corresponding to transmission of a pair of dits (m, n) using a classical channel and subsequent application of a local unitary operation $\hat{U}_{mn}$, qudit 3 holds in the state previously possesed by Alice's qudit 1 (In the ideal case of maximally entangled channel and measurements).

is more convenient to calculate the average fidelity over the set of input states[29]:

$$\langle F \rangle = \frac{1}{V_d} \int \mathrm{d}\Gamma_d F. \tag{2.43}$$

General expressions for $\mathrm{d}\Gamma_d$ and V_d are given in appendix B.

It is possible to show that if only classical resources are employed (no entanglement), then the maximal fidelity one can attain in the protocol is equal to $2/(d+1)$. For this reason such a bound is known as *classical fidelity* [Horodecki, Horodecki e Horodecki 1999, Weinar, Laskowski e Pawłowski 2013].

We devote one chapter of this thesis to present analytical results of fidelity of *qudit* teleportation for a variety of non-ideal scenarios including non-maximally entangled channels and measurements, and noisy environments. Our results are motivated mainly by a recent exploration by Fortes and Rigolin [Fortes e Rigolin 2015], in which it is presented a survey on the diverse manners in which it is possible to increase the fidelity of teleportation under non-ideal scenarios. For a multipartite extension carried out by us under the weak noise regime, see [Cunha et al. 2017].

29 The quality of the protocol may also be analysed by estimating the dispersion of the fidelities around its average value [Bang, Ryu e Kaszlikowski 2018].

As already mentioned, it is usual to evaluate the quality of the teleportation protocol by quantifying how similar is the transferred to the initial state by means of the fidelity. However it has been recently proposed another figure of merit which takes into account only the non-classical contributions to the teleportation process [Cavalcanti, Skrzypczyk e ć 2017]. By using this kind of non-classical teleportation witness, the authors showed that the so called *bound states* are in fact useful to perform quantum teleportation, contrary to previous conclusions obtained in [Horodecki, Horodecki e Horodecki 1999] in which it was employed the fidelity of teleportation. In addition, experimental results recently presented support the ideas introduced above [Carvacho et al. 2018].

The first experimental realizations of the teleportation protocol were carried out in 1997 by Zeilinger's group [Bouwmeester et al. 1997, Pan et al. 2003], in which pairs of entangled photons produced under parametric down-conversion were employed. It is worth mentioning that it is possible to build a Bell measurement apparatus able to discriminate at most two out of the four basis elements using linear optics only, thus in a vast majority of experimental implementations 50% of the entangled pairs are lost, diminishing in this way the efficiency[30]. To date the largest distance it has been possible to teleport a qubit is 1400 km [Ren et al. 2017] in a ground-to-satellite experiment, while the previous record set was 143 km in free space [Ma et al. 2012]. Attaining such distances represents a very important step towards the feasibility of large scale quantum networks and in this way quantum based communication.

Another interesting fact is the relation between nonlocality and teleportation, Cavalcanti and co-workers [Cavalcanti et al. 2013], by using the idea of superactivation of nonlocality[31] found that any state useful for teleportation[32] is also capable of violating a Bell inequality, extending results previously obtained in [Popescu 1994].

30 For an alternative implementation which makes use of a hybrid technique allowing to perform complete Bell measurements see [Takeda et al. 2013].

31 Given a local state $\hat{\rho}$, if k copies of this $\hat{\rho}^{\otimes k}$ may violate a Bell inequality, it is said that the nonlocality of $\hat{\rho}$ can be superactivated [Palazuelos 2012].

32 An entangled state is said to be useful for teleportation whenever its usage leads to a fidelity above the classical threshold [Horodecki, Horodecki e Horodecki 1999].

2.5 Noise and Kraus Operators

The evolution in the state of a quantum system $\hat{\rho}$ after interacting with the environment may be modelled by a completely positive and trace preserving map $\hat{\rho} \rightarrow \hat{\rho}' = \sum_k \hat{E}_k \hat{\rho} \hat{E}_k^\dagger$, where the $\hat{E}_k$'s are known as Kraus operators and satisfy the completeness relation $\sum_k \hat{E}_k^\dagger \hat{E}_k = \hat{1}$ [Nielsen e Chuang 2010].

2.5.1 Noise on qudit systems - Weyl operators

Kraus operators corresponding to bit-flip, phase-flip, bit-phase-flip and depolarizing noise for qubits $(d = 2)$ and qutrits $(d = 3)$ are presented in [Nielsen e Chuang 2010] and [Ramzan 2013] respectively. It is straightforward to show that arbitrary d-dimensional generalizations are proportional to families of Weyl operators $\hat{U}_{jk}$. Such a correspondence is illustrated in figure 4. Moreover the set $\{\hat{U}_{jk}\}$ constitutes a natural basis for the $d \times d$ Hilbert-Schmidt space. As it has already been mentioned, the set of Pauli matrices plus the identity $(\hat{U}_{00} = \hat{1})$ is given by the family of Weyl operators for the particular case $d = 2$. Now we present a brief description of each particular noise and expressions for Kraus operators.

Dit-flip noise

In analogy to bit-flip for the qubit case, this kind of noise considers perturbations that flip the state $|j\rangle$ to one out of the following $|j \oplus 1\rangle$, $|j \oplus 2\rangle$, $\dots$, $|j \oplus d - 1\rangle$, with probability p. The associated Kraus operators are: $\hat{E}_{00} = \sqrt{1-p}\, \hat{U}_{00}$, $\hat{E}_{01} = \sqrt{\frac{p}{d-1}}\, \hat{U}_{01}$, $\dots$, $\hat{E}_{0,d-1} = \sqrt{\frac{p}{d-1}}\, \hat{U}_{0,d-1}$.

d-phase-flip noise

A qudit $|j\rangle$ subject to d-phase-flip noise may with probability p suffer one out of $d-1$ phase-shifts of the form: $\omega_d |j\rangle$, $\omega_d^2 |j\rangle$, $\dots$, $\omega_d^{d-1} |j\rangle$. The corresponding Kraus operators are given by: $\hat{E}_{00} = \sqrt{1-p}\, \hat{U}_{00}$, $\hat{E}_{10} = \sqrt{\frac{p}{d-1}}\, \hat{U}_{10}$, $\dots$, $\hat{E}_{d-1,0} = \sqrt{\frac{p}{d-1}}\, \hat{U}_{d-1,0}$.

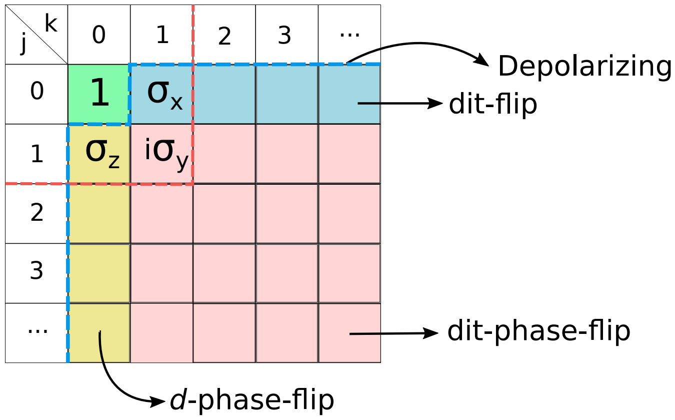

Figure 4 – Weyl operators $\hat{U}_{jk}$ and their relation with Kraus operators for several kinds of noise on d-dimensional systems. The blue row represents dit-flip like operators, the yellow column d-phase-flip like operators and the pink squares are related to matrices corresponding to dit-phase-flip like noise. Note that the three classes mentioned before are employed to define depolarizing noise and the set of Pauli matrices corresponds to the nontrivial operators for $d = 2$.

Dit-phase-flip noise

This is an special case in which a combination of both former kinds of noise may take place, e.g. a qudit suffer a flip and a phase shift at the same time. The related Kraus operators are: $\hat{E}_{00} = \sqrt{1-p}\,\hat{U}_{00}$ and $\hat{E}_{kl} = \frac{\sqrt{p}}{d-1}\,\hat{U}_{kl}$, with $1 \leq j, k \leq d-1$.

Mixed noise

In this case a qudit may with probability p suffer a dit-flip, d-phase-flip or both at the same time. The Kraus operators are given by: $\hat{E}_{00} = \sqrt{1-p}\,\hat{U}_{00}$ and $\hat{E}_{kl} = \sqrt{\frac{p}{d^2-1}}\,\hat{U}_{kl}$, with $0 \leq j, k \leq d-1$, $(j,k) \neq (0,0)$.

Depolarizing noise

A system initially prepared in an arbitrary state evolves to a maximally mixed state $\hat{1}/d$ with probability p. The Kraus operators for this kind of noise are: $\hat{E}_{00} = \sqrt{1 - \frac{d^2-1}{d^2}p}\,\hat{U}_{00}$ and $\hat{E}_{kl} = \frac{\sqrt{p}}{d}\,\hat{U}_{kl}$, with $0 \leq j, k \leq d-1$, $(j,k) \neq (0,0)$.

For the classes of noise mentioned above, the Kraus operators may be written as: $\hat{E}_{mn} = a_{mn}\hat{U}_{mn}$, with coefficients $a_{mn} \in \mathbb{R}$, satisfying $\sum_{mn} a^2_{mn} = 1$. Given an arbitrary system initially prepared in a state $\hat{\rho} = \sum_{\boldsymbol{kl}} \rho_{\boldsymbol{kl}} \ |\boldsymbol{k}\rangle\langle\boldsymbol{l}|$, where N is the number of parts of the system, $\boldsymbol{k} = (k_1, ..., k_N)$ and $0 \leq k_j \leq d-1$, the action of a set of Kraus operators $\hat{E}_{\boldsymbol{kl}} = \hat{E}_{k_1 l_1} \otimes ... \otimes \hat{E}_{k_N l_N} = \prod_{j=1}^{N} a_{k_j l_j} \hat{U}_{k_1 l_1} \otimes ... \otimes \hat{U}_{k_N l_N}$ evolves $\hat{\rho}$ into $\hat{\rho}' = \sum_{\boldsymbol{mn}} \rho'_{\boldsymbol{mn}} \ |\boldsymbol{m}\rangle\langle\boldsymbol{n}|$, with $\rho'_{\boldsymbol{mn}}$ given by:

$$\rho'_{\boldsymbol{mn}} = \sum_{\boldsymbol{kl}} \omega_d^{\boldsymbol{k}\cdot(\boldsymbol{m}-\boldsymbol{n})} \rho_{\boldsymbol{m}\oplus\boldsymbol{l},\boldsymbol{n}\oplus\boldsymbol{l}} \left(\prod_{j=1}^{N} a^2_{k_j l_j} \right). \tag{2.44}$$

In this case, the noise coefficient a_{mn} may be expressed as a superposition of the contributions of each region in figure 4: a_0, noiseless region (green); a_f, flip region (blue); a_p, phase flip region (yellow) and a_c for the region of combination of flip and phase-flip (red). In this way, the squared noise coefficient a^2_{jk} reads:

$$a^2_{jk} = a_0^2 \delta_{j,0}\delta_{k,0} + a_f^2 \delta_{j,0} \sum_{n=1}^{d-1} \delta_{k,n} + a_p^2 \delta_{k,0} \sum_{m=1}^{d-1} \delta_{j,m} + a_c^2 \sum_{m,n=1}^{d-1} \delta_{j,m}\delta_{k,n}. \tag{2.45}$$

Thus we have the following correspondences between noise and reduced coefficients: dit-flip: $a_p = a_c = 0$, d-phase-flip: $a_f = a_c = 0$, dit-phase-flip: $a_p = a_f = 0$ and $a_f = a_p = a_c$ for mixed and depolarizing noise.

The expressions above are useful whenever the Kraus operators are proportional to Weyl matrices. Nevertheless for an important kind of noise, namely *Amplitude Damping* we cannot do that, for this reason we have to turn to the customary computational basis.

2.5.2 Noise on qudit systems - Computational Basis

For a system composed by one part only, we can write a Kraus operator as:

$$\hat{E}_k^{(a)} = \sum_{mn} a_{mn}^{(k)} \, |m\rangle\langle n|$$

and its hermitian conjugate:

$$\hat{E}_k^{(a)\dagger} = \sum_{mn} a_{nm}^{(k)*} \, |m\rangle\langle n| \, ,$$

with operator coefficients $a_{nm}^{(k)} \in \mathbb{C}$, satisfying $\sum_{kn} a_{mn}^{(k)} a_{ln}^{(k)*} = \delta_{ml}$ (due to the completeness relation). The N-party case is a straightforward generalization:

$$\hat{\rho} \to \hat{\rho}' = \sum_{\boldsymbol{k}} \hat{E}_{\boldsymbol{k}}\, \hat{\rho}\, \hat{E}_{\boldsymbol{k}}^{\dagger},$$

where $\dim \boldsymbol{k} = N$, as previously and Kraus operators $\hat{E}_{\boldsymbol{k}}$ given by:

$$\hat{E}_{\boldsymbol{k}} = \hat{E}_{k_1}^{(a_1)} \otimes \hat{E}_{k_2}^{(a_2)} \cdots \otimes \hat{E}_{k_N}^{(a_N)} = \bigotimes_{j=1}^{N} \hat{E}_{k_j}^{(a_j)} = \sum_{\boldsymbol{m},\boldsymbol{n}} \left(\prod_{j=1}^{N} a_{m_j n_j}^{(k_j)} \right) |\boldsymbol{m}\rangle\langle\boldsymbol{n}| \,,$$

substituting, we get:

$$\hat{\rho}' = \sum_{\boldsymbol{m},\boldsymbol{q}} \rho'_{\boldsymbol{m},\boldsymbol{q}} \, |\boldsymbol{m}\rangle\langle\boldsymbol{q}| \,,$$

with modified density operator coefficients $\rho'_{\boldsymbol{m},\boldsymbol{q}}$ given by:

$$\rho'_{\boldsymbol{mq}} = \sum_{\boldsymbol{n},\boldsymbol{p},\boldsymbol{k}} \left(\prod_{j=1}^{N} a_{m_j n_j}^{(k_j)} a_{q_j p_j}^{(k_j)*} \right) \rho_{\boldsymbol{n},\boldsymbol{p}}. \tag{2.46}$$

The coefficients $a_{mn}^{(k)}$ may be easily calculated for each particular case:

$$a_{mn}^{(k)} = \langle m| \, \hat{E}_k \, |n\rangle \,. \tag{2.47}$$

Now let us briefly present the last instance of quantum noise explored in this work:

Amplitude Damping noise:

Amplitude damping noise have been widely used to model loses in two level systems [Nielsen e Chuang 2010]. Recently it has been introduced a d-dimensional generalization [Dutta et al. 2016], with Kraus operators given by:

$$\hat{E}_0 = |0\rangle\langle 0| + \sqrt{1-p} \sum_{j=1}^{d-1} |j\rangle\langle j| \,, \tag{2.48}$$

and

$$\hat{E}_j = \sqrt{p}\, |0\rangle\langle j| \tag{2.49}$$

with $j = 1, \cdots, d-1$. This kind of noise may be interpreted in the following way: A d-dimensional system submitted to interactions with the environment may with probability p lose population in the excited states, leading the system to the ground state $|0\rangle$.

3 PROBABILITY OF VIOLATION AS A QUANTIFIER OF NONLOCALITY

Highlights

The probability of violation is introduced as a quantifier of nonlocality.

The application of this quantifier leads to an agreement between maximal entanglement and maximal nonlocality in several bipartite scenarios of multiple inputs and outcomes.

The anomaly in the nonlocality of pairs of entangled qutrits ceases to happen when the probability of violation is used as a figure of merit.

3.1 Introduction

We have already introduced the concept of quantum nonlocality as a manifestation of the weirdness of quantum predictions when seen under the eyes of a classical observer. In the previous chapter we also mentioned several alternatives to quantify the strength of nonclassical correlations, which interestingly disagree to assess the most nonlocal state under the scenario $(2, 2, d)$ and for $d > 2$, these optimal states are different from the maximally entangled one [Méthot e Scarani 2007]. In this chapter we introduce a measure of nonlocality in which the disagreement between maximal entanglement and nonlocality vanishes for the states in which such an anomaly emerged.

3.2 Motivations

It is clear that entanglement and nonlocality are different resources and the mere presence of the former does not necessarily imply the manifestation of the latter, as we have already seen for Werner states[1]. Nevertheless it is worth to remark that this is a feature exclusive of mixed states, given that due to the Gisin's theorem it is impossible to describe all the correlations produced by *any pure entangled state* by using a LHVM, i.e. in this case a pure entangled state under proper measurements implies nonlocality. Then it is reasonable to expect that the maximally entangled state leads to a stronger manifestation of Bell nonlocality [Lipinska et al. 2018], thus we argue that such a anomaly may occur due to the structure behind the employed figures of merit.

A tenable reasoning about the quantification of nonlocality is that some clue might come from nonlocal hidden variable (NLHV) models capable of reproducing the quantum correlations. For example, one could say that a state $\hat{\rho}$ is more nonlocal than another one $\hat{\varrho}$ if the underlying NLHV model violates local causality in different degrees for these different states. This, however, cannot be inferred from these models in any obvious way. Even for theories relying on finite (superluminal) signaling speed, the relation $I_{max}(\hat{\rho}) > I_{max}(\hat{\varrho}) > S_L$ does not necessarily imply that $v_{\hat{\rho}} > v_{\hat{\varrho}} > c$, where v is the signal velocity for each state and c is the speed of light. This reasoning suggests that all violating states for a particular setting are equally nonlocal, and that the essential information provided by a Bell inequality is of a seemingly Boolean nature, a state being either local or nonlocal with respect to those settings, without gradations. This apparently all-or-nothing picture, however, does not lead to a dead end. On the contrary, it may point to a conceptually simple solution as we will discuss.

An interesting feature exhibited by most of the nonlocality quantifiers mentioned in the previous section is that they consider **only** one measurement setting, say the particular configuration generating maximal violation of the associated Bell inequality. For instance consider the pictorial representation of the Bell value I in function of an experimental parameter θ in figure 5. Note that according to the maximal violation of the Bell inequality, state 1 is more nonlocal than 2, even though the parameter's region leading to the manifestation of nonclassicality is wider for the latter.

Based on the analysis above we propose the following quantifier of nonlo-

[1] The same feature is observed in isotropic states.

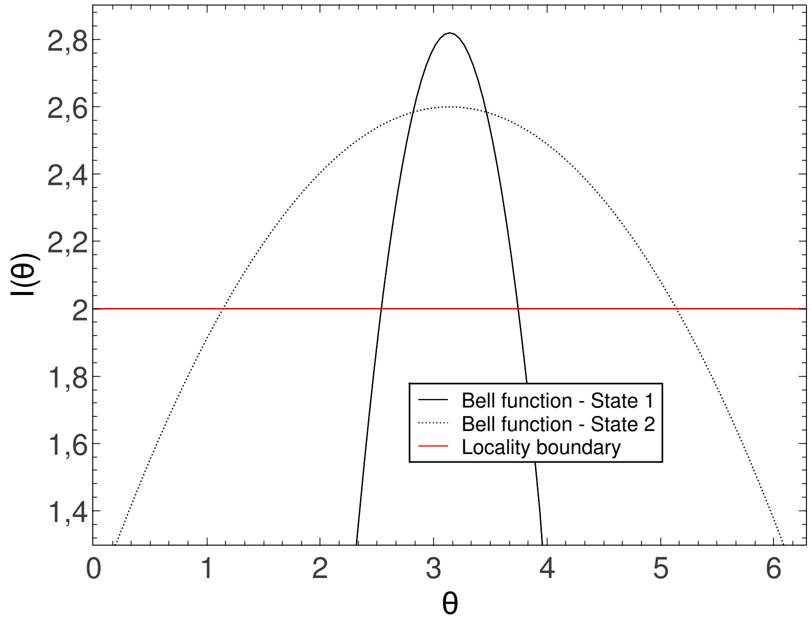

Figure 5 – Pictorial representation of the Bell value $I(\theta)$ associated with two different states in function of the experimental parameter θ (for instance the orientation of detectors in a Stern-Gerlach experiment). Although both states attain maximal violation for $\theta = \pi$, state 1 shows a larger numerical departure from the local boundary, and most remarkably, state 2 presents a wider region of parameters exhibiting Bell nonlocality.

cality.

3.3 The *volume of violation* as a quantifier of nonlocality

Given an entangled state and a specific Bell inequality (and in this way, an experimental scheme with a determinate quantity of parts, inputs and outcomes), the most exhaustive experiment one can go through is to investigate local causality for *all settings*. We are led to state that $\hat{\rho}$ is more nonlocal than $\hat{\varrho}$ if the former violates the inequality, by any extent, for a larger amount of setting parameters than the latter. This statement can be cast in very simple statistical terms: $\hat{\rho}$ is more nonlocal than $\hat{\varrho}$ if, for an unbiased random choice of settings, the probability to observe a violation is larger for $\hat{\rho}$. Following this reasoning, the state 2 is more nonlocal than 1 in figure 5.

To formalize this idea, define the space $\mathcal{X} = \{x_1, ..., x_n\}$ of all possible parameters that determine the settings for a given (preferably tight) Bell inequality I. For a particular state $\hat{\rho}$, let $\Gamma_{\hat{\rho}} \subset \mathcal{X}$ be the set of points leading to violation and $V(\hat{\rho})$ be proportional to the volume of $\Gamma_{\hat{\rho}}$. We say that if $V(\hat{\rho}) > V(\hat{\varrho})$, then $\hat{\rho}$ is more nonlocal than $\hat{\varrho}$, with

$$V_I(\hat{\rho}) \equiv \frac{1}{\mathcal{N}} \int_{\Gamma_\rho} d^n x, \tag{3.1}$$

where $\mathcal{N}$ is a normalization constant. The measure of integration is such that every setting (set of parameters) has equal weight. For instance, one setting corresponding to a direction in space demands two parameters, one polar (φ) and one azimuthal (θ) angle, leading to $d^2x = d\Omega = \sin\theta d\theta d\varphi$. If, on the other hand, the settings are defined by the in-plane angles (φ_j) of n polarisers, e.g., then we simply have $d^n x = d\varphi_1 \cdots d\varphi_n$. We call V the *volume of violation*. We remark that the numeric calculations needed to determine the volume of violation are the paradigmatic problem for which Monte Carlo methods are intended [Landau, Paez e Bordeianu 2008]. The above definition has no relation to the volume of the set of separable states defined in references [Życzkowski et al. 1998, Życzkowski 1999]; the volume of violation is an integration over the experimental parameters that can be varied within the context of a given Bell inequality.

