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L^2 decay for weak solutions of the micropolar equations on $\mathbb{R}^3$

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Tese apresentada ao Programa de Pós-graduação em Matemática da Universidade Federal de Pernambuco como requisito parcial para obtenção do título de Doutora em matemática.

Orientador: Prof. Pablo Gustavo Albuquerque Braz e Silva
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LORENA BRIZZA SOARES FREITAS

L^2 DECAY FOR WEAK SOLUTIONS OF THE MICROPOLAR EQUATIONS ON R^3

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

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ABSTRACT

We obtain decay estimates for solutions of the micropolar fluid equations . Such equations, proposed by A. C. Eringen, generalize the classic model of Navier-Stokes and describe the behavior of fluids with microstructure such as animal blood, liquid crystals, suspensions, among others. For this, we use a method developed by M. Schonbek, known by Fourier Splitting Method. In order to present the method, we first show how it was applied in the context of parabolic conservation laws and the Navier-Stokes equations to obtain decay estimates. Having done this, assuming the existence for solutions of the micropolar fluid system with Dirichlet conditions at infinity and we show the result when the external forces are either null or decay at an appropriate rate. Lastly, through retarded mollifiers and approximate solutions, we guarantee the existence of solutions for the micropolar fluid equations in convenient functional spaces and we prove the desired decay bound.

Keywords: Mathematical Analysis. Partial Differential Equations. Micropolar Fluids. Decay of solutions. Fourier Splitting Method.

RESUMO

Obtemos estimativas de decaimento para as soluções das equações para fluidos micropolares. Tais equações, propostas por A. C. Eringen, generalizam o clássico modelo de Navier-Stokes e descrevem o comportamento de fluidos com microestrutura como sangue de animais, cristais líquidos, suspensões, entre outros. Para tal, utilizamos um método desenvolvido por M. Schonbek, conhecido como Método de Decomposição de Fourier. A fim de apresentar o método, primeiramente mostramos como o mesmo foi aplicado no contexto de leis de conservação parabólicas e das equações de Navier-Stokes para obter estimativas de decaimento. Feito isto, assumindo a existência de soluções para o sistema de fluido micropolar com condições de Dirichlet no infinito, obtemos decaimento no caso em que as forças externas do sistema são nulas ou decaem a uma razão apropriada. Por fim, construindo funções suavizantes e soluções aproximadas, garantimos a existência de soluções das equações de fluido micropolar em espaços funcionais convenientes e provamos a estimativa de decaimento desejada.

Palavras-chave: Análise Matemática. Equações Diferenciais Parciais. Fluido Micropolar. Decaimento de Soluções. Método de Decomposição de Fourier.

SYMBOL LIST

Ω	open subset of $\mathbb{R}^n$, n a positive integer
$\operatorname{supp} u$	the set $\operatorname{supp} u = \overline{\{\boldsymbol{x} \in \Omega : u(\boldsymbol{x}) \neq 0\}}$
C	the set of continuous functions
C^k	the set of functions which have continuous partial derivative of order less than or equal to k
C^∞	the set of functions which have continuous partial derivative of all orders $k \in \mathbb{N}$
$\boldsymbol{C}_0^\infty$	the set of C^∞-function with compact support
$\mathcal{D}(\Omega)$	space of test functions on Ω
$\boldsymbol{L}^p$	$\boldsymbol{L}^p(\mathbb{R}^n) = (L^p(\mathbb{R}^n))^m$, m and n positive integers
$\\|\cdot\\|_p$	$\boldsymbol{L}^p$ norm defined by $\\|\boldsymbol{u}\\|_p = \left(\sum_{j=1}^{3} \\|u_j\\|_p^p\right)^{1/p}$, if $1 \leq p < \infty$
$(\cdot,\cdot)$	the inner product on $\boldsymbol{L}^2$ defined by $(\boldsymbol{u},\boldsymbol{v}) = \sum_{j=1}^{3} \int_{\mathbb{R}^3} u_j \; v_j \, d\boldsymbol{x}$
$\boldsymbol{L}_{loc}^p$	the set of functions which are p-integrable on every compact subset of their domain of definition
$W^{m,p}$	space of functions weakly m-differentiable and p-integrable (Sobolev spaces)
H^m	$W^{m,2}$
$\boldsymbol{u}$	$\boldsymbol{u}(\boldsymbol{x},t) \in \mathbb{R}^3$ the linear velocity
$\boldsymbol{w}$	$\boldsymbol{w}(\boldsymbol{x},t) \in \mathbb{R}^3$ the angular velocity
p	$p(\boldsymbol{x},t) \in \mathbb{R}$ the pressure
$\boldsymbol{f}$, $\boldsymbol{g}$	$\boldsymbol{f}(\boldsymbol{x},t)$, $\boldsymbol{g}(\boldsymbol{x},t) \in \mathbb{R}^3$ the external forces
$\boldsymbol{u}_0$	$\boldsymbol{u}_0(\boldsymbol{x}) \in \mathbb{R}^3$ the initial linear velocity
$\boldsymbol{w}_0$	$\boldsymbol{w}_0(\boldsymbol{x}) \in \mathbb{R}^3$ the initial angular velocity
μ	Newtonian viscosity

μ_r	microrotational viscosity
c_0 , c_a, c_d	coefficients of angular viscosities
$\boldsymbol{\mathcal{V}}$	fields on $\boldsymbol{C}_0^\infty$ with $\operatorname{div} \boldsymbol{v} = 0$
$\boldsymbol{V}$	the closure of $\boldsymbol{\mathcal{V}}$ on $\boldsymbol{H}_0^1$
$\boldsymbol{H}$	the closure of $\boldsymbol{\mathcal{V}}$ on $\boldsymbol{L}^2$
$\boldsymbol{V}'$	the topological dual of $\boldsymbol{V}$
div	divergence operator
curl	rotational operator
∇	gradient operator
Δ	Laplacian operator
$[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]_j$	$\displaystyle\sum_{k=1}^{3} u_k \frac{\partial u_j}{\partial x_k}$, j-th components of $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$
$[(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}]_j$	$\displaystyle\sum_{k=1}^{3} u_k \frac{\partial w_j}{\partial x_k}$, j-th components of $(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}$

CONTENTS

1 INTRODUCTION

Fluid dynamics is an area of extreme importance for many sciences, such as engineering, biology, and physics (see [1]). Due to this, several models of differential equations have been formulated over the years, among them, the classic Navier-Stokes model. However, this model does not consider the influence of particles immersed in the fluid in its flow. Because of this, A. C. Eringen [2] proposed a model for fluids with microstructure called micropolar fluids.

In the present thesis, we investigate the long-time behavior for weak solutions of the micropolar fluid model. More specifically, let T be a positive real number and $\mathcal{Q}_T = \mathbb{R}^3 \times (0,T)$. We use boldface letters to denote vector fields in $\mathbb{R}^n$, as well as to indicate spaces whose elements are of this nature. We consider the system

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\mu+\mu_r)\Delta\boldsymbol{u} + \nabla p - 2\mu_r \operatorname{curl}\boldsymbol{w} = \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - (c_a+c_d)\Delta\boldsymbol{w} - (c_0+c_d-c_a)\nabla(\operatorname{div}\boldsymbol{w}) + 4\mu_r\boldsymbol{w} - 2\mu_r\operatorname{curl}\boldsymbol{u} = \boldsymbol{g}, \\ \boldsymbol{u}\Big|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{w}\Big|_{t=0} = \boldsymbol{w}_0. \end{cases} \tag{1.1}$$

Here, the unknowns $\boldsymbol{u}(\boldsymbol{x},t)\in\mathbb{R}^3$, $p(\boldsymbol{x},t)\in\mathbb{R}$ and $\boldsymbol{w}(\boldsymbol{x},t)\in\mathbb{R}^3$ represent, respectively, the linear velocity, the pressure distribution and the angular velocity of rotation of the fluid particles as functions of the position $\boldsymbol{x}$ and of the time t (see [2], [3], [4]). The functions $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$, $\boldsymbol{w}_0 = \boldsymbol{w}_0(\boldsymbol{x})$, $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t)$ and $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x},t)$ are given and denote, respectively, the initial linear velocity, the initial angular velocity and external forces. The positive constants μ, μ_r , c_0 , c_a and c_d represent viscosity coefficients and satisfy the inequality $c_0 + c_d > c_a$ (see [4]). Without loss of generality to our aim, we will take

$$\mu = \mu_r = 1/2$$

and

$$c_a + c_d = c_0 + c_d - c_a = 1.$$

Thus, system (1.1) can be written as

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p - \operatorname{curl}\boldsymbol{w} = \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \Delta\boldsymbol{w} - \nabla(\operatorname{div}\boldsymbol{w}) + 2\boldsymbol{w} - \operatorname{curl}\boldsymbol{u} = \boldsymbol{g}, \\ \boldsymbol{u}\Big|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{w}\Big|_{t=0} = \boldsymbol{w}_0. \end{cases} \tag{1.2}$$

Observe that, if $\boldsymbol{w} \equiv \boldsymbol{0}$ the system (1.2) reduces to the Navier-Stokes system (see for instance [5] and the references therein).

There are many works concerning the micropolar fluid model. Let us recall some of them

- In [4], the existence of local in time weak solutions for a short time were established (see also [3]).
- The existence of weak solutions was studied in [6, 7] on the Nikolskii spaces context.
- In [8, 9], the existence of strong solutions was established using linearization and the method of successive approximations.
- In [10], a proof of existence and uniqueness of strong solutions using the semi-Galerkin spectral approximations is presented.
- In [11, 12], the authors proved the existence of a small time interval where the fluid variables converge uniformly as the viscosities tend to zero.
- In [13], the existence of local in time semi-strong solutions and global in time strong solutions for the system were studied. Under suitable assumptions, the uniqueness of local semi-strong solutions were also proved.
- In [14], weak solutions with improved regularity were established, so improving the results in [6, 7].

Returning to the purpose of this work, we are interested in proving that under certain assumptions for the solutions of system (1.2), the following estimate

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2} \tag{1.3}$$

holds, for some suitable positive constant C. A similar result was demonstrated for the Navier-Stokes equations [15], using a method known as the "Fourier Splitting Method" developed by M. Schonbek and first applied in the context of parabolic conservation laws [16]. Given that the micropolar fluid model is a generalization of Navier-Stokes equations, we have tried to use the same approach on the system (1.2) to get the bound (1.3).

To present our results, we organized this thesis in six chapters. Chapter 2 contains a review of basic functional analysis concepts and results. The Fourier Splitting Method, as well as examples of its application, can be seen in Chapter 3. Chapters 4 and 5 present our main results, considering either the external forces to be null or to decay at an appropriate rate.

2 BASIC CONCEPTS

The present chapter is devoted to introducing some auxiliary results used in this work. Throughout this chapter, Ω is an open subset of $\mathbb{R}^n$. For more details, see [17–20].

2.1 Spaces of Functions

Definition 2.1. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index, where α_j is a nonnegative integer for all $j \in \mathbb{N}$. We denote by $D^{\boldsymbol{\alpha}}$ the differential operator of order

$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

i.e.

$$D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}.$$

Definition 2.2. Let $u : \Omega \to \mathbb{R}$ be a continuous function. The support of u is the set

$$\operatorname{supp} u = \overline{\{\boldsymbol{x} \in \Omega : u(\boldsymbol{x}) \neq 0\}}.$$

Definition 2.3. The vector space $C_0^\infty(\Omega)$ consists of all functions in $C^\infty(\Omega)$ which have compact support in Ω.

Definition 2.4. A sequence $\{\phi_m\}_{m=1}^\infty \subset C_0^\infty(\Omega)$ is said to be convergent to zero if the following conditions are satisfied

i. there exists a compact set K such that $\operatorname{supp} \phi_m \subset K$ for all $m \in \mathbb{N}$.

ii. For each $\boldsymbol{\alpha} \in \mathbb{N}^n$, the sequence $\{D^{\boldsymbol{\alpha}} \phi_m\}_{m=1}^\infty$ converges to zero uniformly on K.

For a given $\phi \in C_0^\infty(\Omega)$, the sequence $\{\phi_m\}_{m=1}^\infty \subset C_0^\infty(\Omega)$ is said to be convergent to ϕ in $C_0^\infty(\Omega)$ if $\{\phi_m - \phi\}_{m=1}^\infty$ converges to zero in the above sense.

Definition 2.5. The vector space $C_0^\infty(\Omega)$ equipped with the notion of convergence above is denoted by $\mathcal{D}(\Omega)$ and is known as the space of test functions on Ω.

Definition 2.6. A distribution on Ω is a continuous linear functional $T : \mathcal{D}(\Omega) \to \mathbb{R}$.

2.1.1 L^p Spaces

Definition 2.7. Let u be a measurable real function in Ω and $1 \leq p < \infty$. We denote by $L^p(\Omega)$ the Banach space

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} \; ; \; u \text{ is measurable and } \|u\|_p < \infty\}.$$

where

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u(\boldsymbol{x})|^p \, d\boldsymbol{x} \right)^{1/p}.$$

Remark 2.8. If $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x}, \quad u, v : \Omega \to \mathbb{R}.$$

Definition 2.9. For a measurable function $u : \Omega \to \mathbb{R}$, we denote by $L^\infty(\Omega)$ the Banach space

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}\,;\, u \,\text{is measurable and}\, \|u\|_\infty < \infty\}$$

where

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &= \inf\{C > 0\,;\, |u(\boldsymbol{x})| \le C \text{ a.e. on}\, \Omega\} \\ &= \operatorname*{ess\,sup}_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x})|. \end{aligned}$$

Lemma 2.10 (Generalized Young inequality)**.** *Let a, b be two positive real numbers. Consider $1 < p, q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then, for all $\varepsilon > 0$, we have*

$$ab \le \varepsilon a^p + C_\varepsilon b^q,$$

where $C_\varepsilon = \dfrac{p-1}{p^q} \varepsilon^{1-q}$.

Lemma 2.11 (Hölder inequality)**.** *Let $1 \le p, q \le \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\|uv\|_1 \le \|u\|_p \|v\|_q.$$

Lemma 2.12 (Generalized Hölder inequality)**.** *Let $1 \le p_1, \cdots, p_k \le \infty$. Assume that $u_1, u_2, \cdots, u_k$ are functions such that for $i = 1, \cdots, k$*

1. *$u_i \in L^{p_i}(\Omega)$*
2. *$\dfrac{1}{p} = \displaystyle\sum_{i=1}^{k} \dfrac{1}{p_i} \le 1$.*

Then $u = \displaystyle\prod_{i=1}^{k} u_i \in L^p(\Omega)$ and $\|u\|_p \le \displaystyle\prod_{i=1}^{k} \|u_i\|_{p_i}$.

2.1.2 Bochner Spaces

Definition 2.13. Let $1 \le p < \infty$ and a Banach space X with norm $\|\cdot\|_X$. We denote by $L^p(0, T; X)$ the set of mappings $u : [0, T] \to X$ which are strongly measurable and

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \left\| u(t) \right\|_X^p dt \right)^{1/p} < \infty \quad 1 \le p < \infty.$$

If $p = \infty$, we define the space $L^\infty(0,T;X)$ as the set of mappings $u : [0,T] \to X$ which are strongly measurable and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_X < \infty.$$

$L^p(0,T;X)$ and $L^\infty(0,T;X)$ with the norm $\|u\|_{L^p(0,T;X)}$ and $\|u\|_{L^\infty(0,T;X)}$, respectively are Banach spaces.

Lemma 2.14 (Lemma 1.1, Chapter 3, [5])**.** *Let X be a given Banach space and X' its topological dual and let $u, g \in L^1(0,T;X)$. Then, the following conditions are equivalent*

i. u is a.e. a primitive of g, i.e.,

$$u(t) = \xi + \int_0^t g(s)\,ds, \quad \xi \in X, \quad a.e.\ t \in [0,T],$$

ii. For each test function $\phi \in \mathcal{D}((0,T))$,

$$\int_0^T u(t)\phi'(t)\,dt = -\int_0^T g(t)\phi(t)\,dt,$$

iii. For each $\eta \in X'$,

$$\frac{d}{dt}\langle u,\eta\rangle_{X,X'} = \langle g,\eta\rangle_{X,X'},$$

in the distribution sense on $(0,T)$.

If conditions $i - iii$ are satisfied, then u is equal to a continuous function from $[0,T]$ into X a.e.

Lemma 2.15 (Lemma 1.4, Chapter 3, [5])**.** *Let X and Y be two Banach spaces such that $X \subset Y$ with continuous injection. If a function $\phi \in L^\infty(0,T;X)$ is weakly continuous with values in Y, then ϕ is weakly continuous with values in X.*

Lemma 2.16 (Lemma 1.2, Chapter 3, [5])**.** *Let V and H be two Hilbert spaces and V', H' be the dual of V and H respectively, with $V \subset H \equiv H' \subset V'$, where the injections are continuous. If a function u is such that $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V')$, then u is almost everywhere equal to a continuous function from $[0,T]$ into H and the equality*

$$\frac{d}{dt}|u|^2 = 2\langle u_t, u\rangle_{V',V},$$

holds in the distribution sense on $(0,T)$.

Lemma 2.17 (Lemma 2.1, Chapter 3, [5])**.** *Let X_0, X, X_1 be Banach spaces such that $X_0 \subset X \subset X_1$, the injections $X_0 \hookrightarrow X \hookrightarrow X_1$ are continuous, and the injection $X_0 \hookrightarrow X$ is compact. Then, for each $\eta > 0$, there exists some constant $C = C(\eta, X_0, X, X_1)$ such that for all $\nu \in X_0$, we have*

$$\|\nu\|_X \le \eta\|\nu\|_{X_0} + C\|\nu\|_{X_1}.$$

Lemma 2.18 (Aubin-Lions lemma, Theorem 2.1, Chapter 3, [5])**.** *Let X_0, X, X_1 be Banach spaces with X_0, X_1 reflexive. Suppose that injections $X_0 \hookrightarrow X \hookrightarrow X_1$ are continuous, and that the injection $X_0 \hookrightarrow X$ is compact. For $T > 0$ and $\alpha_0, \alpha_1 > 1$, let*

$$\mathcal{Y} = \{v \in L^{\alpha_0}(0, T; X_0) \text{ such that } v_t \in L^{\alpha_1}(0, T; X_1)\},$$

with norm defined by

$$\|v\|_{\mathcal{Y}} = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|v_t\|_{L^{\alpha_1}(0,T;X_1)} .$$

Then, the space $\mathcal{Y}$ is a Banach space and the injection of $\mathcal{Y}$ into $L^{\alpha_0}(0, T; X)$ is compact.

