UNIVERSIDADE FEDERAL DE PERNAMBUCO

CENTRO DE TECNOLOGIA E GEOCIÊNCIAS

DEPARTAMENTO DE ENGENHARIA DE PRODUÇÃO

PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA DE PRODUÇÃO

FRANCE EVELLYN GOMES DE OLIVEIRA

HIGHER-ORDER SEQUENTIAL STABILITIES IN THE GRAPH MODEL FOR CONFLICT RESOLUTION

RECIFE

2018

FRANCE EVELLYN GOMES DE OLIVEIRA

HIGHER-ORDER SEQUENTIAL STABILITIES IN THE GRAPH MODEL FOR CONFLICT RESOLUTION

Dissertação de Mestrado apresentada à UFPE para a obtenção de grau de Mestre como parte das exigências do Programa de Pós-Graduação em Engenharia de Produção.

Área de Concentração: Pesquisa Operacional.

Orientador: Prof. PhD. Leandro Chaves Rêgo.

RECIFE

2018

Catalogação na fonte
Bibliotecária: Rosineide Mesquita Gonçalves Luz / CRB4-1361 (BCTG)

O48h Oliveira, France Evellyn Gomes de.
Higher-order sequential stabilities in the graph model for conflict resolution / France Evellyn Gomes de Oliveira. – Recife: 2018.
64f., il., figs., gráfs., tabs.

Orientador: Prof. Leandro Chaves Rêgo, PhD.
Dissertação (Mestrado) – Universidade Federal de Pernambuco. CTG. Programa de Pós-Graduação em Engenharia de Produção, 2018.
Inclui Referências.

1. Engenharia de Produção. 2. Conflito. 3. Modelo de grafo. 4. Noções de estabilidade. 5. Ameaças credíveis. I. Rêgo, Leandro Chaves (Orientador). II. Título.

658.5 CDD (22.ed) UFPE/BCTG-2018-193

UNIVERSIDADE FEDERAL DE PERNAMBUCO
PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA DE PRODUÇÃO

PARECER DA COMISSÃO EXAMINADORA
DE DEFESA DE DISSERTAÇÃO DE MESTRADO DE

FRANCE EVELLYN GOMES DE OLIVEIRA

"HIGHER-ORDER SEQUENTIAL STABILITIES IN THE GRAPH MODEL FOR CONFLICT RESOLUTION"

ÁREA DE CONCENTRAÇÃO: PESQUISA OPERACIONAL

A comissão examinadora composta pelos professores abaixo, sob a presidência do primeiro, considera a candidata FRANCE EVELLYN GOMES DE OLIVEIRA**, APROVADA.**

Recife, 26 de Fevereiro de 2018.

Prof. LEANDRO CHAVES RÊGO, PhD (UFC)

Prof.ª ANA PAULA CABRAL SEIXAS COSTA, Doutora (UFPE)

Prof. FILIPE COSTA DE SOUZA, Doutor (UPFE)

I dedicate this work first to God, my family and friends for all Love and dedication.

ACKNOWLEDGEMENTS

I would like to acknowledge the God first for having trained me in these 2 years in the Masters in Production Engineering, for keeping me up in all the difficult moments and for always being faithful to me.

My family, especially my parents and my brothers, Marcos Luiz de Oliveira, Edna Gomes de Oliveira, Marcos Victor Gomes de Oliveira and Luiz Vinícius C. de Oliveira for all the affection, dedication, and support provided since the beginning of my educational training, who always gave me the strength to pursue my dreams.

To my other family members, my boyfriend and friends in particular, Heitor de Araujo and Andrea Leite by moments of relaxation, for all the help provided and for always believing in me.

To my advisor, Prof. Dr. Leandro Chaves Rêgo, for the dedication, optimism, availability, attention given and for all the instruction provided to carry out this work.

I would also like to acknowledge CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) for their financial support and for the Programa de Pós-Graduação em Engenharia de Produção, for all the information presented and for the opportunity to meet colleagues and professors who contributed to this work.

“Persistence is the way to success.”

(Charles Chaplin)

ABSTRACT

The Graph Model for Conflict Resolution (GMCR) is based on concepts of Game Theory and Conflict Analysis and is useful for describing and analyzing conflicts. Stability analysis is used in the GMCR to determine possible solutions for the conflict. Several solution concepts have been proposed which accommodate different Decision Makers (DM's) behavior. Some of them are: Nash, General Metarationality (GMR) and Sequential Stability (SEQ). For a state to be Nash stable for a DM, such DM cannot move to a more preferred state in a single step. For GMR and SEQ, while considering moving to a more preferred state, the DM foresees whether the opponent can react leading the conflict to a state not preferred to the current one. What differs GMR and SEQ is that in SEQ, it is not allowed to harm the opponent if it does not benefit from such movement. However, we show by means of an example that there are situations in which to perform such reaction the opponent must be leaving a SEQ state for him, making it non-credible. In order to avoid that problem, we propose new solution concepts for the GMCR, called Higher-order Sequential Stabilities, and explore their relation with other solution concepts commonly used in the GMCR. Additionally, we introduce the concept of Higher-order Sequential Equilibria for coalitional analysis in the GMCR.

Keywords: Conflict. Graph model. Stability notions. Credible threats.

RESUMO

O Modelo de Grafo para Resolução de Conflitos (GMCR) é baseado em conceitos de Teoria dos Jogos e Análise de Conflitos e é útil para descrever e analisar conflitos. A análise de estabilidade é usada no GMCR para determinar possíveis soluções para o conflito. Diversos conceitos de solução foram propostos e acomodam diferentes comportamentos dos Tomadores de Decisão (DM's). Alguns deles são: Nash, Metaracionalidade Geral (GMR) e Estabilidade Sequencial (SEQ). Para um estado ser Nash estável para um DM, esse DM não pode se mover para um estado mais preferido em uma única etapa. Para GMR e SEQ, enquanto se considera mudar para um estado mais preferido, o DM prevê se o oponente pode reagir levando o conflito a um estado não preferido ao atual. O que difere GMR e SEQ é que, na SEQ, não é permitido prejudicar o oponente se ele não se beneficiar de tal movimento. No entanto, mostramos por meio de um exemplo que existem situações em que para realizar tal reação o oponente deve estar deixando um estado SEQ para ele, tornando-o não credível. Para evitar esse problema, propomos novos conceitos de solução para o GMCR, chamados de estabilidade sequencial de ordem superior, e exploramos sua relação com outros conceitos de solução comumente usados no GMCR. Adicionalmente, introduzimos o conceito de Equilíbrios Sequenciais de Alta Ordem para análise de coalizões no GMCR.

Palavras-chave: Conflito. Modelo de grafo. Noções de estabilidade. Ameaças credíveis.

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF ABBREVIATIONS AND ACRONYMS

CG	City Government
CNASH	Coalitional Nash
CGMR	Coalitional General Metarationality or Coalitional General Metarational
CSMR	Coalitional Symmetric Metarationality or Coalitional Symmetric Metarational
CSEQ	Coalitional Sequential Stability or Coalitional Sequentially
CSSEQ	Coalitional Symmetric Sequential Stability or Coalitional Symmetric Sequentially
Cm-SEQ	Coalitional Sequential Stability of Order *m* or *m-th* Order Coalitional Sequentially.
DMs	Decision Makers
DSS	Decision Support System
D	Developer
GMR	General Metarationality or General Metarational
GMDE	Global Market-Driven Economy
GMCR	Graph Model for Conflict Resolution
I	Independent Environmental Expert
IFORS	International Federal of the Operational Research Societies
M	Residents of Majalaya
PO	Property Owner
SEQ	Sequential Stability or Sequentially
m-SEQ	Sequential stability of order *m* or *m-th* order sequentially
SES	Sustainable Ecosystem
SMR	Symmetric Metarationality or Symmetric Metarational
SSEQ	Symmetric Sequential Stability or Symmetric Sequentially
T	Textile Industry
UI	Unilateral Improvement
W	West Java Provincial Government

Contents

1 INTRODUCTION

Decision-making is a common action in people's daily lives. In order to make a decision it is important to perform an analysis of the objective and evaluate the procedures that will help in achieving the main objective. One method that has aided engineers and decision makers is the resolution of conflicts, its importance is due to the fact of increasing social and political influence in engineering decision-making.

Conflict is a process that starts when one of the individuals realizes that its adversaries have negatively affected or negatively affect something that is a concern or interest of the first individual. Such a definition encompasses a variety of existing conflicts within an organization, such as: misalignment of purposes, different interpretation of facts, disagreement caused by expected behavior (ROBBINS, 1990).

It is often inevitable that there are conflicts in situations where humans interact with each other or in groups. Such conflicts can be exemplified as patent disputes between multinational companies, wars between nations, management and labor negotiations, engineering projects, among other possible conflicts.

The main objective of a conflict resolution analysis is to gain a better understanding of the strategic points of a given dispute and thus make more informed and fairer decisions. One of the advantages of using a conflict resolution method is that the result recommended by an stability analysis is reliable in the sense that none of the individuals has advantage of deviating alone from the suggested solution.

Game theory can be used to describe the process of conflict resolution. In 1993, Fang, Hipel & Kilgour proposed the Graph Model for Conflict Resolution (GMCR), which is a model based on ideas from Conflict Analysis and Game Theory. The GMCR has an advantage of being easy to analyze and able to accommodate different behaviors in its stability analysis. The recommendations generated from such analysis can be used by the individuals involved in the conflict, by mediators or by analysts studying the conflict.

The GMCR is used to describe a conflict specifying the agents involved in the conflict, called decision makers (DMs), and the possible scenarios, called states, that may arise in the conflict. Each state is associated with a combination of actions or strategies, one for each DM in the conflict. Thus, DMs can change the conflict state by changing their actions, according to preferences that they have among the possible states and how they foresee that others will react to their moves. The stability analysis of the GMCR consists of investigating which of the possible states are stable for some DM in the sense that such DM has no incentive from moving away from it. If some state is stable for all DMs, then it is called an equilibrium and is appointed as a possible conflict resolution.

Li et al. in 2004 used a new preference structure in the GMCR in which DMs preferences are expressed by a triple of relations $\{\succ_i, \sim_i, U_i\}$, were $s \succ_i s_1$ and $s \sim_i s_1$ are the strict preference and indifference relations, and $s\ U_i\ s_1$ means that DM i is uncertain as to whether he prefers state s to state s_1, prefers s_1 to s, or is indifferent between s and s_1. Rêgo & dos Santos in 2015 presented a generalized GMCR in which it was introduced the possibility that the decision makers had probabilistic preferences among the possible scenarios or states of the conflict. Rêgo & Vieira in 2017a modified the GMCR to model interactive unawareness of DMs about the options available to them in the conflict, where DMs can reason not only about the awareness level of the other DM but also about the awareness level of the other DM regarding his or her own awareness level and so on. Rêgo & Vieira in 2016 generalized a solution concept, called Symmetric Sequential Stability (SSEQ), in the GMCR for conflicts involving n DMs.

The GMCR has been applied by a variety of authors. Kassab, Hipel & Hegazy in 2006 applied a collaborative negotiation methodology in a Decision Support System (DSS) that was named as GMCR II that aims to facilitate the negotiation of multiple agent conflicts in large construction projects. Alamanda et al. in 2015 used the GMCR for the case of waste pollution by the Majalaya textile industry in an area of the Citarum River in Bandung regency.

There are several stability concepts for the analysis of the GMCR which differ in the assumptions about the behavior of the decision makers involved in the conflict. For example, according to the concept of Nash Stability (NASH, 1950; NASH, 1951), each DM assesses whether or not it is possible to take the conflict to a preferable scenario, regardless of possible reactions of the opponents. In the concepts of General Metarationality (GMR) (HOWARD, 1971) and Sequential Stability (SEQ) (FRASER; HIPEL, 1984), the DMs evaluate if the opponents can react taking the conflict to a state that is no better than the current state. The difference between such concepts is given by the fact that in Sequential Stability, the opponents' reactions have to be beneficial to them as well. However, in the Sequential Stability concept, although the opponents' are not allowed to hurt themselves while moving, their reaction might indicate to leave a state that is sequentially stable for them, which does not seem to be credible since according to the solution concept under analysis, a DM should not leave a SEQ stable state. Thus, the SEQ analysis does not treat equally the DM who moves first and the others who move next.

1.1 Justification

For the stability analysis of conflicts, it is important to use solution concepts that better describe DMs behavior. Stability notions which are based on noncredible threats do not seem to be appropriate. For that reason, in this dissertation, we propose new solution concepts for the GMCR by modifying the concept of Sequential Stability to mitigate the problem of noncredible threats found in the Sequential Stability concept. This problem occurs because a DM leaves a state that is already SEQ for him only to punish his opponent. To mitigate this problem we propose notions of stability of different orders.

In a sequentially stable state of order m, the opponent is not allowed to move away from a state that is sequentially stable of order $m-1$ to him. Relationships between the proposed stability concepts were also obtained, together with relationships with other solution concepts commonly used in the GMCR. We also provide an existence result of sequential equilibrium of any odd order in finite conflicts where moves and preferences are transitive. Finally, we propose the concept of Higher-order Sequential equilibria for coalitional analysis in the GMCR.

1.2 Objective

1.2.1 General Objective

We propose refinements of the concept of Sequential Stability in the GMCR for 2 DMs and n-DMs and investigate the relationship of these definitions with five solution concepts commonly used in the GMCR. These refinements are intended to mitigate a problem of noncredible threats that may occur in SEQ.

1.2.2 Specific objectives

To reach the general objective, the specific objectives are:

- Bibliographic review of the Graph Model for Conflict Resolution and of the stability definitions;
- Development of new stability definitions, called Higher-order Sequential Stability, which are based on iterations of the concept of Sequential Stability to avoid noncredible threats;
- Establish relationships between the new concepts proposed and those commonly used in the GMCR;
- Proposal of the concept of Higher-order Sequential Equilibria for coalitional analysis;

- Elaboration and resolution of applications and examples;
- Analysis of results.

1.3 Structure of Work

The remaining sections of this dissertation are organized as follows. In Chapter 2, the GMCR and the stability definitions mentioned previously are revised. In Chapter 3, we present the definitions of Higher-order Sequential Stabilities for a DM for conflicts with 2 DMs, provide some of its properties, we present relationships between the proposed definitions and other solution concepts commonly used in the GMCR and we present three applications to illustrate the proposed solution concepts. In Chapter 4, we present the definitions of Higher-order Sequential Stabilities for a DM for conflicts with n-DMs, provide some of its properties, propose the definition of Higher-order Coalitional Sequential Stability and study its properties and, conclude the chapter presenting applications to illustrate the proposed solution concepts for multiple DMs. Finally, we finish in Chapter 5 with main conclusions and directions for future work.

2 THEORETICAL FOUNDATION AND LITERATURE REVIEW

In this chapter we will present the concepts of a conflict model, the Graph Model for Conflict Resolution. The definitions of stability concepts will also be presented. Finally, we present an illustrative example of a conflict to explain the GMCR.

2.1 GMCR and solution concepts

Strategic conflict can be defined as a decision problem involving multiple DMs, each of which presents distinct preferences regarding possible scenarios or states that could occur as the end result of a conflict (HIPEL; KILGOUR; FANG, 2011).

A conflict model is a systematic structure that aims to encompass the main characteristics of a strategic conflict (HIPEL; KILGOUR; FANG, 2011). From the conflict model, it is possible to carry out analysis in order to determine the possible balances between the possible strategic interactions of DMs. DMs can make use of stability analysis as well as related sensitivity analysis to assist them in decision-making in a conflict. In this section we will present GMCR and its solution concepts.

2.1.1 GMCR

In order to apply the GMCR, it is necessary to understand that it represents a conflict where the DMs involved may change the conflict state by changing actions. These states and transitions are modeled as vertex and arcs of a graph, respectively. The GMCR can incorporate irreversible movements, that is, a DM can change from state s_k to state s_q, but it may not be able to change back from state s_q to state s_k. The GMCR can also describe common movements, that is, more than one DM can cause a conflict to move from one state to another.

Formally, the GMCR was introduced by Kilgour, Hipel & Fang em 1987 and is composed of a directed graph and a preference relation for each DM involved in the conflict over the set of possible states of the conflict. Let $N = \{1,2,\dots,n\}$ be the set of DMs involved in the conflict and $S = \{s_1,s_2,\dots,s_m\}$ be the set of states or possible scenarios of the conflict. A collection of finite directed graphs $D_i = (S,A_i), i \in N$, can be used to model the course of the conflict, where $A_i \subseteq S \times S$ determines for each state to what states DM i can lead the conflict, called reachable states from s in one step (RÊGO; VIEIRA, 2017b). A vertex of a graph is a possible conflict state, and therefore the set of vertices, S, is common to all graphs. If DM i can move unilaterally (in one step) from state s_k to state s_q, then there is an oriented arc from s_k to s_q in A_i.

The possible moves of a DM i can be represented in an efficient way through an accessible list. For $i \in N$, the accessible list for DM i from state $s_k \in S$ is the set $R_i(s_k)$ of all states to which DM i can move (in a single step) from state s_k, formally defined by:

$$R_i(s_k) = \{s_q \in S : (s_k, s_q) \in A_i\}. \tag{2.1}$$

It is common in the literature to assume that $s \notin R_i(s)$, $\forall s \in S$ and $\forall i \in N$, which implies that no state is accessible to itself for any DM.

A binary relation Θ on the set S is a set of ordered pairs of elements of S, i.e. $\Theta \subset S \times S$. Let $\succ_i$ be the (strict) preference relation of DM i over S, so that $s \succ_i s_1$ indicates that DM i strictly prefers state s to state s_1. The weak preference relation, $\succsim_i$, is used to model the possible lack of strict preference, where $s \succsim_i s_1$ means that DM i does not strictly prefer state s_1 to state s. Finally, $\sim_i$ is a binary relation that represents the absence of strict preference in both directions, i.e. $s \sim_i s_1$ if and only if $s \succsim_i s_1$ and $s_1 \succsim_i s$. This latter relation is usually called indifference relation. Assume that the relation $\succ_i$ is irreflexive, so that there is no S such that $s \succ_i s$, and asymmetric, so that it cannot occur both $s_1 \succ_i s_2$ and $s_2 \succ_i s_1$.

