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NEW CONTINUOUS DISTRIBUTIONS APPLIED TO LIFETIME DATA AND SURVIVAL ANALYSIS

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NEW CONTINUOUS DISTRIBUTIONS APPLIED TO LIFETIME DATA AND SURVIVAL ANALYSIS

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CÍCERO RAFAEL BARROS DIAS

NEW CONTINUOUS DISTRIBUTIONS OF PROBABILITY APPLIED TO LIFETIME DATA AND SURVIVAL ANALYSIS

Tese apresentada ao Programa de Pós-Graduação em Estatística da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutor em Estatística.

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(Augostinho da Silva).

Resumo

Análise estatística de dados de tempo de vida é um importante tópico em engenharia, biomedicina, ciências sociais, dentre outras áreas. Existe uma clara necessidade de se estender formas das clássicas distribuições para obter distribuições mais flexíveis com melhores ajustes. Neste trabalho, estudamos e propomos novas distribuições e novas classes de istribuições contínuas. Nós apresentamos o trabalho em três partes independentes. Na primeira, nós estudamos com alguns detalhes um modelo de tempo de vida da classe dos modelos beta generalizados proposto por Eugene; Lee; Famoye (2002). A nova distribuição é denominada de beta Nadarajah-Haghighi, a qual pode ser usada para modelar dados de sobrevivência. Sua função de taxa de falha é bastante flexível podendo ser de diversas formas dependendo dos seus parâmetros. O modelo proposto inclui como casos especiais muitas importantes distribuições discutidas na literatura, tais como as distribuições exponencial, exponential generalizada (GUPTA; KUNDU, 1999), exponencial extendida (NADARAJAH; HAGHIGHI, 2011) e a tipo exponencial (LEMONTE, 2013). Nós fornecemos um tratamento matemático abrangente da nova distribuição e obtemos explícitas expressões para os momentos, funções geratriz de momentos e quantílica, momentos incompletos, estatísticas de ordem e entropias. O método de máxima verossimilhança é usado para estimar os parâmetros do modelo e a matriz de informação observada é derivada. Nós ajustamos o modelo proposto para um conjunto de dados reais para provar a empiricamente sua flexibilidade e potencialidade. Na segunda parte, nós estudamos as propriedades matemáticas gerais de um novo gerador de distribuições contínuas com três parâmetros de forma extras chamada de família de distribuições Marshal-Olkin exponencializada. Nós apresentamos alguns modelos especiais da nova classe e algumas das suas propriedades matemáticas incluindo momentos e função geratriz de momentos. O método de máxima verossimilhança é utilizado para estimação dos parâmetros do modelo. Nós ilustramos a utilidade da nova distribuição por meio de duas aplicações a conjuntos de dados reais. Na terceira parte, nós propomos outra nova classe distribuições baseada na distribuição introduzida por Nadarajah e Haghighi(2011). Nós estudamos algumas propriedades matemáticas dessa nova classe denominada Nadarajah-Haghighi-G (NH-G) família de distribuições. Alguns modelos especiais são apresentados e obtemos explícitas expressões para a função quantília, momentos ordinários e incompletos, função geratriz e estatística de ordem. A estimação dos parâmetros do modelo é explorada por máxima verossimilhança e nós ilustramos a flexibilidade da nova família com duas aplicações a dados reais.

Palavras-chave: Distribuição Marshall-Olkin. Distribuição beta. Distribuição Nadarajah-Haghighi. Distribuições generalizadas. Tempo de vida. Estimação por maxima verossimilhança. Análise de sobrevivência.

Abstract

Statistical analysis of lifetime data is an important topic in engineering, biomedical, social sciences and others areas. There is a clear need for extended forms of the classical distributions to obtain more flexible distributions with better fits. In this work, we study and propose new distributions and new classes of continuous distributions. We present the work in three independents parts. In the first one, we study with some details a lifetime model of the beta generated class proposed by Eugene; Lee; Famoye (2002). The new distribution is called the beta Nadarajah-Haghighi distribution, which can be used to model survival data. Its failure rate function is quite flexible and takes several forms depending on its parameters. The proposed model includes as special models several important distributions discussed in the literature, such as the exponential, generalized exponential (GUPTA; KUNDU, 1999), extended exponential (NADARAJAH; HAGHIGHI, 2011) and exponential-type (LEMONTE, 2013) distributions. We provide a comprehensive mathematical treatment of the new distribution and obtain explicit expressions for the moments, generating and quantile functions, incomplete moments, order statistics and entropies. The method of maximum likelihood is used for estimating the model parameters and the observed information matrix is derived. We fit the proposed model to a real data set to prove empirically its flexibility and potentiality. In the second part, we study general mathematical properties of a new generator of continuous distributions with three extra shape parameters called the exponentiated Marshal-Olkin family. We present some special models of the new class and some of its mathematical properties including moments and generating function. The method of maximum likelihood is used for estimating the model parameters. We illustrate the usefulness of the new distributions by means of two applications to real data sets. In the third part, we propose another new class of distributions based on the distribution introduced by Nadarajah and Haghighi (2011). We study some mathematical properties of this new class called Nadarajah-Haghighi-G (NH-G) family of distributions. Some special models are presented and we obtain explicit expressions for the quantile function, ordinary and incomplete moments, generating function and order statistics. The estimation of the model parameters is explored by maximum likelihood and we illustrate the flexibility of the new family with two applications to real data.

Keywords: Marshall-Olkin distribution. Beta distribution. Nadarajah-Haghighi distribution. Generalized distribution. Lifetime. Maximum likelihood estimation. Survival analysis.

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Chapter 1

Introduction

There has been an increased interest in developing of new classes of distributions by introducing one or more additional shape parameter(s) to the baseline distribution. This parameter(s) induction have attracted the statisticians, mathematicians, scientists, engineers, economists, actuaries, demographers and other applied researchers to find more flexible distributions to obtain best fits.

There exists many generalized classes of distributions such as exponentiated family (EF) of distributions (GUPTA; GUPTA; GUPTA, 1998), beta-generated (Beta-G) family (EUGENE; LEE; FAMOYE, 2002), Kumaraswamy generalized (Kw-G) family (CORDEIRO; DE CASTRO, 2011), exponentiated generalized (Exp-G) family (CORDEIRO; ORTEGA; CUNHA, 2013), gamma generated (GG)family (ZOGRAFOS; BALAKRISHNAN, 2009) and (RISTIĆ; BALAKRISHNAN, 2012) and T-X family (ALZAATREH; LEE; FAMOYE, 2013). These classes or families of distributions have received a great deal of attention in recent years. Below, we show a brief review of major new classes of distributions proposed recently.

For example, if $G(x)$ is the cumulative distribution function (cdf) of the baseline model, then an EF of distributions is defined by taking the ath power of $G(x)$ as

$$F(x) = G(x)^a \tag{1.1}$$

where $a > 0$ is a positive real parameter. The density corresponding to (1.1) can be written as

$$f(x) = a\, g(x) G(x)^{a-1}$$

where $g(x) = dG(x)/dx$ denotes the probability density function (pdf) of a baseline (or parent) distribution.

Now, considering $G(x)$ the baseline cdf depending on a certain parameter vector. The Beta-G family introduced by Eugene; Lee; Famoye (2002) is defined by the cdf and pdf

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1} dt = I_{G(x)}(a,b),$$

and

$$f(x) = \frac{1}{B(a,b)} G(x)^{a-1}[1-G(x)]^{b-1}\, g(x),$$

respectively, where $a > 0$ and $b > 0$ are two additional shape parameters, which control skewness through the relative tail weights, $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta function ratio, $B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\cdot)$is the gamma function.

In turn, for a baseline random variable with cdf $G(x)$ and pdf $g(x)$, the Kw-G class of distributions (CORDEIRO; DE CASTRO, 2011) is denoted by its cdf and pdf as

$$F(x) = 1 - [1 - G(x)^a]^b, x > 0$$

and

$$f(x) = a\, b\, g(x) G(x)^{a-1} [1 - G(x)^a]^{b-1}, x > 0,$$

respectively, where $a > 0$ and $b > 0$ are two additional shape parameters to the G distribution, whose role is partly to govern skewness and to vary tail weights.

Other new class of distributions was defined by Cordeiro; Ortega; Cunha (2013) and called the exponentiated generalized (EG). Let $G(x)$ be a continuous cdf. Then the EG family is given by

$$F(x) = \{1 - [1 - G(x)]^\alpha\}^\beta,$$

where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters. They noted that there is no complicated function in contrast with the Beta-G family, which also includes two extra parameters but involves the beta incomplete function. The pdf of this class has the form

$$f(x) = \alpha\beta\, g(x)[1 - G(x)]^{\alpha-1}\{1 - [1 - G(x)]^\alpha\}^{\beta-1}.$$

For any baseline cumulative distribution function (cdf) $G(x)$, and $x \in \mathbb{R}$, Zografos and Balakrishnan (2009) and Ristić and Balakrishnan (2012) defined the GG distribution with pdf $f(x)$ and cdf $F(x)$ given by

$$f(x) = \frac{g(x)}{\Gamma(a)} \{-\log[1 - G(x)]\}^{a-1}$$

and

$$F(x) = \frac{\gamma\left(a, -\log\left[1 - G(x)\right]\right)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{-\log[1-G(x)]} t^{a-1} \mathrm{e}^{-t} dt,$$

respectively, for $a > 0$, where $g(x) = dG(x)/dx$, $\Gamma(a) = \int_0^\infty t^{a-1}\, \mathrm{e}^{-\mathrm{t}}\mathrm{dt}$ denotes the gamma function, and $\gamma(a,z) = \int_0^z t^{a-1}\, \mathrm{e}^{-\mathrm{t}}\mathrm{dt}$ denotes the incomplete gamma function. The GG distribution has the same parameters of the G distribution plus an additional shape parameter $a > 0$. Each new GG distribution can be obtained from a specified G distribution.

Recently, Alzaatreh; Lee; Famoye (2013) proposed a new technique to derive families of distributions by using any pdf as a generator. This new generator was named as T-X family of distributions and the its cdf is defined as

$$F(x) = \int_a^{W[G(x)]} r(t)dt \tag{1.2}$$

where $G(x)$ is the cdf of a random variable X, $r(t)$ is a pdf of the random variable T defined on $[a,b]$ and $W[G(x)]$ is a function of $G(x)$ so that satisfies the following conditions:

- $W[G(x)] \in [a, b]$;
- $W[G(x)]$ is differentiable and monotonically non-decreasing;
- $W[G(x)] \to a$ as $x \to -\infty$ and $W[G(x)] \to b$ as $x \to \infty$.

Different $W[G(x)]$ will give a new family of distributions and the corresponding pdf associated with (1.2) is

$$f(x) = \left\{ \frac{d}{dx} W[G(x)] \right\} r\{W[G(x)]\}.$$

Finally, others authors proposed recently the Weibull-G class (BOURGUIGNON; SILVA; CORDEIRO, 2014) and the Exponentiated T-X (E-TX) family (ALZAGHAL; FAMOYE; LEE, 2013) of distributions. In the first case, they used the $r(t)$ corresponding to pdf of the Weibull distribution and $W[G(x)] = G(x)/[1 - G(x)]$. On the second situation, was used $W[G(x)] = [-\log(1 - F(x)^c)]$ to obtain the E-TX class of distributions.

This thesis is divided into three parts, composed by three independent papers. So, we decided that, for this thesis, each of the papers fills a distinct chapter. Therefore, each chapter can be read independently of each other, since each is self-contained. Additionally, we emphasize that each chapter contains a thorough introduction to the presented matter, so this general introduction only shows, quite briefly, the context of each chapter. A short overview of the three chapters is presented below.

A lifetime model of the beta generated class is introduced in Chapter 2. The new distribution is called the beta Nadarajah-Haghighi distribution, which can be used to model survival data. Its failure rate function is quite flexible and takes several forms depending on its parameters. The proposed model includes as special models several important distributions discussed in the literature. We provide a comprehensive mathematical treatment of the new distribution and obtain explicit expressions for the moments, generating and quantile functions, incomplete moments, order statistics and entropies. The method of maximum likelihood is used for estimating the model parameters and the observed information matrix is derived. We fit the proposed model to a real data set to prove empirically its flexibility and potentiality.

In Chapter 3, we study general mathematical properties of a new generator of continuous distributions with three extra shape parameters called the exponentiated Marshal-Olkin family. We present some special models of the new class and some of its mathematical properties including moments and generating function. The method of maximum likelihood is used for estimating the model parameters. We illustrate the usefulness of the new distributions by means of two applications to real data sets.

Finally, in Chapter 4, we propose another new class of distributions called Nadarajah-Haghighi-G (NH-G) family of distributions. Some special models are presented and we obtain explicit expressions for the quantile function, ordinary and incomplete moments, generating function and order statistics. The estimation of the model parameters is explored by maximum likelihood and we illustrate the flexibility of the new family with two applications to real data.

Chapter 2

The Beta Nadarajah-Haghighi Distribution

Resumo

Recentemente, há um grande interesse entre estatísticos e pesquisadores aplicados na construção de distribuições flexíveis para se obter melhores modelagens aplicadas a taxas de falha. Nós estudamos um modelo de tempo de vida da família beta generalizada, chamada de distribuição beta Nadarajah-Haghighi, a qual pode ser usada para modelar dados de sobrevivência. Nós sabemos que a função taxa de falha é uma importante quantidade que caracteriza o fenômeno do tempo de vida. A função taxa de falha do novo modelo pode ser constante, decrescente, crescente, forma de banheira e banheira invertida dependendo dos parâmetros. Nós fornecemos um abrangente tratamento matemático da nova distribuição e derivamos expressões explícitas para algumas das suas principais quantidades matemáticas. O método de máxima verossimilhança é usado para estimação dos parâmetros do modelo e uma simulação de Monte Carlo é apresentada para avaliar as estimativas. Nós ajustamos o modelo proposto a dois conjuntos de dados reais e provamos empiricamente sua flexibilidade quando comparado a outras distribuições de tempo de vida.

Keywords: Distribuição Beta. Momentos. Distribuição Nadarajah-Haghighi. Distribuições Generalizadas.

