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# On a class of singular elliptic equation arising in MEMS modeling

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*Tese apresentada ao Departamento de Matemática da Universidade Federal de Pernambuco, como parte dos requisitos para obtenção do título de Doutor em Matemática.*

Orientador: João Marcos Bezerra do Ó

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ESTEBAN PEREIRA DA SILVA

# ON A CLASS OF SINGULAR ELLIPTIC EQUATION ARISING IN MEMS MODELING

Tese apresentada ao do Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

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*ao meu pai*

*"I always did something I was a little not ready to do.
I think that's how you grow."*

*Marissa Mayer - Yahoo's CEO*

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*Esteban Pereira da Silva*

**RESUMO**

Motivados pela recente atenção dada ao estudo das equações que modelam MEMS eletrostáticos, estudamos as soluções radiais de uma classe de equações com não linearidade do tipo inverso do quadrado. De acordo com a escolha dos parâmetros envolvidos no problema, o operador diferencial com o qual lidamos corresponde à forma radial do $p$-laplaciano ($p > 1$) e $k$-hessiano. Provamos a existência de um parâmetro extremal $\lambda^* > 0$ tal que, para $\lambda \in (0, \lambda^*)$, existe uma solução minimal não singular para o problema. Para $\lambda > \lambda^*$ não há solução de nenhum tipo considerado. Estudamos também o comportamento do ramo de soluções minimais e exibimos um método para aproximação numérica destas soluções. No que se refere ao caso $\lambda = \lambda^*$, provamos unicidade de solução e apresentamos um resultado de regularidade. Além disso, apresentamos condições sobre as quais é possível garantir a regularidade da solução crítica ($\lambda = \lambda^*$). Provamos também que sempre que a solução crítica for regular, existe uma outra solução do tipo passo da montanha para $\lambda$ perto de $\lambda^*$.

**Palavras-chave:** Não linearidade singular. Modelagem de MEMS. Equação elíptica quasilinear.

**ABSTRACT**

Motivated by recent works on the study of the equations that model the electrostatic MEMS devices, we study the radial solutions of some quasilinear elliptic equations with nonlinearity of inverse square type. According to the choice of the parameters on the problem, the differential operator which we are dealing with corresponds to the radial form of $p$-Laplacian ($p > 1$) and $k$-Hessian. We prove the existence of an extremal parameter $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$ there exists a minimal solution $u_\lambda$ and for $\lambda > \lambda^*$ there is no solution of any considered kind. Via Shooting Method, we prove uniqueness of solutions for $\lambda$ close to 0. We also study the behavior of the minimal branch of solutions. Concerning the case $\lambda = \lambda^*$, we prove uniqueness of solutions and present a regularity result. In addition, we present conditions over which we can assert regularity for the critical solution with respect to the parameter $\lambda$ for the existence of solutions. Moreover, we prove that whenever the critical solution is regular, there exists other solutions of mountain pass type for $\lambda$ close to $\lambda^*$.

**Keywords:** Singular nonlinearity. MEMS modeling. Quasilinear elliptic equation.

# Contents

# 1 Introduction

This work is devoted to the study a class of radial quasilinear elliptic differential equations involving a singular nonlinearity of an inverse square type. The discontinuity on the nonlinearity brings a difficulty for the application of variational and topological methods. Recently, this class of problems has received much attention due to its applicability on the modeling of electrostatic MEMS (*Micro Electro Mechanical Systems*).

MEMS are micro-devices consisting of electrical and mechanical components combining together on a chip to produce a system of miniature dimensions (between 1 and 100 micrometers, that is 0,001 and 0,1 millimeters, respectively - less than the thickness of a human hair). They are essential components of the modern technology that is currently driving telecommunications, commercial systems, biomedical engineering and space exploration. For more information on the applications and development of the fundamental partial differential equations that model those devices, see [Pelesko e Bernstein 2003, Esposito *et al.* 2010].

We are motivated by recent works on the study of the equations that model a special MEMS component - an electrostatically controlled tunable deformable capacitor - that can be understood as consisting of a thin and deformable microplate whose shape we are representing by $\Omega$ (a bounded domain of $\mathbb{R}^N$), fixed along its boundary, coated with a negligibly thin metallic conducting film and lying above one unit of a parallel rigid grounded plate as shown in Fig. 1.

The membrane deflects towards the conducting plate when a voltage (represented here by $\lambda$) is applied. It may occur that the membrane touches the plate or gets ripped due to the loss of stability between the forces acting on the system, thus, creating a singularity. The modeling is then based on the equilibrium between these forces. On the

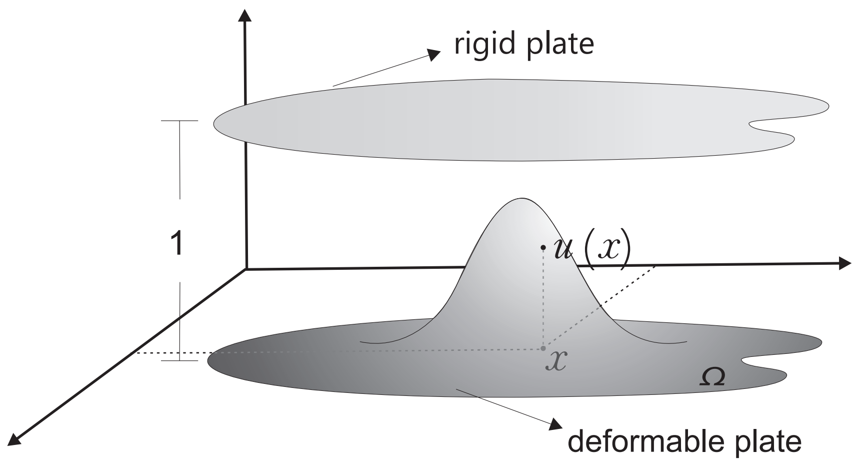


Figure 1.1: Deformable capacitor.

one hand, we have tension due to the stretching (given by the laplacian of the deformation function) and rigidity (given by the bi-laplacian of the deformation function). On the other hand, we have the electrostatic force which, according to the Coulomb's law, is inversely proportional to the square of the distance between the two charged plates. We can also consider the elastic and the electric potential energies associated with the deformation. In the stationary case, a very general model of the deformation $u$ of the membrane is:

$$\begin{cases} \alpha\Delta^2 u \;=\; S(u)\Delta u + \dfrac{\lambda f(x)}{D(u)(1-u)^2} & \text{in } \Omega, \\ 0 \le u < 1 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f$ is a varying dielectric permittivity profile, $\eta$ is the unit outward normal to $\partial\Omega$, the constant $\alpha \ge 0$ represents the membrane width and the terms

$$S(u) = \beta \int_0^1 |\nabla u|^2 dx + \gamma \quad \text{and} \quad D(u) = 1 + \chi \int_0^1 \frac{dx}{(1-u)^2} \quad \text{with} \quad \beta, \gamma, \chi \ge 0$$

represent nonlocal parameters that affect the membrane's deformation (the elastic and the electric potentials respectively).

In the limit case of zero plate thickness, hence for a thin membrane (that is $\alpha = 0$), with zero rigidity and neglecting inertial effects as well as nonlocal effects, that is, in dimensionless constants, we set $\alpha = \beta = \chi = 0$. Then (1.1) reduces to the following semilinear elliptic problem:

$$\begin{cases} \Delta u = \dfrac{\lambda f(x)}{(1-u)^2}, & \text{in } \Omega, \\ 0 < u < 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$S_\lambda$}$$

where, for simplicity, we have set $\gamma = 1$. This problem has been studied in the past in a very general context as we can see in [Guo e Wei 2006, Ghoussoub e Guo 2006/07, Dávila 2008, Esposito 2008, Esposito *et al.* 2007, Guo e Wei 2008, Guo e Wei 2008, Mignot e Puel 1980].

Recently D. Castorina, Esposito and Sciunzi [Castorina *et al.* 2009] studied, for $1 < p \leq 2$, the $p$-MEMS equation (the MEMS equation for the $p$-Laplacian operator), that is,

$$\begin{cases} -\Delta_p u = \lambda h(x) f(u) & \text{in } \Omega, \\ 0 \leq u < 1 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$S_{\lambda,p}$}$$

where $f$ is a non-decreasing positive function, defined on $[0, 1)$ with a singularity at $u = 1$. They established uniqueness results for semistable solutions and stability (in a strict sense) of minimal solutions for $1 < p \leq 2$. In [Castorina *et al.* 2008, Castorina *et al.* 2009, Castorina *et al.* 2011] the authors also prove radial symmetry of the first eigenfunction for the $p$-MEMS problem on a ball. The radial form of the $p$-MEMS equation on a ball $B \in \mathbb{R}^n$ $(n \geq 2)$ is

$$\begin{cases} -(r^{n-1}|u'(r)|^{p-2}u'(r))' = \dfrac{\lambda r^{n-1} f(r)}{(1-u(r))^2}, & r \in (0,1), \\ 0 \leq u(r) < 1, & r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{1.2}$$

## 1.1 Problem formulation

On the spirit of the pioneering work [Crandall e Rabinowitz 1975], we study here the following more general class of quasilinear elliptic equations which includes (1.2) as a particular case, since we consider a continuous variation of the exponents in (1.2), including the radial case for non-integer-dimensional spaces, that is

$$\begin{cases} -(r^{\alpha}|u'(r)|^{\beta}u'(r))' = \dfrac{\lambda r^{\gamma} f(r)}{(1-u(r))^2}, & r \in (0,1), \\ 0 \le u(r) < 1, & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{$P_\lambda$}$$

where

(H.1) $\alpha$, $\beta$ and $\gamma$ are real constants with $\beta > -1$;

(H.2) $f$ is a measurable real function on $(0,1)$;

(H.3)

$$0 < \int_0^r s^{\gamma}|f(s)|\,\mathrm{d}s < +\infty \qquad \text{for every } r \in (0,1).$$

Problems involving the quasilinear operator of the form $Lu := -(r^{\alpha}|u'|^{\beta}u')'$ have been studied in various contexts and applications. It should be mentioned that the operator $Lu$ corresponds to the radial form of various operators for a convenient choice of the parameters $\alpha$, $\beta$ and $\gamma$ namely, We recall that the $k$-Hessian operator, usually represented

| **Operator** | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|
| $p$-Laplacian ($p > 1$) | $n-1$ | $p-2$ | $n-1$ |
| $k$-Hessian | $n-k$ | $k-1$ | $n-1$ |

Table 1.1:

by $S_k(D^2u)$, is defined as the sum of all principal $k \times k$ minors of the Hessian matrix $D^2u$. In particular $S_1(D^2u) = \Delta u$ and $S_N(D^2u) = \det D^2u$, the Monge-Ampère operator.

This class of differential operator has been extensively studied since the papers of D. Joseph and T. Lundgren [Joseph e Lundgren 1972/73], J. Keener and H. Keller [Keener e Keller 1974] and M. Crandall and P. Rabinowitz [Crandall e Rabinowitz 1973, Crandall e Rabinowitz 1975]. In [Mignot e Puel 1980], F. Mignot and J-P. Puel studied regularity results to certain nonlinearities, namely, $g(u) = e^u$, $g(u) = u^m$ with $m > 1$, $g(u) = 1/(1-u)^k$ with $k > 0$.

Recently, J. Jacobsen and K. Schmitt [Jacobsen e Schmitt 2002] studied singular problems involving this class of operators. They determined precise existence and multiplicity results for radial solutions of the Liouville-Bratu-Gelfand problem associated with this class of quasilinear radial operators. In 1996, Clement, de Figueiredo and Mitidieri [Clément *et al.* 1996] studied Brezis-Nirenberg-type problems for this class of quasilinear elliptic operators. For related problems on this subject we also refer to [Dávila 2008, Jacobsen e Schmitt 2004] and the references therein.

## 1.2 Notations and terminology

Before stating our main results and in order to drive properly the approach we used to obtain such results, we would like to introduce a function space, the notions of solution considered and some already know results involving this entities. In our arguments, we are always assuming the hypotheses items (H.1) to (H.3).

Let $\widetilde{\mathrm{X}}$ be the set of the $L^1_{\mathrm{loc}}$ real functions defined on $(0,1)$ with distributional derivative on $L^1_{\mathrm{loc}}$ satisfying

$$\|u\|^{\beta+2} := \int_0^1 r^\alpha |u(r)|^{\beta+2}\,\mathrm{d}r + \int_0^1 r^\alpha |u'(r)|^{\beta+2}\,\mathrm{d}r < \infty. \tag{1.3}$$

If $\beta \geq -1$, equipping $\widetilde{\mathrm{X}}$ with the usual operations makes it a real vector space and (1.3) defines a norm on this space. Every $u \in \widetilde{X}$ is, in particular, absolutely continuous in $(0,1)$. Now, consider the subspace $\hat{X}$ of the elements of $\widetilde{X}$ such that

$$\lim_{r\to 1} u(r) = 0$$

and $X$, the completion of $\hat{X}$ with respect to the equivalent norm

$$\|u\|_X := \left(\int_0^1 r^\alpha |u'(r)|^{\beta+2}\,\mathrm{d}r\right)^{1/(\beta+2)}. \tag{1.4}$$

It is known that $\alpha < \beta + 1$ is a sufficient condition in order to have $X = \hat{X}$. See A. Kufner and B. Opic [Opic e Kufner 1990, Kufner e Opic 1984] for details concerning the proprieties remarked in this paragraph.

Given $u \in X$, we also consider $X_u$ the set of all $w \in L^2((0,1))$ with distributional derivative $w' \in L^2((0,1))$ satisfying

$$\int_0^1 r^\alpha |u'(r)|^\beta (w'(r))^2 \, \mathrm{d}r < \infty.$$

The set $X_u$ is a weighted Sobolev space with the usual operations, with weight $\omega(r) = r^\alpha |u'(r)|^\beta$ and with norm given by

$$\|w\|_u = \left( \int_0^1 r^\alpha |u'(r)|^\beta (w'(r))^2 \, \mathrm{d}r \right)^{1/2}.$$

Actually, $X_u$ is a separable Hilbert space with

$$\langle v, w \rangle_u = \int_0^1 r^\alpha |u'(r)|^\beta v'(r) w'(r) \, \mathrm{d}r.$$

We refer to M. K. V. Murthy and G. Stampacchia [Murthy e Stampacchia, 1968] for the properties introduced above.

Now lets establish the concept of solution used in this work. For that consider the general problem

$$\begin{cases} -(r^\alpha |u'(r)|^\beta u'(r))' = h(r, u(r)), & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{1.5}$$

where $h$ is a measurable and bounded from below function acting over a set $H \subset \mathbb{R}^2$ whose projection onto the first variable is $(0,1)$. We stress that we have focus on the case

$$h(r,z) = \begin{cases} \dfrac{\lambda r^\gamma f(r)}{(1-z)^2} & \text{if} \quad z \neq 1 \\ 0 & \text{if} \quad z = 1. \end{cases}$$

Hereafter, $\varphi_\beta$ will denote a real-valuated function of real variable given by

$$\varphi_\beta(t) = \begin{cases} |t|^{-\beta/(\beta+1)} t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases} \tag{1.6}$$

Due to hypothesis (H.1), $\varphi_\beta$ is an homeomorphism.

**Definition 1** *Notions of solutions for problem* (3.1)*:*

*Integral solution* *We say that* $u$ *is an* integral solution *of problem* $(P_\lambda)$ *if*

$$u \in X \tag{i}$$

$$\int_0^1 \frac{\lambda r^\gamma f(r)}{(1-u(r))^2}\,\mathrm{d}r < \infty \tag{ii}$$

$$-\, r^{\alpha/(\beta+1)} u'(r) = \varphi_\beta \left( \int_0^r \frac{s^\gamma f(s)}{(1-u(s))^2} \right) \mathrm{d}s \qquad \text{for } r \in (0,1) \tag{iii}$$

*We also consider* integral *sub and* super*solutions in analogy with this definition. For instance,* $u$ *is an* integral *subsolution of* $(P_\lambda)$ *if* $u$ *satisfies* (*i*)-(*iii*) *with* "$\le$" *instead of* "$=$" *in* (*iii*).

*Weak solution* *We call* $u$ *a* weak solution *of* $(P_\lambda)$ *whenever*

$$u \in X \tag{i}$$

$$\left| \int_0^1 \frac{\lambda r^\gamma f(r)}{(1-u(r))^2}\, v(r)\,\mathrm{d}r \right| < \infty \qquad \text{for all } v \in X \tag{ii}$$

$$\int_0^1 r^\alpha |u'(r)|^\beta u'(r) v'(r)\,\mathrm{d}r = \int_0^1 \frac{\lambda r^\gamma f(r)}{(1-u(r))^2}\, v(r)\,\mathrm{d}r \qquad \text{for all } v \in X \tag{iii}$$

*If* $u$ *satisfies* (*i*)-(*iii*) *with the* "$\le$" *(respectively* $\ge$*) sign instead of* "$=$" *in* (*iii*) *and* $v \ge 0$*, we say that* $u$ *is a weak supersolution (respectively weak subsolution) of* $(P_\lambda)$*.*

*Regular and singular solution, subsolution or supersolution* *A solution, subsolution or supersolution* $u \in X$ *of* $(P_\lambda)$ *in any sense described above is called regular if it satisfies* $\|u\|_\infty < 1$*, and singular otherwise.*

*Minimal solution* *We call minimal solution a positive solution (in any sense defined above)* $u \in X$ *of* $(P_\lambda)$ *such that for any other positive solution* $v \in X$ *of* $(P_\lambda)$ *one has* $0 \le u(r) \le v(r)$ *for all* $r \in (0,1)$*.*

It is clear that every classical solution (subsolution, supersolution) is also an integral and weak solution (subsolution, supersolution respectively). Another important fact: the definitions of weak solution and integral solution are equivalent (see Proposition 2.1 on page 18).

Given $u \in X$ a regular weak solution of $(P_\lambda)$, we prove (c.f. Proposition 2.4) that

$$0 < u(r) < 1 \quad \text{for all } r \in (0,1).$$

Thus,

$$0 \leq \frac{1}{(1-u(r))^3} < \frac{1}{(1-u(r))^2} \quad \text{for all } r \in (0,1).$$

So it is well defined the operator $\Psi : X_u \to \mathbb{R}$ as

$$\Psi(w) = (\beta+1)\int_0^1 r^\alpha |u'|^\beta (w')^2 \,\mathrm{d}r - 2\lambda \int_0^1 \frac{r^\gamma f(r)}{(1-u)^3} w^2(r)\,\mathrm{d}r.$$

This endows us to formulate the following definition.