For $i \in N$, it is now possible to define the set of unilateral improvements for DM i from state s, as follows:

$$R_i^+(s_k) = \{s_q \in R_i(s_k) : s_q \succ_i s_k\}. \tag{2.2}$$

Let $H \subseteq N$ be a subset of DMs, called coalition, we define $R_H(s) \subseteq S$ as being the set of states that DMs in the coalition H can reach without consecutive movements of the same DM from state s. Formally, $R_H(s)$ is defined inductively as follows. Let $\Omega(s, s_1)$ be the set of all DMs that make a last legal movement in any legal sequence of movements from state s to state s_1, where a sequence of movements is legal if DMs can move move more than once in the sequence but not consecutively. $R_H(s)$ and $\Omega(\cdot)$ are defined as being the smallest sets such that:

(i) If s_1 belongs to $R_i(s)$ and i belongs to H, then s_1 belongs to $R_H(s)$ and i belongs to $\Omega(s, s_1)$;

(ii) If s_1 belongs to $R_H(s)$, i belongs to H, s_2 belong to $R_i(s_1)$, and $\{i\}$ is different from $\Omega(s, s_1)$, then s_2 belongs $R_H(s)$ and i belongs to $\Omega(s, s_2)$.

Similarly, we can define $R_H^+(s) \subseteq R_H(s)$ as the set of states that can be reached by DMs in H by some legal sequence of unilateral improvements from state s. Let $\Omega^+(s,s_1)$ be the set of all DMs that make a last improvement in any legal sequence of unilateral improvements from state s to state s_1. $R_H^+(s)$ and $\Omega^+(\cdot)$ are defined as being the smallest sets such that:

(i) If s_1 belongs to $R_i^+(s)$ and i belongs to H, then s_1 belongs to $R_H^+(s)$ and i belongs to $\Omega^+(s,s_1)$;

(ii) If s_1 belongs to $R_H^+(s)$, i belongs to H, s_2 belong to $R_i^+(s_1)$, and $\{i\}$ is different from $\Omega^+(s,s_1)$, then s_2 belongs $R_H^+(s)$ and i belongs to $\Omega^+(s,s_2)$.

2.1.2 Stability concepts in the GMCR

The study of possible movements and countermovements done by DMs in strategic conflicts is called stability analysis. DMs may behave in different ways in conflict situations and several stability concepts have been proposed to model such variety of behaviors. We now recall some stability concepts that have been used in the GMCR, namely: Nash Stability, General Metarationality (GMR), Symmetric Metarationality (SMR), Sequential Stability (SEQ) and Symmetric Sequential Stability (SSEQ).

For illustrate this study of possible movements and countermovements done by DMs in strategic conflicts we will present the example of the famous game-theory dispute known as the "Prisoner's Dilemma" (AXELROD, 1984) that is very similar to the behavior of decision-makers in a conflict situation. This dispute was used for the purpose of getting ideas about human behavior in conflict resolution decisions (FRASER; HIPEL, 1984).

In the Conflict of the Prisoner's Dilemma two individuals suspected of committing a crime are arrested by the police. But the police do not have enough evidence to condemn them. An exit found by the police is to separate the individuals who were arrested in different rooms and propose the same agreement to them:

- If one of them cooperates (C) with his partner, ie does not confess, and the other individual does not cooperate (D), that is, give away, the individual who betrayed will be free and the one who cooperated, was silent, will receive the penalty of 10 years of jail;
- If both are silent, that is, cooperate with each other, each of the prisoners shall receive the penalty of one year in jail;
- If both betray, that is, betray their partner, each will receive the sentence of 5 years of jail.

Table 1 – Normal form of Prisoners' Dilemma

		DM 2 C	DM 2 D
DM 1	C	CC	CD
DM 1	D	DC	DD

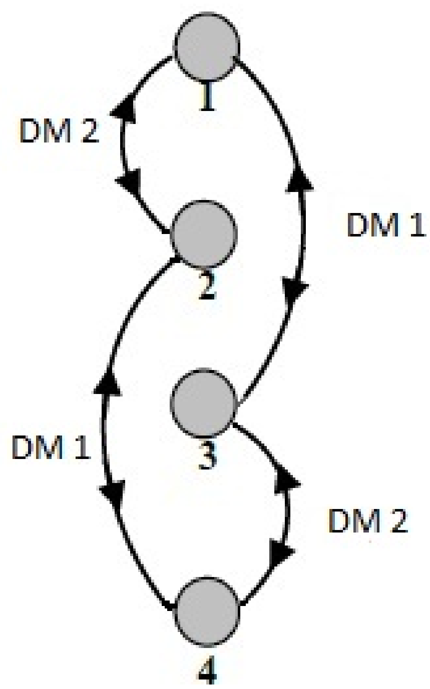
Source: This research (2018)

Each prisoner will have to make the decision without knowing the choice of his partner. In this game, decision makers present 2 strategies, formulating a total of 4 possible scenarios or decision states. The normal form of this game can be represented as follows:

As the normal form of the game shows Table 1, DM 1 controls the strategies of the line, while DM 2 controls the strategies of the column. The pair of letters represents the strategies of DM 1 and DM 2, respectively. For example, the DC state represents the case where the DM 1 does not cooperate (D) and the DM 2 cooperates (C).

This game can be represented using the graph model, where the choices available for each DM can be represented using nodes and arcs as shown in Figure 1, where the decision states are represented by 1 (CC), 2 (CD) , 3 (DC) and 4 (DD).

Figure 1 – Possible movements of DMs in the Prisoners' Dilemma.



Source: This research (2018)

The arcs that connect the nodes (decision states) 1 and 3 represent the possibility of deciding to change from state 1 (CC) to state 3 (DC) or vice versa of DM 1. The same happens with the movement between the decision states 2 and 4, represented by the arcs connecting nodes 2 and 4. In this way it is possible to analyze the decision possibilities of

DM 2.

The preference relation in this conflict for DM 1 is denoted by $3(DC) \succ_1 1(CC) \succ_1 4(DD) \succ_1 2(CD)$ and for DM 2 $2(CD) \succ_2 1(CC) \succ_2 4(DD) \succ_2 3(DC)$. Note that the preferred decision state for DM 1 is state 3 (DC), in this state DM 1 will be free , since he did not cooperate and DM 2 will receive a 10-year prison sentence because he cooperated. For DM 2, the most preferred decision state is state 2 (CD) and the least preferred state is state 3 (DC).

From the graph model for the Prisoners' Dilemma we have seen that DM 1 can change its decision from CC state to DC state, or between CD state and DD state. In the case of DM 2 it can use its strategy to change from CC state to CD state or between DC and DD decision state.

To analyze strategic conflicts, we now recall the most commonly used stability notions in the GMCR literature. For all stability notions, if some state is stable for every DM, it is called an equilibrium according to that stability notion.

2.1.2.1 Nash Stability

Let $i \in N$, state $s \in S$ is Nash stable (or individually rational) for DM i if and only if $R_i^+(s) = \emptyset$. Thus, a state is Nash stable for DM i, if, and only if, DM i cannot unilaterally move to a more preferable state (NASH, 1950; NASH, 1951). Let S_i^{NASH} denote the set of all Nash stable states for DM i.

In the example of the Prisoners' Dilemma, state 3 (DC) is Nash stable for DM 1, since there is no state with more preference than state 3. State 1 (CC) is considered unstable because there is another state that is more preferable for DM 1. Analyzing all decision possibilities for Nash stability in relation to both players, we see that decision state 4 (DD) is the only Nash equilibrium, since no DM can move unilaterally to a better state from state 4. All other states are unstable for at least one DM, which can always improve their situation by failing to cooperate with the other DM.

2.1.2.2 General Metarationality

The General Metarationality (GMR) concept was defined by Howard in 1971. Its purpose is to model the behavior of a DM that analyzes their possible movements in a conservative way, considering all possible reactions to their movements, ignoring their own possible counter-reactions. Let $i \in N$, state $s \in S$ is GMR stable for DM i if, and only if, for all $s_1 \in R_i^+(s)$ there is at least one $s_2 \in R_j(s_1)$ such that $s \succsim_i s_2$. In the case for n-DMs the state $s \in S$ is GMR for a player i if, and only if, $\forall s_1 \in R_i^+(s)$, $\exists s_2 \in R_{(N-i)}(S_1) : s \succsim_i s_2$. Let S_i^{GMR} denote the set of all GMR stable states for DM i.

In the case of the Prisoner's Dilemma, from state 1, DM 1 has a unilateral improvement move to state 3. However, DM 2 can punish DM 1 by moving from state 3 to state 4 which is less preferable to DM 1 than state 1. Thus, state 1 (CC) is GMR stable for DM 1. By symmetry, this state is also GMR stable for DM 2, therefore a GMR equilibrium. By analyzing all the states, it is possible to conclude that states 1 and 4 are equilibria according to the GMR concept. It is noteworthy that if a state is Nash stable for any DM, by definition it is also be GMR (and as we will see SMR, SEQ and SSEQ) stable for this DM.

2.1.2.3 Symmetric Metarationality

A DM is said to use the Symmetric Metarationality stability criterion (SMR) when he considers not only his own possible moves and his opponent's reactions to each of these movements, but also his own counter-reaction. The DM according to this criterion has the capacity to analyze three movements ahead, while according to the criterion of General Metarationality it observes only two movements ahead and according to the criterion of Nash only one movement ahead. This notion was also proposed in 1971 by Howard and it is a stability definition more restrictive than the General Metarationality. Let $i \in N$, state $s \in S$ is SMR for DM i if, and only if, for all $s_1 \in R_i^+(s)$ there exists $s_2 \in R_j(s_1)$ such that $s \succsim_i s_2$ and $s \succsim_i s_3$ for all $s_3 \in R_i(s_2)$. For the case for n-DMs the state $s \in S$ is SMR for a player i if, and only, if $\forall s_1 \in R_i^+(s)$, $\exists s_2 \in R_{(N-i)}(s_1) : s \succsim_i s_2$ and $s \succsim_i s_3$ $\forall s_3 \in R_i(s_2)$. Let S_i^{SMR} denote the set of all SMR stable states for DM i.

For example, in the Prisoner's Dilemma problem when DM 1 moves from decision state 1 to state 3, DM 2 may react moving from state 3 to state 4, where state 4 is not preferred to state 1 by DM 1. In order to escape this punishment, DM 1 can only move from state 4 to state 2, which is also not preferred to state 1 by DM 1. Thus, state 1 is SMR stable for DM 1. State 1 is also SMR stable for DM 2 and, consequently, an SMR equilibrium, together with state 4. It is worth noting that in an SMR stability analysis, the focal DM must consider that his adversary may react hurting himself in order to force the focal DM not to move to a more preferable decision state.

2.1.2.4 Sequential Stability

Defined by Fraser & Hipel (1979, 1984) it is similar to General Metarationality (GMR), but according to SEQ the reactions of the opponent is also beneficial to him, i.e., it is not allowed to harm the opponent if one does not benefit from such move. Let $i \in N$, state $s \in S$ is Sequentially Stable (SEQ) for DM i if, and only if, for all $s_1 \in R_i^+(s)$ there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$. The definition for the case of n-DMs is the state $s \in S$

is SEQ if, and only if, $\forall s_1 \in R_i^+(s), \exists s_2 \in R_{(N-i)}^+(s_1) : s \succsim_i s_2$. Let S_i^{SEQ} denote the set of all SEQ stable states for DM i.

In the case of the Prisoners' Dilemma, while analyzing the SEQ stability of state 1 for DM 1, we see that he has a unilateral improvement move to state 3. On the other hand, DM 2 has a unilateral improvement move from state 3 to state 4, which is less preferred than state 1 by DM 1. Thus, decision state 1 (CC) is SEQ stable for DM 1 and, likewise, SEQ stable for DM 2. Thus, state 1 is a SEQ equilibrium. State 4 is Nash stable for both players and, consequently, also a sequential equilibrium.

2.1.2.5 Symmetric Sequential Stability

Another type of Sequential Stability was proposed by Rêgo & Vieira in 2016. Known as Symmetric Sequential Stability (SSEQ), it defines a kind of Sequential Stability in which a DM, when planning to move, considers not only the reaction of its opponents, but also its own counterattack. It is worth noting that the counterattack is not necessarily a unilateral improvement for the DM, but rather that the resulting state is no better than the current state for each possible counterattack. Let $i \in N$, state $s \in S$ is SSEQ for DM i if, and only if, for all $s_1 \in R_i^+(s)$ there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$ and $s \succsim_i s_3$ for all $s_3 \in R_i(s_2)$. In the case of n-DMs, a state $s \in S$ is Sequentially Symmetric Stable (SSEQ) for DM $i \in N$ if, and only if, for all $s_1 \in R_i^+(s)$ there exists a $s_2 \in R_{(N-i)}^+(S_1)$ such that $s \succsim_i s_2$ and $s \succsim_i s_3$ for all $s_3 \in R_i(s_2)$. Let S_i^{SSEQ} denote the set of all SSEQ stable states for DM i.

Analyzing the SSEQ in the case of the Prisoner Dilemma problem we have a similar result to the SEQ, but it is necessary to analyze the counter-response of the DM 1. Therefore, state 1 of the conflict (CC) is also SSEQ stable for DM 1 and DM 2 and, consequently, a SSEQ equilibrium, since the DMs cannot counteract the punishment of the opponent. In addition, as state 4 (DD) is Nash stable, it is also SSEQ stable. Through the stability analysis, we see that only state 1 and 4 are possible equilibria, state 4 being the one which is equilibrium according to the largest number of stability concepts.

2.1.3 Coalitional Analysis in the GMCR

Coalitional analysis is an extension of the stability analysis to situations where the DMs can act together, being able to realize improvements over the results they would achieve if they were acting alone.

The definitions of coalitional analysis were given by Inohara and Hipel in 2008 and Kilgour et al. in 2001. A coalition is any non-empty set of DMs, $\emptyset \neq H \subseteq N$. Let $\varphi(N)$ be the class of all coalitions of DMs in N and $R_H^{++}(s) = \{s_1 \in S : s_1 \in R_H(s)$ and $s_1 \succ_i s$ for all $i \in$

Table 2 – Stability of a state s for some coalition H in the GMCR

Notion	Definition
CNash	$R_H^{++}(s) = \emptyset$
CGMR	$\forall s_1 \in R_H^{++}(s)$, $\exists s_2 \in R_{\varphi(N-H)}^{++}(s_1)$: $s \succeq_i s_2$ for some $i \in H$
CSMR	$\forall s_1 \in R_H^{++}(s)$, $\exists s_2 \in R_{\varphi(N-H)}^{++}(s_1)$: $s \succeq_i s_2$ for some $i \in H$ and $\forall s_3 \in R_H(s_2)$, $s \succeq_j s_3$ for some $j \in H$
CSEQ	$\forall s_1 \in R_H^{++}(s)$, $\exists s_2 \in R_{\varphi(N-H)}^{++}(s_1)$: $s \succeq_i s_2$ for some $i \in H$
CSSEQ	$\forall s_1 \in R_H^{++}(s)$, $\exists s_2 \in R_{\varphi(N-H)}^{++}(s_1)$: $s \succeq_i s_2$ for some $i \in H$ and $\forall s_3 \in R_H(s_2)$, $s \succeq_j s_3$ for some $j \in H$

Source: This research (2018)

$H\}$ be the set of coalitional improvement moves from s by coalition H. Let $C \subseteq \varphi(N)$ and let $R_C(s)$ be the set of reachable states by class C from s by a legal sequence of movements, where a legal sequence of movements of a class C is one in which no coalition moves twice consecutively. Formally, $R_C(s)$ is defined inductively as follows. Let $\Omega_C(s, s_1)$ be the set of all coalitions that make a last legal movement in any legal sequence of movements from state s to state s_1. $R_C(s)$ and $\Omega_C(\cdot)$ are defined as being the smallest sets such that:

(i) If s_1 belongs to $R_H(s)$ and H belongs to C, then s_1 belongs to $R_C(s)$ and H belongs to $\Omega_C(s, s_1)$;

(ii) If s_1 belongs to $R_C(s)$, H belongs to C, s_2 belong to $R_H(s_1)$, and H is different from $\Omega_C(s, s_1)$, then s_2 belongs $R_C(s)$ and H belongs to $\Omega_C(s, s_2)$.

Similarly, one can define $R_C^{++}(s) \subseteq R_C(s)$ as the set of states that can be reached by coalitions in C by some legal sequence of unilateral improvements from state s. Let $\Omega_C^{++}(s, s_1)$ be the set of all coalitions that make a last improvement in any legal sequence of unilateral improvements from state s to state s_1. $R_C^{++}(s)$ and $\Omega_C^{++}(\cdot)$ are defined as being the smallest sets such that:

(i) If s_1 belongs to $R_H^{+}(s)$ and H belongs to C, then s_1 belongs to $R_C^{++}(s)$ and H belongs to $\Omega_C^{++}(s, s_1)$;

(ii) If s_1 belongs to $R_C^{++}(s)$, H belongs to C, s_2 belong to $R_H^{+}(s_1)$, and H is different from $\Omega_C^{++}(s, s_1)$, then s_2 belongs $R_C^{++}(s)$ and H belongs to $\Omega_C^{++}(s, s_2)$.

Table 2 presents the definitions of CNash, CGMR, CSMR, CSEQ, and CSSEQ stability of a state s for some coalition H for n-DM conflicts. For all these coalitional stability definitions, a state is coalitional stable for a DM, if it is coalitional stable for all coalitions that include such DM.

2.2 Overview about the GMCR literature

Other preference structures have also been developed over time for the GMCR. Li et al. in 2004, present a preference structure for the GMCR that includes uncertain or unknown preferences in comparison of two states - “preferred”, “indifferent” and “unknown”. Ben-Haim & Hipel in 2002 applied the information theory of gaps in order to observe how the changes of uncertainties of preference of DMs could affect the equilibria and the results of stability. Al-Mutairi, Hipel & Kamel in 2008 applied five labels to divide the domain of Fuzzy preferences - “much more preferred”, “more preferred”, “indifferent”, “less preferred” and “much less preferred”.

Rêgo & Santos in 2015, carried out a generalization of GMCR by introducing a possibility for DMs to express their preferences among possible scenarios through probabilistic preferences. They proposed four concepts of solution: 1) α-Nash stability; 2) (α,β)-metarationality; 3) (α,β)-symmetric metarationality; and 4) $(\alpha,\beta,\ )$-sequential stability. These can be used in conflicts with two or more DMs. They also analized applications where it was possible to observe the advantages obtained by allowing users to express their preferences probabilistically.