Abstract

Recently, there has been a great interest among statisticians and applied researchers in constructing flexible distributions for better modeling non-monotone failure rates. We study a lifetime model of the beta generated family, called the beta Nadarajah-Haghighi distribution, which can be used to model survival data. The proposed model includes as special models some important distributions. The hazard rate function is an important quantity characterizing life phenomena. Its hazard function can be constant, decreasing, increasing, upside-down bathtub and bathtub-shaped depending on the parameters. We provide a comprehensive mathematical treatment of the new distribution and derive explicit expressions for some of its basic mathematical quantities. The method of maximum likelihood is used for estimating the model parameters and a small Monte Carlo simulation is conducted to evaluate these estimates. We fit the proposed model to two real data sets to prove empirically its flexibility as compared to other lifetime distributions.

Keywords: Beta Distribution, Moment, Nadarajah-Haghighi Distribution, Generalized Distributions.

2.1 Introduction

Extending continuous univariate distributions by introducing a few extra shape parameters is an essential method to better explore the skewness and tail weights and other properties of the generated distributions. Following the latest trend, applied statisticians are now able to construct more generalized distributions, which provide better goodness-of-fit measures when fitted to real data rather than by using the classical distributions.

The exponential distribution is perhaps the most widely applied statistical distribution for problems in reliability and survival analysis. This model was the first lifetime model for which statistical methods were extensively developed in the lifetime literature. A generalization of the exponential distribution was recently proposed by Nadarajah and Haghighi (NH) (2011). Its cumulative distribution function (cdf) is given by

$$G(x) = 1 - \exp[1 - (1 + \lambda x)^{\alpha}], \quad x > 0, \tag{2.1}$$

where $\lambda > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. The probability density function (pdf) and the hazard rate function (hrf) corresponding to (2.1) are given by

$$g(x) = \alpha\,\lambda(1 + \lambda x)^{\alpha-1} \exp[1 - (1 + \lambda x)^{\alpha}] \tag{2.2}$$

and

$$h(x) = \alpha\,\lambda(1 + \lambda x)^{\alpha-1},$$

respectively.

Then, if Y follows the NH distribution, we shall denote by $Y \sim NH(\alpha, \lambda)$. The exponential distribution is a special case of the NH model when $\alpha = 1$. Nadarajah and Haghighi (2011) pointed out that the its hrf can be monotonically increasing for $\alpha > 1$, monotonically decreasing for $\alpha < 1$ and, for $\alpha = 1$, it becomes constant. They also presented some motivations for introducing this distribution.

The first motivation is based on the relationship between the pdf in (2.2) and its hrf. The NH density function can be monotonically decreasing and its hrf can be increasing. The gamma, Weibull and exponentiated exponential (EE) distributions do not allow for an increasing failure function when their corresponding densities are monotonically decreasing. The second motivation is related with the ability (or the inability) of the NH distribution to model data that have their mode fixed at zero. The gamma, Weibull and EE distributions are not suitable for situations of this kind. The third motivation is based on the following mathematical relationship: if Y is a Weibull random variable with shape parameter α and scale parameter λ, then the density in equation (2.2) is the same as that of the random variable $Z = Y - \lambda - 1$ truncated at zero, that is, the NH distribution can be interpreted as a truncated Weibull distribution.

In this paper, we propose a new model called the *beta Nadarajah-Haghighi* (BNH) distribution, which contains as sub-models the exponential, generalized exponential (GE) (GUPTA; KUNDU, 1999), beta exponential (BE) (NADARAJAH; KOTZ, 2006), NH and exponentiated NH (ENH) (LEMONTE, 2013) distributions. These special cases are given in Table 2.1. Besides extending these five distributions, the advantage of the new model, in addition to the advantages of the NH distribution, lies in the great flexibility of its pdf and hrf. Thus, the new model provides a good alternative to many existing life distributions in modeling positive real data sets. As we will show later, the hrf of the BNH distribution can exhibit the classical four forms (increasing, decreasing, unimodal and bathtub-shaped) depending on its shape parameters. We obtain some basic mathematical properties and discuss maximum likelihood estimation of the model parameters.

The paper is outlined as follows. In Section 2.2, we define the BNH distribution and provide plots of the density and hazard rate functions. We derive a useful linear representation in Section 2.3. In Sections 2.4, 2.5 and 2.6, we obtain explicit expressions for the moments, quantile function and moment generating function (mgf), respectively. Incomplete moments and mean deviations are determined in Sections 2.7 and 2.8, respectively. In Section 2.9, we present the Rényi and Shannon entropies. The BNH order statistics are investigated in Section 2.10. Maximum likelihood estimation and a small simulation study are addressed in Sections 2.11 and 2.12. Two empirical applications to real data are illustrated in Section 2.13. Finally, Section 2.14 offers some concluding remarks.

Table 2.1: Special models of the BNH distribution.

α	a	b	Reduced Distribution
1	-	-	BE distribution (NADARAJAH; KOTZ, 2006)
1	-	1	GE distribution (KUNDU; GUPTA, 1998)
-	1	-	ENH distribution (LEMONTE, 2013)
-	1	1	NH distribution (NADARAJAH; HAGHIGHI, 2011)
1	1	1	exponential distribution

Source: Author’s elaboration.

2.2 The BNH distribution

Several ways of generating new distributions from classic ones were developed recently. Eugene; Lee; Famoye (2002) proposed the beta family of distributions. They demonstrated that its density function is a generalization of the density function of the order statistics of a random sample from a parent G distribution and studied some general properties. This class of generalized distributions has received considerable attention in recent years. In particular, taking $G(x)$ to be the density function of the normal distribution, they defined and studied the beta normal distribution, highlighting its great flexibility in modeling not only symmetric heavy-tailed distributions, but also skewed and bimodal distributions.

Nadarajah and Gupta (2004), Nadarajah and Kotz (2004 and 2006) , Lee; Famoye; Olumolade (2007) and Akinsete; Famoye; Lee (2008) defined the beta Fréchet, beta Gumbel, beta exponential, beta Weibull and beta Pareto distributions by taking $G(x)$ to be the cdf of the Fréchet, Gumbel, exponential, Weibull and Pareto distributions, respectively. More recently, Barreto-Souza; Santos; Cordeiro (2010), Pescim *et al.* (2010) and Cordeiro and Lemonte (2011) proposed the beta generalized exponential, beta generalized half-normal and beta Birnbaum-Saunders distributions, respectively.

The generalization of the NH distribution is motivated by the work of Eugene; Lee; Famoye (2002). Let $G(x)$ be the baseline cdf depending on a certain parameter vector. In order to have greater flexibility in modeling observed data, they defined the beta family by the cdf and pdf

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt = I_{G(x)}(a,b), \tag{2.3}$$

and

$$f(x) = \frac{1}{B(a,b)} G(x)^{a-1}[1-G(x)]^{b-1}\, g(x),$$

respectively, where $a > 0$ and $b > 0$ are two additional shape parameters, which control skewness through the relative tail weights, $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta function ratio, $B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\cdot)$is the gamma function.

Then, the cdf of the BNH distribution is given by

$$F(x;\theta) = \mathrm{I}_{1-\exp[1-(1+\lambda x)^{\alpha}]}(a,b), \quad x > 0, \tag{2.4}$$

where $\alpha > 0$, $\lambda > 0$, $a > 0$, $b > 0$ and $\theta = (\alpha, \lambda, a, b)^{\top}$.

The corresponding density and hazard rate functions to (2.4) are given by

$$f(x;\theta) = \frac{\alpha\,\lambda}{\mathrm{B}(a,b)}(1+\lambda x)^{\alpha-1}\{1-\exp[1-(1+\lambda x)^{\alpha}]\}^{a-1}\{\exp[1-(1+\lambda x)^{\alpha}]\}^{b} \tag{2.5}$$

and

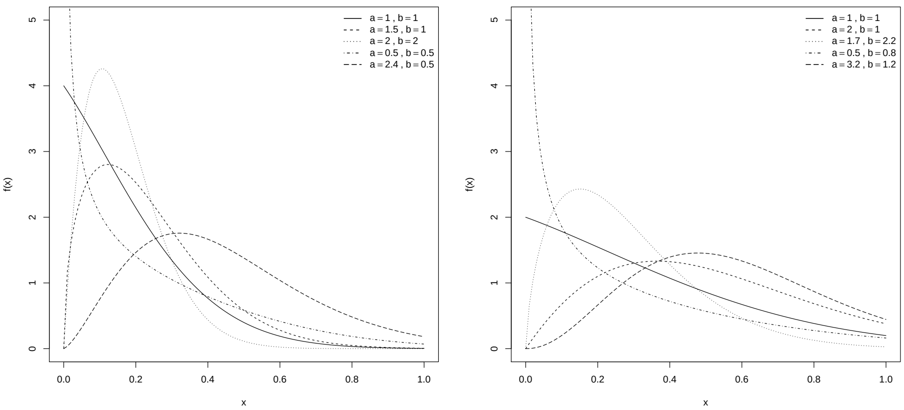
$$h(x;\theta) = \frac{\alpha\,\lambda(1+\lambda x)^{\alpha-1}\{1-\exp[1-(1+\lambda x)^{\alpha}]\}^{a-1}\{\exp[1-(1+\lambda x)^{\alpha}]\}^{b}}{\mathrm{B}(a,b) - \mathrm{B}_{1-\exp[1-(1+\lambda x)^{\alpha}]}(a,b)}. \tag{2.6}$$

A random variable X following (2.4) is denoted by $X \sim \text{BNH}(\boldsymbol{\theta})$. Simulating the BNH random variable is relatively simple. Let Y be a random variable distributed according to the usual beta distribution given by (2.3) with parameters a and b. Then, using the inverse transformation method, the random variable X can be expressed as

$$X = \frac{1}{\lambda}\left\{[1 - \log(1 - Y)]^{\frac{1}{\alpha}} - 1\right\}.$$

Plots of the density and hazard rate functions for selected parameters a and b, including the special case of the NH distribution, are displayed in Figures 2.1 and 2.2, respectively.

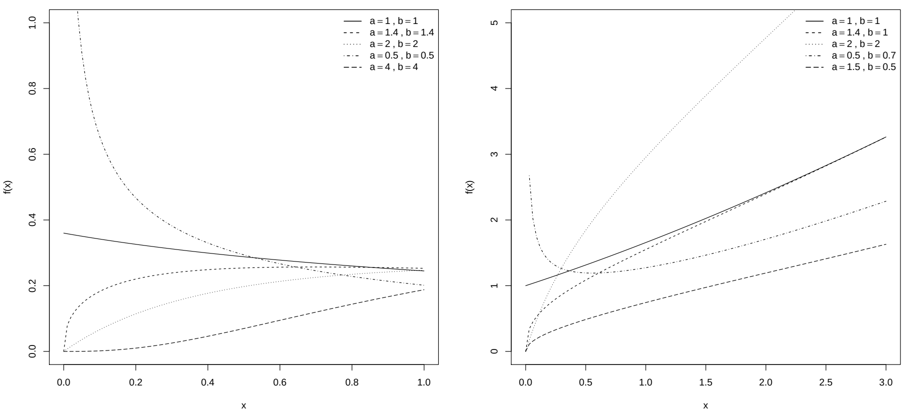
Figure 2.1: Plots of the BNH density function for some parameter values.



(a) $\alpha = 2.0$ and $\lambda = 2.0$ (b) $\alpha = 0.5$ and $\lambda = 0.5$

Source: Author's elaboration.

Figure 2.2: Plots of the BNH hazard function for some parameter values.



(a) $\alpha = 0.4$ and $\lambda = 0.9$ (b) $\alpha = 2.5$ and $\lambda = 0.4$

Source: Author's elaboration.

We now explore the asymptotics behaviors of the cumulative, density and hazard functions. First, as $x \to 0$, equations (2.4), (2.5) and (2.6) are given by

$$
\begin{aligned}
F(x) &\sim \frac{(\alpha\,\lambda\,x)^a}{a\,B(a,b)} && \text{as} \quad \mathrm{x} \to 0,\\
f(x) &\sim \frac{(\alpha\,\lambda)^a\,x^{a-1}}{B(a,b)} && \text{as} \quad \mathrm{x} \to 0,\\
h(x) &\sim \frac{(\alpha\,\lambda)^a\,x^{a-1}}{B(a,b)} && \text{as} \quad \mathrm{x} \to 0.
\end{aligned}
$$

Second, the asymptotics of equations (2.4), (2.5) and (2.6) as $x \to \infty$ are given by

$$
\begin{aligned}
1 - F(x) &\sim \frac{\exp[-b(\lambda\,x)^\alpha]}{b\,B(a,b)} && \text{as} \quad \mathrm{x} \to \infty,\\
f(x) &\sim \frac{\alpha\,\lambda^\alpha x^{\alpha-1}\exp[-(\lambda\,x)^\alpha]}{B(a,b)} && \text{as} \quad \mathrm{x} \to \infty,\\
h(x) &\sim b\,\alpha\,\lambda^\alpha\,x^{\alpha-1} && \text{as} \quad \mathrm{x} \to \infty.
\end{aligned}
$$

2.3 Linear representations

In this section, we provide linear representations for the cdf and pdf of X. For $|z| < 1$ and $b > 0$ a real non-integer number, the generalized binomial expansion holds

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} \frac{(-1)^i\,\Gamma(b)}{i!\,\Gamma(b-i)}\,z^i.$$

Let

$$w_i = w_i(a,b) = \frac{(-1)^i\,\Gamma(a+b)}{(a+i)\,i!\,\Gamma(a)\,\Gamma(b-i)}.$$

Applying this identity in equation (2.3) gives the linear representation

$$F(x) = \sum_{i=0}^{\infty} w_i\,H_{a+i}(x),$$

where $H_a(x) = G(x)^a$ denotes the exponentiated-G (exp-G) cumulative distribution and $G(x)$ is obtained from (2.1). By differentiating the last equation, the BNH pdf can be expressed as a linear representation

$$f(x) = \sum_{i=0}^{\infty} w_i\,h_{a+i}(x), \tag{2.7}$$

where $h_a(x) = aG(x)^{a-1}g(x)$ denotes the exponentiated NH (ENH) density function with power parameter $a > 0$. The properties of exponentiated distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1995) for exponentiated Weibull, Gupta; Gupta; Gupta (1998) for exponentiated Pareto, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah (2005) for exponentiated Gumbel, Lemonte (2013) for ENH, among several others. Equation (2.7) allows that some mathematical properties such as ordinary and incomplete moments, generating function and mean deviations of the BNH distribution can be derived from those quantities of the ENH distribution.