An interesting question we have not tackled is whether the volume of violation satisfies the set of natural requirements an operational measure of non-locality must fulfil. Nevertheless Lipinska and co-workers have recently derived several analytical results supporting this idea [Lipinska et al. 2018].

3.3.1 Probability of violation of local realism

An interesting feature of the proposed measure is that $V(\hat{\rho})$ may be interpreted either as the fraction of the volume of the set of parameters that leads to violation of the given Bell inequality, or as the probability of violation of a Bell inequality if Alice and Bob carry out local random isotropic measurements (up to permutations of obsevables) [Liang et al. 2010, Wallman, Liang e Bartlett 2011]. A similar approach was recently developed by Atkin and Zohren [Atkin e Zohren 2015], in which the measurement settings are kept fixed, equal to the optimal

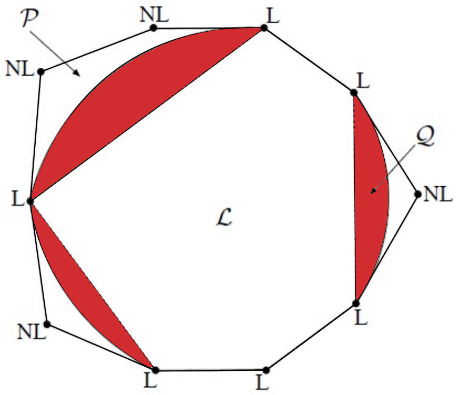

Figure 6 – Schematic representation of the space of behaviours $\mathcal{P}$, quantum set $\mathcal{Q}$ and local polytope $\mathcal{L}$. L and NL indicate local and nonlocal extemal points. The red region corresponds to the fraction of the quantum set which cannot be explained by using a local hidden variables model. Its volume compared to that of the quantum set may be conceived as the probability of violation of a given state [De Rosier et al. 2017].

set and it is evaluated how the probabilty of violation varies with the number of outcomes of the measurements for several ensembles of random pure states. In the present work, instead of the measurement settings, the state of interest is fixed and it is investigated the dependence of nonlocality with the dimension for random uniformly distributed measurements.

It is worth mentioning that the only analytical results regarding probability of violation of local realism under random unbiased measurements p_v, currently in the literature (to the knowledge of the author) are limited to the scenario $(2, 2, 2)$ for the CHSH inequality $p_v = 2(\pi - 3)$ [Liang et al. 2010] and the first Bell inequality $p_v = 1/3$ [Parisio 2016], for an entangled pair of qubits in a singlet state. Both calculations were possible due to the high degree of symmetry of the singlet.

3.3.2 Probability of violation in the space of behaviors

A more fundamental definition should not invoke a particular Bell inequality, but rather the set of conditional probabilities $P(ab|xy)$, where a and b are outcomes and x and y inputs for Alice and Bob observables respectively. This amounts to an integration similar to eq. 3.1, but over the exterior, no-signalling part of the local polytope (see red region in figure 6). More formally, it can be written as:

$$p_v = \frac{\mathrm{vol}(\mathcal{L}^c \cap \mathcal{Q})}{\mathrm{vol}(\mathcal{Q})}, \tag{3.2}$$

where vol(A) is the Haar-measure associated volume to the set A and $\mathcal{L}^c$ indicates the complement of the local polytope. Although representing an exponentially hard computational problem, it has already been considered in the multipartite/multiple inputs context [De Rosier et al. 2017], in which the authors find several interesting results. Remarkably, they were able to show numerically that within this context, as the quantity of inputs increases, it becomes easier for a given state to demonstrate Bell nonlocality, even for states with a low amount of entanglement. Some of these results have also been obtained analytically in [Lipinska et al. 2018].

Although this approach has an implicit fundamental relevance, due to the difficulty to perform measurement operations capable of exploring the whole behaviours space, it seems impractical from an experimental point of view. For this reason in this thesis we give emphasis on the calculation of the probability of violation by employing the volume of violation as originally presented in equation 3.1.

3.4 Probability of violation - Scenario $(2, m, 2)$

Let us explore some initial results on the families of tight Bell inequalities involving two parts and two outcomes mentioned in the previous chapter.

Alice and Bob share an entangled state

$$|\psi_\alpha\rangle = \alpha\,|00\rangle + \sqrt{1-\alpha^2}\,|11\rangle\,, \tag{3.3}$$

where $\alpha \in [0,1]$ and $\{|0\rangle\,, |1\rangle\}$ are eigenstates of the z-spin operator $\hat{S}_z = (\hbar/2)\hat{\sigma}_z$. In this case we have to deal with $2 \times m$ independent measurement directions $(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m)$ and $(\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m)$ for Alice and Bob, respectively. Each measurement direction may be parametrized with spherical coordinates $(\theta_{k_j}, \varphi_{k_j})$, for

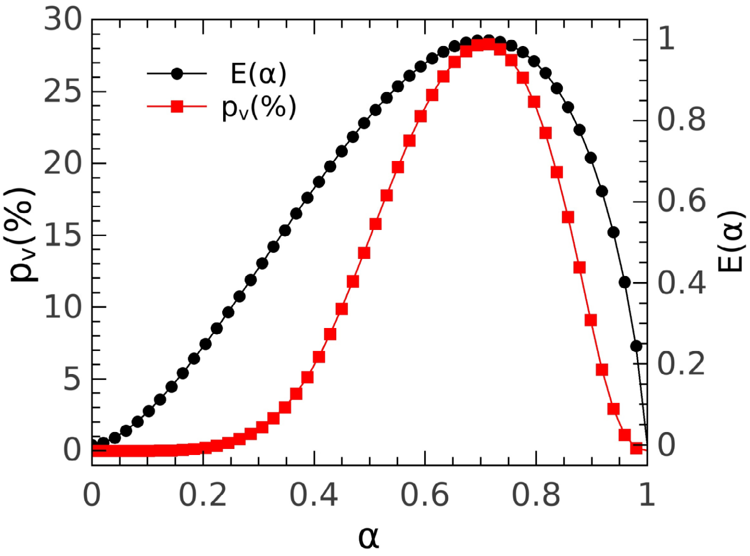

Figure 7 – *Red squares*: Probability of violation of the CHSH inequality by an entangled pair of qubits in the state $|\psi_\alpha\rangle = \alpha\,|00\rangle + \sqrt{1-\alpha^2}\,|11\rangle$ under local isotropic random spin-$1/2$ measurements. *Black circles*: Normalized amount of entanglement. Note that the probability of violation is maximized by the state with the larger amount of entanglement, $\alpha = 1/\sqrt{2} \approx 0.7071$. Note: Data obtained after 10^8 runs.

this the integration measure is given by $d^n x = d\Omega_{a_1}\cdots d\Omega_{a_m} d\Omega_{b_1}\cdots d\Omega_{b_m}$, with $d\Omega_{k_j} = \sin\theta_{k_j} d\theta_{k_j} d\varphi_{k_j}$.

In each step of the procedure a set of $4\times m$ angles are sampled uniformly on the unit sphere are generated and substituted into the expressions for joint probabilities (Eqns. A.6, A.7, and A.8 in appendix A), which are then used to evaluate the corresponding Bell function. The volume of violation is equal to the fraction of the times such an inequality is not satisfied. In order to calculate the probability of violation we have to take into account the different permutations of local observables which could have led to violation of the same inequality. For the $(2,m,2)$ scenario, it is not hard to show that such a factor is equal to $(m!)^2$.

We have carried our calculations for the scenario $m = 2$ (CHSH), the results are presented in figure 7. The trivial cases $\alpha = 0$ and $\alpha = 1$ exhibit a null probability of violation as expected, and surprisingly, $p_v(\alpha)$ presents a monotonic behaviour with respect to the amount of entanglement of the state, attaining a

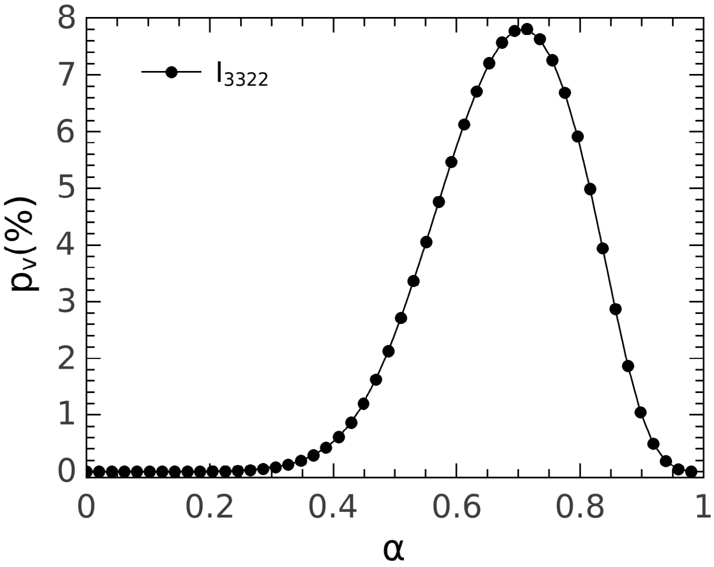

Figure 8 – Probability of violation of the I_{3322} inequality by an entangled pair of qubits in the state $|\psi_\alpha\rangle = \alpha\,|00\rangle + \sqrt{1-\alpha^2}\,|11\rangle$ under local random isotropic spin-$1/2$ measurements. As in the previous case, p_v is maximized by the maximally entangled state, $\alpha = 1/\sqrt{2} \approx 0.7071$. Note: Data obtained after 10^9 runs in approximately 24 hours on a standard laptop.

maximum of $p_v \approx 28.3\%$ for $\alpha = 1/\sqrt{2}$, in agreement with previous analytical results [Liang et al. 2010].

In addition, we calculated the probability of violation for the same state (eq. 3.3), for the cases in which Alice and Bob have three, four and five measurement choices, obtaining results qualitatively identical to those from the previous treatment. In figures 8 and 9 we plot the probability of violation of the I_{3322} and I_{4422} inequalities as a function of the parameter α. Once again it is possible to observe an coincidence between maximal nonlocality and entanglement, however this time the highest probability of violation attains lower values in comparison with the CHSH case: $p_v^{m=3} \approx 7.8\%$, $p_v^{m=4} \approx 0.4\%$ and $p_v^{m=5} \approx 8 \cdot 10^{-7}\%$. Which could in principle seem to be in contradiction with results of [De Rosier et al. 2017, Lipinska et al. 2018] indicating that p_v increases with m. However we must note that our approach does not take in consideration all the facets of the local polytope, but only those described by the I_{mm22} inequality.

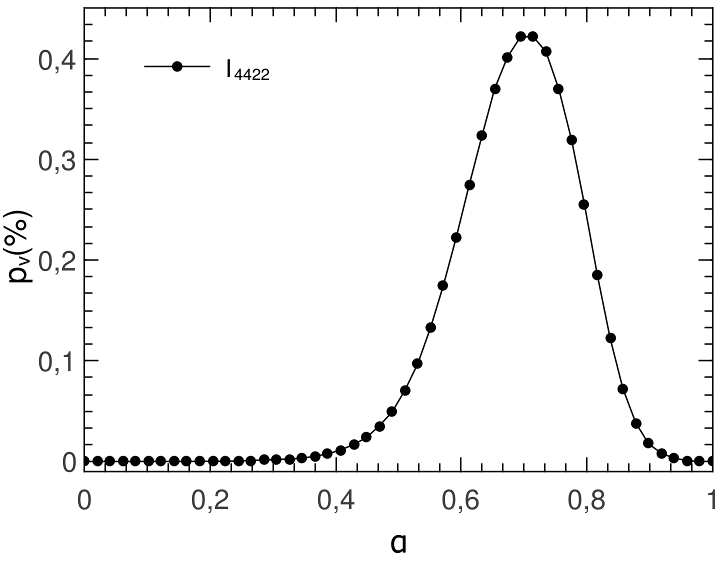

Figure 9 – Probability of violation of the I_{4422} inequality by an entangled pair of qubits in the state $|\psi_\alpha\rangle = \alpha\,|00\rangle + \sqrt{1-\alpha^2}\,|11\rangle$ under local random isotropic spin-$1/2$ measurements. p_v is maximized by the maximally entangled state, $\alpha = 1/\sqrt{2} \approx 0.7071$. Note: Data obtained after 10^9 runs.

So far, the probability of violation gives no sensible new information in comparison to the maximum of Bell functions, yet it is consistent with our expectations on what should be a nonlocality measure in the safe terrain of two entangled qubits. This agreement between p_v and I_{max} ceases to happen when two higher-dimensional systems are considered, even in the pure case.

3.5 No anomaly in the nonlocality of pairs of entangled qutrits

Now consider two entangled qutrits, i.e., a composite system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\dim \mathcal{H}_B = \dim \mathcal{H}_B = 3$. As mentioned in the previous chapter, any arbitrary state of two qutrits (three-level quantum systems) may be decomposed in the Schmidt basis as

$$|\Psi\rangle = \sum_{m=0}^{2} \alpha_m |m\rangle_A \otimes |m\rangle_B, \tag{3.4}$$

where $\{|0\rangle_i, |1\rangle_i, |2\rangle_i\}$ is an orthonormal basis in $\mathcal{H}_i$.

A very interesting point regarding qutrit systems is that contrary to the two dimensional case, here spin-1 measurements are not capable of revealing the whole richness of the Hilbert space[2] [Kaszlikowski et al. 2000, Kaszlikowski 2000], then a more sophisticated approach must be taken in consideration. An alternative is given by the so called *multiport beam splitters and phase shifters* (MBSPS) scheme. Under this, each of the parties can execute one out of two d-outcome projective measurements ($x, y = 1, 2$), consisting in diagonal phase-shift unitary operations with $U_{mm} = e^{i\phi_x^m}$ (Alice) and $U_{nn} = e^{i\varphi_x^n}$ (Bob), followed by discrete Fourier transforms U_{FT} and U_{FT}^* on Alice's and Bob's subsystems respectively, and then a projection onto the original basis [Kaszlikowski et al. 2000, Żukowski, Zeilinger e Horne 1997, Durt, Kaszlikowski e Żukowski 2001]. A picture of the experimental implementation is illustrated in figure 10.

For $d = 3$, the CGLMP inequality (Eq. 2.37) reads:

$$\begin{aligned} I_3 = P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1) + \\ - P(A_1 = B_1 - 1) - P(B_1 = A_2) - P(A_2 = B_2 - 1) - P(B_2 = A_1 - 1) \leq 2. \end{aligned} \tag{3.5}$$

For details of the reduction of the CGLMP inequality under MBSPS measurements, we refer the reader to appendix C.

It is important to note that the MBSPS scheme does not exhaust the $(2, 2, d)$ CGLMP scenario, however, we obtain a great simplification by remaining within MBSPS realizations, which are often employed in CGLMP-tests. In addition, this was exactly the considered situation when the anomaly in the nonlocality of two qutrits was first reported [Acín et al. 2002]. It has also been conjectured that the optimal settings are contained in the MBSPS scenario [Durt, Kaszlikowski e Żukowski 2001], which has been proved in the two-qutrit case numerically in [Navascués, Pironio e Acín 2007] and analytically in [Yang et al. 2014]. The optimal phases for a general d-outcome MBSPS setup are given by: $\phi_1^j = 0$, $\phi_2^j = \frac{\pi}{d} j$, $\varphi_1^j = \frac{\pi}{2d} j$ and $\varphi_2^j = -\frac{\pi}{2d} j$, providing the highest violation for maximally entangled states $\left|\Psi_{me}^d\right\rangle$ and those which maximally violate the CGLMP inequality

2 For instance, the Hilbert space associated to a three level system can be explored by the usage of $U(3)$ transformations.

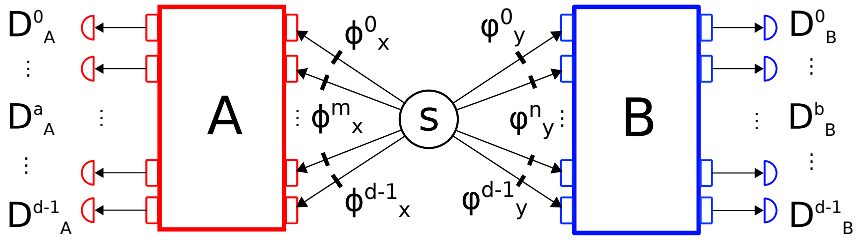

Figure 10 – Multiport beam splitters and phase shifters scheme. A pair of entangled photons leave the k-th channel of a source S in opposite directions with probability α_k^2. After, each of them pass through a polarizer characterized by the phases ϕ_x^k and φ_y^k, for Alice's and Bob's photons respectively. Then every photon gets into the k-th input of an unbiased multiport beam splitter, leaving by any output with the same probability $1/d$ regardless the input it has come. After many runs of the experiment and construction of joint probabilities it is not hard to show that for some pure states it is impossible to describe some results by using a LHVM [Zukowski, Zeilinger e Horne 1997].

$\left|\Psi_{mv}^d\right\rangle$. Let us introduce a family of states which encloses both of them:

$$\left|\Psi_\gamma\right\rangle = \frac{1}{\sqrt{2+\gamma^2}}(\left|00\right\rangle + \gamma\left|11\right\rangle + \left|22\right\rangle). \tag{3.6}$$

It is straightforward to see that $\gamma = 1$ recovers the state $\left|\Psi_{me}\right\rangle$. On the other side, $\gamma_{mv} = \left(\sqrt{11} - \sqrt{2}\right)/2 \approx 0.7923$, as first found in [Acín et al. 2002], leads to the state which maximally violates the CGLMP inequality for $d = 3$ by a value $I_3(\Psi_{mv}) = 1 + \sqrt{11/3} \approx 2.915$, in contrast with the value $I_3(\Psi_{me}) \approx 2.873$ attained by the maximally entangled state of two qutrits. Thus, the use of the maximal violation of a Bell inequality as a figure of merit to assess how nonlocal a given state is leads to a disagreement between maximal nonlocality and entanglement. This, along with results from other quantifiers constitute the so called *anomaly of nonlocality* [Méthot e Scarani 2007]. We argue that such a divergence may be due to the way the amount of nonlocality of a given state is usually assessed, then we proceed to employ the probability of violation for this particular case.

Under a MBSPS measurement scheme the experimental parameters are 12 in-plane polarization angles, hence the integration measure may be written as

$$d^n x = \prod_{x,y=1}^{2} \prod_{j,k=0}^{2} d\phi_x^j d\varphi_y^k,$$

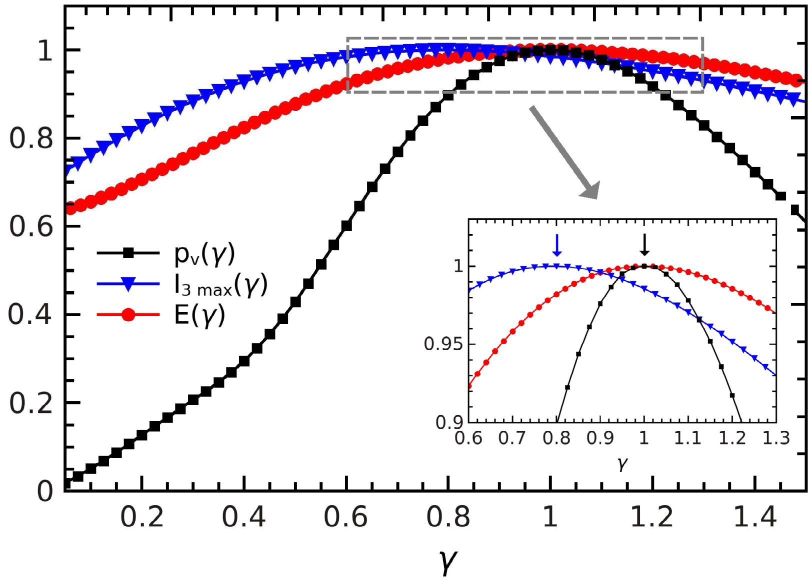

Figure 11 – *Red circles*: Entropy of entanglement, *Blue triangles* I_{max}, and *Black squares*: p_v as functions of γ for the family of states in equation 3.6. All quantities are normalized such that their maximal value is 1. The inset shows a zoom in of the region marked by the rectangle in dashed lines.

thus the total volume is $V(\mathcal{X}) = (2\pi)^{12}$. In order to calculate the probability of violation, we have to take into account that the permutation among observables gives a factor equal to 4. In this way, p_v may be calculated as:

$$p_v(|\Psi\rangle) = \frac{4}{(2\pi)^{12}} \int_{\Gamma_{|\Psi\rangle}} \prod_{x,y=1}^{2} \prod_{j,k=0}^{2} d\phi_x^j d\varphi_y^k, \tag{3.7}$$

where $\Gamma_{|\Psi\rangle}$ corresponds to the subset of $\mathcal{X}$ for which the measurement parameters lead to violation of the inequality $I_3 \leq 2$ for a fixed state $|\Psi\rangle$.

The calculation of the term $\int_{\Gamma_{|\Psi\rangle}} \prod_{x,y=1}^{2} \prod_{j,k=0}^{2} d\phi_x^j d\varphi_y^k$ in Eq. 3.7 is performed via Monte Carlo method. First of all we fix the state $|\Psi\rangle$, then in each iteration we pick 12 measurement angles at random from a uniform distribution between 0 and 2π and after we substitute into the CGLMP function. As the number of iterations grows, the fraction of the times the inequality is violated gets closer to the value of the integration.