In the GMCR literature, there are models that are able to represent different points of view of the DMs regarding the ongoing conflict. According to Obeidi, Kilgour & Hipel in 2009a, one way to allow decision makers to independently describe the conflict is by modeling the effects of future scenario emotions considered by DMs, this model is known as Perceptual GMCR. This type of preference structure enables large improvements in the GMCR analysis, minimizing the complexity of describing realistic models, enhancing the application of GMCR modeling algorithms in realistic conflicts that are composed of perception and emotion.

In a conflict, the existence of discrepant perceptions of DMs, often caused by asymmetric information between the DMs or by negative emotions, are modeled in the systems of perceptive graph models. Its objective is to make predictions related to possible responses in order to reveal the dependence of these predictions on the variability in the awareness of a DM. (OBEIDI; KILGOUR; HIPEL, 2009b)

Rêgo & Vieira in 2017a, proposed a modification in the GMCR to accommodate the iterative unawareness of DMs in relation to the options available in a conflict. This model allows a DM to reason about the awareness level of his opponents and about his own level of awareness about his opponents awareness levels and so on. The authors also performed a generalization of the standard solution concepts and an application in a hypothetical conflict in a war situation.

In the next chapter, we present the notions of Higher-order Sequential Stability and the relation of this new concept of stability with the concepts presented in this chapter.

Three applications are performed to demonstrate the usefulness of the proposed model, where two of them are classical games in Game Theory.

3 HIGHER-ORDER SEQUENTIAL STABILITIES FOR BILATERAL CONFLICTS

In this chapter, we present an example that motivated us proposing the notions of Higher-order Sequential Stabilities, by showing an inconsistency in the notion of Sequential Stability. We will also present definitions, properties, relationships among Higher-order Sequential Stabilities and other solution concepts in the GMCR and applications. A preliminary version of the results of this chapter was presented at the 21st Conference of the International Federation of Operational Research Societies (IFORS), in July 2017, in Quebec City, Canada. The complete results are under review at the journal IEEE Transactions on systems, man and Cybernetics: Systems.

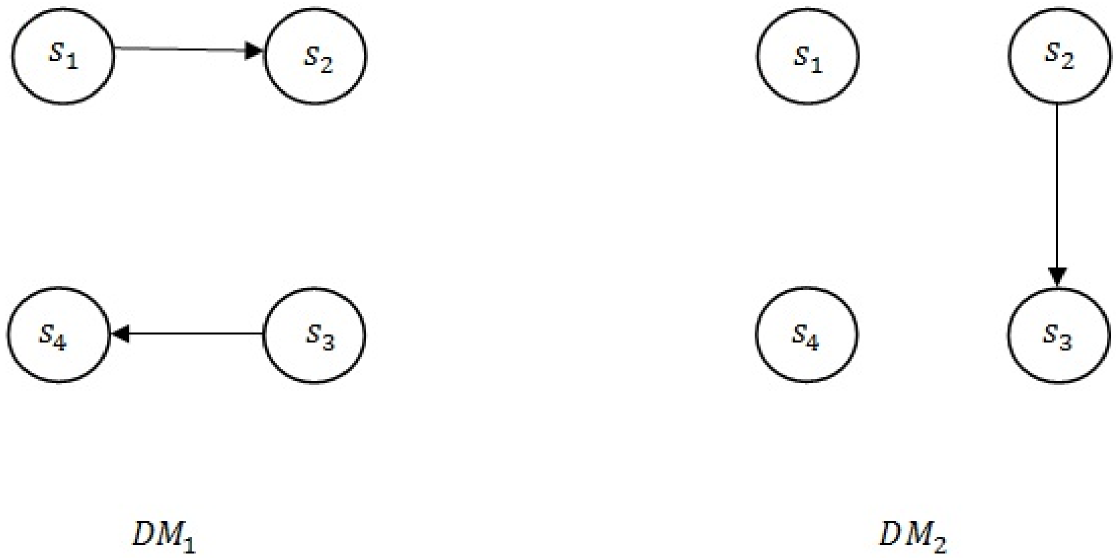
3.1 A motivational example

The following example presents a conflict where there exists a SEQ stable state for one DM (focal DM) which relies on a reaction to the focal DM that is not credible. Such reaction is not credible, in the sense that the state at which the opponent of the focal DM moves, even though not being Nash stable, it is SEQ stable for the opponent.

Consider the conflict illustrated in Figure 2, where the state space is defined by $S = \{s_1, s_2, s_3, s_4\}$ with two decision makers, DM 1 and DM 2. Suppose that $R_1(s_1) = \{s_2\}, R_1(s_3) = \{s_4\}, R_2(s_2) = \{s_3\}$ and $R_1(s_2) = R_1(s_4) = R_2(s_1) = R_2(s_3) = R_2(s_4) = \emptyset$. Also, consider the following preference relations: $s_2 \succ_1 s_1 \succ_1 s_4 \succ_1 s_3$ and $s_3 \succ_2 s_2 \succ_2 s_4 \succ_2 s_1$.

Note that state s_1 is SEQ and SSEQ for DM 1, since the only unilateral improvement from s_1 for DM 1 is state s_2 and DM 2 has a unilateral improvement from s_2 that takes the conflict to s_3 and s_3 is worse than s_1 for DM 1 and DM 1 can only move away from s_3 to s_4 which is also not preferred to s_1 for DM 1. However, if we are adopting the criterion that a DM will not move away from a state that is SEQ to him, the punishment of DM 2 going from state s_2 to state s_3 is not credible, since state s_2 is SEQ for DM 2. In order to see that, note that the unique unilateral improvement from s_2 for DM 2 is state s_3 and DM 1 has a unilateral improvement from s_3 that takes the conflict to s_4 and s_4 is worse than s_2 for DM 2. This example motivates us to propose new solution concepts for the GMCR by introducing the concept of Higher-order Sequential Stabilities to avoid the problem of noncredible Sequential Stabilities.

Figure 2 – GMCR which illustrates a noncredible sequentially stable state.



Source: This research (2018)

3.2 Higher-order Sequential Stabilities

From the analysis of the example shown in Subsection 3.1, we want to define a solution concept that avoids DMs leaving from states which are SEQ for them both when they are taking the position of a focal DM or when they are reacting to the focal DM. In order to avoid having a circular definition, we need to define what we call a second order SEQ stability notion, where the first order SEQ is the usual SEQ notion. Intuitively, a state is second order SEQ stable for a DM if for every unilateral improvement that he has, his opponent may react benefiting himself leaving a state which is not first order SEQ stable for him such that the resulting state is not preferable for the focal DM to the present state.

However, a similar problem may happen in the second order SEQ stability since although the opponent of the focal DM cannot leave a first order SEQ stable state, he may leave a second order SEQ stable state for him. That fact motivated us to define the notion of SEQ stability for various orders. In the following definition, assume that S_i^{1-SEQ} is the set of all (first order) SEQ stable states for DM i, i.e., $S_i^{1-SEQ} = S_i^{SEQ}$.

Definition 3.2.1. For an integer m such that $m \geq 2$, let S_i^{m-SEQ} be the set of all states that satisfy m-th order sequential stability for DM i, where state s is m-th order sequentially stable for DM i if for every state $s_1 \in R_i^+(s)$, $s_1 \notin S_j^{(m-1)-SEQ}$ and there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$.

Therefore, for each DM i, there exists a sequence of sets $(S_i^{m-SEQ})_{m=1}^{\infty}$ consisting of the states which satisfies the various orders of SEQ stability for DM i. As usual for sequences of sets, we can define two limiting sets, which are known as $liminf$ and $limsup$ of the sequence. They are formally given by:

$$S_i^{inf-SEQ} = \bigcup_{l=1}^{\infty} \bigcap_{m=l}^{\infty} S_i^{m-SEQ} \tag{3.1}$$

and

$$S_i^{sup-SEQ} = \bigcap_{l=1}^{\infty} \bigcup_{m=l}^{\infty} S_i^{m-SEQ}. \tag{3.2}$$

It is well-known that $S_i^{inf-SEQ}$ consists of the states which belong to all, except finitely many, sets in the sequence $(S_i^{m-SEQ})_{m=1}^{\infty}$ and that $S_i^{sup-SEQ}$ consists of the states which belong to infinitely many sets in the sequence $(S_i^{m-SEQ})_{m=1}^{\infty}$.

Analyzing the stability of states from the motivational example according to the proposed concept, we can verify that $S_1^{NASH} = \{s_2, s_4\} \subseteq S_1^{m-SEQ}$ and $S_2^{NASH} = \{s_1, s_3, s_4\} \subseteq S_2^{m-SEQ}$, for all $m \geq 1$. Then, note that for all $m \geq 1$, $s_3 \notin S_1^{m-SEQ}$, since from s_3 DM 1 has a unilateral improvement move to s_4 and, from s_4, DM 2 has no move available to punish DM 1. On the other hand, $s_1 \in S_1^{SEQ}$ (resp., $s_2 \in S_2^{SEQ}$) since from the unique unilateral improvement from such state for DM 1, s_2, (resp., DM 2, s_3), there exists a unilateral improvement for DM 2 (resp., DM 1) leading the conflict to state s_3 (resp., s_4), which is worse than the initial one for DM 1 (resp., DM 2). Since for all $m \geq 1$, $s_3 \notin S_1^{m-SEQ}$, it follows that $s_2 \in S_2^{m-SEQ}$, for all $m \geq 2$. This last result implies that $s_1 \notin S_1^{m-SEQ}$ for all $m \geq 2$. Thus, we have that $S_1^{inf-SEQ} = S_1^{sup-SEQ} = \{s_2, s_4\}$ and $S_2^{inf-SEQ} = S_2^{sup-SEQ} = S$. Although in this example the sets $S_i^{inf-SEQ}$ and $S_i^{sup-SEQ}$ are identical, we show in Theorem 3.3.2 that there are conflicts in which this equality does not hold.

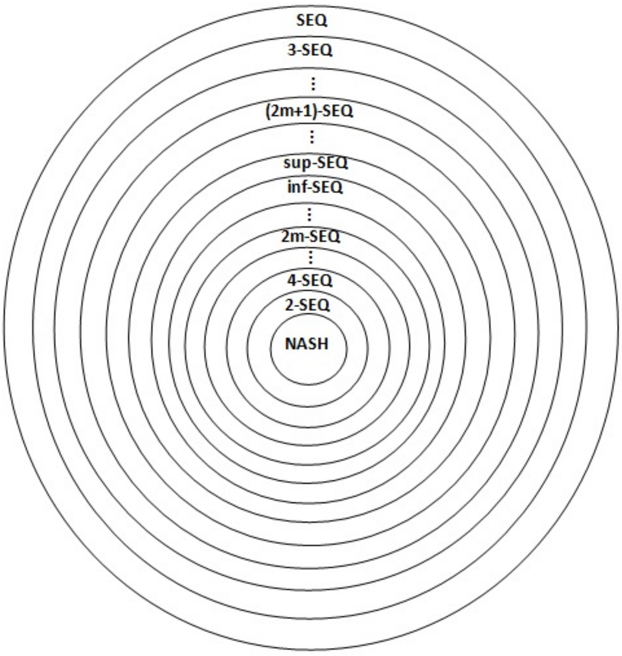
In the following section, we study properties satisfied by the proposed Higher-order Sequential Stabilities concepts.

3.3 Properties of Higher-order Sequential Stabilities

Theorem 3.3.1 shows us how the sets of m-th order sequentially stable states, for different m values, relate to one another. Figure 3 illustrates the result of this theorem.

Theorem 3.3.1. Let m and m_1 be any positive integers. For $i \in N$, if m_1 is odd, then $S_i^{m-SEQ} \subseteq S_i^{m_1-SEQ}$, for all $m \geq m_1$ and, otherwise, if m_1 is even, it follows that $S_i^{m_1-SEQ} \subseteq S_i^{m-SEQ}$, for all $m \geq m_1$.

Figure 3 – Relationship among m-SEQ stability for different m values.



Source: This research (2018)

Proof: We prove the theorem by mathematical induction on m_1.

As $S_i^{1-SEQ} = S_i^{SEQ}$, for $m_1 = 1$, we need to prove that if $s \in S_i^{m-SEQ}$, for $m \geq 1$, then $s \in S_i^{SEQ}$. For $m = 1$, the result is trivially true. Assume that $m \geq 2$ and that $s \in S_i^{m-SEQ}$, then for all $s_1 \in R_i^+(s)$, $s_1 \notin S_j^{(m-1)-SEQ}$ and there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$. Thus, for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$. Therefore, $s \in S_i^{SEQ}$.

For $m_1 = 2$, we need to prove that if $s \in S_i^{2-SEQ}$, then $s \in S_i^{m-SEQ}$, for $m \geq 2$. Assume that $s \in S_i^{2-SEQ}$, then for all $s_1 \in R_i^+(s)$, $s_1 \notin S_j^{SEQ}$ and there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$. Then, using the case $m_1 = 1$, we have that $S_j^{(m-1)-SEQ} \subseteq S_j^{SEQ}$, for $m - 1 \geq 1$. Thus, it follows that for all $s_1 \in R_i^+(s)$, $s_1 \notin S_j^{(m-1)-SEQ}$, for $m \geq 2$, and there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$. Therefore, $s \in S_i^{m-SEQ}$, for $m \geq 2$.

Now assume the following inductive hypothesis:

(I1) $\forall i \in N$, $S_i^{m-SEQ} \subseteq S_i^{(m_1-1)-SEQ}$, for all $m \geq (m_1 - 1)$ and $m_1 - 1$ odd, and
(I2) $\forall i \in N$, $S_i^{(m_1-1)-SEQ} \subseteq S_i^{m-SEQ}$, for all $m \geq (m_1 - 1)$ and $m_1 - 1$ even.

First, suppose that m_1 is odd. Then, assume that $s \in S_i^{m-SEQ}$, for $m \geq m_1$. Thus, it follows that for all $s_1 \in R_i^+(s)$, $s_1 \notin S_j^{(m-1)-SEQ}$ and there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$. As $m_1 - 1$ is even and $m - 1 \geq m_1 - 1$, I2 implies that for all $s_1 \in R_i^+(s)$, $s_1 \notin S_j^{(m_1-1)-SEQ}$ and there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$. Therefore, $s \in S_i^{m_1-SEQ}$.

Second, suppose that m_1 is even. Then, assume that $s \in S_i^{m_1-SEQ}$. Thus, it follows that for all $s_1 \in R_i^+(s)$, $s_1 \notin S_j^{(m_1-1)-SEQ}$ and there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$. Let $m \geq m_1$. As $m_1 - 1$ is odd and $m-1 \geq m_1 - 1$, I1 implies that for all $s_1 \in R_i^+(s)$, $s_1 \notin S_i^{(m-1)-SEQ}$ and there exists $s_2 \in R_j^+(s)$ such that $s \succsim_i s_2$. Therefore, $s \in S_i^{m-SEQ}$. □

The next result shows that $S_i^{inf-SEQ}$ consists of states which are higher-order SEQ stable for DM i for at least one even order and that $S_i^{sup-SEQ}$ consists of the states which are higher-order SEQ stable for DM i for every odd order.

Corollary 3.3.1.1. $S_i^{inf-SEQ} = \bigcup_{m=1}^{\infty} S_i^{2m-SEQ}$ and $S_i^{sup-SEQ} = \bigcap_{m=1}^{\infty} S_i^{(2m-1)-SEQ}$.

Proof: Using the result of Theorem 3.3.1, Equations 3.1 and 3.2 can be simplified. First, Theorem 3.3.1 implies that $\bigcap_{m=l}^{\infty} S_i^{m-SEQ} = S_i^{l-SEQ}$ if l is even and $\bigcap_{m=l}^{\infty} S_i^{m-SEQ} = S_i^{(l+1)-SEQ}$ if l is odd. Therefore, $\bigcup_{l=1}^{\infty}\bigcap_{m=l}^{\infty} S_i^{m-SEQ} = \bigcup_{m=1}^{\infty} S_i^{2m-SEQ}$, as desired.

Second, Theorem 3.3.1 implies that $\bigcup_{m=l}^{\infty} S_i^{m-SEQ} = S_i^{l-SEQ}$ if l is odd and $\bigcup_{m=l}^{\infty} S_i^{m-SEQ} = S_i^{(l+1)-SEQ}$ if l is even. Therefore, $\bigcap_{l=1}^{\infty}\bigcup_{m=l}^{\infty} S_i^{m-SEQ} = \bigcap_{m=1}^{\infty} S_i^{(2m-1)-SEQ}$, as desired. □

It is well-known that $S_i^{inf-SEQ} \subseteq S_i^{sup-SEQ}$. Theorem 3.3.2 shows that the reversed inclusion is not valid in the present context.

Theorem 3.3.2. There exist conflicts such that $S_i^{inf-SEQ} \neq S_i^{sup-SEQ}$.