2.4 Moments

Hereafter, let $Y_{a+i} \sim$ ENH($a+i$) denotes the ENH random variable with power parameter $a+i$. The sth integer moment of X follows from (2.7) as

$$E(X^s) = \sum_{i=0}^{\infty} w_i \, E(Y_{a+i}^s),$$

where the sth moment of Y_{a+i} can be obtained from

$$E(Y_{a+i}^s) \quad = \quad \lambda^{-s} \, (a+i) \sum_{j,k,r=0}^{\infty} \frac{(-1)^{j+r} \binom{s}{j} \binom{\frac{s-j}{\alpha}}{k} \binom{a+i-1}{r} k!}{(1+r)^{k+1}}.$$

Then, the sth integer moment of X can be expressed as

$$E(X^s) \quad = \quad \lambda^{-s} \sum_{i,j,k,r=0}^{\infty} \frac{(-1)^{i+j+r} \binom{s}{j} \binom{\frac{s-j}{\alpha}}{k} \binom{a+i-1}{r} k! \, \Gamma(a+b)}{(1+r)^{k+1} \, i! \, \Gamma(a) \Gamma(b-i)}.$$

2.5 Quantile function

The quantile function (qf) of X is given by

$$Q(u) = F^{-1}(u) = \lambda^{-1} [1 - \log(1 - I_u^{-1}(a,b))]^{1/\alpha} - 1, \, 0 < u < 1, \tag{2.8}$$

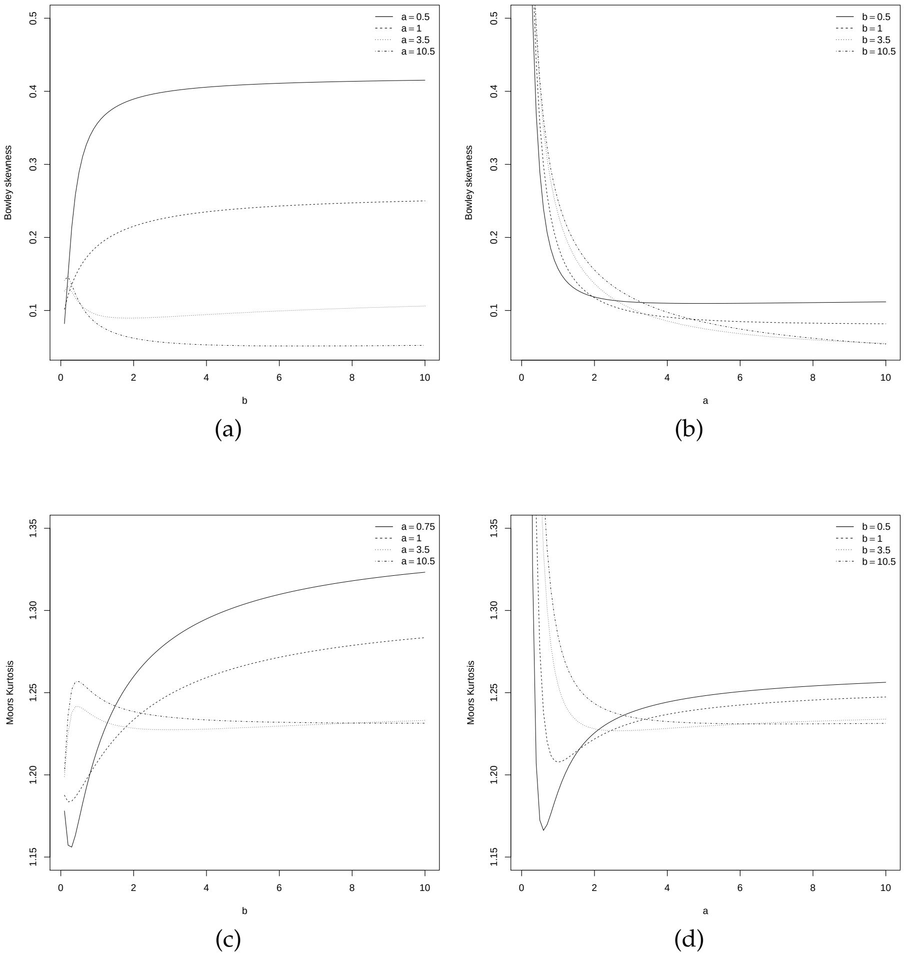
where $I_u^{-1}(a,b)$ is the inverse of the incomplete beta function. The shortcomings of the classical kurtosis measure are well-known. There are many heavy-tailed distributions for which this quantity is infinite. So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existence of classical kurtosis for many generalized distributions. The Bowley's skewness is based on quartiles (Kenney and Keeping, 1962)

$$B = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}$$

and the Moors' kurtosis (Moors, 1998) is based on octiles

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)},$$

Figure 2.3: Plots of the Bowley skewness and Moors kurtosis for some parameter values.



Source: Author's elaboration.

where $Q(\cdot)$ is obtained from (2.8). Plots of the skewness and kurtosis for some choices of the parameter b as functions of a, and for some choices of a as functions of b, for $\alpha = 2.0$ and $\lambda = 3.0$, are displayed in Figure 2.3.

The inverse of the incomplete beta function $I_u^{-1}(a, b)$ can be expressed as a power series from the Wolfram website
http://functions.wolfram.com/06.23.06.0004.01

$$I_u^{-1}(a, b) = \sum_{i=1}^{\infty} q_i \, [a\, B(a, b)\, u]^{i/a}.$$

Here, $q_1 = 1$ and the other q_i's (for $i \geq 2$) can be obtained from the recurrence equation

$$q_i = \frac{1}{i^2 + (a-2)i + (1-a)} \Big\{ (1-\delta_{i,2}) \sum_{r=2}^{i-1} q_r \, q_{i+1-r} \, [r(1-a)(i-r) - r(r-1)] \\ + \sum_{r=1}^{i-1} \sum_{s=1}^{i-r} q_r \, q_s \, q_{i+1-r-s} \, [r(r-a) + s(a+b-2)(i+1-r-s)] \Big\},$$

where $\delta_{i,2} = 1$ if $i = 2$ and $\delta_{i,2} = 0$ if $i \neq 2$. Using the generalized power series, we can write

$$Q(u) = \sum_{k=1}^{\infty} \beta_k \, \left[I_u^{-1}(a,b) \right]^k,$$

where $\beta_1 = 1/[\alpha \, \lambda]$, $\beta_2 = 1/[2 \, \alpha^2 \, \lambda]$, $\beta_3 = (\alpha^2 + 1)/(6\alpha^3 \, \lambda)$, $\beta_4 = (8 \, \alpha^4 + 5 \, \alpha^3 + 10 \, \alpha^2 + 1)/[120 \, \alpha^5 \, \lambda]$, etc. Then,

$$\begin{aligned} Q(u) &= \sum_{k=1}^{\infty} \beta_k \left(\sum_{i=1}^{\infty} q_i \, [a \, B(a,b) \, u]^{i/a} \right)^k \\ &= \sum_{k=1}^{\infty} \beta_k \, u^{\frac{k}{a}} \left(\sum_{i=0}^{\infty} q_{i+1} \, [a \, B(a,b) \, u]^{i/a} \right)^k \\ &= \sum_{k=1}^{\infty} \beta_k \, u^{\frac{k}{a}} \left(\sum_{i=0}^{\infty} \lambda_i \, u^{i/a} \right)^k, \end{aligned} \tag{2.9}$$

where $\lambda_i = q_{i+1} \, [a \, B(a,b)]^{i/a}$ for $i \geq 0$. We use throughout the paper a result of Gradshteyn and Ryzhik (2007) for a power series raised to a positive integer k (for $k \geq 1$)

$$\left(\sum_{i=0}^{\infty} \lambda_i \, v^i \right)^k = \sum_{i=0}^{\infty} c_{k,i} \, v^i, \tag{2.10}$$

where $c_{k,0} = \lambda_0^k$ and the coefficients $c_{k,i}$ (for $i = 1, 2, \ldots$) are obtained from the recurrence equation

$$c_{k,i} = (i \, \lambda_0)^{-1} \sum_{m=1}^{i} [m(k+1) - i] \, \lambda_m \, c_{k,i-m}. \tag{2.11}$$

Clearly, $c_{k,i}$ can be determined from the quantities $\lambda_0, \ldots, \lambda_i$. Based on equation (2.10), we can write

$$Q(u) = \sum_{k=1}^{\infty} \beta_k \, u^{\frac{k}{a}} \sum_{i=0}^{\infty} c_{k,i} \, u^{i/a} = \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \beta_k \, c_{k,i} \, u^{(i+k)/a} = \sum_{l=1}^{\infty} e_l \, u^{l/a}, \tag{2.12}$$

where (for $l \geq 1$) $e_l = \sum_{(k,i) \in I_l} \beta_k \, c_{k,i}$ and

$$I_l = \{(i,k) | l = i + k, k = 1, 2, \cdots, i = 0, 1, \cdots \}.$$

Let $W(\cdot)$ be any integrable function in the positive real line. We can write

$$\int_0^\infty W(x)\, f(x;\theta)dx = \int_0^1 W\left(\sum_{l=1}^\infty e_l\, u^{l/a}\right) du. \tag{2.13}$$

Equations (2.12) and (2.13) are the main results of this section since we can obtain from them various BNH mathematical quantities. Established algebraic expansions to determine these quantities based on equation (2.13) can be more efficient then using numerical integration of the density (2.5), which can be prone to rounding off errors among others. For the majority of these quantities we can substitute ∞ in the sum by a moderate number as twenty. In fact, several of them can follow by using the hight-hand integral for special $W(\cdot)$ functions, which are usually more simple than if they are based on the left-hand integral.

2.6 Generating Function

Here, we provide a formula for the mgf $M(t) = \mathrm{E}(\mathrm{e}^{tX})$ of X. Thus, $M(t)$ is given by

$$M_X(t) = \int_0^1 \exp\left[t\, Q(u)\right]\, du = \sum_{n=0}^\infty \frac{t^n}{n!} \int_0^1 Q(u)^n\, du.$$

However

$$Q(u)^n = \left(\sum_{l=1}^\infty e_l\, u^{l/a}\right)^n = u^{n/a}\left(\sum_{l=0}^\infty e_{l+1}\, u^{l/a}\right)^n = \sum_{l=0}^\infty p_{n,l}\, u^{(n+l)/a}$$

where $p_{n,0} = {e_1}^n$ and the coefficients $p_{n,i}$ (for $i = 1, 2, \ldots$) come from the recurrence equation

$$p_{n,i} = (i\, e_1)^{-1} \sum_{m=1}^{i} \left[m(n+1) - i\right] e_{m+1}\, p_{n,i-m}.$$

Finally, we obtain

$$M_X(t) = a \sum_{n,l=0}^\infty \frac{p_{n,l}\ t^n}{n!\, (n+l+a)}.$$

An alternative expression for $E(X^n)$ follows as

$$E(X^n) = a \sum_{l=0}^\infty \frac{p_{n,l}}{(n+l+a)}.$$

2.7 Incomplete Moments

The nth incomplete moment of X is defined as $m_r(y) = E(X^r | X > y) = \int_y^\infty x^r f(x)dx$. It can be immediately derived from the moments of Y having the ENH distribution. Thus, from equation (2.7), we can write $m_r(y)$ as

$$\begin{aligned} m_r(y) \quad &= \quad \sum_{i,j=0}^\infty \sum_{k=0}^r w_i \frac{(-1)^{s+j-k}\, \mathrm{e}^{(j+1)}}{(j+1)^{k/\alpha} + 1} \binom{a+i-1}{j} \binom{r}{k} \\ &\times \quad \Gamma\left(\frac{k}{\alpha} + 1, (j+1)(1+\lambda\, y)^\alpha\right). \end{aligned}$$

An alternative expression for $m_r(y)$ takes the form

$$
\begin{aligned}
m_r(y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{r} \frac{(-1)^{r+i-j}\,\mathrm{e}^{(b+i)}}{\lambda^r\,B(a,b)\,(b+i)^{j/\alpha+1}} \binom{a-1}{i}\binom{r}{j} \\
&\times\ \Gamma\left(\frac{j}{\alpha}+1,(b+i)(1+\lambda\,x)^{\alpha}\right).
\end{aligned}
$$

2.8 Mean Deviations

The deviations from the mean and the median are usually used as measures of spread in a population. Let $\mu = E(X)$ and M be the median of the BNH distribution, respectively. The mean deviations about the mean and about the median of X can be calculated as

$$
\delta_1 = E(|x-\mu|) = 2\mu\,F(\mu) - 2\,m_1(\mu) \quad \text{and} \quad \delta_2 = E(|x-M|) = \mu - 2\,m_1(M)
$$

respectively, $F(\mu)$ follows (2.4) and $m_1(q) = \int_0^q t\,f(t)dt$. The function $m_1(q)$ can be expressed as

$$
\begin{aligned}
m_1(q) &= \sum_{i=0}^{\infty}\sum_{j=0}^{1} \frac{(-1)^{1+i-j}\,\mathrm{e}^{(b+i)}}{\lambda\,B(a,b)(b+i)^{j/\alpha+1}} \binom{a-1}{i}\binom{1}{j} \\
&\times\ \left[\Gamma\left(\frac{j}{\alpha}+1\right) - \Gamma\left(\frac{j}{\alpha}+1,(b+i)(1+\lambda\,x)^{\alpha}\right)\right].
\end{aligned}
$$

Based on the mean deviations, we can construct Lorenz and Bonferroni curves, which are important in several areas such as economics, reliability, demography and actuary. For a given probability π, the Bonferroni and Lorenz curves are defined by $B(\pi) = m_1(q)/(\pi\,\mu)$ and $L(\pi) = m_1(q)/\mu$, respectively, where $q = F^{-1}(\pi) = Q(\pi)$ can be obtained from (2.8).