**Definition 2** *A solution $u$ of $(P_\lambda)$ is* semi-stable *if $\Psi(w) \geq 0$ for all $w \in X_u$. Consider*

$$\mu_1(u) := \inf\{\Psi(w) : w \in X_u \text{ and } \|w\|_u = 1\}, \tag{1.7}$$

*we say that a solution $u$ of $(P_\lambda)$ is* stable *if $\mu_1(u) > 0$ and* unstable *if $\mu_1(u) < 0$.*

## 1.3 Outline

This work is divided in two chapters. In the first chapter, we prove the existence of a constant $\lambda^*$ such that for $\lambda \in (0, \lambda^*)$ there exists a minimal classical solution $u_\lambda$ and for $\lambda > \lambda^*$ there are no solutions of any kind. In addition, we provide some estimates for $\lambda^*$. Via Shooting Method, we prove uniqueness of solution for $\lambda$ close to 0. Concerning qualitative properties of solutions, we prove that the function $\lambda \mapsto u_\lambda$ is increasing in the pointwise sense. Furthermore, we provide a way to compute the minimal solutions by approximation. The main result of this chapter concerns the uniqueness for the solution of $(P_{\lambda^*})$ ($(P_\lambda)$-problem for $\lambda = \lambda^*$) for $\beta > -1$, which improves the results in [Castorina *et al.* 2008] since our arguments can be also applied to the $p$-Laplace operator even for $p > 2$ and when $f(r)$ is a general function under suitable assumptions.

In Chapter 2, we prove the existence of semi-stable solutions. Actually, we prove semi-stability as an additional property of the minimal solutions which existence is guaranteed by Theorem 3.7. The existence of a solution different from the minimal one depends on the regularity of $u^*$. By using the Mountain Pass Theorem, we become able to ensure the existence of unstable solutions.

# 2 Early Comments on the solutions of $(P_\lambda)$

First of all, we would like to mention that integral solutions and weak solutions are the same (Proposition 2.1). Thence we will use just the term weak solution to denote these functions, aware of, in fact, they satisfy both, the integral solution and the weak solution conditions.

**Proposition 2.1** *Given $u$ a weak solution then $u$ is an integral solution and conversely.*

*Proof.* Let $u$ be a weak solution of $(P_\lambda)$, consider for each $r \in (0,1)$ and $\varepsilon > 0$ the continuous map

$$v_\varepsilon(s) = \begin{cases} 1 & \text{if} \quad 0 \le s \le r, \\ -s/\varepsilon + r/\varepsilon + 1 & \text{if} \quad r < s < r+\varepsilon, \\ 0 & \text{if} \quad s \ge r+\varepsilon. \end{cases}$$

Since $v_\varepsilon \in X$, we can take it as a test function, obtaining:

$$-\frac{1}{\varepsilon}\int_r^{r+\varepsilon} s^\alpha |u'(s)|^\beta u'(s)\,\mathrm{d}s = \int_0^r \frac{\lambda s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s + \int_r^{r+\varepsilon} \frac{\lambda s^\gamma f(s)}{(1-u(s))^2} v_\varepsilon(s)\,\mathrm{d}s. \tag{2.1}$$

Observe that

$$\left|\frac{\lambda r^\gamma f(r)}{(1-u(r))^2} v_\varepsilon(r)\chi_{[r,r+\varepsilon]}(r)\right| \le \left|\frac{\lambda r^\gamma f(r)}{(1-u(r))^2}\right|,$$

where $\chi_{[r,r+\varepsilon]}$ is the characteristic function of the set $[r, r+\varepsilon]$ on $(0,1)$. Moreover,

$$\frac{\lambda r^\gamma f(r)}{(1-u(r))^2} v_\varepsilon(r)\chi_{[r,r+\varepsilon]}(r) \to 0$$

pointwise almost everywhere as $\varepsilon \to 0$. Then we can apply the Dominated Convergence Theorem with $\varepsilon \to 0$ in (2.1) obtaining

$$r^\alpha |u'(r)|^\beta u'(r) = \int_0^r \frac{\lambda s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s.$$

For the converse part, first observe that, as a simple application of the Hölder inequality, one has

$$\left| \int_0^1 r^\alpha |u'(r)|^\beta u'(r) v'(r)\,\mathrm{d}r \right| < \infty \quad \text{ for all } u, v \in X.$$

Thus, if $u$ an integral solution of $(P_\lambda)$, we can integrate from 0 to 1 the integral solution condition multiplied by $v'(r)$, where $v \in X$, obtaining

$$-\int_0^1 \left( \int_0^r \frac{\lambda r^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s \right) v'(r)\,\mathrm{d}r = \int_0^1 r^\alpha |u'(r)|^\beta u'(r) v'(r)\,\mathrm{d}r < \infty. \tag{2.2}$$

Integrating the left hand side of (2.2) by parts we conclude that $u$ is a weak solution of $(P_\lambda)$.

## 2.1 First approach on existence of integral solutions

In this first considerations, we are assuming the hypothesis (H.1) to (H.3). In addition, we assume

(H.4) $\eta \in L^{1/(\beta+1)}((0,1)) \quad$ and $\quad \eta \neq 0$,

where $\eta$ denote the function $\eta : (0,1] \to \mathbb{R}$ given by

$$\eta(r) := \frac{1}{r^\alpha} \int_0^r s^\gamma |f(s)|\,\mathrm{d}s. \tag{2.3}$$

**Remark 2.2** *Hypothesis (H.2) to (H.4) are verified for instance if*

$$\gamma > -1, \quad f \in C([0,1]) \quad \textit{and} \quad \alpha < \gamma + \beta + 2. \tag{2.4}$$

*In this case,*

$$|\eta(r)|^{1/(\beta+1)} = O(r^{(\gamma-\alpha+1)/(\beta+1)}) \quad \textit{and} \quad \frac{\gamma - \alpha + 1}{\beta + 1} > -1.$$

*The assumption* (2.4) *still includes the cases on Table 1.1.*

If $u$ is an integral solution of $(P_\lambda)$, then it also satisfies

$$u(r) = \int_r^1 \varphi_\beta \left( \frac{1}{s^\alpha} \int_0^s \frac{\lambda t^\gamma f(t)}{(1-u(t))^2} \, \mathrm{d}t \right) \mathrm{d}s, \tag{2.5}$$

where $\varphi_\beta$ denote the function $\varphi_\beta : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_\beta(s) := \begin{cases} s|s|^{-\beta/(\beta+1)} & \text{if} \quad s \neq 0, \\ 0 & \text{if} \quad s = 0. \end{cases} \tag{N.2}$$

For $\beta > -1$, $\varphi_\beta$ is an increasing homomorphism from $\mathbb{R}$ to itself.

Our first approach on Problem $(P_\lambda)$ is to study the equation (2.5) and then, we investigate under which assumptions solutions of (2.5) are solutions of $(P_\lambda)$ in the classical sense.

**Theorem 2.3** *Assume the hypothesis (H.1) to (H.4). Consider the set*

$$B = \left\{ u \in C([0,1]) : \|u\|_\infty < \frac{\beta+1}{\beta+3} \right\}.$$

*Then, there exists an unique $u \in B$ which solves* (2.5), *provided*

$$|\lambda| < \frac{1}{\|\eta\|_{1/\beta+1}} \left( \frac{\beta+1}{\beta+3} \right)^{\beta+1} \left( \frac{2}{\beta+3} \right)^2. \tag{2.6}$$

*Moreover, $u$ is an integral solution of $(P_\lambda)$ with*

$$u \in C([0,1]) \cap C^1((0,1]).$$

*If $f$ is continuous and $\gamma > 0$, then*

$$u \in C([0,1]) \cap C^2((0,1]).$$

*If $f$ is continuous, $\gamma > 0$ and $\alpha < \gamma + \beta + 2$ then*

$$u \in C^1([0,1]) \cap C^2((0,1]).$$

*Proof.* For every $\delta \in (0,1)$, it is possible to choose $\lambda$ small enough with which it is well defined the function $Z : B_\delta \to B_\delta$, where

$$B_\delta = \{u \in C([0,1]); \|u\|_\infty \leq \delta\}$$

and

$$(Z(u))(r) = \int_r^1 \varphi_\beta \left( \frac{1}{s^\alpha} \int_0^s \frac{\lambda t^\gamma f(t)}{(1-u(t))^2} \,\mathrm{d}t \right) \mathrm{d}s.$$

In fact, observe that

$$\|Z(u)\|_\infty \le \left( \frac{|\lambda| \|\eta\|_{1/(\beta+1)}}{(1-\|u\|_\infty)^2} \right)^{1/(\beta+1)}.$$

Then, $Z(u) \in C([0,1])$. Moreover, for $u \in B_\delta$, we have

$$\|Z(u)\|_\infty \le \left( \frac{|\lambda| \|\eta\|_{1/(\beta+1)}}{(1-\delta)^2} \right)^{1/(\beta+1)}. \tag{2.7}$$

Thus, we have $\|Z(u)\|_\infty \le \delta$ provided

$$|\lambda| \le \lambda_1(\delta) := \frac{1}{\|\eta\|_{1/(\beta+1)}} \delta^{\beta+1}(1-\delta)^2.$$

We can choose $\lambda$ smaller in order to assure that $Z$ is a contraction on $B_\delta$ with respect to the norm $\|\cdot\|_\infty$. Indeed, for arbitrary $u, v \in B_\delta$, we have

$$|Z(u) - Z(v)| \le \left( |\lambda| \|\eta\|_{1/(\beta+1)} \right)^{1/(\beta+1)} \left| \frac{1}{(1-\|u\|_\infty)^{2/(\beta+1)}} - \frac{1}{(1+\|v\|_\infty)^{2/(\beta+1)}} \right|. \tag{2.8}$$

The real function $h(z) := (1-z)^{-2/(\beta+1)}$ defined on $[-\delta, \delta]$ satisfies

$$|h(z_1) - h(z_2)| \le \frac{2}{(\beta+1)(1-\delta)^{(\beta+3)/(\beta+1)}} |z_1 - z_2| \quad \text{for every} \quad z_1, z_2 \in [-\delta, \delta]. \tag{2.9}$$

Taking $z_1 = \|u\|_\infty$ and $z_2 = -\|v\|_\infty$ in (2.9) and applying in (2.8), we get

$$\|Z(u) - Z(v)\|_\infty \le \frac{2}{\beta+1} \left( \frac{|\lambda| \|\eta\|_{1/(\beta+1)}}{(1-\delta)^{\beta+3}} \right)^{1/(\beta+1)} \|u - v\|_\infty.$$

Thus, $Z$ is a contraction on $B_\delta$ with respect to the norm $\|\cdot\|_\infty$ provided

$$|\lambda| < \lambda_2(\delta) := \frac{1}{\|\eta\|_{1/(\beta+1)}} \left( \frac{\beta+1}{2} \right)^{\beta+1} (1-\delta)^{\beta+3}.$$

The biggest $\lambda$ for which $Z$ is a contraction is

$$\lambda_0 = \max_{\delta \in (0,1)} \{ \min\{\lambda_1(\delta), \lambda_2(\delta)\} \} = \frac{1}{\|\eta\|_{1/\beta+1}} \left( \frac{\beta+1}{\beta+3} \right)^{\beta+1} \left( \frac{2}{\beta+3} \right)^2,$$

which is attained for

$$\delta = \frac{\beta+1}{\beta+3}.$$

We conclude, by the Banach fixed-point Theorem, that $Z$ has a fixed point provided

Observe that the term

$$\frac{1}{s^\alpha}\int_0^s \frac{\lambda t^\gamma f(t)}{(1-u(t))^2}\,\mathrm{d}t \quad \text{is continuous for} \quad s\in(0,1]\,.$$

Thus,

$$u \in C([0,1]) \cap C^1((0,1]\,).$$

If $f$ is continuous and $\gamma > 0$, then the mentioned term is differentiable for $s \in (0,1]$. Thus,

$$u \in C([0,1]) \cap C^2((0,1]\,).$$

If $f$ is continuous, $\gamma > 0$ and $\alpha < \gamma + \beta + 2$ then, applying the L'Hôpital's rule, we get

$$u \in C^1([0,1]) \cap C^2((0,1]\,).$$

■

## 2.2 First approach on regularity of integral solutions

In this section, we are assuming the hypothesis (H.1) to (H.3). In addition, we assume

(H.5) $f$ is positive on $(0,1)$.

Lets look to equation (2.5) rewritten as follows:

$$u(r) = \varphi_\beta(\lambda)\int_r^1 \varphi_\beta(s^{-\alpha})G(u,s)\,\mathrm{d}s, \tag{2.10}$$

and

$$G(u,r) := \varphi_\beta\left(\int_0^r \frac{s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s\right). \tag{2.11}$$

Due to (H.5), the term $G$ defines a monotone function on $(0,1)\times B$ with respect to $r \in (0,1)$ and $u$ with the pointwise order.

The following proposition, among others results, asserts that, if $u$ is an integral solution of $(P_\lambda)$, then it can not cross the value 1.

**Proposition 2.4** *Suppose (H.1) to (H.3) and (H.5). Then, any integral solution of* $(P_\lambda)$ *is decreasing, precisely,*

$$u'(r) < 0 \quad \textit{for} \quad r \in (0,1). \tag{2.12}$$

*Moreover,*

$$u \in C([0,1]), \quad u(0) = \|u\|_\infty \leq 1 \quad \textit{and} \quad u(r) < 1 \textit{ for } r \in (0,1) \tag{2.13}$$

*and*

1. *If* $\|u\|_\infty < 1$*, then* $u \in C^2((0,1]) \cap C([0,1])$*.*

2. *If* $\gamma > \alpha - 1$ *and* $\|u\|_\infty < 1$*, then* $u \in C^2((0,1]) \cap C^{1,\mu}([0,1])$ *and* $u'(0) = 0$*.*

3. *If* $\gamma > \alpha + \beta$ *and* $\|u\|_\infty < 1$*, then we can consider* $u \in C^2([0,1]) \cap C^{1,\mu}([0,1])$*. In addition,* $u'(0) = u''(0) = 0$*.*

*Proof.* Looking at (2.10), we can easily see that $u \in C((0,1))$. Since $G(u,r)$ is positive for $r \in (0,1)$, it follows that $u'(r) < 0$ for $r \in (0,1)$, that is, $u$ is decreasing. Now, assume the existence of $\hat{r} \in (0,1)$ with $u(\hat{r}) = 1$. Thus, we can consider

$$r_0 = \inf\{r \in (0,1) : u(r) = 1\}.$$

Observe that $r_0 > 0$ since $u \in X$. Moreover, it follows by the continuity of $u$ that $u(r_0) = 1$. Let $\delta > 0$ be such that $[r_0 - \delta, r_0 + \delta] \subset (0,1)$. Since $u$ is decreasing, we have $u(r) \neq 0$ for all $r \in [r_0 - \delta, r_0 + \delta] \setminus \{r_0\}$. Moreover, since $r^{-\alpha}$ and $G(u,r)$ are bounded for $u$ fixed and $r$ varying in $[r_0 - \delta, r_0 + \delta]$, we have

$$|1 - u(r)| = |u(r_0) - u(r)| = \left| \int_r^{r_0} \varphi_\beta(\lambda s^{-\alpha}) G(u,s) \mathrm{d}s \right| \leq C|r_0 - r|,$$

where

$$C := \inf\{r^\gamma f(r) : r \in [r_0 - \delta,\, r_0 + \delta]\} > 0.$$

It follows that

$$\int_{r_0-\delta}^{r_0+\delta} \frac{r^\gamma f(r)}{(1-u(r))^2} \mathrm{d}r \geq C \int_{r_0-\delta}^{r_0+\delta} \frac{1}{(r-r_0)^2} \mathrm{d}r = +\infty,$$

which is in a contradiction with the definition of integral solution once we require

$$\int_0^1 \frac{r^\gamma f(r)}{(1-u(r))^2}\mathrm{d}r < \infty.$$

Thus we conclude that (2.13) holds.

To verify item *1* we just have to use (2.13) and the facts that $r^\gamma f(r)(1-u(r))^{-2}$ is continuous for $r \in (0,1]$ and $\varphi_\beta$ is a diffeomorphism on $(0,\infty)$.

In order to prove item *2*, suppose $\|u\|_\infty < 1$ and denote

$$g(r) := \int_0^r \frac{s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s.$$

By L'Hôpital's Rule, we have

$$\lim_{r\to 0} g(r) = \lim_{r\to 0} \frac{r^\gamma f(r)}{\alpha r^{\alpha-1}\left(1-u(r)\right)^2}. \tag{2.14}$$

We can see that $g$ is a continuous function over $[0,1]$ with

$$g(0) = f(0)/\alpha(1-u(0))^2 \quad \text{if} \quad \gamma = \alpha - 1 \quad \text{or} \quad g(0) = 0 \quad \text{if} \quad \gamma > \alpha - 1.$$

In both cases we conclude that $u \in C^1([0,1])$. We emphasize that if $\gamma > \alpha - 1$, we have $u'(0) = 0$.

To prove the Hölder continuity of $u'$, since $\varphi_\beta$ is Hölder continuous, it is sufficient to prove that $g$ is Hölder continuous. Since $g$ is differentiable in $(0,1]$,

$$g(r) - g(s) = \int_s^r g'(t)\,\mathrm{d}t \quad \text{for every} \quad r,s \in (0,1].$$

Using Hölder inequality with $p > 1$ and $1/p + 1/q = 1$, we have

$$|g(r) - g(s)| \le |r-s|^{1/p}\left(\int_s^r |g'(t)|^q\,\mathrm{d}t\right)^{1/q} \quad \text{for every} \quad r,s \in (0,1]. \tag{2.15}$$

Since

$$g'(r) = \frac{\alpha}{r^{\alpha+1}}\int_0^r \frac{\lambda s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s - r^{\gamma-\alpha}\frac{\lambda f(r)}{(1-u(r))^2},$$

using L'Hôpital's rule we get

$$\lim_{r\to 0} g'(r) = \lim_{r\to 0}\left(\frac{\alpha}{\alpha+1} - 1\right) r^{\gamma-\alpha}\frac{\lambda f(r)}{(1-u(r))^2},$$

that is, $g'(r) = O(r^{\gamma-\alpha})$ as $r \to 0$ and the integral in (2.15) is bounded if $(\gamma-\alpha)q > -1$ or equivalently

$$\gamma - \alpha + 1 > 1 - \frac{1}{q} = \frac{1}{p}. \tag{2.16}$$

Thus, if $\gamma > \alpha - 1$, we can find $p > 1$ satisfying (2.16) and we conclude that $u'$ is Hölder continuous on $[0, 1]$ with exponent

$$\mu < \begin{cases} \gamma - \alpha + 1 & \text{if } -1 < \beta < 0, \\ \dfrac{\gamma - \alpha + 1}{\beta + 1} & \text{if } \beta \geq 0. \end{cases}$$

Now we are going to verify that $u'$ is differentiable at 0. Note that

$$\lim_{r\to 0} \frac{\varphi_\beta\left(\lambda g(r)\right)}{r} = \lim_{r\to 0} \varphi_\beta \left(\frac{\lambda g(r)}{r^{\beta+1}}\right) = \varphi_\beta \left(\lambda \lim_{r\to 0} \frac{g(r)}{r^{\beta+1}}\right).$$

Again by the L'Hôpital's Rule, $\lim_{r\to 0} g(r)/r^{\beta+1}$ exists when $\gamma \geq \alpha + \beta$ and vanishes to zero when $\gamma > \alpha + \beta$. This completes the proof.