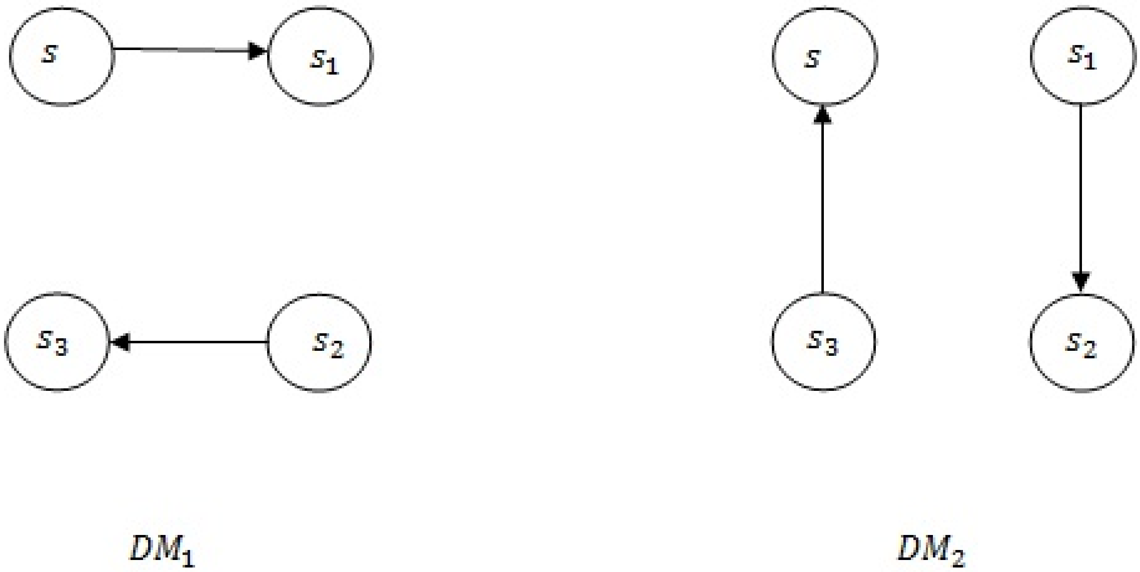
Proof: We show that result by means of an example of a conflict, where there exists a state s in $S_i^{sup-SEQ} - S_i^{inf-SEQ}$ for some DM i, i.e., $s \in S_i^{m-SEQ}$, for every odd m, and $s \notin S_i^{m-SEQ}$, for every even m, for some DM i. Consider a GMCR with 4 States s, s_1, s_2, s_3 and 2 DMs such that $R_1(s) = \{s_1\}$, $R_1(s_2) = \{s_3\}$, $R_2(s_1) = \{s_2\}$, $R_2(s_3) = \{s\}$, $s_1 \succ_1 s$, $s_3 \succ_1 s_2$, $s_2 \succ_2 s_1$ and $s \succ_2 s_3$. This conflict is illustrated in Figure 4.

Clearly, $s_1, s_3 \in S_1^{NASH}$ and $s, s_2 \in S_2^{NASH}$. We now show by induction that $s, s_2 \in (S_1^{sup-SEQ} - S_1^{inf-SEQ})$ and $s_1, s_3 \in (S_2^{sup-SEQ} - S_2^{inf-SEQ})$. It is easy to see that $s, s_2 \in S_1^{SEQ}$ (resp, $s_1, s_3 \in S_2^{SEQ}$) since from the unique unilateral improvement from such states for DM 1 (resp, DM 2) there exists a unilateral improvement for DM 2 (resp, DM 1) leading the conflict to a state no better than the initial one for DM 1 (resp, DM 2). Thus, it follows that $s, s_2 \notin S_1^{2-SEQ}$ and $s_1, s_3 \notin S_2^{2-SEQ}$.

For the induction hypothesis, assume that $s, s_2 \in (S_1^{(2m-1)-SEQ} - S_1^{2m-SEQ})$ and $s_1, s_3 \in (S_2^{(2m-1)-SEQ} - S_2^{2m-SEQ})$. Let us show that $s, s_2 \in (S_1^{(2m+1)-SEQ} - S_1^{(2m+2)-SEQ})$ and $s_1, s_3 \in (S_2^{(2m+1)-SEQ} - S_2^{(2m+2)-SEQ})$.

Since $s, s_2 \in S_1^{SEQ}$ and $s_1, s_3 \notin S_2^{2m-SEQ}$, it follows that $s, s_2 \in S_1^{(2m+1)-SEQ}$. Similarly, since $s_1, s_3 \in S_2^{SEQ}$ and $s, s_2 \notin S_1^{2m-SEQ}$, it follows that $s_1, s_3 \in S_2^{(2m+1)-SEQ}$. Since $s, s_2 \in S_1^{(2m+1)-SEQ}$, it follows that $s_1, s_3 \notin S_2^{(2m+2)-SEQ}$. Since $s_1, s_3 \in S_2^{(2m+1)-SEQ}$, it follows that $s, s_2 \notin S_1^{(2m+2)-SEQ}$. □

Figure 4 – Conflict with a State that is m-SEQ Stable Only for Odd m.



Source: This research (2018)

Theorem 3.3.3 shows that if transitivity of preferences and movements hold for both DMs, then there exists at least one state that is a $m-SEQ$ equilibrium for every odd m. Therefore, the notion of $sup-SEQ$ is a refinement of the notion of SEQ, where the existence result is still valid.

Theorem 3.3.3. If transitivity of preferences and movements hold, then the set $S_1^{sup-SEQ} \cap S_2^{sup-SEQ}$ is non-empty.

Proof: Suppose the theorem is false. Then for every state there exists at least one DM for which such state is $m-SEQ$ unstable for some m odd. Without loss of generality, assume that there are $m-SEQ$ unstable states for DM i for some m odd. By transitivity of preferences and since S is finite, there exists some state s_0 which is $m-SEQ$ unstable for DM i and there is no other $m-SEQ$ unstable state s for DM i such that $s \succ_i s_0$.

Since s_0 is $m-SEQ$ unstable for DM i, there exists $s_1 \in R_i^+(s_0)$ such that:

(a) $s_1 \in S_j^{(m-1)-SEQ}$ or

(b) for every $s_2 \in R_j^+(s_1), s_2 \succ_i s_0$.

Let us consider case (a) first. By definition of s_0, s_1, must be $m-SEQ$ stable for DM i and since m is odd $S_j^{(m-1)-SEQ} \subseteq S_j^{m-SEQ}$. Thus, s_1 is an $m-SEQ$ equilibrium, a contradiction.

Now consider case (b). Since s_1 is $m-SEQ$ stable for DM i ,it must be $m-SEQ$ unstable for DM j. Thus, let $s_2^* \in R_j^+(s_1)$ be such that there is no $s_3 \in R_j^+(s_1)$ such that

$s_3 \succ_j s_2^*$. The existence of s_2^* follows from transitivity of preferences. Since $s_2^* \in R_j^+(s_1)$, case (b) implies that $s_2^* \succ_i s_0$. Therefore, s_2^* is $m-SEQ$ stable for DM i. Thus, s_2^* is $m-SEQ$ unstable for DM j. Thus, there exists $s_3 \in R_j^+(s_2^*)$. Transitivity of movements and preferences implies that $s_3 \in R_j^+(s_1)$ and since $s_3 \succ_j s_2^*$, we have a contradiction to the definition of s_2^*. □

3.4 An alternative definition

An alternative way to overcome the problem of noncredible threats in the SEQ analysis is to use the following alternative definition for SEQ stability of order m, where the state in which the opponent of the focal DM moves is not $(m-1)-SEQ$ for the opponent, however it also is not $k-SEQ$ for the opponent for every $0<k<m$. To formalize that, in the following definition, assume that $S_i^{1-SEQ^2}$ is the set of all (second kind first order) SEQ states for DM i.

Definition 3.4.1. For an integer m such that $m \geq 2$, let $S_i^{m-SEQ^2}$ be the set of all states that satisfy second kind m-th order sequential stability for DM i, where state s is second kind m-th order sequentially stable for DM i if for every state $s_1 \in R_i^+(s)$, $s_1 \notin \cup_{k=1}^{m-1} S_j^{k-SEQ^2}$ and there exists $s_2 \in R_j^+(s_1)$ such that $s \succsim_i s_2$.

Theorem 3.4.1 shows that this alternative definition is a particular case of the original definition of m-SEQ stability.

Theorem 3.4.1. Let $i \in N$. The set of second kind m-th order sequentially stable states for DM i is equal to the set of 2-nd order sequentially stable states for DM i, for every $m \geq 2$, i.e., $S_i^{m-SEQ^2} = S_i^{2-SEQ}$, for every $m \geq 2$.

Proof: We show the result by induction on m. Since by definition, $\forall i \in N$, $S_i^{1-SEQ^2} = S_i^{1-SEQ} = S_i^{SEQ}$, it follows that $S_i^{2-SEQ^2} = S_i^{2-SEQ}$, $\forall i \in N$. Therefore, by Theorem 3.3.1, $S_i^{2-SEQ^2} \subseteq S_i^{1-SEQ^2}$, $\forall i \in N$.

Assume for the induction hypothesis that for all $i \in N$, $S_i^{m-SEQ^2} = S_i^{2-SEQ}$, for $m \geq 2$. Thus, $\cup_{k=1}^{m} S_j^{k-SEQ^2} = S_j^{1-SEQ}$, which implies that $S_i^{(m+1)-SEQ^2} = S_i^{2-SEQ}$. □

Given this result, from now on, we focus only on the original definition of m-SEQ stability.

3.5 Relationships among Higher-order Sequential Stabilities and other solution concepts in the GMCR

In this section the relationships between Higher-order SEQs and other GMCR solution concepts are studied.

Theorem 3.5.1. The following relations between Higher-order SEQs and standard stability notions of the GMCR hold:

(a) For $i \in N$, if $s \in S_i^{NASH}$, then $s \in S_i^{m-SEQ}$, for all $m \geq 1$.

(b) For $i \in N$, if $s \in S_i^{m-SEQ}$ for some $m \geq 1$, then $s \in S_i^{SEQ}$.

(c) For $i \in N$, if $s \in S_i^{m-SEQ}$ for some $m \geq 1$, then $s \in S_i^{GMR}$.

(d) For $i \in N$, $S_i^{SMR} \not\subseteq S_i^{m-SEQ}$, for all $m \geq 1$.

(e) For $i \in N$, $S_i^{SSEQ} \not\subseteq S_i^{m-SEQ}$, for all $m \geq 2$.

(f) For $i \in N$, $S_i^{m-SEQ} \not\subseteq S_i^{SMR}$, for all $m \geq 1$.

(g) For $i \in N$, $S_i^{m-SEQ} \not\subseteq S_i^{SSEQ}$, for all $m \geq 1$.

Proof:

For part (a), if $s \in S_i^{NASH}$, then $R_i^+(s) = \emptyset$. Therefore, it follows that $s \in S_i^{m-SEQ}$, for all $m \geq 1$.

For part (b), since by definition $S_i^{1-SEQ} = S_i^{SEQ}$, it follows from Theorem 3.3.1 that $S_i^{m-SEQ} \subseteq S_i^{SEQ}$, for all $m \geq 1$.

For part (c), since SEQ implies GMR, the result follows from part (b).

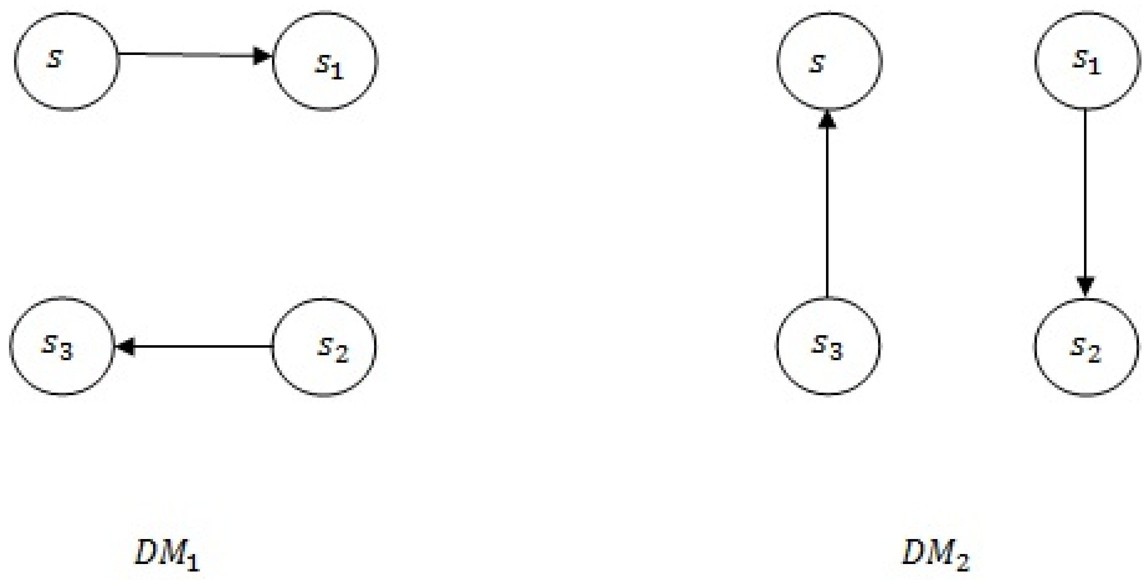
For part (d), since, by part (b), $m-SEQ$ implies SEQ and since SMR does not imply SEQ, it follows that SMR does not imply $m-SEQ$.

For part (e), by Theorem 3.3.1, it is enough to show that $S_i^{SSEQ} \not\subseteq S_i^{3-SEQ}$. For that, consider a GMCR with 2 DMs and 4 states s, s_1, s_2 and s_3 such that $R_1(s) = \{s_1\}$, $R_1(s_2) = \{s_3\}$, $R_2(s_1) = \{s_2\}$, $R_2(s_3) = \{s\}$, $s_1 \succ_1 s$, $s_3 \succ_1 s_2$, $s \succ_2 s_1$ and $s_2 \succ_2 s_1$. Figure 5 illustrates this conflict.

Note that s is $SSEQ$ stable for DM 1, since from s the unique unilateral improvement for DM 1 is s_1 and from s_1 DM 2 has a unilateral improvement s_2 which is no better than s for DM 1 and from s_2 DM 1 can only reach s_3 which is no better than s.

However, we claim that s is not $3-SEQ$ stable for DM 1. In order to verify that, note first that s_2 is not SEQ stable for DM 1, since from s_2 DM 1 has a unilateral improvement s_3 and from s_3, DM 2 does not have any unilateral improvement move. Second note that s_1, the unique unilateral improvement for DM 1 from s, is 2-SEQ stable for DM 2, since from s_1 DM 2 can move to s_2, which is not SEQ stable for DM 1, and from s_2 DM 1 has a unilateral improvement s_3, which is no better than s_1 for DM 2. Therefore, s is not $3-SEQ$ stable for DM 1.

Figure 5 – Conflict with a State that is m-SEQ Stable Only for Odd m.



Source: This research (2018)

For part (f), by Theorem 3.3.1, it is enough to show that $S_i^{2-SEQ} \not\subseteq S_i^{SMR}$. For that, consider the example that was used in proof of letter (e). We showed that s_1 is $2-SEQ$ stable for DM 2.

To show that s_1 is SMR unstable for DM 2, note that, from s_1, DM 2 has a unique unilateral improvement going to s_2, but DM 1 can punish DM 2 moving from s_2 to s_3. However, DM 2 can escape this punishment moving from s_3 to s, and s is preferable to s_1 for DM 2. Therefore, s_1 is not SMR stable for DM 2.

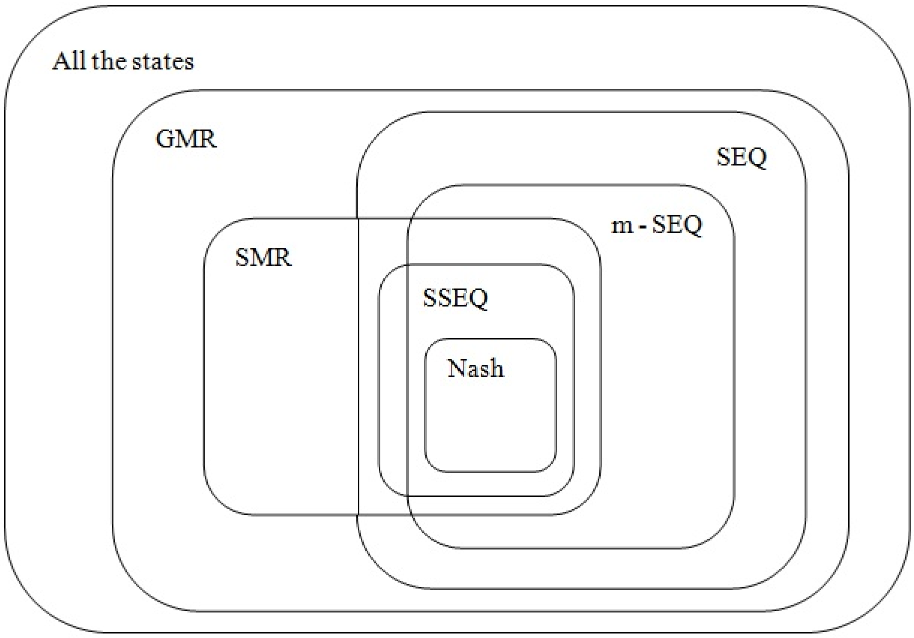
For part (g), note that it follows from part (f). Since $SSEQ$ implies SMR and by part (f), $m-SEQ$ does not imply SMR, it follows that $m-SEQ$ cannot imply $SSEQ$. □

Figure 6 illustrates the relationships described in Theorem 3.5.1.

3.6 Applications

In this section, we present three applications to illustrate the usefulness of the m-SEQ stability concept. The first two applications are the classical game theory examples Matching Pennies and Rock, Scissors and Paper. The third application is the representation of a conflict that describes the confrontation of values between a Global Market-Driven economy and a Sustainable Ecosystem philosophy that was first modeled using the GMCR by Hipel & Obeidi in 2005.

Figure 6 – Relationships described in Theorem 3.5.1.



Source: This research (2018)

3.6.1 Matching Pennies

A zero-sum game is one in which one player's gain is the other's loss. A classic example of a zero-sum game is the Matching Pennies game (GIBBONS, 1992). It has the property that it does not have a Nash equilibrium in pure strategies, since in every possible situation, one of the players has an incentive for changing strategies.

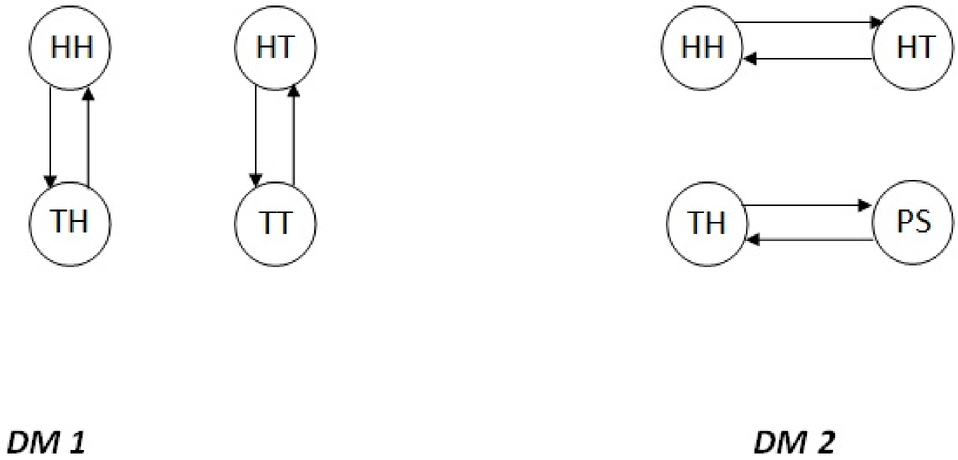
The Matching Pennies game involves two players who move simultaneously. Suppose that each player has a penny and must choose whether to display it with heads(H) or tails(T) facing up. If the pennies match, the first player wins and gets the other's penny, but if the pennies do not match, the second player wins and gets the penny.