2.9 Rényi entropy and Shannon entropy

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi (RÉNYI, 1961) and Shannon (SHANNON, 1951) entropies. The Rényi entropy of a random variable with pdf $f(x)$ is defined as

$$
I_R(\gamma) = \frac{1}{1-\gamma}\log\left(\int_0^{\infty} f^{\gamma}(x)dx\right),
$$

for $\gamma > 0$ and $\gamma \neq 1$. The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is a special case of the Rényi entropy when $\gamma \uparrow 1$. For the BNH model direct calculation yields

$$
\begin{aligned}
&\mathrm{E}\{-\log[f(X)]\} = -\log(\alpha\lambda) + \log[B(a,b)] + (1-\alpha)\,\mathrm{E}\{\log[1+\lambda X]\} \\
-\ \ & b\,\mathrm{E}\{1-(1+\lambda X)^{\alpha}\} + (1-a)\,\mathrm{E}\left\{\log\left[1-\mathrm{e}^{1-(1+\lambda X)^{\alpha}}\right]\right\}.
\end{aligned}
$$

First, we define and compute

$$A(a_1,a_2;\alpha,\lambda,b)=\int_0^{\infty}(1+\lambda x)^{a_1}\mathrm{e}^{b[1-(1+\lambda x)^{\alpha}]}\left[1-\mathrm{e}^{1-(1+\lambda x)^{\alpha}}\right]^{a_2}dx.$$

Using the binomial expansion, we have

$$A(a_1,a_2;\alpha,\lambda,b) \;=\; \frac{\lambda}{\alpha}\sum_{i,j=0}^{\infty}(-1)^i\binom{a_2}{i}\binom{-1+\frac{a_2+1}{\alpha}}{j}\frac{\Gamma(j+1)}{(b+i)^{j+1}}$$

Proposition:

Let X be a random variable with pdf (2.5). Then,

$$\mathrm{E}\left\{\log\left[1+\lambda X\right]\right\} \;=\; \frac{\alpha\lambda}{B(a,b)}\left.\frac{\partial A(\alpha+t-1,a-1;\alpha,\lambda,b)}{\partial t}\right|_{t=0},$$

$$\mathrm{E}\left\{1-(1+\lambda X)^{\alpha}\right\} \;=\; 1-\frac{\alpha\lambda}{B(a,b)}\,A(2\alpha-1,a-1,;\alpha,\lambda,b)$$

and

$$\mathrm{E}\left\{\log\left[1-\mathrm{e}^{1-(1+\lambda X)^{\alpha}}\right]\right\} \;=\; \frac{\alpha\lambda}{B(a,b)}\left.\frac{\partial A(\alpha-1,a+t-1;\alpha,\lambda,b)}{\partial t}\right|_{t=0}.$$

The simplest formula for the Shannon entropy of X is given by

$$\begin{aligned}
\mathrm{E}\left\{-\log[f(X)]\right\} \;=\; & -\log(\alpha\lambda)+\log\left[B(a,b)\right] \\
& +\;(1-\alpha)\frac{\alpha\lambda}{B(a,b)}\left.\frac{\partial A(\alpha+t-1,a-1;\alpha,\lambda,b)}{\partial t}\right|_{t=0} \\
& -\;b\left[1-\frac{\alpha\lambda}{B(a,b)}\,A(2\alpha-1,a-1,;\alpha,\lambda,b)\right] \\
& +\;(1-a)\frac{\alpha\lambda}{B(a,b)}\left.\frac{\partial A(\alpha-1,a+t-1;\alpha,\lambda,b)}{\partial t}\right|_{t=0}.
\end{aligned}$$

After some algebraic developments, we obtain an alternative expression for $I_R(\gamma)$

$$\begin{aligned}
I_R(\gamma) \;=\; & \frac{\gamma}{1-\gamma}\log(\alpha\lambda)-\frac{\gamma}{1-\gamma}\log\left[B(a,b)\right] \\
& +\;\frac{1}{1-\gamma}\log\left[\frac{\lambda}{\alpha}\sum_{i,j=0}^{\infty}(-1)^i\binom{\gamma(a-1)}{i}\binom{(\gamma-1)(1-\frac{1}{\alpha})}{j}\frac{\Gamma(j+1)}{(b+i)^{j+1}}\right].
\end{aligned}$$

2.10 Order Statistics

Let $X_1,\ldots,X_n$ be a random sample of size n from BNH(a,b,α,λ). Then, the pdf and cdf of the ith order statistic, say $X_{i:n}$, are given by

$$\begin{aligned}
f_{i:n}(x) \;&=\; \frac{f(x)}{B(i,n-i+1)}F^{i-1}(x)(1-F(x))^{n-i} \\
&=\; \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}(-1)^m\binom{n-i}{m}f(x)F^{i+m-1}(x), \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned} F_{i:n}(x) &= \int_0^x f_{i:n}(t)dt \\ &= \frac{1}{B(i,n-i+1)} \sum_{m=0}^{n-i} \frac{(-1)^m}{m+i} \binom{n-i}{m} F^{i+m}(x), \end{aligned} \tag{2.15}$$

where $F^{i+m}(x) = \left[\sum_{r=0}^{\infty} b_r\, G(x)^r\right]^{i+m}$. Using (2.10) and (2.11), equations (2.14) and (2.15) can be written as

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)} \sum_{m=0}^{n-i} \sum_{r=1}^{\infty} \frac{(-1)^m r c_{i+m,r}}{m+i} g(x) G^{r-1}(x),$$

and

$$F_{i:n}(x) = \frac{1}{B(i,n-i+1)} \sum_{m=0}^{n-i} \sum_{r=0}^{\infty} \frac{(-1)^m c_{i+m,r}}{m+i} G^r(x).$$

Therefore, the sth moment of $X_{i:n}$ follows as

$$\begin{aligned} E(X_{i:n}^s) &= \frac{1}{B(i,n-i+1)} \sum_{m=0}^{n-i} \sum_{r=1}^{\infty} \frac{(-1)^m c_{i+m,r}}{m+i} \int_0^{+\infty} t^s g(t) G^{r-1}(t) dt \\ &= \frac{1}{B(i,n-i+1)} \sum_{m=0}^{n-i} \sum_{r=1}^{\infty} \frac{(-1)^m c_{i+m,r}}{m+i} \\ &\times\ \alpha \sum_{l=0}^{r-1} \sum_{i_1=0}^{s} \sum_{i_2=0}^{\infty} (-1)^{l+s-i_1} \lambda^{-s+1} \binom{r-1}{l} \binom{s}{i_1} \binom{\frac{i_1}{\alpha}}{i_2} \frac{\Gamma(1+i_2)}{(1+i_2)^{l+1}}. \end{aligned}$$

2.11 Estimation

Here, we determine the maximum likelihood estimates (MLEs) of the model parameters of the new family from complete samples only. Let $x_1, \ldots, x_n$ be observed values from the BNH distribution with parameters a, b, α and λ. Let $\Theta = (a, b, \alpha, \lambda)^\top$ be the parameter vector. The total log-likelihood function for Θ is given by

$$\begin{aligned} \ell &= \ell(\Theta) = n \log(\alpha\lambda) - n \log\left[B(a,b)\right] + (1 - \frac{1}{\alpha}) \\ &\times \sum_{i=1}^{n} \log(1-t_i) + (a-1) \sum_{i=1}^{n} \log(1-\mathrm{e}^{t_i}) + b \sum_{i=1}^{n} t_i, \end{aligned} \tag{2.16}$$

where $t_i = 1 - (1 + \lambda x_i)^\alpha$.

Numerical maximization of (2.16) can be performed by using the RS method (RIGBY; STASINOPOULOS, 2005), which is available in the gamlss package of the R, SAS (Proc NLMixed) or the Ox program, sub-routine MaxBFGS (DOORNIK,2009) or by solving the nonlinear likelihood equations obtained by differentiating (2.16).

The components of the score function $U_n(\Theta) = (\partial\ell_n/\partial a, \partial\ell_n/\partial b, \partial\ell_n/\partial\alpha, \partial\ell_n/\partial\lambda)$ are

$$U_a = \frac{\partial\ell}{\partial a} = -n\psi(a) + n\psi(a+b) + \sum_{i=1}^{n} \log(1-\mathrm{e}^{t_i}),$$

$$U_b = \frac{\partial\ell}{\partial b} = -n\psi(b) + n\psi(a+b) + \sum_{i=1}^{n} t_i,$$

$$U_\alpha = \frac{n}{\alpha} - \frac{1}{\alpha^2}\sum_{i=1}^{n} log(1-t_i) - (1-\frac{1}{\alpha})\sum_{i=1}^{n}\frac{t_i^{(\alpha)}}{1-t_i} + (1-a)\sum_{i=1}^{n}\frac{t_i^{(\alpha)}\mathrm{e}^{t_i}}{1-\mathrm{e}^{t_i}} + b\sum_{i=1}^{n} t_i^{(\alpha)}$$

and

$$U_\lambda = \frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - (1-\frac{1}{\alpha})\sum_{i=1}^{n}\frac{t_i^{(\lambda)}}{1-t_i} + (1-a)\sum_{i=1}^{n}\frac{t_i^{(\lambda)}\mathrm{e}^{t_i}}{1-\mathrm{e}^{t_i}} + b\sum_{i=1}^{n} t_i^{(\lambda)},$$

where $t_i^{(\alpha)} = -(1+\lambda x_i)^\alpha \log[1+\lambda x_i]$ and $t_i^{(\lambda)} = -\alpha x_i(1+\lambda x_i)^{\alpha-1}$. Setting these equations to zero, $U_a = U_b = U_\alpha = U_\lambda = 0$, and solving them simultaneously yields the MLE $\widehat{\Theta}$ of Θ.

For interval estimation on the model parameters, we require the observed information matrix $J_n(\theta)$ whose elements $U_{rs} = \partial^2\ell/\partial r\partial s$ (for $r, s = a, b, \alpha, \lambda$) can be obtained from the authors upon request. Under standard regularity conditions, we can approximate the distribution of $\sqrt{n}(\widehat{\Theta} - \Theta)$ by the multivariate normal $N_r(0, K(\theta)^{-1})$ distribution, where $K(\theta) = \lim_{n\to\infty} n^{-1}J_n(\theta)$ is the unit information matrix and r is the number of parameters of the new distribution.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct likelihood ratio (LR) statistics for testing some sub-models of the BNH distribution. For example, we may use LR statistics to check if the fit using the BNH distribution is statistically "superior" to the fits using the ENH, NH, E, GE, BE distributions for a given data set. In any case, considering the partition $\Theta = (\Theta_1^T, \Theta_2^T)^T$, tests of hypotheses of the type $H_0 : \Theta_1 = \Theta_1^{(0)}$ versus $H_A : \Theta_1 \neq \Theta_1^{(0)}$ can be performed using the LR statistic $w = 2\{\ell(\widehat{\Theta}) - \ell(\widetilde{\Theta})\}$, where $\widehat{\Theta}$ and $\widetilde{\Theta}$ are the estimates of Θ under H_A and H_0, respectively. Under the null hypothesis H_0, $w \xrightarrow{d} \chi_q^2$, where q is the dimension of the vector Θ_1 of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ_q^2 distribution. Often with lifetime data and reliability studies, one encounters censoring. A very simple random censoring mechanism very often realistic is one in which each individual i is assumed to have a lifetime X_i and a censoring time C_i, where X_i and C_i are independent random variables. Suppose that the data consist of n independent observations $x_i = \min(X_i, C_i)$ and $\delta_i = I(X_i \le C_i)$ is such that $\delta_i = 1$ if X_i is a time to event and $\delta_i = 0$ if it is right censored for $i = 1, \ldots, n$. The censored likelihood $L(\Theta)$ for the model parameters is

$$L(\Theta) \propto \prod_{i=1}^{n} [f(x_i; a, b, \alpha, \lambda)]^{\delta_i}\ [S(x_i; a, b, \alpha, \lambda)]^{1-\delta_i}, \tag{2.17}$$

where $S(x; a, b, \alpha, \lambda) = 1 - F(x; a, b, \alpha, \lambda)$ is the survival function obtained from (2.4) and $f(x; a, b, \alpha, \lambda)$ is given by (2.5). We maximize the log-likelihood (2.17) in the same way as described before.

2.12 Simulation

In this section, we conduct Monte Carlo simulation studies to assess on the finite sample behavior of the MLEs of α, λ, a and b. All results were obtained from 5000 Monte Carlo replications and the simulations were carried out the `R` programming language. In each replication, a random sample of size n is drawn from the BNH(α, λ, a, b) distribution and the BFGS method has been used by the authors for maximizing the total log-likelihood function $l(\theta)$. The BNH random number generation was performed using the inversion method. The true parameter values used in the data generating processes are $\alpha = 1.5, \lambda = 2, a = 0.5$ and $b = 2.5$. The Table 2.2 reports the empirical means, bias the mean squared errors (MSE) of the corresponding estimators for sample sizes $n = 25, 50, 100, 200$ and 400. From these figures in this table, we note that, as the sample size increases, the empirical biases and mean squared errors decrease in all the cases analyzed, as expected.

Table 2.2: Empirical means, bias and mean squared errors

n	Parameter	Mean	Bias	MSE
25	α	2,1882	0,6882	1,7969
	λ	2,4169	0,4169	0,8079
	a	0,5192	0,0192	0,0201
	b	2,1575	-0,3425	1,6367
50	α	1,8886	0,3886	0,8238
	λ	2,2029	0,2029	0,3293
	a	0,5116	0,0116	0,0089
	b	2,2749	-0,2251	0,8922
100	α	1,7211	0,2211	0,4281
	λ	2,1077	0,1077	0,1942
	a	0,5056	0,0056	0,004
	b	2,3704	-0,1296	0,5281
200	α	1,6029	0,1029	0,1628
	λ	2,0396	0,0396	0,074
	a	0,5040	0,004	0,0018
	b	2,4378	-0,0622	0,2087
400	α	1,5676	0,0676	0,1229
	λ	2,0329	0,0329	0,0614
	a	0,5020	0,002	0,0009
	b	2,4656	-0,0344	0,1636

Source: Author's elaboration.

2.13 Applications

In this section, we present two applications of the new distribution for two real data sets to illustrate its potentiality. We compared the fits of the BNH distribution with some of its special cases and other models such as beta Weibull (BW) (FAMOYE; LEE; OLUMOLADE, 2008), exponetiated Weibull (EW) (MUDHOLKAR; SRIVASTAVA; FREIMER, 1995), Weibull

(W), generalized exponential (GE) (GUPTA; KUNDU, 1999) and exponential (E) distributions given by:

- BW: $f(x) = \frac{c\lambda^c}{B(a,b)} x^{c-1} \exp[-b(\lambda x)^c]\{1 - \exp[-(\lambda x)^c]\}^{a-1}$;
- EW: $f(x) = c\, a\lambda(\lambda x)^{c-1} \exp[-(\lambda x)^c]\{1 - \exp[-(\lambda x)^c]\}^{a-1}$;
- W: $f(x) = c\lambda^c\, x^{c-1} \exp[-(\lambda x)^c]$;
- GE: $f(x) = \alpha\lambda \exp[-\lambda x](1 - \exp[-\lambda x])^{\alpha-1}$;
- E: $f(x) = \lambda \exp[-\lambda x]$.

First, we consider an uncensored data set corresponding to the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). The second data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (LEE, 1992). The required numerical evaluations are carried out using the AdequacyModel package of the R software. Table 2.3 provides some descriptive measures for the two data sets.