■

## 2.3 Existence of the Pull-in Voltage

Here, on the light of Proposition 2.4, we prove the existence of a critical parameter $\lambda^*$ for the existence and non-existence of minimal solutions of $(P_\lambda)$ with respect to the parameter $\lambda$. On the following, we obtain an upper bound for the set of parameters $\lambda$ for which $(P_\lambda)$ possesses solution.

**Proposition 2.5** *Suppose (H.1) to (H.3) and (H.5). If $(P_\lambda)$ has an integral solution $u \in X$, then*

$$\int_0^1 \varphi_\beta(r^{-\alpha}) G(r,0)\,\mathrm{d}r < \infty$$

*and*

$$\lambda \leq \left(\varphi_\beta^{-1}\left(\int_0^1 \varphi_\beta(r^{-\alpha}) G(r,0)\,\mathrm{d}r\right)\right)^{-1}.$$

*Proof.* The assumption (H.3) implies that $g(r) := G(r,0)$ is a well defined function on $(0,1)$. Suppose that $(P_\lambda)$ admits an integral solution $u \geq 0$. Then, we have

$$\begin{aligned} 1 \geq u(0) &= \varphi_\beta(\lambda) \int_0^1 \varphi_\beta(r^{-\alpha}) G(r,u)\,\mathrm{d}r \\ &\geq \varphi_\beta(\lambda) \int_0^1 \varphi_\beta(r^{-\alpha}) G(r,0)\,\mathrm{d}r. \end{aligned}$$

We used in the estimate above that $G$ is an increasing function with respect to its second variable and $u \geq 0$.

■

**Remark 2.6** We emphasize that in Proposition 2.5 we have proved that problem $(P_\lambda)$ has no solution, even in the integral sense, if

$$\lambda > \left( \varphi_\beta^{-1} \left( \int_0^1 \varphi_\beta(r^{-\alpha}) G(r,0)\,\mathrm{d}r \right) \right)^{-1}.$$

Then we can define

$$\lambda^* := \sup\{\lambda > 0 : (P_\lambda) \text{ has a classical solution}\}, \tag{2.17}$$

where classical means that $u \in C^2(0,1] \cap C([0,1])$ and solves $(P_\lambda)$. We can also define

$$\lambda^{**} := \sup\{\lambda > 0 : (P_\lambda) \text{ has an integral solution}\}. \tag{2.18}$$

If follows immediately

$$\lambda^* \leq \lambda^{**} < \left( \varphi_\beta^{-1} \left( \int_0^1 \varphi_\beta(r^{-\alpha}) G(r,0)\,\mathrm{d}r \right) \right)^{-1}. \tag{2.19}$$

Indeed, we prove in the next Section that

$$\lambda^* = \lambda^{**}.$$

## 2.4 A Shooting Method approach

In this section, we are supposing hypothesis from (H.1) to (H.5). Using the Shooting Method, we prove existence and uniqueness for solution of $(P_\lambda)$ for every $\lambda$ in a neighborhood of 0. We also assert the continuous dependence of the solutions with respect to the parameter $\lambda$. In other words, there exists a branch of solutions passing through 0. An immediate consequence of it is that, if the branch of solutions bifurcates in the line $\lambda = \lambda^*$ then, it stops or bifurcates again before $\lambda = 0$.

In order to study problem $(P_\lambda)$, consider for some $\tau \in (0,1)$ the auxiliary problem

$$\begin{cases} u'(r) = -\varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,u), \\ u(0) = \tau, \end{cases} \tag{$P_{\lambda,\tau}$}$$

where

- $\tilde{f} : (0,+\infty) \to \mathbb{R}$ is the extension of $f$ given by

$$\tilde{f}(r) = \begin{cases} r^\gamma f(r) & \text{for } 0 \leq r < 1 \\ 0 & \text{for } r > 1. \end{cases} \tag{2.20}$$

- $\tilde{\varphi} : (0,+\infty) \to \mathbb{R}$ is a $C^1$-function satisfying

$$\tilde{\varphi}(r) = \begin{cases} \varphi_\beta(r^{-\alpha}) & \text{for } 0 < r < 1 \\ 0 & \text{for } r > 2. \end{cases} \tag{2.21}$$

- The nonlocal term $\tilde{G}$ is given by

$$\tilde{G}(r,u) := \varphi_\beta\left(\int_0^r \frac{\tilde{f}(s)}{(1-u(s))^2}\,\mathrm{d}s\right). \tag{2.22}$$

Observe that assumption (H.3) implies

$$\int_0^{+\infty} \tilde{\varphi}(r)\tilde{G}(r,0)\,\mathrm{d}r < \infty. \tag{2.23}$$

**Proposition 2.7** *Denote*

$$\tilde{C} := \left(\varphi_\beta^{-1}\left(\frac{1}{2}\int_0^{+\infty} \tilde{\varphi}(r)\tilde{G}(r,0)\,\mathrm{d}r\right)\right)^{-1} > 0.$$

*If* $0 < \lambda < \tilde{C}$, *then problem* $(P_{\lambda,\tau})$ *possesses a unique classical solution* $u = u_{\lambda,\tau}$ *such that* $u(1) = 0$ *for some* $\tau \in (0,1)$.

Observe that, for $\tau \in (0,1)$, a solution $u$ of Problem $P_{\lambda,\tau}$ satisfying $u(1)=0$ is automatically a solution of Problem $P_\lambda$. Writing

$$C := \left(\varphi_\beta^{-1}\left(\frac{1}{2}\int_0^1 \varphi_\beta\left(\frac{1}{r^\alpha}\int_0^r s^\gamma f(s)\mathrm{d}s\right)\mathrm{d}r\right)\right)^{-1},$$

despite $C < \tilde{C}$, the constant $\tilde{C}$ can be taken as much close of $C$ as we need for a convenient choice of the term $\tilde{\varphi}$. Then, we have the following result.

**Theorem 2.8** *Problem$(P_\lambda)$ has a unique regular solution provided*

$$\lambda < \left(\varphi_\beta^{-1}\left(\frac{1}{2}\int_0^1 \varphi_\beta\left(\frac{1}{r^\alpha}\int_0^r s^\gamma f(s)\mathrm{d}s\right)\mathrm{d}r\right)\right)^{-1}.$$

It remains to prove Proposition 2.7. In order to do that, we need the following result.

**Lemma 2.9** *For any $\tau \in (0,1)$ and $\lambda > 0$, problem $(P_{\lambda,\tau})$ possesses a unique positive solution $u_{\lambda,\tau}$ defined on the maximal interval $[0,\omega_\tau)$, where $\omega_\tau \in \mathbb{R}\cup\{+\infty\}$. Moreover, if $\omega_\tau \in \mathbb{R}$ then*

$$\lim_{r\to\omega_\tau} u_\tau(r) = 0.$$

*Furthermore, the family of solutions $(u_{\lambda,\tau})_{\tau\in(0,1)}$ is strictly increasing with respect to the parameter $\tau$, that is, given $\underline{\tau} < \overline{\tau}$, then $\omega_{\underline{\tau}} \le \omega_{\overline{\tau}}$ and*

$$u_{\underline{\tau}}(r) < u_{\overline{\tau}}(r), \qquad \textit{for any} \qquad r \in [0,\omega_{\underline{\tau}}).$$

*Proof. Part I (existence and uniqueness of local solutions).* As in the proof of the classical Picard's Theorem, we can obtain by Banach Fixed Point Theorem the existence and uniqueness of a local solution for the following nonlocal ODE problem:

$$\begin{cases} u'(r) = -\varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,u), \\ u(r_0) = \tau, \end{cases} \tag{2.24}$$

where $r_0 \in [0,+\infty)$. Moreover, the length of the interval where such solution is defined does not depend on $r_0$.

Ideed, fix $\varepsilon > 0$ such that $\tau+\varepsilon < 1$ and choose a constant $\delta$ satisfying

$$0 < \delta < \min\left\{\frac{\varepsilon}{\varphi_\beta(\lambda)MN}, \frac{(\beta+1)(1-(\tau+\varepsilon))^{3/(\beta+1)}}{2M\varphi_\beta(2P\lambda)}\right\}, \tag{2.25}$$

where $M, N$ and $P$ are the following positive constants

$$M = \max_{r\in[0,+\infty)} \tilde{\varphi}(r), \qquad N = \max_{r\in[0,+\infty)} \tilde{G}(r, \tau+\varepsilon) \quad \text{and} \quad P := \int_0^{+\infty} \tilde{f}(r)\,\mathrm{d}r. \tag{2.26}$$

Now consider for such $\delta$ the Banach space $C_\delta$ of the continuous real valuated functions defined over the interval $[r_0, r_0+\delta]$ endowed with the uniform convergence norm. Consider also the function $I : B \to C_\delta$, where $B := \{u \in C_\delta : \|u\|_\infty \leq \tau + \varepsilon\}$, given by

$$[I(u)](r) := \tau - \int_{r_0}^{r} \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(u, r)\,\mathrm{d}r.$$

The operator $I$ is well defined due to assumption (2.23).

Endowing $C_\delta$ with the pointwise partial order, we see that $\tilde{G}$ is decreasing with respect to the variable $u$. Then

$$\tilde{G}(u, r) \leq \tilde{G}(\tau+\varepsilon, r) \quad \text{for all } u \in B.$$

It follows from the triangular inequality and (2.26) that,

$$\begin{aligned}\|I(u)\|_\infty &\leq \tau + \int_{r_0}^{r_0+\delta} \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(u, r)\,\mathrm{d}r \\ &\leq \tau + \delta\varphi_\beta(\lambda)MN,\end{aligned}$$

which together with (2.25) becomes

$$\|I(u)\|_\infty < \tau + \varepsilon$$

Then $I$ is a well defined application on $B$ into itself.

Given $(u, v) \in B \times B$, consider $w : [0, 1] \to B$ given by

$$w(\theta) = \theta u + (1-\theta)v$$

and $\xi : [0, 1] \times [0, 1] \to \mathbb{R}$ given by

$$\xi(\theta, r) := I(w(\theta))(r).$$

It is easy to see that $\xi$ is differentiable with respect to $\theta$ with

$$\frac{\partial}{\partial\theta}\xi(\theta, r) := \int_{r_0}^{r} \varphi_\beta(\lambda)\tilde{\varphi}(s)\varphi_\beta'\left(\int_0^s \frac{\tilde{f}(t)}{(1-w(\theta)(t))^2}\,\mathrm{d}t\right)\left(\int_0^s \frac{2\tilde{f}(t)(u(t)-v(t))}{(1-w(\theta)(t))^3}\,\mathrm{d}t\right)\mathrm{d}s.$$

Observing that

$$z\varphi_\beta'(z) = \frac{1}{\beta+1}\varphi_\beta(|z|)$$

and using (2.26), we see that

$$\left|\frac{\partial}{\partial\theta}\xi(\theta,r)\right| \leq \frac{\delta 2M\varphi_\beta(P\lambda)\|u-v\|_\infty}{(\beta+1)(1-(\tau+\varepsilon))^{3/(\beta+1)}}.$$

Since, $\xi(0,r) = I(v)(r)$ and $\xi(1,r) = I(u)(r)$, applying Mean Value Theorem on the variable $\theta$, we get

$$|I(u)(r) - I(v)(r)| \leq \frac{\delta 2M\varphi_\beta(P\lambda)}{(\beta+1)(1-(\tau+\varepsilon))^{3/(\beta+1)}}\|u-v\|_\infty.$$

It follows from the choice made in (2.25) that $I$ is a contraction. We conclude by the Banach Fixed Point Theorem that problem (2.24) admits a unique local solution defined on the interval $[r_0, r_0+\delta]$ for every $r_0 \geq 0$.

If $r_0 > 0$, choose $\delta > 0$ satisfying (2.25) and such that $0 < r_0 - \delta$. Observe that the previous argument can be analogously followed if we change $C_\delta$ by the set

$$\tilde{C}_\delta := \{u : [r_0-\delta, r_0] \to \mathbb{R} : u \text{ is continuous }\}.$$

Then we conclude that in this case there exists also a unique local solution of (2.24) defined on the interval $[r_0-\delta, r_0]$.

*Part 2 (existence, uniqueness and monotonicity with respect to the parameter $\tau$ of global solutions).* Given $\tau > 0$ and using the results in *Part I*, it follows from standard continuation argument that $(P_{\lambda,\tau})$ admits a unique positive solution $u_\tau$ defined on a maximal interval of the form $[0, \omega_\tau)$. It may happen $\omega_\tau = +\infty$ or $\omega_\tau < +\infty$ and

$$\lim_{r\to\omega_\tau} u_\tau(r) = 0.$$

Given $0 < \underline{\tau} < \overline{\tau}$, suppose that there exists $r_0 \in [0, \omega_{\underline{\tau}})$ such that $r_0 \notin [0, \omega_{\overline{\tau}})$. It follows that $\omega_{\overline{\tau}} < +\infty$ and $\omega_{\overline{\tau}} < \omega_{\underline{\tau}}$. So we can consider $u_{\underline{\tau}}(\omega_{\overline{\tau}})$, which is positive. Since $u_{\underline{\tau}}(0) = \underline{\tau} < \overline{\tau} = u_{\overline{\tau}}(0)$ and $u_{\overline{\tau}}$ is continuous, there exists $r_0 \in (0, \omega_{\overline{\tau}})$ such that $u_{\underline{\tau}}(r_0) = u_{\overline{\tau}}(r_0)$, that is, the curves $u_{\underline{\tau}}$ and $u_{\overline{\tau}}$ intersect each other. Consider the first intersection point and still denote it by $r_0$. The existence of such point is in contradiction with the possibility of applying a continuation argument on the left side of $r_0$ as it is asserted on *Part I*. Thus we conclude that $\omega_{\underline{\tau}} \leq \omega_{\overline{\tau}}$. The analogous argument can be used to prove that

$$u_{\underline{\tau}}(r) < u_{\overline{\tau}}(r), \quad \text{for any} \quad r \in [0, \omega_{\underline{\tau}}) \,.$$

■

Now we are able to prove Proposition 2.7.

*Proof.* [Proof of Proposition 2.7] Consider the following sets

$$\overline{S} := \{\tau \in (0,1) : u_\tau \text{ exists on } [0,1) \text{ and satisfies } u_\tau(1) > 0\}$$
$$\underline{S} := \{\tau \in (0,1) : \omega_\tau < 1\}.$$

First we will prove that $\overline{S}$ is nonempty. Due to ((H.3)), we can choose $\tau$ and $\varepsilon$ satisfying

$$2\int_0^{+\infty} \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,0)\,\mathrm{d}s < \tau < 1 - \varepsilon < 1. \tag{2.27}$$

Suppose that $\omega_\tau \in (0,1)$. Since $u_\tau(0) = \tau$ and $u_\tau(\omega_\tau) = 0$, and using the continuity of $u_\tau$, there exists $r_1 \in (0, \omega_\tau) \subset (0,1)$ such that $u_\tau(r_1) = \tau/2$. Observe that $G$ is increasing with respect to the first variable, namely, and $u_\tau > 0$, we have

$$u_\tau(r_1) - u_\tau(0) = -\int_0^{r_1} \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r, u_\tau)\,\mathrm{d}s \geq -\int_0^{r_1} \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,0)\,\mathrm{d}s,$$

This, together with (2.27) carries

$$-\frac{\tau}{2} = u_\tau(r_1) - u(0) > -\frac{\tau}{2},$$

which is an absurd. Thus, it follows that $u_\tau(r) > \tau/2$ for all $r \in (0,1)$.

Now we prove that $\underline{S}$ is nonempty. Assume by contradiction that $\underline{S} = \emptyset$. It is clear that in this case, we have $\omega_\tau \geq 1$. Since we are supposing that $f$ is positive on $(0,1)$, it follows that $\tilde{f}$ has the same property and, consequently, $\tilde{G}$ is positive on $(0,+\infty) \times C([0,+\infty))$. Choose $\tau > 0$ sufficiently small such that

$$\int_0^{1/2} \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,0)\,\mathrm{d}r > \tau.$$

In this way, we have that

$$u_\tau(1/2) - u_\tau(0) = -\int_0^{1/2} \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,u_\tau)\,\mathrm{d}r \leq -\int_0^{1/2} \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,0)\,\mathrm{d}r < -\tau.$$

Finally, we find

$$u_\tau(1/2) < \tau - \tau = 0,$$

which is a contradiction because $u_\tau(r) > 0$ for all $r \in (0, \omega_\tau)$.

We claim that $\tau^* := \inf \overline{S}$ does not belong to $\overline{S}$. Indeed, suppose by contradiction that $\tau^* \in \overline{S}$. Thus $\omega_{\tau^*} > 1$ and $\ell := u_{\tau^*}(1) > 0$. Because of the continuous dependence of the solutions with respect to the parameter $\tau$, there exists $\tau \in (0, \tau^*)$ sufficiently close to $\tau^*$ such that $u_\tau(1) > \ell/2$. Thus, $\tau \in \overline{S}$, which is a contradiction with the definition of $\tau^*$.