Thus, a conflict described the Matching Pennies game has four states: HH, TT, TH and HT. Figure 7 illustrates this conflict using the GMCR. The preferences of the DMs are given by $HH \sim_1 TT \succ_1 HT \sim_1 TH$ and $HT \sim_2 TH \succ_2 HH \sim_2 TT$.

In Table 3, the stable states for each DM are represented, according to the most usual definitions of stability and also according to $m-SEQ$. Each cell in the array specifies for which DMs, if any, the column state is stable according to the stability definition of the corresponding line.

We can observe in Table 3 that the all the states are SEQ equilibrium, but this result is only true for $m-SEQ$, if m is odd.

Figure 7 – Possibility of movement's decision in the game Matching Pennies.



Source: This research (2018)

Table 3 – Stable states according to some stability definitions Matching Pennies game

Stability Definition	HH	HT	TH	TT
Nash	1	2	2	1
GMR	1,2	1,2	1,2	1,2
SMR	1	2	2	1
SEQ	1,2	1,2	1,2	1,2
SSEQ	1	2	2	1
Odd m-SEQ	1,2	1,2	1,2	1,2
Even m-SEQ	1	2	2	1

Source: This research (2018)

3.6.2 Rock, Paper and Scissors

The Rock, Paper, Scissors game is a classic of Game Theory and is similar to the Matching Pennies game in the sense that it is also a zero-sum game and it does not have any Nash equilibrium in pure strategies (OSBORNE, 2004).

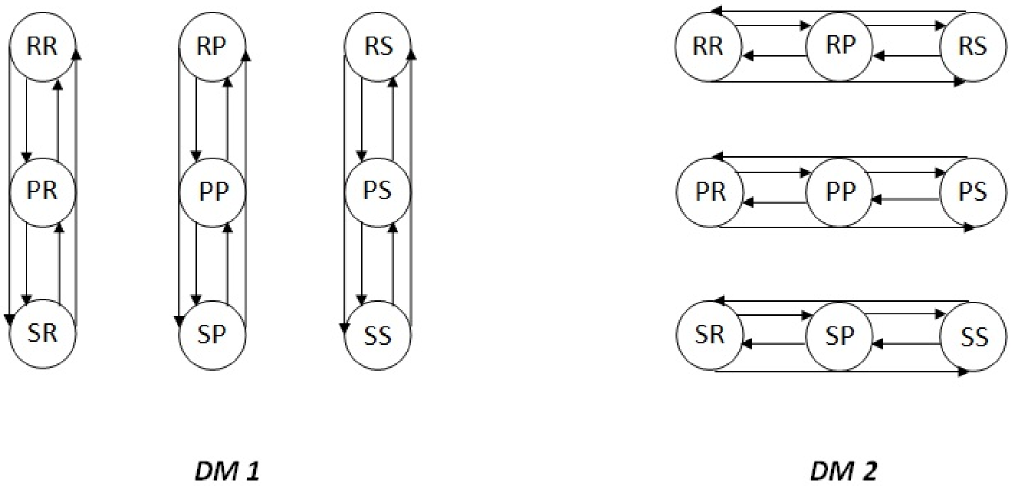
Rock, Paper, Scissors game is usually played between two decision makers, where each player's strategy space is $\{R, P, S\}$. Each DM simultaneously forms one of three shapes with an extended hand. These shapes are Rock (closed hand), Paper (open hand) and Scissors (two open fingers forming a V).

A DM who decides to play Rock will beats another DM who has chosen Scissors ("rock crushes scissors"), but will lose to one who has played Paper ("paper covers rock"); a play of Paper will lose to a play of Scissors ("scissors cut paper"). If both DMs choose the same shape, the game is tied and is usually immediately replayed to break the tie.

Let us analyze a conflict that is described by the game Rock, Paper, Scissors using the GMCR. It has nine possible states which describe the actions chosen by the two

DMs involved. Figure 8 illustrate this conflict. The preferences of the DMs are given by $RS \sim_1 PR \sim_1 SP \succ_1 RR \sim_1 PP \sim_1 SS \succ_1 SR \sim_1 RP \sim_1 PS$ and $SR \sim_2 RP \sim_2 PS \succ_2 RR \sim_2 PP \sim_2 SS \succ_2 RS \sim_2 PR \sim_2 SP$. Table 4 presents the results of the stability analysis of the Rock, Paper and Scissors game.

Figure 8 – Possibility of movement's decision in the game Rock, Paper and Scissors game



Source: This research (2018)

TABLE 4 – Stable states according to some stability definitions in Rock, Paper, Scissors game

Stability Definition	RR	PR	SR	RP	PP	SP	RS	PS	SS
Nash		1	2	2		1	1	2	
GMR	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2
SMR		1	2	2		1	1	2	
SEQ	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2
SSEQ		1	2	2		1	1	2	
Odd m-SEQ	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2
Even m-SEQ		1	2	2		1	1	2	

Source: This research (2018)

We can observe in Table 4 that again all states are SEQ equilibrium, but this result is only true for m-SEQ if m is odd.

3.6.3 Conflict of Values

Hipel & Obeidi in 2005 studied using the GMCR a conflict generated by the confrontation of values of a Global Market-Driven Economy (GMDE) and a Sustainable Ecosystem (SES) philosophy. They showed that the environment and social patterns will continue to deteriorate if the entrenched positions and related value systems of both fields persist.

The game is represented by two DMs, GMDE and SES. The GMDE has three options: Influence states to adopt market-oriented economic policies; Promote the ideals of globalization and internationalization through a media that stresses the efficiency and prosperity of societies that are part of global free trade agreements; and Reform the WTO so that the environment is treated as a public trust and not as a commodity. SES has three main options: Foster public education that promotes environmental integrity and social responsibility, as well as warns of the dangers of succumbing to values of GMDE; Lobby governments to incorporate environmental, ecosystem and other social concerns into free trade agreements; and Put pressure on trade negotiators to consider more societal concerns on their agenda.

Figure 9 presents the two DMs in the Conflict of Values model followed by the options under their control. A "Y" opposite an option indicates "Yes", the option is selected by the DM controlling it, while "N" corresponds to "No", the option is not taken. It is possible to note that the GMDE is influencing governments and promoting the ideals of free trade agreements, but not considering reforms. The SES is doing everything it can: educating the public, lobbying governments and pressuring negotiators by demonstrating whenever they conduct a meeting. Tension between the two opponents is escalating.

Figure 9 – Decision Makers and Options in the Conflict of Values.

Decision Makers	Options	Status Quo State	
GMDE			
	1. Influence	Y	GMDE's Strategy
	2. Promote	Y	
	3. Reform	N	
SES			
	4. Educate	Y	SES's Strategy
	5. Lobby	Y	
	6. Pressure	Y	

Source: (HIPEL; OBEIDI, 2005)

In this game there are mathematically 64 (2^6) possible states in the Conflict of Values model shown in Figure 9. In the modeling stage, unviable state removals are performed to obtain the list of feasible states. In the Conflict of Values model, there are two reasons for infeasibility. Some options are mutually exclusive and cannot be selected at the same time. Therefore, the GMDE will not simultaneously select reform and influence options, nor will it choose to reform and promote together. In addition, there are infeasible states whenever the three options influence, promote, and reform are taken. It may also occur that, if GMDE chooses to reform, SES will not pressure trade negotiators. Figure 10 illustrates the remaining 36 feasible states in the conflict, where state 36 corresponds to the status quo state.

Figure 10 – Feasible States in the Conflict of Values.

DMs	Options	States																	
GMDE		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	1. Influence	N	Y	N	Y	N	N	Y	N	Y	N	N	Y	N	Y	N	N	Y	N
	2. Promote	N	N	Y	Y	N	N	N	Y	Y	N	N	N	Y	Y	N	N	N	Y
	3. Reform	N	N	N	N	Y	N	N	N	N	Y	N	N	N	N	Y	N	N	N
SES																			
	4. Educate	N	N	N	N	N	Y	Y	Y	Y	Y	N	N	N	N	N	Y	Y	Y
	5. Lobby	N	N	N	N	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y	Y	Y
	6. Pressure	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
		States																	
GMDE		19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
	1. Influence	Y	N	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y	N	Y
	2. Promote	Y	N	N	N	Y	Y	N	N	Y	Y	N	N	Y	Y	N	N	Y	Y
	3. Reform	N	Y	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
SES																			
	4. Educate	Y	Y	N	N	N	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y	Y
	5. Lobby	Y	Y	N	N	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y	Y	Y
	6. Pressure	N	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

Source: (HIPEL; OBEIDI, 2005)

The preferences of each DM are presented in Figure 11 referring to the most preferred to least preferred states. States in parentheses are equally preferred states for a DM and higher states in a column are more preferred than those lower. For example, the GMDE equally prefers states in the first set: $(3, 4)$, but the states in this set are more preferred than states in the second set: $(13, 14)$, and so on.

Figure 11 – Preference Ranking of States in the Conflict of Values.

GMDE		SES
(3,4)	More preferred ↑	(10,20)
(13,14)		(5,15)
(8,9)		(16,18)
(18,19)		(33,35)
(1,2)		(6,8)
(11,12)		(25,27)
(6,7)		(11,13)
(16,17)		(29,31)
(23,24)		(1,3)
(31,32)		(21,23)
(27,28)		(17,19,34,36)
(35,36)		(7,9,26,28)
(21,22)		(12,14,30,32)
(29,30)		(2,4,22,24)
(25,26)		
(33,34)		
(5,10,15,20)	Less preferred ↓	

Source: (HIPEL; OBEIDI, 2005)

Analyzing this conflict, we can verify that the use of SEQ implies the existence of a noncredible punishment in the analysis. In order to see that, note that from state 25, the

DM SES can move to states $1, 6, 11, 16, 21, 29$, and 33. Of these states only states $6, 16$ and 33 are preferable to state 25 by DM SES, as can be seen by the Ranking of Preferences. However analyzing these 3 states we see that:

- From state 6, DM GMDE can improve going to state 9 which is not preferable to state 25 for DM SES;
- From state 16, DM GMDE can improve by going to state 19 which is not preferable to state 25 for DM SES;
- From state 33, DM GMDE can improve by going to state 36 which is not preferable to state 25 for DM SES.

Therefore, state 25 is SEQ for DM SES. Let us now verify that state 25 is not 2-SEQ for DM SES, by showing that state 6 is SEQ for DM GMDE, which implies that the punishment that DM GMDE would enforce in state 6 is noncredible. To see that state 6 is SEQ stable for DM GMDE, note that from 6 DM GMDE can only move to states 7, 8, 9 and 10. Of these states only states 8 and 9 are preferable to state 6 by DM GMDE. However analyzing these 2 states we see that:

- From state 8, DM SES can improve going to state 35 which is not preferable to state 6 for DM GMDE;
- From state 9, DM SES can improve by going to state 36 which is not preferable to state 6 for DM GMDE;

In the next chapter we present the definitions of Higher-order Sequential Stabilities for a DM for conflicts with $n-DMs$, provide some of its properties, and the definition of Higher-order Coalitional Sequential Stability and study its properties.

4 HIGHER-ORDER SEQUENTIAL STABILITIES FOR MULTILATERAL CONFLICTS

Many important conflict situations involve multiple parties. In such cases, where there are more than 2-DMs in the conflict, while analyzing the stabilities for a focal DM, he must foresee the sanctions that the other DMs may together achieve through a sequence of movements. This sequence of movements of the opponents is usually restrained so that in the sequence a given DM may even move more than once, but not twice consecutively, that is, the DMs should alternate as they move.

As in the case of a bilateral conflict, the movements in the sanctions considered, while analyzing the SEQ stability of a given state for a focal DM, must be unilateral improvement moves. However, there is nothing in the definition of SEQ stability that prevents the DMs from moving away from states which are SEQ stable for them, what makes some of the sanctions noncredible.

Our purpose in this chapter, is to extend the results of the previous chapter for multilateral conflicts. First, we define what is a $m-th$ order credible legal sequence of unilateral improvements for a coalition of DMs. Then, we define the notion of $m-th$ order sequential stability. Our next step, is to derive the properties of this new notion proposed relating it to other commonly used stability notions used in the GMCR with multiple DMs. We also introduce the concept of Higher-order Sequential Stability in coalitional analysis. Finally, we illustrate the usefulness of the proposed model through two real-world conflict situations.

4.1 Higher-order Sequential Stabilities for n DMs

Our first step in the analysis of Higher-order Sequential Stabilities for n DMs is to identify what are the possible credible sanctions that the opponents of a focal DM may impose on him or her. By credibility, we mean that DMs should not leave states which are sequentially stable for them. In order to avoid a circular definition, we need to define various orders of sequential stability, as in the case of bilateral conflicts.

For a coalition $H \subseteq N$ and $m \geq 2$, let $R_H^{+m}(s)$ be the set of states achievable by the DMs in H through a m-order credible legal sequence of unilateral improvements from state s, where the sequence of unilateral improvements is legal if DMs cannot move twice consecutively in the sequence and it is m-order credible if no DM leaves a state which $(m-1)$-SEQ stable for him or her.

In what follows, we assume that $R_H^{+1}(s) = R_H^{+}(s)$ and that $\Omega_H^{+1}(\cdot) = \Omega_H^{+}(\cdot)$. Thus, a first order credible move means that no DM leaves a Nash stable state to sanction another DM. For $m \geq 2$, let us define $R_H^{+m}(s)$ formally by induction. Note that, for $m \geq 2$, the definition of $R_H^{+m}(s)$ depends on the definition of $S_i^{(m-1)-SEQ}$. Thus, to complete the induction process, we define S_i^{m-SEQ} in terms of $R_H^{+m}(s)$ for $m \geq 1$, as follows:

Definition 4.1.1. For an integer m such that $m \geq 1$, let S_i^{m-SEQ} be the set of all states satisfying the sequential stability of m order for DM i, where the state s is $m-$th order sequentially stable for DM i if for each state s_1 belonging to $R_i^{+}(s)$, there exists s_2 belonging to $R_{N-i}^{+m}(s_1)$ such that s_2 is not preferable to s for DM i.

Thus, since $R_H^{+1}(s) = R_H^{+}(s)$, it follows that $S_i^{1-SEQ} = S_i^{SEQ}$, as in the case of bilateral conflicts. Moreover, the intuition for a state s to be 2-SEQ stable, for DM i, is that for every possible unilateral improvement move that DM i has, s_1, the opponents may achieve through a legal sequence of unilateral improvements, where no DM leaves a SEQ stable state for himself, a state s_2 which is no better that s for DM i. Thus, the sequence that leads to a sanction is credible in the sense that no DM is moving away from a SEQ stable state. However, in such a sequence of movements the opponents may move away from a 2-SEQ stable state, leading to a non-credibility of higher order. Therefore, we propose sequentially stability for several orders so that in the movements made by DMs to sanction a focal DM, no DM leaves a SEQ stable state of the immediately lower order.

Thus, the various orders of sequential stability for a given DM i are given by the sequence of sets $(S_i^{m-SEQ})_{m=1}^{\infty}$. As in the case of bilateral conflicts, two limiting sets, known as *liminf* and *limsup*, of this sequence can be defined as follows:

$$S_i^{inf-SEQ} = \bigcup_{l=1}^{\infty} \bigcap_{m=l}^{\infty} S_i^{m-SEQ} \tag{4.1}$$

and

$$S_i^{sup-SEQ} = \bigcap_{l=1}^{\infty} \bigcup_{m=l}^{\infty} S_i^{m-SEQ}. \tag{4.2}$$

As before, $S_i^{inf-SEQ}$ is the set of states which belong to all, except finitely many, sets in the sequence $(S_i^{m-SEQ})_{m=1}^{\infty}$ and $S_i^{sup-SEQ}$ is the set of states which belong to infinitely many sets in the sequence $(S_i^{m-SEQ})_{m=1}^{\infty}$.

Next, we analyze which properties are satisfied by the proposed Higher-order Sequential Stability notions for multilateral conflicts.

4.2 Properties of Higher-order Sequential Stabilities for multilateral conflicts

Theorem 4.2.1 shows us how the sets of m-th order sequentially stable states, for different m values, relate to one another.

Theorem 4.2.1. Let m and m_1 be any positive integers. For $i \in N$, if m_1 is odd, then $R_H^{+m}(s) \subseteq R_H^{+m_1}(s)$ and $S_i^{m-SEQ} \subseteq S_i^{m_1-SEQ}$, for all $m \geq m_1$ and, otherwise, if m_1 is even, it follows that $R_H^{+m_1}(s) \subseteq R_H^{+m}(s)$ and $S_i^{m_1-SEQ} \subseteq S_i^{m-SEQ}$, for all $m \geq m_1$.

Proof: We prove the theorem by mathematical induction on m_1.

Let us consider first the case $m_1 = 1$. Then, we first show that $R_H^{+m}(s) \subseteq R_H^{+1}(s) = R_H^{+}(s)$ and that $\Omega_H^{+m}(s,\cdot) \subseteq \Omega_H^{+1}(s,\cdot) = \Omega_H^{+}(s,\cdot)$, for every $H \subseteq N$, $s \in S$ and $m \geq 1$. For $m = 1$, the result is trivially true. Thus, let us consider the case $m > 1$. We prove that by induction on the minimum number of moves made by DMs in the coalition H, l^*, that is necessary to reach a state $s_1 \in R_H^{+m}(s)$ by a m-order credible legal sequence of unilateral improvements. For $l^* = 1$, if $s_1 \in R_H^{+m}(s)$, then it follows that $s_1 \in R_i^{+}(s)$, for some $i \in H$, and s does not belong to $S_i^{(m-1)-SEQ}$ and $i \in \Omega_H^{+m}(s,s_1)$. Thus, we have that $s_1 \in R_H^{+}(s)$ and $i \in \Omega_H^{+}(s,s_1)$.