2.1.3 Sobolev Spaces

Definition 2.19. Let $\Omega \subset \mathbb{R}^n$ be an open set, $u, v \in L^1_{loc}(\Omega)$, and a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$ of length $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We say that v is the α^{th}-weak partial derivative of u, denoted by

$$D^{\boldsymbol{\alpha}} u = v,$$

if

$$\int_\Omega u(\boldsymbol{x})\, D^{\boldsymbol{\alpha}}\phi(\boldsymbol{x})\, d\boldsymbol{x} = (-1)^{|\alpha|} \int_\Omega v(\boldsymbol{x})\, \phi(\boldsymbol{x})\, d\boldsymbol{x}, \tag{2.1}$$

for all test functions $\phi \in C_0^\infty(\Omega)$.

Definition 2.20. Fix $p \in \mathbb{R}$ with $1 \le p \le \infty$ and m a positive integer. The Sobolev space $W^{m,p}(\Omega)$ consists of all functions $u \in L^p(\Omega)$ such that for each multi-index α with $|\boldsymbol{\alpha}| \le m$, the derivative $D^{\boldsymbol{\alpha}}u$ exists in the weak sense and belongs to $L^p(\Omega)$. If $u \in W^{m,p}(\Omega)$, we define its norm by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \int_\Omega |D^{\boldsymbol{\alpha}} u|^p dx \right)^{1/p}, \text{ if } 1 \le p < \infty,$$

and

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\boldsymbol{\alpha}| \le m} \operatorname*{ess\,sup}_{\Omega} |D^{\boldsymbol{\alpha}} u|.$$

Remark 2.21. If $p = 2$ and $m \in \mathbb{N}$, the Sobolev space $W^{m,2}(\Omega)$ is usually denoted by $H^m(\Omega)$. One can prove that $H^m(\Omega)$ is a Hilbert space with the norm associated to the inner product

$$\langle u, v \rangle = \sum_{|\alpha| \le m} (D^{\boldsymbol{\alpha}} u, D^{\boldsymbol{\alpha}} v)\,,$$

where $(\cdot, \cdot)$ is the inner product on $L^2(\Omega)$.

Definition 2.22. Let $1 \le p \le \infty$ and m a positive integer. The space $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. If $p = 2$, we write $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Definition 2.23. Let $1 \leq p < \infty$, $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $m \in \mathbb{N}$. The space $W^{-m,q}(\Omega)$ is the topological dual of $W_0^{m,p}(\Omega)$. If $q = 2$, we write $H^{-m}(\Omega) = W^{-m,2}(\Omega)$.

Lemma 2.24 (Lemma 6.7, Chapter 6, [21])**.** *Let Ω be an open and bounded subset of $\mathbb{R}^3$ and $u \in L^2(0,T;\boldsymbol{H}_0^1(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))$. Then $u \in L^4(0,T;\boldsymbol{L}^3(\Omega))$.*

2.2 Other results

Definition 2.25. $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be a Calderón-Zygmund kernel if satisfies the following conditions

(i) $|K(\boldsymbol{x},\boldsymbol{y})| \leq \dfrac{C}{|\boldsymbol{x}-\boldsymbol{y}|^n}$,

(ii) $|K(\boldsymbol{x},\boldsymbol{y}) - K(\bar{\boldsymbol{x}},\boldsymbol{y})| + |K(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x},\bar{\boldsymbol{y}})| \leq \dfrac{C}{|\boldsymbol{x}-\boldsymbol{y}|^{n+1}}$,

(iii) $K(\boldsymbol{x},\boldsymbol{y}) = -K(\boldsymbol{y},\boldsymbol{x})$,

for some constant $C > 0$.

Lemma 2.26 (Calderón-Zygmund Theorem, [22])**.** *Let T be a singular integral operator, i.e.,*

$$T(u)(\boldsymbol{x}) = \int_{\mathbb{R}^n} K(\boldsymbol{x},\boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y},$$

where K is a Calderón-Zygmund kernel. If $u \in L^p(\mathbb{R}^n)$, with $1 < p < \infty$, then $T(u) \in L^p(\mathbb{R}^n)$ and

$$\|T(u)\|_p \leq C\, \|u\|_p,$$

where C depends only on p and on the dimension n.

Lemma 2.27 (Proposition 3.2, [23])**.** *Let M be a Hermitian matrix with all eigenvalues $\{\lambda_i\}_{i=1}^n$ positive. Then,*

$$\|e^{-Mt}\| \leq e^{-(\min_i \lambda_i)t}, \tag{2.2}$$

for all $t \geq 0$.

Proof. Since M is a Hermitian matrix, it is diagonalizable. So, consider $\beta = \{v_1, \cdots, v_n\}$ a basis of eigenvectors for M with respective eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$. Then β is also a basis of eigenvectors for e^{-Mt} with respective eigenvalues $\{e^{-\lambda_1 t}, \cdots, e^{-\lambda_n t}\}$. Let $w \in \mathbb{C}^n$ such that

$$w = \alpha_1 v_1 + \cdots + \alpha_n v_n,$$

where $\alpha_i \in \mathbb{C}$ for all $i = 1, \cdots, n$. Take $\|w\| = \max_i |\alpha_i|$. Then for $t \geq 0$, one has

$$\left\|e^{-Mt}w\right\| = \left\|\sum_{i=1}^{n} \alpha_k e^{-\lambda_k t} v_k\right\| \leq \|w\| \left(\max_{i=1,\cdots,n} \left|e^{-\lambda_i t}\right|\right) = \|w\|\, e^{-(\min_i \lambda_i)t}.$$

Thus, the inequality (2.2) is proved. □

3 THE FOURIER SPLITTING METHOD AND DECAY ESTIMATES

This chapter is dedicated to explain the Fourier Splitting Method and to show how it can be applied to obtain decay estimates for parabolic conservation laws and Navier-Stokes equations.

3.1 Fourier Splitting Method

Definition 3.1. If $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, the Fourier Transform of u, denoted by $\widehat{u}$ is defined as

$$\mathscr{F}\{u\}(\boldsymbol{\xi}) = \widehat{u}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} u(\boldsymbol{x})\, d\boldsymbol{x}, \quad i = \sqrt{-1},$$

with $\boldsymbol{\xi} \in \mathbb{R}^n$.

The next theorem is a fundamental tool to apply the Fourier splitting method.

Theorem 3.2 (Plancherel theorem, Chapter 4, [17])**.** *If* $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*, then* $\widehat{u} \in L^2(\mathbb{R}^n)$ *and*

$$\|\widehat{u}\|_2 = (2\pi)^{n/2} \, \|u\|_2.$$

Proof. Note that if $v \in L^1(\mathbb{R}^n)$, then $\widehat{v} \in L^\infty(\mathbb{R}^n)$. Indeed,

$$\begin{aligned} |\widehat{v}(\boldsymbol{\xi})| &= \left| \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} v(\boldsymbol{x})\, d\boldsymbol{x} \right| \\ &\le \int_{\mathbb{R}^n} |e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} v(\boldsymbol{x})|\, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} |v(\boldsymbol{x})|\, d\boldsymbol{x} < \infty. \end{aligned}$$

Thus, if $v, w \in L^1(\mathbb{R}^n)$, then $\widehat{v}, \widehat{w} \in L^\infty(\mathbb{R}^n)$. Besides that, we have

$$\int_{\mathbb{R}^n} v(\boldsymbol{x})\, \widehat{w}(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\mathbb{R}^n} \widehat{v}(\boldsymbol{\xi})\, w(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{3.1}$$

By direct calculus, we have the identity

$$\int_{\mathbb{R}^n} e^{i\,\boldsymbol{\xi}\cdot\boldsymbol{x} - t|\boldsymbol{x}|^2}\, d\boldsymbol{x} = \left(\frac{\pi}{t}\right)^{n/2} e^{-\frac{|\xi|^2}{4t}}, \, \forall\, t > 0.$$

So, from equality (3.1) for each $\varepsilon > 0$, we obtain

$$\int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi}) e^{-\varepsilon|\boldsymbol{\xi}|^2}\, d\boldsymbol{\xi} = \left(\frac{\pi}{\varepsilon}\right)^{n/2} \int_{\mathbb{R}^n} w(\boldsymbol{x}) e^{-\frac{|\boldsymbol{x}|^2}{4\varepsilon}}\, d\boldsymbol{x}. \tag{3.2}$$

Now, let $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and set $v(x) := \overline{u}(-x)$, where $\overline{u}$ is the conjugate complex of u. Define $w \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ by

$$w(\boldsymbol{y}) = (2\pi)^n (u * v)(\boldsymbol{y}) = (2\pi)^n \int_{\mathbb{R}^n} u(\boldsymbol{x})\, v(\boldsymbol{y} - \boldsymbol{x})\, d\boldsymbol{x}.$$

Observe that,

$$\begin{aligned}
\widehat{w}(\boldsymbol{\xi}) &= (2\pi)^n \widehat{(u*v)}(\boldsymbol{\xi}) \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \int_{\mathbb{R}^n} u(\boldsymbol{y})\, v(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x} \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{y}}\, u(\boldsymbol{y}) \left(\int_{\mathbb{R}^n} e^{-i\,(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\xi}}\, v(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{x} \right) d\boldsymbol{y} \\
&= (2\pi)^n \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{y}}\, u(\boldsymbol{y})\, \widehat{v}(\boldsymbol{\xi})\, d\boldsymbol{y} \\
&= (2\pi)^n\, \widehat{u}(\boldsymbol{\xi})\, \widehat{v}(\boldsymbol{\xi}).
\end{aligned}$$

Thus, $\widehat{w} \in L^\infty(\mathbb{R}^n)$. But,

$$\widehat{v}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\, \overline{u}(-\boldsymbol{x})\ d\boldsymbol{x} = \overline{\widehat{u}}(\boldsymbol{\xi}).$$

So, $\widehat{w} = (2\pi)^n |\widehat{u}|^2$. Since w is continuous,

$$\lim_{\varepsilon\to 0} \left(\frac{\pi}{\varepsilon}\right)^{n/2} \int_{\mathbb{R}^n} w(\boldsymbol{x}) e^{-\frac{|\boldsymbol{x}|^2}{4\varepsilon}}\ d\boldsymbol{x} = (2\pi)^n w(\mathbf{0}).$$

From identity (3.2), we deduce

$$\int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi})\ d\boldsymbol{\xi} = (2\pi)^n w(\mathbf{0}).$$

Hence,

$$\int_{\mathbb{R}^n} |\widehat{u}|^2\, d\boldsymbol{\xi} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{w}(\boldsymbol{\xi})\, d\boldsymbol{\xi} = w(\mathbf{0}) = (2\pi)^n \int_{\mathbb{R}^n} u(\boldsymbol{x}) v(-\boldsymbol{x})\, d\boldsymbol{x} = (2\pi)^n \int_{\mathbb{R}^n} |u|^2\, d\boldsymbol{x},$$

and the desired identity follows. □

Lemma 3.3 (Hausdorff-Young inequality, Chapter V, [24])**.** *Let* $1 \le p \le 2$. *If* $u \in L^p(\mathbb{R}^n)$, *then*

$$\|\widehat{u}\|_q \le C\, \|u\|_p\,,$$

where q *is the conjugate of* p *and* $C = C(n,p,q) > 0$.

The Fourier Splitting Method was first applied to parabolic conservation laws in [25] and it can be used as follows. Let $\boldsymbol{u}(\boldsymbol{x},t) \in \mathbb{R}^3$ be a function satisfying the following *energy inequality*

$$\frac{d}{dt}\|\boldsymbol{u}\|_2^2 \le -C\|\nabla \boldsymbol{u}\|_2^2 + E(t), \tag{3.3}$$

where the $E(t)$ satisfies

$$E(t) \le C_1(t+1)^{-(\alpha+1)},$$

for some constants C_1 and α. If for $C_0 > 0$

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| \le C_0, \tag{3.4}$$

for all $\boldsymbol{\xi} \in S(t)$, where

$$S(t) = \left\{ \boldsymbol{\xi} \in \mathbb{R}^n : |\boldsymbol{\xi}| \leq \left(\frac{n}{C(t+1)} \right)^{1/2} \right\},$$

then

$$\|\boldsymbol{u}(\cdot, t)\|_2^2 \leq \widetilde{C}(t+1)^{-\alpha_0}, \tag{3.5}$$

where $\alpha_0 = \min(\alpha, n/2)$. and the constant $\widetilde{C}$ depends only on α, n, C, C_0, C_1 and on $\|\boldsymbol{u}_0\|_2$.

In [26], we can find its proof and some examples in which the method can be applied. Let us see two of them: Parabolic Conservation Laws and the Navier-Stokes equations.

3.2 Known Decay Results

3.2.1 Parabolic Conservation Laws

Consider the Cauchy problem

$$\begin{cases} u_t + \displaystyle\sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j(u) = \varepsilon \Delta u, & \boldsymbol{x} \in \mathbb{R}^n,\ t > 0, \\ \qquad\qquad u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^n, \end{cases} \tag{3.6}$$

where $u = u(\boldsymbol{x}, t) \in \mathbb{R}$ and $\boldsymbol{f} = (f_1, \cdots, f_n) : \mathbb{R} \to \mathbb{R}^n$ is a smooth map.

Theorem 3.4 (Theorem 2.1, [25]). *If $f_j \in C^1(\mathbb{R}, \mathbb{R})$, $u_0 \in L^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n) \cap C_0^1(\mathbb{R}^n)$, and u is a solution of system* (3.6)*, then*

$$\|u(\cdot, t)\|_2^2 \leq C(t+1)^{-n/2}, \tag{3.7}$$

where the constant C depends only on ε, n and u_0.

Proof. Here, we suppose that the Cauchy problem (3.6) has a solution. Multiplying the equation (3.6) by u and integrating by parts, we get

$$\frac{d}{dt} \int_{\mathbb{R}^n} u^2 \, d\boldsymbol{x} = -2\varepsilon \int_{\mathbb{R}^n} |\nabla u|^2 \, d\boldsymbol{x}. \tag{3.8}$$

By Theorem 3.2, we have

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\boldsymbol{\xi} = -2\varepsilon \int_{\mathbb{R}^n} \left|\widehat{\nabla u}\right|^2 \, d\boldsymbol{\xi}. \tag{3.9}$$

Then,

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\boldsymbol{\xi} = -2\varepsilon \int_{\mathbb{R}^n} |\boldsymbol{\xi}|^2 \, |\hat{u}|^2 \, d\boldsymbol{\xi}. \tag{3.10}$$

Now, we can use the Fourier splitting method. Define,

$$A(t) = \left\{ \boldsymbol{\xi} \in \mathbb{R}^n : |\boldsymbol{\xi}| > \left(\frac{n}{2\varepsilon(t+1)} \right)^{\frac{1}{2}} \right\}. \tag{3.11}$$

So,

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\boldsymbol{\xi} \leq -2\varepsilon \int_{A(t)} |\boldsymbol{\xi}|^2 \, |\hat{u}|^2 \, d\boldsymbol{\xi}. \tag{3.12}$$

Since $\boldsymbol{\xi} \in A(t)$, one obtains

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\boldsymbol{\xi} \leq -\frac{n}{t+1} \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\boldsymbol{\xi} + \frac{n}{t+1} \int_{S(t)} |\hat{u}|^2 \, d\boldsymbol{\xi}, \tag{3.13}$$

where $S(t) := A(t)^c$. Multiplying inequality (3.13) by $(t+1)^n$, we get

$$(t+1)^n \frac{d}{dt} \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\boldsymbol{\xi} + n(t+1)^{n-1} \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\boldsymbol{\xi} \leq n(t+1)^{n-1} \int_{S(t)} |\hat{u}|^2 \, d\boldsymbol{\xi}. \tag{3.14}$$

Since $u_0 \in L^1(\mathbb{R}^n)$, one has $\|u(\cdot,t)\|_1 \leq \|u_0\|_1$. In particular, $\|\widehat{u}(\cdot,t)\|_\infty \leq \|u_0\|_1$. Therefore,

$$\frac{d}{dt} \left[(t+1)^n \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\boldsymbol{\xi} \right] \leq \|u_0\|_1^2 \, n(t+1)^{n-1} \omega_n \left[\frac{2\varepsilon(t+1)}{n} \right]^{-n/2}, \tag{3.15}$$

where ω_n is the volume of the n-dimensional unit sphere. The desired bound is obtained by integrating inequality (3.15) over $[0,t]$. $\square$

Remark 3.5. (3.7) is also valid if $u_0 \in L^1(\mathbb{R}^n)$ only.

3.2.2 Navier-Stokes Equations

Let us establish the $\boldsymbol{L}^2$ decay of the solutions for the Navier-Stokes equations on $\mathbb{R}^n$

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{0}, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}). \end{cases} \tag{3.16}$$

Theorem 3.6 (Theorem 2.1, [15])**.** *Let $\boldsymbol{u} : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}^n$ and $p : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}$, be smooth functions, where $\boldsymbol{u}$ vanishes at infinity. If $\boldsymbol{u}$ and p satisfy the system* (3.16) *with $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\mathbb{R}^n) \cap \boldsymbol{L}^1(\mathbb{R}^n)$, then*

$$\left\| \boldsymbol{u}(\cdot,t) \right\|_2^2 \leq C(t+1)^{-n/2+1},$$

where the positive constant C depends only on $\|\boldsymbol{u}_0\|_1$, $\|\boldsymbol{u}_0\|_2$, and on n.