Results of calculations are plotted in fig. 11. In order to facilitate the analysis, we compare normalized results of $p_v(\gamma)$, entropy of entanglement $E(\gamma)$ and the maximum of I_3. The maxima of entropy and probability of violation coincide exactly at $\gamma = 1$, as can be seen in the inset, while I_{3max} attains its maximum at $\gamma = \gamma_{mv}$. This shows that the anomaly in the nonlocality of two entangled qutrits does not exist, if one adopts the probability of violation (or equivalently volume of violation) as the measure of nonlocality.

It is easy to understand what is going on. Although $|\Psi^3_{mv}\rangle$ presents a more pronounced maximum of I_3 in comparison to $|\Psi^3_{me}\rangle$, the nonlocality of the former is less robust, for, as we get farther away from the optimal setting in $\mathcal{X}$, $I_3(\gamma_{mv})$ falls off faster than $I_3(\gamma = 1)$. This effect on the volume of violation is clearly illustrated in Fig. 12, where two-dimensional sections $\phi^0_1 - \varphi^2_2$ of Γ are shown for $|\Psi^3_{mv}\rangle$ [Fig. 12(b)] and for $|\Psi^3_{me}\rangle$ [Fig. 12(a)]. The other parameters are set as $\phi^0_2 = \phi^1_2 = \pi/6$, and $\varphi^j_1 = 0$, the remaining angles keeping the optimal values. In this particular example, the violation area for $\gamma = 1$ is about 14% larger than that for $\gamma = \gamma_{mv}$. The scales are identical in both figures.

We argue that, given a state, a Bell inequality, and a particular setting, there should be no gradations of nonlocality, with the inequality functioning as a witness. However, by "tracing over the settings", attributing equal weight to all those that violate the inequality and weight zero to those that do not lead to violations, we showed that it is possible to quantify Bell nonlocality in a consistent way. In particular, within the context of our proposal, there is no discrepancy between maximally entangled and maximally nonlocal states, at least for entangled qutrits. Note moreover that as we will show in the next chapter this feature persists in systems of higher dimensions, at least up to $d = 7$. In addition, in [De Rosier et al. 2017], the authors calculate the probability of violation of the same states considered here under the context of the space of behaviours (eq. 3.2) obtaining qualitative similar results, and in this way supporting our idea.

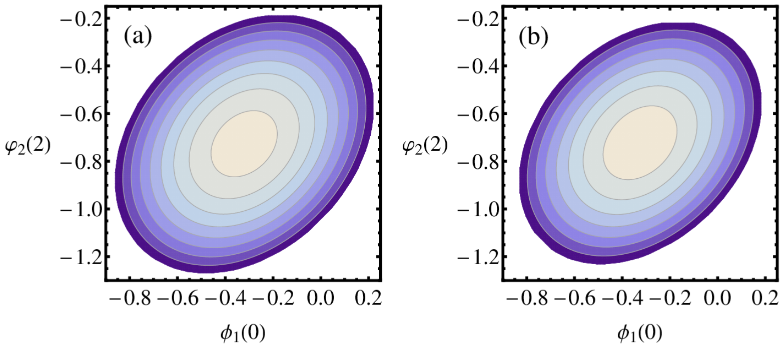

Figure 12 – Sections $\phi_1^0 - \varphi_2^2$ of the 12-dimensional space $\mathcal{X}$. The regions inside the ovals represent configurations leading to violation of the CGLMP inequality. In addition it is worth to mention that some parameters (ϕ_j^x, φ_y^k) were set away from the optimal values. The area of violation for (a) $\gamma = 1$ is 14% larger than that for (b) $\gamma = 0.792$.

4 PROBABILITY OF VIOLATION IN HIGHER DIMENSIONS

Highlights

By using the probability of violation, it is shown that there is no anomaly of nonlocality at least up to $d = 7$.

The amount of nonlocality decreases with the local dimension of subsystems d.

4.1 Introduction

The violation of Bell inequalities constitutes one of the most impressive confirmations of the nonlocal character of quantum theory. Presently, the majority of the state-of-the-art experiments in the field involve two qubits in the context of the CHSH inequality. However, it became clear that the use of systems of higher dimensionality, or *qudits*, may lead to new, interesting phenomena and improvements in the efficiency of some practical tasks [Mischuck e Mølmer 2013, Strauch 2011, Lanyon et al. 2009, Ralph, Resch e Gilchrist 2007, Durt et al. 2004]. In particular, it has been proven that loophole-free Bell tests are more feasible if qudits are employed [Vértesi, Pironio e Brunner 2010] (See [Lo et al. 2016] for recent experimental developments.). The nonlocality of pairs of entangled qudits have been used to certify high dimensional entanglement and in the study of robustness against noise, imperfect state preparation and measurements [Dada et al. 2011, Weiss et al. 2016, Dutta et al. 2016, Polozova e Strauch 2016].

A more specific, but important question refers to the macroscopic limit. Pioneering works, addressing two spin-s particles, revealed a tendency toward local, classical behaviours as $s \to \infty$ [Mermin 1980, Mermin e Schwarz 1982], in the sense that the range of parameters for which nonclassicality arises vanishes as

$1/s$ (however the considered inequalities are not tight). Complementarily, Gisin and Peres [Gisin e Peres 1992] showed that for particular choices of measurement parameters (under the context of the CHSH inequality), it is always possible to obtain violation, but never above the Tsirelson bound.

The authors of [Kaszlikowski et al. 2000] employed the resistance to noise as a nonlocality quantifier, and numerically calculated it for maximally entangled states of two qudits up to $d = 9$, each subject to one out of two local measurements characterized by a MBSPS scheme, they found that the resistance to white noise increases with the dimension d. Presently, it is acknowledged that, although physically relevant, resistance to noise is not a good measure of nonlocality [Acín et al. 2002, De Rosier et al. 2017]. Also in this context, a surprising result is that the nonlocality of a system of N qubits tends to increase with N, provided that the ability to individually address each qubit is preserved [De Rosier et al. 2017].

This chapter is devoted to present results on calculations of the probability of violation to quantify the nonlocality of two entangled qudits, addressing a specific experimental situation, i. e., a fixed Bell scenario (CGLMP) and the set of observables which are accessible in a particular experimental realization, namely, multiport beam splitters and phase shifters (MBSPS) and then we contrast them with results obtained by our collaborators who investigated the same set of states in a more fundamental perspective, by calculating the probability of violation directly in the full space of joint probabilities (the space of behaviours). While the first approach corresponds to a situation that can be exhaustively investigated within a single experimental preparation, it also inherits the bias associated with the choice of a particular facet of the local polytope. We consider dimensions d with $2 \leq d \leq 7$. The second approach is conceptually more powerful, since it takes into account all possible Bell inequalities (with a certain number of observables per party), however, the probabilities of violation calculated in the space of behaviours cannot possibly be determined by a single experimental setup. We discuss, both the common points and the differences between the two approaches.

4.2 Probability of violation for qudits in a MBSPS scheme and the CGLMP inequality

Our main goal in this chapter is to compare the amount of nonlocality for several entangled states of two qudits from the point of view of the probability of violation of the CGLMP inequality. Among the states of interest we have: maximally entangled states, low-rank states (states with a number of non-vanishing Schmidt coefficients lower than the dimension of the associated Hilbert space), and states which maximally violate the CGLMP inequality.

It is usual to find the maximal value of CGLMP functions in the literature but not the states leading to them. In order to calculate the state coefficients associated to $\left|\Psi_{mv}^{d}\right\rangle$ we applied the multidimensional Newton-Raphson method, maximizing the violation of the CGLMP inequality (equation C.8) under the optimal phases introduced in the previous chapter.

Let us parametrize the state as

$$|\Psi_{mv}^{d}\rangle = \frac{1}{\sqrt{\sum \gamma_{jd}^{2}}} \sum_{j=0}^{d-1} \gamma_{jd} |jj\rangle, \tag{4.1}$$

where $\gamma_{jd} \in \mathbb{R}$. Numerical values of γ_{jd} and maximal violation of CGLMP inequality for Ψ_{mv}^{d} and Ψ_{me}^{d} up to $d = 7$ are summarized in table 1.

Table 1 – Approximate values of coefficients γ_{jd} leading to maximal violation of CGLMP inequality and corresponding maximum value of the CGLMP function $I_{d}^{max}\left(\left|\Psi_{mv}^{d}\right\rangle\right)$ up to $d \leq 7$. Values for γ_{0d} are not presented because those are all equal to one.

d	γ_{1d}	γ_{2d}	γ_{3d}	γ_{4d}	γ_{5d}	γ_{6d}	$I_{max}^{d}\left(\Psi_{mv}^{d}\right)$	$I_{max}^{d}\left(\Psi_{me}^{d}\right)$
2	1.000	-	-	-	-	-	2.8284	2.8284
3	0.792	1.000	-	-	-	-	2.9148	2.8729
4	0.739	0.739	1.000	-	-	-	2.9727	2.8962
5	0.719	0.660	0.719	1.000	-	-	3.0157	2.9105
6	0.709	0.626	0.626	0.709	1.000	-	3.0497	2.9202
7	0.705	0.607	0.581	0.607	0.705	1.000	3.0776	2.9272

The calculation of the probability of violation in this case is carried out in the same fashion as in the last part of the previous chapter, by using the results of appendix C and generalizing expression 3.7, for arbitrary d:

$$p_v(|\Psi\rangle) = \frac{4}{(2\pi)^{4d}} \int_{\Gamma_{|\Psi\rangle}} \prod_{x,y=1}^{2} \prod_{j,k=0}^{d-1} d\phi_x^j d\varphi_y^k. \tag{4.2}$$

The results presented in this chapter have been obtained via Monte Carlo integrations, corresponding to several runs of a Bell experiment using uniform random measurement configurations on a definite quantum state. Calculations of the probability of violation of pairs of qudits in maximally entangled states (MES) and maximally violating states (MVS) under the CGLMP inequality and MBSPS measurements were carried out up to $d = 7$. The results are shown in a monolog plot in figure 13. As it can be seen, the higher the dimension, the lower the probability of violation. In this way it is possible to conclude that the nonlocal content of a quantum entangled state of two qudits under the considered scenario and the CGLMP inequality exponentially decreases with the dimensionality of the system, which is in agreement with the notion of restoration of classical features in the limit of high quantum numbers. However, we stress that the CGLMP scenario refers to two observables per party, no matter the value of d. We found that the exponential decay behaviour assumes a particularly simple form if we use 2π as the basis (this is a natural basis in MBSPS scenarios). The points are well described by:

$$p_v(d) \sim (2\pi)^{-d}, \tag{4.3}$$

where $p_v(d)$ refers to the maximally entangled state (MES) of two qudits with d levels each. In figure 13, these points are represented by (red) triangles, and the upper continuous line corresponds to the best fitting with $p_v(d) \sim (2\pi)^{-1.04d}$. The squares correspond to the states that yield the maximal numeric violation of the CGLMP inequality. Except for $d = 2$ (for which equal probabilities are obtained), the MES present a higher probability in comparison with the maximally violating states. The probability of violation for the MVS's drops off approximately as $p_v(d) \sim (2\pi)^{-1.07d}$. This extends the conclusion of the previous chapter, showing that there is no anomaly in the nonlocality of two entangled qudits up to $d = 7$, at least in the CGLMP scenario, when p_v is used as a figure of merit.

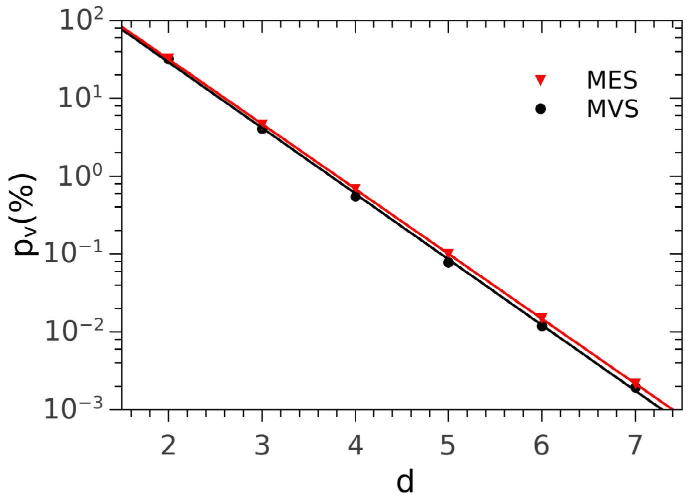

Figure 13 – Monolog plot of the probabilities of violation (in percents) of the maximally entangled state (MES) and maximally violating state (MVS) as a function of the dimension d (CGLMP inequality and MBSPS measurements). The amount nonlocality decreases exponentially with the dimension. Note that apart from the qubit case $(d = 2)$, the MES presents more nonlocality than the MVS. Results obtained after 10^{10} runs of the code on each point

The MES and MVS coincide for $d = 2$, and $p_v(d = 2) \approx 0.32$, which shows that the restriction to MBSPS measurements increases the probability of violation. As already mentioned, for general measurements, the probability of violation is around 0.28 for maximally entangled states, since the CGLMP and the CHSH inequalities are equivalent for $d = 2$. Furthermore, a similar result appears when, in the CHSH scenario, the parties previously agree on one of the measurement directions. With this the inequality becomes the first Bell inequality, for which $p_v = 1/3 \approx 0.33$ [Parisio 2016].

Now we address another interesting case: low rank states.

4.3 Rank vs Dimensionality under MBSPS

Regarding two qudits, MES are also maximally symmetric. However, one can consider maximally symmetric states (MSS) with Schmidt ranks k, such that $k < d$, which are not maximally entangled. In this case, the inequivalence between

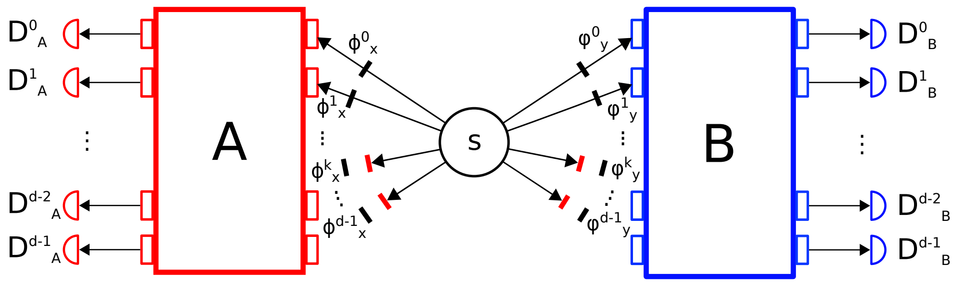

Figure 14 – Multiport beam splitters and phase shifters setup for low rank states. Note that the source has $d-k$ output channels blocked on each side. For this setup the state coefficients α_j must be interpreted as the probability amplitude of a photon to reach j-th phase shifter.

MSS's and states that maximize p_v reappears for the CGLMP inequality. In spite of the balancedness of states like $(|00\rangle + |11\rangle + \cdots + |(k-1),(k-1)\rangle)/\sqrt{r}$ due to the fact that the basis kets $|kk\rangle, \cdots, |d-1,d-1\rangle$ are missing, they are not maximally nonlocal, in the CGLMP scenario. However, this doesn't constitute a true anomaly, since the symmetric low rank states cannot be considered maximally entangled. The investigation of states with lower ranks will provide a clear illustration of how different can the results be when a single Bell inequality is considered instead of the full space of behaviours.

As a starting point we carried out a numerical search of k-rank states violating the CGLMP inequality for d-outcomes. We considered the experimental scenario given in figure 14. The state of the system before the photons reach the phase shifters may be written as $|\phi\rangle = \sum_{j=0}^{k-1} \alpha_j |jj\rangle$, with:

$$\alpha_j = \begin{cases} \cos\theta_0 & \text{for } j=0 \\ \sin\theta_0 \ \ldots \ \sin\theta_{j-1}\cos\theta_j & \text{for } 1 \leq j \leq k-2 \\ \sin\theta_0 \ \ldots \ \sin\theta_{k-3}\sin\theta_{k-2} & \text{for } j = k-1, \end{cases} \tag{4.4}$$

with $0 < \theta_j \leq \pi/2$.

Results of the search are summarized in table 2 for states up to $k = 7$ under a CGLMP scenario and $d \leq 12$. The most interesting observation is that while rank-2 and rank-3 states only violate the CGLMP inequality up to $d = 2$ and $d = 5$ respectively, states with rank larger than $k = 4$ are capable of violating the CGLMP inequality at least up to $d = 12$.

Table 2 – Violation of the CGLMP inequality for d-outcomes by a k-rank state under a MBSPS scheme (see fig. 14). The symbol "✓"("–") denotes the cases in which we were (not) able to observe violation of the CGLMP inequality (no matter by how much), and "✓"indicates the trivial case $k = d$.

		d										
		2	3	4	5	6	7	8	9	10	11	12
k	2	✓	–	–	–	–	–	–	–	–	–	–
	3		✓	✓	✓	–	–	–	–	–	–	–
	4			✓	✓	✓	✓	✓	✓	✓	✓	✓
	5				✓	✓	✓	✓	✓	✓	✓	✓
	6					✓	✓	✓	✓	✓	✓	✓
	7						✓	✓	✓	✓	✓	✓

As an example, let us consider the family of states (with zero as the coefficient of $|33\rangle$):

$$\cos\theta_0|00\rangle + \sin\theta_0\cos\theta_1|11\rangle + \sin\theta_0\sin\theta_1|22\rangle. \tag{4.5}$$

In Fig. 15 we plot p_v for the above rank-3 states with $d = 4$, as a function of θ_0 and θ_1. The balanced state ($\alpha_j = 1/\sqrt{3}$) is identified by the cross, while the two states that maximize the probability of violation are given by $(\theta_0, \theta_1) \approx (0.864, 0.604)$,

$$\approx 0.647\,|00\rangle + 0.628\,|11\rangle + 0.431\,|22\rangle\,; \tag{4.6}$$

and $(\theta_0, \theta_1) \approx (1.126, 0.798)$ (equivalent to the above state with $|00\rangle \leftrightarrow |22\rangle$), $p_v \approx 0.224 \times p_v$(MES), where p_v(MES), refers to the full rank maximally entangled state.

Analogous calculations were carried out for $r = 3$ and $d = 5$, results are presented in figure 16. In this case the states with larger probability of violation corresponds to $(\theta_0, \theta_1) \approx (0.840, 0.585)$ and $(\theta_0, \theta_1) \approx (1.120, 0.809)$,

$$\approx 0.667\,|00\rangle + 0.621\,|11\rangle + 0.411\,|22\rangle\,, \tag{4.7}$$

with $p_v \approx 1.88 \cdot 10^{-3}$ (approximately 2.4% of the probability of violation of the $d = 5$ maximally entangled state, under the CGLMP context for $d = 5$). A very interesting observation here is that the balanced state (represented by a cross in the figure) does not lead to violation of the CGLMP inequality.

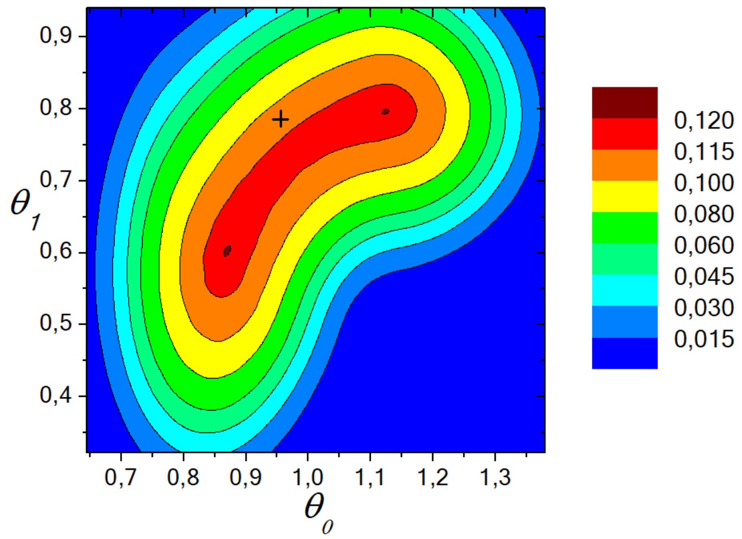

Figure 15 – Probability of violation (%) for rank-3 states with $d = 4$ in the context of the CGLMP inequality. The cross corresponds to the state $(|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$, and the lower-left darker spot corresponds to state (4.6). Results obtained after 10^8 runs of the code on each point.

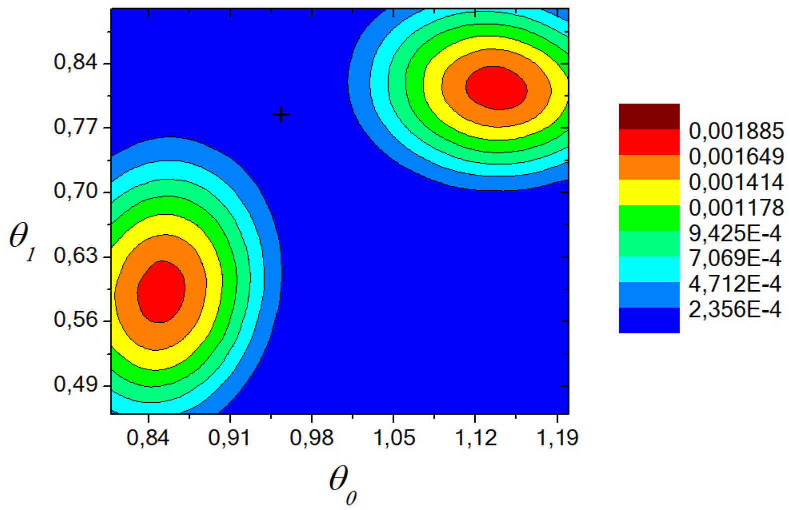

Figure 16 – Probability of violation (%) for rank-3 states under the CGLMP inequality for $d = 5$. As before, the cross corresponds to the state $(|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$, but interestingly, in this case the probability of violation is null. Results obtained after 10^8 runs of the code on each point.