Table 2.3: Descriptives statistics

Statistics	Real data sets	
	Data Set 1	Data Set 2
Mean	9.3656	46.3289
Median	6.3950	40.0000
Variance	110.4250	1244.4644
Skewness	3.2866	1.0432
Kurtosis	18.4831	3.4021
Minimum	0.0800	0.3000
Maximum	79.0500	154.0000

Source: (LEE; WANG, 2003) and (LEE, 1992)

The MLEs of the model parameters for the fitted distributions and the Cramér-von Mises (W^*) and Arderson-Darling (A^*) statistics are given in Tables 2.4. These test statistics are described by Chen and Balakrishnan (1995). They are used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit.

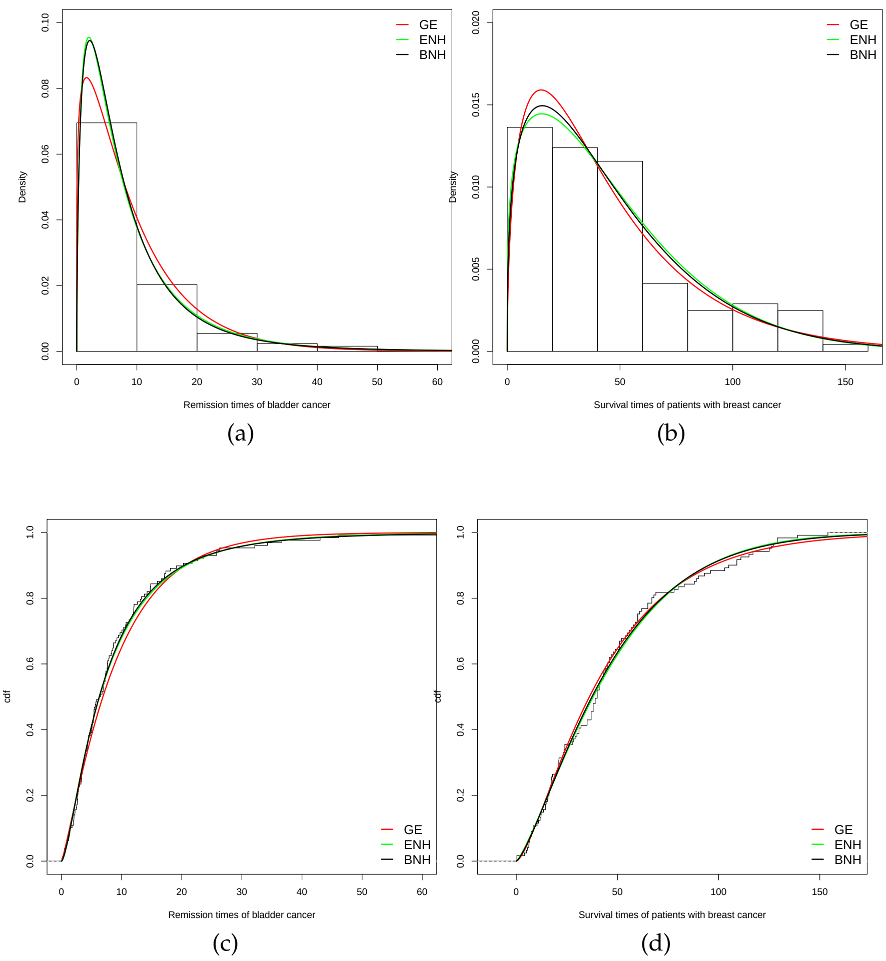
We note that the BNH model fits the first data set better than the others models according to these statistics A^* and W^*. On the other hand, the second data set is better fitted by the BNH and Weibull distributions according to these statistics. Therefore, for both data sets, the BNH distribution can be chosen as the best distribution. Plots of the estimated pdf and cdf of the BNH, ENH, NH and GE models fitted are given in Figure 2.4

Table 2.4: MLEs (standard errors in parentheses) and A* and W* statistics.

Data Set	Distribution	Estimates				A*	W*
1(n=128)	BNH (α, λ, a, b)	0.1643	0.0649	1.5848	21.6176	**0.2022**	**0.0302**
		(0.3236	0.05939	0.2859	56.0143)		
	BW (α, β, a, b)	2.7346	0.9074	0.6662	0.3219	0.2882	0.0436
		(1.599	1.5103	0.2450	0.4363)		
	ENH(α, λ, c)	0.6372	0.3444	1.6885		0.2779	0.0421
		(0.1173	0.1752	0.3646)			
	EW (α, β, c)	2.7964	0.2989	0.6544		0.2884	0.0437
		(1.2631	0.1687	0.1346)			
	NH (α, λ)	0.9226	0.1217			0.6138	0.1017
		(0.1514	0.0344)				
	W (α, β)	0.1046	1.0478			0.7864	0.1314
		(0.0093	0.0675)				
	GE (λ, c)	0.1212	1.2179			0.6741	0.1122
		(0.0135	0.1488)				
	E (λ)	0.1068				0.7159	0.1192
		(0.0094)					
2(n=121)	BNH (α, λ, a, b)	1.4783	0.0090	1.3645	1.7799	**0.3985**	0.0537
		(1.1933	0.00580	0.4293	3.3848)		
	BW (α, β, a, b)	0.8184	2.0818	1.4783	0.0104	0.4418	0.05887
		(0.3705	2.2577	0.4305	0.0077)		
	ENH(α, λ, c)	1.6630	0.0119	1.2657		0.4251	0.0567
		(0.5787	0.0062	0.1877)			
	EW (α, β, c)	0.8131	0.0174	1.4761		0.4491	0.0599
		(0.3345	0.0045	0.3820)			
	NH (α, λ)	2.5705	0.0061			0.5443	0.0748
		(0.7452	0.0021)				
	W (α, β)	0.0199	1.3053			0.4013	**0.0536**
		(0.0014	0.0934)				
	GE (λ, c)	0.0278	1.5179			0.4288	0.0615
		(0.0029	0.1927)				
	E (λ)	0.0216				0.4146	0.0585
		(0.0019)					

Source: Author's elaboration.

Figure 2.4: Plots of the estimated pdf and cdf of the BNH, ENH and GE models.



Source: Author's elaboration.

2.14 Concluding remarks

Continuous univariate distributions have been extensively used over the past decades for modeling data in several fields such as environmental and medical sciences, engineering, demography, biological studies, actuarial, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. In this paper, we propose a new distribution called the beta Nadarajah-Haghighi (BNH) distribution, which generalizes the Nadarajah-Haghighi distribution. Further, the proposed distribution includes as special models other well-known distributions in the statistical literature. We study some of its mathematical and statistical properties. We provide explicit expressions for the moments, incomplet moments, mean deviations, Rényi and Shannon entropies. The new model provide a good alternative to many existing life distributions in modeling positive real data sets. The hazard rate function of the BNH model can be constant, decreasing, increasing, upside-down bathtub and bathtub-shaped. The model parameters are estimated by maximum likelihood and the expected information matrix is derived. The usefulness of the new model is illustrated in two applications to real data using goodness-of-fit tests. Both applications have shown that the new model is superior to other fitted models. Therefore, the BNH distribution is an alternative model to the beta Weibull, exponentiated Weibull, Weibull, generalized exponential, extended exponential distributions and exponentiated Nadarajah-Haghighi distributions. We hope that the proposed model may be interesting for a wider range of statistical applications.

Chapter 3

Exponentiated Marshall-Olkin Family of Distributions

Resumo

Nós estudamos propriedades matemáticas gerais de uma nova classe de distribuições contínuas com três parâmetros de forma adicionais chamada de *exponentiated Marshal-Olkin* família de distribuições. Além disso, apresentamos alguns modelos especiais da nova classe e investigamos a forma e expressõ es explícitas para os momentos ordinários e incompletos, função quantílica e geratriz de momentos, *probability weighted moments*, curvas de Bonferroni e Lonrenz, entropias de Shannon e Renyi e estatísticas de ordem. Adicionalmente, nós exploramos a estimação dos parâmetros do modelo por máxima verossimilhança e provamos empiricamente a potencialidade da nova família com a aplicação a dois conjuntos de dados reais.

Keywords: Distribuições Generalizadas, máxima verossimilhança, momentos, estatística de ordem, Marshal-Olkin.

Abstract

We study general mathematical properties of a new class of continuous distributions with three extra shape parameters called the exponentiated Marshal-Olkin family of distributions. Further, we present some special models of the new class and investigate the shapes and derive explicit expressions for the ordinary and incomplete moments, quantile and generating functions, probability weighted moments, Bonferroni and Lorenz curves, Shannon and Rényi entropies and order statistics. Further, we discuss the estimation of the model parameters by maximum likelihood and prove empirically the potentiality of the family by means of two applications to real data.

Keywords: Generalized exponential geometric distribution, Generated family, Maximum likelihood, Moment, Order statistic.

3.1 Introduction

In the past few years, several ways of generating new distributions from classic ones were developed and discussed. Jones (2004) studied a family of distributions that arises naturally from the distribution of the order statistics. The beta-generated family introduced by Eugene; Lee; Famoye (2002) was further studied by Jones (2004). Zografos and Balakrishnan (2009) proposed the gamma-generated family of distributions. Later, Cordeiro and de Castro (2011) defined the Kumaraswamy family. Recently, Alzaatreh; Lee; Famoye (2013) proposed a new technique to derive wider families by using any probability density function (pdf) as a generator. This generator called the T-X family of distributions has cumulative distribution function (cdf) defined by

$$F(x) = \int_a^{W[G(x)]} r(t)dt \tag{3.1}$$

where $G(x)$ is the cdf of a random variable X, $r(t)$ is the pdf of a random variable T defined on $[a, b]$ and $W[G(x)]$ is a function of $G(x)$, which satisfies the following conditions:

- $W[G(x)] \in [a, b]$;
- $W[G(x)]$ is differentiable and monotonically non-decreasing;
- $W[G(x)] \to a$ as $x \to -\infty$ and $W[G(x)] \to b$ as $x \to \infty$.

Following Alzaatreh; Lee; Famoye (2013) and replacing $r(t)$ by the generalized exponential-geometric (GEG) density function (SILVA; BARRETO-SOUZA; CORDEIRO, 2010), where $T \in [0, \infty)$, and using $W[G(x)] = -\log[1 - G(x)]$, we define the cdf of a new wider family by

$$F(x) \quad = \quad \int_0^{-\log[1-G(x;\boldsymbol{\xi})]} \frac{\alpha\lambda(1-p)\mathrm{e}^{-\lambda t}[1-\mathrm{e}^{-\lambda t}]^{\alpha-1}}{[1-p\,\mathrm{e}^{-\lambda t}]^{\alpha+1}}\mathrm{d}t = \left\{\frac{1-[1-G(x;\boldsymbol{\xi})]^{\lambda}}{1-p[1-G(x;\boldsymbol{\xi})]^{\lambda}}\right\}^{\alpha}, \tag{3.2}$$

where $G(x; \boldsymbol{\xi})$ is the baseline cdf depending on a parameter vector $\boldsymbol{\xi}$ and $\alpha > 0$, $\lambda > 0$ and $p < 1$ are three additional shape parameters. For each baseline G, the *exponentiated Marshall-Olkin-G* ("EMO-G" for short) distribution is defined by the cdf (3.2). The EMO family includes as special cases the *exponentiated generalized class of distributions* (CORDEIRO; ORTEGA; CUNHA,, 2013), the proportional and reversed hazard rate models, the Marshall-Olkin family and other sub-families. Some special models are listed in Table 3.1, where $G(x) = G(x; \boldsymbol{\xi})$.

This paper is organized as follows. In Section 3.2, we define the new family of distributions and provide a physical interpretation. Five of its special distributions are discussed in Section 3.3. In Section 3.4, the shape of the density and hazard rate functions are described analytically. Two useful linear mixtures are provided in Section 3.5. In Section 3.6, we derive a power series for the quantile function (qf). In Section 3.7, we provide two general formulae for the moments. The incomplete moments are investigated in Section 3.8. In Section 3.9, we derive

Table 3.1: Some special models.

α	λ	p	$G(x)$	Reduced distribution
-	-	0	$G(x)$	Exponentiated Generalized Class (CORDEIRO *et al.*, 2013)
1	-	-	$G(x)$	Marshal-Olkin family of distributions (MARSHALL; OLKIN, 2007)
1	-	0	$G(x)$	Proportional hazard rate model (GUPTA; GUPTA, 2007)
-	1	0	$G(x)$	Proportional reversed hazard rate model (GUPTA; GUPTA, 2007)
1	1	0	$G(x)$	$G(x)$
1	-	-	$1-\mathrm{e}^{-x}$	Exponential - Geometric distribution (ADAMIDIS; LOUKAS, 1998)
-	-	-	$1-\mathrm{e}^{-x}$	Generalized Exponential - Geometric distribution (SILVA *et al.*, 2010)
1	-	-	$1-\mathrm{e}^{-\beta x^{\gamma}}$	Weibull-Geometric distribution (BARRETO-SOUZA *et al.*, 2011)
-	-	-	$1-\mathrm{e}^{-\beta x^{\gamma}}$	Exponentiated Weibull-Geometric distribution (MAHMOUDI, 2012)

Source: Author's elaboration.

the moment generating function (mgf). In Sections 3.10 we determine the mean deviations. In Section 3.11, we present general expressions for the Rényi and Shannon entropies. Section 3.12 deals with the order statistics and their moments. Estimation of the model parameters by maximum likelihood is performed in Section 3.13. Applications to two real data sets illustrate the performance of the EMO family in Section 3.14. The paper is concluded in Section 3.15.

3.2 The new family

The density function corresponding to (3.2) is given by

$$f(x) \quad = \quad \alpha\,\lambda\,(1-p)\,g(x;\boldsymbol{\xi})\,[1-G(x;\boldsymbol{\xi})]^{\lambda-1}\,\frac{\{1-[1-G(x;\boldsymbol{\xi})]^{\lambda}\}^{\alpha-1}}{\{1-p[1-G(x;\boldsymbol{\xi})]^{\lambda}\}^{\alpha+1}}, \tag{3.3}$$

where $g(x;\boldsymbol{\xi})$ is the baseline pdf. This density function will be most tractable when the functions $G(x)$ and $g(x)$ have simple analytic expressions. Hereafter, a random variable X with density function (3.3) is denoted by $X \sim$ EMO-G$(p,\alpha,\lambda,\boldsymbol{\xi})$. Henceforth, we can omit sometimes the dependence on the baseline vector $\boldsymbol{\xi}$ of parameters and write simply $G(x)=G(x;\boldsymbol{\xi})$, $f(x)=f(x;\alpha,\lambda,\beta,\boldsymbol{\xi})$, etc.