Proof. Multiplying equation $(3.16)_1$ by $\boldsymbol{u}$, and using the incompressibility condition $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\boldsymbol{u}|^2 \, d\boldsymbol{x} = -2 \int_{\mathbb{R}^n} |\nabla \boldsymbol{u}|^2 \, d\boldsymbol{x}.$$

By Theorem 3.2,

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\widehat{\boldsymbol{u}}|^2 \; d\boldsymbol{\xi} = -2\int_{\mathbb{R}^n} |\boldsymbol{\xi}|^2|\widehat{\boldsymbol{u}}|^2 \, d\boldsymbol{\xi}.$$

Define

$$S(t) = \left\{ \boldsymbol{\xi} \in \mathbb{R}^n : |\boldsymbol{\xi}| \leq \left(\frac{n}{2(t+1)} \right)^{\frac{1}{2}} \right\}.$$

So,

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\widehat{\boldsymbol{u}}|^2 \, d\boldsymbol{\xi} \leq -2\int_{S(t)^c} |\boldsymbol{\xi}|^2|\widehat{\boldsymbol{u}}|^2 \, d\boldsymbol{\xi}.$$

Then,

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\widehat{\boldsymbol{u}}|^2 \, d\boldsymbol{\xi} + \frac{n}{t+1}\int_{\mathbb{R}^n} |\widehat{\boldsymbol{u}}|^2 \; d\boldsymbol{\xi} \leq \frac{n}{t+1}\int_{S(t)} |\widehat{\boldsymbol{u}}|^2 \, d\boldsymbol{\xi}. \tag{3.17}$$

Now, we will to prove the inequality

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi}, t)| \leq C|\boldsymbol{\xi}|^{-1}, \quad \text{if} \quad \boldsymbol{\xi} \in S(t), \tag{3.18}$$

for some constant $C > 0$. Applying the Fourier Transform to equation $(3.16)_1$ and integrating by parts, we get

$$\widehat{\boldsymbol{u}}_t + |\boldsymbol{\xi}|^2\widehat{\boldsymbol{u}} = -\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p\}(\boldsymbol{\xi}). \tag{3.19}$$

Multiplying (3.19) by $e^{|\boldsymbol{\xi}|^2 t}$, we obtain

$$e^{|\boldsymbol{\xi}|^2 t}(\widehat{\boldsymbol{u}}_t + |\boldsymbol{\xi}|^2\widehat{\boldsymbol{u}}) = e^{|\boldsymbol{\xi}|^2 t}(-\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p\}(\boldsymbol{\xi})). \tag{3.20}$$

Integrating (3.20) from 0 to t,

$$e^{|\boldsymbol{\xi}|^2 t}\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}_0}(\boldsymbol{\xi}) + \int_0^t e^{|\boldsymbol{\xi}|^2 s}(-\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p\}(\boldsymbol{\xi}))\, ds.$$

Thus,

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi}, t)| \leq e^{-|\boldsymbol{\xi}|^2 t}|\widehat{\boldsymbol{u}_0}(\boldsymbol{\xi})| + \int_0^t e^{-|\boldsymbol{\xi}|^2(t-s)}|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p\}(\boldsymbol{\xi})|\, ds. \tag{3.21}$$

Now, we have to estimate the term $|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p\}(\boldsymbol{\xi})|$. Integrating by parts and using the fact that $\operatorname{div}\boldsymbol{u} = 0$, we have

$$|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})| \leq \sum_{j,k=1}^{n}\int_{\mathbb{R}^n} |u_k u_j|\, |\xi_k|\, d\boldsymbol{x}.$$

By Hölder inequality, we get

$$\|u_k u_j(\cdot, t)\|_1 \leq \|u_k(\cdot, t)\|_2\|u_j(\cdot, t)\|_2 \leq \|\boldsymbol{u}_0\|_2^2.$$

So,

$$|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|.$$

To estimate the term $\mathscr{F}\{\nabla p\}(\boldsymbol{\xi})$, apply the divergence operator in $(3.16)_1$ to get

$$\Delta p = -\sum_{j,k=1}^{n}\frac{\partial^2}{\partial x_j \partial x_k}(u_j u_k).$$

So,

$$\mathscr{F}\{\Delta p\}(\boldsymbol{\xi}) = \sum_{j,k=1}^{n} \xi_j \xi_k \widehat{u_j u_k}.$$

Since $\mathscr{F}\{\Delta p\}(\boldsymbol{\xi}) = -|\boldsymbol{\xi}|^2 \widehat{p}$, one obtains

$$|\widehat{p}| \leq C.$$

From

$$\mathscr{F}\{\nabla p\}(\boldsymbol{\xi}) = i\, \boldsymbol{\xi}\, \mathscr{F}\{p\},$$

we then get

$$|\mathscr{F}\{\nabla p\}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|.$$

Then,

$$|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p\}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|. \tag{3.22}$$

From (3.21) and the inequality (3.22), we get

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| \leq e^{-|\boldsymbol{\xi}|^2 t}|\widehat{\boldsymbol{u}_0}(\boldsymbol{\xi})| + C\int_0^t e^{-|\boldsymbol{\xi}|^2(t-s)}|\boldsymbol{\xi}|\, ds.$$

Then, since $\boldsymbol{u}_0 \in \boldsymbol{L}^1$ we have

$$|\widehat{\boldsymbol{u}_0}(\boldsymbol{\xi})| \leq C.$$

So, for $\boldsymbol{\xi} \in S(t)$

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1}. \tag{3.23}$$

Thus, from the bounds (3.17) and (3.23), we get

$$\frac{d}{dt}\left[(t+1)^n \int_{\mathbb{R}^n} |\widehat{\boldsymbol{u}}|^2\, d\boldsymbol{\xi}\right] \leq C\, n\, (t+1)^{n-1} \int_{S(t)} |\boldsymbol{\xi}|^{-2}\, d\boldsymbol{\xi}.$$

By a change of coordinates and integration by parts, one has

$$\begin{aligned} \frac{d}{dt}\left[(t+1)^n \int_{\mathbb{R}^n} |\widehat{\boldsymbol{u}}|^2 d\boldsymbol{\xi}\right] &\leq \frac{C\omega_n}{n-2}(t+1)^{n-1}\left[\frac{n}{2(t+1)}\right]^{n/2-1} \\ &\leq C(t+1)^{n/2}, \end{aligned} \tag{3.24}$$

where ω_n is the volume of the n-dimensional unit sphere. Since $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\mathbb{R}^n)$, integrating inequality (3.24) with respect to t we have

$$(t+1)^n \int_{\mathbb{R}^n} |\widehat{\boldsymbol{u}}|^2 d\boldsymbol{\xi} \leq C(t+1)^{n/2+1}.$$

It follows from Theorem 3.2 that

$$\int_{\mathbb{R}^n} |\boldsymbol{u}|^2 d\boldsymbol{x} \leq C(t+1)^{-n/2+1} \tag{3.25}$$

□

4 DECAY RESULTS FOR THE MICROPOLAR SYSTEM: FORMAL PROOFS

In this chapter, assuming suitable hypothesis for the external forces $\boldsymbol{f}$ and $\boldsymbol{g}$, we prove the estimate

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2+\left\|\boldsymbol{w}(\cdot,t)\right\|_2^2\leq C(t+1)^{-1/2},$$

for solutions of the Cauchy problem in $\mathbb{R}^3\times(0,+\infty)$

$$\begin{cases} \boldsymbol{u}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\Delta\boldsymbol{u}+\nabla p-\operatorname{curl}\boldsymbol{w}=\boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u}=0, \\ \boldsymbol{w}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}-\Delta\boldsymbol{w}-\nabla(\operatorname{div}\boldsymbol{w})+2\boldsymbol{w}-\operatorname{curl}\boldsymbol{u}=\boldsymbol{g}, \\ \boldsymbol{u}\Big|_{t=0}=\boldsymbol{u}_0, \quad \boldsymbol{w}\Big|_{t=0}=\boldsymbol{w}_0, \end{cases} \tag{4.1}$$

complemented with Dirichlet conditions at infinity, i.e.,

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x},t)\to\boldsymbol{0} \quad \text{as} \quad |\boldsymbol{x}|\to\infty,\ t>0, \\ \boldsymbol{w}(\boldsymbol{x},t)\to\boldsymbol{0} \quad \text{as} \quad |\boldsymbol{x}|\to\infty,\ t>0, \end{cases} \tag{4.2}$$

Section 4.1 is devoted to the case $\boldsymbol{f}=\boldsymbol{g}=\boldsymbol{0}$. In Section 4.2, assuming that the external forces decay in a suitable rate, we prove similar estimates for $\boldsymbol{L}^2$ norms of $\boldsymbol{u}$ and $\boldsymbol{w}$.

4.1 Null External Forces

The main result of this section is

Theorem 4.1. *Let $(\boldsymbol{u},p,\boldsymbol{w})$ be a smooth solution of problem (4.1)-(4.2), with $\boldsymbol{f}=\boldsymbol{g}=\boldsymbol{0}$. If $\boldsymbol{u}_0$, $\boldsymbol{w}_0\in\boldsymbol{L}^1(\mathbb{R}^3)\cap\boldsymbol{L}^2(\mathbb{R}^3)$, with $\operatorname{div}\boldsymbol{u}_0=0$, then there exists a constant $C>0$ such that, for all $t\geq0$,*

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2+\left\|\boldsymbol{w}(\cdot,t)\right\|_2^2\leq C(t+1)^{-1/2}.$$

The constant C depends only on the $\boldsymbol{L}^1$ and $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$.

Before proving Theorem 4.1, we show some auxiliary results.

Lemma 4.2. *Under the assumptions of Theorem 4.1, there exists $C>0$ such that*

$$|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})|+|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi})|+|\mathscr{F}\{\nabla p\}(\boldsymbol{\xi})|\leq C|\boldsymbol{\xi}|,$$

for all $t\geq0$ and $\boldsymbol{\xi}\in\mathbb{R}^3$ with $|\boldsymbol{\xi}|\neq0$. The constant C depends only on $\|\boldsymbol{u}_0\|_2$ and $\|\boldsymbol{w}_0\|_2$.

Proof. We begin by estimating $|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})|$. Let $[\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})]_j$ the j-th coordinate of $\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})$, i.e.,

$$[\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})]_j=\int_{\mathbb{R}^3}e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\sum_{k=1}^3 u_k\frac{\partial u_j}{\partial x_k}\right)\,d\boldsymbol{x}.$$

Integrating by parts, one obtains

$$\begin{aligned}[\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})]_j &= -\int_{\mathbb{R}^3} u_j \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} u_k\right)\, d\boldsymbol{x} \\ &= -\int_{\mathbb{R}^3} u_j \sum_{k=1}^{3} \left[e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(-i\xi_k u_k + \frac{\partial u_k}{\partial x_k}\right)\right] d\boldsymbol{x}.\end{aligned}$$

Since $\operatorname{div}\boldsymbol{u} = 0$, we get

$$[\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})]_j = i\int_{\mathbb{R}^3} u_j \sum_{k=1}^{3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \xi_k u_k \, d\boldsymbol{x}.$$

Thus,

$$|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})| \leq \sum_{j,\,k=1}^{3} \int_{\mathbb{R}^3} |u_k\, u_j||\xi_k|\, d\boldsymbol{x}.$$

Analogously, we get

$$|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi})| \leq \sum_{j,\,k=1}^{3} \int_{\mathbb{R}^3} |u_k\, w_j||\xi_k|\, d\boldsymbol{x}.$$

By Hölder inequality and estimate (4.25), we conclude that

$$\|u_k\, u_j(\cdot,t)\|_1 \leq \|u_k(\cdot,t)\|_2\, \|u_j(\cdot,t)\|_2 \leq \|\boldsymbol{u}(\cdot,t)\|_2^2 \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 \leq C$$

and

$$\|u_k\, w_j(\cdot,t)\|_1 \leq \|u_k(\cdot,t)\|_2\, \|w_j(\cdot,t)\|_2 \leq \|\boldsymbol{u}(\cdot,t)\|_2 \|\boldsymbol{w}(\cdot,t)\|_2 \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 \leq C.$$

Then,

$$|\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}| \quad \text{and} \quad |\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|, \tag{4.3}$$

where C depends only on the $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$. Now, note that

$$\mathscr{F}\{\nabla p\}(\boldsymbol{\xi}) = i\,\boldsymbol{\xi}\,\mathscr{F}\{p\}.$$

Taking the divergence of Eq. $(4.1)_1$, one gets

$$\Delta p = -\operatorname{div}\left[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = -\sum_{j,k=1}^{3} \frac{\partial^2}{\partial x_j \partial x_k}(u_j u_k). \tag{4.4}$$

Applying the Fourier transform to the identity (4.4), one obtains

$$-|\boldsymbol{\xi}|^2\, \widehat{p} = -i\,\boldsymbol{\xi}\cdot\, \mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi}) = \sum_{j,\,k=1}^{3} \xi_j\, \xi_k\, \mathscr{F}\{u_j\, u_k\}(\boldsymbol{\xi}).$$

Since $\mathscr{F}\{u_j\, u_k\} \in L^\infty$ and

$$|\mathscr{F}\{u_j\, u_k\}(\boldsymbol{\xi})| \leq \|u_j\, u_k(\cdot,t)\|_1 \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 \leq C,$$

one gets $|\boldsymbol{\xi}|^2\,|\,\widehat{p}\,| \leq C\,|\boldsymbol{\xi}|^2$. Thus,

$$|\,\widehat{p}\,| \leq C, \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}.$$

Consequently,

$$|\mathscr{F}\{\nabla p\}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|. \tag{4.5}$$

Due to the estimates (4.3) and (4.5), one gets the desired bound. □

Proposition 4.3. *Let $\mathcal{K} \subset \mathbb{R}^3$ be a compact set. Under the assumptions of Theorem 4.1, we have*

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1}, \quad \textit{for all} \;\; t \geq 0 \;\; \textit{and} \;\; \boldsymbol{\xi} \in \mathcal{K}, \quad \textit{with} \;\; \boldsymbol{\xi} \neq \mathbf{0},$$

where $C > 0$ depends only on the $\boldsymbol{L}^1$ and $\boldsymbol{L}^2$ norms of the initial data and on $\mathcal{K}$.

Remark 4.4. Compare with the estimate (3.23).

Proof. Let us take the Fourier transform in equations $(4.1)_1$ and $(4.1)_3$.

i. Fourier Transform of $\boldsymbol{u}_t$ and $\boldsymbol{w}_t$

Since $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$, we have

$$\boldsymbol{u}_t = ((u_1)_t, (u_2)_t, (u_3)_t) \quad \text{and} \quad \boldsymbol{w}_t = ((w_1)_t, (w_2)_t, (w_3)_t)\,.$$

Then,

$$\mathscr{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi}) = \left(\mathscr{F}\{(u_1)_t\}(\boldsymbol{\xi}), \mathscr{F}\{(u_2)_t\}(\boldsymbol{\xi}), \mathscr{F}\{(u_w)_t\}(\boldsymbol{\xi})\right),$$

and

$$\mathscr{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi}) = \left(\mathscr{F}\{(w_1)_t\}(\boldsymbol{\xi}), \mathscr{F}\{(w_2)_t\}(\boldsymbol{\xi}), \mathscr{F}\{(w_3)_t\}(\boldsymbol{\xi})\right).$$

Let $[\mathscr{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi})]_j$ and $[\mathscr{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi})]_j$ the j-th coordinate of $\mathscr{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi})$ and $\mathscr{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi})$, respectively. Observe that

$$[\mathscr{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi})]_j = \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}(u_j)_t\,d\boldsymbol{x} = (\widehat{u_j})_t(\boldsymbol{\xi}),$$

and

$$[\mathscr{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi})]_j = \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}(w_j)_t\,d\boldsymbol{x} = (\widehat{w_j})_t(\boldsymbol{\xi}).$$

Thus,

$$\mathscr{F}\{\boldsymbol{u}_t\}(\boldsymbol{\xi}) = \widehat{\boldsymbol{u}}_t, \quad \text{and} \quad \mathscr{F}\{\boldsymbol{w}_t\}(\boldsymbol{\xi}) = \widehat{\boldsymbol{w}}_t. \tag{4.6}$$

ii. Fourier Transform of $\Delta\boldsymbol{u}$ and $\Delta\boldsymbol{w}$

Note that

$$\Delta\boldsymbol{u} = (\Delta u_1, \Delta u_2, \Delta u_3) \quad \text{and} \quad \Delta\boldsymbol{w} = (\Delta w_1, \Delta w_2, \Delta w_3)\,.$$

So,

$$\mathscr{F}\{\Delta\boldsymbol{u}\}(\boldsymbol{\xi}) = \left(\mathscr{F}\{\Delta u_1\}(\boldsymbol{\xi}), \mathscr{F}\{\Delta u_2\}(\boldsymbol{\xi}), \mathscr{F}\{\Delta u_3\}(\boldsymbol{\xi})\right),$$

and

$$\mathscr{F}\{\Delta \boldsymbol{w}\}(\boldsymbol{\xi}) = \left(\mathscr{F}\{\Delta w_1\}(\boldsymbol{\xi}), \mathscr{F}\{\Delta w_2\}(\boldsymbol{\xi}), \mathscr{F}\{\Delta w_3\}(\boldsymbol{\xi})\right).$$

Hence,

$$[\mathscr{F}\{\Delta \boldsymbol{u}\}(\boldsymbol{\xi})]_j = \sum_{k=1}^{3} \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \frac{\partial^2 u_j}{\partial x_k^2}\, d\boldsymbol{x} = -\sum_{k=1}^{3} \int_{\mathbb{R}^3} \xi_k^2 e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} u_j(\boldsymbol{x})\, d\boldsymbol{x}. = -|\boldsymbol{\xi}|^2 \widehat{u}_j,$$

and

$$[\mathscr{F}\{\Delta \boldsymbol{w}\}(\boldsymbol{\xi})]_j = \sum_{k=1}^{3} \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \frac{\partial^2 w_j}{\partial x_k^2}\, d\boldsymbol{x} = -\sum_{k=1}^{3} \int_{\mathbb{R}^3} \xi_k^2 e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} w_j(\boldsymbol{x})\, d\boldsymbol{x}. = -|\boldsymbol{\xi}|^2 \widehat{w}_j.$$

Consequently,

$$\mathscr{F}\{\Delta \boldsymbol{u}\}(\boldsymbol{\xi}) = -|\boldsymbol{\xi}|^2 \widehat{\boldsymbol{u}} \quad \text{and} \quad \mathscr{F}\{\Delta \boldsymbol{w}\}(\boldsymbol{\xi}) = -|\boldsymbol{\xi}|^2 \widehat{\boldsymbol{w}}. \tag{4.7}$$

iii. Fourier Transform of curl $\boldsymbol{u}$ and curl $\boldsymbol{w}$

We define

$$\operatorname{curl} \boldsymbol{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

and

$$\operatorname{curl} \boldsymbol{w} = \left(\frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_3}, \frac{\partial w_1}{\partial x_3} - \frac{\partial w_3}{\partial x_1}, \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\right),$$