For the induction hypothesis, assume that for any $s_1 \in R_H^{+m}(s)$ which can be reached by a m-order credible legal sequence of unilateral improvements by DMs in H in at most l^*-1 steps, then $R_H^{+m}(s_1) \subseteq R_H^{+1}(s_1) = R_H^{+}(s_1)$ and that $\Omega_H^{+m}(s,s_1) \subseteq \Omega_H^{+1}(s,s_1) = \Omega_H^{+}(s,s_1)$. Thus, if $s_2 \in R_H^{+m}(s)$ and can be reached by a m-order credible legal sequence of unilateral improvements by DMs in H in exactly l^* steps, it follows that there exists $s_1 \in R_H^{+m}(s)$ which can be reached by a m-order credible legal sequence of unilateral improvements by DMs in H in exactly l^*-1 steps, $s_2 \in R_i^{+}(s_1)$, $i \in H$, s_1 does not belong to $S_i^{(m-1)-SEQ}$, $\{i\}$ is different from $\Omega^{+m}(s,s_1)$ and $i \in \Omega^{+m}(s,s_2)$. Therefore, by the induction hypothesis, we have that there exists $s_1 \in R_H^{+}(s)$ such that $s_2 \in R_i^{+}(s_1)$, $i \in H$ and $\{i\}$ is different from $\Omega^{+}(s,s_1)$, which implies that $s_2 \in R_H^{+1}(s)$ and $i \in \Omega^{+1}(s,s_2)$.

Therefore, if $s \in S_i^{m-SEQ}$, for $m \geq 1$, it follows that for all $s_1 \in R_i^{+}(s)$, there exists $s_2 \in R_{N-i}^{+m}(s_1)$ such that $s \succsim_i s_2$. Thus, since $R_{N-i}^{+m}(s_1) \subseteq R_{N-i}^{+1}(s_1)$, it follows that for all $s_1 \in R_i^{+}(s)$, there exists $s_2 \in R_{N-i}^{+1}(s_1)$ such that $s \succsim_i s_2$. Thus, $s \in S_i^{1-SEQ} = S_i^{SEQ}$.

For $m_1 = 2$, we first show that $R_H^{+2}(s) \subseteq R_H^{+m}(s)$ and that $\Omega_H^{+2}(s,\cdot) \subseteq \Omega_H^{+m}(s,\cdot)$, for every $H \subseteq N$, $s \in S$ and $m \geq 2$. We prove that by induction on the minimum number of moves made by DMs in the coalition H, l^*, that is necessary to reach a state $s_1 \in R_H^{+2}(s)$ by a 2-order credible legal sequence of unilateral improvements. For $l^* = 1$, if $s_1 \in R_H^{+2}(s)$, then it follows that $s_1 \in R_i^{+}(s)$, for some $i \in H$, and s does not belong to S_i^{SEQ} and $i \in \Omega_H^{+2}(s,s_1)$. By the case $m_1 = 1$, we have that $s_1 \in R_i^{+}(s)$, for some $i \in H$, and s does not belong to $S_i^{(m-1)-SEQ}$, for $m \geq 2$, which implies that $s_1 \in R_H^{+m}(s)$ and $i \in \Omega_H^{+m}(s,s_1)$.

For the induction hypothesis, assume that for any $s_1 \in R_H^{+2}(s)$ which can be reached by a 2-order credible legal sequence of unilateral improvements by DMs in H in at most l^*-1 steps, then $R_H^{+2}(s_1) \subseteq R_H^{+m}(s_1)$ and that $\Omega_H^{+2}(s,s_1) \subseteq \Omega_H^{+m}(s,s_1)$. Thus, if $s_2 \in R_H^{+2}(s)$ and can be reached by a 2-order credible legal sequence of unilateral improvements by DMs in H in exactly l^* steps, it follows that there exists $s_1 \in R_H^{+2}(s)$ which can be reached by a 2-order credible legal sequence of unilateral improvements by DMs in H in exactly l^*-1 steps, $s_2 \in R_i^+(s_1)$, $i \in H$, s_1 does not belong to S_i^{SEQ}, $\{i\}$ is different from $\Omega^{+2}(s,s_1)$ and $i \in \Omega^{+2}(s,s_2)$. Therefore, by the case $m_1 = 1$ and the induction hypothesis, we have that there exists $s_1 \in R_H^{+m}(s)$ such that $s_2 \in R_i^+(s_1)$, $i \in H$, s_1 does not belong to $S_i^{(m-1)-SEQ}$ and $\{i\}$ is different from $\Omega^{+m}(s,s_1)$, which implies that $s_2 \in R_H^{+m}(s)$ and $i \in \Omega^{+m}(s,s_2)$.

Therefore, if $s \in S_i^{2-SEQ}$, it follows that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-i}^{+2}(s_1)$ such that $s \succsim_i s_2$. Thus, since $R_{N-i}^{+2}(s_1) \subseteq R_{N-i}^{+m}(s_1)$, for $m \geq 2$, it follows that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-i}^{+m}(s_1)$ such that $s \succsim_i s_2$. Thus, $s \in S_i^{m-SEQ}$, for $m \geq 2$.

Now assume the following inductive hypothesis:

(I1) $\forall i \in N$, $R_H^{+m}(s) \subseteq R_H^{+(m_1-1)}(s)$ and $S_i^{m-SEQ} \subseteq S_i^{(m_1-1)-SEQ}$, for all $m \geq (m_1-1)$ and m_1-1 odd, and
(I2) $\forall i \in N$, $R_H^{+(m_1-1)}(s) \subseteq R_H^{+m}(s)$ and $S_i^{(m_1-1)-SEQ} \subseteq S_i^{m-SEQ}$, for all $m \geq (m_1-1)$ and m_1-1 even.

First, suppose that m_1 is odd. We now show that $R_H^{+m}(s) \subseteq R_H^{+m_1}(s)$ and that $\Omega_H^{+m}(s,\cdot) \subseteq \Omega_H^{+m_1}(s,\cdot)$, for every $H \subseteq N$, $s \in S$ and $m \geq m_1$. We prove that by induction on the minimum number of moves made by DMs in the coalition H, l^*, that is necessary to reach a state $s_1 \in R_H^{+m}(s)$ by a m-order credible legal sequence of unilateral improvements. For $l^* = 1$, if $s_1 \in R_H^{+m}(s)$, then it follows that $s_1 \in R_i^+(s)$, for some $i \in H$, and s does not belong to $S_i^{(m-1)-SEQ}$ and $i \in \Omega_H^{+m}(s,s_1)$. Thus, by (I2), we have that $s_1 \in R_i^+(s)$, for some $i \in H$, and s does not belong to $S_i^{(m_1-1)-SEQ}$, which implies that $s_1 \in R_H^{+m_1}(s)$ and $i \in \Omega_H^{+m_1}(s,s_1)$.

For the induction hypothesis, assume that for any $s_1 \in R_H^{+m}(s)$ which can be reached by a m-order credible legal sequence of unilateral improvements by DMs in H in at most l^*-1 steps, then $R_H^{+m}(s_1) \subseteq R_H^{+m_1}(s_1)$ and that $\Omega_H^{+m}(s,s_1) \subseteq \Omega_H^{+m_1}(s,s_1)$. Thus, if $s_2 \in R_H^{+m}(s)$ and can be reached by a m-order credible legal sequence of unilateral improvements by DMs in H in exactly l^* steps, it follows that there exists $s_1 \in R_H^{+m}(s)$ which can be reached by a m-order credible legal sequence of unilateral improvements by DMs in H in exactly l^*-1 steps, $s_2 \in R_i^+(s_1)$, $i \in H$, s_1 does not belong to $S_i^{(m-1)-SEQ}$, $\{i\}$ is different from $\Omega^{+m}(s,s_1)$ and $i \in \Omega^{+m}(s,s_2)$. Therefore, by (I2), we have that there exists $s_1 \in R_H^{+m_1}(s)$ such that $s_2 \in R_i^+(s_1)$, $i \in H$, s_1 does not belong to $S_i^{(m_1-1)-SEQ}$ and $\{i\}$ is different from $\Omega^{+m_1}(s,s_1)$, which implies that $s_2 \in R_H^{+m_1}(s)$ and $i \in \Omega^{+m_1}(s,s_2)$.

Therefore, if $s \in S_i^{m-SEQ}$, it follows that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-i}^{+m}(s_1)$ such that $s \succsim_i s_2$. Thus, since for $m \geq m_1$, $R_{N-i}^{+m}(s_1) \subseteq R_{N-i}^{+m_1}(s_1)$, it follows that for all

$s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-i}^{+m_1}(s_1)$ such that $s \succsim_i s_2$. Thus, $s \in S_i^{m_1-SEQ}$.

Finally, suppose that m_1 is even. We now show that $R_H^{+m_1}(s) \subseteq R_H^{+m}(s)$ and that $\Omega_H^{+m_1}(s,\cdot) \subseteq \Omega_H^{+m}(s,\cdot)$, for every $H \subseteq N$, $s \in S$ and $m \geq m_1$. We prove that by induction on the minimum number of moves made by DMs in the coalition H, l^*, that is necessary to reach a state $s_1 \in R_H^{+m_1}(s)$ by a m_1-order credible legal sequence of unilateral improvements. For $l^* = 1$, if $s_1 \in R_H^{+m_1}(s)$, then it follows that $s_1 \in R_i^+(s)$, for some $i \in H$, and s does not belong to $S_i^{(m_1-1)-SEQ}$ and $i \in \Omega_H^{+m_1}(s,s_1)$. By (I1), we have that $s_1 \in R_i^+(s)$, for some $i \in H$, and s does not belong to $S_i^{(m-1)-SEQ}$, for $m \geq m_1$, which implies that $s_1 \in R_H^{+m}(s)$ and $i \in \Omega_H^{+m}(s,s_1)$.

For the induction hypothesis, assume that for any $s_1 \in R_H^{+m_1}(s)$ which can be reached by a m_1-order credible legal sequence of unilateral improvements by DMs in H in at most l^*-1 steps, then $R_H^{+m_1}(s_1) \subseteq R_H^{+m}(s_1)$ and that $\Omega_H^{+m_1}(s,s_1) \subseteq \Omega_H^{+m}(s,s_1)$. Thus, if $s_2 \in R_H^{+m_1}(s)$ and can be reached by a m_1-order credible legal sequence of unilateral improvements by DMs in H in exactly l^* steps, it follows that there exists $s_1 \in R_H^{+m_1}(s)$ which can be reached by a m_1-order credible legal sequence of unilateral improvements by DMs in H in exactly l^*-1 steps, $s_2 \in R_i^+(s_1)$, $i \in H$, s_1 does not belong to $S_i^{(m_1-1)-SEQ}$, $\{i\}$ is different from $\Omega^{+m_1}(s,s_1)$ and $i \in \Omega^{+m_1}(s,s_2)$. Therefore, by (I1) and the induction hypothesis (on l^*), we have that there exists $s_1 \in R_H^{+m}(s)$ such that $s_2 \in R_i^+(s_1)$, $i \in H$, s_1 does not belong to $S_i^{(m-1)-SEQ}$ and $\{i\}$ is different from $\Omega^{+m}(s,s_1)$, which implies that $s_2 \in R_H^{+m}(s)$ and $i \in \Omega^{+m}(s,s_2)$.

Therefore, if $s \in S_i^{m_1-SEQ}$, it follows that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-i}^{+m_1}(s_1)$ such that $s \succsim_i s_2$. Thus, since for $m \geq m_1$, $R_{N-i}^{+m_1}(s_1) \subseteq R_{N-i}^{+m}(s_1)$, it follows that for all $s_1 \in R_i^+(s)$, there exists $s_2 \in R_{N-i}^{+m}(s_1)$ such that $s \succsim_i s_2$. Thus, $s \in S_i^{m-SEQ}$, for $m \geq m_1$. □

Figure 3 still illustrates the result of this theorem.

The next result shows that $S_i^{inf-SEQ}$ consists of states which are Higher-order SEQ for DM i for at least one even order and that $S_i^{sup-SEQ}$ consists of the states which are Higher-order SEQ for DM i for every odd order.

Corollary 4.2.1.1. $S_i^{inf-SEQ} = \bigcup_{m=1}^{\infty} S_i^{2m-SEQ}$ and $S_i^{sup-SEQ} = \bigcap_{m=1}^{\infty} S_i^{(2m-1)-SEQ}$.

Proof: Using the result of Theorem 4.2.1, the proof of Corollary 4.2.1.1 is identical to the proof of Corollary 3.3.1.1. □

Theorem 4.2.2 shows that the set of $m-SEQ$ stable states, for m odd, is non-empty, if preferences, $\succ_i$, and movements, R_i, are transitive, for every DM i. It is easy to see that if transitivity of preferences and movements holds, then for every coalition $H \subseteq N$, if $s_1 \in R_H^{+m}(s)$, then $R_H^{+m}(s_1) \subseteq R_H^{+m}(s)$.

Theorem 4.2.2. If transitivity of movements and transitivity of preferences holds, then $S^{m-SEQ} \neq \emptyset$, for every m odd.

Proof: Suppose that Theorem 4.2.2 is false. Then, there exists some m odd such that every state is not $m-SEQ$ for at least one DM $i \in N$. Suppose that there is some state $m-SEQ$ unstable for DM i_1 and let s_0 be a state such that there is not other not $m-SEQ$ unstable state for DM i_1 that is preferable to s_0 for DM i_1.

Then, there exists $s_1 \in R^{+}_{i_1}(s_0)$, such that for all $s_2 \in R^{+m}_{N-\{i_1\}}(s_1)$, $s_2 \succ_{i_1} s_0$. Since, $s_1 \succ_{i_1} s_0$, by definition of s_0, we have that s_1 is $m-SEQ$ for DM i_1 and therefore is $m-SEQ$ unstable for some DM i_2, $i_2 \in N-\{i_1\}$. Thus, by definition of s_1, we have that if $s \in R^{+m}_{N-\{i_1\}}(s_1) \cup \{s_1\}$, then $s \succ_{i_1} s_0$, and, consequently, is $m-SEQ$ stable for DM i_1.

We now show by induction the following fact. For $k \geq 1$, there are distinct DMs $i_1, i_2, \ldots, i_{k+1}$ and a state s_k such that s_k is not $m-SEQ$ for DM i_{k+1} and if $s \in R^{+m}_{N-\{i_1,i_2,\ldots,i_k\}}$ $(s_k) \cup \{s_k\}$, then s is $m-SEQ$ for DMs $i_1, i_2, \ldots, i_k$. Since the set of DMs is finite and the results holds for every k, we have a contradiction. For $k=1$, we have already shown the fact. Now, suppose that the fact is true for some k and let us show that it is also true for $k+1$. Let t_{k+1} denote a state in $R^{+m}_{N-\{i_1,i_2,\ldots,i_k\}}(s_k) \cup \{s_k\}$ that is $m-SEQ$ unstable for DM i_{k+1} such that there is no other t state in $R^{+m}_{N-\{i_1,i_2,\ldots,i_k\}}(s_k) \cup \{s_k\}$ which is not $m-SEQ$ for DM i_{k+1}, such that $t \succ_{i_{k+1}} t_{k+1}$. The existence of t_{k+1} is guaranteed by the fact that s_k is not $m-SEQ$ for DM i_{k+1} and that $\succ_{i_{k+1}}$ is transitive.

By definition of $m-SEQ$, there exists $s_{k+1} \in R^{+}_{i_{k+1}}(t_{k+1})$ such that for every $s \in R^{+m}_{N-\{i_{k+1}\}}(s_{k+1})$, we have that $s \succ_{i_{k+1}} t_{k+1}$.

By transitivity of preferences and movements and the fact that t_{k+1} is not $m-SEQ$ for DM i_{k+1}, $s_{k+1} \in R^{+m}_{N-\{i_1,i_2,\ldots,i_k\}}(s_k)$. Since $s_{k+1} \succ_{i_{k+1}} t_{k+1}$, s_{k+1} is $m-SEQ$ for DM i_{k+1} and, since, $s_{k+1} \in R^{+}_{N-\{i_1,i_2,\ldots,i_k\}}(s_k)$, by the induction hypothesis, it is $m-SEQ$ for DMs $i_1, i_2, \ldots, i_k$. If s_{k+1} is $m-SEQ$ for DMs in $N-\{i_1,i_2,\ldots,i_{k+1}\}$, the proof is finished, since it would be a $m-SEQ$ equilibrium. Otherwise, there is some DM $i_{k+2} \in N-\{i_1,i_2,\ldots,i_{k+1}\}$ such that s_{k+1} is not $m-SEQ$ for DM i_{k+2}. It remains to show that if $s \in R^{+m}_{N-\{i_1,i_2,\ldots,i_{k+1}\}}(s_{k+1}) \cup \{s_{k+1}\}$, then s is $m-SEQ$ for DMs $i_1, i_2, \ldots, i_{k+1}$. We already showed that if $s = s_{k+1}$, then this is true. Now observe that

$$R^{+m}_{N-\{i_1,i_2,\ldots,i_{k+1}\}}(s_{k+1}) \subseteq R^{+m}_{N-\{i_1,i_2,\ldots,i_k\}}(s_{k+1}) \subseteq R^{+m}_{N-\{i_1,i_2,\ldots,i_k\}}(s_k).$$

Therefore, by the induction hypothesis, if $s \in R^{+m}_{N-\{i_1,i_2,\ldots,i_{k+1}\}}(s_{k+1})$, then $s \in R^{+m}_{N-\{i_1,i_2,\ldots,i_k\}}(s_k)$ and s is $m-SEQ$ for DMs $i_1, i_2, \ldots, i_k$. It remains to show that s is $m-SEQ$ for DM i_{k+1}. Since $R^{+m}_{N-\{i_1,i_2,\ldots,i_{k+1}\}}(s_{k+1}) \subseteq R^{+m}_{N-\{i_{k+1}\}}(s_{k+1})$, it follows that $s \succ_{i_{k+1}} t_{k+1}$, which implies, by definition of t_{k+1}, that s is $m-SEQ$ stable for DM i_{k+1}. □

Theorem 4.2.3 shows that the definition proposed in this chapter is a generalization of that proposed in Chapter 3.