4.4 Probability of violation on the space of behaviors

The results presented in this chapter are part of a work in collaboration with Anna de Rosier and Wiesław Laskowski from the University of Gdànsk in Poland, and Tamás Vértesi from the Institute for Nuclear Research in Hungary [Fonseca et al. 2018]. Part of their contribution is the calculation of the probability of violation for the same states we worked but from a more general perspective, the space of behaviours (see section 3.3.2 and equation 3.2). Several results are qualitatively very similar, for instance the probability of violation also suffers a drop with the dimension d, but in this case the decrease is more moderate, showing an almost linear behaviour compared to the exponential decay of the nonlocality observed within the MBSPS and the CGLMP inequality context[1] (see Fig. 4 in [Fonseca et al. 2018]). It may be due to the fact that there is no restriction to a particular Bell inequality because all relevant scenarios with a fixed number of observables per party are simultaneously considered. There also no anomaly shows up.

Differently from what we observed in the CGLMP-MBSPS scenario, it was found that balanced states with any rank larger than 1, present a nonvanishing probability of violation. For instance, for $r = 2$ and $d = 6$ it was found that 0.173% of the possible behaviours are outside the local polytope, while for $r = d = 6$ this percentage is about 9.3%.

Another interesting feature is the strong enhancement in the ability to detect nonlocality by increasing the number of observables per party. In the simplest case of two entangled qubits, this amounts to a change from $p_v \approx 28.3\%$ to $p_v \approx 78.2\%$ for MES when m is increased from 2 to 3. For $d = r = 5$, the probabilities of violation for 2 and 3 observables per party are 12.7% and 56.5%, respectively. As we have already mentioned, similar results have been recently reported in a variety of contexts [Lipinska et al. 2018, De Rosier et al. 2017]. Particularly, in [Lipinska et al. 2018] it is shown that for any pure bipartite entangled state, p_v tends to unity whenever the number of measurement choices

[1] Nevertheless, Barasinski and Nowotarski have recently presented results of calculations of the volume of violation associated to the CGLMP inequality in a slightly different context, in which an *anomaly* is observed for $d \geq 8$ [Barasiński e Nowotarski 2018].

(of the two parties) tends to infinity.

Finally, they addressed the family of states in Eq. 4.5, this time considering all possible behaviours. The results for the probability of violation show a much more symmetric shape than the corresponding plot, restricted to the CGLMP-MBSPS scenario (figure 15). In this case they were able to show that the balanced state with $r = 3$ ($\alpha_j = 1/\sqrt{3}$), under the scenario $d = 4$ attains the maximum of nonlocality, concluding thus that the apparent asymmetry revealed in figure 15 is mainly due to the bias introduced by the choice of a particular facet of the local polytope. Since the number of relevant Bell inequalities grows with the dimension, the effect of this bias tend to increase with d.

4.5 Closing remarks

The goal of the present chapter was to study quantum nonlocality in bipartite systems of high dimensionality. The results presented in Fig. 13 showed that the extent of nonlocality decreases with the dimension of the qudits in both, the CGLMP scenario and in the space of behaviours ($d \leq 7$). The decay being exponential for the particular Bell inequality we addressed and much slower, at most linear, when all possible behaviours are considered. It was additionally shown that, within both approaches, no anomaly of nonlocality showed up, with p_v as the figure of merit.

The qualitative agreement between the two approaches ceases to hold when maximally symmetric states of lower rank ($r < d$) are considered. While in the fixed Bell scenario we observed that the balanced states are not maximally nonlocal, we found numerical evidence that, whenever the entire local polytope is considered this is no longer true. This may be understood as an effect of the increasing (as d grows) bias introduced by the choice of a particular facet. This is a further indication that the probability of violation defined in the space of behaviours is a more fundamental quantity as compared to the volume of violation of a particular Bell scenario.

The regime of large d may be, at least in some sense, considered as a classical limit, and then, we should observe local behaviours as the dominant ones. However, we may as well conceive the classical limit as a large gathering of two-level systems, which leads to an apparent contradiction. It has been shown that the probability of violation strongly increases with the number N of qubits, and two observables per party [González-Guillén et al. 2016, De Rosier et al. 2017]. In fact, random states of 5 qubits typically present $p_v > 0.99$ [De Rosier et al. 2017] and nonlocality becomes completely dominant for large N. We remark that this is not a loose comparison because there is an isomorphism between the Hilbert space of a system with N qubits (for simplicity we assume N to be even) and the Hilbert space of two qudits with $d = 2^{N/2}$ levels, each. How do we get opposite trends in the limit $N \to \infty$, and consequently in the limit $d \to \infty$? The point is that, in both cases, we have two observables per party, but this amounts to quite different physical situations. In the N-qubit case we have two observables per qubit, say $A_1, A_2; B_1, B_2; C_1, C_2;$ etc. Since each observable is dichotomous, we have 4 possibilities involving the choice of observables and potential outcomes for every qubit. This leads to a total of $4^N = 2^{2N}$ independent possibilities. In the case of 2 qudits with dimension $d = 2^{N/2}$ we only have four observables: $\mathcal{A}_1, \mathcal{A}_2; \mathcal{B}_1, \mathcal{B}_2$, each with $2^{N/2}$ outputs, leading to a total of $4 \times 2^{N/2} \times 2^{N/2} = 2^{N+2}$ possibilities. So, the four many-output observables in the latter case are not sufficient to compensate for the $2N$ dichotomic observables in the former situation. Of course, in practice, it may become increasingly hard to address individual qubits in the large-N regime.

Here we conclude the presentation of results regarding nonlocality. The second part of this thesis is devoted to present some results related to the noise influence in the quality of protocols in quantum information.

5 USING THREE-PARTITE GHZ STATES FOR PARTIAL QUANTUM ERROR-DETECTION IN ENTANGLEMENT-BASED PROTOCOLS

Highlights

It is shown that the use of a three-partite GHZ state and measurements instead of their EPR counterparts leads to an increase in the efficiency under certain noisy scenarios.

The idea is extended to any protocol using entangled states and measurements.

It is provided a generalization for N-GHZ states and measurements and it is concluded that the optimal number of qubits is only three.

5.1 Introduction

It has been just over two decades since the appearance of error-correction schemes for quantum systems first introduced in a seminal paper by Shor [Shor 1995] and further extended 1 year later by Steane [Steane 1996]. Nowadays these protocols represent a cornerstone in quantum information science (QIS) due to the role played toward the possibility of building quantum devices large enough to be able to improve processing capacity and information storage stability when compared to classical counterparts [Terhal 2015]. Shor's work inspired several theoretical extensions and experimental realizations, and today it represents a very active area in quantum information. For a deeper exploration and recent progresses, we refer the reader to [Terhal 2015, Devitt, Munro e Nemoto 2013, Raussendorf 2012].

Entanglement is one of the essential elements in quantum error correction codes (QEC) and hence on the feasibility of quantum computation [Horodecki et al. 2009]. Besides QEC applications, entanglement is also a key resource for a large variety of tasks in QIS [Vedral e Plenio 1998, Brandao e Plenio 2008], among the most known we find: Ekert's quantum key distribution [Ekert 1991], superdense coding [Bennett e Wiesner 1992] and the teleportation protocol [Bennett et al. 1993]. In fact, an entire quantum computer can be conceived where entanglement provides all the basic structure [Raussendorf e Briegel 2001].

Due to the importance of these protocols, several efforts have been put toward their implementation under more realistic frameworks, i.e., considering the effect of interactions with the environment. In reference [Taketani, Melo e Filho 2012], the action of noise in the teleportation protocol is contemplated and an optimal protocol is derived. A set of strategies to improve the fidelity in quantum teleportation under different kinds of noise is proposed in [Fortes e Rigolin 2015]. In addition, several schemes comparing multipartite channels were considered in [Jung et al. 2008].

In the present chapter we introduce a scheme for partial error detection concerning protocols based on bipartite entanglement between qubits. We show that this is possible by literally replacing EPR states by tripartite Greenberger–Horne–Zeilinger (GHZ) states [Greenberger, Horne e Zeilinger 1989, Greenberger et al. 1990], and whenever needed, instead of EPR measurements we use the GHZ basis. Our procedure, inspired by some results reported in [Cunha et al. 2017], follows the basic ideas of reference [Grassl, Beth e Pellizzari 1997], using an ancillary system which allows for detection of the noise incidence and post-selection of the desired outputs, a process that has been considered as a viable mean of computation [Knill 2005]. On the one hand, our protocol is in a certain way limited, for it only permits a partial detection of one kind of noise, say bit-flip; on the other hand, it is far less expensive in terms of resources: while the QEC proposed by Shor [Shor 1995] demands two extra qubits to detect bit-flip in each qubit of memory, our proposal demands only one to reveal noise on two qubits. Furthermore, our process does not demand an adjacent computation to be implemented; instead, it is only necessary to adjust some steps of the existing task.

The chapter is organized as follows: First we describe the approach used to model the effect of noise on the system in terms of the Kraus operators, and the domain of validity of the model. In Sect. 5.2, we compare some protocols using EPR states to the case when these are replaced by GHZ counterparts. In both cases, we consider perfect realization (perfect in the sense of production of states and completion of ideal measurements) and the presence of noise. In Sect. 5.4, the ideas exposed previously are extended in order to cover any protocol using pairs of entangled qubits and EPR measurements. Section 5.5 is devoted to show the optimality of the protocol for the case of $N = 3$. In the last section, the main results are discussed and some conclusions are given.

5.2 Bit flip noise

In general terms, it is possible to include the noise effect on a quantum system by employing several approaches. In this thesis, we are not interested in the dynamics of the system in a detailed way; thus, we can use the formalism of Kraus operators [Nielsen e Chuang 2010]. This approach provides a practical way to describe several types of errors that may take place during the experimental implementation of quantum protocols. In this chapter, our main concern is the study of a system affected by bit-flip noise; under which a qubit initially prepared in a state $|j\rangle$ is modified as:

$$|j\rangle \rightarrow |j \oplus 1\rangle$$

where the symbol "$\oplus$" denotes sum modulo 2. Given a N-partite system, if the k-th qubit may with probability p be affected, the Kraus operators read:

$$\hat{A}_0 = \sqrt{1-p}\,\hat{\mathbb{1}}_1 \otimes \cdots \otimes \hat{\mathbb{1}}_N, \quad \hat{A}_1 = \sqrt{p}\,\hat{\mathbb{1}}_1 \otimes \cdots \otimes \hat{\sigma}_x^{(k)} \otimes \cdots \otimes \hat{\mathbb{1}}_N.$$

In the Kraus formalism, the quantum state evolution after the interaction with the environment may be described as a map:

$$\hat{\rho} \in \mathcal{B}(\mathcal{H}) \longmapsto \varepsilon(\hat{\rho}) \in \mathcal{B}(\mathcal{H}), \tag{5.1}$$

where $\mathcal{B}(\mathcal{H})$ is the space of the bounded operators on the Hilbert space $\mathcal{H}$. More explicitly, $\varepsilon(\hat{\rho})$ is given by:

$$\varepsilon(\hat{\rho}) = \sum_j \hat{A}_j \hat{\rho} \hat{A}_j^\dagger, \tag{5.2}$$

in this way, when a single qubit described by $\hat{\rho} = |\phi\rangle\langle\phi|$ is affected by bit-flip noise, we have

$$\varepsilon(\hat{\rho}) = (1-p)\,|\phi\rangle\langle\phi| + p\hat{\sigma}_x\,|\phi\rangle\langle\phi|\,\hat{\sigma}_x^\dagger. \tag{5.3}$$

For a composite system of N qubits subjected to bit-flip noise acting locally in each subsystem, we have:

$$\varepsilon(\hat{\rho}) = \sum_{j_1,\ldots,j_N=0}^{1} \hat{A}_{j_1} \otimes \cdots \otimes \hat{A}_{j_N}\ \hat{\rho}\ \hat{A}_{j_1}^\dagger \otimes \cdots \otimes \hat{A}_{j_N}^\dagger. \tag{5.4}$$

Let us assume that every part of the system may be affected with equal probability p, and moreover, we restrict ourselves to the weak noise regime, i.e., the probability is low enough in order to ensure that events in which we have at least two qubits affected are very unlikely compared to those where there is only one. In this way, after some calculations, Eq. 5.4 is reduced to:

$$\begin{aligned}\varepsilon(\hat{\rho}) \approx (1-Np)\hat{\rho} + p\Big\{ &\left(\hat{\sigma}_x \otimes \hat{\mathbb{1}} \otimes \cdots \otimes \hat{\mathbb{1}}\right) \hat{\rho} \left(\hat{\sigma}_x \otimes \hat{\mathbb{1}} \otimes \cdots \otimes \hat{\mathbb{1}}\right) + \\ &+ \left(\hat{\mathbb{1}} \otimes \hat{\sigma}_x \otimes \cdots \otimes \hat{\mathbb{1}}\right) \hat{\rho} \left(\hat{\mathbb{1}} \otimes \hat{\sigma}_x \otimes \cdots \otimes \hat{\mathbb{1}}\right) + \ \cdots \ + \\ &+ \left(\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \cdots \otimes \hat{\sigma}_x\right) \hat{\rho} \left(\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \cdots \otimes \hat{\sigma}_x\right) \Big\} + \mathcal{O}(p^2) f(\hat{\rho}),\end{aligned} \tag{5.5}$$

where $f(\hat{\rho})$ represents higher order perturbations on the initially prepared state $\hat{\rho}$.

Note however that from this point we additionally restrict ourselves to the case in which Bob is capable of protecting his qubit perfectly. In this way, the term $\left(\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \cdots \otimes \hat{\sigma}_x\right) \hat{\rho} \left(\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \cdots \otimes \hat{\sigma}_x\right)$ does not appear in forthcoming calculations[1].

[1] It can be experimentally seen as the case in which the source of entangled particles is in Bob's side and then he sends them to Alice's location.

5.3 Comparison between strategies

In this section, we provide a comparative overview between two possible strategies to develop two very important and well-known protocols in QIS: quantum teleportation and superdense coding. We present the protocols in two scenarios: the first one corresponding to the traditional way, using EPR pairs weakly subject to bit-flip noise and EPR measurements. In the second scenario, we replace all EPR states present in the system by noisy three-qubit GHZ states and GHZ measurements.

5.3.1 Quantum Teleportation

Proposed initially by Bennet et al. and collaborators [Bennett et al. 1993] and posteriorly experimentally implemented [Bouwmeester et al. 1997], the quantum teleportation protocol represents a very important subject because it illustrates how quantum mechanics can be used to develop new types of communication technologies [Pirandola et al. 2015], and remarkably, in recent times two realizations that make the protocol feasible in the context of global communications [Yin et al. 2017, Ren et al. 2017] have been reported.

Let us start analizing the traditional scenario using EPR pairs, and in the following, we consider the teleportation scheme using a GHZ state as the channel.

Teleportation using EPR states and measurements

Recall the EPR basis $\{|\psi_{mn}\rangle\}$, whose elements are given by:

$$|\psi_{mn}\rangle = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} (-1)^{mj} |j, j \oplus n\rangle , \tag{5.6}$$

where $m, n \in \{0, 1\}$. In the same way, the projector of the (m, n) EPR state is given by:

$$\hat{\Pi}_{mn} \equiv |\psi_{mn}\rangle\langle\psi_{mn}| . \tag{5.7}$$

As already mentioned, the set $\{\hat{\Pi}_{mn}\}$ forms a complete basis for the two-qubit Hilbert space (also known as Bell basis) and any element $|\psi_{jk}\rangle$ may be obtained by application of Pauli matrices on the state $|\psi_{00}\rangle$:

$$|\psi_{mn}\rangle = (\hat{\sigma}_z^m \otimes \hat{\sigma}_x^n)\,|\psi_{00}\rangle\,, \tag{5.8}$$

where $\hat{\sigma}_\mu^k$ indicates k times the "μ" Pauli matrix.

The goal of the teleportation protocol is to send an *a priori* unknown state $|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ from one part; let us say Alice whose qubit state lie on the Hilbert space $\mathcal{H}_A$, to a distant part, hereafter Bob, possessing a qubit on the Hilbert space $\mathcal{H}_B$. Initially, Alice and Bob share an EPR state; thus, the total quantum state of the system is described by:

$$\hat{\rho}_o = |\Psi\rangle\langle\Psi|_A \otimes |\psi_{00}\rangle\langle\psi_{00}|_{AB}\,,$$

$$\hat{\rho}_o = \frac{1}{2}\sum_{jkmn=0}^{1} \alpha_j\alpha_k^*\,|jm\rangle\langle kn|_A \otimes |m\rangle\langle n|_B\,.$$

Decomposing Alice's part in the Bell basis, using the relation between the computational and Bell basis $|mn\rangle = \sum_k (-1)^{km}\,|\psi_{k,m\oplus n}\rangle\,/\sqrt{2}$ and after some calculations, it is possible to show that:

$$\hat{\rho}_o = \frac{1}{4}\sum_{mn=0}^{1} \hat{\Pi}_{mn}^{(A)} \otimes \left(\hat{\sigma}_x^n\hat{\sigma}_z^m\,|\Psi\rangle\langle\Psi|_B\,\hat{\sigma}_z^m\hat{\sigma}_x^n\right) + \hat{\rho}_{null}. \tag{5.9}$$

$\hat{\rho}_{null}$ corresponds to the non-diagonal part of $\hat{\rho}_o$ (i.e., terms proportional to $|\psi_{kl}\rangle\langle\psi_{mn}|$ with $k \neq m$ and $l \neq n$ on the Alice's part of the system). In forthcoming decompositions, we use the same notation.

In the next step, Alice performs a projective measurement in the EPR basis and according to her output, she tells Bob how to adjust his state with one out of the set of operations $\{\mathbb{1}, \sigma_x, \sigma_z, \sigma_z\sigma_x\}$. We can represent the teleportation protocol by the transformation $T : \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}(\mathcal{H}_\mathcal{B}) \mapsto \mathcal{B}(\mathcal{H}_\mathcal{B})$, where $\mathcal{B}(\mathcal{H}_\mathcal{B})$ is the space of the bounded operators on $\mathcal{H}_\mathcal{B}$:

$$T(\hat{C}) = \mathrm{Tr}_A\left\{\sum_{mn=0}^{1}\left(\hat{\Pi}_{mn} \otimes \hat{\sigma}_z^m\hat{\sigma}_x^n\right)\hat{C}\left(\hat{\Pi}_{mn} \otimes \hat{\sigma}_x^n\hat{\sigma}_z^m\right)\right\}, \tag{5.10}$$

where Tr_A represents the partial trace over Alice's part. It is straightforward to show that the application of the transformation T on the initial state $\hat{\rho}_o$ gives:

$$T(\hat{\rho}_o) = |\Psi\rangle\langle\Psi|_B\,, \tag{5.11}$$

as required by the teleportation protocol.

Now we assume that before the teleportation process, a small amount of bit-flip noise affects Alice's part. Thus, the state of the system (Alice + Bob) is modified as $\hat{\rho}_o \to \hat{\varrho}$:

$$\hat{\varrho} = (1-2p)\hat{\rho}_o + p\Big\{ \left(\hat{\sigma}_x \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}\right) \hat{\rho}_o \left(\hat{\sigma}_x \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}\right) + \left(\hat{\mathbb{1}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{1}}\right) \hat{\rho}_o \left(\hat{\mathbb{1}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{1}}\right) \Big\} + \\ + \mathcal{O}(p^2) f(\hat{\rho}_o),$$

where $f(\hat{\rho}_o)$ is an operation on the initial state $\hat{\rho}_o$, which is not interesting under the scope of the present work. By noticing that $\hat{\sigma}_x^k \otimes \hat{\mathbb{1}}\,|\psi_{mn}\rangle = (-1)^{km}\,|\psi_{m,n\oplus k}\rangle$ and $\hat{\mathbb{1}} \otimes \hat{\sigma}_x^k\,|\psi_{mn}\rangle = |\psi_{m,n\oplus k}\rangle$, and after some calculations the density operator reduces to

$$\hat{\varrho} = (1-2p)\hat{\rho}_d + \frac{p}{2}\sum_{mn=0}^{1} \hat{\Pi}^{(A)}_{m,n\oplus 1} \otimes \left(\hat{\sigma}_x^n \hat{\sigma}_z^m\,|\Psi\rangle\langle\Psi|_B\,\hat{\sigma}_z^m \hat{\sigma}_x^n\right) + \hat{\varrho}_{null}, \tag{5.12}$$

where $\hat{\rho}_d$ represents the non-null part of $\hat{\rho}_o$ (in Eq. 5.9).

Note above that the effect of the noise in the final state is to produce an extra term where the adjustment to be performed according to the output of the measurement is not the correct one. Proceeding with the protocol will have:

$$T(\hat{\varrho}) = (1-2p)\,|\Psi\rangle\langle\Psi| + 2p(\hat{\sigma}_x\,|\Psi\rangle\langle\Psi|\,\hat{\sigma}_x). \tag{5.13}$$

Hence, only with probability $(1-2p)$ Bob's part ends up in the desired state.

Now we turn our attention to the teleportation protocol, but instead of EPR states, in this case we use three-partite GHZ states and measurements.