So, integrating by parts we get

$$\begin{aligned}
[\mathscr{F}\{\operatorname{curl} \boldsymbol{u}\}(\boldsymbol{\xi})]_1 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) d\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} -i e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \left(\xi_2 u_3 - \xi_3 u_2\right) d\boldsymbol{x} \\
&= -i\left(\xi_2 \widehat{u_3} - \xi_3 \widehat{u_2}\right),
\end{aligned}$$

$$\begin{aligned}
[\mathscr{F}\{\operatorname{curl} \boldsymbol{u}\}(\boldsymbol{\xi})]_2 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) d\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} -i e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \left(\xi_3 u_1 - \xi_1 u_3\right) d\boldsymbol{x} \\
&= -i\left(\xi_3 \widehat{u_1} - \xi_1 \widehat{u_3}\right),
\end{aligned}$$

$$\begin{aligned}
[\mathscr{F}\{\operatorname{curl} \boldsymbol{u}\}(\boldsymbol{\xi})]_3 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) d\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} -i e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}} \left(\xi_1 u_2 - \xi_2 u_1\right) d\boldsymbol{x} \\
&= -i\left(\xi_1 \widehat{u_2} - \xi_2 \widehat{u_1}\right),
\end{aligned}$$

$$\begin{aligned}
[\mathscr{F}\{\operatorname{curl}\boldsymbol{w}\}(\boldsymbol{\xi})]_1 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_3}{\partial x_2}-\frac{\partial w_2}{\partial x_3}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} -ie^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\xi_2 w_3-\xi_3 w_2\right)d\boldsymbol{x}\\
&= -i\left(\xi_2\widehat{w_3}-\xi_3\widehat{w_2}\right),
\end{aligned}$$

$$\begin{aligned}
[\mathscr{F}\{\operatorname{curl}\boldsymbol{w}\}(\boldsymbol{\xi})]_2 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_1}{\partial x_3}-\frac{\partial w_3}{\partial x_1}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} -ie^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\xi_3 w_1-\xi_1 w_3\right)d\boldsymbol{x}\\
&= -i\left(\xi_3\widehat{w_1}-\xi_1\widehat{w_3}\right),
\end{aligned}$$

$$\begin{aligned}
[\mathscr{F}\{\operatorname{curl}\boldsymbol{w}\}(\boldsymbol{\xi})]_3 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right)d\boldsymbol{x}\\
&= \int_{\mathbb{R}^3} -ie^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\xi_1 w_2-\xi_2 w_1\right)d\boldsymbol{x}\\
&= -i\left(\xi_1\widehat{w_2}-\xi_2\widehat{w_1}\right).
\end{aligned}$$

Therefore,

$$\mathscr{F}\{\operatorname{curl}\boldsymbol{u}\}(\boldsymbol{\xi}) = -i\begin{pmatrix}0 & -\xi_3 & \xi_2\\ \xi_3 & 0 & -\xi_1\\ -\xi_2 & \xi_1 & 0\end{pmatrix}\begin{pmatrix}\widehat{u_1}\\ \widehat{u_2}\\ \widehat{u_3}\end{pmatrix}, \tag{4.8}$$

and

$$\mathscr{F}\{\operatorname{curl}\boldsymbol{w}\}(\boldsymbol{\xi}) = -i\begin{pmatrix}0 & -\xi_3 & \xi_2\\ \xi_3 & 0 & -\xi_1\\ -\xi_2 & \xi_1 & 0\end{pmatrix}\begin{pmatrix}\widehat{w_1}\\ \widehat{w_2}\\ \widehat{w_3}\end{pmatrix}. \tag{4.9}$$

iv. Fourier Transform of $\operatorname{div}\boldsymbol{w}$

From the following identity

$$\nabla(\operatorname{div}\boldsymbol{w}) = \left(\frac{\partial^2 w_1}{\partial x_1^2}+\frac{\partial^2 w_2}{\partial x_1\partial x_2}+\frac{\partial^2 w_3}{\partial x_2\partial x_3},\frac{\partial^2 w_1}{\partial x_1\partial x_2}+\frac{\partial^2 w_2}{\partial x_2^2}+\frac{\partial^2 w_3}{\partial x_2\partial x_3},\frac{\partial^2 w_1}{\partial x_1\partial x_3}+\frac{\partial^2 w_2}{\partial x_3\partial x_2}+\frac{\partial^2 w_3}{\partial x_3^2}\right),$$

one obtains

$$\begin{aligned}
[\mathscr{F}\{\nabla(\operatorname{div}\boldsymbol{w})\}(\boldsymbol{\xi})]_1 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial^2 w_1}{\partial x_1^2}+\frac{\partial^2 w_2}{\partial x_1\partial x_2}+\frac{\partial^2 w_3}{\partial x_1 x_3}\right)d\boldsymbol{x}\\
&= \xi_1^2\widehat{w_1}+\xi_1\xi_2\widehat{w_2}+\xi_1\xi_3\widehat{w_3},
\end{aligned}$$

$$\begin{aligned}
[\mathscr{F}\{\nabla(\operatorname{div}\boldsymbol{w})\}(\boldsymbol{\xi})]_2 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial^2 w_1}{\partial x_1\partial x_2}+\frac{\partial^2 w_2}{\partial x_2^2}+\frac{\partial^2 w_3}{\partial x_2\partial x_3}\right)d\boldsymbol{x}\\
&= \xi_1\xi_2\widehat{w_1}+\xi_2^2\widehat{w_2}+\xi_2\xi_3\widehat{w_3},
\end{aligned}$$

$$\begin{aligned}
[\mathscr{F}\{\nabla(\operatorname{div}\boldsymbol{w})\}(\boldsymbol{\xi})]_3 &= \int_{\mathbb{R}^3} e^{-i\,\boldsymbol{\xi}\cdot\boldsymbol{x}}\left(\frac{\partial^2 w_1}{\partial x_1\partial x_3}+\frac{\partial^2 w_2}{\partial x_3\partial x_2}+\frac{\partial^2 w_3}{\partial x_3^2}\right)d\boldsymbol{x}\\
&= \xi_1\xi_3\widehat{w_1}+\xi_2\xi_3\widehat{w_2}+\xi_3^3\widehat{w_3}.
\end{aligned}$$

Thus,

$$[\mathscr{F}\{\nabla(\operatorname{div}\boldsymbol{w})\}(\boldsymbol{\xi})] = -\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{pmatrix}\begin{pmatrix} \widehat{w_1} \\ \widehat{w_2} \\ \widehat{w_3} \end{pmatrix}. \tag{4.10}$$

From the identities (4.6), (4.7), (4.8), (4.9), and (4.10), we rewrite equations $(4.1)_1$ and $(4.1)_3$ as

$$\widehat{\boldsymbol{u}}_t + |\boldsymbol{\xi}|^2\widehat{\boldsymbol{u}} - iL(\boldsymbol{\xi})\widehat{\boldsymbol{w}} = \boldsymbol{G}_1(\boldsymbol{\xi},t), \;\; \boldsymbol{\xi}\in\mathbb{R}^3,\, t>0, \tag{4.11}$$

and

$$\widehat{\boldsymbol{w}}_t + \Big(|\boldsymbol{\xi}|^2+2\Big)\,\widehat{\boldsymbol{w}} - iL(\boldsymbol{\xi})\widehat{\boldsymbol{u}} - P(\boldsymbol{\xi})\widehat{\boldsymbol{w}} = \boldsymbol{G}_2(\boldsymbol{\xi},t), \;\; \boldsymbol{\xi}\in\mathbb{R}^3,\, t>0, \tag{4.12}$$

where $L(\boldsymbol{\xi})$ and $P(\boldsymbol{\xi})$ are the real matrices

$$L(\boldsymbol{\xi}) := \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}, \quad P(\boldsymbol{\xi}) := -\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{pmatrix},$$

and $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ are defined by

$$\begin{cases} \boldsymbol{G}_1(\boldsymbol{\xi},t) := -\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p\}(\boldsymbol{\xi}), \\ \boldsymbol{G}_2(\boldsymbol{\xi},t) := -\mathscr{F}\{(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi}). \end{cases} \tag{4.13}$$

Now, setting $\boldsymbol{y} := (\boldsymbol{u},\boldsymbol{w})$ and $\boldsymbol{G} := (\boldsymbol{G}_1,\boldsymbol{G}_2)$, we can rewrite system (4.11)-(4.12) as

$$\widehat{\boldsymbol{y}}_t + A(\boldsymbol{\xi})\widehat{\boldsymbol{y}} = \boldsymbol{G}(\boldsymbol{\xi},t), \tag{4.14}$$

where $A(\boldsymbol{\xi})$ is the Hermitian matrix

$$A(\boldsymbol{\xi}) := \begin{pmatrix} |\boldsymbol{\xi}|^2 I & B(\boldsymbol{\xi}) \\ B(\boldsymbol{\xi}) & R(\boldsymbol{\xi}) + (|\boldsymbol{\xi}|^2+2)\,I \end{pmatrix}, \tag{4.15}$$

with $B(\boldsymbol{\xi}) := -i\,L(\boldsymbol{\xi})$ and $R(\boldsymbol{\xi}) := -P(\boldsymbol{\xi})$. It follows from Lemma 2.27 that there exists a constant $K>0$, independent of $\boldsymbol{\xi}$, such that

$$\|e^{-tA(\boldsymbol{\xi})}\| \le e^{-K|\boldsymbol{\xi}|^2 t}, \;\; \forall\, t\ge 0 \;\; \text{and} \;\; |\boldsymbol{\xi}|\neq 0, \tag{4.16}$$

where $\|\cdot\|$ denotes the Euclidean norm of a matrix. Multiplying equation (4.14) by $e^{tA(\boldsymbol{\xi})}$, we obtain

$$\frac{d}{dt}\left[e^{tA(\boldsymbol{\xi})}\,\widehat{\boldsymbol{y}}\right] = e^{tA(\boldsymbol{\xi})}\boldsymbol{G}(\boldsymbol{\xi},t).$$

Integrating with respect to time, we get

$$\widehat{\boldsymbol{y}}(\boldsymbol{\xi},t) = e^{-tA(\boldsymbol{\xi})}\,\widehat{\boldsymbol{y}_0}(\boldsymbol{\xi}) + \int_0^t e^{-(t-s)A(\boldsymbol{\xi})}\,\boldsymbol{G}(\boldsymbol{\xi},s)\,ds,$$

where $\widehat{\boldsymbol{y}_0} = \widehat{\boldsymbol{y}}(\cdot,0) = (\widehat{\boldsymbol{u}_0},\widehat{\boldsymbol{w}_0})$. Using the5 bound (4.16), one gets

$$|\widehat{\boldsymbol{y}}(\boldsymbol{\xi},t)| \le e^{-K|\boldsymbol{\xi}|^2 t}\,|\widehat{\boldsymbol{y}_0}(\boldsymbol{\xi})| + \int_0^t e^{-K|\boldsymbol{\xi}|^2(t-s)}\,|\boldsymbol{G}(\boldsymbol{\xi},s)|\,ds. \tag{4.17}$$

By Lemma 4.2, we have that $|\boldsymbol{G}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|$, for all $t \geq 0$ and $\boldsymbol{\xi} \neq \boldsymbol{0}$, where C depends only on $\|\boldsymbol{u}_0\|_2$ and $\|\boldsymbol{w}_0\|_2$. Thus, by inequality (4.17), we conclude that

$$|\widehat{\boldsymbol{y}}(\boldsymbol{\xi},t)| \leq e^{-K|\boldsymbol{\xi}|^2 t}\, |\widehat{\boldsymbol{y}_0}(\boldsymbol{\xi})| + C\int_0^t e^{-K|\boldsymbol{\xi}|^2(t-s)}\, |\boldsymbol{\xi}|\, ds. \tag{4.18}$$

Since $\boldsymbol{y}_0 = \boldsymbol{y}(\cdot,0) = (\boldsymbol{u}_0,\boldsymbol{w}_0) \in \boldsymbol{L}^1 \times \boldsymbol{L}^1$, it follows that $\widehat{\boldsymbol{y}_0} \in \boldsymbol{L}^\infty \times \boldsymbol{L}^\infty$, that is, we have

$$|\widehat{\boldsymbol{y}_0}(\boldsymbol{\xi})| \leq |\widehat{\boldsymbol{u}_0}(\boldsymbol{\xi})| + |\widehat{\boldsymbol{w}_0}(\boldsymbol{\xi})| \leq \|\boldsymbol{u}_0\|_1 + \|\boldsymbol{w}_0\|_1 \leq C,$$

for all $\boldsymbol{\xi}$ in a compact set $\mathcal{K}$ and some positive constant C depending on the $\boldsymbol{L}^1$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$. Hence, after calculating the integral in the right-hand side of inequality (4.18), we obtain

$$|\widehat{\boldsymbol{y}}(\boldsymbol{\xi},t)| \leq C\, e^{-K|\boldsymbol{\xi}|^2 t} + \frac{C}{K|\boldsymbol{\xi}|}\left(1 - e^{-K|\boldsymbol{\xi}|^2 t}\right),$$

for all $t \geq 0$, $\boldsymbol{\xi} \neq \boldsymbol{0}$ and some constant $C \in \mathbb{R}^+$ depending only on the $\boldsymbol{L}^1$ and $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$. Since $\mathcal{K} \subset \mathbb{R}^3$ is a compact set, we find that

$$|\widehat{\boldsymbol{y}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1}.$$

This completes the proof of Proposition 4.3. □

Proof of Theorem 4.1. Multiplying equation $(4.1)_1$ by $\boldsymbol{u}$, we have

$$(\boldsymbol{u}_t,\boldsymbol{u}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{u}) - (\Delta\boldsymbol{u},\boldsymbol{u}) + (\nabla p,\boldsymbol{u}) - (\operatorname{curl}\boldsymbol{w},\boldsymbol{u}) = 0. \tag{4.19}$$

Let us compute each term of Eq. (4.19). First of all

$$(\boldsymbol{u}_t,\boldsymbol{u}) = \frac{1}{2}\frac{d}{dt}(\boldsymbol{u},\boldsymbol{u}) = \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2. \tag{4.20}$$

Integrating by parts, using the incompressibility condition, and the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned} ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{u}) &= \int_{\mathbb{R}^3}\sum_{j,k=1}^{3} u_k\,\frac{\partial u_j}{\partial x_k}\,u_j\,d\boldsymbol{x} \\ &= -\int_{\mathbb{R}^3}\sum_{j,k=1}^{3} u_j\,\frac{\partial u_j}{\partial x_k}\,u_k\,d\boldsymbol{x} \\ &= -(\boldsymbol{u},(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}). \end{aligned} \tag{4.21}$$

Consequently, $((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{u}) = 0$. Moreover,

$$\begin{aligned} (\Delta\boldsymbol{u},\boldsymbol{u}) &= \int_{\mathbb{R}^3}\sum_{j,k=1}^{3} \frac{\partial^2 u_j}{\partial x_k^2}\,u_j\,d\boldsymbol{x} \\ &= -\int_{\mathbb{R}^3}\sum_{j,k=1}^{3}\left(\frac{\partial u_j}{\partial x_k}\right)^2 d\boldsymbol{x} \\ &= -(\nabla\boldsymbol{u},\nabla\boldsymbol{u}) \\ &= -\|\nabla\boldsymbol{u}\|_2^2. \end{aligned} \tag{4.22}$$

Finally, since $\operatorname{div} \boldsymbol{u} = 0$, we get

$$\begin{aligned} (\nabla p, \boldsymbol{u}) &= \int_{\mathbb{R}^3} \sum_{j=1}^{3} \frac{\partial p}{\partial x_j} u_j \, d\boldsymbol{x} \\ &= -\int_{\mathbb{R}^3} \sum_{j=1}^{3} p \, \frac{\partial u_j}{\partial x_j} \, d\boldsymbol{x} \\ &= -(p, \operatorname{div} \boldsymbol{u}) \\ &= 0. \end{aligned} \tag{4.23}$$

Then, using the Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \|\nabla \boldsymbol{u}\|_2^2 &= (\operatorname{curl} \boldsymbol{w}, \boldsymbol{u}) = (\boldsymbol{w}, \operatorname{curl} \boldsymbol{u}) \\ &\leq \|\boldsymbol{w}\|_2 \, \|\operatorname{curl} \boldsymbol{u}\|_2 = \|\boldsymbol{w}\|_2 \, \|\nabla \boldsymbol{u}\|_2 \\ &\leq \|\boldsymbol{w}\|_2^2 + \frac{1}{4}\|\nabla \boldsymbol{u}\|_2^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_2^2 + \|\nabla \boldsymbol{w}\|_2^2 + \|\operatorname{div} \boldsymbol{w}\|_2^2 + 2\|\boldsymbol{w}\|_2^2 &= (\operatorname{curl} \boldsymbol{u}, \boldsymbol{w}) \\ &\leq \|\operatorname{curl} \boldsymbol{u}\|_2 \, \|\boldsymbol{w}\|_2 = \|\nabla \boldsymbol{u}\|_2 \, \|\boldsymbol{w}\|_2 \\ &\leq \|\boldsymbol{w}\|_2^2 + \frac{1}{4}\|\nabla \boldsymbol{u}\|_2^2. \end{aligned}$$

Adding up these two inequalities, manipulating the terms in the right-hand side, we find the following *energy inequality*

$$\frac{d}{dt}\left(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{w}\|_2^2\right) \leq -\left(\|\nabla \boldsymbol{u}\|_2^2 + \|\nabla \boldsymbol{w}\|_2^2\right). \tag{4.24}$$

Integrating estimate (4.24) with respect to time in $[0, t]$, one gets

$$\|\boldsymbol{u}(\cdot, t)\|_2^2 + \|\boldsymbol{w}(\cdot, t)\|_2^2 \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2, \;\; \forall\, t \geq 0. \tag{4.25}$$

Note that from inequality (4.25) one has $\boldsymbol{u}$, $\boldsymbol{w} \in \boldsymbol{L}^2$. Hence, the transforms $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{w}}$ are well defined. Applying the Plancherel theorem to bound (4.24), we obtain

$$\frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2\right) \leq -\left(\|\widehat{\nabla \boldsymbol{u}}\|_2^2 + \|\widehat{\nabla \boldsymbol{w}}\|_2^2\right) = -\int_{\mathbb{R}^3} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi}.$$