Theorem 4.2.3. If $n=2$, them s is m-th order sequentially stable for DM i according to the definition for multilateral conflicts if and only if, it is m-th order sequentially stable for DM i according to the definition for bilateral conflicts.

Proof: If $n=2$, them $N-i=\{j\}$. Moreover, $s_2 \in R^{+m}_{\{j\}}(s_1)$ if and only if $s_1 \notin S_j^{(m-1)-SEQ}$ and $s_2 \in R_j^{+}(s_1)$. Therefore, both definitions coincide. □

4.3 An alternative definition

An alternative way to overcome the problem of noncredible threats in the SEQ analysis is to use the following alternative definition for SEQ of order m, where the state in which the opponent of the focal DM moves is not only when is not $(m-1)$-SEQ for the opponent, but it is also when is not k-SEQ for the opponent for every $0<k<m$. To formalize that, consider the following definition:

Definition 4.3.1. For an integer m such that $m \geq 2$, let $S_i^{m-SEQ^2}$ be the set of all states that satisfy second kind m-th order sequential stability for DM i, where state s is second kind m-th order sequentially stable for DM i if for every state $s_1 \in R_i^{+}(s)$, there exists $s_2 \in \cap_{k=1}^{m} R^{+k}_{N-i}(s_1)$ such that $s \succsim_i s_2$.

Theorem 4.3.1 shows that this alternative definition is a particular case of the original definition of m-SEQ stability.

Theorem 4.3.1. Let $i \in N$. The set of second kind m-th order sequentially stable states for DM i is equal to the set of 2-nd order sequentially stable states for DM i, for every $m \geq 2$, i.e., $S_i^{m-SEQ^2} = S_i^{2-SEQ}$, for every $m \geq 2$.

Proof:

By Theorem 4.2.1, we have that for any $m \geq 2$, $\cap_{k=1}^{m} R^{+k}_{N-i}(s_1) = R^{+2}_{N-i}(s_1)$. Therefore, $S_i^{m-SEQ^2} = S_i^{2-SEQ}$, as desired. □

Finally, based on this result, from now on, we are only interested on the original definition of $m-SEQ$ stability for multiple DMs.

4.4 Relationships among Higher-order Sequential Stabilities and other solution concepts in the GMCR for multiple DMs

In this section the relationships between higher-order SEQ and other GMCR solution concepts for conflicts involving multiple DMs are studied.

Theorem 4.4.1. The following relations between Higher-order SEQ and standard stability notions of the GMCR hold:

(a) For $i \in N$, if $s \in S_i^{NASH}$, then $s \in S_i^{m-SEQ}$, for all $m \geq 1$.

(b) For $i \in N$, if $s \in S_i^{m-SEQ}$ for some $m \geq 1$, then $s \in S_i^{SEQ}$.

(c) For $i \in N$, if $s \in S_i^{m-SEQ}$ for some $m \geq 1$, then $s \in S_i^{GMR}$.

(d) For $i \in N$, $S_i^{SMR} \not\subseteq S_i^{m-SEQ}$, for all $m \geq 1$.

(e) For $i \in N$, $S_i^{SSEQ} \not\subseteq S_i^{m-SEQ}$, for all $m \geq 2$.

(f) For $i \in N$, $S_i^{m-SEQ} \not\subseteq S_i^{SMR}$, for all $m \geq 1$.

(g) For $i \in N$, $S_i^{m-SEQ} \not\subseteq S_i^{SSEQ}$, for all $m \geq 1$.

Proof: Identical to the proof of Theorem 3.5.1. □

Figure 6 still illustrates the relationships described in Theorem 4.4.1 for conflicts with multiple DMs.

4.5 Coalitional Analysis

In this section, we introduce the concept of Higher-order Sequential Equilibria in coalitional analysis for the Graph Model for Conflict Resolution. Our first step is to identify what are the possible credible sanctions that the opponents of a focal coalition may impose on it. By credibility, we mean that coalitions should not leave states which are sequentially stable for them. In order to avoid a circular definition, we need to define various orders of coalitional sequential stability, as in the case of bilateral and multilateral conflicts.

For a coalition $C \in \varphi(N)$ and $m \geq 2$, let $R_C^{++m}(s)$ be the set of states achievable by the coalitions in C through a m-order credible legal sequence of unilateral improvements from state s, where the sequence of unilateral improvements is legal if coalitions cannot move twice consecutively in the sequence and it is m-order credible if no coalition leaves a state which $(m-1)$-SEQ stable for it.

In what follows, we assume that $R_C^{++1}(s) = R_C^{++}(s)$ and that $\Omega_C^{++1}(\cdot) = \Omega_C^{++}(\cdot)$. Thus, a first order credible move means that no coalition leaves a coalitional Nash stable state to sanction another coalition. For $m \geq 2$, let us define $R_C^{++m}(s)$ formally by induction.

Note that, for $m \geq 2$, the definition of $R_C^{++m}(s)$ depends on the definition of $S_H^{(m-1)-SEQ}$. Thus, to complete the induction process, we define S_H^{m-SEQ} in terms of $R_C^{++m}(s)$ for $m \geq 1$, as follows:

Definition 4.5.1. For an integer m such that $m \geq 1$, let S_H^{m-SEQ} be the set of all states satisfying the coalitional sequential stability of m order for coalition H, where the state s

is $m-$th order coalitional sequentially stable for coalition H if for each state s_1 belonging to $R_H^{++}(s)$, there exists s_2 belonging to $R_{\varphi(N-H)}^{++m}(s_1)$ such that s_2 is not preferable to s for DM i for some $i \in H$.

Thus, since $R_C^{++1}(s) = R_C^{++}(s)$, it follows that $S_H^{1-SEQ} = S_H^{SEQ}$. Moreover, the intuition for a state s to be 2-SEQ, for coalition H, is that for every possible unilateral improvement move that coalition H has, s_1, the opponents may achieve through a legal sequence of unilateral improvements, where no coalition leaves a coalitional SEQ state for itself, a state s_2 which is no better that s for DM i, where i belongs to H. Thus, the sequence that leads to a sanction is credible in the sense that no coalition is moving away from a coalitional SEQ state. However, in such a sequence of movements the opponents may move away from a coalitional 2-SEQ state, leading to a non-credibility of Higher-order. Therefore, we propose coalitional sequential stability for several orders so that in the movements made by coalitions to sanction a focal coalition, no coalition leaves a coalitional SEQ state of the immediately lower order.

Thus, the various orders of coalitional sequential stability for a given coalition H are given by the sequence of sets $(S_H^{m-SEQ})_{m=1}^{\infty}$. As in the case of bilateral and multilateral conflicts, two limiting sets, known as $liminf$ and $limsup$, of this sequence can be defined as follows:

$$S_H^{inf-SEQ} = \bigcup_{l=1}^{\infty} \bigcap_{m=l}^{\infty} S_H^{m-SEQ} \tag{4.3}$$

and

$$S_H^{sup-SEQ} = \bigcap_{l=1}^{\infty} \bigcup_{m=l}^{\infty} S_H^{m-SEQ}. \tag{4.4}$$

As before, $S_H^{inf-SEQ}$ is the set of states which belong to all, except finitely many, sets in the sequence $(S_H^{m-SEQ})_{m=1}^{\infty}$ and $S_H^{sup-SEQ}$ is the set of states which belong to infinitely many sets in the sequence $(S_H^{m-SEQ})_{m=1}^{\infty}$.

Next, we analyze which properties are satisfied by the proposed Higher-order Coalitional Sequential Stability notions for coalitional analysis.

4.6 Properties of Higher-order Coalitional Sequential Stabilities for coalitional analysis

Theorem 4.6.1 shows us how the sets of $m-th$ order coalitional sequentially stable states, for different m values, relate to one another.

Theorem 4.6.1. Let m and m_1 be any positive integers. For $i \in N$, if m_1 is odd, then $R_C^{++m}(s) \subseteq R_C^{++m_1}(s)$ and $S_H^{m-SEQ} \subseteq S_H^{m_1-SEQ}$, for all $m \geq m_1$ and, otherwise, if m_1 is even, it follows that $R_C^{++m_1}(s) \subseteq R_C^{++m}(s)$ and $S_H^{m_1-SEQ} \subseteq S_H^{m-SEQ}$, for all $m \geq m_1$.

Proof: The proof for the Theorem 4.6.1 by mathematical induction is similar with the proof of Theorem 4.2.1. The only necessary changes are to replace i for H and H for C in the previous proof.

The next result shows that $S_H^{inf-SEQ}$ consists of states which are Higher-order Coalitional SEQ for coalition H for at least one even order and that $S_H^{sup-SEQ}$ consists of the states which are Higher-order Coalitional SEQ for coalition H for every odd order.

Corollary 4.6.1.1. $S_H^{inf-SEQ} = \bigcup_{m=1}^{\infty} S_H^{2m-SEQ}$ and $S_H^{sup-SEQ} = \bigcap_{m=1}^{\infty} S_H^{(2m-1)-SEQ}$.

Proof: Using the result of Theorem 4.6.1, the proof of Corollary 4.6.1.1 is identical to the proof of Corollary 3.3.1.1. □

4.6.1 Relationships among Higher-order Coalitional Sequential Stabilities and other coalitional solution concepts in the GMCR

In this section the relationships between Higher-order Coalitional SEQ and other GMCR coalitional solution concepts are studied.

Theorem 4.6.2. The following relations between Higher-order Coalitional SEQ and standard coalitional stability notions of the GMCR hold:

(a) For $H \in \varphi(N)$, if $s \in S_H^{NASH}$, then $s \in S_H^{m-SEQ}$, for all $m \geq 1$.

(b) For $H \in \varphi(N)$, if $s \in S_H^{m-SEQ}$ for some $m \geq 1$, then $s \in S_H^{SEQ}$.

(c) For $H \in \varphi(N)$, if $s \in S_H^{m-SEQ}$ for some $m \geq 1$, then $s \in S_H^{GMR}$.

(d) For $H \in \varphi(N)$, $S_H^{SMR} \not\subseteq S_H^{m-SEQ}$, for all $m \geq 1$.

(e) For $H \in \varphi(N)$, $S_H^{SSEQ} \not\subseteq S_H^{m-SEQ}$, for all $m \geq 2$.

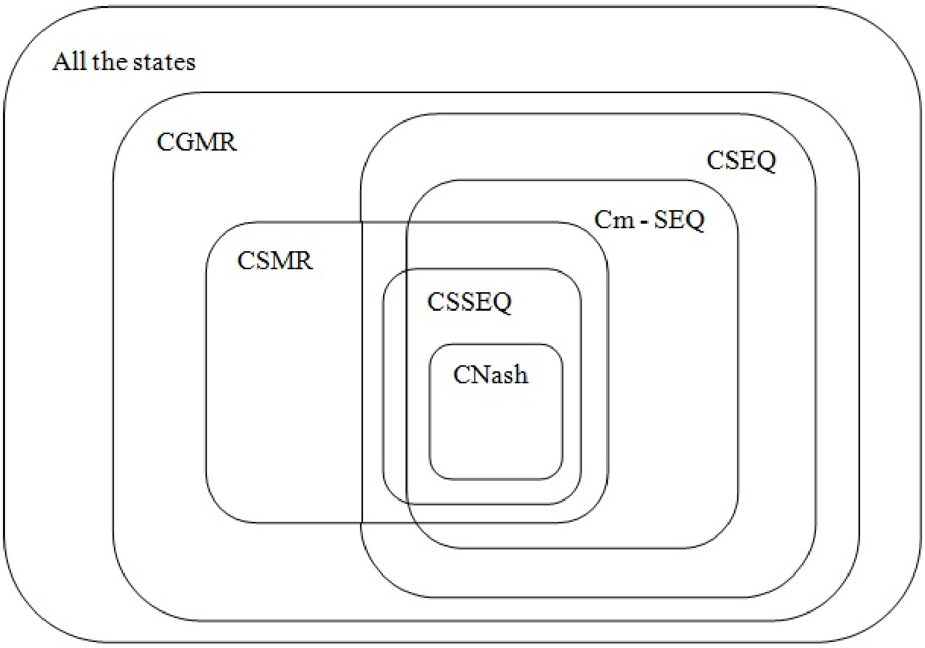
(f) For $H \in \varphi(N)$, $S_H^{m-SEQ} \not\subseteq S_H^{SMR}$, for all $m \geq 1$.

(g) For $H \in \varphi(N)$, $S_H^{m-SEQ} \not\subseteq S_H^{SSEQ}$, for all $m \geq 1$.

Proof: Identical to the proof of Theorem 3.5.1.

Figure 12 illustrates the relationships described in Theorem 4.6.2 for coalitional analysis of conflicts with multiple DMs.

Figure 12 – relationships described in Theorem 4.6.2.



Source: This research (2018)

4.7 Applications

Two applications are modeled in this section to illustrate how to analyze the $m-SEQ$ concept in conflicts with multiple DMs. The first application was studied by Alamanda et al.in 2015, called the Majalaya's Textile Industries Waste Pollution Conflict. This conflict concerns the waste pollution in the Upstream of Citarum River in Indonesia. It models the optimal solution based on the preferences for each DM in the conflict, namely the government of West Java province, the residents of Majalaya, the textile industry, and the independent environmental expert in the city of Bandung.

The second application is the Private Brownfield Renovation conflict which was analyzed by Walker, Boutilier & Hipel in 2010. In particular, we analyze an acquisition conflict, where one party attempts to besides buying the property as cheaply as possible, also obtaining as much benefits as possible from the local government. On the other hand, the property owner tries to sell the property at the highest possible price. The analysis of the conflict shows how a friendly interaction of the decision makers in a conflict can cause them to come to a consensus which benefits all.

While analyzing both conflicts, we point out some different results found regarding the stability of some states compared to the original analysis. Besides pointing out the mistakes, we show the importance of analyzing the credibility of the sanctions that may be imposed on the focal DM.

4.7.1 Majalaya's Textile Industries Waste Pollution Conflict

This application has the objective of examining the conflicts in the textile industry waste disposal in the upstream region of Citarum River by using the GMCR. The study was realized in the years of 2010, 2011, 2013, and 2014 in the Majalaya textile industry areas located in the upstream part of Citarum River, Majalaya sub-district, Bandung regency in Indonesia.

In this game, there are 4 players, the Textile Industry (T), the West Java Provincial Government (W), the Residents of Majalaya (M) and the Independent Environmental Expert (I). According to Alamanda et al.in 2015, the DMs have the following options:

- Textile Industry (T)
  - Maximizing function of the Waste-water Treatment Plant.
- West Java Provincial Government (W)
  - Strict punishment the textile industry which proven to dump waste into the upstream watersheds of Citarum river without processing;
  - Doing 3-Re (Revision, Relocation and Recreation).
- The Residents of Majalaya (M)
  - Demonstrating anarchically;
  - Moving to a place safer from waste of textile industry.
- The Independent Environmental Expert (I)
  - Doing negative publicity about the dangers of the waste of textile industry in medias (printed and electronic).

The feasible states of the Majalaya Conflict are shown in Figure 13. Each DM can change states by changing his own option keeping the options of the other DMs fixed.

Figure 13 – Feasible state of Majalaya conflict.

Option	Scenario												
	1	2	3	4	5	6	7	8	9	10	11	12	13
Textile Industry													
1. Maximizing the function of the Wastewater Treatment Plant (WWTP)	N	N	N	N	N	Y	N	N	N	N	N	N	N
West Java provincial government													
2. Give strict punishment to the textile industry which proved to dump wastes into the upstream area of Citarum river without processing	N	Y	N	N	Y	N	N	N	N	Y	N	Y	Y
3. Doing 3Re (Revision, Relocation and Recreation)	N	Y	N	Y	N	N	N	Y	Y	N	Y	Y	Y
Residents of Majalaya													
4. Doing Anarchist Demo	Y	N	Y	Y	N	N	Y	Y	Y	N	Y	N	N
5. Moving to a place safer from the waste of textile industry	N	Y	N	Y	N	N	Y	N	N	N	Y	N	Y
Independant Environmental Experts													
6. Conducting negative publicity about the dangers of textile industry waste in medias (printed and electronic)	Y	N	N	N	Y	N	Y	N	Y	N	Y	Y	Y

Source: (ALAMANDA et al., 2015)

According to (ALAMANDA et al., 2015) the DM's preferences are given by:

- The preference for DM T are:
  $6 \succ_T 4 \succ_T 2 \succ_T 13 \succ_T 10 \succ_T 11 \succ_T 7 \succ_T 12 \succ_T 5 \succ_T 3 \succ_T 1 \succ_T 8 \succ_T 9$;
- The preference for DM W are:
  $13 \succ_W 2 \succ_W 5 \succ_W 11 \succ_W 4 \succ_W 9 \succ_W 8 \succ_W 6 \succ_W 12 \succ_W 7 \succ_W 1 \succ_W 10 \succ_W 3$;
- The preference for DM M are:
  $12 \succ_M 5 \succ_M 10 \succ_M 6 \succ_M 9 \succ_M 7 \succ_M 8 \succ_M 3 \succ_M 13 \succ_M 11 \succ_M 2 \succ_M 4 \succ_M 1$;
- The preference for DM I are:
  $11 \succ_I 13 \succ_I 7 \succ_I 2 \succ_I 4 \succ_I 12 \succ_I 10 \succ_I 6 \succ_I 5 \succ_I 9 \succ_I 8 \succ_I 1 \succ_I 3$.

The stability of each state is indicated using letters. For the Nash stable state, (r) is used, for the sequential stable state (s) is used, for the m-th order sequential stable of state (ms) is used. If a state is unstable for a particular concept x, (notx) is used. In Table 5, for each state and each DM, we list the unilateral moves available (UMs), the unilateral improvements (UIs), the sanctions the opponents of the focal DM can impose on him or her for each UI and the credible sanctions the opponents of the focal DM can impose on him or her for each UI.