Teleportation using GHZ states and measurements

First let us recall the three partite GHZ basis elements and related projectors:

$$|\phi_{kmn}\rangle = \frac{1}{\sqrt{2}}\sum_{j=0}^{1}(-1)^{kj}\,|j, j\oplus m, j\oplus n\rangle\,, \tag{5.14}$$

and

$$\hat{\Pi}'_{kmn} \equiv |\phi_{kmn}\rangle\langle\phi_{kmn}|\,, \tag{5.15}$$

where $k, m, n \in \{0, 1\}$. Analogously to the EPR basis, any element $|\phi_{kmn}\rangle$ may be obtained by application of Pauli matrices on the state $|\phi_{000}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$:

$$|\phi_{kmn}\rangle = \left(\hat{\sigma}_z^k \otimes \hat{\sigma}_x^m \otimes \hat{\sigma}_x^n\right)|\phi_{000}\rangle\,. \tag{5.16}$$

Now Alice and Bob share three entangled qubits prepared in a GHZ state $|\phi_{000}\rangle$: two qubits are held by Alice and the other one by Bob. Alice's objective is again to send an unknown qubit state $|\Psi\rangle$ to Bob. The initial state of the four qubits reads:

$$\hat{\rho}'_o = |\Psi\rangle\langle\Psi|_A \otimes |\phi_{000}\rangle\langle\phi_{000}|_{AAB}\,,$$

$$\hat{\rho}'_o = \frac{1}{2}\sum_{jkmn=0}^{1}\alpha_j\alpha_k^*\,|jmm\rangle\langle knn|_A \otimes |m\rangle\langle n|_B\,. \tag{5.17}$$

We can use the relation $|kmn\rangle = \sum_j (-1)^{jk}\,|\phi_{j,k\oplus m,k\oplus n}\rangle/\sqrt{2}$ to rewrite Eq. 5.17 in the following way:

$$\hat{\rho}'_o = \frac{1}{4}\sum_{mn=0}^{1}\hat{\Pi}'^{(A)}_{mnn} \otimes \left(\hat{\sigma}_x^n\hat{\sigma}_z^m\,|\Psi\rangle\langle\Psi|_B\,\hat{\sigma}_z^m\hat{\sigma}_x^n\right) + \hat{\rho}'_{null}. \tag{5.18}$$

From the projector on Alice's part, we can see that instead of eight, there are only four possible outputs for a measurement in the GHZ basis, and the desired operation to accomplish the teleportation process can be described by:

$$T'(\hat{C}) = \mathrm{Tr}_A\left\{\sum_{mn=0}^{1}\left(\hat{\Pi}'_{mnn}\otimes\hat{\sigma}_z^m\hat{\sigma}_x^n\right)\hat{C}\left(\hat{\Pi}'_{mnn}\otimes\hat{\sigma}_x^n\hat{\sigma}_z^m\right)\right\}, \tag{5.19}$$

It is straightforward to show that $T'(\hat{\rho}'_o) = |\Psi\rangle\langle\Psi|_B$, as expected.

As in the previous case, before the teleportation protocol takes process, we consider bit-flip noise acting on all Alice's qubits under a weak regime. Thus, we have $\hat{\rho}'_o \to \hat{\varrho}'$:

$$\begin{aligned}\hat{\varrho}' = (1-3p)\hat{\rho}'_o + p\Big\{ &\left(\hat{\sigma}_x \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}\right) \hat{\rho}'_o \left(\hat{\sigma}_x \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}\right) + \\ &+ \left(\hat{\mathbb{1}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}\right) \hat{\rho}'_o \left(\hat{\mathbb{1}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}}\right) + \\ &+ \left(\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{1}}\right) \hat{\rho}'_o \left(\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{1}}\right) \Big\} + \mathcal{O}(p^2) f'(\hat{\rho}'_o).\end{aligned}$$

In this case, we take into account the following relations: $\hat{\sigma}_x^j \otimes \hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \left|\phi_{k,m,n}\right\rangle = (-1)^{jk} \left|\phi_{k,m\oplus j,n\oplus j}\right\rangle$, $\hat{\mathbb{1}} \otimes \hat{\sigma}_x^j \otimes \hat{\mathbb{1}} \left|\phi_{k,m,n}\right\rangle = \left|\phi_{k,m\oplus j,n}\right\rangle$ and $\hat{\mathbb{1}} \otimes \hat{\mathbb{1}} \otimes \hat{\sigma}_x^j \left|\phi_{k,m,n}\right\rangle = \left|\phi_{k,m,n\oplus j}\right\rangle$. Substituting, we have:

$$\begin{aligned}\hat{\varrho}' &= (1-3p)\hat{\rho}'_d + \\ &+ \frac{p}{4} \sum_{mn=0}^{1} \left(\hat{\Pi}'^{(A)}_{m,n\oplus 1,n\oplus 1} + \hat{\Pi}'^{(A)}_{m,n\oplus 1,n} + \hat{\Pi}'^{(A)}_{m,n,n\oplus 1}\right) \otimes \left(\hat{\sigma}_x^n \hat{\sigma}_z^m \left|\Psi\right\rangle\!\left\langle\Psi\right|_B \hat{\sigma}_z^m \hat{\sigma}_x^n\right) + \hat{\varrho}'_{null},\end{aligned}$$

where as previously, $\hat{\rho}'_d$ is the diagonal part of $\hat{\rho}'_o$ in Eq. 5.18.

While in the ideal case we would only get four outputs, now we observe four additional measurement results $\{\left|\phi_{001}\right\rangle, \left|\phi_{010}\right\rangle, \left|\phi_{101}\right\rangle, \left|\phi_{110}\right\rangle\}$ due to the incidence of noise. At this point we cannot proceed as in the traditional error syndrome, because we can only determine when noise has affected one of the GHZ qubits in Alice, but we are not capable of identifying on which one specifically. However, we can take advantage of this partial knowledge to perform a post-selection during the protocol: discarding Bob's qubits whenever "undesired" outputs are obtained. In other words, the efficiency is decreased in favour of the precision of the protocol.

Following this process and performing the proper renormalization, the final state reads:

$$\hat{\varrho}_F = \frac{1-3p}{1-2p} |\Psi\rangle\langle\Psi| + \frac{p}{1-2p} \hat{\sigma}_x \left|\Psi\right\rangle\!\left\langle\Psi\right| \hat{\sigma}_x, \tag{5.20}$$

which is better than the result obtained in the protocol using EPR states (eq. 5.13) for $p < 1/4$ i.e. under a weak noise regime.

5.3.2 Superdense Coding

Here, we analyze the protocol of superdense coding, theoretically proposed in [Bennett e Wiesner 1992] and experimentally implemented in [Mattle et al. 1996], whose importance lies on its capability of reaching a compression factor of 2 per message. The scenario consists in two parts: Alice (the sender), sharing an EPR state $|\psi_{00}\rangle$ with Bob (the receiver).

Through the application of the operations $\hat{\sigma}_x^{a_2}\hat{\sigma}_z^{a_1}$, Alice encodes a two-bit message (a_1, a_2) on her qubit which is afterward physically sent to Bob. The receiver (now holding both qubits) performs a measurement in the EPR basis and recovers the original message according to his output: $|\psi_{a_1,a_2}\rangle$.

If nevertheless before Bob's measurement the system is affected by bit-flip noise, then under the regime of weak noise, the resulting state is given by:

$$\hat{\varrho} = (1-2p)\hat{\Pi}_{a_1,a_2} + 2p\hat{\Pi}_{a_1,a_2\oplus 1}, \tag{5.21}$$

thus only a fraction of $1-2p$ outputs will lead to a proper interpretation of the original message.

It is possible to think on a different procedure to carry out this protocol. Assume that Alice and Bob share a GHZ state $|\phi_{000}\rangle_{ABB}$ instead of an EPR state. Alice encodes her message in the same way as before and sends her qubit to Bob who performs a measurement in the GHZ basis. He recovers Alice's string by associating the output $|\phi_{k,m,n}\rangle$ with the string (k, m). Notice that, so far, the outputs with $m \neq n$ are not possible; however when the action of noise on the system is considered (before Bob's measurement), the system is described by:

$$\hat{\varrho}' = (1-3p)\hat{\Pi}'_{a_1,a_2,a_2} + p\left(\hat{\Pi}'_{a_1,a_2\oplus 1,a_2\oplus 1} + \hat{\Pi}'_{a_1,a_2\oplus 1,a_2} + \hat{\Pi}'_{a_1,a_2,a_2\oplus 1}\right). \tag{5.22}$$

In this case, after performing a measurement in the GHZ basis, Bob will receive a fraction of $1-3p$ of correct signal. However, he is capable of detecting that a flip has happened by observing the detection of “disallowed outputs" ($\hat{\Pi}'_{k,m,n}$ for $m \neq n$). He can thus discard those results and increase to a rate of $(1-3p)/(1-2p)$ accurate strings, which is better than $1-2p$ in the weak noise regime.

5.4 General protocol

5.4.1 Any protocol involving EPR pairs may be performed using GHZ states

In this section, our goal is to demonstrate that any protocol involving an EPR pair and measurements can be performed using a GHZ state and measurements. With this result, we show that by using GHZ configurations it is possible to detect the presence of weak bit-flip noise and reach a higher precision by post-selection.

Assume we are given a task to be performed using an EPR pair ($|\psi_{mn}\rangle$) and an arbitrary subsystem described by the density operator $\hat{\rho}_S \in \mathcal{B}(\mathcal{H}_S)$. The initial state of the system as a whole reads:

$$\hat{\rho}_o = \hat{\rho}_S \otimes \hat{\Pi}_{mn}, \tag{5.23}$$

for $\hat{\Pi}_{mn} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, with $\mathcal{H}_j$ the Hilbert space associated to the qubits of the EPR state.

In order to perform the task in question, eventually some operation T will be performed on $\hat{\rho}_o$ transforming into a new state $\hat{\rho}_f$. Hence at some point we must have:

$$T(\hat{\rho}_o) = \hat{\rho}_f. \tag{5.24}$$

In general, such a transformation may be written as:

$$T(\hat{\sigma}) = \mathrm{Tr}_{\mathcal{H}_Q}\left(\sum_{k=1}^{m'} \hat{V}_k \hat{\sigma} \hat{V}_k^\dagger\right) \Gamma_{m'}^{-1}, \tag{5.25}$$

where $\Gamma_{m'} = \mathrm{Tr}\left\{\sum_{k=1}^{m'} \hat{V}_k \hat{\sigma} \hat{V}_k^\dagger\right\}$ and $\sum_k^m \hat{V}_k^\dagger \hat{V}_k = \hat{\mathbb{1}}$, for a given $m' \leq m$, under an arbitrary Hilbert space $\mathcal{H}_Q$ satisfying: $\mathcal{H}_Q \subset \mathcal{H}_S \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$.

We can always add an extra qubit $\hat{\rho}_{anc} = |0\rangle\langle 0| \in \mathcal{B}(\mathcal{H}_{anc})$, sometimes called *ancilla* and define the extended state $\hat{\rho}_o'$ as follows:

$$\hat{\rho}_o' = (\hat{\mathbb{1}}_S \otimes \hat{\mathbb{1}}_1 \otimes \hat{C}_{not})\hat{\rho}_o \otimes \hat{\rho}_{anc}(\hat{\mathbb{1}}_S \otimes \hat{\mathbb{1}}_1 \otimes \hat{C}_{not}^\dagger) = \hat{\rho}_S \otimes \hat{\Pi}_{mnn}', \tag{5.26}$$

where $\hat{C}_{not}$ represents the CNOT operation acting on $\mathcal{H}_2 \otimes \mathcal{H}_{anc}$.

The transformation defined previously (Eq. 5.25) may be generalized to the extended Hilbert space by introducing the operators $\hat{V}'_k \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_{anc})$ in the following way:

$$\hat{V}'_k = (\hat{V}_k \otimes \hat{\mathbb{1}}_{anc})(\hat{\mathbb{1}}_S \otimes \hat{\mathbb{1}}_1 \otimes \hat{C}^\dagger_{not}), \tag{5.27}$$

likewise, the transformation T' holds:

$$T'(\hat{\sigma}) = \mathrm{Tr}_{\mathcal{H}_{Q'}} \left(\sum_{k=1}^{m'} \hat{V}'_k \hat{\sigma} \hat{V}'^\dagger_k \right), \tag{5.28}$$

which implies that

$$T(\hat{\rho}_o) = T'(\hat{\rho}'_o) = \hat{\rho}_f. \tag{5.29}$$

In conclusion, any task making use of EPR pairs may be performed using the same number of GHZ states.

5.4.2 Noise detection

Now we focus on tasks using EPR states that involve detecting some elements of an EPR basis $\{|\psi_{mn}\rangle\}$, i.e., T is of the form:

$$T(\hat{\rho}) = \mathrm{Tr}_{\mathcal{H}_Q} \left\{ \sum_{mn} \left(\hat{U}_{mn} \otimes \hat{\Pi}_{mn} \right) \hat{\rho} \left(\hat{U}^\dagger_{mn} \otimes \hat{\Pi}_{mn} \right) \right\} \tag{5.30}$$

where $\hat{U}_{mn}$ are arbitrary unitary matrices. It is important to remark that we assume the measurement acting on at least one of the qubits in the EPR state.

Consider the incidence of bit-flip noise on the qubits involved in the measurement. Under the weak noise regime, we have:

$$\begin{aligned} T(\hat{\varrho}_o) = (1-2p)\,\mathrm{Tr}_{\mathcal{H}_Q} & \left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}_{mn} \right) \hat{\rho}_o \left(\hat{U}^\dagger_{mn} \otimes \hat{\Pi}_{mn} \right) \right\} + \\ & + 2p\,\mathrm{Tr}_{\mathcal{H}_Q} \left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}_{m,n\oplus 1} \right) \hat{\rho}_o \left(\hat{U}^\dagger_{mn} \otimes \hat{\Pi}_{m,n\oplus 1} \right) \right\}, \end{aligned} \tag{5.31}$$

$$T(\hat{\varrho}_o) = (1-2p)\hat{\rho}_f + 2p\,\mathrm{Tr}_{\mathcal{H}_Q} \left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}_{m,n\oplus 1} \right) \hat{\rho}_o \left(\hat{U}^\dagger_{mn} \otimes \hat{\Pi}_{m,n\oplus 1} \right) \right\}. \tag{5.32}$$

As we can see, the effect of bit-flip noise on the qubits involved in the measurement is to mix the labels and in this way decrease the precision of the operation.

Following the results of the last section, we are able to modify the transformation above in order to replace EPR states by GHZ states. Thus, it is straightforward to show that the corresponding operation T' is:

$$T'(\hat{C}) = \mathrm{Tr}_{\mathcal{H}_{Q'}} \left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}'_{mnn} \right) \hat{C} \left(\hat{U}^{\dagger}_{mn} \otimes \hat{\Pi}'_{mnn} \right) \right\}. \tag{5.33}$$

On the other hand, if we look at the GHZ version of the protocol, we have:

$$\begin{aligned} T'(\hat{\varrho}'_o) = (1-3p)\, \mathrm{Tr}_{\mathcal{H}_{Q'}} &\left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}'_{mnn} \right) \hat{\rho}'_o \left(\hat{U}^{\dagger}_{mn} \otimes \hat{\Pi}'_{mnn} \right) \right\} + \\ + p\, \mathrm{Tr}_{\mathcal{H}_{Q'}} &\left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}'_{m,n\oplus1,n\oplus1} \right) \hat{\rho}'_o \left(\hat{U}^{\dagger}_{mn} \otimes \hat{\Pi}'_{m,n\oplus1,n\oplus1} \right) \right\} + \\ + p\, \mathrm{Tr}_{\mathcal{H}_{Q'}} &\left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}'_{m,n\oplus1,n} \right) \hat{\rho}'_o \left(\hat{U}^{\dagger}_{mn} \otimes \hat{\Pi}'_{m,n\oplus1,n} \right) \right\} + \\ + p\, \mathrm{Tr}_{\mathcal{H}_{Q'}} &\left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}'_{m,n,n\oplus1} \right) \hat{\rho}'_o \left(\hat{U}^{\dagger}_{mn} \otimes \hat{\Pi}'_{m,n,n\oplus1} \right) \right\}. \end{aligned} \tag{5.34}$$

It is clear from the above equation that the effect of noise in this scenario is richer. We observe as in the EPR protocol a mixing of the detections labels; however, it happens in a smaller proportion, and to compensate that effect, new outcomes become possible. As those are supposed to appear, we can always perform a post-selection by eliminating those outcomes. This can be done in the generic operation $\tilde{T}'$:

$$\begin{aligned} \tilde{T}'(\hat{\varrho}'_o) = \Gamma \cdot \Bigg\{ (1-3p)\, \mathrm{Tr}_{\mathcal{H}_{Q'}} &\left[\sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}'_{mnn} \right) \hat{\rho}'_o \left(\hat{U}^{\dagger}_{mn} \otimes \hat{\Pi}'_{mnn} \right) \right] + \\ + p\, \mathrm{Tr}_{\mathcal{H}_{Q'}} &\left[\sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}'_{m,n\oplus1,n\oplus1} \right) \hat{\rho}'_o \left(\hat{U}^{\dagger}_{mn} \otimes \hat{\Pi}'_{m,n\oplus1,n\oplus1} \right) \right] \Bigg\}, \end{aligned} \tag{5.35}$$

where Γ is a normalization factor, which depends on the unexpected new outputs and is equal to $1-2p$.

5.5 General protocol using N-partite GHZ states

A natural point remaining to be explored in this work is whether the improvement in the protocols observed here is only a manifestation of a non-ideal version of some error-correction scheme, whose imperfection might arise from the low size of the employed ancilla (0.5 qubits of the ancilla per qubit in the original protocol). This can be verified by generalizing this protocol to the case where more than one qubit is attached to the original system, i.e., replacing EPR pairs and measurements by N-partite GHZ states and measurements, with $N > 2$. Such a generalization is straightforward and first the N-partite GHZ state is defined as follows:

$$|\phi_{\vec{\mu}}\rangle = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} (-1)^{j\mu_0} |j, j \oplus \mu_1, \ldots, j \oplus \mu_{N-1}\rangle \,, \tag{5.36}$$

where $\vec{\mu} = (\mu_0, \ldots, \mu_{N-1})$, with $\mu_j \in \{0, 1\}$. In analogy to the previous cases, this state may also be obtained from local application of Pauli operators on the state $|\phi_{0,\ldots,0}\rangle$, as:

$$|\phi_{\vec{\mu}}\rangle = (\hat{\sigma}_z^{\mu_0} \otimes \hat{\sigma}_x^{\mu_1} \otimes \cdots \otimes \hat{\sigma}_x^{\mu_{N-1}}) \, |\phi_{0,\ldots,0}\rangle \,. \tag{5.37}$$

In the same way, the projector reads:

$$\hat{\Pi}''_{\vec{\mu}} = |\phi_{\vec{\mu}}\rangle\langle\phi_{\vec{\mu}}| \,. \tag{5.38}$$

Thus, the operation defined in Eq. 5.33 in this case reads:

$$T''(\hat{C}) = \mathrm{Tr}_{\mathcal{H}_{Q''}} \left\{ \sum_{mn=0}^{1} \left(\hat{U}_{mn} \otimes \hat{\Pi}''_{m,n,n,\ldots,n} \right) \hat{C} \left(\hat{U}^{\dagger}_{mn} \otimes \hat{\Pi}''_{m,n,n,\ldots,n} \right) \right\} . \tag{5.39}$$

In the presence of bit-flip noise, under the weak-noise regime, we have:

$$\begin{aligned}\tilde{T}''(\hat{\varrho}''_o) = \Gamma \cdot \Bigg\{ &(1-Np)\,\mathrm{Tr}_{\mathcal{H}_{Q''}}\left[\sum_{mn=0}^{1}\left(\hat{U}_{mn}\otimes\hat{\Pi}''_{m,n,\dots,n}\right)\hat{\rho}'_o\left(\hat{U}^{\dagger}_{mn}\otimes\hat{\Pi}''_{m,n,\dots,n}\right)\right]+\\ &+p\,\mathrm{Tr}_{\mathcal{H}_{Q''}}\left[\sum_{mn=0}^{1}\left(\hat{U}_{mn}\otimes\hat{\Pi}''_{m,n\oplus1,n\oplus1,\dots,n\oplus1}\right)\hat{\rho}''_o\left(\hat{U}^{\dagger}_{mn}\otimes\hat{\Pi}''_{m,n\oplus1,n\oplus1,\dots,n\oplus1}\right)\right]+\\ &+p\,\mathrm{Tr}_{\mathcal{H}_{Q''}}\left[\sum_{mn=0}^{1}\left(\hat{U}_{mn}\otimes\hat{\Pi}''_{m,n\oplus1,n,\dots,n}\right)\hat{\rho}''_o\left(\hat{U}^{\dagger}_{mn}\otimes\hat{\Pi}''_{m,n\oplus1,n,\dots,n}\right)\right]+\\ &+\quad\cdots\quad+\\ &+p\,\mathrm{Tr}_{\mathcal{H}_{Q''}}\left[\sum_{mn=0}^{1}\left(\hat{U}_{mn}\otimes\hat{\Pi}''_{m,n,n,\dots,n\oplus1}\right)\hat{\rho}''_o\left(\hat{U}^{\dagger}_{mn}\otimes\hat{\Pi}''_{m,n,n,\dots,n\oplus1}\right)\right]\Bigg\}. \end{aligned} \tag{5.40}$$

Therefore, after post-selection we get an efficiency of $(1-Np)/\left[1-(N-1)p\right]$. Now it is natural to ask whether increasing the number of parties above $N=3$ enhances the performance of the protocol, i.e. for what values of N^* and arbitrary p, the following condition holds:

$$\frac{1-N^*p}{1-(N^*-1)p} > \frac{1-3p}{1-2p},$$

however this is only possible for $N^* < 3$. In this way, we have shown that the best strategy is to employ three-partite GHZ states and measurements.

5.6 Discussion and Conclusion

We presented a protocol for error detection in some entanglement-based tasks, in which the key ingredient is the replacement of EPR pairs by GHZ states. First, we showed its performance in two well-known tasks: teleportation and superdense coding. Then, we considered a general task under the effect of weak bit-flip noise, and we have demonstrated that it is alway possible to increase its efficiency by using our protocol. Additionally, this process is less expensive than many other protocols, as it only demands one ancillary qubit per pair of qubits in the system. Nonetheless, it is important to remark that such an enhancement is not for free; in fact, measurements in the EPR basis must be replaced by measurements in the GHZ basis. Furthermore, we have proven that our proposal is not equivalent to any QEC by showing that it cannot be improved by increasing the size of the ancilla.