Now, we will use the Fourier Splitting Method described in Chapter 3 . Let

$$S(t) = \left\{\boldsymbol{\xi} \in \mathbb{R}^3 \,:\, |\boldsymbol{\xi}| \leq r(t)\right\}, \tag{4.26}$$

where $r(t) = \left[\dfrac{3}{t+1}\right]^{1/2}$. Then,

$$\begin{aligned} \frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2\right) &\leq -\int_{S(t)^c} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi} - \int_{S(t)} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi} \\ &\leq -\int_{S(t)^c} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi} \\ &\leq -\frac{3}{t+1}\int_{S(t)^c} \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi}, \end{aligned}$$

since $-|\boldsymbol{\xi}|^2 \leq -\dfrac{3}{t+1}$ if $\boldsymbol{\xi} \in S(t)^c$. Thus, since $\mathbb{R}^3 = S(t) \cup S(t)^c$, we have

$$\frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2\right) \leq -\frac{3}{t+1}\int_{\mathbb{R}^3}\left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi} + \frac{3}{t+1}\int_{S(t)}\left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi},$$

or

$$\frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2\right) + \frac{3}{t+1}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2\right) \leq \frac{3}{t+1}\int_{S(t)}\left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi}.$$

Multiplying the above inequality by the integrating factor $(t+1)^3$, we get

$$\frac{d}{dt}\left[(t+1)^3(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2)\right] \leq 3(t+1)^2\int_{S(t)}\left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi}. \tag{4.27}$$

By Proposition 4.3, we conclude that

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)|^2 + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)|^2 \leq C|\boldsymbol{\xi}|^{-2},\ \forall\,\boldsymbol{\xi} \in S(t)\backslash\{\mathbf{0}\} \ \text{ and } \ \forall\, t \geq 0, \tag{4.28}$$

where $C > 0$ is a constant depending only on the $\boldsymbol{L}^1$ and $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$. By inequalities (4.27) and (4.28), one obtains

$$\frac{d}{dt}\left[(t+1)^3(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2)\right] \leq C(t+1)^2\int_{S(t)}|\boldsymbol{\xi}|^{-2} d\boldsymbol{\xi}.$$

A straightforward calculation via spherical coordinates gives

$$\int_{S(t)}|\boldsymbol{\xi}|^{-2} d\boldsymbol{\xi} = 4\pi\, r(t) = 4\sqrt{3}\,\pi(t+1)^{-1/2}.$$

Here, $r(t)$ is the radius of the ball $S(t)$ (see (4.26)). Hence,

$$\frac{d}{dt}\left[(t+1)^3(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2)\right] \leq C(t+1)^{3/2}.$$

Integrating in time yields

$$(t+1)^3(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2) \leq \|\widehat{\boldsymbol{u}_0}\|_2^2 + \|\widehat{\boldsymbol{w}_0}\|_2^2 + C\left[(t+1)^{5/2} - 1\right],$$

where $\widehat{\boldsymbol{u}_0} = \widehat{\boldsymbol{u}}(\cdot,0)$ and $\widehat{\boldsymbol{w}_0} = \widehat{\boldsymbol{w}}(\cdot,0)$. Using Theorem 3.2, one finds

$$(t+1)^3(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{w}\|_2^2) \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + C(t+1)^{5/2} \leq C(t+1)^{5/2},\ \forall\, t \geq 0.$$

Therefore,

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 \leq C(t+1)^{-1/2},\ \forall\, t \geq 0.$$

This completes the proof of Theorem 4.1. □

4.2 General External Forces

Now we deal with the case when the external forces $\boldsymbol{f}$ and $\boldsymbol{g}$ do not vanish. We assume $\boldsymbol{f}$ and $\boldsymbol{g}$ to satisfy

$$\operatorname{div} \boldsymbol{f}(t) = 0, \quad \forall\, t \geq 0, \tag{4.29}$$

$$\|\boldsymbol{f}(\cdot,t)\|_2 + \|\boldsymbol{g}(\cdot,t)\|_2 \leq K_1(t+1)^{-3/2}, \quad \forall\, t \geq 0, \tag{4.30}$$

$$|\widehat{\boldsymbol{f}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{g}}(\boldsymbol{\xi},t)| \leq K_2|\boldsymbol{\xi}|, \quad \forall\, t \geq 0 \ \text{ and } \ \boldsymbol{\xi} \in \mathbb{R}^3, \tag{4.31}$$

for some $K_1,\ K_2 > 0$.

Remark 4.5. Under the hypothesis (4.30), we have that $\boldsymbol{f},\, \boldsymbol{g} \in L^1(0,\infty;\boldsymbol{L}^2(\mathbb{R}^3))$.

Theorem 4.6. *Let* $(\boldsymbol{u},p,\boldsymbol{w})$ *be a smooth solution of problem* (4.1)-(4.2)*. If* $\boldsymbol{u}_0, \boldsymbol{w}_0 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$*, with* $\operatorname{div} \boldsymbol{u}_0 = 0$ *and* $\boldsymbol{f},\, \boldsymbol{g}$ *satisfying* (4.29)-(4.31)*, then*

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2},$$

where $C \in \mathbb{R}^+$ *depends only on* K_1*,* K_2 *and on the* $\boldsymbol{L}^1$ *and* $\boldsymbol{L}^2$ *norms of* $\boldsymbol{u}_0$ *and* $\boldsymbol{w}_0$*.*

Proof. In this case, the energy estimate (4.24) is replaced by

$$\frac{d}{dt}\left(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{w}\|_2^2\right) \leq -\left(\|\nabla \boldsymbol{u}\|_2^2 + \|\nabla \boldsymbol{w}\|_2^2\right) + 2\,|(\boldsymbol{f},\boldsymbol{u})| + 2\,|(\boldsymbol{g},\boldsymbol{w})|. \tag{4.32}$$

Integrating inequality (4.32) over $[0,t]$, one gets

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2 \leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + 2\int_0^t \Big(|(\boldsymbol{f},\boldsymbol{u})| + |(\boldsymbol{g},\boldsymbol{w})|\Big)ds. \tag{4.33}$$

Initially, note that $\boldsymbol{u}(\cdot,t)$ and $\boldsymbol{w}(\cdot,t) \in \boldsymbol{L}^2(\mathbb{R}^3)$. Indeed, if $\boldsymbol{f}$ and $\boldsymbol{g}$ are smooth with $\|\boldsymbol{f}(\cdot,t)\|_2 > 0$ and $\|\boldsymbol{g}(\cdot,t)\|_2 > 0$, then the functions F and G defined by

$$F(t) := |(\boldsymbol{f}(t),\boldsymbol{u}(t))|\, \|\boldsymbol{f}(t)\|_2^{-1} \quad \text{and} \quad G(t) := |(\boldsymbol{g}(t),\boldsymbol{w}(t))|\, \|\boldsymbol{g}(t)\|_2^{-1},$$

are continuous (see [27]). Hence, by the estimate (4.33) and Cauchy-Schwarz and Young inequalities, one gets

$$\begin{aligned}
F^2(t) + G^2(t) &\leq \|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{w}(t)\|_2^2 \\
&\leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + 2\int_0^t (F(s)\,\|\boldsymbol{f}(s)\|_2 + G(s)\,\|\boldsymbol{g}(s)\|_2)ds \\
&\leq \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + \int_0^\infty (\|\boldsymbol{f}(s)\|_2 + \|\boldsymbol{g}(s)\|_2)ds + \int_0^t (F^2(s)\,\|\boldsymbol{f}(s)\|_2 + G^2(s)\,\|\boldsymbol{g}(s)\|_2)ds \\
&\leq \widetilde{C} + \int_0^t (F^2(s) + G^2(s))\,(\|\boldsymbol{f}(s)\|_2 + \|\boldsymbol{g}(s)\|_2)ds,
\end{aligned}$$

with $\widetilde{C}$ a positive constant depending only on the $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ and on K_1. By Gronwall inequality, we have that

$$F(t) \leq C \quad \text{and} \quad G(t) \leq C,$$

where the constant $C \in \mathbb{R}^+$ depends only on the $\boldsymbol{L}^2$ norms of the initial data and on K_1. In particular,

$$|(\boldsymbol{f}(t), \boldsymbol{u}(t))| \leq C \, \|\boldsymbol{f}(t)\|_2 \qquad \text{and} \qquad |(\boldsymbol{g}(t), \boldsymbol{w}(t))| \leq C \, \|\boldsymbol{g}(t)\|_2. \tag{4.34}$$

Using the hypotheses on $\boldsymbol{f}$ and $\boldsymbol{g}$ and bounds (4.33)-(4.34), we conclude that $\boldsymbol{u}(\cdot, t)$, $\boldsymbol{w}(\cdot, t) \in \boldsymbol{L}^2(\mathbb{R}^3)$. Using Theorem 3.2, the inequality (4.32) can be rewritten as

$$\frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2\right) \leq -\int_{\mathbb{R}^3} |\boldsymbol{\xi}|^2 \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi} + H(t), \tag{4.35}$$

where $H(t) := 16\pi^3\Big(|(\boldsymbol{f}(t), \boldsymbol{u}(t))| \, + \, |(\boldsymbol{g}(t), \boldsymbol{w}(t))|\Big)$. By the hypothesis under the external forces (4.30) and the inequalities in (4.34), one gets

$$H(t) \leq C \, (t+1)^{-3/2},$$

where $C \in \mathbb{R}^+$ depends only on the $\boldsymbol{L}^2$ norm of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ and on the constant K_1. From inequality (4.35) and repeating the arguments in the proof of Theorem 4.1, we obtain

$$\frac{d}{dt}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2\right) + \frac{3}{(t+1)}\left(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2\right) \leq \frac{3}{(t+1)} \int_{S(t)} \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi} + C(t+1)^{-3/2},$$

where $S(t)$ is the ball defined in (4.26). Multiplying the inequality above by $(t+1)^3$, one finds

$$\frac{d}{dt}\Big[(t+1)^3(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2)\Big] \leq 3(t+1)^2 \int_{S(t)} \left(|\widehat{\boldsymbol{u}}|^2 + |\widehat{\boldsymbol{w}}|^2\right) d\boldsymbol{\xi} + C(t+1)^{3/2}. \tag{4.36}$$

Repeating the arguments used in Lemma 4.2 and Proposition 4.3, together with the hypothesis (4.29) and (4.31), we get

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi}, t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi}, t)| \leq C|\boldsymbol{\xi}|^{-1}, \ \forall \ \boldsymbol{\xi} \in S(t)\backslash\{\boldsymbol{0}\} \ \text{ and } \ t \geq 0, \tag{4.37}$$

where the positive constant C depends only on the $\boldsymbol{L}^1$ and $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ and on the constant K_2. Thus, by inequalities (4.36) and (4.37), we have

$$\frac{d}{dt}\left[(t+1)^3(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2)\right] \leq C(t+1)^2 \int_{S(t)} |\boldsymbol{\xi}|^{-2} \, d\boldsymbol{\xi} + C(t+1)^{3/2}.$$

Using spherical coordinates, one gets

$$\frac{d}{dt}\left[(t+1)^3(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2)\right] \leq C(t+1)^{3/2}.$$

Integrating in time from 0 to t, one obtains

$$(t+1)^3(\|\widehat{\boldsymbol{u}}\|_2^2 + \|\widehat{\boldsymbol{w}}\|_2^2) \leq (\|\widehat{\boldsymbol{u}_0}\|_2^2 + \|\widehat{\boldsymbol{w}_0}\|_2^2) + C(t+1)^{5/2}.$$

It follows from Plancherel theorem that

$$(t+1)^3(\|\boldsymbol{u}\|_2^2 + \|\boldsymbol{w}\|_2^2) \leq \left(\|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2\right) + C(t+1)^{5/2}.$$

Therefore

$$\|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{w}(t)\|_2^2 \leq C(t+1)^{-1/2}.$$

This completes the proof of Theorem 4.6. □

If $\boldsymbol{u}(\cdot,t)$ and $\boldsymbol{w}(\cdot,t) \in \boldsymbol{L}^1(\mathbb{R}^3)$, then the following decay rate can be obtained for solutions of system (4.1):

Corollary 4.7. *Let* $(\boldsymbol{u}, p, \boldsymbol{w})$ *be a smooth solution of problem* (4.1)-(4.2)*. Let* $\boldsymbol{u}_0$*,* $\boldsymbol{w}_0$*,* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *be as in Theorem 4.6. If* $\|\boldsymbol{u}(\cdot,t)\|_1 + \|\boldsymbol{w}(\cdot,t)\|_1 \leq K_3$*, then*

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2},$$

where C *depends only on* K_1*,* K_2*,* K_3 *and on the* $\boldsymbol{L}^1$ *and* $\boldsymbol{L}^2$ *norms of* $\boldsymbol{u}_0$ *and* $\boldsymbol{w}_0$*.*

Proof. As $\boldsymbol{u}(\cdot,t)$, $\boldsymbol{w}(\cdot,t) \in \boldsymbol{L}^1(\mathbb{R}^3)$ and $\widehat{\boldsymbol{u}}$, $\widehat{\boldsymbol{w}} \in \boldsymbol{L}^\infty$, we have

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)| \leq \|\boldsymbol{u}(\cdot,t)\|_1 + \|\boldsymbol{w}(\cdot,t)\|_1 \leq K_3.$$

Just repeat the proof of Theorem 4.6 using the estimate above. □

5 DECAY RESULTS FOR THE MICROPOLAR SYSTEM: RIGOROUS ARGUMENTS

In Chapter 4, we have showed some decay theorems for a smooth solution of the problem

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p - \operatorname{curl}\boldsymbol{w} = \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \Delta\boldsymbol{w} - \nabla(\operatorname{div}\boldsymbol{w}) + 2\boldsymbol{w} - \operatorname{curl}\boldsymbol{u} = \boldsymbol{g}, \\ \boldsymbol{u}\Big|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{w}\Big|_{t=0} = \boldsymbol{w}_0, \end{cases} \tag{5.1}$$

with

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x},t) \to \boldsymbol{0} \quad \text{as} \quad |\boldsymbol{x}| \to \infty,\ t > 0, \\ \boldsymbol{w}(\boldsymbol{x},t) \to \boldsymbol{0} \quad \text{as} \quad |\boldsymbol{x}| \to \infty,\ t > 0, \end{cases} \tag{5.2}$$

assuming its existence. The proposal of the current chapter is to make such results rigorous, guaranteeing the existence of the weak solution for system (5.1)-(5.2) and then, proving the bound

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 \le C(t+1)^{-1/2}, \quad \forall\, t \ge 0.$$

To do so, let T be an arbitrary positive number. Recall that we are using the following notation

$$\begin{gathered} \mathcal{Q}_T = \mathbb{R}^3 \times (0,T), \\ \boldsymbol{\mathcal{V}} := \left\{ \boldsymbol{v} \in \boldsymbol{C}_0^\infty(\mathbb{R}^3) \,/\, \operatorname{div}\boldsymbol{v} = 0 \text{ in } \mathbb{R}^3 \right\}, \\ \boldsymbol{H} := \text{the closure of } \boldsymbol{\mathcal{V}} \text{ in } \boldsymbol{L}^2(\mathbb{R}^3), \\ \boldsymbol{V} := \text{the closure of } \boldsymbol{\mathcal{V}} \text{ in } \boldsymbol{H}^1(\mathbb{R}^3), \\ \boldsymbol{V}' := \text{the topological dual of } \boldsymbol{V}. \end{gathered}$$

Definition 5.1. Let $\boldsymbol{u}_0 \in \boldsymbol{H}$, $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$, $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$. We say that $(\boldsymbol{u},\boldsymbol{v})$ is a weak solution of the problem (5.1)-(5.2) if

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H}),$$

$$\boldsymbol{w} \in L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)),$$

and $(\boldsymbol{u},\boldsymbol{w})$ satisfies

$$\begin{cases} (\boldsymbol{u}_t,\boldsymbol{\varphi}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{\varphi}) + (\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}) = (\operatorname{curl}\boldsymbol{w},\boldsymbol{\varphi}) + (\boldsymbol{f},\boldsymbol{\varphi}) \\ (\operatorname{div}\boldsymbol{u},\boldsymbol{\varphi}) = 0 \\ (\boldsymbol{w}_t,\boldsymbol{\phi}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{\phi}) + (\nabla\boldsymbol{w},\nabla\boldsymbol{\phi}) + (\operatorname{div}\boldsymbol{w},\operatorname{div}\boldsymbol{\phi}) + 2\,(\boldsymbol{w},\boldsymbol{\phi}) \\ \qquad\qquad = (\operatorname{curl}\boldsymbol{u},\boldsymbol{\phi}) + (\boldsymbol{g},\boldsymbol{\phi})\,, \\ \boldsymbol{u}\Big|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{w}\Big|_{t=0} = \boldsymbol{w}_0 \quad \text{weakly in } \boldsymbol{L}^2(\mathbb{R}^3), \end{cases} \tag{5.3}$$

for all $\boldsymbol{\varphi} \in \boldsymbol{V}$ and $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\mathbb{R}^3)$.

In the next section we construct approximate solutions to the problem (5.1) and present several lemmas established in [4, 28, 29]. Finally, in Section 5.2, we prove the $\boldsymbol{L}^2$ decay for the approximate solutions of the system (5.1) and the previous estimate follows.

5.1 Approximate Solutions and Auxiliary Results

Following the ideas introduced by L. Caffarelli, R. Kohn and L. Nirenberg in [28], we define the retarded mollifiers as

Definition 5.2 (Chapter 3, Section 1.4, [3])**.** For $\boldsymbol{u} \in L^2(0, T; \boldsymbol{V})$ and $\delta > 0$, the retarded mollifier of $\boldsymbol{u}$ is defined by

$$\Psi_\delta(\boldsymbol{u})(\boldsymbol{x}, t) := \frac{1}{\delta^4} \iint\limits_{\mathbb{R}^4} \psi\left(\frac{\boldsymbol{y}}{\delta}, \frac{\tau}{\delta}\right) \widetilde{\boldsymbol{u}}(\boldsymbol{x} - \boldsymbol{y}, t - \tau)\, d\boldsymbol{y}\, d\tau,$$

where $\psi(\boldsymbol{x}, t) \in C^\infty(\mathbb{R}^4)$ is such that

$$\psi \geq 0, \qquad \iint\limits_{\mathbb{R}^4} \psi(\boldsymbol{x}, t)\, d\boldsymbol{x}\, dt = 1,$$
$$\operatorname{supp} \psi \subseteq \{(\boldsymbol{x}, t) \in \mathbb{R}^4 ;\, |\boldsymbol{x}|^2 < t,\, 1 < t < 2\},$$

and $\widetilde{\boldsymbol{u}} : \mathbb{R}^4 \to \mathbb{R}^3$ is a extension of $\boldsymbol{u}$, that is

$$\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t) := \begin{cases} \boldsymbol{u}(\boldsymbol{x}, t), & \text{if} \quad (\boldsymbol{x}, t) \in \mathcal{Q}_T, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

Remark 5.3. Note that $\Psi_\delta(\boldsymbol{u})$ is a smooth function, i.e., $\Psi_\delta(\boldsymbol{u}) \in \boldsymbol{C}^\infty(\mathbb{R}^3 \times [0, T]; \mathbb{R}^3)$, whose values at time t depend only on the values of $\boldsymbol{u}$ at times $\tau \in (t - 2\delta, t - \delta)$.