A physical interpretation of the EMO-G distribution can be given as follows. Consider a system formed by α independent components having the Marshall-Olkin cdf given by

$$H(x)=\frac{1-[1-G(x)]^{\lambda}}{1-p[1-G(x)]^{\lambda}}.$$

Suppose the system fails if all of the α components fail and let X denote the lifetime of the entire system. Then, the cdf of X is

$$F(x)=H(x)^{\alpha}=\left\{\frac{1-[1-G(x;\boldsymbol{\xi})]^{\lambda}}{1-p[1-G(x;\boldsymbol{\xi})]^{\lambda}}\right\}^{\alpha}.$$

The hazard rate function (hrf) of X becomes

$$h(x) = \frac{\alpha\,\lambda\,(1-p)\,g(x;\boldsymbol{\xi})\,[1-G(x;\boldsymbol{\xi})]^{\lambda-1}\,\{1-[1-G(x;\boldsymbol{\xi})]^{\lambda}\}^{\alpha-1}}{\{\{1-p[1-G(x;\boldsymbol{\xi})]^{\lambda}\}^{\alpha}-\{1-[1-G(x;\boldsymbol{\xi})]^{\lambda}\}^{\alpha}\}\,\{1-p[1-G(x;\boldsymbol{\xi})]^{\lambda}\}}. \quad (3.4)$$

The EMO family of distributions is easily simulated by inverting (3.2): if u has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation

$$X = G^{-1}\left[1-\left(\frac{1-u^{1/\alpha}}{1-p\,u^{1/\alpha}}\right)^{1/\lambda}\right] \quad (3.5)$$

follows the density function (3.3).

3.3 Special EMO distributions

For $p = 0$, we obtain, as a special case of (3.3), the exponentiated generalized class (CORDEIRO; ORTEGA; CUNHA, 2013) of distributions , which provides greater flexibility of its tails and can be applied in many areas of engineering and biology. Here, we present some special cases of the EMO family since it extends several useful distributions in the literature. For all cases listed below, $p \in (0,1)$, $\alpha > 0$ and $\lambda > 0$. These cases are defined by taking $G(x)$ and $g(x)$ to be the cdf and pdf of a specified distribution. The general form of the pdf of the special EMO distributions can be expressed as:

$$f(x) = q(\theta_1,\theta_2,\ldots,\theta_m)g(x)\,[1-G(x)]^{\lambda-1}\,\frac{\left\{1-[1-G(x)]^{\lambda}\right\}^{\alpha-1}}{\left\{1-p\,[1-G(x)]^{\lambda}\right\}^{\alpha+1}},$$

where $q(\theta_1,\theta_2,\ldots,\theta_m)$ is defined as a function of m parameters of the special EMO distribution. We list some special EMO distributions in Table 3.2, where the letters N, Fr, Ga, B, and Gu stand for the normal, Fréchet, gamma, beta and Gumbel baselines, respectively.

For the EMON distribution, the parameter σ has the same dispersion property as in the normal density. For the EMOB distribution, the beta distribution corresponds to the limiting case: $p \to 0$ and $\alpha = \lambda = 1$. For the EMOFr distribution, we have the classical Fréchet distribution when $p = 0$ and $\alpha = \lambda = 1$. The Kumaraswamy beta (KwB) and Kumaraswamy-gamma (KwGa) distributions can be obtained from the EMOB and EMOGa models when $p \to 0$. Plots of these EMOG density functions are displayed in Figures 3.1 to 3.5.

Table 3.2: Special EMO distributions

Distribution	$q(\cdot)$	$G(x)$	$g(x)$
$EMON(p,\alpha,\lambda,\mu,\sigma^2)$	$\frac{\alpha\lambda(1-p)}{\sigma}$	$\phi\left(\frac{x-\mu}{\sigma}\right)$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$
$EMOFr(p,\alpha,\lambda,\beta,\sigma)$	$\alpha\lambda(1-p)\beta$	$\exp\left\{-\left(\frac{\sigma}{x}\right)^{\beta}\right\}$	$\sigma^{\beta}x^{-\beta-1}\exp\left\{-\left(\frac{\sigma}{x}\right)^{\beta}\right\}$
$EMOGa(p,\alpha,\lambda,a,b)$	$\frac{\alpha\lambda(1-p)b^{a}}{\Gamma(a)}$	$\frac{\gamma(a,bx)}{\Gamma(a)}$	$\frac{b^{a}}{\Gamma(a)}x^{a-1}e^{-bx}$
$EMOB(p,\alpha,\lambda,a,b)$	$\frac{\alpha\lambda(1-p)}{B(a,b)}$	$\frac{1}{B(a,b)}\int_{0}^{x} w^{a-1}(1-w)^{b-1}dw$	$\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$
$EMOGu(p,\alpha,\lambda,\mu,\sigma)$	$\frac{\alpha\lambda(1-p)}{\sigma}$	$\exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}$	$\exp\left\{-\exp\left[-\frac{(x-\mu)}{\sigma}\right]-\frac{(x-\mu)}{\sigma}\right\}$

Source: Author's elaboration.

3.4 Asymptotic and Shapes

Proposition 1. *Let $a = \inf\{x|\, G(x) > 0\}$. The asymptotics of equations (3.2), (3.3) and (3.4) when $x \to a$ are given by*

$$\begin{aligned} F(x) &\sim [\lambda\, G(x)]^{\alpha}\,, \\ f(x) &\sim \alpha\, \lambda^{\alpha}\, g(x)\, G(x)^{\alpha-1}, \\ h(x) &\sim \alpha\, \lambda^{\alpha}\, g(x)\, G(x)^{\alpha-1}. \end{aligned}$$

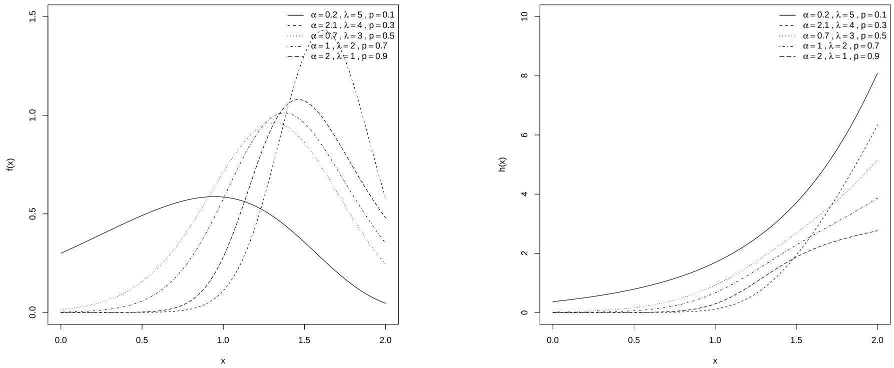
Proposition 2. *The asymptotics of equations (3.2), (3.3) and (3.4) when $x \to \infty$ are given by*

$$\begin{aligned} 1-F(x) &\sim \alpha\, \bar{G}(x)^{\lambda}, \\ f(x) &\sim \alpha\, \lambda\, g(x)\, \bar{G}(x)^{\lambda-1}, \\ h(x) &\sim \frac{\lambda\, g(x)}{\bar{G}(x)}. \end{aligned}$$

The shapes of the density and hazard rate functions can be described analytically. The critical points of the EMO-G density function are the roots of the equation

$$\begin{aligned} \frac{g'(x)}{g(x)} \quad + \quad (1-\lambda)\frac{g(x)}{1-G(x)} &= \lambda g(x)[1-G(x)]^{\lambda-1} \\ \times \quad \left\{\frac{1-\alpha}{1-[1-G(x)]^{\lambda}} + \frac{p(\alpha+1)}{1-p[1-G(x)]^{\lambda}}\right\}&. \end{aligned} \tag{3.6}$$

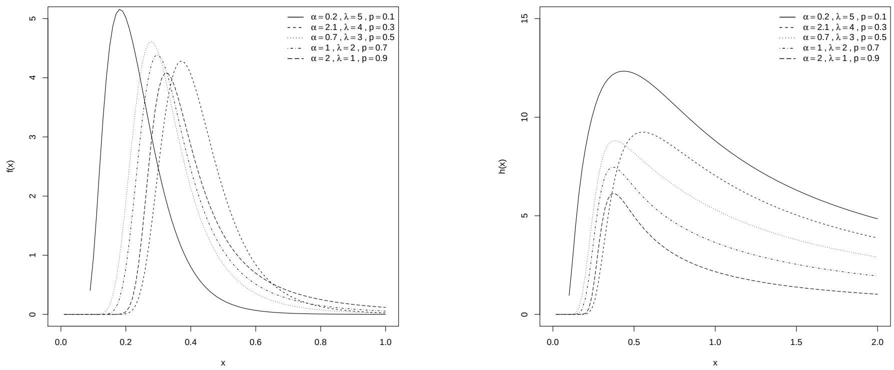
Figure 3.1: Plots of the EMON pdf and EMON hrf for some parameter values.



(a) $\mu = 0.5$ and $\sigma = 0.5$ (b) $\mu = 0.5$ and $\sigma = 0.5$

Source: Author's elaboration.

Figure 3.2: Plots of the EMOFr pdf and EMOFr hrf for some parameter values.



(a) $\beta = 0.5$ and $\sigma = 1.5$ (b) $\beta = 0.5$ and $\sigma = 1.5$

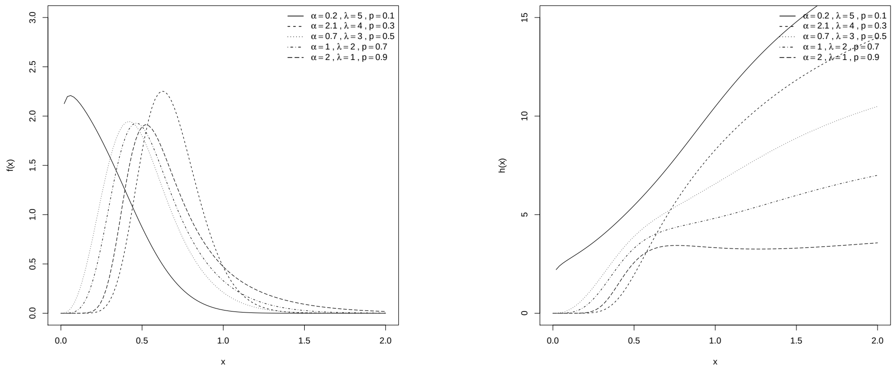
Source: Author's elaboration.

There may be more than one root to (3.6). Let $\lambda(x) = d^2 f(x)/dx^2$. We have

$$
\begin{aligned}
\lambda(x) = {} & \frac{g''(x)g(x) - g'(x)^2}{g^2(x)} + (1-\lambda)\frac{g'(x)[1-G(x)] + g^2(x)}{[1-G(x)]^2} + \lambda(\alpha - 1) \\
& \times \left\{ g'(x)\frac{[1-G(x)]^{\lambda-1}}{1-[1-G(x)]^{\lambda}} - (\lambda-1)g^2(x)\frac{[1-G(x)]^{\lambda-2}}{1-[1-G(x)]^{\lambda}} - \lambda g^2(x)\frac{[1-G(x)]^{2\lambda-2}}{\{1-[1-G(x)]^{\lambda}\}^2} \right\} \\
& - p\lambda(\alpha+1)\left\{ g'(x)\frac{[1-G(x)]^{\lambda-1}}{1-p[1-G(x)]^{\lambda}} - (\lambda-1)g^2(x)\frac{[1-G(x)]^{\lambda-2}}{1-p[1-G(x)]^{\lambda}} \right. \\
& \left. - p\lambda g^2(x)\frac{[1-G(x)]^{2\lambda-2}}{\{1-p[1-G(x)]^{\lambda}\}^2} \right\}.
\end{aligned}
\tag{3.7}
$$

If $x = x_0$ is a root of (3.7) then it corresponds to a local maximum if $\lambda(x) > 0$ for all $x < x_0$ and $\lambda(x) < 0$ for all $x > x_0$. It corresponds to a local minimum if $\lambda(x) < 0$ for all $x < x_0$ and

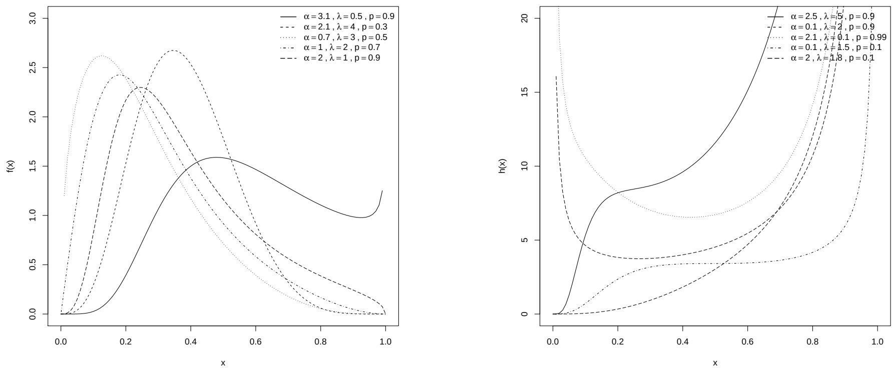
Figure 3.3: Plots of the EMOGa pdf and EMOGa hrf for some parameter values.



(a) $a = 5.5$ and $b = 5.5$ (b) $a = 5.5$ and $b = 5.5$

Source: Author's elaboration.

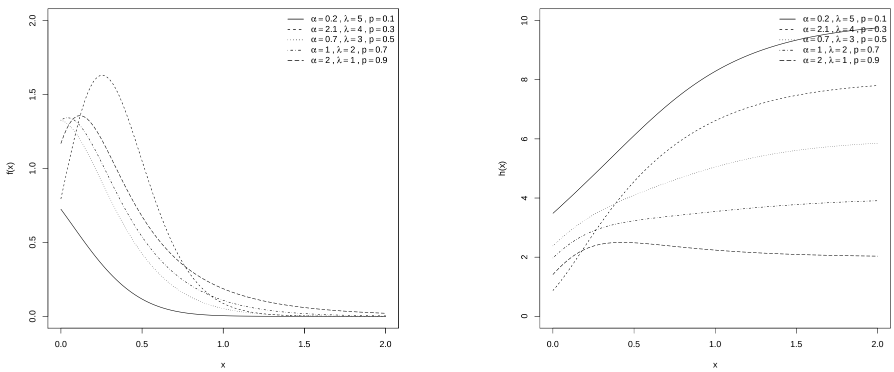
Figure 3.4: Plots of the EMOB pdf and EMOB hrf for some parameter values.