We have also that $\tau_* := \sup \underline{S}$ does not belong to $\underline{S}$. Suppose by contradiction that $\tau_* \in \overline{S}$, that is, $\omega_{\tau_*} < 1$. Take an arbitrary $\delta > 0$ satisfying

$$\delta < \int_{\omega_{\tau_*}}^1 \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,0)\,\mathrm{d}r\,\mathrm{d}r.$$

Due to the continuous dependence of $u_\tau$ with respect to the parameter $\tau$, we can take $\tau > \tau_*$ such that $0 < u_\tau(\tau_*) < \delta$. According to de definition of $\tau_*$ we have $\omega_\tau > 1$ and $\omega_\tau(1) > 0$. Thus,

$$\begin{aligned} u_\tau(1) &= u_\tau(\omega_{\tau^*}) - \int_{\omega_{\tau^*}}^1 \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,\omega_\tau)\,\mathrm{d}r \\ &< \delta - \int_{\omega_{\tau^*}}^1 \varphi_\beta(\lambda)\tilde{\varphi}(r)\tilde{G}(r,0)\,\mathrm{d}r \\ &< \delta - \delta < 0. \end{aligned}$$

Finally we prove that $\sup \overline{S} = \inf \underline{S}$. It follows from Lemma 2.9 that if $\tau_1 \in \overline{S}$ and $\tau_2 \in \underline{S}$ then $\tau_1 < \tau_2$. So we already know that $\sup \overline{S} \leq \inf \underline{S}$. Suppose $\overline{\tau} := \sup \overline{S} < \underline{\tau} := \inf \underline{S}$. Since $\underline{\tau}, \overline{\tau}$ belong neither to $\overline{S}$ nor to $\underline{S}$, we find that $\omega_{\underline{\tau}} = \omega_{\overline{\tau}} = 1$, which is a contradiction together with Lemma 2.9, whereby we have $\omega_{\underline{\tau}} < \omega_{\overline{\tau}}$.

■

# 3 The branch of minimal solutions

## 3.1 A sub- and super-solution argument

Now, we present a suitable sub- and supersolution procedure to approach problem $(P_\lambda)$ (Proposition 3.2). Using such a technique we can prove that $(P_\lambda)$ admits classical solution whenever $\lambda \in (0, \lambda^*)$ and no solution, even in the integral sense, for $\lambda > \lambda^*$, where $\lambda^*$ is defined in (2.17). In other words, writing

$$\Lambda^* := \{\lambda > 0 : (P_\lambda) \text{ has a classical solution}\},$$

we prove that $\Lambda$ is a nondegenerate and bounded interval.

On the following we consider the general problem

$$\begin{cases} -(r^\alpha |u'(r)|^\beta u'(r))' = h(r, u(r)), & r \in (0,1), \\ \qquad u'(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $h$ is a real-valued continuous function acting over a set $H \subset \mathbb{R}^2$ which projection over the first variable is $(0, 1)$.

In this case we can also consider the following notions:

**Definition 3.1 (Kind of solutions of Problem (3.1))**

Integral solution: $u \in X$ is an *integral solution* of problem (3.1) provided that $(r, u(r)) \in H$ for every $r \in (0, 1)$, $h(\cdot, u) \in L^1((0, 1))$ and

$$-r^\alpha |u'(r)|^\beta u'(r) = \int_0^r h(s, u(s))\, \mathrm{d}s \quad \text{a.e. in} \quad (0,1). \tag{3.2}$$

Integral super-solution: We say that $\overline{u} \in X$ is an *integral super-solution* of problem (3.1) when $(r, \overline{u}(r)) \in H$ for every $r \in (0,1)$, $h(\cdot, \overline{u}) \in L^1((0,1))$ and

$$-r^\alpha |\overline{u}'(r)|^\beta \overline{u}'(r) \geq \int_0^r h(s, \overline{u}(s))\ \mathrm{d}s \qquad \text{a.e. in} \qquad (0,1).$$

Integral sub-solution: In the same way, a function $\underline{u} \in X$ is said to be an *integral sub-solution* of (3.1) if $(r, \underline{u}(r)) \in H$ for every $r \in (0,1)$, $h(\cdot, \underline{u}) \in L^1((0,1))$ and

$$-r^\alpha |\underline{u}'(r)|^\beta \underline{u}'(r) \leq \int_0^r h(s, \underline{u}(s))\ \mathrm{d}s \qquad \text{a.e. in} \qquad (0,1).$$

Minimal solution: We call minimal solution a positive solution (in any sense defined above) $u \in X$ of (3.1) such that for any other positive solution $v \in X$ of (3.1) one has $0 \leq u(r) \leq v(r)$ for all $r \in (0,1)$.

Before we state the sub- and supersolution method suitable for our study, we point out some facts about the particular case when $h$ is constant with respect to the second variable, namely,

$$\begin{cases} -(r^\alpha |u'(r)|^\beta u'(r))' = g(r) \quad r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{3.3}$$

where $\alpha$ and $\beta$ are real constants and $g$ is a continuous and integrable real function over the interval $(0,1)$ satisfying the following conditions. Consider

$$\tilde{g}(r) := \frac{1}{r^\alpha} \int_0^r g(s)\ \mathrm{d}s,$$

suppose $\beta > -1$ and let

$$\lim_{s \to 0} \tilde{g}(s) = 0, \tag{3.4}$$

which are hypotheses on $\alpha$ and $g$. In view of this, one can define a function $u : [0,1] \to \mathbb{R}$ given by

$$u(r) := \int_r^1 \varphi_\beta \left( \frac{1}{s^\alpha} \int_0^s g(t)\ \mathrm{d}t \right)\ \mathrm{d}s \qquad \text{for } r \in (0,1).$$

In fact, due to (3.4), $\tilde{g}$ is continuous on $[0,1]$. Since $\beta > -1$ then, $\varphi_\beta$ defines a homeomorphism from $\mathbb{R}$ into itself. Thus $\varphi_\beta \circ \tilde{g}$ is continuous on $[0,1]$ and consequently $u$ is a well defined element of $C^1[0,1]$. Observe that $u$ satisfies

$$-r^\alpha |u'(r)|^\beta u'(r) = \int_0^r g(s)\ \mathrm{d}s \qquad \text{pointwise in } (0,1).$$

Observe, in particular, that $u$ is an integral solution of $(P_\lambda)$ for $h(r,z) = g(r)$.

Now we are ready to present the method of sub- and supersolution.

**Proposition 3.2 (Method of sub- and supersolution)** *Suppose that $\beta > -1$, that $h$ is continuous on $H$ and nondecreasing on the second variable, and assume that there exist $\underline{u} \leq \overline{u}$, an integral subsolution and an integral supersolution, respectively, of* (3.1) *with*

$$\lim_{s\to 0} \frac{1}{s^\alpha} \int_0^s h(t, \underline{u}(t))\mathrm{d}t = \lim_{s\to 0} \frac{1}{s^\alpha} \int_0^s h(t, \overline{u}(t))\mathrm{d}t = 0.$$

*Then there exists an integral solution $u \in X$ of* (3.1) *satisfying*

$$\underline{u} \leq u \leq \overline{u}.$$

*Moreover, $u$ is given by*

$$u(r) := \lim_{k\to\infty} u_k(r),$$

*where $(u_k) \subset X$ is a sequence given recursively as follows. We set $u_0 = \underline{u}$ and define $u_{k+1}$ as the unique solution of the problem*

$$\begin{cases} -(r^\alpha |u'_{k+1}(r)|^\beta u'_{k+1}(r))' = \ h(r, u_k(r)) \quad \text{in } (0,1), \\ u'_{k+1}(0) = u_{k+1}(1) = 0, \end{cases} \tag{3.5}$$

*which can be explicitly written as*

$$u_{k+1}(r) = \int_r^1 \varphi_\beta \left( \frac{1}{s^\alpha} \int_0^s h(t, u_k(t))\, \mathrm{d}t \right) \mathrm{d}s \tag{3.6}$$

*In particular, $u_k \to u$ in $X$.*

*Proof.* We divide the proof into three parts.

**Claim 3.3** The sequence $(u_k)$ is well defined and

$$\underline{u} \leq u_k \leq \overline{u} \quad \text{for all} \quad k \in \mathbb{N}.$$

Suppose that the claim is true for some $k \in \mathbb{N}$. Since $h(r, \overline{u}(r)) \in L^1(0,1)$ and

$$h(r, \underline{u}(r)) \leq h(r, u_k(r)) \leq h(r, \overline{u}(r)),$$

it follows that $h(r,u_k(r)) \in L^1(0,1)$. Observe that,

$$\lim_{s\to 0}\frac{1}{s^\alpha}\int_0^s h(r,\underline{u}(r))\,\mathrm{d}t \le \lim_{s\to 0}\frac{1}{s^\alpha}\int_0^s h(r,u_k(r))\,\mathrm{d}t \le \lim_{s\to 0}\frac{1}{s^\alpha}\int_0^s h(r,\overline{u}(r))\,\mathrm{d}t = 0.$$

Then, taking $g := h(\cdot,u_k)$ in (3.3), since such $g$ is continuous on $(0,1)$, the conditions on $\alpha,\beta$ and $g$ in the beginning of this section are satisfied. So we conclude that $u_{k+1}$ given by (3.6) is a well defined element of $C^1([0,1])$.

Since $\varphi_\beta$ is increasing for $\beta > -1$, we have

$$\begin{aligned} u_{k+1}(r) &= \int_r^1 \varphi_\beta\left(\frac{1}{s^\alpha}\int_0^s h(t,u_k(t))\,\mathrm{d}t\right)\mathrm{d}s \\ &\le \int_r^1 \varphi_\beta\left(\frac{1}{s^\alpha}\int_0^s h(t,\overline{u}(t))\,\mathrm{d}t\right)\mathrm{d}s \le \overline{u}(r) \end{aligned}$$

and analogously,

$$\begin{aligned} u_{k+1}(r) &= \int_r^1 \varphi_\beta\left(\frac{1}{s^\alpha}\int_0^s h(t,u_k(t))\,\mathrm{d}t\right)\mathrm{d}s \\ &\ge \int_r^1 \varphi_\beta\left(\frac{1}{s^\alpha}\int_0^s h(t,\underline{u}(t))\,\mathrm{d}t\right)\mathrm{d}s \ge \underline{u}(r). \end{aligned}$$

Whence we conclude that

$$\underline{u} \le u \le \overline{u}.$$

Observe also that

$$\begin{aligned} u_{k+1}'(r) &= -\varphi_\beta\left(\frac{1}{r^\alpha}\int_0^r h(s,u_k(s))\,\mathrm{d}s\right) \\ &\le -\varphi_\beta\left(\frac{1}{r^\alpha}\int_0^r h(s,\underline{u}(s))\,\mathrm{d}s\right) = \underline{u}(r) \end{aligned}$$

and

$$\begin{aligned} u_{k+1}'(r) &= -\varphi_\beta\left(\frac{1}{r^\alpha}\int_0^r h(s,u_k(s))\,\mathrm{d}s\right) \\ &\ge -\varphi_\beta\left(\frac{1}{r^\alpha}\int_0^r h(s,\overline{u}(s))\,\mathrm{d}s\right) \ge \overline{u}'(r), \end{aligned}$$

that is,

$$\overline{u}' \le u_{k+1}' \le \underline{u}'$$

pointwise on $[0,1]$. Then

$$r^\alpha|\overline{u}'(r)|^{\beta+2} \le r^\alpha|u_{k+1}'(r)|^{\beta+2} \le r^\alpha|\underline{u}'(r)|^{\beta+2}$$

pointwise on $(0,1]$. Thus we have that $u_{k+1} \in X$. Since the claim is automatically true for $k = 0$, we conclude by an induction argument that $(u_k)$ is well defined and $\underline{u} \leq u_k \leq \overline{u}$ for all $k \in \mathbb{N}$.

**Claim 3.4** The sequence $(u_k)$ is non-decreasing.

We proved above that $u_1(r) \geq \underline{u}(r) = u_0(r)$. Taking $k \in \mathbb{N}$ such that our claim is true up to $k$, we have

$$\begin{aligned} u_{k+1}(r) &= \int_r^1 \varphi_\beta\left(\frac{1}{s^\alpha}\int_0^s h(t, u_k(t))\,\mathrm{d}t\right)\mathrm{d}s \\ &\geq \int_r^1 \varphi_\beta\left(\frac{1}{s^\alpha}\int_0^s h(t, u_{k-1}(t))\,\mathrm{d}t\right)\mathrm{d}s = u_k(r). \end{aligned}$$

Then our claim is valid inductively for every $k \in \mathbb{N}$.

**Claim 3.5** The sequence $(u_k)$ converges in $X$ to an integral solution $u$ of (3.1). Moreover, for any $r \in (0,1]$, we have

$$u(r) := \lim_{k\to\infty} u_k(r). \tag{3.7}$$

Due to the arguments above, the function $u : [0,1] \to \mathbb{R}$ given by (3.7) is well defined. By the monotone convergence theorem, $r^{\alpha/(\beta+2)}u(r) \in L^{\beta+2}((0,1))$ and

$$\int_0^1 r^\alpha |u_k(r)|^{\beta+2}\,\mathrm{d}r \to \int_0^1 r^\alpha |u(r)|^{\beta+2}\,\mathrm{d}r.$$

Now, using the Brezis-Lieb Lemma (see [Brézis e Lieb 1983]), we have

$$\int_0^1 r^\alpha |u_k(r) - u(r)|^{\beta+2} \to 0.$$

The monotonicity of $(u_k)$ with respect to $k$ implies a monotonicity property for the sequence of the corresponding derivatives $(u_k')$. In fact, for every $k \geq 1$,

$$\begin{aligned} u_{k+1}'(r) &= -\varphi_\beta\left(\frac{1}{r^\alpha}\int_0^r h(s, u_k(s))\,\mathrm{d}s\right) \\ &\leq -\varphi_\beta\left(\frac{1}{r^\alpha}\int_0^r h(s, u_{k-1}(s))\,\mathrm{d}s\right) = u_k'(r). \end{aligned}$$

Thus, for almost every $r \in [0,1]$, we can define

$$v(r) := \lim_{k\to\infty} u_k'(r).$$

Observe that $\left(r^\alpha |u_k'|^{\beta+2}(r)\right)$ is a non-decreasing sequence and, as a consequence of the monotone convergence theorem,

$$\int_0^1 r^\alpha |u_k'(r)|^{\beta+2}\,\mathrm{d}r \to \int_0^1 r^\alpha |v(r)|^{\beta+2}\,\mathrm{d}r, \qquad \text{as } k \to \infty.$$

Furthermore, the sequence $\left(r^{\alpha/(\beta+2)} u_k'(r)\right)$ is bounded in $L^{\beta+2}((0,1))$, since

$$\int_0^1 r^\alpha |u_k'(r)|^{\beta+2}\,\mathrm{d}r \le \int_0^1 r^\alpha |\overline{u}'(r)|^{\beta+2}\,\mathrm{d}r < \infty.$$

Then, by Brezis-Lieb lemma [Brézis e Lieb 1983], we have

$$\int_0^1 r^\alpha |u_k'(r) - v(r)|^{\beta+2} \to 0.$$

Thus $(u_k)$ is a Cauchy sequence in the complete space $X$. Since the norms defined in (1.4) and (1.3) are equivalent and $v(0) = \lim_{k\to\infty} u_k'(0) = 0$, we conclude that

$$\int_1^r v(s)\,\mathrm{d}s = u(r) \in X.$$

For almost every $r \in [0,1]$, we have

$$\begin{aligned} -r^\alpha |u'(r)|^\beta u'(r) &= -\lim_{k\to\infty} r^\alpha |u_{k+1}'(r)|^\beta u_{k+1}'(r) \\ &= \lim_{k\to\infty} \int_0^r h(s, u_k(s))\,\mathrm{d}s \\ &= \int_0^r h(s, u(s))\,\mathrm{d}s. \end{aligned}$$

Due to the monotonicity property of $h(r,z)$ for $z \in I$, it follows that

$$\int_0^1 h(r, u(r))\,\mathrm{d}r \le \int_0^1 h(r, \overline{u}(r))\,\mathrm{d}r < \infty.$$

Then we conclude that $u$ is an integral solution of (3.1).

■

## 3.2 Describing the branch of minimal solutions

Our description of the branch of minimal solutions is given by means of the three following results. The first result asserts that there exists a critical value $\lambda^* > 0$ for the existence of positive solutions for $(P_\lambda)$. More precisely, $(P_\lambda)$ admits classical solution whenever $\lambda \in (0, \lambda^*)$ and none, even in the integral sense, for $\lambda > \lambda^*$. In other words, let

$$\Lambda := \{\lambda > 0 : (P_\lambda) \text{ has a classical solution}\}$$

we have that $\Lambda$ is a non-degenerate and bounded interval. The boundedness of set $\Lambda$ was proved in Section 2.3 (Proposition 2.5). This result can be written as follows.

**Theorem 3.6** *There exists a finite pull-in voltage $\lambda^* > 0$ such that*

1. *If $0 \le \lambda < \lambda^*$, there exists at least one regular solution for $(P_\lambda)$;*

2. *If $\lambda = \lambda^*$, there exists at least one integral solution for $(P_\lambda)$;*

3. *If $\lambda > \lambda^*$, there is no solution of any kind for $(P_\lambda)$.*

*Moreover, we have the lower bound*

$$0 < \left(\frac{\beta+1}{\beta+3}\right)^{\beta+1}\left(1 - \frac{\beta+1}{\beta+3}\right)^2 C \le \lambda^* \le C, \tag{3.8}$$

*where $C = C(\alpha, \beta, \gamma, f)$ is the following well defined constant:*

$$C := \left(\varphi_\beta^{-1}\left(\int_0^1 \varphi_\beta\left(\frac{1}{s^\alpha}\int_0^s t^\gamma f(t)\,\mathrm{d}t\right)\,\mathrm{d}s\right)\right)^{-1},$$

*and*

$$\varphi_\beta(s) := \begin{cases} s|s|^{-\beta/(\beta+1)} & \text{if} \quad s \neq 0, \\ 0 & \text{if} \quad s = 0. \end{cases}$$

We also prove the existence of a branch of minimal solutions. Each minimal solution can be obtained as the limit of a recursive sequence, which enables one to make numerical approximation.

**Theorem 3.7 (Existence of the minimal branch)** *For each $\lambda \in (0, \lambda^*)$, problem $(P_\lambda)$ admits a unique positive minimal solution, $u_\lambda$, which is obtained as the limit of the following sequence: $u_0 = 0$ and $u_n$ is the solution of*

$$\begin{cases} -r^{-\gamma}(r^\alpha |u'_{n+1}|^\beta u'_{n+1})' = \dfrac{\lambda f}{(1-u_n)^2}, \quad r \in (0,1), \\ u'_{n+1}(0) = u_{n+1}(1) = 0 \end{cases} \tag{$P_\lambda(n)$}$$

*Moreover, each minimal solution is regular and the function* $\lambda \mapsto u_\lambda$ *is increasing in* $(0, \lambda^*)$, *that is, if* $\lambda_1 < \lambda_2$ *then* $u_{\lambda_1} \le u_{\lambda_2}$ *and* $u_{\lambda_1} \neq u_{\lambda_2}$.