Lemma 5.4 (Lemma A.8, [28])**.** *If $\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{V})$, then*

$$\operatorname{div} \Psi_\delta(\boldsymbol{u}) = 0, \tag{5.4}$$

and

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^3} |\Psi_\delta(\boldsymbol{u})|^2\, d\boldsymbol{x} \leq C \operatorname*{ess\,sup}_{t \in (0, T)} \int_{\mathbb{R}^3} |\boldsymbol{u}|^2\, d\boldsymbol{x}, \tag{5.5}$$

where $C \in \mathbb{R}^+$ is an universal constant.

Proof. Since $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\operatorname{div} \widetilde{\boldsymbol{u}} := \begin{cases} \operatorname{div} \boldsymbol{u}, & \text{if} \quad (\boldsymbol{x}, t) \in \mathcal{Q}_T, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\operatorname{div} \widetilde{\boldsymbol{u}} \equiv 0$ and assertion (5.4) is proved. Inequality (5.5) follows from the identity

$$\Psi_\delta(\boldsymbol{u}) \equiv \widetilde{\boldsymbol{u}} * \Phi_\delta,$$

where $*$ denotes the convolution operator and $\Phi_\delta : \mathbb{R}^4 \to \mathbb{R}$ is such that

$$\Phi_\delta(\boldsymbol{x}, t) := \delta^{-4}\psi\left(\delta^{-1}\boldsymbol{x}, \delta^{-1}t\right),$$

and by Young inequality for convolutions,

$$\|\Psi_\delta(\boldsymbol{u})\|_2 = \|\widetilde{\boldsymbol{u}} * \Phi_\delta\|_2 \le C\|\widetilde{\boldsymbol{u}}\|_2,$$

and we prove inequality (5.5). □

Now, let $N \in \mathbb{N}$ and for some fixed $T > 0$ consider $\delta = T/N$ in Definition 5.2. We define approximate solutions $(\boldsymbol{u}^N, p^N, \boldsymbol{w}^N)$ for the system (5.1) as solutions of the problem

$$\begin{cases} \boldsymbol{u}_t^N + \left(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla\right)\boldsymbol{u}^N - \Delta\boldsymbol{u}^N + \nabla p^N - \operatorname{curl}\boldsymbol{w}^N = \boldsymbol{f} \ \text{ in } \ \mathcal{Q}_T, \\ \operatorname{div}\boldsymbol{u}^N = 0 \ \text{ in } \ \mathcal{Q}_T, \\ \boldsymbol{w}_t^N + \left(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla\right)\boldsymbol{w}^N - \Delta\boldsymbol{w}^N - \nabla(\operatorname{div}\boldsymbol{w}^N) + 2\boldsymbol{w}^N - \operatorname{curl}\boldsymbol{u}^N = \boldsymbol{g} \ \text{ in } \ \mathcal{Q}_T, \\ \boldsymbol{u}^N(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{w}^N(\boldsymbol{x},0) = \boldsymbol{w}_0(\boldsymbol{x}) \ \text{ in } \ \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} p^N d\boldsymbol{x} = 0 \ \text{ a.e. in } \ [0,T], \end{cases} \tag{5.6}$$

with

$$\begin{cases} \boldsymbol{u}^N(\boldsymbol{x},t) \to \boldsymbol{0} \quad \text{as} \quad |\boldsymbol{x}| \to \infty, \ t > 0, \\ \boldsymbol{w}^N(\boldsymbol{x},t) \to \boldsymbol{0} \quad \text{as} \quad |\boldsymbol{x}| \to \infty, \ t > 0. \end{cases} \tag{5.7}$$

Here, $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$.

Let us review some auxiliary results for the problems (5.1) and (5.6) (see [28, Appendix] and [3, Chapter 3]).

Lemma 5.5. *Suppose $\boldsymbol{F} \in L^2(0,T;\boldsymbol{V}')$, $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$, and p is a distribution such that*

$$\boldsymbol{u}_t - \Delta\boldsymbol{u} + \nabla p = \boldsymbol{F}, \tag{5.8}$$

in the sense of distributions on $\mathcal{Q}_T$. Moreover, assume $\boldsymbol{G} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$, $\boldsymbol{w} \in L^2(0,T;\boldsymbol{H}_0^1(\mathbb{R}^3))$, and

$$\boldsymbol{w}_t - \Delta\boldsymbol{w} - \nabla(\operatorname{div}\boldsymbol{w}) = \boldsymbol{G}. \tag{5.9}$$

Then,

$$\boldsymbol{u}_t \in L^2(0,T;\boldsymbol{V}'), \qquad \boldsymbol{w}_t \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3)), \tag{5.10}$$

$$\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 = (\boldsymbol{u}_t, \boldsymbol{u}), \qquad \frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 = (\boldsymbol{w}_t, \boldsymbol{w}), \tag{5.11}$$

in the sense of distributions, and

$$\boldsymbol{u} \in C([0,T];\boldsymbol{H}), \qquad \boldsymbol{w} \in C([0,T];\boldsymbol{L}^2(\mathbb{R}^3)), \tag{5.12}$$

after a possible modification on a set of measure zero. Solutions of the equation (5.8) *are unique in the space* $L^2(0,T;\boldsymbol{V})$ *for a given initial data* $\boldsymbol{u}_0 \in \boldsymbol{H}$*. Similarly, solutions of the equation* (5.9) *are unique in the space* $L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3))$ *for a given initial data* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$*.*

Proof. We will only prove the results for $\boldsymbol{u}$, since the proof of the results for $\boldsymbol{w}$ is analogous. Multiplying the equation (5.8) by $\boldsymbol{v} \in \boldsymbol{V}$ in $\boldsymbol{L}^2(\mathbb{R}^3)$, we get

$$(\boldsymbol{u}_t, \boldsymbol{v}) - (\Delta \boldsymbol{u}, \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) = \langle \boldsymbol{F}, \boldsymbol{v} \rangle. \tag{5.13}$$

Since $\operatorname{div} \boldsymbol{v} = 0$, we have

$$\frac{d}{dt}(\boldsymbol{u}, \boldsymbol{v}) + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \langle \boldsymbol{F}, \boldsymbol{v} \rangle. \tag{5.14}$$

Now, observe that as

$$\boldsymbol{V} \ni \boldsymbol{v} \mapsto (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$$

is a linear and continuous map on $\boldsymbol{V}$, then there exists $\mathbf{A}\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}')$ such that

$$\langle \mathbf{A}\boldsymbol{u}, \boldsymbol{v} \rangle = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}).$$

Thus, we can rewrite the identity (5.14) as

$$\frac{d}{dt}\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{F} - \mathbf{A}\boldsymbol{u}, \boldsymbol{v} \rangle. \tag{5.15}$$

Since $\boldsymbol{F}, \mathbf{A}\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}')$, we conclude that

$$\boldsymbol{u}_t \in L^2(0,T;\boldsymbol{V}'). \tag{5.16}$$

Moreover, from (5.16) and using Lemma 2.16 for $\boldsymbol{V}$, $\boldsymbol{V}'$, $\boldsymbol{H}$, we get (5.11) and (5.12). To prove the uniqueness of $\boldsymbol{u}$, let us assume that $(\boldsymbol{u}^1, p^1)$ and $(\boldsymbol{u}^2, p^2)$ are two solutions of (5.8) with initial data $\boldsymbol{u}_0$ and external force $\boldsymbol{F}$, i.e.,

$$\boldsymbol{u}_t^1 - \Delta \boldsymbol{u}^1 + \nabla p^1 = \boldsymbol{F}, \qquad \boldsymbol{u}^1(0) = \boldsymbol{u}_0,$$

and

$$\boldsymbol{u}_t^2 - \Delta \boldsymbol{u}^2 + \nabla p^2 = \boldsymbol{F}, \qquad \boldsymbol{u}^2(0) = \boldsymbol{u}_0.$$

Define $\boldsymbol{u} = \boldsymbol{u}^1 - \boldsymbol{u}^2$ and $p = p^1 - p^2$. Then, $\boldsymbol{u}$ belongs to the same spaces as $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ and satisfies

$$\boldsymbol{u}_t - \Delta \boldsymbol{u} + \nabla p = \mathbf{0}, \qquad \boldsymbol{u}(0) = \mathbf{0}. \tag{5.17}$$

Multiplying the equation (5.17) by $\boldsymbol{u}$, from the first identity of (5.11) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 + \|\nabla \boldsymbol{u}\|_2^2 = 0.$$

Integrating from 0 to t, $0 < t \leq T$, one has

$$\|\boldsymbol{u}(t)\|^2 \leq \|\boldsymbol{u}(0)\|^2 = 0.$$

Therefore, $\boldsymbol{u}^1 = \boldsymbol{u}^2$. □

Lemma 5.6 (Lemma 1.2.1, [3])**.** *If* $\boldsymbol{v}, \boldsymbol{w} \in C^\infty(\bar{\mathcal{Q}}_T; \mathbb{R}^3)$, $\operatorname{div} \boldsymbol{v} = 0$, $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$, *and* $\boldsymbol{u}_0 \in \boldsymbol{H}$, *then there exists a unique function* $\boldsymbol{u}$,

$$\boldsymbol{u} \in C([0,T];\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{5.18}$$

and a distribution p *on* $\mathcal{Q}_T$ *such that equation*

$$\boldsymbol{u}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p - \operatorname{curl}\boldsymbol{w} = \boldsymbol{f} \tag{5.19}$$

holds in the sense of distributions on $\mathcal{Q}_T$.

Proof. The existence of $\boldsymbol{u}$ and p is proved in Theorem 5.8 and assertion (5.18) follows directly by repeating arguments of the proof of Lemma 5.5 and by Lemma 2.16. □

Lemma 5.7 (Lemma 1.3.1, [3])**.** *If* $\boldsymbol{u} \in C^\infty(\bar{\mathcal{Q}}_T; \mathbb{R}^3)$, $\operatorname{div} \boldsymbol{u} = 0$, $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$, *and* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$, *then there exists a unique function* $\boldsymbol{w}$,

$$\boldsymbol{w} \in C([0,T];\boldsymbol{L}^2(\mathbb{R}^3)) \cap L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)), \quad \boldsymbol{w}(0) = \boldsymbol{w}_0, \tag{5.20}$$

such that equation

$$\boldsymbol{w}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \Delta\boldsymbol{w} - \nabla(\operatorname{div}\boldsymbol{w}) + 2\boldsymbol{w} - \operatorname{curl}\boldsymbol{u} = \boldsymbol{g}$$

holds in the sense of distributions on $\mathcal{Q}_T$.

Proof. Using Lemma 2.16 and by the the proof of Lemma 5.6. □

Theorem 5.8. *If* $\boldsymbol{u}_0 \in \boldsymbol{H}$, $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$, $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$, *and* $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$, *then there exists a weak solution* $(\boldsymbol{u}, p, \boldsymbol{w})$ *of the problem* (5.1) *such that*

$$\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}) \tag{5.21}$$

and

$$\boldsymbol{w} \in L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)) \cap L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)). \tag{5.22}$$

Proof. Let us show the existence of a weak solution for the system (5.1) using the Faedo-Galerkin method. Consider $(\boldsymbol{\phi}_k)_{k\in\mathbb{N}} \subset \boldsymbol{V}$ a basis of eigenfunctions for the Stokes operator $\widetilde{\Delta} := -P\Delta$, where P is the orthogonal projection from $\boldsymbol{L}^2(\mathbb{R}^3)$ onto $\boldsymbol{H}$, and $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset \boldsymbol{H}_0^1(\mathbb{R}^3)$ is a basis of eigenfunctions for the Lamé operator $L := -\Delta - \nabla\operatorname{div}$ (see [10]). For each $m \in \mathbb{N}$, we define an approximate solution for problem (5.1) as follows:

$$\begin{aligned}
\boldsymbol{u}_m(\boldsymbol{x},t) &= \sum_{k=1}^m h_{km}(t)\boldsymbol{\phi}_k(\boldsymbol{x}), \quad h_{km}(t) \in \mathbb{R}, \\
\boldsymbol{w}_m(\boldsymbol{x},t) &= \sum_{k=1}^m \bar{h}_{km}(t)\boldsymbol{\varphi}_k(\boldsymbol{x}), \quad \bar{h}_{km}(t) \in \mathbb{R},
\end{aligned}$$

where the pair $(\boldsymbol{u}_m, \boldsymbol{w}_m)$ is the solution of the approximate problem

$$\begin{cases} (\partial_t \boldsymbol{u}_m, \boldsymbol{\phi}_k) + ((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{\phi}_k) - (\Delta \boldsymbol{u}_m, \boldsymbol{\phi}_k) = (\mathrm{curl}\, \boldsymbol{w}_m, \boldsymbol{\phi}_k) + (\boldsymbol{f}, \boldsymbol{\phi}_k), \quad k = 1, \cdots, m, \\ (\partial_t \boldsymbol{w}_m, \boldsymbol{\varphi}_k) + ((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{w}_m, \boldsymbol{\varphi}_k) - (\Delta \boldsymbol{w}_m, \boldsymbol{\varphi}_k) - (\nabla \mathrm{div}\, \boldsymbol{w}_m, \boldsymbol{\varphi}_k) + 2(\boldsymbol{w}_m, \boldsymbol{\varphi}_k) \\ \qquad\qquad = (\mathrm{curl}\, \boldsymbol{u}_m, \boldsymbol{\varphi}_k) + (\boldsymbol{g}, \boldsymbol{\varphi}_k), \quad k = 1, \cdots, m, \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m}, \quad \boldsymbol{w}_m(0) = \boldsymbol{w}_{0m}. \end{cases} \tag{5.23}$$

It is not difficult to prove that the nonlinear system of ordinary differential equations (5.23) has only one solution $(\boldsymbol{u}_m, \boldsymbol{w}_m)$ defined in a interval $[0, T_m)$, with $T_m \in (0, T)$. Actually it is possible to extend this solution to $[0, T]$ (for more details, see [10]). Now, multiplying both sides of the equations $(5.23)_1$ and $(5.23)_2$ by h_{km} and $\bar{h}_{km}$, respectively, adding up the corresponding equations from $k = 1, \cdots, m$, and using Cauchy-Schwarz and Young inequalities, we get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|_2^2 + \frac{1}{2}\|\nabla \boldsymbol{u}_m\|_2^2 \le \frac{1}{2}\|\boldsymbol{w}_m\|_2^2 + \|\boldsymbol{f}(t)\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}},$$

and

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_m\|_2^2 + \|\nabla \boldsymbol{w}_m\|_2^2 + \|\mathrm{div}\, \boldsymbol{w}_m\|_2^2 + \|\boldsymbol{w}_m\|_2^2 \le \frac{1}{4}\|\nabla \boldsymbol{u}_m\|_2^2 + \|\boldsymbol{g}(t)\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}.$$

Add the above inequalities, we have

$$\frac{d}{dt}\left(\|\boldsymbol{u}_m\|_2^2 + \|\boldsymbol{w}_m\|_2^2\right) + \frac{1}{2}\left(\|\nabla \boldsymbol{u}_m\|_2^2 + \|\nabla \boldsymbol{w}_m\|_2^2\right) \le 2\left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}\right). \tag{5.24}$$

Integrating the inequality (5.24) with respect to time on the interval $[0, s]$, with $s > 0$, we get

$$\begin{aligned} \|\boldsymbol{u}_m(s)\|_2^2 + \|\boldsymbol{w}_m(s)\|_2^2 &\le \|\boldsymbol{u}_{0m}\|_2^2 + \|\boldsymbol{w}_{0m}\|_2^2 + 2\int_0^s \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}\right) dt \\ &\le \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + 2\int_0^T \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}\right) dt. \end{aligned}$$

Then, it follows from Young inequality and the assumptions of $\boldsymbol{u}_0$, $\boldsymbol{w}_0$, $\boldsymbol{f}$, and $\boldsymbol{g}$ that

$$\|\boldsymbol{u}_m(s)\|_2^2 + \|\boldsymbol{w}_m(s)\|_2^2 \le C + C\int_0^s \left(\|\boldsymbol{u}_m\|_{\boldsymbol{V}}^2 + \|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}^2\right) dt.$$

So, by using Gronwall's inequality, we have

$$\sup_{s \in [0,T]} \left(\|\boldsymbol{u}_m(s)\|_2^2 + \|\boldsymbol{w}_m(s)\|_2^2\right) \le C, \tag{5.25}$$

Now, integrating the estimate (5.24) from 0 to T, we obtain

$$\begin{aligned} \|\boldsymbol{u}_m(T)\|_2^2 \; &+ \; \|\boldsymbol{w}_m(T)\|_2^2 + \frac{1}{2}\int_0^T \left(\|\nabla \boldsymbol{u}_m(t)\|_2^2 + \|\nabla \boldsymbol{w}_m(t)\|_2^2\right) dt \\ &\le \; \|\boldsymbol{u}_{0m}\|_2^2 + \|\boldsymbol{w}_{0m}\|_2^2 + 2\int_0^s \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}\right) dt \\ &\le \; \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + 2\int_0^T \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}_m\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}_m\|_{\boldsymbol{H}^1}\right) dt. \end{aligned} \tag{5.26}$$

Let us prove the existence of $\boldsymbol{u}$ satisfying the assertion (5.21). For this, observe that it follows from the inequality (5.25) that

$$\{\boldsymbol{u}_m\} \text{ stays bounded in } L^\infty(0,T;\boldsymbol{H}), \tag{5.27}$$

and

$$\{\boldsymbol{w}_m\} \text{ stays bounded in } L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)). \tag{5.28}$$

From the bound (5.26), it follows that

$$\{\boldsymbol{u}_m\} \text{ is bounded in } L^2(0,T;\boldsymbol{V}), \tag{5.29}$$

and

$$\{\boldsymbol{w}_m\} \text{ stays bounded in } L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)). \tag{5.30}$$