(a) $a = 2.0$ and $b = 1.5$ (b) $a = 2.0$ and $b = 1.5$

Source: Author's elaboration.

Figure 3.5: Plots of the EMOGu pdf for some parameter values.



(a) $\mu = 0.5$ and $\sigma = 1.5$ (b) $\mu = 0.5$ and $\sigma = 1.5$

Source: Author's elaboration.

$\lambda(x) > 0$ for all $x > x_0$. It gives a point of inflexion if either $\lambda(x) > 0$ for all $x \neq x_0$ or $\lambda(x) < 0$ for all $x \neq x_0$.

The critical points of the hazard rate function (hrf) of X are obtained from the equation

$$\frac{g'(x)}{g(x)} + (1-\lambda)\frac{g(x)}{1-G(x)} + \lambda(\alpha-1)\frac{g(x)[1-G(x)]^{\lambda-1}}{1-[1-G(x)]^{\lambda}} - \lambda p \frac{g(x)[1-G(x)]^{\lambda-1}}{1-p[1-G(x)]^{\lambda}}$$
$$-\frac{p\alpha\lambda g(x)[1-G(x)]^{\lambda-1}\left\{1-[1-G(x)]^{\lambda}\right\}^{\alpha-1} - \lambda\alpha g(x)[1-G(x)]^{\lambda-1}\left\{1-p[1-G(x)]^{\lambda}\right\}^{\alpha-1}}{\{1-p[1-G(x)]^{\lambda}\}^{\alpha} - \{1-[1-G(x)]^{\lambda}\}^{\alpha}} = 0. \tag{3.8}$$

There may be more than one root to (3.8). Let $\tau(x) = d^2 \log[h(x)]/dx^2$. If $x = x_0$ is a root of (3.8) then it refers to a local maximum if $\tau(x) > 0$ for all $x < x_0$ and $\tau(x) < 0$ for all $x > x_0$. It corresponds to a local minimum if $\tau(x) < 0$ for all $x < x_0$ and $\tau(x) > 0$ for all $x > x_0$. It gives an inflexion point if either $\tau(x) > 0$ for all $x \neq x_0$ or $\tau(x) < 0$ for all $x \neq x_0$.

3.5 Linear mixtures

We can demonstrate that the cdf (3.2) of X admits the expansion

$$F(x) = \sum_{k=0}^{\infty} b_k\, H_k(x;\boldsymbol{\xi}), \tag{3.9}$$

where $b_k = \sum\limits_{i,j=0}^{\infty} w_{i,j,k}$,

$$w_{i,j,k} = w_{i,j,k}(\alpha,\lambda,p) = (-1)^{i+j+k}\binom{-\alpha}{i}\binom{\alpha}{j}\binom{(i+j)\lambda}{k}p^i,$$

and $H_k(x;\boldsymbol{\xi}) = G(x;\boldsymbol{\xi})^k$ denotes the exponentiated-G ("exp-G") cdf with power parameter k.

The density function of X can be expressed as an infinite linear mixture of exp-G density functions

$$f(x) = \sum_{k=0}^{\infty} b_{k+1}\, h_{k+1}(x;\boldsymbol{\xi}), \tag{3.10}$$

where (for $k \geq 0$) $h_{k+1}(x;\boldsymbol{\xi}) = (k+1)\, g(x;\boldsymbol{\xi})\, G(x;\boldsymbol{\xi})^k$ denotes the density function of the random variable $Y_{k+1} \sim$ exp-G$(k+1)$. Equation (3.10) reveals that the EMO-G density function is a linear mixture of exp-G density functions. Thus, some of its mathematical properties can be derived directly from those properties of the exp-G distribution. For example, the ordinary and incomplete moments and moment generating function (mgf) of X can be obtained from those quantities of the exp-G distribution. Some structural properties of the exp-G distributions are well-defined by Mudholkar and Hutson (1996), Gupta and Kundu (2001) and Nadarajah and Kotz (2006), among others.

The formulae derived throughout the paper can be easily handled in most symbolic computation software plataforms such as Maple, Mathematica and Matlab. These platforms have currently the ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration. The infinity limit in these sums can be substituted by a large positive integer such as 20 or 30 for most practical purposes.

3.6 Quantile power series

We obtain explicit expressions for the moments and generating function of the EMO family using a power series for the qf $x = Q(u) = F^{-1}(u)$ of X by expanding (3.5). If the G qf, say $Q_G(u)$, does not have a closed-form expression, this function can usually be expressed as a power series

$$Q_G(u) = \sum_{i=0}^{\infty} a_i\, u^i, \tag{3.11}$$

where the coefficients a_i's are suitably chosen real numbers depending on the parameters of the parent distribution. For several important distributions such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions but it can be expanded as in equation (3.11).

We use throughout the paper a result of Gradshteyn (2000) for a power series raised to a positive integer n (for $n \geq 1$)

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{3.12}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are easily determined from the recurrence equation, with $c_{n,0} = a_0^n$,

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}. \tag{3.13}$$

Clearly, $c_{n,i}$ can be easily evaluated numerically from $c_{n,0}, \ldots, c_{n,i-1}$ and then from the quantities $a_0, \ldots, a_i$.

Next, we derive an expansion for the argument of $Q_G(\cdot)$ in equation (3.5)

$$A = 1 - \frac{(1 - u^{1/\alpha})^{1/\lambda}}{(1 - p\, u^{1/\alpha})^{1/\lambda}}.$$

Using the the generalized binomial expansion four times since $u \in (0, 1)$, we can write

$$A = \sum_{r,s,t=0}^{\infty} \sum_{m=0}^{t} (-1)^{r+s+t+m}\, p^r \binom{-\lambda^{-1}}{r} \binom{\lambda^{-1}}{s} \binom{(r+s)\alpha^{-1}}{t} \binom{t}{m} u^m$$

Then, the qf of X can be expressed from (3.5) as

$$Q(u) = Q_G\left(\sum_{m=0}^{\infty} \delta_m\, u^m\right), \tag{3.14}$$

where

$$\delta_m = \begin{cases} 1 - \sum_{r,s,t=0}^{\infty} (-1)^{r+s+t}\, p^r \binom{-\lambda^{-1}}{r} \binom{-\lambda^{-1}}{s} \binom{(r+s)\alpha^{-1}}{t}, & m = 0, \\ \sum_{r,s,t=0}^{\infty} (-1)^{r+s+t+m}\, p^r \binom{-\lambda^{-1}}{r} \binom{\lambda^{-1}}{s} \binom{(r+s)\alpha^{-1}}{t} \binom{t}{m}, & m > 0. \end{cases}$$

By combining (3.11) and (3.14), we have

$$Q(u) = \sum_{i=0}^{\infty} a_i \left(\sum_{m=0}^{\infty} \delta_m \, u^m \right)^i ,$$

and then using (3.12) and (3.13),

$$Q(u) = \sum_{m=0}^{\infty} e_m \, u^m, \tag{3.15}$$

where $e_m = \sum_{i=0}^{\infty} a_i \, d_{i,m}$, $d_{i,0} = \delta_0^i$ and, for $m > 1$,

$$d_{i,m} = (m \, \delta_0)^{-1} \sum_{n=1}^{m} [n(i+1) - m] \, \delta_n \, d_{i,m-n}.$$

Equation (3.15) is the main result of this section. It allows to obtain various mathematical quantities for the EMO-G family as can be seen in the next sections. Note that

$$Q(u)^r = \left(\sum_{m=0}^{\infty} e_m u^m \right)^r = \sum_{m=0}^{\infty} f_{r,m} \, u^m, \tag{3.16}$$

where $f_{r,m}$ is obtained from the e_m's using (3.13).

The effects of the shape parameters on the skewness and kurtosis can be determined from quantile measures. The shortcomings of the classical kurtosis measure are well-known. The Bowley skewness (KENNEY; KEEPING, 1962) is one of the earliest skewness measures defined by the average of the quartiles minus the median divided by half the interquartile range, namely

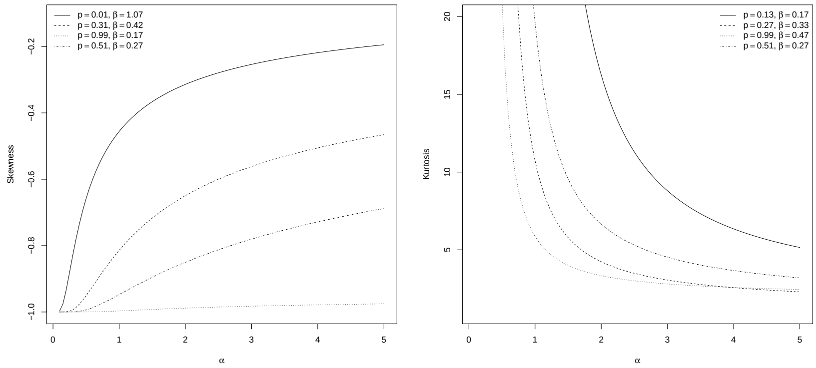
$$B = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

Since only the middle two quartiles are considered and the outer two quartiles are ignored, this adds robustness to the measure. The Moors kurtosis (MOORS, 1998) is based on octiles

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}.$$

These measures are less sensitive to outliers and they exist even for distributions without moments. In Figures 3.6 and 3.7, we plot the measures B and M for the EMOFr and EMON distributions (discussed in Section 3.3), respectively. These plots reveal how both measures B and M vary on the shape parameters.

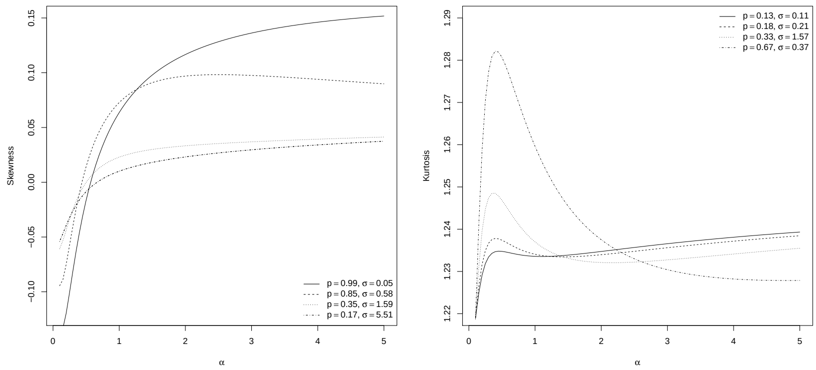
Figure 3.6: Skewness (a) and Kurtosis (b) of the EMOFr distribution.



(a) $\sigma = 1.0$ and $\lambda = 1.0$ (b) $\sigma = 1.0$ and $\lambda = 1.0$

Source: Author's elaboration.

Figure 3.7: Skewness (a) and Kurtosis (b) of the EMON distribution.



(a) $\mu = 3.3$ and $\lambda = 1.0$ (b) $\mu = 3.3$ and $\lambda = 1.0$

Source: Author's elaboration.

3.7 Moments

Hereafter, we shall assume that G is the cdf of a random variable Z and that F is the cdf of a random variable X having density function (3.3). The rth ordinary moment of X can be obtained from the (r, k)th Probability Weighted Moment (PWM) of Z defined by

$$\tau_{r,k} = \mathrm{E}[Z^r\, G(Z)^k] = \int_{-\infty}^{\infty} z^r\, G(z)^k\, g(z) dz. \tag{3.17}$$

In fact, we have

$$\mathrm{E}(X^r) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\, \tau_{r,k}. \tag{3.18}$$

Thus, the moments of any EMO-G distribution can be expressed as an infinite linear combination of the PWMs of G. A second formula for $\tau_{r,k}$ can be based on the parent qf $Q_G(u) = G^{-1}(u)$. Setting $G(x) = u$, we obtain

$$\tau_{r,k} = \int_0^1 Q_G(u)^r \, u^k du, \tag{3.19}$$

where the integral follows from (3.16) as

$$\mathrm{E}(X^r) = \int_0^1 Q(u)^r du = \sum_{m=0}^{\infty} \frac{f_{r,m}}{m+1}. \tag{3.20}$$

The PWMs for some well-known distributions will be determined in the following sections using alternatively equations (3.17) and (3.19).

The central moments (μ_s) and cumulants (κ_s) of X can be obtained from equations (3.18) and (3.20) as

$$\mu_s = \sum_{j=0}^{s} (-1)^j \binom{s}{j} \mu_1'^s \mu_{s-j}', \qquad \kappa_s = \mu_s' - \sum_{j=1}^{s-1} \binom{s-1}{j-1} \kappa_j \mu_{s-j}',$$

where $\kappa_1 = \mu_1'$. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The pth descending factorial moment of X is

$$\mu_{(p)}' = \mathrm{E}[X^{(p)}] = \mathrm{E}[X(X-1) \times \cdots \times (X-p+1)] = \sum_{k=0}^{p} s(p,k) \mu_k',$$

where $s(r,k) = (k!)^{-1} [d^k x^{(r)} / dx^k]_{x=0}$ is the Stirling number of the first kind. So, we can obtain the factorial moments from the ordinary moments given before.

3.8 Incomplete moments

The nth incomplete moment of X is defined as $m_r(y) = \int_{-\infty}^{y} x^r f(x) dx$. For empirical purposes, the shape of many distributions can be usefully described by the incomplete moments. Here, we propose two methods to determine the incomplete moments of the new family. First, we can express $m_r(y)$ as

$$m_r(y) = \sum_{k=0}^{\infty} (k+1) \, b_{k+1} \int_0^{G(y;\boldsymbol{\xi})} Q_G(u)^r \, u^k \, du. \tag{3.21}$$

The integral in (3.21) can be evaluated at least numerically for most baseline distributions.

A second method for the incomplete moments of X follows from (3.21) using equations (3.12) and (3.13). We obtain

$$m_r(y) = \sum_{k,m=0}^{\infty} \frac{(k+1) \, b_{k+1} \, c_{r,m}}{m+k+1} \, G(y;\boldsymbol{\xi})^{m+k+1}. \tag{3.22}$$

Equations (3.21) and (3.22) are the main results of this section.