It is a curious fact that the replacement of an EPR pair by a GHZ state as a resource leads to an increase in the precision of some task that may be performed between two remote parts, for instance the protocol of teleportation. That is because if we consider the bipartition involved in this protocol, the three-partite GHZ and the EPR state have the same entropy of entanglement [Horodecki et al. 2009], and thus, they should represent the same resource with the same potential (which should be the maximum possible) for such a task.

An interesting question is whether a similar process can be performed for tasks demanding a higher number of entangled parts using another relevant classes of quantum states, such as cluster states, for instance.

6 QUDIT TELEPORTATION UNDER NOISY ENVIRONMENTS

Highlights

It is presented a generalization of the teleportation scheme proposed Fortes and Rigolin [Fortes e Rigolin 2015], for arbitrary dimension d.

Several special cases are studied in detail.

It is shown that the scenarios in which is possible to *fight noise with noise* [Fortes e Rigolin 2015] are more restricted for arbitrary d when compared with the case of qubits.

6.1 Introduction

From its proposal in 1993 [Bennett et al. 1993] and even nowadays [Pirandola et al. 2015], the teleportation protocol represents one of the most known and widely studied applications of quantum entanglement. Under this, if one has a source of pairs of maximally entangled qudits and a measurement apparatus capable of discriminating the d^2 elements of the generalized Bell basis, then it is possible to send an arbitrary qudit state between two locations even without prior knowledge of it. Nevertheless, in real life experiments one has to deal with the unavoidable interaction of the parties involved with the environment, and in this way leading to losses of quantum resources responsible for the improvement of the task when compared to the usage of a classical channel. In addition to concentrating experimental efforts to isolate the system, one may also adapt the scheme of measurements and operations on the qudits in order to improve the performance of the protocol (see [Taketani, Melo e Filho 2012], and references therein). From another point of view, Fortes and Rigolin [Fortes e Rigolin 2015] (hereafter FR15) have recently shown that in some particular cases the presence of noise can enhance the fidelity of qubit teleportation, even reaching values above the classical value for noise fractions close to one. In this chapter we present an extension of FR15 for systems of arbitrary dimension d, for the families of noise presented in section 2.5.1. First we present the derivation of a general expression for the average fidelity of teleportation under the families of noise whose Kraus operators are proportional to Weyl matrices, then we explore some special cases. In the final section we derive an expression for the fidelity which takes into account amplitude damping noise and briefly present some results.

6.2 Teleportation protocol

Alice's aim is to send an arbitrary state $|\phi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle$ to Bob, through a quantum channel $\hat{\rho}_{ch}$. We assume that Alice and Bob share many copies of an entangled pair initially prepared in a pure state $|\psi\rangle$. Thus we can write $\hat{\rho}_{ch} = |\psi\rangle\langle\psi|$, with $|\psi\rangle = \sum_{k=0}^{d-1} \gamma_k |kk\rangle$. For convenience, we express the initial state of the system as

$$\hat{\rho} = |\phi\rangle\langle\phi| \otimes \hat{\rho}_{ch} = \sum_{\substack{j_1 j_2 j_3 \\ k_1 k_2 k_3 = 0}}^{d-1} \rho_{j_1 j_2 j_3 k_1 k_2 k_3} |j_1 j_2 j_3\rangle\langle k_1 k_2 k_3| \,. \tag{6.1}$$

As we mentioned in the first chapter (Sec. 2.5), the presence of noise tends to change the state of the system ($\hat{\rho} \to \hat{\rho}'$). If we assume that noise acts locally, then we can express the modified state coefficients $\rho'_{\boldsymbol{m},\boldsymbol{n}}$ as in equations 2.44 and 2.46 in the Weyl and computational basis, respectively.

Alice performs a joint measurement in a basis whose elements are given by $|\Phi_{\mu\nu}\rangle = \sum_{k=0}^{d-1} \omega_d^{k\mu} \beta_k |k, k \oplus \nu\rangle$, and then sends her result (μ, ν) through a classical channel to Bob, who applies the operation $\hat{U}_{\mu\nu}$ on his qudit. The usual way to quantify how close are two states is given by the fidelity (see Eq. 2.42). In the ideal case (no noise and maximally entangled channel and measurements $\beta_k = \gamma_k = 1/\sqrt{d}$), Bob's qudit recovers the state initially sent by Alice, $|\phi\rangle$, and in this way $F = 1$. Let us examine some particular cases, starting our discussion with the kinds of noise which can be decomposed in the Weyl basis.

6.3 Fidelity and Weyl-like noises

After some algebra, the fidelity of teleportation $F = \sum_{\mu\nu} \mathrm{Tr}\left\{|\phi\rangle\langle\phi|\,\hat{\rho}_{\mu\nu}\right\}$ takes the form:

$$F = \sum_{\substack{jkmn\\ \mu\nu=0}}^{d-1} \alpha_m \alpha_n^* \beta_j \beta_k^* \omega_d^{\mu(n-m+j-k)} \rho'_{k,k\oplus\nu,n\oplus\nu,j,j\oplus\nu,m\oplus\nu}, \tag{6.2}$$

using equation (2.44), the expression above reduces to

$$F = \sum_{\substack{jk\mu\nu p_1 p_2 p_3\\ q_1 q_2 q_3=0}}^{d-1} \alpha_{j\oplus q_2\ominus q_3} \alpha^*_{k\oplus q_2\ominus q_3} \alpha^*_{j\oplus q_1} \alpha_{k\oplus q_1} \omega_d^{(k-j)(p_1+p_2+p_3)} \times$$
$$\times\, \beta_j \beta_k^* \gamma_{k\oplus\nu\oplus q_2} \gamma^*_{j\oplus\nu\oplus q_2} a^2_{p_1 q_1} b^2_{p_2 q_2} c^2_{p_3 q_3}, \tag{6.3}$$

where, $a_{p_1q_1}$, $b_{p_2q_2}$ and $c_{p_3q_3}$ are the noise coefficients corresponding to the input, and the channel qudits respectively. F typically depends on the input state coefficients α_j, for this reason it is more convenient to calculate the average fidelity over the set of input states.

Using the result of appendix B (eq. B.11) and after some calculations, the average fidelity takes the form:

$$\langle F\rangle = \frac{1}{d+1}\left\{1 + \sum_{\substack{jk\nu p_1 p_2 p_3\\ q_1 q_2 q_3=0}}^{d-1} \omega_d^{(k-j)(p_1+p_2+p_3)} \beta_j \beta_k^* \gamma_{k\oplus\nu\oplus q_2} \gamma^*_{j\oplus\nu\oplus q_2} a^2_{p_1 q_1} b^2_{p_2 q_2} c^2_{p_3 q_3} \delta_{q_2, q_1\oplus q_3}\right\}. \tag{6.4}$$

By substituting a_{jk} (Eq. 2.45) and using analogous expressions for the noise coefficients of the channel qudits b_{jk} and c_{jk} into equation 6.4, and after some steps, the fidelity of teleportation becomes:

$$\langle F\rangle = \frac{1}{d+1}\Bigg(1+d\Big\{b_p^2\Big[a_0^2c_0^2+(d-1)a_f^2c_f^2\Big]+\Big[a_p^2c_0^2+a_0^2c_p^2+(d-1)\left(a_f^2c_c^2+a_c^2c_f^2\right)\Big]\times$$
$$\times\Big[b_0^2+(d-2)b_p^2\Big]+\Big[(d-2)b_0^2+(d^2-3d+3)b_p^2\Big]\Big[a_p^2c_p^2+(d-1)a_c^2c_c^2\Big]\Big\}+$$
$$+d(d-1)\Big\{\Big[(d-2)b_f^2+(d^2-3d+3)b_c^2\Big]\Big[a_p^2c_c^2+a_c^2c_p^2+(d-2)a_c^2c_c^2\Big]+b_c^2\Big[a_f^2c_0^2+a_0^2c_f^2+(d-2)a_f^2c_f^2\Big]+$$
$$+\Big[b_f^2+(d-2)b_c^2\Big]\Big[a_c^2c_0^2+a_0^2c_c^2+a_f^2c_p^2+a_p^2c_f^2+(d-2)\left(a_f^2c_c^2+a_c^2c_f^2\right)\Big]\Big\}+$$
$$+\left(b_0^2-b_p^2\right)\Big[\left(a_0^2-a_p^2\right)\left(c_0^2-c_p^2\right)+(d-1)\left(a_f^2-a_c^2\right)\left(c_f^2-c_c^2\right)\Big]\Big(1+(d+1)f_Q\Big)+$$
$$+\left(b_f^2-b_c^2\right)\Big[\left(a_0^2-a_p^2\right)\left(c_f^2-c_c^2\right)+\left(a_f^2-a_c^2\right)\left(c_0^2-c_p^2\right)+(d-2)\left(a_f^2-a_c^2\right)\left(c_f^2-c_c^2\right)\Big]\tilde{f}\Bigg), \tag{6.5}$$

where f_Q is the quantum contribution to the fidelity of teleportation in the absence of noise

$$f_Q = \frac{2}{d+1}\sum_{j>k,\nu=0}^{d-1}\mathrm{Re}\left\{\beta_j\beta_k^*\gamma_{k\oplus\nu}\gamma_{j\oplus\nu}^*\right\} \tag{6.6}$$

and $\tilde{f}$ is related to the channel and measurement coefficients as:

$$\tilde{f} = \sum_{\substack{jk\nu=0\\ q=1}}^{d-1}\beta_j\beta_k^*\gamma_{k\oplus\nu\oplus q}\gamma_{j\oplus\nu\oplus q}^*. \tag{6.7}$$

When the entanglement of the channel and measurements is maximum, it is straightforward to show that these functions attain maximum values of $f_Q = (d-1)/(d+1)$ and $\tilde{f} = d(d-1)$.

It is worth mentioning that for the case of qubits ($d=2$), the expression for $\langle F\rangle$ above reproduces all the results obtained in FR15, with the exception of those regarding *amplitude damping* noise. However we were able to find a general relation considering this class of noise as well by expressing the associated Kraus operators in the computational basis, as we will show in the end of the chapter.

Now we consider several important examples, starting with the noiseless case.

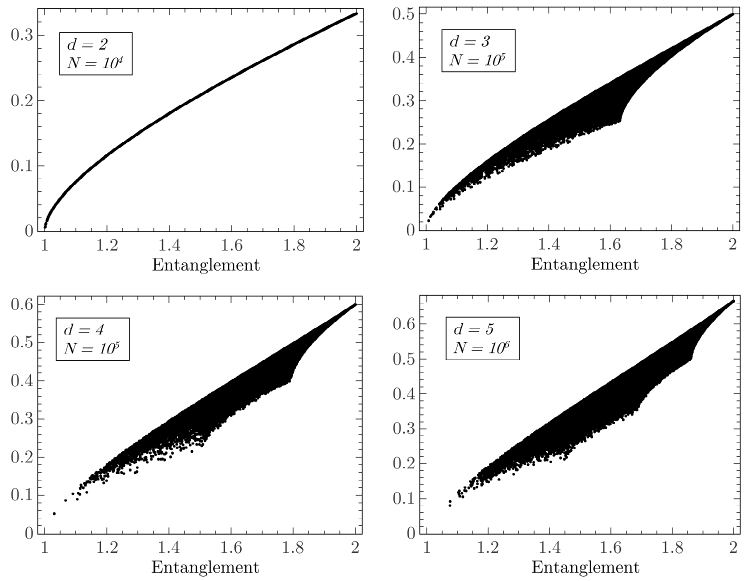

Figure 17 – Quantum contribution to the fidelity of teleportation f_Q (Eq. 6.6), as a function of the total amount of entanglement of the system (channel + measurement, each normalized to 1) for a maximally entangled measurement basis and a set of N random states of two entangled qudits in the channel, for several values of d. Note that $d = 2$ is the only case in which f_Q shows a monotonical behavior in function of the entanglement. In the other cases, if the amount of entanglement of the channel is known we can only determine bounds of f_Q.

6.4 Noise-Free environment

When there is no noise acting on the system, the noise coefficients are reduced to $a_{p_j q_j} = \delta_{p_j,0}\delta_{q_j,0}$, or equivalently: $a_0 = 1$, $a_f = 0$, $a_p = 0$ and $a_p = 0$ (the same relations valid for coefficients $b_{p_j q_j}$, $c_{p_j q_j}$, b_j and c_k). Substituting in Eq. 6.5, the fidelity takes the form:

$$\langle F \rangle = \frac{2}{d+1}\left(1 + \sum_{j>k,\nu=0}^{d-1} \mathrm{Re}\left\{\beta_j \beta_k^* \gamma_{k\oplus\nu} \gamma_{j\oplus\nu}^*\right\}\right). \tag{6.8}$$

In this expression the classical and quantum contributions to the fidelity are made clear. It is straightforward to see that if we have maximal entanglement in the channel and measurements ($\beta_j = \gamma_k = 1/\sqrt{d}$), then the fidelity reaches a maximum value of 1, as expected.

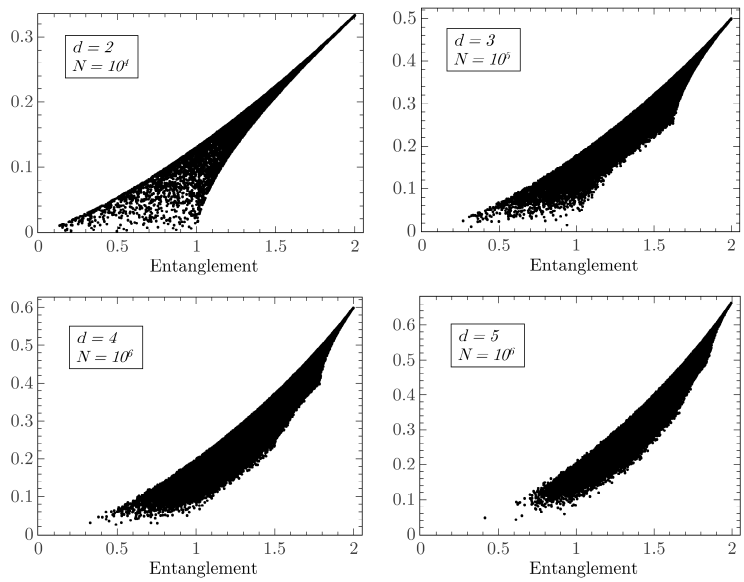

Figure 18 – Quantum contribution to the fidelity of teleportation f_Q (Eq. 6.6), as a function of the total amount of entanglement of the system (channel + measurement, each normalized to 1) for a set of N random states of two entangled qudits for the channel and measurement basis, for several values of d. As in the preceding case, the total amount of entanglement of the system determines bounds of the quantum fidelity only.

In order to have a qualitative picture of how the amount of entanglement of the channel and measurement basis are related to the quantum contribution to the fidelity of teleportation, we produced a sample of random entangled states uniformly distributed on the space of the Schmidt basis[1]. Results of the quantum contribution to the fidelity of teleportation f_Q as a function of the total amount of entanglement in the channel and measurements (normalized to 2), for the case in which neither the channel nor the measurement are maximally entangled are presented in figure 17. Results corresponding to both components random are shown In figure 18. From the results we see that if one is able to determine the amount of entanglement, we can evaluate bounds of quantum fidelity.

In addition to the technical limitations in the preparation of the system

1 We produced random states uniformly distributed in the basis $\{|00\rangle, |11\rangle, \cdots |d-1, d-1\rangle\}$. Note that this parametrization does not cover the whole space of pure states for two-qudits. Nevertheless for our purposes it is enough.

and realization of measurements in maximally entangled states, noise is an unavoidable feature of real experiments, for this reason it is very important to establish strategies which lead to the improvement of the final results. In the following sections we explore the influence of protecting one or more qudits from noise on the fidelity of teleportation.

6.5 Noise in one qudit

In this section we consider the case in which two qudits are fully protected from noise, e.g. an experiment in which the production of pairs of entangled qudits is carried out in Alice's location and Bob's qudit is affected by interacting with the environment during the transportation process.

By direct substitution into the expression for fidelity (eq. 6.4), it is easy to see that when noise is acting on one qudit only, the final result does not depend on the qudit affected. Then, giving continuity to the example given above, we consider that the affected qudit is that on Bob's location. In this case the general expression for the fidelity of teleportation is reduced to:

$$\langle F\rangle = \frac{1}{d+1}\Big\{1 + dc_p^2 + c_0^2 - c_p^2 + (d+1)(c_0^2 - c_p^2)f_Q\Big\}, \tag{6.9}$$

which does not depend on the coefficient c_f, for this reason the fidelity corresponding to dit-flip and dit-phase-flip noises attain the same values:

$$\langle F_F\rangle = \langle F_{FP}\rangle = \frac{2}{d+1}\left(1-\frac{p}{2}\right) + f_Q(1-p). \tag{6.10}$$

For d-phase-flip noise, the fidelity is reduced to:

$$\langle F_P\rangle = \frac{2}{d+1} + f_Q\left(1-\frac{d}{d-1}p\right), \tag{6.11}$$

the classical fidelity is not affected because phase shifts are exclusive elements of quantum systems. This feature will be explored in more detail in the following subsection.

The corresponding fidelities under mixed and depolarizing noise respectively are:

$$\langle F_M\rangle = \frac{2}{d+1}\left(1-\frac{d}{2(d+1)}p\right) + f_Q\left(1-\frac{d^2}{d^2-1}p\right), \tag{6.12}$$

and

$$\langle F_D \rangle = \frac{2}{d+1}\left(1 - \frac{d-1}{2d}p\right) + f_Q\,(1-p)\,. \tag{6.13}$$

Our results for $\langle F_F \rangle$, $\langle F_P \rangle$ and $\langle F_D \rangle$ are in agreement with t/se of FR15 for qubits, as expected.

It is important to note that when we have a maximally entangled channel and measurements $[f_Q = (d-1)/(d+1)]$, the fidelities $\langle F_f \rangle$, $\langle F_p \rangle$, and $\langle F_{fp} \rangle$ are all equal to:

$$\langle F \rangle = 1 - \frac{d}{d+1}p, \tag{6.14}$$

and the corresponding to depolarizing reduces to:

$$\langle F_D \rangle = 1 - \frac{d-1}{d}p. \tag{6.15}$$

6.5.1 Optimization of fidelity under d-phase-flip noise in one qudit

Besides the fact that the classical fidelity is not affected by the presence of d-phase-flip noise on one qudit, it is possible to find other interesting features. By analysing the expression for fidelity (eq. 6.11), we see that above a noise threshold $p^* = (d-1)/d$, the quantum contribution becomes negative. This situation may be overcome if we make a phase addition in the measurement basis, as pointed out in FR15 for $d = 2$. Without loss of generality and in order to simplify calculations, assume a channel initially prepared in a maximally entangled state and a measurement basis maximally entangled with arbitrary phases ϕ_j: $\beta_j = \exp\{i\phi_j\}/\sqrt{d}$, with $\phi_0 = 0$. The fidelity is reduced to:

$$\langle F_P \rangle = \frac{2}{d+1}\left\{1 + \frac{1}{d}\left(1 - \frac{pd}{d-1}\right)\left(\sum_{k=1}^{d-1}\cos\phi_k + \sum_{k>l=1}^{d-1}\cos(\phi_l - \phi_k)\right)\right\}. \tag{6.16}$$

Thus the problem is reduced to an optimization procedure in which we search for extremal values (maximum when $p < p^*$ and minimum for $p > p^*$) of the quantum contribution to the fidelity. We carried out analytical calculations up to $d = 3$, obtaining the following results: For noise fractions below the treshold p^*, the whole set of phases are null, as expected. For $p > p^*$, we got $\phi_1 = \pi$ for $d = 2$ and $(\phi_1, \phi_2) = (2\pi/3, 4\pi/3)$ for $d = 3$. The resulting fidelities are plotted in

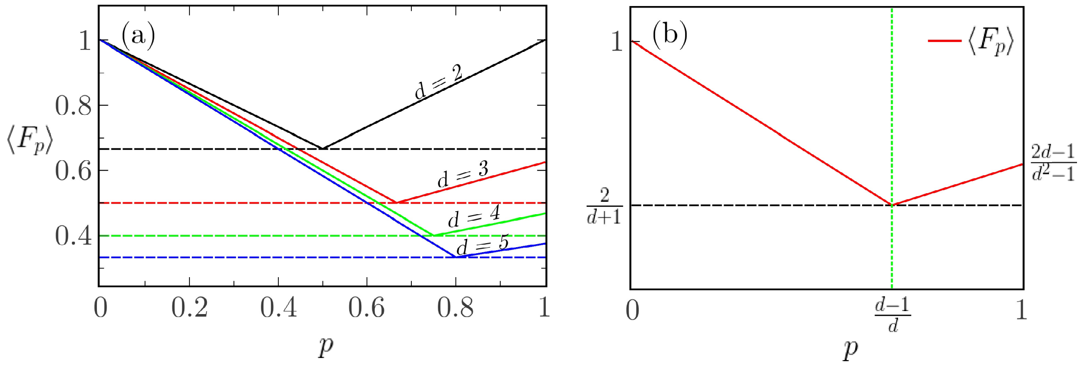

Figure 19 – *(Solid):* Optimal fidelity of teleportation for the case in which only one of the qudits of the system may suffer d-phase-flip noise. *(Dashed):* Classical fidelity. (a) Calculations of $\langle F_p \rangle$ for $2 \leq d \leq 5$. (b) Optimal fidelity of teleportation for arbitrary dimension d.

figure 19.(a). Furthermore we performed numerical calculations, from these we were able to infer the following expression for fidelity:

$$\langle F_P \rangle = \begin{cases} 1 - \frac{dp}{d+1} & \text{for } p < p^* \\ \\ \frac{dp+d-1}{d^2-1} & \text{for } p > p^*. \end{cases} \tag{6.17}$$

In conclusion, if somehow Alice and Bob are capable of measuring the amount of noise on the affected qudit, then she can improve the fidelity of the teleported state by choosing one out of two measurement basis. The results are summarized in Fig. 19.b. As it can be seen, the best improvement is attained by systems composed by qubits. As FR15 have shown, this feature can be exploited if we permit a part of the system to be affected by phase-flip noise[2]. Unfortunately this is not the case for arbitrary dimension, for as it can be seen in Fig. 19.a., the recovery becomes lower as we increase d.