Assertions (5.27) and (5.28) guarantee the existence of $\boldsymbol{u}$, $\boldsymbol{w}$, and a subsequence $m' \to \infty$, such that

$$\boldsymbol{u}_{m'} \overset{*}{\rightharpoonup} \boldsymbol{u}_* \text{ in } L^\infty(0,T;\boldsymbol{H}), \tag{5.31}$$

and

$$\boldsymbol{w}_{m'} \rightharpoonup^* \boldsymbol{w}_* \text{ on } L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)). \tag{5.32}$$

From assertions (5.29) and (5.30), we have

$$\{\boldsymbol{u}_{m'}\} \text{ stays bounded in } L^2(0,T;\boldsymbol{V}), \tag{5.33}$$

and

$$\{\boldsymbol{w}_{m'}\} \text{ stays bounded in } L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)). \tag{5.34}$$

So, there exists $\boldsymbol{u}_* \in L^2(0,T;\boldsymbol{V})$, $\boldsymbol{w}_* \in L^2(0,T;\boldsymbol{H}^1_0(\mathbb{R}^3))$, and a subsequence, for simplicity denoted again by $\boldsymbol{u}_{m'}$ and $\boldsymbol{w}_{m'}$, such that

$$\boldsymbol{u}_{m'} \rightharpoonup \boldsymbol{u} \text{ in } L^2(0,T;\boldsymbol{V}), \tag{5.35}$$

and

$$\boldsymbol{w}_{m'} \rightharpoonup \boldsymbol{w} \text{ on } \boldsymbol{L}^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)). \tag{5.36}$$

In particular, for each $\boldsymbol{\nu} \in L^2(0,T;\boldsymbol{V})$

$$\int_0^T (\boldsymbol{u}_{m'}(t),\boldsymbol{\nu}(t))\,dt \to \int_0^T (\boldsymbol{u}_*(t),\boldsymbol{\nu}(t))\,dt, \tag{5.37}$$

and

$$\int_0^T (\boldsymbol{w}_{m'}(t),\boldsymbol{\nu}(t))\,dt \to \int_0^T (\boldsymbol{w}_*(t),\boldsymbol{\nu}(t))\,dt, \tag{5.38}$$

for each $\boldsymbol{\nu} \in L^2(0,T;\boldsymbol{H}^1_0(\mathbb{R}^3))$. Then, from the convergences (5.31) and (5.32), we get

$$\int_0^T (\boldsymbol{u}(t) - \boldsymbol{u}_*(t),\boldsymbol{\nu}(t))\,dt = 0, \quad \forall\,\boldsymbol{\nu} \in L^2(0,T;\boldsymbol{V}), \tag{5.39}$$

and

$$\int_0^T (\boldsymbol{w}(t) - \boldsymbol{w}_*(t), \boldsymbol{\nu}(t))\, dt = 0, \quad \forall\, \boldsymbol{\nu} \in L^2(0,T;\boldsymbol{H}_0^1(\mathbb{R}^3)). \tag{5.40}$$

Consequently,

$$\boldsymbol{u} = \boldsymbol{u}_* \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H}),$$

and

$$\boldsymbol{w} = \boldsymbol{w}_* \in L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)),$$

and the existence of $(\boldsymbol{u},\boldsymbol{w})$ is proved. Finally, let us show the existence of the pressure p. Consider $\boldsymbol{u}$ and $\boldsymbol{w}$ satisfying $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$, $\boldsymbol{w} \in L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)) \cap L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3))$, and satisfying for each $\boldsymbol{v} \in \boldsymbol{V}$,

$$\frac{d}{dt}(\boldsymbol{u},\boldsymbol{v}) - (\Delta \boldsymbol{u},\boldsymbol{v}) - (\operatorname{curl} \boldsymbol{w},\boldsymbol{v}) - \langle \tilde{\boldsymbol{f}},\boldsymbol{v}\rangle = 0, \tag{5.41}$$

where $\tilde{\boldsymbol{f}} = \boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. Set

$$\boldsymbol{U}(t) = \int_0^t \boldsymbol{u}(s)\, ds, \quad \boldsymbol{W}(t) = \int_0^t \boldsymbol{w}(s)\, ds, \quad \boldsymbol{F}(t) = \int_0^t \tilde{\boldsymbol{f}}(s)\, ds. \tag{5.42}$$

So, $\boldsymbol{U} \in C([0,T];\boldsymbol{V})$, $\boldsymbol{W} \in C([0,T];\boldsymbol{H}^1(\mathbb{R}^3))$, and $\boldsymbol{F} \in \boldsymbol{C}([0,T];\boldsymbol{V}')$. Then, integrating the identity (5.41) on t, we have for $\boldsymbol{v} \in \boldsymbol{V}$,

$$\langle \boldsymbol{u}(t) - \boldsymbol{u}_0 - \Delta\boldsymbol{U}(t) - \operatorname{curl}\boldsymbol{W}(t) - \boldsymbol{F}(t), \boldsymbol{v}\rangle = 0, \quad \forall\, t \in [0,T]. \tag{5.43}$$

It follows from Proposition 1.1 and Proposition 1.2, Chapter I of [5], that there exists a function $P(t) \in L^2(\mathbb{R}^3)$ such that

$$\boldsymbol{u}(t) - \boldsymbol{u}_0 - \Delta\boldsymbol{U}(t) - \operatorname{curl}\boldsymbol{W}(t) - \boldsymbol{F}(t) = -\nabla P(t). \tag{5.44}$$

Moreover, $P \in C([0,T];L^2(\mathbb{R}^3))$. Then, differentiating equation (5.44) with respect to t, in the sense of distribution in $\mathcal{Q}_T$ and setting

$$p = \frac{\partial P}{\partial t},$$

we obtain

$$\boldsymbol{u}_t - \Delta\boldsymbol{u} - \operatorname{curl}\boldsymbol{w} - \boldsymbol{f} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p \quad \text{in } \mathcal{Q}_T.$$

This finishes the proof of existence for the pressure p. □

Lemma 5.9 (Lemma A.4, [28])**.** *Suppose* $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$, $\operatorname{div}\boldsymbol{f} = 0$, *and* $\boldsymbol{u}_0 \in \boldsymbol{H}$*. Under the same hypotheses of the Lemma 5.6, the pressure satisfies*

$$\Delta p = -\operatorname{div}\left[(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\right] = -\sum_{j,k=1}^{3} \frac{\partial^2}{\partial x_j \partial x_k}(v_j u_k),$$

and $p \in L^{5/3}(\mathcal{Q}_T)$*, for all* $T > 0$.

Proof. Taking the divergence of

$$\boldsymbol{u}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p - \operatorname{curl}\boldsymbol{w} = \boldsymbol{f},$$

the identity

$$\Delta p = -\operatorname{div}\left[(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\right] = -\sum_{i,j=1}^{3}\frac{\partial^2}{\partial x_i\partial x_j}(v_iu_j)$$

follows. Thus,

$$p(\boldsymbol{x},t) = \frac{1}{4\pi}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\left(\frac{\partial^2}{\partial y_j\partial y_k}(v_ju_k)\right)(\boldsymbol{y})\,d\boldsymbol{y}.$$

So, from Lemma 2.26 we have

$$\int_{\mathbb{R}^3}|p|^q\;d\boldsymbol{x} \le C(q)\int_{\mathbb{R}^3}(|\boldsymbol{v}||\boldsymbol{u}|)^q\,d\boldsymbol{x},$$

for $1 < q < \infty$. In particular, for $\boldsymbol{u}$ as in Lemma 5.6, we get

$$\int_0^T\int_{\mathbb{R}^3}|p|^{5/3}\,d\boldsymbol{x}\,dt \le C\int_0^T\int_{\mathbb{R}^3}|\boldsymbol{u}|^{10/3}\,d\boldsymbol{x}\,dt.$$

Since

$$\int_{\mathbb{R}^3}|\boldsymbol{u}|^{10/3}\;d\boldsymbol{x} \le C\left(\int_{\mathbb{R}^3}|\nabla\boldsymbol{u}|^2\,d\boldsymbol{x}\right)\left(\int_{\mathbb{R}^3}|\boldsymbol{u}|^2\,d\boldsymbol{x}\right)^{2/3},$$

then

$$\int_0^T\int_{\mathbb{R}^3}|p|^{5/3}\,d\boldsymbol{x}\,dt < \infty,$$

i.e., $p \in L^{5/3}(\mathcal{Q}_T)$ for all $T > 0$. □

Lemma 5.10. *Suppose* $\boldsymbol{f}, \boldsymbol{g} \in L^2(0,T;\boldsymbol{H}^{-1}(\mathbb{R}^3))$, $\operatorname{div}\boldsymbol{f} = 0$, $\boldsymbol{u}_0 \in \boldsymbol{H}$, *and* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3)$. *Let* $(\boldsymbol{u}^N, p^N, \boldsymbol{w}^N)$, $N \in \mathbb{N}$, *be the unique solution of the problem* (5.6)-(5.7). *Then*

$$\boldsymbol{u}^N \to \boldsymbol{u}\begin{cases}\text{strongly in } \ \boldsymbol{L}^2(\mathcal{Q}_T),\\ \text{weakly in } \ L^2(0,T;\boldsymbol{V}),\\ \text{weakly-star in } \ L^\infty(0,T;\boldsymbol{H}),\end{cases}$$

$$p^N \to p \ \text{ weakly in } \ L^{5/3}(\mathcal{Q}_T),$$

$$\boldsymbol{w}^N \to \boldsymbol{w}\begin{cases}\text{strongly in } \ \boldsymbol{L}^2(\mathcal{Q}_T),\\ \text{weakly in } \ L^2(0,T;\boldsymbol{H}_0^1(\mathbb{R}^3)),\\ \text{weakly-star in } \ L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)),\end{cases}$$

$$\Psi_\delta(\boldsymbol{u}^N) \to \boldsymbol{u} \ \text{ strongly in } \ \boldsymbol{L}^2(\mathcal{Q}_T),$$

$$\lim_{N\to\infty}\boldsymbol{u}^N(0) = \boldsymbol{u}_0, \quad \lim_{N\to\infty}\boldsymbol{w}^N(0) = \boldsymbol{w}_0,$$

and $(\boldsymbol{u},\boldsymbol{w})$ *is a weak solution for the micropolar equations* (5.1) *with forces* $\boldsymbol{f}$ *and* $\boldsymbol{g}$, *pressure* p *and initial data* $\boldsymbol{u}_0$ *and* $\boldsymbol{w}_0$.

Proof. The existence of the solution $(\boldsymbol{u}^N, p^N, \boldsymbol{w}^N)$ of system (5.6)-(5.7) follows applying Theorem 5.8 inductively on each time interval $(k\delta, (k+1)\delta)$, $0 \le k \le N-1$. It remains to show the convergences above. Through analogous computations as in the proof of Theorem 4.1, we obtain the energy estimate

$$\frac{d}{dt}\left(\|\boldsymbol{u}^N\|_2^2 + \|\boldsymbol{w}^N\|_2^2\right) + \frac{1}{2}\left(\|\nabla \boldsymbol{u}^N\|_2^2 + \|\nabla \boldsymbol{w}^N\|_2^2\right) \le 2\left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}^N\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}^N\|_{\boldsymbol{H}^1}\right), \tag{5.45}$$

for $t \in (0,T)$. Integrating the estimate (5.45) in s, with $s > 0$, one gets

$$\begin{aligned}\|\boldsymbol{u}^N(s)\|_2^2 + \|\boldsymbol{w}^N(s)\|_2^2 &\le \|\boldsymbol{u}_{0m}\|_2^2 + \|\boldsymbol{w}_{0m}\|_2^2 + 2\int_0^s \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}^N\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}^N\|_{\boldsymbol{H}^1}\right) dt \\ &\le \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + 2\int_0^T \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}^N\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}^N\|_{\boldsymbol{H}^1}\right) dt.\end{aligned}$$

On the other hand, integrating the inequality (5.45) from 0 to T, one obtains

$$\begin{aligned}\|\boldsymbol{u}^N(T)\|_2^2 + \|\boldsymbol{w}^N(T)\|_2^2 &+ \frac{1}{2}\int_0^T \left(\|\nabla \boldsymbol{u}^N(t)\|_2^2 + \|\nabla \boldsymbol{w}^N(t)\|_2^2\right) dt \\ &\le \|\boldsymbol{u}_0\|_2^2 + \|\boldsymbol{w}_0\|_2^2 + 2\int_0^T \left(\|\boldsymbol{f}\|_{\boldsymbol{V}'}\|\boldsymbol{u}^N\|_{\boldsymbol{V}} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1}}\|\boldsymbol{w}^N\|_{\boldsymbol{H}^1}\right) dt.\end{aligned}$$

Therefore, we conclude that

$$\{\boldsymbol{u}^N\} \text{ stays bounded in } L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}) \tag{5.46}$$

and

$$\{\boldsymbol{w}^N\} \text{ is bounded in } L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)) \cap L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)). \tag{5.47}$$

Now, denote by $\boldsymbol{V}_2$ the closure of $\boldsymbol{\mathcal{V}}$ in $\boldsymbol{H}^2(\mathbb{R}^3)$ and by $\boldsymbol{V}_2'$ its dual space. Multiplying the equation $(5.6)_1$ by $\boldsymbol{\nu} \in \boldsymbol{\mathcal{V}}$, we obtain

$$\langle \boldsymbol{u}_t^N, \boldsymbol{\nu}\rangle = \langle \Delta \boldsymbol{u}^N, \boldsymbol{\nu}\rangle - \langle (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N, \boldsymbol{\nu}\rangle + \langle \operatorname{curl} \boldsymbol{w}^N, \boldsymbol{\nu}\rangle + \langle \boldsymbol{f}, \boldsymbol{\nu}\rangle. \tag{5.48}$$

Since $\boldsymbol{H}_0^1(\mathbb{R}^3) \subset \boldsymbol{L}^6(\mathbb{R}^3)$, we have, by Lemma 2.24, that

$$\begin{aligned}\left|\langle (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N, \boldsymbol{\nu}\rangle\right| &= \left|\int_0^T\int_{\mathbb{R}^3} (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N\,\boldsymbol{\nu}\,d\boldsymbol{x}\,dt\right| \\ &= \left|\int_0^T\int_{\mathbb{R}^3} (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{\nu}\,\boldsymbol{u}^N\,d\boldsymbol{x}\,dt\right| \\ &\le \int_0^T \|\Psi_\delta(\boldsymbol{u}^N)\|_3\,\|\boldsymbol{u}^N\|_2\,\|\nabla\boldsymbol{\nu}\|_6\,dt \\ &\le C\,\|\Psi_\delta(\boldsymbol{u}^N)\|_{L^4(0,T;\boldsymbol{L}^3)}\|\boldsymbol{u}^N\|_{L^4(0,T;\boldsymbol{L}^2)}\,\|\boldsymbol{\nu}\|_{L^2(0,T;\boldsymbol{V}_2)}.\end{aligned} \tag{5.49}$$

So, it follows from the properties of the retarded mollifier Ψ_δ, from (5.46) and from Lemma 2.24, that

$$\|(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N\|_{L^2(0,T;\boldsymbol{V}_2')} \le C,$$

where C is a positive constant independent of N. Similarly, we have

$$\|(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{w}^N\|_{L^2(0,T;\boldsymbol{H}^{-2}(\mathbb{R}^3))}\leq C,$$

where $C>0$ is independent of N. Thus,

$$\{\boldsymbol{u}_t^N\} \text{ stays bounded in } L^2(0,T;\boldsymbol{V}_2'). \tag{5.50}$$

and

$$\{\boldsymbol{w}_t^N\} \text{ is bounded in } L^2(0,T;\boldsymbol{H}^{-2}(\mathbb{R}^3)). \tag{5.51}$$

Therefore, by Lemma 2.18, we have

$$\{\boldsymbol{u}^N\} \text{ and } \{\boldsymbol{w}^N\} \text{ are in a compact subset of } \boldsymbol{L}^2(\mathcal{Q}_T). \tag{5.52}$$

On the other hand, using Lemmas 5.4 and 5.9, we get

$$\{p^N\} \text{ is bounded in } L^{5/3}(\mathcal{Q}_T). \tag{5.53}$$

So, by (5.46)-(5.47) and (5.50)-(5.53), we conclude that there exists a subsequence, again denoted by $(\boldsymbol{u}^N, p^N, \boldsymbol{w}^N)$, converging to the limit $(\boldsymbol{u}, p, \boldsymbol{w})$ in the following sense:

$$\boldsymbol{u}^N\to\boldsymbol{u}\begin{cases}\text{strongly in } \ \boldsymbol{L}^2(\mathcal{Q}_T),\\ \text{weakly in } \ L^2(0,T;\boldsymbol{V}),\\ \text{weakly-star in } \ L^\infty(0,T;\boldsymbol{H}),\end{cases} \tag{5.54}$$

$$p^N\to p \ \text{ weakly in } \ L^{5/3}\left(\mathcal{Q}_T\right), \tag{5.55}$$

$$\boldsymbol{w}^N\to\boldsymbol{w}\begin{cases}\text{strongly in } \ \boldsymbol{L}^2(\mathcal{Q}_T),\\ \text{weakly in } \ L^2(0,T;\boldsymbol{H}^1(\mathbb{R}^3)),\\ \text{weakly-star in } \ L^\infty(0,T;\boldsymbol{L}^2(\mathbb{R}^3)).\end{cases} \tag{5.56}$$

From the definition of Ψ_δ and by (5.54)-(5.56), one obtains

$$\Psi_\delta(\boldsymbol{u}^N)\to\boldsymbol{u} \ \text{ strongly in } \ \boldsymbol{L}^2(\mathcal{Q}_T).$$

Finally, by assertions (5.50) and (5.51), we conclude that the functions $\{\boldsymbol{u}^N\}$ and $\{\boldsymbol{w}^N\}$) are uniformly continuous from $(0,T)$ to $\boldsymbol{V}_2'$ and $\boldsymbol{H}^{-2}(\mathbb{R}^3)$), respectively. Therefore,

$$\left(\boldsymbol{u}^N(0),\boldsymbol{w}^N(0)\right)\xrightarrow[N\to\infty]{}(\boldsymbol{u}(0),\boldsymbol{w}(0))=(\boldsymbol{u}_0,\boldsymbol{w}_0)\,.$$

This completes the proof. □

5.2 Main Results

The first result considers the case of null external forces.

Theorem 5.11. *Let* $\boldsymbol{u}_0 \in \boldsymbol{H} \cap \boldsymbol{L}^1(\mathbb{R}^3)$ *and* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$. *There exists a weak solution* $(\boldsymbol{u}, p, \boldsymbol{w})$ *of problem* (5.1) *with* $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$ *such that*

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2},$$

where $C \in \mathbb{R}^+$ *is a constant depending only on the* $\boldsymbol{L}^1$ *and* $\boldsymbol{L}^2$ *norms of* $\boldsymbol{u}_0$ *and* $\boldsymbol{w}_0$.