On the critical problem with respect to the variable $\lambda$, we prove that problem $(P_{\lambda^*})$ has a unique integral solution that can be considered as the pointwise limit of the family of solutions $(u_\lambda)_\lambda$, namely

$$u^*(r) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(r), \quad r \in (0,1).$$

**Theorem 3.8** *The function* $u^*$ *is an integral solution of* $(P_\lambda)$ *with* $\lambda = \lambda^*$. *Moreover, if* $\lambda = \lambda^*$ *then* $(P_\lambda)$ *has a unique integral solution, consequently* $u^*$ *is a minimal solution of* $(P_{\lambda^*})$.

### 3.2.1 *Proof of Theorem 3.6, item 1*

First of all, we are going to prove the existence of a solution of $(P_\lambda)$ for $\lambda$ near to 0. Let $v \in C^2([0,1])$ be the unique solution of (3.3) for $g(r) = r^\gamma f$, take $\delta > 0$ small enough to ensure that $\overline{u} := \delta v$ satisfies $\|\overline{u}\|_\infty < 1$, that is, $0 < \delta < \|v\|_\infty^{-1}$. Note that

$$-(r^\alpha |\overline{u}'|^\beta \overline{u}')' = \delta^{\beta+1}(r^\alpha |v'|^\beta v')' = \delta^{\beta+1} r^\gamma f(r) = \delta^{\beta+1}(1 - \|\overline{u}\|_\infty)^2 \frac{r^\gamma f(r)}{(1 - \|\overline{u}\|_\infty)^2}.$$

Since $(1-z)^{-2}$ is increasing on $(-\infty, 0)$, it follows that

$$-(r^\alpha |\overline{u}'|^\beta \overline{u}')' \ge \delta^{\beta+1}(1 - \|\overline{u}\|_\infty)^2 \frac{r^\gamma f(r)}{(1-\overline{u})^2} \ge \lambda \frac{r^\gamma f(r)}{(1-\overline{u})^2},$$

whenever

$$0 < \lambda \le \delta^{\beta+1}(1 - \|\overline{u}\|_\infty)^2 = \delta^{\beta+1}(1 - \delta \|v\|_\infty)^2.$$

So we have found a super-solution of $(P_\lambda)$. Since zero is a sub-solution of $(P_\lambda)$, we can conclude that $(P_\lambda)$ admits an integral solution $u \in X$ (cf. Proposition 3.2) satisfying

$$0 \le u \le \overline{u}.$$

This implies $\|u\|_\infty \le \|\overline{u}\|_\infty < 1$. Then, by Proposition 2.4 $u$ is a classical solution for $(P_\lambda)$. Finally, observing that

$$\left( \frac{\beta+1}{(\beta+3)\|v\|_\infty} \right)^{\beta+1} \left( 1 - \frac{\beta+1}{\beta+3} \right)^2 = \max_{0<\delta<\|v\|_\infty^{-1}} \delta^{\beta+1}(1 - \delta |v|_\infty)^2,$$

we get (3.8).

Given $\bar{\lambda} \in \Lambda$, a classical solution $\overline{u}$ of $(P_{\bar{\lambda}})$ is a super-solution of $(P_\lambda)$ for every $\lambda \in (0, \bar{\lambda})$. By Proposition 3.2 we assert the existence of a classical solution of $(P_\lambda)$ for $\lambda \in (0, \bar{\lambda})$. Then we conclude that $\Lambda$ is in fact an interval.

### 3.2.2 *Proof of Theorem 3.6, item 3 and proof of Theorem 3.7*

First we need to establish a helpful Lemma which will be used many times in our upcoming arguments. On the following, we denote

$$G(u,r) := \varphi_\beta\left(\int_0^r \frac{s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s\right)$$

where $\varphi_\beta(z) := |z|^{-\beta/(\beta+1)}z$.

**Lemma 3.9** *Given two continuous functions $u_1, u_2 : [0,1] \to \mathbb{R}$, we consider*

$$u_t := (1-t)u_1 + tu_2.$$

*Suppose*

$$u_t(r) < 1 \quad \textit{for every} \quad (t,r) \in [0,1] \times (0,1)$$

*and $G(u_t, r)$ is well defined for the same range of values. Then we have*

$$G(u_t, r) \leq (1-t)G(u_1, r) + tG(u_2, r) \quad \textit{for} \quad (t,r) \in [0,1] \times [0,1].$$

*Moreover, if $u_1(r) < u_2(r)$ strictly on $(\delta_1, \delta_2) \subset [0,1]$, then there exists for each $t \in (0,1)$ a positive constant $c = c(t)$ such that*

$$G(u_t, r) + c \leq (1-t)G(u_1, r) + tG(u_2, r) \quad \textit{for every} \quad r \in (\delta_1, 1]\,.$$

*Proof.* For $t \in [0,1]$, consider

$$\xi(t,r) := G(u_t, r) - tG(u_2, r) - (1-t)G(u_1, r).$$

Observe that $\xi(0,r) \equiv \xi(1,r) \equiv 0$. Moreover, since $\varphi_\beta$ is differentiable for $\beta > -1$ and the integrands in the definition of $\xi$ are continuous for every $(t,s) \in [0,1] \times [0,r]$ and

$r \in (0,1)$ as well as its derivative with respect to $t$, it follows by Leibniz's rule that $\xi$ is derivable with respect to $t$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t,r) = G(u_1,r) - G(u_2,r) + \frac{2}{\beta+1}\int_0^r \frac{s^\gamma f(s)(u_2(s)-u_1(s))}{(1-u_t(s))^3}\,\mathrm{d}s \left|\int_0^r \frac{s^\gamma f(s)}{(1-u_t(s))^2}\,\mathrm{d}s\right|^{-\beta/(\beta+1)}.$$

Applying again the same argument, we can calculate

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\xi(t,r) = -\frac{4\beta}{(\beta+1)^2}\left|\int_0^r \frac{s^\gamma f(s)}{(1-u_t(s))^2}\,\mathrm{d}s\right|^{-(2\beta+1)/(\beta+1)} \left(\int_0^r \frac{s^\gamma f(s)(u_2(s)-u_1(s))}{(1-u_t(s))^3}\,\mathrm{d}s\right)^2 + \frac{6}{\beta+1}\left|\int_0^r \frac{s^\gamma f(s)}{(1-u_t(s))^2}\,\mathrm{d}s\right|^{-\beta/(\beta+1)} \int_0^r \frac{s^\gamma f(s)}{(1-u_t(s))^4}(u_2(s)-u_1(s))^2\,\mathrm{d}s,$$

or equivalently,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\xi(t,r) = \zeta(t,r)\left(3\eta_1(t,r) - \frac{2\beta}{\beta+1}\eta_2(t,r)\right), \tag{3.9}$$

where

$$\begin{cases} \zeta(t,r) := \dfrac{2}{\beta+1}\left|\displaystyle\int_0^r \frac{s^\gamma f(s)}{(1-u_t(s))^2}\,\mathrm{d}s\right|^{-(2\beta+1)/(\beta+1)}, \\ \eta_1(t,r) := \displaystyle\int_0^r \frac{s^\gamma f(s)}{(1-u_t(s))^2}\,\mathrm{d}s \int_0^r \frac{s^\gamma f(s)(u_2(s)-u_1(s))^2}{(1-u_t(s))^4}\,\mathrm{d}s, \\ \eta_2(t,r) := \left(\displaystyle\int_0^r \frac{s^\gamma f(s)(u_2(s)-u_1(s))}{(1-u_t(s))^3}\,\mathrm{d}s\right)^2. \end{cases}$$

Observe that $\zeta(t,r) \geq 0$ for all $(t,r) \in [0,1]\times[0,1]$. In fact, the term inside the parenthesis in (3.9) is also nonnegative for all $\beta > -1$ since

$$\frac{2\beta}{\beta+1} > -2,$$

and, by the Cauchy-Schwarz inequality, we have

$$\left(\int_0^r \frac{s^\gamma f(s)(u_2(s)-u_1(s))}{(1-u_t(s))^3}\,\mathrm{d}s\right)^2 = \left(\int_0^r \frac{(s^\gamma f(s))^{1/2}}{1-u_t(s)} \frac{(s^\gamma f(s))^{1/2}(u_2(s)-u_1(s))}{(1-u_t(s))^2}\,\mathrm{d}s\right)^2 \leq \int_0^r \frac{s^\gamma f(s)}{(1-u_t(s))^2}\,\mathrm{d}s \int_0^r \frac{s^\gamma f(s)(u_2(s)-u_1(s))^2}{(1-u_t(s))^4}\,\mathrm{d}s. \tag{3.10}$$

That is, $\xi''$ is non-negative and we conclude that $\xi$ is nonpositive.

If $u_1(r) < u_2(r)$ strictly on $(\delta_1, \delta_2) \subset [0, 1]$, then the terms $\eta_1$ and $\eta_2$ are strictly positive on $[0, 1] \times (\delta_1, 1]$ consequently, $\xi''$ is strictly positive for $(t, r) \in [0, 1] \times (\delta_1, 1]$ and we finally conclude that, $\xi$ is strictly negative for $(t, r) \in (0, 1) \times (\delta_1, 1]$.

■

The first part of Theorem 3.7 is a consequence of the sub- and super-solution method (cf. Proposition 3.2). For the second part, consider $\lambda_1, \lambda_2 \in (0, \lambda^{**})$ with $\lambda_1 < \lambda_2$, and the respective minimal integral solutions $u_1$, $u_2$ of problems $(P_{\lambda_1})$ and $(P_{\lambda_2})$. Observe that $u_2 \geq 0$, $u_2 \neq 0$ and that $u_2$ is a super-solution of $(P_{\lambda_1})$. Then, $(P_{\lambda_1})$ admits a non-negative solution $u \neq 0$ satisfying $u \leq u_2$. By the definition of minimal solution we must have $u_1 \leq u_2$. We can not have $u_1 = u_2$ because this implies $\lambda_1 = \lambda_2$.

Now, consider

$$\lambda_0 = \sup\{\lambda \in \Lambda : \text{ the associated minimal solution } u_\lambda \text{ is regular}\}.$$

Suppose $\lambda_0 < \lambda^{**}$ and take $\lambda_1 \in (0, \lambda_0)$ and $\lambda_2 \in (\lambda_0, \lambda^{**})$. Consider also the minimal solutions $u_1$ and $u_2$ associated with $(P_{\lambda_1})$ and $(P_{\lambda_2})$ respectively. Choose $t \in (0, 1)$ such that

$$\lambda_1^{1/(\beta+1)} < \lambda_0^{1/(\beta+1)} < t\lambda_1^{1/(\beta+1)} + (1-t)\lambda_2^{1/(\beta+1)} < \lambda_2^{1/(\beta+1)}$$

and set

$$\lambda = \left(t\lambda_1^{1/(\beta+1)} + (1-t)\lambda_2^{1/(\beta+1)}\right)^{\beta+1} > \lambda_0$$

and $v = tu_1 + (1+t)u_2$. We see from Lemma 3.9 that

$$\begin{aligned} -r^{\alpha/(\beta+1)}v' &= t\lambda_1^{1/(\beta+1)}G(u_1) + (1-t)\lambda_2^{(\beta+1)}G(u_2) \\ &\geq \lambda^{1/(\beta+1)}G(v) \end{aligned}$$

Thus, $v$ is a super-solution of $(P_\lambda)$. Let $u$ be a integral solution of $(P_\lambda)$, ensured by the sub- and super-solution method. Since $\|v\|_\infty \leq t\|u_1\|_\infty + (1-t)\|u_2\|_\infty < 1$ and $0 \leq u \leq v$, we deduce that $u$ is a regular solution. This is in contradiction with the definition of $\lambda_0$. Therefore, we have proved that $\lambda_0 = \lambda^* = \lambda^{**}$, and consequently we also have proved item *3* of Theorem 3.6.

### 3.2.3 *Proof of Theorem 3.6, item 2 and proof of Theorem 3.8*

Here the study of the critical problem $(P_\lambda)$ with $\lambda = \lambda^*$:

$$\begin{cases} -(r^\alpha|u'|^\beta u')' = \dfrac{\lambda^* r^\gamma f(r)}{(1-u)^2}, & r \in (0,1), \\ 0 \le u(r) \le 1, & r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{$P_{\lambda^*}$}$$

We prove that problem $(P_{\lambda^*})$ has a unique solution for $-1 < \beta$, as stated in Theorem 3.8. Since for each fixed $r \in [0,1]$, the function $\lambda \mapsto u_\lambda(r)$ is nondecreasing (see Theorem 3.7) and bounded by 1 (see Proposition 2.4), we have that the function $u^* : [0,1] \to \mathbb{R}$ given by

$$u^*(r) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(r), \quad r \in [0,1],$$

is well defined. Then, by using an argument similar to the one in the last part of the proof of Proposition 3.2, we can conclude, that $u^*$ is an integral solution of $(P_{\lambda^*})$ and, consequently, we have proved item *(2)* of Theorem 3.6. Now, since we have proved the existence of solution for $(P_{\lambda^*})$, consider an arbitrary solution $v$. Due to the monotonicity property of the the term $G$, we have that $v$ is an integral supersolution of $(P_\lambda)$ for every $0 < \lambda < \lambda^*$. Then $u_\lambda(r) \le v(r)$ for every $r \in [0,1]$, where $u_\lambda$ is the minimal solution of $(P_\lambda)$. In the limit case, it follows that

$$u^* \le v$$

in the pointwise sense. Therefore, $u^*$ is minimal.

For the uniqueness part, suppose that there exists a solution $v$ of $(P_{\lambda^*})$ with $v \not\equiv u^*$. Since $u^*$ is minimal and no solution has values larger than 1, we know that

$$u^*(r) \le v(r) < 1 \quad \text{for} \quad r \in (0,1).$$

Denote by $(\delta_1, \delta_2) \subset [0,1]$ the open interval on which

$$u^*(r) < v(r) \quad \text{for} \quad r \in (\delta_1, \delta_2).$$

Consider

$$\overline{u} := \frac{u^* + v}{2}$$

and observe that

$$\overline{u}(r) < 1 \quad \text{for} \quad r \in (0,1).$$

Applying Lemma 3.9 with $t = 1/2$, we guarantee the existence of a positive constant $c$ such that

$$\begin{cases} -\overline{u}'(r) \geq \varphi_\beta(\lambda^* r^{-\alpha}) G(\overline{u}, r), & r \in (0,1), \\ -\overline{u}'(r) \geq \varphi_\beta(\lambda^* r^{-\alpha}) G(\overline{u}, r) + c, & r \in [\delta_1, 1]. \end{cases}$$

Thus,

$$-\overline{u}'(r) \geq \varphi_\beta(\lambda^* r^{-\alpha}) G(\overline{u}, r) + \xi(r) \quad \text{for} \quad r \in (0,1),$$

where $\xi$ is a non zero continuous function supported on $[\delta_1, 1]$ and satisfying

$$\xi(r) \leq c \quad \text{and} \quad \xi(1) > 0.$$

So $\overline{u}$ is a supersolution for the problem

$$\begin{cases} -u'(r) = \varphi_\beta(\lambda^* r^{-\alpha}) G(u, r) + \xi(r), & r \in (0,1), \\ u(1) = 0. \end{cases} \tag{3.11}$$

Since the zero function is a subsolution of (3.11), we conclude via Proposition 3.2 that (3.11) has an integral solution $u_0$ satisfying

$$0 \leq u_0 \leq \overline{u} \leq 1.$$

Considering

$$\psi(r) := \int_r^1 \xi(r)\, \mathrm{d}r,$$

it is easy to see, in particular, that $u$ is a solution of (3.11) if and only if $u - \psi$ is a solution of $(P_\lambda)$.

Denote

$$\eta(r) := \frac{u_0(r)}{\psi(r)}.$$

Since

$$\lim_{r \to 1} u_0(r) = \lim_{r \to 1} \psi(r) = 0,$$

we can apply L'Hôpital's rule to obtain

$$\lim_{r \to 1} \eta(r) = \lim_{r \to 1} \frac{u_0'(r)}{\psi'(r)} = \frac{\varphi_\beta(\lambda^*) G(u_0, 1) + \xi(1)}{\xi(1)}.$$

Thus, $\eta$ defines a continuous function over $[0,1]$. Then we can take $\delta > 0$ sufficiently small such that

$$\delta u_0 \leq \psi. \tag{3.12}$$

Choose $\varepsilon > 0$ sufficiently small such that

$$\frac{\varphi_\beta(\lambda^* + \varepsilon)}{\varphi_\beta(\lambda^*)} - 1 < \delta$$

and define

$$w_\varepsilon = \frac{\varphi_\beta(\lambda^* + \varepsilon)}{\varphi_\beta(\lambda^*)} u_0 - \psi.$$

Due to (3.12) we see that

$$u_0 - w_\varepsilon = \psi + \left(1 - \frac{\varphi_\beta(\lambda^* + \varepsilon)}{\varphi_\beta(\lambda^*)}\right) u_0 > \psi - \delta u_0 \geq 0,$$

that is,

$$w_\varepsilon \leq u_0 \leq 1.$$

Since $G$ is monotone on the variable $u$ for $u \leq 1$, it follows that

$$\begin{aligned} -w'_\varepsilon(r) &= \frac{\varphi_\beta(\lambda^* + \varepsilon)}{\varphi_\beta(\lambda^*)} \varphi_\beta(\lambda^* r^{-\alpha}) G(u_0, r) + \left(\frac{\varphi_\beta(\lambda^* + \varepsilon)}{\varphi_\beta(\lambda^*)} - 1\right) G_0(u_0, r) \\ &\geq \varphi_\beta((\lambda^* + \varepsilon) r^{-\alpha}) G(w_\varepsilon, r). \end{aligned}$$

Thus, $w_\varepsilon$ is a super solution of $(P_\lambda)$ for $\lambda = \lambda^* + \varepsilon$. Moreover, $w_\varepsilon \geq 0$. Then, via Proposition 3.2 there exists a solution of $(P_\lambda)$ for $\lambda > \lambda^*$, which contradicts the definition of $\lambda^*$.

# 4 Existence of a bifurcation point

## 4.1 Stability of minimal solutions

Now, in order to get additional properties of solutions of $(P_\lambda)$, we present a sub- and supersolution result which enables us to obtain a weak solution of $(P_\lambda)$ as a minimum of a functional in an appropriate set.