Having in hand the unilateral moves of DMs and the preferences of each DM, it is possible to find the results presented in the Table 5. Observing the Table 5 of the results

TABLE 5 – Stability Analysis Result of the Majalaya's Textile Industries Waste Pollution

Textile Industry (T)													
Stability	r	r	r	r	r	r	r	r	r	r	r	r	r
State Ranking	6	4	2	13	10	11	7	12	5	3	1	8	9
UMs													
UIs													
West Java Provincial Government (W)													
Stability	r	r	r	r	r	r	r	r	2s	nots	nots	r	nots
State Ranking	13	2	5	11	4	9	8	6	12	7	1	10	3
UMs									5	11	9		8
UIs									5	11	9		8
Sanction									(I,10)				
Credible Sanction									(I,10)				
Residents of Majalaya (M)													
Stability	r	r	r	r	r	r	r	r	nots	nots	r	nots	nots
State Ranking	12	5	10	6	9	7	8	3	13	11	2	4	1
UMs									12	9		8	7
UIs									12	9		8	7
Independent Environmental Experts (I)													
Stability	r	r	r	2s	2s	r	r	r	nots	r	nots	r	
State Ranking	11	13	7	2	4	12	10	6	5	9	8	1	3
UMs				13	11				10		9		1
UIs				13	11				10		9		1
Sanction				(M, 12) and (M, 12, W,5)	(M, 9)								(M,7), (M,11, W,11) and (M,9, W,9, M,9)
Credible Sanction				(M, 12)	(M, 9)								(M,7), (M,11, W,11) and (M,9, W,9, M,9)

Source: This research (2018)

of the analysis of the Majalaya's conflict, we can note about the concepts of Nash and sequential stability, that for DM T, none of the states presented any credible sanction, since, for this DM, all the states are stable in Nash. However, some divergent results were found in what was reported by Alamanda et al. in 2015. In the case of DM W state 1 is not SEQ, since it can move to 9 that is not credible, however in the article state 1 it does not move to any state being then Nash. In state 10 the author states that he can move to state 9 however he will not be SEQ, but state 10 is Nash because there is no improvement from it. For DM M state 13 is not SEQ because it can move to state 12 and DM 12 moves to 5 and from 5 DM I can move to 10 which is less preferable than 13 for the DM M. In this way the author incorrectly states that it is SEQ, however it is a not SEQ state. In the case of states 4 and 1 for DM M and 5 for DM I the author mentions that they are sequentially stable states, but in fact they are not SEQ.

In relation to the results of m -SEQ. In this case, all SEQ states are also 2-SEQ, since there is always a credible sanction, the only sanction would not involve W going from 12 in the analysis of state 2 to DM I, but there is another sanction that is believable, so the state is 2-SEQ.

4.7.2 Private Brownfield Renovation Conflict

The conflict in Private Brownfield Renovation is the acquisition of the brownfield property, which involves a developer (D) or real estate company, the property owner (PO) and the city government (CG). The property owner and the city government try to entice the buyer into purchasing the property. DM D wants to buy the property for the lowest possible price and to obtain as much benefits from DM CG as possible, while DM PO tries to increase the property price as much as possible.

There are six options for the DMs involved. The property owner can either sell high or low, the city Government can offer incentives and the developer has the option to buy. Moreover, both the property owner and the developer have the option to walk away from the negotiation. If they choose that option, they cannot move back to negotiation and the conflict moves to state 13 no matter what are the options taken by the other DMs. Since some options are exclusive, not all of the 2^6 possible combinations of options taken are feasible. The options for each DM and the feasible states are displayed in Figure 14. Each DM can change the conflict state by changing his own options, keeping the options of the other DMs fixed. The only irreversible option is when DM PO or DM D decide to walk away.

Figure 14 – Feasible States in the Acquisition Conflict.

PO	Sell high	N	Y	N	N	Y	N	N	Y	N	N	Y	N	—	—
	Sell low	N	N	Y	N	N	Y	N	N	Y	N	N	Y	—	—
	Walk	N	N	N	N	N	N	N	N	N	N	N	N	—	Y
CG	Incentives	N	N	N	Y	Y	Y	N	N	N	Y	Y	Y	—	—
Developer (D)	Buy	N	N	N	N	N	N	Y	Y	Y	Y	Y	Y	—	—
	Walk	N	N	N	N	N	N	N	N	N	N	N	N	Y	—
	State ID	1	2	3	4	5	6	7	8	9	10	11	12	13	

Source: (WALKER; BOUTILIER; HIPEL, 2010)

Figure 15 shows the preferences over the states for each DM. Each DM's state's ranking is listed from left to right from most to least preferred, the equally preferred states are in parenthesis within the state ranking.

Figure 15 – Ranking of states of the Acquisition Conflict.

DM	State rankings
PO	(8, 11), (1, 2, 4, 5, 7, 10), (3, 6, 13), (9, 12)
CG	(8, 9), (11, 12), (1, 2, 3, 7), (4, 5, 6, 10), 13
D	12, 9, 11, 10, 6, 5, (3, 4), (1, 2, 7, 13), 8

Source: (WALKER; BOUTILIER; HIPEL, 2010)

Based on preference rankings and on the unilateral moves available for DMs, the acquisition conflict can be analyzed.

Regarding the concepts of Nash and Sequential Stability, we found some results different from what was reported by Walker, Boutilier & Hipel in 2010. For example, state 10 is sequentially stable for DM PO since from the unique UI 11 for him, there exists a legal sequence of UIs where DM CG moves to state 8 followed by an UI from DM D leading the conflict to state 13, which is worse than state 10 for DM PO. Walker, Boutilier & Hipel misreported that state 10 does not satisfy Sequential Stability for DM PO. Moreover, Walker, Boutilier & Hipel also reported that states 6 and 10 were sequentially unstable for DM CG. However, from state 6, the unique UI for DM CG is state 3. But from state 3, there exists a legal sequence of UIs where DM D moves to state 9 followed by a move to state 13 of DM PO and state 13 is worse than state 6 for DM CG. From state 10, the unique UI for DM CG is state 7. But from state 7, there exists a legal sequence of UIs where DM PO moves to state 8 followed by a move to state 13 of DM D and state 13 is worse than state 10 for DM CG. Therefore, state 10 is a sequential equilibrium for the conflict.

Regarding the concept, of m-th order sequential stability, we have that in the Table 6, we can see that: (i) states 7 and 11 are 2-SEQ equilibria, (ii) although state 10 is a sequential equilibrium, it is not m-SEQ stable for DM PO and DM CG, for $m > 1$, because

TABLE 6 – Stability Analysis Result of the Brownfield Conflict for the DM PO

Property Owner (PO)					
Stability	State Ranking	UMs	UIs	Sanction	Credible Sanction
r	(8	7,9,13			
r	11)	10,12,13			
r	(1	2,3,13			
r	2	1,3,13			
r	4	5,6,13			
r	5	4,6,13			
2s	7	8,9,13	8	(D,13)	(D,13)
s, but ns for m>1	10)	11,12,13	11	(CG,8,D,13)	None
ns	(3	1,2,13	1,2	No Sanction for 1 or 2	
ns	6	4,5,13	4,5	No Sanction for 4, but for 5 (D,11,CG,8,D,13)	None for 4 or 5
r	13)				
ns	(9	7,8,13	7,8,13	No Sanction for 7, 8 or 13	
ns	12)	10,11,13	10,11,13	No Sanction for 10, 11 or 13	

Source: This research (2018)

the sanction to DM PO is not credible since it involves DM CG leaving state 11 which is 2-SEQ stable for DM CG and the sanction to DM CG is not credible since it involves DM PO leaving state 7 which is 2-SEQ stable for DM PO. In the Table 7 : (iii) states 11 and 12 are 2-SEQ stable for DM CG, (iv) although state 6 is a sequentially stable for DM CG, it is not m-SEQ stable for DM CG, for $m > 1$, because the sanctions are not credible since they involve DM D leaving state 3 which is 2-SEQ stable for DM D. Finally, in the Table 8: (v) states 3 and 6 are 2-SEQ stable for DM D, (vi) although state 5 is a sequentially stable for DM D, it is not m-SEQ stable for DM D, for $m > 1$, because the sanction is not credible since it involves DM CG leaving state 11 which is 2-SEQ stable for DM CG, and (vii) state 4 is SEQ and m-SEQ for DM D for $m > 2$, however it is not 2-SEQ for DM D, since the sanction where DM CG moves from state 10 to state 7 is only credible for $m > 2$, since this state is SEQ stable, but not m-SEQ stable for DM CG, for $m > 1$.

TABLE 7 – Stability Analysis Result of the Brownfield Conflict for the DM CG

City Government (CG)					
Stability	State Ranking	UMs	UIs	Sanction	Credible sanction
r	(8	11			
r	9)	12			
2s	(11	8	8	(D,2) or (D,13)	(D,2) or (D,13)
2s	12)	9	9	(PO,7), (PO,13)	(PO,7), (PO,13)
r	(1	4			
r	2	5			
r	3	6			
r	7)	10			
ns	(4	1	1	No Sanction	
ns	5	2	2	No Sanction	
s, but ns for m>1	6	3	3	(D,9,PO,13), (D,9,PO,8,D,13)	None
s, but ns for m>1	10)	7	7	(PO,8,D,13)	None
r	13				

Source: This research (2018)

TABLE 8 – Stability Analysis Result of the Brownfield Conflict for the DM D

Developer (D)					
Stability	State Ranking	UMs	UIs	Sanction	Credible Sanction
r	12	6,13			
r	9	3,13			
r	11	5,13			
r	10	4,13			
2s	6	12,13	12	(PO,13), (PO,10,CG,7), (PO,10,CG,7,PO,8), (PO,11,CG,8)	(PO,13)
s, but ns for m>1	5	11,13	11	(CG,8)	None
2s	(3	9,13	9	(PO,13), (PO,8), (PO,7)	(PO,13), (PO,8), (PO,7)
s, n2s, but ms for m>2	4)	10,13	10	(PO,11,CG,8), (CG,7), (CG,7,PO,8)	(CG,7) is only credible for m>2
r	(1	7,13			
r	2	8,13			
r	7	1,13			
r	13)				
ns	8	2,13	2,13	No Sanction for 2 or 13	

Source: This research (2018)

5 *CONCLUSION*

In this dissertation, we present some new definitions of stability, called sequential stability of order m (m-SEQ) in the GMCR for conflicts with 2 DMs and n DMs. We also propose the definition of higher-order coalitional sequential stability for coalitional analysis. The m-SEQ stability, which is a refinement of SEQ stability, was proposed to mitigate a problem of incredible threats that occur in SEQ. We present some properties of the proposed concepts. In particular, we showed existence of m-SEQ equilibria in finite GMCRs where preferences and moves are transitive, if m is odd. We also present some results about the relationships of m-SEQ with five other solution concepts commonly used in the GMCR.

For bilateral conflicts, three applications were made to demonstrate the efficiency of the $m-SEQ$ concept. Two classic applications of Game Theory, the Matching Pennies, and The game Rock, Scissors and Paper, in which all states were SEQ, but the result was only true for $m-SEQ$, if m was odd, and the Conflict of values between a Global Market-Driven economy and a Sustainable Ecosystem philosophy, where it was possible to verify that SEQ stability implied the existence of an incredible punishment in the analysis.

In order to broaden the results of the Chapter 3, new definitions were made for the case of the sequential stability of the order m (m-SEQ) in the GMCR for conflicts with n DMs, which generalized the definitions for conflicts with 2 DMs. To present the generalized concept, the idea of a credible legal sequence of m-th order of unilateral improvements for a coalition of DMs was proposed. Properties and relationships of m-SEQ stability with other notions of stability commonly used in the GMCR with multiple DMs were presented.

We introduce the concept of Higher-order Sequential Equilibria for coalitional analysis in the GMCR. Various orders of coalitional sequential stability were defined, as in the case of bilateral and multilateral conflicts. Additionally, we also present the relationships of coalitional m-SEQ with other solutions concepts used in the coalitional analysis in the GMCR in the literature.

To demonstrate the utility of the sequential stability model of the m (m-SEQ) order in the GMCR for conflicts with n DMs, the new definitions were applied in real-world conflict situations: Private Brownfield Renovation Conflict and Majalaya's Textile Industries

Waste Pollution Conflict.

Finally, from the results it was possible to perform refinements in the concept of Sequential Stability for several decision makers, being able to make improvements related to stability analysis.

5.1 Future Works

In future research, we intend to apply the $m-$SEQ with other Preference Structures that were proposed to be used in the GMCR, such as Uncertain Preferences (LI et al., 2004), Probabilistic Preference (RÊGO; SANTOS, 2015), Upper and Lower Probabilistic Preferences (SANTOS; RÊGO, 2014), Fuzzy Preferences (BASHAR; KILGOUR; HIPEL, 2012) and Grey Preference (KUANG et al., 2015).We also intend to determine matrix methods to efficiently find m-SEQ stability in a conflict.

REFERENCE

AL-MUTAIRI, M. S.; HIPEL, K. W.; KAMEL, M. S. *Fuzzy preferences in conflicts*. Journal of Systems Science and Systems Engineering, Springer, v. 17, n. 3, p. 257–276, 2008. 26

ALAMANDA, D. T. et al. *Conflict resolution using graph model for conflict resolution: Case study majalaya's textile industries waste pollution.* Telkom University, Bandung, Indonesia, 2015. 15, 53, 54, 55, 57

AXELROD, R. *The Evolution of Cooperation*. [S.l.]: New York, 1984. 20

BASHAR, M. A.; KILGOUR, D. M.; HIPEL, K. W. *Fuzzy preferences in the graph model for conflict resolution*. IEEE Transactions on Fuzzy Systems, IEEE, v. 20, n. 4, p. 760–770, 2012. 63

BEN-HAIM, Y.; HIPEL, K. W. *The graph model for conflict resolution with information- gap uncertainty in preferences*. Applied Mathematics and Computation, Elsevier, v. 126, n. 2, p. 319–340, 2002. 26

FANG, L.; HIPEL, K. W.; KILGOUR, D. M. *Interactive decision making: the graph model for conflict resolution*. [S.l.]: John Wiley & Sons, 1993. v. 3. 14

FRASER, N. M.; HIPEL, K. W. *Solving complex conflicts*. IEEE Transactions on Systems, Man, and Cybernetics, IEEE, v. 9, n. 12, p. 805–816, 1979. 23

FRASER, N. M.; HIPEL, K. W. *Conflict analysis: models and resolutions*. [S.l.]: North-Holland, 1984. v. 11. 15, 20, 23

GIBBONS, R. *A primer in game theory*. [S.l.]: Harvester Wheatsheaf, 1992. 37

HIPEL, K. W.; KILGOUR, D. M.; FANG, L. *The graph model for conflict resolution*. [S.l.]: Wiley Online Library, 2011. 18

HIPEL, K. W.; OBEIDI, A. *Trade versus the environment: Strategic settlement from a systems engineering perspective*. Systems Engineering, Wiley Online Library, v. 8, n. 3, p. 211–233, 2005. 36, 39, 40, 41

HOWARD, N. *Paradoxes of rationality: games, metagames, and political behavior*. [S.l.]: MIT press New York, 1971. 15, 22, 23

INOHARA, T.; HIPEL, K. W. *Coalition analysis in the graph model for conflict resolution*. Systems Engineering, Wiley Online Library, v. 11, n. 4, p. 343–359, 2008. 24

KASSAB, M.; HIPEL, K.; HEGAZY, T. *Conflict resolution in construction disputes using the graph model*. Journal of construction engineering and management, American Society of Civil Engineers, v. 132, n. 10, p. 1043–1052, 2006. 15

KILGOUR, D. M.; HIPEL, K. W.; FANG, L. *The graph model for conflicts*. Automatica, Elsevier, v. 23, n. 1, p. 41–55, 1987. 18

KILGOUR, D. M. et al. *Coalition analysis in group decision support*. Group Decision and Negotiation, Springer, v. 10, n. 2, p. 159–175, 2001. 24

KUANG, H. et al. *Grey-based preference in a graph model for conflict resolution with multiple decision makers*. IEEE Transactions on Systems, Man, and Cybernetics: Systems, IEEE, v. 45, n. 9, p. 1254–1267, 2015. 63

LI, K. W. et al. *Preference uncertainty in the graph model for conflict resolution*. IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans, IEEE, v. 34, n. 4, p. 507–520, 2004. 15, 25, 63

NASH, J. *Non-cooperative games*. Annals of mathematics, JSTOR, p. 286–295, 1951. 15, 22

NASH, J. F. et al. *Equilibrium points in n-person games*. Proceedings of the national academy of sciences, USA, v. 36, n. 1, p. 48–49, 1950. 15, 22

OBEIDI, A.; KILGOUR, D. M.; HIPEL, K. W. *Perceptual graph model systems*. Group Decision and negotiation, Springer, v. 18, n. 3, p. 261–277, 2009. 26

OBEIDI, A.; KILGOUR, D. M.; HIPEL, K. W. *Perceptual stability analysis of a graph model system*. IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans, IEEE, v. 39, n. 5, p. 993–1006, 2009. 26

OSBORNE, M. *An Introduction to Game Theory*. [S.l.]: Oxford University Press, 2004. 38

RÊGO, L. C.; SANTOS, A. M. dos. *Probabilistic preferences in the graph model for conflict resolution*. IEEE Transactions on Systems, Man, and Cybernetics: Systems, IEEE, v. 45, n. 4, p. 595–608, 2015. 15, 26, 63

RÊGO, L. C.; VIEIRA, G. I. A. *Symmetric sequential stability in the graph model for conflict resolution with multiple decision makers*. Group Decision and Negotiation, Springer, p. 1–18, 2016. 15, 24

RÊGO, L. C.; VIEIRA, G. I. A. *Interactive unawareness in the graph model for conflict resolution*. IEEE Transactions on Systems, Man, and Cybernetics: Systems, IEEE, 2017. 15, 26

RÊGO, L. C.; VIEIRA, G. I. A. *Symmetric sequential stability in the graph model for conflict resolution with multiple decision makers*. Group Decision and Negotiation, Springer, v. 26, n. 4, p. 775–792, 2017. 19

ROBBINS, S. *Organization Theory: Structure,Design and Applications*. [S.l.]: Englewood Cliffs, New Jersey: Prentice- Hall,Inc., 1990. 14

SANTOS, A. M. dos; RÊGO, L. C. *Graph model for conflict resolution with upper and lower probabilistic preferences*. In: The proccedings of the 14th international conference on group decision and negotiation. [S.l.: s.n.], 2014. v. 1, p. 208–215. 63

WALKER, S. B.; BOUTILIER, T.; HIPEL, K. W. *Systems management study of a private brownfield renovation*. Journal of Urban Planning and Development, American Society of Civil Engineers, v. 136, n. 3, p. 249–260, 2010. 53, 58