2 Nevertheless as it can be easily seen, such a recovery in the fidelity of qubit teleportation reported in [Fortes e Rigolin 2015] may be explained by the fact that an increase in the noise fraction p leads to a suppression of phase-flip noise effect of the Kraus operators on the final state for $p > 1/2$.

6.6 Noise in more than one qudit

In this section we treat the case in which protection may be applied in at most one of the qudits. Again, in order to have a better insight on the results, we assume maximum entanglement in the channel and measurements.

Before examining some cases in detail let us summarize some general observations regarding dit-flip (F), d-phase-flip (P), dit-phase-flip (FP), mixed (M) and depolarizing (D) noises on the teleportation protocol:

The fidelity of teleportation does not depend on how noise is distributed on the qudits[3], thus for instance a situation in which two qudits may be affected is equivalent to that of having the input protected only i.e. $\langle F_{X,\varnothing,Y}\rangle = \langle F_{X,Y,\varnothing}\rangle = \langle F_{\varnothing,X,Y}\rangle$, where $\langle F_{X,Y,Z}\rangle$ denotes the fidelity when X, Y and Z kinds of noise are acting on the input, Alice's and Bob's qudits composing the channel respectively, and we use the symbol "$\varnothing$" to indicate a noise-free qudit.

Furthermore it is possible to observe some symmetries when we consider dit-flip, d-phase-flip and dit-phase-flip noises: $\langle F_{F,F,X}\rangle = \langle F_{P,P,X}\rangle$ with $X = \{\varnothing, FP, M, D\}$, $\langle F_{X',Y',F}\rangle = \langle F_{X',Y',P}\rangle = \langle F_{X',Y',FP}\rangle$ for $X' \neq Y' = \{\varnothing, M, D\}$ and $\langle F_{X,Y,\varnothing}\rangle = \langle F_{X,Z,\varnothing}\rangle$ for $X \neq Y \neq Z = \{F, P, FP\}$. Explicit expressions are not presented here, however all of them may be obtained by direct substitution in the general equation for $\langle F\rangle$.

Among the considered cases, in the scenario $F, F, \varnothing$ (or equivalently $P, P, \varnothing$) there is always a quantum contribution to the fidelity, with $d = 2$ as an extremal case in which $\langle F\rangle = 1$, when the fraction of noise is maximal (see figures 20 and 21).

6.6.1 Noise thresholds

In order to compare the efficiency of the protocol for different schemes we calculated the maximum fraction of noise in which there is still a quantum contribution to the fidelity, p^*. As in the previous analysis, we assume maximal entanglement in the channel and measurements.

3 This result holds for any case: one, two or three qudits affected.

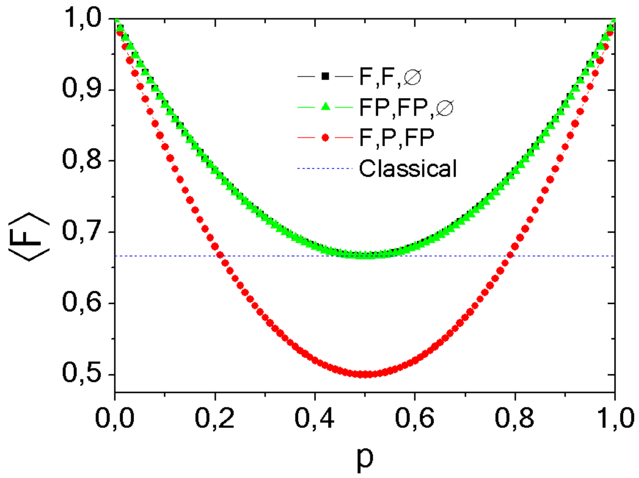

Figure 20 – Fidelity of teleportation in function of the fraction of noise for the cases in which it is possible to *fight noise with noise* [Fortes e Rigolin 2015] without using a measurement basis change, for $d = 2$. In order to facilitate the visualization of results we have done $p_a = p_b = p$ $(= p_c$, whenever noise is acting on the third qubit).

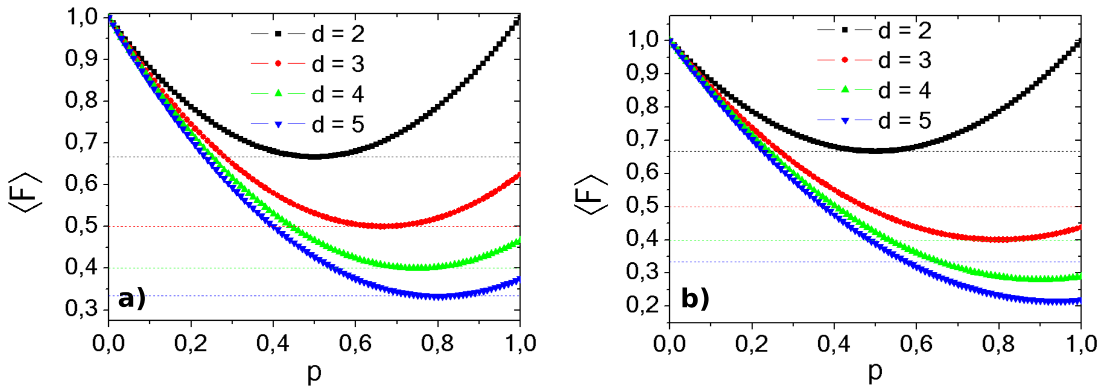

Figure 21 – a) Scenario $F, F, \varnothing$ (or $P, P, \varnothing$). These are the only found instances in which it is possible to fight noise by using noise without changing the measurement basis. b) Scenario $FP, FP, \varnothing$, for this the only instance exhibiting the possibility of fighting noise with noise is the case of qubits. In order to facilitate the visualization of results we have done $p_a = p_b = p$.

Scenario $(X, \varnothing, \varnothing)$

For the simplest case where there is noise in one qudit only, it is straightforward to show that $p^* = (d-1)/d$ for dit-flip, d-phase-flip, dit-phase-flip and mixed noise, while $p^* = d/(d+1)$ for depolarizing.

Scenario $(X, Y, \varnothing)$

Without considering the cases in which errors can be suppressed by adding noise to another part of the system, we carried out a search of extremal values for noise thresholds under this scenario. For the worst situation $(X \neq Y \in \{F, P, FP\})$ we found a noise threshold of $p^* = 1 - \frac{1}{\sqrt{d}}$ and for the best one $(X = Y \in \{F, P, FP\})$, $p^* = 1 - \frac{1}{d}$, for $d = 2$ and $X = Y \in \{F, P\}$ for $d > 2$ (see also Fig. 21).

Scenario (X, Y, Z)

Finally, for the cases in which all qudits can be affected we were able to calculate the maximal noise fraction for the extremal pictures: The worst situation is reached for the scenario $X \neq Y \neq Z \in \{F, P, FP\}$ with a noise threshold given by

$$p^* = \frac{d-1}{(d-1)^2-1}\left\{d - 1 - 2^{-\frac{1}{3}}\left[\sqrt[3]{(d-2)^2 - 2 - (d-2)\sqrt{d(d-4)}} + \right.\right.$$
$$\left.\left. + \sqrt[3]{(d-2)^2 - 2 + (d-2)\sqrt{d(d-4)}}\right]\right\}. \quad (6.18)$$

For the best situation we have ($X = Y = Z \in \{F, P, FP\}$, for $d = 2$ and $X = Y \in \{F, P\}$ for $d > 2$) with $p^* = 1 - \frac{1}{d}$.

A summary of the relevant noise thresholds for several dimensions is given in figure 22. From this we can see that the situations leading to the highest and lowest noise thresholds are those associated with only one and all qudits affected (by different kinds of noise) respectively, as expected. Nevertheless we found an extra feature for the (F, F, F) scenario: even though the fidelity of teleportation has the strongest decay, the noise threshold is the same as in the case of two protected qudits under the same kind of noise.

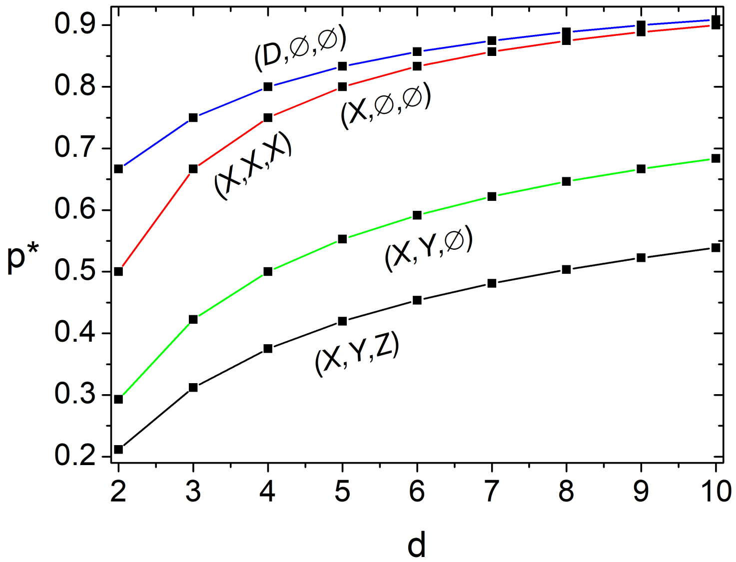

Figure 22 – Summary of extremal noise thresholds for several cases. The best scenarios are those corresponding to two protected qudits $(\varnothing, \varnothing, X)$ with X denoting any noise, or the same noise on all qudits (X, X, X) with $X = F, P$ (and FP for $d = 2$). Note that the case $(D, \varnothing, \varnothing)$ leads to the best performance, however it does not make much sense to compare it with the other instances due to the fact that noise fraction for depolarizing has a very different meaning. The worst case is that in which different kinds of noise (F, P or FP) act on each qudit.

Particular results for the cases in which there is only one kind of noise on the system are shown in figure 23 for three cases: one, two and no protected qudits. It is important to remark that results corresponding to d-phase-flip noise are not presented because those have exactly the same behaviour as for the dit-flip noise case. Note also that we restrict ourselves to the case in which all qudits may be affected with the same probability.

6.7 Fidelity of teleportation - Amplitude damping noise

Due to the structure of the Kraus operators related to *amplitude damping noise* it is not possible to decompose them by using Weyl matrices, for this reason we use the traditional computational basis (see section 2.5.2).

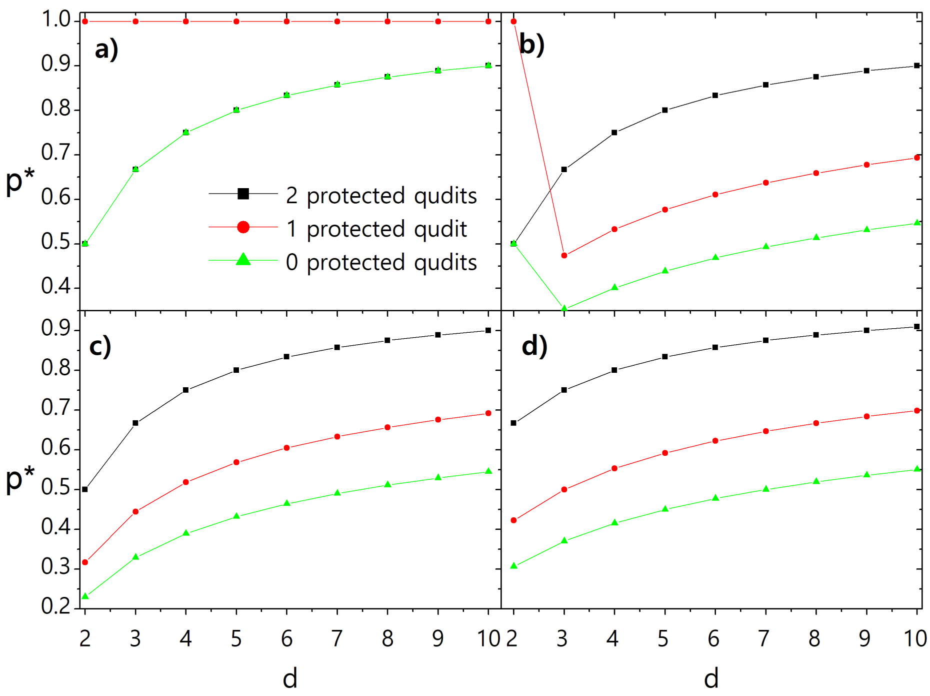

Figure 23 – Maximum fraction of noise p^* for which the fidelity of teleportation has a quantum contribution in function of the dimension d, in a scenario where the qudits of the system may be affected (each with the same probability) by only one kind of noise: a) dit-flip (d-phase-flip), b) dit-phase-flip, c) mixed, and d) depolarizing noise. Note that with the exception of dit-flip and bit-phase-flip noises, the scenarios in which two qudits are protected always lead to the best performance, as expected.

Let us calculate the fidelity of teleportation. Substituting Eq. 2.46 in 6.2, we have

$$F = \sum_{\substack{jkmn\mu\nu \\ n_1 n_2 p_1 p_2 \\ k_1 k_2 k_3}} \alpha_m \alpha_n^* \alpha_{n_1} \alpha_{p_1}^* \beta_j \beta_k^* \gamma_{n_2} \gamma_{p_2}^* \omega_d^{\mu(n-m+j-k)} a_{k,n_1}^{(k_1)} b_{k\oplus\nu,n_2}^{(k_2)} c_{n\oplus\nu,n_2}^{(k_3)} a_{j,p_1}^{(k_1)*} b_{j\oplus\nu,p_2}^{(k_2)*} c_{m\oplus\nu,p_2}^{(k_3)*}.$$

Analogously to the previous treatment, using the results of Appendix B and after

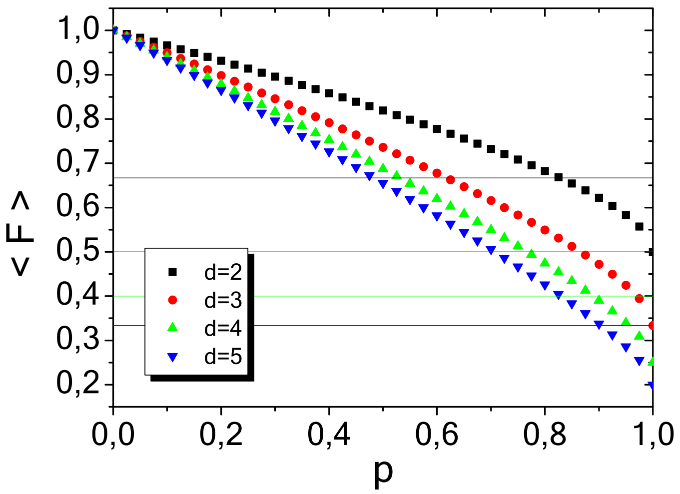

Figure 24 – *(Dots):* Fidelity of qubit teleportation under amplitude damping noise acting on the input qudit and, *(Lines):* classical fidelities, for several values of d.

some calculations the average fidelity of teleportation holds

$$\langle F\rangle = \frac{1}{d+1}\sum_{\substack{km\nu n_1 n_2\\ p_1 k_1 k_2 k_3}} \gamma_{n_2}\gamma^*_{p_2} a^{(k_1)}_{k,n_1} b^{(k_2)}_{k\oplus\nu,n_2} c^{(k_3)*}_{m\oplus\nu,p_2}\Big[|\beta_k|^2 a^{(k_1)*}_{k,n_1} b^{(k_2)*}_{k\oplus\nu,p_2} c^{(k_3)}_{m\oplus\nu,n_2} + \\ + \beta_{k\oplus m\oplus(-n_1)}\beta^*_k a^{(k_1)*}_{k\oplus m\oplus(-n_1),m} b^{(k_2)*}_{k\oplus m\oplus(-n_1)\oplus\nu,p_2} c^{(k_3)}_{n_1\oplus\nu,n_2}\Big]. \quad (6.19)$$

Beside contemplating the relevant case of amplitude damping noise, it is important to mention that from this expression we can obtain all the results presented in previous sections.

In Figs. 24, 25 and 26, we have plotted the fidelity of teleportation for the possible scenarios involving amplitude damping noise only. Note that, protecting two qubits from noise leads to the higher increase in the noise threshold when compared to the other two scenarios. From the figures it is also possible to observe that the difference between fidelities diminishes and the noise threshold attains larger values as the dimension grows, even for the case in which the three qudits are affected.

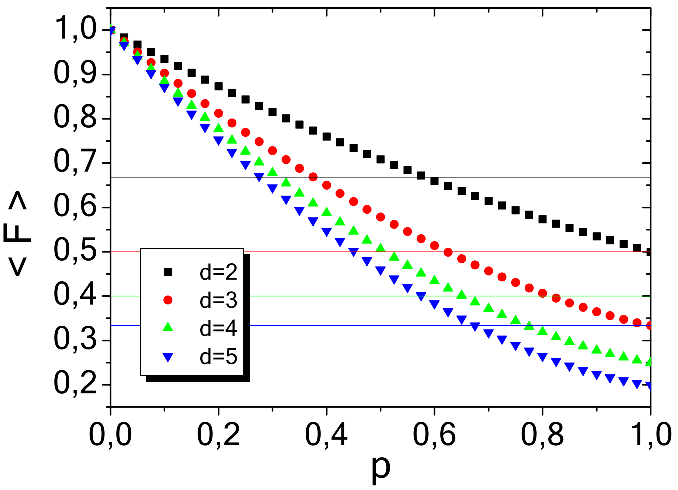

Figure 25 – *(Dots):* Fidelity of qubit teleportation under amplitude damping noise acting on Alice's qudits and, *(Lines):* classical fidelities, for several values of d. Note that in order to facilitate the visualization of results we have done $p_a = p_b = p_c = p$.

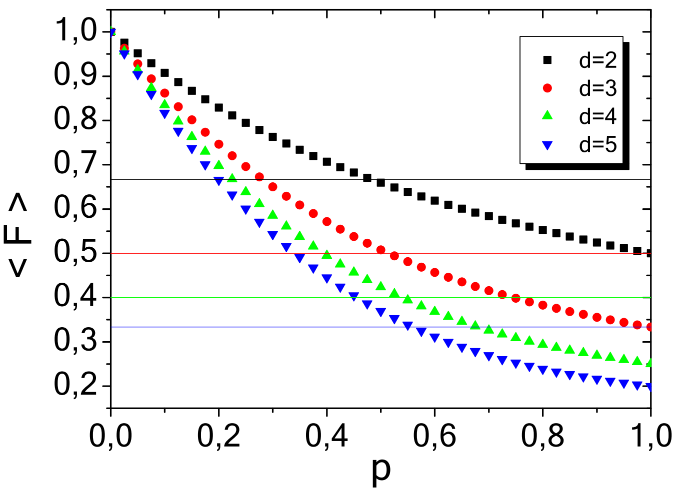

Figure 26 – *(Dots):* Fidelity of qubit teleportation under amplitude damping noise acting on all qudits and, *(Lines):* classical fidelities, for several values of d. Note that in order to facilitate the visualization of results we have done $p_a = p_b = p_c = p$.

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APPENDIX A – CALCULATION OF JOINT PROBABILITIES

Joint probability for systems under spin-$\frac{S}{2}$ measurements

The eigenvalue equation for a spin-$\frac{S}{2}$ measurement ($S \in \mathbb{Z}^+$), in an arbitrary direction $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ may be written as

$$\hat{\boldsymbol{S}} \cdot \boldsymbol{n} \left| \hat{\boldsymbol{S}} \cdot \boldsymbol{n}, s \right\rangle = s\hbar \left| \hat{\boldsymbol{S}} \cdot \boldsymbol{n}, s \right\rangle = s\hbar \left| s \right\rangle_{\boldsymbol{n}}, \tag{A.1}$$

with $\hat{\boldsymbol{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$, where $\hat{S}_j$ indicates the spin-$\frac{S}{2}$ operator in the j direction, and $s = \left\{-\frac{S}{2}, -\frac{S}{2}+1, \cdots, \frac{S}{2}-1, \frac{S}{2}\right\}$. After solving A.1 for $|s\rangle_{\boldsymbol{n}}$, by convenience we relabel s to start from zero. In this way we can express the eigenvectors $|s\rangle_{\boldsymbol{n}}$ as

$$|s\rangle_{\boldsymbol{n}} = \sum_{j=0}^{S} \beta_{sj}^{(\boldsymbol{n})} |j\rangle . \tag{A.2}$$

Given a N-partite system characterized by the density matrix $\hat{\rho} = \sum_{\boldsymbol{jk}} \rho_{\boldsymbol{jk}} |\boldsymbol{j}\rangle\langle\boldsymbol{k}|$, each part subject to spin-$\frac{S}{2}$ measurements, we can calculate the joint probability of the outcomes s_1, s_2, $\cdots$, s_N in each one of the subsystems, given that they chose to perform the measurements labelled by x_1, x_2, $\cdots$, x_N respectively,

$$P(\boldsymbol{s}|\boldsymbol{x}) = P(s_1, s_2, \cdots, s_N | x_1, x_2, \cdots, x_N)$$

$$P(\boldsymbol{s}|\boldsymbol{x}) = \mathrm{Tr}\left[\left(|a_1\rangle\langle a_1|_{x_1} \otimes \cdots \otimes |a_N\rangle\langle a_N|_{x_N} \right) \hat{\rho} \right],$$

$$P(\boldsymbol{s}|\boldsymbol{x}) = \mathrm{Tr}\left(|\boldsymbol{s}\rangle\langle\boldsymbol{s}|_{\boldsymbol{x}} \ \hat{\rho} \right),$$

$$P(\boldsymbol{s}|\boldsymbol{x}) = \sum_{\boldsymbol{jk}} \rho_{\boldsymbol{jk}} \left(\prod_{m=1}^{N} \beta_{s_m,k_m}^{(x_m)} \ \beta_{s_m,j_m}^{(x_m)*} \right). \tag{A.3}$$

In the same way we can calculate marginal probabilities of a specific part of the system j getting the outcome s_j, given the input choice x_j:

$$P(s_j|x_j) = \mathrm{Tr}\left[\left(\hat{1}_1 \otimes \cdots \otimes |s_j\rangle\langle s_j|_{x_j} \otimes \cdots \otimes \hat{1}_N \right) \hat{\rho} \right],$$

substituting and after some steps, we get

$$P(s_j|x_j) = \sum_{\boldsymbol{k},m,n} \rho_{k_1,\cdots,k_{j-1},n,k_{j+1},\cdots,k_N;k_1,\cdots,k_{j-1},m,k_{j+1},\cdots,k_N} \beta^{(x_j)}_{s_j,m} \beta^{(x_j)*}_{s_j,n}. \tag{A.4}$$

In the next subsections we provide particular expressions for two cases worked in this thesis: spin-$\frac{1}{2}$ and spin-1 measurements.