Hereafter we are using the following notations. For $v, u \in X$ with $v \leq u$, we define the interval $M_{v,u}$ in $X$ setting

$$M_{v,u} := \{w \in X : v \leq w \leq u\}.$$

Given $\underline{u} \leq \overline{u}$, regular sub- and supersolution respectively of $(P_\lambda)$, we also define the functional $I : M_{\underline{u},\overline{u}} \to \mathbb{R}$ as

$$I(u) := \frac{1}{\beta+2}\|u\|_X^{\beta+2} - \int_0^1 F(u,r)\,\mathrm{d}r,$$

where

$$F(u,r) := \int_{u(r)}^{\underline{u}(r)} \frac{\lambda r^\gamma f(r)}{(1-s)^2}\,\mathrm{d}s = \frac{\lambda r^\gamma f(r)}{1-u(r)} - \frac{\lambda r^\gamma f(r)}{1-\underline{u}(r)}. \tag{4.1}$$

We omitted the dependence on $\underline{u}$ and $\overline{u}$ in order to avoid overburdening the notation. Note that $I$ is well defined. Indeed, since $\|u\|_\infty < 1$ for every $u \in M_{\underline{u},\overline{u}}$ and $r \in (0,1)$, then $F$ is continuous on $r$. Moreover,

$$0 \leq \int_0^1 \frac{\lambda r^\gamma f(r)}{1-v(r)}\mathrm{d}r \leq \int_0^1 \lambda r^\gamma f(r) \max\{1, (1-v(r))^{-2}\}\mathrm{d}r < \infty \quad \text{for} \quad v = u, \underline{u}.$$

**Proposition 4.1** *Let $\underline{u}$ and $\overline{u}$ be a regular weak subsolution and a regular weak supersolution respectively for $(P_\lambda)$, satisfying $\underline{u} \leq \overline{u}$. Then there exists $u_0 \in M_{\underline{u},\overline{u}}$ such* *that*

$$I(u_0) = \min_{u \in M_{\underline{u},\overline{u}}} I(u).$$

*Moreover, $u_0$ is a weak solution of $(P_\lambda)$.*

*Proof.* Given $u \in M_{\underline{u},\overline{u}}$, we have

$$0 \leq \frac{1}{1-u} - \frac{1}{1-\underline{u}} = \frac{u-\underline{u}}{(1-u)(1-\underline{u})} \leq \frac{u-\underline{u}}{(1-\overline{u})^2}.$$

Multiplying the inequality above by $\lambda r^\gamma f(r)$ and integrating, we get

$$0 \leq \int_0^1 F(u,r)\,\mathrm{d}r \leq \int_0^1 \frac{\lambda r^\gamma f(r)}{(1-\overline{u}(r))^2}(u(r)-\underline{u}(r))\,\mathrm{d}r.$$

Then, by the weak supersolution condition we have

$$0 \leq \int_0^1 F(u,r)\,\mathrm{d}r \leq \int_0^1 r^\alpha |\overline{u}'(r)|^\beta \overline{u}'(r)(u(r)-\underline{u}(r))'\,\mathrm{d}r.$$

Applying Hölder inequality we settle

$$0 \leq \int_0^1 F(u,r)\,\mathrm{d}r \leq \|\overline{u}\|_X^{\beta+1}\|u-\underline{u}\|_X.$$

Finally we conclude

$$I(u) = \frac{1}{\beta+2}\|u\|_X^{\beta+2} - \int_0^1 F(u,r)\,\mathrm{d}r \geq \frac{1}{\beta+2}\|u\|_X^{\beta+2} - \|\overline{u}\|_X^{\beta+1}\|u-\underline{u}\|_X,$$

from which we get that $I$ is coercive and bounded from below.

Given $u_n \in M_{\underline{u},\overline{u}}$ with $u_n \rightharpoonup u$, since $X$ is reflexive, $u_n \to u$ almost everywhere on $(0,1)$ and $F(u_n,r)$ is continuous on $r \in (0,1)$. Therefore, we have $F(u_n,r) \to F(u,r)$ almost everywhere for $r \in (0,1)$ and consequently,

$$\int_0^1 F(u_n,r)\,\mathrm{d}r \to \int_0^1 F(u,r)\,\mathrm{d}r.$$

In particular, $I$ is lower weakly semi-continuous. The previous observations and the weakly compactness of $M_{\underline{u},\overline{u}}$ make us to conclude that $I$ attains its infimum on $M_{\underline{u},\overline{u}}$.

Now we are going to prove that minimizers of $I$ on $M_{\underline{u},\overline{u}}$ are weak solutions of $(P_\lambda)$. Let $u \in M_{\underline{u},\overline{u}}$ be a minimizer of $I$ and consider for $\varepsilon > 0$ and $v \in X$

$$v_\varepsilon = \min\{\overline{u}, \max\{\underline{u}, u+\varepsilon v\}\} = u + \varepsilon v - b_\varepsilon + p_\varepsilon$$

with $b_\varepsilon = (u + \varepsilon v - \overline{u})^+$ and $p_\varepsilon = (u + \varepsilon v - \underline{u})^-$. Since $v_\varepsilon \in M_{\underline{u},\overline{u}}$ and $u + t(v_\varepsilon - u) \in M_{\underline{u},\overline{u}}$ for all $t \in [0,1]$, we have that $I$ is differentiable at $u$ on the direction $v_\varepsilon - u$.

Given $\overline{t} \in [0,1]$ there exists $\underline{t} \in [0,\overline{t}]$ such that

$$0 \leq \frac{1}{\overline{t}} \left( I \left( u + \overline{t}\,(v_\varepsilon - u) \right) - I(u) \right) = I' \left( u + \underline{t}\,(v_\varepsilon - u) \right) (v_\varepsilon - u) \, .$$

Then, by the continuity of $I'$, taking $\overline{t} \to 0$, we have

$$0 \leq I'(u)(v_\varepsilon - u). \tag{4.2}$$

Since

$$I'(u)(v_\varepsilon - u) = \varepsilon I'(u)v - I'(u)b_\varepsilon + I'(u)p_\varepsilon,$$

looking at (4.2) we get

$$I'(u)v \geq \frac{1}{\varepsilon} \left( I'(u)b_\varepsilon - I'(u)p_\varepsilon \right). \tag{4.3}$$

Now, let us verify that $\liminf_{\varepsilon \to 0} \varepsilon^{-1} I'(u) b_\varepsilon \geq 0$. Let $\overline{u}$ be a supersolution of $(P_\lambda)$. Since $b_\varepsilon \geq 0$, we must have

$$I'(\overline{u})b_\varepsilon \geq 0.$$

It follows that,

$$\begin{aligned} I'(u)b_\varepsilon \geq \left(I'(u) - I'(\overline{u})\right) b_\varepsilon &= \int_0^1 r^\alpha \left( |u'(r)|^\beta u'(r) - |\overline{u}'(r)|^\beta \overline{u}'(r) \right) b_\varepsilon'(r)\, \mathrm{d}r \\ &\quad - \int_0^1 \lambda r^\gamma f(r) \left[ \frac{1}{(1-u(r))^2} - \frac{1}{(1-\overline{u}(r))^2} \right] b_\varepsilon(r)\, \mathrm{d}r \\ &\geq \int_0^1 r^\gamma \left( |u'|^\beta u' - |\overline{u}'|^\beta \overline{u}' \right) b_\varepsilon'(r)\, \mathrm{d}r, \end{aligned}$$

where the last inequality is a consequence of the monotonicity of $1/(1-s)^2$. Therefore, we get

$$I'(u)b_\varepsilon \geq \int_{l_\varepsilon} r^\gamma \left( |u'(r)|^\beta u'(r) - |\overline{u}'(r)|^\beta \overline{u}'(r) \right) \left( u'(r) + \varepsilon v'(r) - \overline{u}'(r) \right) \mathrm{d}r \tag{4.4}$$

with $l_\varepsilon = \{ r \in [0,1] : u(r) < \overline{u}(r) \leq u(r) + \varepsilon v(r) \}$.

Since the function $|s|^\beta s$ is increasing, given $x, y \in \mathbb{R}$ with $x \geq y$, we have $|x|^\beta x \geq |y|^\beta y$. Thus,

$$(|x|^\beta x - |y|^\beta y)(x - y) \geq 0 \quad \text{ for all } x, y \in \mathbb{R}.$$

Using this inequality in (4.4) with $x = u'$ and $y = v'$, we obtain

$$I'(u)b_\varepsilon \geq \varepsilon \int_{l_\varepsilon} \left(|u'(r)|^\beta u'(r) - |\overline{u}'(r)|^\beta \overline{u}'(r)\right) v'(r)\, \mathrm{d}r,$$

which converges to zero as $\varepsilon \to 0$, since the measure of $l_\varepsilon$ goes to zero. Analogously, we can obtain $I(u)p_\varepsilon \geq 0$ and consequently by (4.3) we achieve $I(u)v \leq 0$. Since $v \in X$ is arbitrary, we also must have $I(u)(-v) \leq 0$. Thus we conclude $I(u)v = 0$ for every $v \in X$, that is, $u$ is a weak solution of $(P_\lambda)$.

■

The assumption $\|\underline{u}\|_\infty < 1$ in Proposition 4.1 reveals to us that only regular solutions can be pointed out by this result. Singular solutions may be considered by a limiting process. On the following, we use Proposition 4.1 to obtain additional properties of the minimal solutions.

**Proposition 4.2 (Existence of semi-stable solutions)** *For $\lambda \in (0, \lambda^*)$, the minimal solution of $(P_\lambda)$ is semi-stable.*

*Proof.* Given $\lambda \in (0, \lambda^*)$ and $u$ the associated minimal weak solution (see Theorem 3.7), since $\|u\|_\infty < 1$, we know $u \in M_{\underline{u},\overline{u}}$ where $\underline{u} \equiv -1$ is a weak subsolution of $(P_\lambda)$ and $\overline{u} = u$ is, in particular, a weak supersolution of the same problem. According to Proposition 4.1, there exists a solution $v$ of $(P_\lambda)$ which minimizes the functional $I$ with $\underline{u}$ and $\overline{u}$ chosen as above. In particular

$$\underline{u} \leq v \leq \overline{u} = u$$

and

$$0 \leq u \leq v,$$

since $u$ is minimal. Thus $u = v$, that is, $u$ minimizes $I$ on $M_{\underline{u},\overline{u}}$ where $\underline{u} \equiv -1$ and $\overline{u} = u \geq 0$.

Consider the set

$$X_u^\infty := \{v \in X_u : \|v\|_\infty < \infty\}. \tag{4.5}$$

$X_u^\infty$ is a dense subspace of $X$ and $v \in C[0,1]$ for every $v \in X_u^\infty$. Given $v \in X_u^\infty$ with $v \geq 0$, there exists $t_0 > 0$ such that $\|u + tv\|_\infty < 1$ for every $t \in [-t_0, 0]$. Thus $u + tv \in M_{\underline{u},\overline{u}}$ for every $t \in [-t_0, 0]$. Consider function $\eta : [-t_0, 0] \to \mathbb{R}$ as

$$\eta(t) := I(u + tv) = \frac{1}{\beta + 2}\|u + tv\|_X^{\beta+2} - \int_0^1 F(u + tv, r)\,\mathrm{d}r,$$

We see by the Leibniz integral rule that $\eta$ is twice differentiable on $[-t_0, 0]$ with

$$\eta'(t) = \int_0^1 r^\alpha |u'(r) + tv'(r)|^\beta (u'(r) + tv'(r))v'(r)\,\mathrm{d}r - \int_0^1 \frac{\lambda r^\gamma f(r)v(r)}{(1 - (u(r) + tv(r)))^2}\,\mathrm{d}r.$$

and

$$\eta''(t) = (\beta + 1)\int_0^1 r^\alpha |u'(r) + tv'(r)|^\beta (v'(r))^2\,\mathrm{d}r - \int_0^1 \frac{2\lambda r^\gamma f(r)v^2(r)}{(1 - (u(r) + tv(r)))^3}\,\mathrm{d}r.$$

Since $u$ is a solution of $(P_\lambda)$, we have $\eta'(0) = 0$. Since $u$ is an absolute minimizer of $I$ in $M_{\underline{u},\overline{u}}$, we have $\eta''(0) = 0$, that is, $u$ satisfies

$$(\beta + 1)\int_0^1 r^\alpha |u'(r)|^\beta (v'(r))^2\mathrm{d}r - \int_0^1 \frac{2\lambda r^\gamma f(r)v^2(r)}{(1 - u(r))^3}\,\mathrm{d}r \geq 0.$$

By density argument and due to the arbitrariness of $v$, we conclude that $u$ is a semi-stable solution.

■

**Proposition 4.3** *Calling*

$$R_u(v) = \frac{1}{\|v\|_u}\left((\beta + 1)\int_0^1 r^\alpha |u'(r)|^\beta (v'(r))^2\,\mathrm{d}r - 2\lambda\int_0^1 \frac{r^\gamma f(r)}{(1 - u(r))^3}v^2(r)\,\mathrm{d}r\right),$$

*we have that*

$$\inf_{v \in X_u \setminus \{0\}} R_u(v)$$

*is attained for some $\phi_1 \in X_u$.*

*Proof.* Consider a minimizing sequence $v_n$ of $\mu_1(u)$, that is, $R(v_n) \to \mu_1(u)$ as $n \to \infty$, with $\|v_n\| = 1$. Since $u$ is regular, then the term $r^\gamma f(r)/(1 - u(r))^3$ defines a bounded function on $[0, 1]$ (continuous, in fact). Moreover, the sequence $(\|v_n\|_u)_n$ is bounded since $(R(v_n))_n$ is bounded. In fact,

$$R_u(v_n) \leq C_1 \quad \Rightarrow \quad (\beta+1)\int_0^1 r^\alpha |u'(r)|^\beta (v_n'(r))^2\,\mathrm{d}r \leq C + 2\lambda\int_0^1 \frac{r^\gamma f(r)}{(1 - u(r))^3}v_n^2(r)\,\mathrm{d}r \leq C_2,$$

where $C_1$ and $C_2$ are upper bounds for $R_u(v_n)$ and $r^\gamma f(r)/(1-u(r))^3$ respectively. Then, up to a subsequence, we get that

$$v_n \rightharpoonup \phi_1 \quad \text{weakly in } X_u.$$

Since $X_u$ is compactly embedding in $L^2$, we have that

$$v_n \to \phi_1 \quad \text{strongly in } L^2((0,1)).$$

In particular,

$$\|\phi_1\|_2 = 1.$$

Since,

$$\int_0^1 \frac{r^\gamma f(r)}{(1-u(r))^3} v^2(r)\,\mathrm{d}r \le C_2 \int_0^1 v_n^2(r)\,\mathrm{d}r,$$

we see that this term is continuous in $L^2((0,1))$. We also have that $\|\cdot\|_u$ is lower semi-continuous in $X_u$. Then $R_u(\phi_1) = \mu_1(u)$, that is, $\mu_1$ is attained at $\phi_1$. Moreover, due to Riesz's Representation Theorem, $\phi_1$ solves the problem

$$\begin{cases} r^\alpha |u'|^\beta \phi_1'' - \dfrac{2\lambda r^\gamma f(r)}{(1-u(r))^3}\phi_1 = \mu_1\phi_1 \quad r \in (0,1), \\ \phi_1'(0) = \phi_1(1) = 0. \end{cases}$$

in the weak sense.

■

**Proposition 4.4 (Existence of stable solutions)** *For $\lambda \in (0,\lambda^*)$, and $\beta \ge 0$ the minimal solution of $(P_\lambda)$ is stable.*

*Proof.* Suppose that for some $\lambda_0 \in (0,\lambda^*)$, $\mu_1(u_{\lambda_0}) = 0$. Take $\lambda \in (\lambda_0,\lambda^*)$ and consider $u_0 := u_{\lambda_0}$ and $u := u_\lambda$ the minimal solution associated with $\lambda_0$ and $\lambda$ respectively. Consider also $\phi_0, \phi \in X_u$ the minimizers of $\mu_1(u_0)$ and $\mu_1(u)$ respectively.

We know from Theorem 3.7 that

$$u_0 \le u \quad \text{and} \quad |u'| \le |u_0'|.$$

It follows

$$\int_0^1 r^\alpha |u_0'(r)|^\beta (\phi_0'(r))^2\,\mathrm{d}r \ge \int_0^1 r^\alpha |u'(r)|^\beta (\phi_0'(r))^2\,\mathrm{d}r. \tag{4.6}$$

and

$$
\begin{aligned}
(\beta+1)\int_0^1 r^\alpha |u_0'(r)|^\beta (\phi_0'(r))^2\,\mathrm{d}r &= 2\lambda_0 \int_0^1 \frac{r^\gamma f(r)}{(1-u_0(r))^3}(\phi_0(r))^2\,\mathrm{d}r \\
&\le 2\lambda_0 \int_0^1 \frac{r^\gamma f(r)}{(1-u(r))^3}(\phi_0(r))^2\,\mathrm{d}r \qquad (4.7)\\
&< 2\lambda \int_0^1 \frac{r^\gamma f(r)}{(1-u(r))^3}(\phi_0(r))^2\,\mathrm{d}r.
\end{aligned}
$$

In other hand

$$
0 \le \mu_1(u) \le (\beta+1)\int_0^1 r^\alpha |u'(r)|^\beta (\phi_0'(r))^2\,\mathrm{d}r - 2\lambda \int_0^1 \frac{r^\gamma f(r)}{(1-u(r))^3}(\phi_0(r))^2\,\mathrm{d}r. \qquad (4.8)
$$

Putting together (4.7) and (4.8) we obtain

$$
\int_0^1 r^\alpha |u_0'(r)|^\beta (\phi_0'(r))^2\,\mathrm{d}r < \int_0^1 r^\alpha |u'(r)|^\beta (\phi_0'(r))^2\,\mathrm{d}r.
$$

This is in contradiction with (4.6).

■

Now we present the main result of this section. This result is a characterization of the minimal solutions and will be very useful in our upcoming arguments.