In order to prove Theorem 5.11, we first obtain a decay estimate for the approximate solutions $(\boldsymbol{u}^N, p^N, \boldsymbol{w}^N)$ through the same arguments as in the proof of Theorem 4.1.

Theorem 5.12. *Suppose* $\boldsymbol{u}_0 \in \boldsymbol{H} \cap \boldsymbol{L}^1(\mathbb{R}^3)$ *and* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$. *Let* $(\boldsymbol{u}^N, p^N, \boldsymbol{w}^N)$, $N \in \mathbb{N}$, *be a solution of problem* (5.6)-(5.7). *Then*

$$\left\|\boldsymbol{u}^N(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}^N(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2},$$

where $C > 0$ *is a constant which depends only on the* $\boldsymbol{L}^2$ *and* $\boldsymbol{L}^1$ *norms of* $\boldsymbol{u}_0$ *and* $\boldsymbol{w}_0$.

Proof. For simplicity, we write $\boldsymbol{u}^N = \boldsymbol{u}$, $p^N = p$ and $\boldsymbol{w}^N = \boldsymbol{w}$. Due to Lemma 5.5, Lemma 5.6, and Lemma 5.7, we have that H and Q defined by

$$H(t) := \|\boldsymbol{u}(\cdot,t)\|_2^2 + \|\boldsymbol{w}(\cdot,t)\|_2^2, \qquad Q(t) := (\boldsymbol{u}, \boldsymbol{u}_t) + (\boldsymbol{w}, \boldsymbol{w}_t),$$

satisfy H, $Q \in L^2_{loc}(\mathbb{R})$, and

$$\frac{1}{2}\frac{d}{dt}H(t) = Q(t) \tag{5.57}$$

as distributions. Moreover, H is absolutely continuous and identity (5.57) can be interpreted in the classical sense. Consequently, to prove Theorem 5.12, it is enough to show that

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1} \tag{5.58}$$

holds for all $\boldsymbol{\xi} \in S(t)$ (see (4.26)) and then just follow the same steps as in the proof of Theorem 4.1 to finish the proof. We establish estimate (5.58) in the following

Proposition 5.13. *Let* $\boldsymbol{u}_0 \in \boldsymbol{H} \cap \boldsymbol{L}^1(\mathbb{R}^3)$ *and* $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$. *Let* $\boldsymbol{v}(\cdot,t) \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{C}^\infty(\mathbb{R}^3)$ *with* $\operatorname{div}\boldsymbol{v} = 0$. *If* $(\boldsymbol{u}, p, \boldsymbol{w})$ *is the unique solution of the problem*

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p - \operatorname{curl}\boldsymbol{w} = \boldsymbol{0}, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \boldsymbol{w}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} - \Delta\boldsymbol{w} - \nabla(\operatorname{div}\boldsymbol{w}) + 2\boldsymbol{w} - \operatorname{curl}\boldsymbol{u} = \boldsymbol{0}, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \boldsymbol{w}(\boldsymbol{x},0) = \boldsymbol{w}_0(\boldsymbol{x}), \end{cases} \tag{5.59}$$

then for $\boldsymbol{\xi} \in \mathcal{K}$, $\mathcal{K} \subset \mathbb{R}^3$ *a compact set, it holds*

$$|\widehat{\boldsymbol{u}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{w}}(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}|^{-1}, \text{ a.e. in } t, \tag{5.60}$$

where the constant $C > 0$ *depends on* $\mathcal{K}$, *on the* $\boldsymbol{L}^1$ *and* $\boldsymbol{L}^2$ *norms of* $\boldsymbol{u}_0$ *and* $\boldsymbol{w}_0$, *and on the* $\boldsymbol{L}^2$ *norm of* $\boldsymbol{v}$.

Proof. Following the ideas in the proof of Proposition 4.3, we write the system (5.59) as

$$\widehat{\boldsymbol{y}}_t + A(\boldsymbol{\xi})\widehat{\boldsymbol{y}} = \boldsymbol{G}(\boldsymbol{\xi}, t)$$

in the sense of distributions, where $\boldsymbol{y} := (\boldsymbol{u}, \boldsymbol{w})$, $\boldsymbol{G} := (\boldsymbol{G}_1, \boldsymbol{G}_2)$,

$$\boldsymbol{G}_1(\boldsymbol{\xi}, t) := -\mathscr{F}\{\nabla p + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\}(\boldsymbol{\xi}), \quad \boldsymbol{G}_2(\boldsymbol{\xi}, t) := -\mathscr{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\}(\boldsymbol{\xi}),$$

and $A(\boldsymbol{\xi})$ is the Hermitian matrix defined in (4.15). Then, to prove the proposition we only need to show that

$$|\boldsymbol{G}_1(\boldsymbol{\xi}, t)| + |\boldsymbol{G}_2(\boldsymbol{\xi}, t)| \leq C|\boldsymbol{\xi}| \tag{5.61}$$

and repeat the arguments used in the proof of Theorem 4.1. To obtain suitable bounds for $\boldsymbol{G}_1, \boldsymbol{G}_2$, we first show that $\boldsymbol{G}_1, \boldsymbol{G}_2$ are well defined. First of all, by Lemma 5.9 we have

$$\int_0^t \int_{\mathbb{R}^3} |p(\boldsymbol{x}, s)|^{5/3} d\boldsymbol{x}\, ds \leq C.$$

Thus

$$\int_{\mathbb{R}^3} |p(\boldsymbol{x}, t)|^{5/3} d\boldsymbol{x} \leq C \quad \text{a.e. in } t.$$

Hence, by Lemma 3.3, we get

$$\|\widehat{p}(\cdot, t)\|_{5/2} \leq C\|p(\cdot, t)\|_{5/3}.$$

It follows that $\widehat{p}(\cdot, t) \in L^{5/2}(\mathbb{R}^3)$ a.e. in t. Moreover, by Lemma 5.9,

$$\Delta p = -\mathrm{div}\,[(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}] = -\sum_{j,k=1}^{3} \frac{\partial^2}{\partial x_j \partial x_k}(v_j u_k). \tag{5.62}$$

So, taking the Fourier transform of the identity (5.62), we obtain

$$-|\boldsymbol{\xi}|^2\, \widehat{p} = -i\,\boldsymbol{\xi}\, \mathscr{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\} = \sum_{j,k=1}^{3} \xi_j \xi_k \mathscr{F}\{u_j v_k\}.$$

Thus,

$$|\boldsymbol{\xi}|^2\, |\widehat{p}| = \left|\sum_{j,k=1}^{3} \xi_j \xi_k \mathscr{F}\{u_j v_k\}\right| \leq \sum_{j,k=1}^{3} |\xi_j \xi_k| \|u_j v_k\|_1.$$

Since u_j and v_k are in $L^2(\mathbb{R}^3)$, it follows that

$$|\widehat{p}| \leq C.$$

On the other hand, as

$$\mathscr{F}\{\nabla p\}(\boldsymbol{\xi}) = i\,\boldsymbol{\xi}\,\widehat{p},$$

we conclude that

$$|\mathscr{F}\{\nabla p\}(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|, \tag{5.63}$$

where C is a constant depending only on the $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{v}$.

To be able to obtain a bound as in inequality (5.61) for the terms $\mathscr{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\}$ and $\mathscr{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\}$, we must show that it makes sense in $L^2(\mathcal{Q}_T)$, i.e., $\mathscr{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\}$ and $\mathscr{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\}\in L^2(\mathcal{Q}_T)$. But, this fact follows from Lemmas 5.6 and 5.7, since

$$\int\int_{\mathbb{R}^3}|(\boldsymbol{v}\cdot\nabla)u_j|^2\,d\boldsymbol{x}\,dt\leq\sum_{k=1}^{3}\int\int_{\mathbb{R}^3}\left|v_k\frac{\partial u_j}{\partial x_k}\right|^2 d\boldsymbol{x}\,dt\leq\int\int_{\mathbb{R}^3}|\nabla\boldsymbol{u}|^2\,d\boldsymbol{x}\,dt\leq C,$$

$$\int\int_{\mathbb{R}^3}|(\boldsymbol{v}\cdot\nabla)w_j|^2\,d\boldsymbol{x}\,dt\leq\sum_{k=1}^{3}\int\int_{\mathbb{R}^3}\left|v_k\frac{\partial w_j}{\partial x_k}\right|^2 d\boldsymbol{x}\,dt\leq\int\int_{\mathbb{R}^3}|\nabla\boldsymbol{w}|^2\,d\boldsymbol{x}\,dt\leq C.$$

Since $\operatorname{div}\boldsymbol{v}=0$ and v_k, u_j and w_j are in $L^2(\mathbb{R}^3)$, we have

$$\left|[\mathscr{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\}]_j\right|=\left|i\sum_{k=1}^{3}\xi_k\mathscr{F}\{u_jv_k\}\right|\leq|\boldsymbol{\xi}|\sum_{k=1}^{3}|\mathscr{F}\{u_jv_k\}|\leq C|\boldsymbol{\xi}|, \tag{5.64}$$

$$\left|[\mathscr{F}\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\}]_j\right|=\left|i\sum_{k=1}^{3}\xi_k\mathscr{F}\{w_jv_k\}\right|\leq|\boldsymbol{\xi}|\sum_{k=1}^{3}|\mathscr{F}\{w_jv_k\}|\leq C|\boldsymbol{\xi}|, \tag{5.65}$$

where $C>0$ is a constant depending only on the $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$, $\boldsymbol{w}_0$ and $\boldsymbol{v}$. Combining inequalities (5.63), (5.64) and (5.65), one obtains (5.61). Now, we use the inequality (5.61) to show that (5.60) holds. Set

$$\boldsymbol{\varphi}(\boldsymbol{\xi},t):=e^{tA(\boldsymbol{\xi})}\widehat{\boldsymbol{y}}.$$

Then

$$\boldsymbol{\varphi}_t(\boldsymbol{\xi},t)=e^{tA(\boldsymbol{\xi})}\boldsymbol{G}(\boldsymbol{\xi},t), \tag{5.66}$$

in the sense of distributions. By estimate (5.61), it follows that $e^{-tA(\boldsymbol{\xi})}\boldsymbol{G}\in\boldsymbol{L}^\infty_{loc}(\mathbb{R})$. Thus, identity (5.66) can be defined in the classical sense a.e. in t. Therefore,

$$\boldsymbol{\varphi}(\boldsymbol{\xi},t)=\widehat{\boldsymbol{y}}(\boldsymbol{\xi},0)+\int_0^t e^{sA(\boldsymbol{\xi})}\boldsymbol{G}(\boldsymbol{\xi},s)\,ds$$

and estimate (5.60) is proved. This completes the proof of Proposition 5.13. □

Now, we just consider $\boldsymbol{v}=\Psi_\delta(\boldsymbol{u})$ in Proposition 5.13, repeat the arguments in the proof of Theorem 4.1 and apply Lemma 5.4 to show that the constant C depends only on the $\boldsymbol{L}^2$ and $\boldsymbol{L}^1$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$. This finishes the proof. □

Proof of Theorem 5.11. Let $(\boldsymbol{u},\boldsymbol{w})$ be the weak solution obtained as the strong limit in $L^2(\mathcal{Q}_T)$ of the $(\boldsymbol{u}^N,\boldsymbol{w}^N)$ through Lemma 5.10. Then

$$\lim_{N\to\infty}\int_{\mathbb{R}^3}|\boldsymbol{u}^N(\boldsymbol{x},t)-\boldsymbol{u}(\boldsymbol{x},t)|^2\,d\boldsymbol{x}=0=\lim_{N\to\infty}\int_{\mathbb{R}^3}|\boldsymbol{w}^N(\boldsymbol{x},t)-\boldsymbol{w}(\boldsymbol{x},t)|^2\,d\boldsymbol{x}.$$

By Theorem 5.12, we have

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|_2 &\leq \|\boldsymbol{u}^N(\cdot,t)\|_2+\|(\boldsymbol{u}-\boldsymbol{u}^N)(\cdot,t)\|_2\leq C(t+1)^{-1/4}+\|(\boldsymbol{u}-\boldsymbol{u}^N)(\cdot,t)\|_2,\\ \|\boldsymbol{w}(\cdot,t)\|_2 &\leq \|\boldsymbol{w}^N(\cdot,t)\|_2+\|(\boldsymbol{w}-\boldsymbol{w}^N)(\cdot,t)\|_2\leq C(t+1)^{-1/4}+\|(\boldsymbol{w}-\boldsymbol{w}^N)(\cdot,t)\|_2.\end{aligned}$$

Taking $N \to \infty$, one obtains

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2},$$

where C is a positive constant depending only on the $\boldsymbol{L}^1$ and $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$. □

For the case where the forces $\boldsymbol{f}$ and $\boldsymbol{g}$ do not vanish, we have

Corollary 5.14. *Suppose* $\boldsymbol{u}_0 \in \boldsymbol{H} \cap \boldsymbol{L}^1(\mathbb{R}^3)$, $\boldsymbol{w}_0 \in \boldsymbol{L}^2(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$, *and the external forces satisfy*

$$\boldsymbol{f} \in L^2(0,\infty;\boldsymbol{V}'),$$

$$\boldsymbol{g} \in L^2(0,\infty;\boldsymbol{H}^{-1}(\mathbb{R}^3)),$$

$$\operatorname{div}\boldsymbol{f} = 0,$$

$$\|\boldsymbol{f}(\cdot,t)\|_2 + \|\boldsymbol{g}(\cdot,t)\|_2 \leq K_1(t+1)^{-3/2}, \quad \forall\, t \geq 0,$$

$$|\widehat{\boldsymbol{f}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{g}}(\boldsymbol{\xi},t)| \leq K_2|\boldsymbol{\xi}|, \quad \forall\, t \geq 0 \;\text{ with }\; \boldsymbol{\xi} \in \mathbb{R}^3,$$

for some positive constants K_1 *and* K_2. *Then, the unique solution* $(\boldsymbol{u}^N, p^N, \boldsymbol{w}^N)$, $N \in \mathbb{N}$, *of problem* (5.6)-(5.7) *satisfies*

$$\left\|\boldsymbol{u}^N(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}^N(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2},$$

where the positive constant C *depends only on the* $\boldsymbol{L}^1$ *and* $\boldsymbol{L}^2$ *norms of the initial data and on the positive constants* K_1, K_2.

Proof. Set

$$\boldsymbol{G}_1(\boldsymbol{\xi},t) := \mathscr{F}\{\boldsymbol{f} - \nabla p^N - (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N\}(\boldsymbol{\xi}), \quad \boldsymbol{G}_2(\boldsymbol{\xi},t) := \mathscr{F}\{\boldsymbol{g} - (\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{w}^N\}(\boldsymbol{\xi}).$$

We will show that

$$|\boldsymbol{G}_1(\boldsymbol{\xi},t)| + |\boldsymbol{G}_2(\boldsymbol{\xi},t)| \leq C|\boldsymbol{\xi}| \tag{5.67}$$

and then we repeat the proof of Theorem 5.12. The same arguments as in the proof of Theorem 5.12 yield

$$|\mathscr{F}\{\nabla p^N\}| + |\mathscr{F}\{(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{u}^N\}| + |\mathscr{F}\{(\Psi_\delta(\boldsymbol{u}^N)\cdot\nabla)\boldsymbol{w}^N\}| \leq C|\boldsymbol{\xi}|,$$

and, since

$$|\widehat{\boldsymbol{f}}(\boldsymbol{\xi},t)| + |\widehat{\boldsymbol{g}}(\boldsymbol{\xi},t)| \leq K_2|\boldsymbol{\xi}|, \forall\, t \geq 0 \;\text{ and }\; \boldsymbol{\xi} \in \mathbb{R}^3,$$

bound (5.67) follows. This ends the proof of Corollary 5.14. □

The next corollary is the final result of this work

Corollary 5.15. *Let $\boldsymbol{u}_0$, $\boldsymbol{w}_0$, $\boldsymbol{f}$ and $\boldsymbol{g}$ be as in Corollary 5.14. Let $(\boldsymbol{u}, \boldsymbol{w})$ be a weak solution of the micropolar equations* (5.1) *obtained as the $\boldsymbol{L}^2$ limit of the solutions $(\boldsymbol{u}^N, \boldsymbol{w}^N)$ of system (5.6). Then*

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2},$$

where the constant $C \in \mathbb{R}^+$ depends only on $\boldsymbol{f}$, $\boldsymbol{g}$ and the $\boldsymbol{L}^1$ and on $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$.

Proof. By Lemma 5.10, one has

$$\lim_{N\to\infty}\int_{\mathbb{R}^3} |\boldsymbol{u}^N(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t)|^2\, d\boldsymbol{x} = 0 = \lim_{N\to\infty}\int_{\mathbb{R}^3} |\boldsymbol{w}^N(\boldsymbol{x},t) - \boldsymbol{w}(\boldsymbol{x},t)|^2\, d\boldsymbol{x}$$

and, by Corollary 5.14, one gets

$$\begin{aligned}
\|\boldsymbol{u}(\cdot,t)\|_2 &\leq \|\boldsymbol{u}^N(\cdot,t)\|_2 + \|(\boldsymbol{u}-\boldsymbol{u}^N)(\cdot,t)\|_2 \leq C(t+1)^{-1/4} + \|(\boldsymbol{u}-\boldsymbol{u}^N)(\cdot,t)\|_2,\\
\|\boldsymbol{w}(\cdot,t)\|_2 &\leq \|\boldsymbol{w}^N(\cdot,t)\|_2 + \|(\boldsymbol{w}-\boldsymbol{w}^N)(\cdot,t)\|_2 \leq C(t+1)^{-1/4} + \|(\boldsymbol{w}-\boldsymbol{w}^N)(\cdot,t)\|_2.
\end{aligned}$$

Finally, taking $N \to \infty$, one obtains

$$\left\|\boldsymbol{u}(\cdot,t)\right\|_2^2 + \left\|\boldsymbol{w}(\cdot,t)\right\|_2^2 \leq C(t+1)^{-1/2},$$

where the positive constant C depends only on $\boldsymbol{f}$, $\boldsymbol{g}$ and on the $\boldsymbol{L}^1$ and $\boldsymbol{L}^2$ norms of $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$. The proof of Corollary 5.15 is finished. □

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