Joint probability for a bipartite system under spin-$\frac{1}{2}$ measurements

Consider a bipartite spin-$\frac{1}{2}$ system initially prepared in a state $\hat{\rho} = \sum_{jklm} \rho_{jklm} |jk\rangle\langle lm|$. and related eigenvectors given by

$$|\pm\rangle_{\boldsymbol{n}} = \cos\left(\frac{\theta_{\boldsymbol{n}}}{2}\right) |+\rangle \pm \sin\left(\frac{\theta_{\boldsymbol{n}}}{2}\right) \exp\{i\phi_{\boldsymbol{n}}\} |-\rangle , \tag{A.5}$$

where $\hat{\boldsymbol{S}} = \frac{\hbar}{2}(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ and $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. By making the following relabeling: $|+\rangle \to |0\rangle$ and $|-\rangle \to |1\rangle$, we can write the eigenvector $|\pm\rangle_{\boldsymbol{n}}$ more compactly as:

$$|a\rangle_{\boldsymbol{n}} = \sum_{j=0}^{1} \beta^{(\boldsymbol{n})}_{aj} |j\rangle ,$$

where $\beta^{(\boldsymbol{n})}_{a,0} = \cos\left(\frac{\theta_{\boldsymbol{n}}}{2}\right)$ and $\beta^{(\boldsymbol{n})}_{a,1} = (-1)^a \sin\left(\frac{\theta_{\boldsymbol{n}}}{2}\right) \exp\{i\phi_{\boldsymbol{n}}\}$. Substituting, the joint probability take the form:

$$P(ab|xy) = \sum_{jkmn=0}^{1} \rho_{jmkn} \beta^{(x)}_{a,k} \beta^{(x)*}_{a,j} \beta^{(y)}_{b,n} \beta^{(y)*}_{b,m}. \tag{A.6}$$

Marginal probabilities $P(a|x)$ and $P(b|y)$ hold:

$$P(a|x) = \mathrm{Tr}\left[\left(|a\rangle\langle a|_x \otimes \hat{1}_B \right) \hat{\rho} \right],$$

$$P(a|x) = \sum_{jkm=0}^{1} \rho_{jkmk} \beta^{(x)}_{a,m} \beta^{(x)*}_{a,j}, \tag{A.7}$$

and

$$P(b|y) = \mathrm{Tr}\left[\left(\hat{1}_A \otimes |b\rangle\langle b|_y \right) \; \hat{\rho} \right],$$

$$P(b|y) = \sum_{jkm=0}^{1} \rho_{jkjm} \; \beta_{b,m}^{(y)} \; \beta_{b,k}^{(y)*}. \tag{A.8}$$

Joint probability for systems under spin-1 measurements

For $S = 1$, the spin operators are given by:

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \hat{S}_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then, the operator related to a spin-1 measurement in an arbitrary direction $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is given by:

$$\hat{\boldsymbol{S}} \cdot \boldsymbol{n} = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} \sqrt{2}\cos\theta & \exp\{-i\phi\}\sin\theta & 0 \\ \exp\{i\phi\}\sin\theta & 0 & \exp\{-i\phi\}\sin\theta \\ 0 & \exp\{i\phi\}\sin\theta & -\sqrt{2}\cos\theta \end{bmatrix}.$$

After diagonalization, the set of eigenvalues of $\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$ is $\{-1, 0, 1\}$, and associated eigenvectors [Zettili 2009]:

$$|{-1}\rangle_{\boldsymbol{n}} = \frac{1}{2} \begin{bmatrix} (1-\cos\theta)\exp\{-i\phi\} \\ -\frac{2}{\sqrt{2}}\sin\theta \\ (1+\cos\theta)\exp\{i\phi\} \end{bmatrix}, \quad |0\rangle_{\boldsymbol{n}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\exp\{-i\phi\}\sin\theta \\ \sqrt{2}\cos\theta \\ \exp\{i\phi\}\sin\theta \end{bmatrix},$$

$$|1\rangle_{\boldsymbol{n}} = \frac{1}{2} \begin{bmatrix} (1+\cos\theta)\exp\{-i\phi\} \\ \frac{2}{\sqrt{2}}\sin\theta \\ (1-\cos\theta)\exp\{i\phi\} \end{bmatrix}.$$

Relabelling, we can write the eigenvectors compactly as:

$$|a\rangle_{\boldsymbol{n}} = \sum_{j=0}^{2} \beta_{aj}^{(\boldsymbol{n})} \, |j\rangle \,,$$

with $a = \{0, 1, 2\}$. Finally, the joint probability of outcomes a and b in Alice and Bob, given that their system was initially prepared as $\hat{\rho} = \sum_{jklm} \rho_{jklm} |jk\rangle\langle lm|$ and chose to measure in the directions labelled by x and y, respectively is given by:

$$P(ab|xy) = \sum_{jkmn=0}^{2} \rho_{jmkn}\, \beta_{a,k}^{(x)}\, \beta_{a,j}^{(x)*} \beta_{b,n}^{(y)}\, \beta_{b,m}^{(y)*}. \tag{A.9}$$

Finally, if the system is in a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, with $|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |jj\rangle$:

$$P(ab|xy) = \sum_{jk=0}^{2} \alpha_j \alpha_k^*\, \beta_{a,k}^{(x)}\, \beta_{a,j}^{(x)*} \beta_{b,k}^{(y)}\, \beta_{b,j}^{(y)*}. \tag{A.10}$$

Joint Probability for a bipartite system under general unitaries and CGLMP inequality

Given a bipartite system initially described by the state $\hat{\rho} = \sum_{jklm} \rho_{jklm} |jk\rangle\langle lm|$ under the action of the unitaries $\hat{U}_x = \sum_{mn=0}^{d-1} U_{mn}^{(x)} |m\rangle\langle n|$ and $\hat{U}_y = \sum_{pq=0}^{d-1} U_{pq}^{(y)} |p\rangle\langle q|$ on the first and second parts respectively, the joint probability $P(ab|xy)$ is equal to:

$$P(ab|xy) = \mathrm{Tr}\left(|ab\rangle\langle ab|\, \hat{U}_x \otimes \hat{U}_y\, \hat{\rho}\, \hat{U}_x^\dagger \otimes \hat{U}_y^\dagger \right),$$

After some calculations, we have

$$P(ab|xy) = \sum_{jklm=0}^{d-1} \rho_{jklm} U_{aj}^{(x)} U_{bk}^{(y)} U_{al}^{(x)*} U_{bm}^{(y)*}.$$

Particularly for the case of a system initially prepared in a pure state $|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |jj\rangle$, the joint probability holds:

$$P(ab|xy) = \sum_{jk=0}^{d-1} \alpha_j \alpha_k^* U_{aj}^{(x)} U_{bj}^{(y)} U_{ak}^{(x)*} U_{bk}^{(y)*}. \tag{A.11}$$

This expression will be useful in some parts of this thesis due to the fact that our focus is mainly pure states.

CGLMP inequality

After some calculaions the CGLMP inquality (eq. 2.37) holds:

$$I_d = \sum_{k=0}^{[\frac{d}{2}]-1} \left(1 - \frac{2k}{d-1}\right) \sum_{j=0}^{d-1} \sum_{x,y=1}^{2} (-1)^{y(x-1)} \left\{\mathcal{P}_1 - \mathcal{P}_2\right\}, \tag{A.12}$$

with $\mathcal{P}_1 = P\left(j \oplus \left[(-1)^{x(y-1)}k\right], j|x,y\right)$ and $\mathcal{P}_2 = P\left(j, j \oplus \left[(-1)^{x(y-1)}(k \oplus 1)\right] |x,y\right)$.

Using eq. A.11, joint probabilities $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively hold:

$$\mathcal{P}_1 = \sum_{mn=0}^{d-1} \alpha_m \alpha_n^* U^{(x)}_{j\oplus(-1)^{x(y-1)}k,j} U^{(y)}_{j,m} U^{(x)*}_{j\oplus(-1)^{x(y-1)}k,n} U^{(y)*}_{j,n}. \tag{A.13}$$

$$\mathcal{P}_2 = \sum_{mn=0}^{d-1} \alpha_m \alpha_n^* U^{(x)}_{j,m} U^{(y)}_{j\oplus(-1)^{x(y-1)}(k\oplus1),m} U^{(x)*}_{j,n} U^{(y)*}_{j\oplus(-1)^{x(y-1)}(k\oplus1),n}. \tag{A.14}$$

Thus the CGLMP inequality for a bipartite system initially prepared in a pure state $|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |jj\rangle$, under the action of a set of local unitary operations $\{\hat{U}_x, \hat{U}_y\}$ reduces to:

$$I_d = \sum_{k=0}^{[\frac{d}{2}]-1} \left(1 - \frac{2k}{d-1}\right) \sum_{jmn=0}^{d-1} \alpha_m \alpha_n^* \sum_{xy=1}^{2} (-1)^{y(x-1)} \Bigg(U^{(x)}_{j\oplus(-1)^{x(y-1)}k,j} U^{(y)}_{j,m} \times \\ \times U^{(x)*}_{j\oplus(-1)^{x(y-1)}k,n} U^{(y)*}_{j,n} - U^{(x)}_{j,m} U^{(y)}_{j\oplus(-1)^{x(y-1)}(k\oplus1),m} U^{(x)*}_{j,n} U^{(y)*}_{j\oplus(-1)^{x(y-1)}(k\oplus1),n} \Bigg) \leq 2, \tag{A.15}$$

APPENDIX B – THE SPACE OF PURE STATES

Volume of the space of pure qudit states

Any arbitrary pure state of a qudit $|\phi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle$ may be parametrized as:

$$\alpha_j = \begin{cases} \cos\theta_0 & \text{for } j = 0 \\ \sin\theta_0 \ \dots \ \sin\theta_{j-1}\cos\theta_j \ \exp\{i\phi_j\} & \text{for } 1 \leq j \leq d-2 \\ \sin\theta_0 \ \dots \ \sin\theta_{d-2} \ \exp\{i\phi_{d-1}\} & \text{for } j = d-1, \end{cases} \tag{B.1}$$

with $0 < \theta_j \leq \pi/2$ and $0 < \phi_j \leq 2\pi$.

Under this parametrization the volume element of the space $\mathrm{d}\Gamma_d$ holds [Caves 2001, Bengtsson e Zyczkowski 2007, Życzkowski e Sommers 2001]:

$$\mathrm{d}\Gamma_d = \sin^{2d-3}\theta_0 \dots \sin\theta_{d-2}\cos\theta_0 \dots \cos\theta_{d-2}\mathrm{d}\theta_0 \dots \mathrm{d}\theta_{d-2}\mathrm{d}\phi_1 \dots \mathrm{d}\phi_{d-1},$$

in a compact form:

$$\mathrm{d}\Gamma_d = \prod_{j=0}^{d-2} \sin^{2d-2j-3}\theta_j \cos\theta_j \ \mathrm{d}\theta_j \ \mathrm{d}\phi_{j+1}. \tag{B.2}$$

The total volume $V_d = \int \mathrm{d}\Gamma_d$ may be easily calculated and is equal to:

$$V_d = \frac{\pi^{d-1}}{(d-1)!} \tag{B.3}$$

Generation of random uniform qudit states

There are plenty of methods to generate random pure qudit states (for several examples see [Maziero 2016]). Nevertheless, here we use a simple procedure that takes advantage of the parametrization presented before.

Due to the form of the volume element (eq. B.2), the generation of a uniformly distributed sample of random pure states is not as simple as producing random angles θ_m and ϕ_n and after substituting in equation (B.1). Instead, we have to carry out a change of variables such that the volume element may be written as $\mathrm{d}\Gamma_d \propto \prod_k \mathrm{d}r_k \mathrm{d}\phi_{k+1}$. After some calculations, we get the following relation:

$$\theta_k = \operatorname{asin}\left(r_k^{\frac{1}{2(d-k-1)}} \right), \tag{B.4}$$

where $0 \le r_k \le 1$. Note that there is no necessity of such an operation for the variables ϕ_j. At the end, the production of random uniform pure qudit states is reduced to the generation of random numbers $\{r_k, \phi_j\}$ uniformly distributed and subsequent substitution in equations (B.4) and (B.1).

Calculation of $\langle \alpha_j \alpha_k^* \alpha_l \alpha_m^* \rangle$

In principle, the calculation of $\langle \alpha_j \alpha_k^* \alpha_l \alpha_m^* \rangle = \frac{1}{V_d} \int \mathrm{d}\Gamma_d \; \alpha_j \alpha_k^* \alpha_l \alpha_m^*$ may seem a difficult task, however we can take advantage of some symmetries to make things easier. First of all, note that the volume element $\mathrm{d}\Gamma_d$ does not depend explicitly on the phases ϕ_j. Moreover, the state coefficients α_j are proportional to $\exp(i\phi_j)$, then the only way in which the integration does not vanish is having both: the coefficient and its conjugate inside the argument in order to cancel the corresponding phases. In this way we must have: $\langle \alpha_j \alpha_k^* \alpha_l \alpha_m^* \rangle \propto \big(\delta_{jk}\delta_{lm} + \delta_{jm}\delta_{kl}\big)$. Let us determine the proportionality constant. For simplicity we only show calculations for $\langle |\alpha_0|^4 \rangle$, nevertheless as the generated states are uniformly distributed, then any choice is equivalent. The integration reads:

$$\left\langle |\alpha_0|^4 \right\rangle = \frac{(d-1)!}{\pi^{d-1}} \int \prod_{j=0}^{d-2} \sin^{2d-2j-3}\theta_j \cos\theta_j \; \mathrm{d}\theta_j \; \mathrm{d}\phi_{j+1} \cos^4\theta_0 \tag{B.5}$$

$$\left\langle |\alpha_0|^4 \right\rangle = 2^{d-1}(d-1)! \int \prod_{j=0}^{d-2} \sin^{2d-2j-3}\theta_j \cos\theta_j \; \mathrm{d}\theta_j \cos^4\theta_0 \tag{B.6}$$

$$\left\langle |\alpha_0|^4 \right\rangle = 2^{d-1}(d-1)! \int \sin^{2d-3}\theta_0 \cos^5\theta_0 \mathrm{d}\theta_0 \prod_{j=1}^{d-2} \int \sin^{2d-2j-3}\theta_j \cos\theta_j \; \mathrm{d}\theta_j .$$

It is not hard to show that integrations of the kind above have the following solutions:

$$I_m^n = \int_0^{\pi/2} \sin^m x \cos^{n+1} x \, \mathrm{d}x = \sum_{k=0}^{n/2} \frac{(-1)^k \left(\frac{n}{2}\right)!}{k! \left(\frac{n}{2} - k\right)!} \frac{1}{2k + m + 1}, \tag{B.7}$$

for $n = 0, 2, 4, \cdots$ and $m > 0$. Thus, the expectation value above is reduced to:

$$\left\langle |\alpha_0|^4 \right\rangle = 2^{d-1}(d-1)! \; I_{2d-3}^4 \prod_{j=1}^{d-2} I_{2d-2j-3}^0. \tag{B.8}$$

From Eq. B.7, we have $I_{2d-2j-3}^0 = \frac{1}{2(d-j-1)}$ and $I_{2d-3}^4 = \frac{1}{(d+1)d(d-1)}$. Then

$$\left\langle |\alpha_0|^4 \right\rangle = 2^{d-1}(d-1)! \frac{1}{(d+1)d(d-1)} \prod_{j=1}^{d-2} \frac{1}{2(d-j-1)}. \tag{B.9}$$

This expression reduces to

$$\left\langle |\alpha_0|^4 \right\rangle = \frac{2}{d(d+1)}. \tag{B.10}$$

Back to the general case, it is straightforward to see that the proportionality factor must be equal to $\frac{1}{d(d+1)}$, and in this way for the general case we have

$$\langle \alpha_j \alpha_k^* \alpha_l \alpha_m^* \rangle = \frac{1}{V_d} \int \mathrm{d}\Gamma_d \; \alpha_j \alpha_k^* \alpha_l \alpha_m^* = \frac{1}{d(d+1)} \left(\delta_{jk}\delta_{lm} + \delta_{jm}\delta_{kl} \right). \tag{B.11}$$

This result has been very useful in the calculation of reduced expressions for the average fidelity of teleportation in chapter 6.

APPENDIX C – CGLMP INEQUALITY - MULTIPORT BEAM SPLITTERS AND PHASE SHIFTERS

Recall the CGLMP inequality:

$$I_d = \sum_{k=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1}\right) \left\{\mathcal{B}_k - \mathcal{B}_{-(k+1)}\right\} \leq 2, \tag{C.1}$$

where $\mathcal{B}_k = P(A_1 = B_1+k)+P(B_1 = A_2+k+1)+P(A_2 = B_2+k)+P(B_2 = A_1+k)$.

Any probability term $P(A_x = B_y + k)$ may be written in function of joint probabilities as:

$$\begin{aligned} P(A_x = B_y + k) &= \sum_{j=0}^{d-1} P(A_x = j \oplus k, B_y = j) \\ &= \sum_{j=0}^{d-1} P_{xy}(j \oplus k, j), \end{aligned}$$

thus, $\mathcal{B}_k$ holds:

$$\mathcal{B}_k = \sum_{x,y=1}^{2} \sum_{j=0}^{d-1} P_{xy}(j \oplus \kappa_{xyk}, j \oplus \lambda_{xyk}),$$

with non vanishing coefficients κ_{xyk} and λ_{xyk} given by: $\kappa_{11k} = \kappa_{22k} = \lambda_{12k} = k$, and $\lambda_{21k} = k + 1$.

After some calculations, the joint probability of Alice and Bob to obtain the a-th and b-th outputs respectively, is given that their measurement choices were x and y is given by:

$$P^Q(ab|xy) = \frac{1}{d^2} + \frac{2}{d^2} \sum_{m>n=0}^{d-1} \mathrm{Re}(\alpha_m \alpha_n^*) \cos \Delta_{xy}^{mn}(a, b), \tag{C.2}$$

with

$$\Delta_{xy}^{mn}(a, b) = \phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n + \frac{2\pi}{d}(m - n)\big(a \oplus (-b)\big). \tag{C.3}$$

Moreover, joint probabilities for the MBSPS setup (equation C.2) satisfy the following symmetry property:

$$\sum_{j=0}^{d-1} P_{xy}^{QM}(j \oplus k, j \oplus l) = d \cdot P_{xy}^{QM}(k,l), \tag{C.4}$$

taking this into account, it is easy to see that the term $\mathcal{B}_k - \mathcal{B}_{-(k+1)}$ in the CGLMP inequality reduces to:

$$\mathcal{B}_k - \mathcal{B}_{-(k+1)} = \frac{2}{d} \sum_{x,y=1}^{2} \sum_{m>n}^{d-1} \mathrm{Re}(\alpha_m \alpha_n^*) \Big\{ \cos \Delta\beta_{xy}^{mn}(\kappa_{xyk}, \lambda_{xyk}) \\ - \cos \Delta\beta_{xy}^{mn}\big(\kappa_{xy(-k-1)}, \lambda_{xy(-k-1)}\big) \Big\}.$$

Using trigonometrical identities, the CGLMP function I_d takes the form:

$$I_d = \sum_{x,y=1}^{2} \sum_{m>n=0}^{d-1} C_{xy}^{mn} \sin\left(\frac{\pi}{d}(m-n)\right) \times \\ \times \left\{ \cos\left(\phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n\right) + A_{xy}^{mn} \sin\left(\phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n\right) \right\},$$

with:

$$A_{xy}^{mn} = (-1)^{x(1+y)+1} \cot\left(\frac{\pi}{d}(m-n)\right) \tag{C.5}$$

and

$$C_{xy}^{mn} = \frac{4\,\mathrm{Re}(\alpha_m \alpha_n^*)}{d} (-1)^{y(1+x)} \mathcal{C}_{mn}, \tag{C.6}$$

where:

$$\mathcal{C}_{mn} = \sum_{k=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1}\right) \sin\left(\frac{\pi}{d}(m-n)(2k+1)\right), \tag{C.7}$$

By using the harmonic addition theorem, the CGLMP function for quantum joint probabilities under a measurement scheme based on multiport beam splitters and phase shifters characterized by a set of angles (ϕ_x^n, φ_y^m) reduces to:

$$I_d = \sum_{x,y=1}^{2} \sum_{m>n=0}^{d-1} C_{xy}^{mn} \cos\big(\phi_x^m + \varphi_y^m - \phi_x^n - \varphi_y^n + \Psi_{xy}^{mn}\big), \tag{C.8}$$

with amplitude C_{xy}^{mn} given by C.6 and phase coefficient:

$$\Psi_{xy}^{mn} = (-1)^{x(1+y)} \left[\frac{\pi}{2} - \frac{\pi}{d}(m-n) \right]. \tag{C.9}$$

This simplification of the CGLMP inequality under MBSPS measurements has been very helpful in the reduction of the computational time involved in numerical calculations.