**Theorem 4.5** *The minimal solution of* $(P_\lambda)$ *is semi-stable. For* $\beta > 0$ *and* $\lambda \in (0, \lambda^*)$ *it is in fact stable and any stable solution is minimal.*

*Proof.* Let $u$ be a semi-stable solution of $(P_\lambda)$ and $u_\lambda$ be the minimal solution of the same problem. Since $u_\lambda \le u$, it follows

$$
-r^\alpha |u_\lambda'(r)|^\beta u_\lambda'(r) = \int_0^r \frac{\lambda s^\gamma f(s)}{(1-u_\lambda(s))^2}\,\mathrm{d}s \le \int_0^r \frac{\lambda s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s = -r^\alpha |u'(r)|^\beta u'(r),
$$

for all $r \in (0,1)$. Thereby we have

$$
u' \le u_\lambda'.
$$

Suppose $u_\lambda(r) < u(r)$ strictly on $(\delta_1, \delta_2) \subset [0,1]$ and set

$$
u_t(r) := tu(r) + (1-t)u_\lambda(r) \quad \text{ and } \quad u_t' = \frac{\mathrm{d}}{\mathrm{d}r}u_t.
$$

According to Lemma 3.9, we have, for every $(t,r) \in [0,1] \times (0,1]$,

$$
\begin{aligned}
r^{\alpha/(\beta+1)}u_t'(r) + G(u_t, r) &\le r^{\alpha/(\beta+1)}u_t'(r) + (1-t)G(u_\lambda, r) + tG(u) \\
&= r^{\alpha/(\beta+1)}u_t'(r) - (1-t)r^{\alpha/(\beta+1)}u_\lambda'(r) - tr^{\alpha/(\beta+1)}u'(r) = 0.
\end{aligned}
$$

In addition, also from Lemma 3.9, analogously as above, we get

$$r^{\alpha/(\beta+1)}u_t'(r) + G(u_t)(r) < 0 \quad \text{for every } (t,r) \in [0,1] \times (\delta_1, 1]\,.$$

Therefore,

$$-\, r^{\alpha}|u_t'(r)|^{\beta}u_t'(r) \geq \int_0^r \frac{\lambda s^{\gamma} f(s)}{(1-u_t(s))^2}\,\mathrm{d}s \quad \text{for every } (t,r) \in [0,1] \times (0,1] \tag{4.9}$$

and

$$-\, r^{\alpha}|u_t'(r)|^{\beta}u_t'(r) > \int_0^r \frac{\lambda s^{\gamma} f(s)}{(1-u_t(s))^2}\,\mathrm{d}s \quad \text{for every } (t,r) \in (0,1) \times (\delta_1,1]\,. \tag{4.10}$$

In particular, $u_t$ is an integral supersolution of $(P_\lambda)$ but it is not a solution for any $t \in (0,1)$. Thus, multiplying (4.9) by $u' - u_\lambda'$, which is non-positive, and integrating on $[0,1]$ we get, in view of (4.10),

$$\int_0^1 r^{\alpha}|u_t'(r)|^{\beta}u_t'(r)(u(r)-u_\lambda(r))'\,\mathrm{d}r > \int_0^1 \frac{\lambda r^{\gamma} f(r)}{(1-u_t(r))^2}(u(r)-u_\lambda(r))\,\mathrm{d}r \quad \text{for all } t \in (0,1).$$

Consider the function $\psi : [0,1] \to \mathbb{R}$ settled by

$$\psi(t) := \int_0^1 r^{\alpha}|u_t'(r)|^{\beta}u_t'(r)(u(r)-u_\lambda(r))'\,\mathrm{d}r - \int_0^1 \frac{\lambda r^{\gamma} f(r)}{(1-u_t(r))^2}(u(r)-u_\lambda(r))\,\mathrm{d}r,$$

we have

$$\psi(t) > 0 \quad \text{for all } t \in (0,1) \quad \text{and} \quad \psi(0) = \psi(1) = 0. \tag{4.11}$$

Then,

$$\psi'(0) \geq 0 \quad \text{and} \quad \psi'(1) \leq 0. \tag{4.12}$$

Observe that

$$\psi'(t) = (\beta+1)\int_0^1 r^{\alpha}|u_t'(r)|^{\beta}(u'(r)-u_\lambda'(r))^2\,\mathrm{d}r - \int_0^1 \frac{2\lambda s^{\gamma} f(r)}{(1-u_t(r))^3}(u(r)-u_\lambda(r))^2\,\mathrm{d}r.$$

Since $u$ is a semi-stable solution, we have $\psi'(1) \geq 0$. This, together with (4.12), gives $\psi'(1) = 0$. If $\beta > 0$, then $\psi'(t)$ is strictly decreasing and consequently, $\psi'(t) > 0$ for every $t \in (0,1)$. This is a contradiction with (4.11).

■

## 4.2 Regularity of the extremal solution

Let $u$ be a weak solution of $(P_\lambda)$, that is, $u \in X$ satisfying

$$u'(r) = \varphi_\beta(\lambda r^{-\alpha}) G(r,u) \quad \text{for} \quad r \in (0,1).$$

Let us denote

$$w := \ln\left(\frac{1}{1-u}\right). \tag{4.13}$$

We see that

$$w' = \frac{u'}{1-u}.$$

Thus, we conclude that $w$ satisfies

$$\begin{cases} w'(r) = (1-u(r))^{-1}\varphi_\beta(\lambda r^{-\alpha})G(r,u) & \text{for } r \in (0,1) \\ w(1) = 0. \end{cases} \tag{4.14}$$

or equivalently

$$\begin{cases} r^\alpha \varphi_\beta^{-1}(w'(r)) = \lambda \varphi_\beta^{-1}\left(\frac{1}{1-u(r)}\right) \int_0^r \frac{s^\gamma f(s)}{(1-u(s))^2}\, \mathrm{d}s & \text{for } r \in (0,1) \\ w(1) = 0. \end{cases} \tag{4.15}$$

Observe from (4.13) that $w$ is bounded if and only if $\|u\|_\infty < 1$. Thus, our task will be to discuss about under which conditions we can ensure that $w$ is bounded.

In our upcoming arguments we make use of an embedding result on weighted Lebesgue spaces. Given $q \geq 1$, we denote by $L^q_\theta([0,1])$ the Banach space of Lebesgue measurable functions $u : [0,1] \to \mathbb{R}$ such that

$$\|u\|_{L^q_\theta} = \left(\int_0^1 r^\theta |u(r)|^q \, \mathrm{d}r\right)^{1/q} < \infty.$$

With this notation, we invoke the following proposition.

**Proposition A (One-dimensional Hardy Inequality)** *Given $u \in X$ then there exists $C > 0$ such that*

$$\left(\int_0^1 r^\theta |u(r)|^q \, \mathrm{d}r\right) \leq C \left(\int_0^1 r^\alpha |u'(r)|^{\beta+2} \, \mathrm{d}r\right)^{1/(\beta+2)}$$

*provided that*

*1.* $1 \leq \beta + 2 \leq q < \infty$ *and*

   *(a)* $\alpha > \beta + 1$, $\theta \geq q\dfrac{\alpha - \beta - 1}{\beta + 2} - 1$, *or*

   *(b)* $\alpha \leq \beta$, $\theta > -1$,

*2.* $1 \leq q < \beta + 2 < \infty$ *and*

   *(a)* $\alpha > \beta + 1$, $\theta > q\dfrac{\alpha - \beta - 1}{\beta + 2} - 1$, *or*

   *(b)* $\alpha \leq \beta$, $\theta > -1$.

*If* $\beta < \alpha - 1$ *then* $X$ *is compactly embedding in* $L^q_\theta$ *for* $1 < q < q^*$, *where*

$$q^* = \frac{(\gamma + 1)(\beta + 2)}{\alpha - \beta - 1}.$$

We refer to A. Kufner and B. Opic [Opic e Kufner 1990] for the proprieties remarked here.

**Proposition 4.6** *Given* $u \in X$ *satisfying*

$$r^\alpha \varphi_\beta^{-1}(u'(r)) = r^{\alpha/(\beta+2)} F(r), \tag{4.16}$$

*if there exist constants* $\delta_1, \delta_2 > 0$ *and* $\tau > (\beta + 2)/(\beta + 1)$ *such that* $F \in L^q_\theta((0, 1))$ *for*

$$\theta = \frac{1 - \delta_1}{\tau} \qquad \text{and} \qquad q = \frac{\tau(\beta + 2)^2}{\tau(\beta + 1) - (\beta + 2)} + \delta_2, \tag{4.17}$$

*then,* $u$ *is bounded.*

*Proof.* The proof follows the Moser's iteration argument due to De Giorgi-Nash. Given $q$ satisfying (4.17), choose $\theta$ such that the pair $(\theta, q)$ satisfies the conditions on Proposition A. Now, for $k > 0$, consider

$$v(r) := \begin{cases} \operatorname{sign}(u)(|u| - k) & \text{if } r^{\theta/q}|u| > k, \\ 0 & \text{it } r^{\theta/q}|u| \leq k \end{cases} \tag{4.18}$$

and denote

$$A(k) := \left\{ r \in (0, 1) : r^{\theta/q}|u| > k \right\}. \tag{4.19}$$

Since $v' = u'$ in $A(k)$ and $v' \equiv 0$ in $(0, 1) \setminus A(k)$, we see that $v \in X'$. Multiplying (4.16) by $u'$ and integrating on $(0, 1)$, we get

$$\int_{A(k)} r^\alpha |u'(r)|^{\beta+2} \mathrm{d}r = \int_{A(k)} r^{\alpha/(\beta+2)} F(r) u'(r)\, \mathrm{d}r. \tag{4.20}$$

Applying the Hölder inequality on the right hand side of (4.20) we obtain

$$\int_{A(k)} F(r) r^{\alpha/(\beta+2)} u'(r)\,\mathrm{d}r$$
$$\leq C_0 \|F\|_{L^q_\theta} \left( \int_{A(k)} r^\alpha |u'(r)|^{\beta+2}\,\mathrm{d}r \right)^{1/(\beta+2)} |A(k)|^{(\beta+1)/(\beta+2)-1/q-1/\tau}, \quad (4.21)$$

where

$$C_0 := \left( \int_0^1 r^{-\theta\tau}\,\mathrm{d}r \right)^{1/\tau}.$$

Observe that $C_0$ is well defined since $-\theta\tau > -1$. It follows from (4.20) and (4.21) that

$$\left( \int_{A(k)} r^\alpha |u'(r)|^{\beta+2}\,\mathrm{d}r \right)^{(\beta+1)/(\beta+2)} \leq C_0 \|F\|_{L^q_\theta} |A(k)|^{(\beta+1)/(\beta+2)-1/q-1/\tau}. \quad (4.22)$$

Notice that for $0 < k < h$

$$|A(h)|^{1/q}(h-k) = \left( \int_{A(h)} (h-k)^q\,\mathrm{d}r \right)^{1/q}$$
$$\leq \left( \int_{A(h)} \left( r^{\theta/q}|u(r)| - k \right)^q\,\mathrm{d}r \right)^{1/q}$$

and then

$$\begin{aligned} |A(h)|^{1/q}(h-k) &\leq \left( \int_{A(h)} r^\theta \left|\operatorname{sign}(u(r))\left(|u(r)|-k\right)\right|^q\,\mathrm{d}r \right)^{1/q} \\ &= \left( \int_{A(h)} r^\theta |v(r)|^q \mathrm{d}r \right)^{1/q}. \end{aligned} \quad (4.23)$$

We know from Proposition A that

$$\left( \int_0^1 r^\theta |v(r)|^q \mathrm{d}r \right)^{1/q} \leq C \left( \int_0^1 r^\alpha |v'(r)|^{\beta+2} \mathrm{d}r \right)^{1/(\beta+2)} \quad (4.24)$$

for some constant $C > 0$. Applying (4.24) in (4.23), we get

$$|A(h)| \leq C^q \frac{1}{(h-k)^q} \left( \int_0^1 r^\alpha |v'(r)|^{\beta+2} \mathrm{d}r \right)^{q/(\beta+2)} \quad (4.25)$$

Now, using (4.22), estimate (4.25) becomes

$$|A(h)| \leq C_2^q \frac{1}{(h-k)^q} \|F\|_q^{q/(\beta+1)} |A(k)|^{q/(\beta+2)-1/(\beta+1)-q/\tau(\beta+1)}, \quad (4.26)$$

with

$$C_2 := C C_0^{1/(\beta+1)} \|F\|_q^{1/(\beta+1)}.$$

**Claim 4.7** *Let $\eta : [0,\infty) \to [0,\infty)$ be a nonincreasing function. If there exist $q > 0$, $p > 1$, $k_0 \geq 0$ and $M > 0$ such that*

$$\eta(h) \leq \left(\frac{M}{h-k}\right)^q (\eta(k))^p \quad \text{for} \quad h > k > k_0. \tag{4.27}$$

*Then there exists $k^* \geq 0$ such that $\eta(k_0 + k^*) = 0$, where $k^* = M2^{p/(p-1)}(\eta(k))^{(p-1)/q}$.*

We can apply Claim 4.7 with $\eta(h) = |A(h)|$, $k_0 = 0$, $M = C\|F\|_q^{1/(\beta+1)}$ and

$$p = \frac{q}{\beta+2} - \frac{1}{\beta+1} - \frac{q}{\tau(\beta+1)} > 1.$$

Observe that $\eta(k_0) = 1$. Thus we conclude that there exists $k^* > 0$ such that $\eta(k^*) = 0$ namely

$$\|u\|_\infty \leq k^*.$$

To finish the proof, it remains to prove Claim 4.7.

*Proof of Claim* (4.7). Consider the sequence

$$k_n := k_0 + \left(1 - \frac{1}{2^n}\right) k^* \tag{4.28}$$

where $k^*$ is a constant to be determined. Observe that $k_n \to k_0 + k^*$. We can prove by induction that

$$\eta(k_n) \leq \frac{\eta(k_0)}{r^n} \quad \text{for all} \quad n \in \mathbb{N} \tag{4.29}$$

where $r > 1$ is a constant to be properly chosen. Letting $n \to \infty$ on (4.29) we get $\eta(k_0 + k^*) = 0$ and the claim is proved.

Now, lets us prove estimate (4.29). First, observe that (4.29) is trivially valid for $n = 0$. Suppose that (4.29) is valid for a fixed $n$. We see from (4.28) that

$$k_n - k_{n-1} = \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) k_0 = \frac{k^*}{2^n}. \tag{4.30}$$

It follows from (4.27) together with (4.30) that

$$\eta(k_{n+1}) \leq \left(\frac{M2^{n+1}}{k^*}\right)^q |\eta(k_n)|^p. \tag{4.31}$$

Applying the induction hypothesis (4.29) in (4.31) we get

$$\eta(k_{n+1}) \leq \left(\frac{M2^{n+1}}{k^*}\right)^q \left(\frac{|\eta(k_0)|}{r^n}\right)^p. \tag{4.32}$$

Observe that

$$\left(\frac{M2^{n+1}}{k^*}\right)^q \left(\frac{|\eta(k_0)|}{r^n}\right)^p = \frac{\eta(k_0)}{r^{n+1}} \left(\frac{M2^{n+1}}{k^*}\right)^q \frac{1}{r^{n(p-1)-1}} |\eta(k_0)|^{p-1}. \tag{4.33}$$

Then, choosing $r = 2^{q/(p-1)}$ we get

$$\eta(k_{n+1}) \leq \frac{\eta(k_0)}{r^{n-1}} \left(\frac{M}{k^*}\right)^q 2^{pq/(p-1)} |\eta(k_0)|^{p-1}.$$

From this we see that (4.29) is valid substituting $n$ by $n+1$, provided that $k^*$ satisfies

$$\left(\frac{M}{k^*}\right)^q 2^{pq/(p-1)} |\eta(k_0)|^{p-1} \leq 1,$$

that is,

$$k^* \geq M 2^{p/(p-1)} |\eta(k_0)|^{(p-1)/q}.$$

■

In the light of Preposition 4.6, the task of proving the regularity of the critical solution $u^*$ comes down to prove some estimate on $F$.

**Proposition 4.8** *Given $u$ a semi-stable solution of $(P_\lambda)$ then,*

$$\int_0^1 r^\alpha \left| \left( \frac{1}{(1-u(r))^q} \right)' \right|^{\beta+2} \mathrm{d}r < \infty \tag{4.34}$$

*provided that*

$$\begin{cases} 0 < q < \dfrac{2\beta}{(\beta-1)(\beta+2)} & \text{if } \beta > 1, \\ \\ 0 < q & \text{if } \beta \leq 1. \end{cases}$$

*Proof.* Given $k \in (0,1)$, consider $T_k u$ the truncation of $u$ on the level $1-k$, that is,

$$T_k u(r) = \begin{cases} 1-k & \text{if } u > 1-k, \\ u(r) & \text{if } u < 1-k. \end{cases}$$

For $p > 0$, we set

$$v := \frac{1}{(1-T_k u(r))^p} - 1.$$

Observe that $v \in X$. Then we can take $v$ as a test function on the weak formulation of $(P_\lambda)$. Since $|u'|^\beta u'(T_k u)' = |(T_k u)'|^{\beta+2}$, we have that

$$p \int_0^1 r^\alpha \frac{|(T_k u)'|^{\beta+2}}{(1 - T_k u(r))^{p+1}} \, \mathrm{d}r = \int_0^1 \frac{\lambda r^\gamma f(r)}{(1 - u(r))^2} \left( \frac{1}{(1 - T_k u(r))^p} - 1 \right) \mathrm{d}r. \tag{4.35}$$

Observe also that

$$\frac{1}{1 - T_k u} \leq \frac{1}{1 - u}.$$

Then we can see from (4.35) that

$$\int_0^1 r^\alpha \frac{|(T_k u)'|^{\beta+2}}{(1 - T_k u(r))^{p+1}} \, \mathrm{d}r \leq \frac{1}{p} \int_0^1 \frac{\lambda r^\gamma f(r)}{(1 - u(r))^3} \frac{1}{(1 - T_k u(r))^{p-1}} \, \mathrm{d}r + C_1, \tag{4.36}$$

where

$$C_1 = -\frac{1}{p} \int_0^1 \frac{\lambda r^\gamma f(r)}{(1 - u(r))^2} \, \mathrm{d}r$$

does not depend on $k$.

Using

$$(a + b)^2 \leq 2(a^2 + b^2)$$

with

$$a = \frac{1}{(1 - T_k u(r))^{(p-1)/2}} - 1 \quad \text{and} \quad b = 1$$

we get

$$\int_0^1 r^\alpha \frac{|(T_k u)'|^{\beta+2}}{(1 - T_k u(r))^{p+1}} \, \mathrm{d}r \leq \frac{2}{p} \int_0^1 \frac{\lambda r^\gamma f(r)}{(1 - u(r))^3} \left( \frac{1}{(1 - T_k u(r))^{(p-1)/2}} - 1 \right)^2 \mathrm{d}r + C_2, \tag{4.37}$$

where

$$C_2 = C_1 + \frac{2}{p} \int_0^1 \frac{\lambda r^\gamma f(r)}{(1 - u(r))^3} \, \mathrm{d}r$$

is a constant which does not depend on $k$.

Taking

$$v := \frac{1}{(1 - T_k u(r))^q} - 1$$

in the semi-stability condition, we get

$$\int_0^1 \frac{\lambda r^\gamma f(r)}{(1 - u(r))^3} \left( \frac{1}{(1 - T_k u(r))^q} - 1 \right)^2 \mathrm{d}r \leq \frac{q(\beta + 1)}{2} \int_0^1 r^\alpha \frac{|(T_k u)'|^{\beta+2}}{(1 - T_k u(r))^{2(q+1)}} \, \mathrm{d}r. \tag{4.38}$$

It follows, from (4.37) and (4.38) with $q = (p-1)/2$, the estimate

$$\int_0^1 \frac{r^\alpha |(T_k u)'|^{\beta+2}}{(1-T_k u(r))^{p+1}}\,\mathrm{d}r \le \frac{(\beta+1)(p-1)}{2p}\int_0^1 \frac{r^\alpha |(T_k u)'|^{\beta+2}}{(1-T_k u(r))^{p+1}}\,\mathrm{d}r + C_2. \tag{4.39}$$

Thus, we can conclude

$$\int_0^1 \frac{r^\alpha |(T_k u)'|^{\beta+2}}{(1-T_k u(r))^{p+1}}\,\mathrm{d}r < C_4 \tag{4.40}$$

for some positive constant $C_4$ uniformly on the variable $k$, provided that

$$\begin{cases} 0 < p < \dfrac{\beta+1}{\beta-1} & \text{if } \beta > 1, \\ \\ 0 < p & \text{if } \beta \le 1. \end{cases} \tag{4.41}$$

Observe also that

$$\int_0^1 r^\alpha \left| \left( \frac{1}{(1-T_k u(r))^q} \right)' \right|^{\beta+2} \mathrm{d}r = q^{\beta+2} \int_0^1 r^\alpha \frac{|(T_k u)'(r)|^{\beta+2}}{(1-T_k u(r))^{(q+1)(\beta+2)}}\,\mathrm{d}r.$$

Thus, for $q > 0$ and according with (4.40) with $p = (q+1)(\beta+2)-1$, we see that

$$\int_0^1 r^\alpha \left| \left( \frac{1}{(1-T_k u(r))^q} \right)' \right|^{\beta+2} \mathrm{d}r < C_5, \tag{4.42}$$

where $C_5 = q^{\beta+2} C_4$ does not depend on $k$, provided

$$\begin{cases} 0 < q < \dfrac{2\beta}{(\beta-1)(\beta+2)} & \text{if } \beta > 1, \\ \\ 0 < q & \text{if } \beta \le 1. \end{cases} \tag{4.43}$$

Taking the supremum in $k$ on the expression (4.42) we obtain

$$\int_0^1 r^\alpha \left| \left( \frac{1}{(1-u(r))^q} \right)' \right|^{\beta+2} \mathrm{d}r < \infty \tag{4.44}$$

under condition (4.43).

■

Now we are ready to present the main result of this section.

**Theorem 4.9** *The critical solution $u^*$ is regular provided that*

$$\alpha < \frac{2(\beta+2)((2\beta+1)(\beta+2)^2 - (\beta+1)^2)}{(\beta+1)^2 + (\beta+2)^2 + 1}.$$

*Proof.* Observe that $w$, defined in (4.13), satisfies (4.16) for

$$F(r) := r^{-\alpha/(\beta+2)} \left(\frac{1}{1-u(r)}\right)^{\beta+1} \int_0^r \frac{\lambda s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s \tag{4.45}$$

and

$$F(r) \le r^{-\alpha/(\beta+2)} \left(\frac{1}{1-u(r)}\right)^{\beta} \int_0^r \frac{\lambda s^\gamma f(s)}{(1-u(s))^3}\,\mathrm{d}s. \tag{4.46}$$

According to Proposition 4.6, in order to prove that $w$ is bounded, we need to find $N > 0$ such that $F \in L^p((0,1))$ for

$$p = \frac{\beta+2}{\beta+1} + \beta + 2 + N.$$

Since

$$\int_0^r \frac{s^\gamma f(s)}{(1-u(s))^2}\,\mathrm{d}s \le (1-u(r)) \int_0^r \frac{s^\gamma f(s)}{(1-u(s))^3}\,\mathrm{d}s,$$

it is sufficient to prove that

$$\int_0^1 r^\theta \left(\frac{1}{(1-u(r))^p}\right)^q \mathrm{d}r < \infty.$$

for

$$\theta = -\alpha\left(\frac{\beta+2}{\beta+1} + \frac{N}{\beta+2}\right)$$

and

$$pq = \beta\left(\frac{\beta+2}{\beta+1} + \beta + 2 + N\right).$$

In view of Proposition 4.8, we intend to choose

$$v(r) = \frac{1}{(1-u(r))^p}$$

in Proposition A, where

$$p = \frac{5}{(\beta-1)(\beta+2)} + 1 - \frac{\beta-1}{|\beta-1|}\delta$$

for some $\delta > 0$. Then we need to check the conditions of Proposition A for

$$q = \frac{(2\beta+1)(\beta+2)^2 + (\beta+1)(\beta+2)N}{(\beta-1)(\beta+2) + 5 - \tilde{\delta}},$$

where

$$\tilde{\delta} = |\beta-1|(\beta+2)\delta$$

and $\theta$ given as above.

First, observe that

$$(\beta-1)(\beta+2)+5=\frac{(\beta-1)^2+(\beta+2)^2+1}{2}>1.$$

Then, the variable $q$ may assume any value

$$1<q<\frac{(2\beta+1)(\beta+2)^2+(\beta+1)(\beta+2)N}{(\beta-1)(\beta+2)+5}$$

provided we choose a suitable $N$ in order to make possible the choice of such $q$. Setting

$$N=-\frac{\beta-1}{\beta+2}$$

we get $\theta=0$ and

$$q=\frac{(\beta+1)^2+(\beta+2)^2+1}{2((2\beta+1)(\beta+2)^2-(\beta+1)^2)}.$$

Finally, the condition

$$\theta>q\frac{\alpha-\beta-1}{\beta+2}-1$$

imposes some restriction over $\alpha$, namely

$$\alpha<\frac{2(\beta+2)((2\beta+1)(\beta+2)^2-(\beta+1)^2)}{(\beta+1)^2+(\beta+2)^2+1}.$$

■

## 4.3 Multiplicity of solutions

In this section we prove the existence of nonminimal solutions for $\lambda$ close to $\lambda^*$ when $u^*$ is a regular solution. According with Theorem 4.5, such solutions are automatically unstable. In the following we are assuming

(H.6) $M:=\max_{r\in(0,1)}\{r^\gamma f(r)\}<\infty$.

In order to avoid the singularity on $(P_\lambda)$ for $u=1$, we analyze the perturbed problem

$$\begin{cases} -(r^\alpha |u'|^\beta u')' = \lambda r^\gamma f(r) g_\varepsilon(r,u), & r \in (0,1), \\ 0 \le u(r) \le 1, & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{4.47}$$

where

$$g_\varepsilon(r,z) = \begin{cases} \dfrac{1}{(1-z)^2}, & \text{for } u \le 1-\varepsilon, \\ \dfrac{f(1-\varepsilon)}{(1-\varepsilon)^{\theta-\gamma}} \dfrac{r^{\theta-\gamma}}{f(r)} \left( \dfrac{1}{\varepsilon^2} - \dfrac{2(1-\varepsilon)}{p\varepsilon^2} + \dfrac{2}{p\varepsilon^2(1-\varepsilon)^{p-1}} z^p \right), & \text{for } u \ge 1-\varepsilon \end{cases} \tag{4.48}$$

is a continuous function over $(0,1) \times \mathbb{R}$. Moreover, $g_\varepsilon$ is increasing and twice differentiable on the second variable and satisfies

$$0 \le g_\varepsilon(u) \le C_\varepsilon(1 + r^\theta |u|^p). \tag{4.49}$$

Choose $\theta$ and $p$ satisfying the conditions on Proposition A in order to have $X$ compactly embedding in $L^p_\theta$ and $\varepsilon$ small in order to have $\|u\|_\infty < 1 - \varepsilon$. This way, solutions $u$ of (4.47) are automatically solutions of $(P_\lambda)$ and conversely.

The proof of the existence of a branch of solutions for (4.47) apart from that induced by $(P_\lambda)$, the minimal branch, relies on the following standard version of the Mountain Pass Theorem .

**Theorem A (Mountain-Pass Theorem)** *Let $J$ be a $C^1$-functional defined on a Banach space $E$ that satisfies the Palais-Smale condition, that is, any sequence $(u_n) \subset E$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ in $E^*$ is relatively compact in $E$. Assume the following conditions:*

1. *There exists a neighborhood $B$ of some $u \in E$ and a constant $\sigma > 0$ such that*

$$J(h) \ge J(u) + \sigma \quad \text{for all} \quad h \in \partial B.$$

2. *There exists $w \notin B$ such that $J(w) \le J(u)$.*

*Define*

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = u, \gamma(1) = w\}$$

*then there exists* $v \in E$ *such that* $J'(v) = 0$ *and* $J(v) = c$*, where*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \{J(\gamma(t))\}.$$

In the following, we prove that the functional $J_{\varepsilon,\lambda} : X \to \mathbb{R}$ given by

$$J_{\varepsilon,\lambda}(u) := \frac{1}{\beta+2} \int_0^1 r^\alpha |u'(r)|^{\beta+2} \mathrm{d}r - \int_0^1 \lambda r^\gamma f(r) G_\varepsilon(r,u) \mathrm{d}r, \tag{4.50}$$

with

$$G_\varepsilon(r,u) := \int_{-\infty}^u g_\varepsilon(r,s) \mathrm{d}s$$

for $\varepsilon > 0$ sufficiently small, satisfies the hypothesis of Theorem A. The term in (4.50) is well defined due to (4.49).

**Lemma 4.10** *The minimal solution* $u_\lambda$ *of* $(P_\lambda)$ *is a strict local minimum of* $J_{\varepsilon,\lambda}$*.*

*Proof.* Since for $0 < \varepsilon < 1 - \|u_\lambda\|$

$$(J''(u_\lambda)w, w) = \Psi(w) \quad \text{for all } w \in X,$$

and $u_\lambda$ is a stable solution of $(P_\lambda)$. Thus,

$$(J''(u_\lambda)w, w) > 0 \quad \text{for all } w \in X.$$

Then, $u$ is a local minimum with respect to the topology of $X$.

∎

Now, we prove that the functional $J_{\varepsilon,\lambda}$ satisfies the the Palais-Smale condition on $X$.

**Lemma 4.11** *If the sequence* $(w_n) \subset X$ *satisfies*

$$J_{\varepsilon,\lambda_n}(w_n) \le C \quad \textit{and} \quad J'_{\varepsilon,\lambda_n}(w_n) \to 0 \textit{ in } X, \tag{4.51}$$

*for* $\lambda_n \to \lambda > 0$*, then the sequence* $(w_n)$ *admits a convergent subsequence in* $X$*.*

*Proof.* By (4.51) we have as $n \to +\infty$,

$$\int_0^1 r^\alpha |w_n'|^{\beta+2} \mathrm{d}r - \lambda_n \int_0^1 r^\gamma f(r) g_\varepsilon(r, w_n) w_n \, \mathrm{d}r = o(\|w_n\|_X).$$

Using the subcritical growth (4.49) of $g_\varepsilon$ and the inequality

$$\theta G_\varepsilon(r, u) \leq u g_\varepsilon(r, u) \quad \text{for } u \geq M_\varepsilon \tag{4.52}$$

where $M_\varepsilon$ is sufficiently large and $\theta = (p+3)/2$, we obtain

$$\begin{aligned} \frac{1}{\beta+2} \int_0^1 r^\alpha |w_n'(r)|^{\beta+2} \mathrm{d}r - \lambda_n \int_0^1 r^\gamma f(r) G_\varepsilon(r, w_n) \, \mathrm{d}r &= \left( \frac{1}{\beta+2} - \frac{1}{\theta} \right) \int_0^1 r^\alpha |w_n'(r)|^{\beta+2} \mathrm{d}r \\ &- \lambda_n \int_0^1 r^\gamma f(r) \left( \frac{1}{\theta} w_n g_\varepsilon(w_n) G_\varepsilon(r, w_n) \right) \mathrm{d}r + o(\|w_\varepsilon\|). \end{aligned}$$

Furthermore,

$$\begin{aligned} \left( \frac{1}{\beta+2} - \frac{1}{\theta} \right) \int_0^1 r^\alpha |w_n'(r)|^{\beta+2} \mathrm{d}r - \lambda_n \int_0^1 r^\gamma f(r) \left( \frac{1}{\theta} w_n g_\varepsilon(w_n) G_\varepsilon(r, w_n) \right) \mathrm{d}r + o(\|w_\varepsilon\|) \\ \geq \left( \frac{1}{\beta+2} - \frac{1}{\theta} \right) \int_0^1 r^\alpha |w_n'(r)|^{\beta+2} \mathrm{d}r + o(\|w_\varepsilon\|) - C_\varepsilon \\ + \lambda_n \int_{w_n \geq M_\varepsilon} r^\gamma f(r) \left( \frac{1}{\theta} w_n g_\varepsilon(r, w_\varepsilon) - G_\varepsilon(r, w_n) \right) \mathrm{d}r. \end{aligned}$$

Hence, $\sup_{n \in \mathbb{N}} \|w_n\| < +\infty$.

Since $p$ is subcritical, the compactness of the embedding $X \hookrightarrow L^{p+1}((0,1))$ provides that, up to a subsequence, $w_n \to w$ weakly in in $X$ and strongly in $L^{p+1}((0,1))$ for some $w \in X$. By (4.51)

$$\int_0^1 r^\alpha |u'|^{\beta+1} \, \mathrm{d}r = \lambda \int_0^1 r^\gamma f(r) g_\varepsilon(r, w) w \, \mathrm{d}r,$$

and by (4.49), we deduce that

$$\begin{aligned} \int_0^1 r^\alpha |w_n'(r) - w'(r)|^{\beta+1} \, \mathrm{d}r &= \int_0^1 r^\alpha |w_n'(r)|^{\beta+1} \, \mathrm{d}r - \int_0^1 r^\alpha |w'(r)|^{\beta+1} \, \mathrm{d}r + o(1) \\ &= \lambda_n \int_0^1 r^\gamma f(r) g_\varepsilon(r, w_n) w_n \, \mathrm{d}r - \lambda \int_0^1 r^\gamma f(r) g_\varepsilon(r, w) w \, \mathrm{d}r + o(1) \to 0 \end{aligned}$$

as $n \to +\infty$, and the lemma is proved.

■

It remains to prove that $J_{\varepsilon,\lambda}$ has a mountain pass geometry in $X$.

**Lemma 4.12** *There exists $w \in X$ such that*

$$J_{\varepsilon,\lambda}(w) \leq J_{\varepsilon,\lambda}(u_\lambda)$$

*for every $\lambda$ on a neighborhood of $\lambda^*$.*

*Proof.* Since $f \not\equiv 0$, fix some small interval $I \subset (0,1)$ of radius $2a$, $a > 0$, so that

$$\int_I r^\gamma f(r)\,\mathrm{d}r > 0.$$

Take a cutoff function $\chi$ so that $\chi = 1$ on $B_a$ and $\chi = 0$ outside $B_{2r}$. Let $w_\varepsilon = (1-\varepsilon)\chi \in X$. We have that

$$J_{\varepsilon,\lambda}(w_\varepsilon) \leq \frac{(1-\varepsilon)^2}{\beta+2}\int_0^1 r^\alpha|\chi'|^{\beta+2}\mathrm{d}r - \frac{\lambda}{\varepsilon^2}\int_0^1 r^\gamma f(r)\mathrm{d}r \to -\infty$$

as $\varepsilon \to 0$ and uniformly for $\lambda$ bounded away from 0. Since

$$\begin{aligned} J_{\varepsilon,\lambda}(u_\lambda) = \frac{1}{\beta+2}\int_0^1 r^\alpha|u_\lambda'(r)'|^{\beta+2}\mathrm{d}r - \lambda\int_0^1 \frac{r^\gamma f(r)}{1-u_\lambda}\,\mathrm{d}r \\ \to \frac{1}{\beta+2}\int_0^1 r^\alpha|(u^*)'(r)|^{\beta+2}\mathrm{d}r - \lambda^*\int_0^1 \frac{r^\gamma f(r)}{1-u^*}\,\mathrm{d}r \end{aligned}$$

as $\lambda \to \lambda^*$, we can get for $\varepsilon > 0$ small that the inequality

$$J_{\varepsilon,\lambda}(w_\varepsilon) \leq J_{\varepsilon,\lambda}(u_\lambda)$$

holds for $\lambda$ close to $\lambda^*$.

■

We finish this section with its main result.

**Theorem 4.13** *If $u^*$ is regular then there exist $\delta > 0$ such that for any $\lambda \in (\lambda^* - \delta, \lambda^*)$ there exists a classical solution $U_\lambda$ of (4.47), different from $u_\lambda$.*

*Proof.* The hypotheses on Theorem A are clearly satisfied. due to Lemmas 4.10, 4.11 and 4.12. Consider

$$c_{\varepsilon,\lambda} = \inf_{\gamma\in\Gamma}\max_{u\in\gamma} J_{\varepsilon,\lambda}(u)$$

where

$$\Gamma = \{\gamma : [0,1] \to X : \gamma \text{ is continuous and } \gamma(0) = u_\lambda,\ u(1) = w_\varepsilon\}.$$

We can then use Theorem A to get a solution $v_{\varepsilon,\lambda}$ of (4.47) for $\lambda$ close to $\lambda^*$. By Theorem 3.7 we get the uniform convergence of $v_{\varepsilon,\lambda}$ as $\lambda \nearrow \lambda^*$.

On the other hand, the convexity of $g_\varepsilon$ ensures that problem (4.47) has a unique solution at $\lambda = \lambda^*$, which is nothing but $u^*$, the extremal solution of $(P_\lambda)$. This allows us to deduce that $v_{\varepsilon,\lambda} \to u^*$ in $C([0,1])$ as $\lambda \nearrow \lambda^*$, which also implies that $v_{\varepsilon,\lambda} \leq 1 - \varepsilon$ for $\lambda$ close to $\lambda^*$. Therefore, $v_{\varepsilon,\lambda}$ is the second solution for $(P_\lambda)$ bifurcating from $u^*$.

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