3.9 Generating function

Here, we provide three formulae for the mgf $M(s) = E(\mathrm{e}^{s\,X})$ of X. A first formula for $M(s)$ comes from equation (3.10) as

$$M(s) = \sum_{k=0}^{\infty} b_{k+1}\, M_{k+1}(s), \tag{3.23}$$

where $M_{k+1}(s)$ is the generating function of the exp-G$(k+1)$ distribution. Hence, $M(s)$ can be determined from an infinite linear combination of the exp-G generating functions.

A second formula for $M(s)$ can be derived from equation (3.10) as

$$M(s) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\, \rho_k(s), \tag{3.24}$$

where

$$\rho_k(s) = \int_0^1 \exp\left[s\, Q_G(u)\right] u^k du. \tag{3.25}$$

We can derive the mgfs of several EMO distributions directly from equations (3.24) and (3.25). For example, the mgfs of the exponentiated Marshall-Olkin exponential (EMOE) (such that $\lambda s < 1$) and EMO-standard logistic (for $s < 1$) distributions are given by

$$M(s) = \sum_{k=0}^{\infty} (k+1\,)\, b_{k+1}\;\; B(k+1, 1-\lambda s) \quad \text{and} \quad M(s) = \sum_{k=0}^{\infty} (k+1\,)\, b_{k+1}\, B(s+k+1, 1-s),$$

respectively.

3.10 Mean deviations

The mean deviations about the mean ($\delta_1 = E(|X - \mu_1'|)$) and about the median ($\delta_2 = E(|X - M|)$) of X can be expressed as

$$\delta_1 = 2\mu_1'\, F\left(\mu_1'\right) - 2m_1\left(\mu_1'\right) \qquad \text{and} \qquad \delta_2 = \mu_1' - 2m_1(M), \tag{3.26}$$

respectively, $F(\mu_1')$ is easily evaluated from equation (3.2),

$$M = Q_G\left[1 - \left(\frac{1-2^{-1/\alpha}}{1-p\times 2^{-1/\alpha}}\right)^{1/\lambda}\right]$$

is the median of X, $\mu_1' = \mathrm{E}(X)$ comes from (3.18) and $m_1(y)$ is the first incomplete moment of X determined from (3.22) with $r = 1$.

Next, we provide three alternative ways to compute δ_1 and δ_2. A general equation for $m_1(z)$ is given by (3.21). A second general formula for $m_1(z)$ can be obtained from (3.10) as

$$m_1(z) = \sum_{k=0}^{\infty} b_{k+1}\, J_{k+1}(z), \tag{3.27}$$

where

$$J_{k+1}(z) = \int_{-\infty}^{z} x\, h_{k+1}(x) dx. \tag{3.28}$$

Equation (3.28) is the basic quantity to compute the mean deviations for the exp-G distributions. A simple application of (3.27) and (3.28) can be conducted to the exponentiated Marshall-Olkin Weibull (EMOW) distribution. The exponentiated Weibull density function (for $x > 0$) with power parameter $k+1$, shape parameter c and scale parameter β is given by

$$h_{k+1}(x) = c\,(k+1)\,\beta^c\, x^{c-1}\, \exp\left\{-(\beta x)^c\right\}\, \left[1-\exp\left\{-(\beta x)^c\right\}\right]^k,$$

and then

$$J_{k+1}(z) \quad = \quad c\,(k+1)\,\beta^c \sum_{r=0}^{\infty} (-1)^r \binom{k}{r} \int_0^z x^c\, \exp\left\{-(r+1)(\beta x)^c\right\} dx.$$

The last integral reduces to the incomplete gamma function

$$J_{k+1}(z) = c\,(k+1)\,\beta^c \sum_{r=0}^{\infty} (-1)^r \binom{k}{r}\, \gamma\left(c+1, (r+1)(\beta z)^c\right),$$

where $\gamma(a,x) = \int_0^x w^{a-1}\, \mathrm{e}^{-w} \mathrm{d}w$.

A third general formula for $m_1(z)$ can be derived by setting $u = G(x)$ in (3.10)

$$m_1(z) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\; T_k(z), \tag{3.29}$$

where $T_k(z)$ is given by

$$T_k(z) = \int_0^{G(z)} Q_G(u)\, u^k du. \tag{3.30}$$

Applications of these equations are straightforward to obtain Bonferroni and Lorenz curves. These curves are defined (for a given probability π) by $B(\pi) = m_1(q)/(\pi\, \mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $q = F^{-1}(\pi) = Q(\pi)$ comes from the qf of X for a given probability π.

3.11 Entropies

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi (1961) and Shannon (1951). The Rényi entropy of a random variable with pdf $f(x)$ is defined as

$$I_R(c) = \frac{1}{1-c} \log\left(\int_0^{\infty} f(x)^c dx\right),$$

for $c > 0$ and $c \neq 1$. The Shannon entropy of a random variable X is defined by $E\left\{-\log\left[f(X)\right]\right\}$. It is the special case of the Rényi entropy when $c \uparrow 1$. Direct calculation yields

$$\begin{aligned}
\mathrm{E}\left\{-\log\left[f(X)\right]\right\} &= -\log\left[\alpha\lambda(1-p)\right] - \mathrm{E}\left\{\log\left[g(X;\boldsymbol{\xi})\right]\right\} \\
&- (\lambda-1)\,\mathrm{E}\left\{\log\left[1-G(x;\boldsymbol{\xi})\right]\right\} \\
&- (\alpha-1)\,\mathrm{E}\left[\log\left\{1-\left[1-G(x;\boldsymbol{\xi})\right]^{\lambda}\right\}\right] \\
&+ (\alpha+1)\,\mathrm{E}\left[\log\left\{1-p\left[1-G(x;\boldsymbol{\xi})\right]^{\lambda}\right\}\right].
\end{aligned}$$

After some algebraic manipulations, we obtain:

Proposition 3. *Let X be a random variable having density* (3.3)*. Then,*

$$\mathrm{E}\left\{\log\left[1-G(X)\right]\right\} = \frac{\alpha(1-p)}{\lambda}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+1}p^{i}\binom{-\alpha-1}{i}\binom{\alpha-1}{j}}{(i+j+1)^{2}},$$

$$\mathrm{E}\left[\log\left\{1-\left[1-G(x;\boldsymbol{\xi})\right]^{\lambda}\right\}\right] = \alpha(1-p)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}p^{i}\binom{-\alpha-1}{i}\left[\frac{\partial}{\partial t}\binom{t+\alpha-1}{j}\Big|_{t=0}\right]}{i+j+1} \tag{3.31}$$

and

$$\mathrm{E}\left[\log\left\{1-p\left[1-G(x;\boldsymbol{\xi})\right]^{\lambda}\right\}\right] = \alpha(1-p)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}p^{i}\binom{\alpha-1}{j}\left[\frac{\partial}{\partial t}\binom{t-\alpha-1}{i}\Big|_{t=0}\right]}{i+j+1}.$$

The simplest formula for the entropy of X is given by

$$\begin{aligned}
\mathrm{E}\left\{-\log[f(X)]\right\} &= -\log[\alpha\lambda(1-p)] - \mathrm{E}\left\{\log[g(X;\boldsymbol{\xi})]\right\} \\
&+\frac{\alpha(1-p)(1-\lambda)}{\lambda}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+1}p^{i}\binom{-\alpha-1}{i}\binom{\alpha-1}{j}}{(i+j+1)^{2}} \\
&+\alpha(1-\alpha)(1-p)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}p^{i}\binom{-\alpha-1}{i}\left[\frac{\partial}{\partial t}\binom{t+\alpha-1}{j}\Big|_{t=0}\right]}{i+j+1} \\
&+\alpha(\alpha+1)(1-p)\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}p^{i}\binom{\alpha-1}{j}\left[\frac{\partial}{\partial t}\binom{t-\alpha-1}{i}\Big|_{t=0}\right]}{i+j+1}.
\end{aligned}$$

After some algebraic developments, we obtain an alternative expression for $I_R(c)$

$$I_R(c) = \frac{c}{1-c}\log[\alpha\lambda(1-p)] + \frac{1}{1-c}\log\left[\sum_{i,j=0}^{\infty} w_{i,j}^{*}\, I(\gamma,\lambda,i,j)\right],$$

where

$$w^*_{i,j} = (-1)^{i+j}\, p^i \binom{-c(\alpha+1)}{i} \binom{c(\alpha-1)}{j},$$

and

$$I(\gamma,\lambda,i,j) = \int_{-\infty}^{\infty} g(x)^{\gamma}\, \bar{G}(x)^{(\lambda-1)c+\lambda(i+j)}\, dx.$$

3.12 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the EMO-G distribution. Let $X_{i:n}$ be the ith order statistic. From equations (3.9) and (3.10), the pdf of $X_{i:n}$ can be written as

$$f_{i:n}(x) = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[\sum_{r=0}^{\infty} b_{r+1}\, r\, G(x)^{r-1}\, g(x)\right] \left[\sum_{k=0}^{\infty} b_k\, G(x)^{a+k}\right]^{j+i-1},$$

where $K = n!/[(i-1)!\,(n-i)!]$. Using (3.12) and (3.13), we can write

$$\left[\sum_{k=0}^{\infty} b_k\, G(x)^k\right]^{j+i-1} = \sum_{k=0}^{\infty} e_{j+i-1,k}\; G(x)^k,$$

where $e_{j+i-1,0} = b_0^{j+i-1}$ and

$$e_{j+i-1,k} = (k\, b_0)^{-1} \sum_{m=1}^{k} [m(j+i)-k]\, b_m\, e_{j+i-1,k-m}.$$

Hence,

$$f_{i:n}(x) = \sum_{k=0}^{\infty} d_k\, h_{k+1}(x), \tag{3.32}$$

where $d_k = K \sum_{j=0}^{n-i} \sum_{m=0}^{k} b_{m+1} e_{j+i-1,k-m}$.

Equation (3.32) is the main result of this section. It reveals that the pdf of the EMO-G order statistics is a linear combination of exp-G densities. So, several mathematical quantities of the EMO-G order statistics such as ordinary, incomplete and factorial moments, mgf, mean deviations and several others can be obtained from those quantities of the exp-G distribution.

3.13 Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals for the model parameters. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters for this family from complete samples only by maximum likelihood. Here, we determine the MLEs of the parameters of the new family of distributions from complete samples only.

Let $x_1, \ldots, x_n$ be the observed values from the EMOG distribution with parameters p, α, λ and $\boldsymbol{\xi}$. Let $\theta = (p, \alpha, \lambda, \boldsymbol{\xi})^\top$ be the $r \times 1$ parameter vector. The total log-likelihood function for θ is given by

$$\begin{aligned}
\ell_n &= \ell_n(\Theta) = n \log \alpha + n \log \lambda + n \log(1-p) + \sum_{i=1}^{n} \log\left[g(x_i; \boldsymbol{\xi})\right] \\
&+ (\lambda - 1) \sum_{i=1}^{n} \log\left[1 - G(x_i; \boldsymbol{\xi})\right] + (\alpha - 1) \sum_{i=1}^{n} \log\left\{1 - [1 - G(x_i; \boldsymbol{\xi})]^{\lambda}\right\} \\
&- (\alpha + 1) \sum_{i=1}^{n} \log\left\{1 - p[1 - G(x_i; \boldsymbol{\xi})]^{\lambda}\right\}. \qquad (3.33)
\end{aligned}$$

The maximized log-likelihood can be either directly by using the NLMIXED procedure in SAS or the sub-routine MaxBFGS in the Ox program (DOORNIK, 2009) or by solving the nonlinear likelihood equations obtained by differentiating (3.33). The components of the score function

$$U_n(\theta) = (\partial \ell_n / \partial p, \partial \ell_n / \partial \alpha, \partial \ell_n / \partial \lambda, \partial \ell_n / \partial \boldsymbol{\xi})^\top$$

are given by

$$\begin{aligned}
\frac{\partial \ell_n}{\partial p} &= (\alpha + 1) \sum_{i=1}^{n} \frac{[1 - G(x_i; \boldsymbol{\xi})]^{\lambda}}{1 - p[1 - G(x_i; \boldsymbol{\xi})]^{\lambda}} - \frac{n}{1-p}, \\
\frac{\partial \ell_n}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left\{1 - [1 - G(x_i; \boldsymbol{\xi})]^{\lambda}\right\} - \sum_{i=1}^{n} \log\left\{1 - p\,[1 - G(x_i; \boldsymbol{\xi})]^{\lambda}\right\}, \\
\frac{\partial \ell_n}{\partial \lambda} &= \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left[1 - G(x_i; \boldsymbol{\xi})\right] - (\alpha - 1) \sum_{i=1}^{n} \frac{[1 - G(x_i; \boldsymbol{\xi})]^{\lambda} \log\left[1 - G(x_i; \boldsymbol{\xi})\right]}{1 - [1 - G(x_i; \boldsymbol{\xi})]^{\lambda}} \\
&\quad + p\,(\alpha + 1) \sum_{i=1}^{n} \frac{[1 - G(x_i; \boldsymbol{\xi})]^{\lambda} \log\left[1 - G(x_i; \boldsymbol{\xi})\right]}{1 - p[1 - G(x_i; \boldsymbol{\xi})]^{\lambda}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \boldsymbol{\xi}} &= \sum_{i=1}^{n} \frac{g^{(\xi)}(x_i; \boldsymbol{\xi})}{g(x_i; \boldsymbol{\xi})} - (\lambda - 1) \sum_{i=1}^{n} \frac{G^{(\xi)}(x_i; \boldsymbol{\xi})}{1 - G(x; \boldsymbol{\xi})} \\
&\quad + \lambda(\alpha - 1) \sum_{i=1}^{n} \frac{g^{(\xi)}(x; \boldsymbol{\xi})[1 - G(x_i; \boldsymbol{\xi})]^{\lambda - 1}}{1 - [1 - G(x_i; \boldsymbol{\xi})]^{\lambda}} - p\lambda(\alpha + 1) \\
&\quad \times \sum_{i=1}^{n} \frac{g^{(\xi)}(x_i; \boldsymbol{\xi})[1 - G(x_i; \boldsymbol{\xi})]^{\lambda - 1}}{1 - p[1 - G(x_i; \boldsymbol{\xi})]^{\lambda}},
\end{aligned}$$

where $h^{(\xi)}(\cdot)$ means the derivative of the function h with respect to ξ. For interval estimation of the model parameters, we can derive the observed information matrix $J_n(\theta)$, whose elements can be obtained from the authors upon request. Let $\widehat{\theta}$ be the MLE of θ. Under standard regularity conditions (COX; HINKLEY, 1974), we can approximate the distribution of $\sqrt{n}(\widehat{\theta} - \theta)$ by the multivariate normal $N_r(0, K(\theta)^{-1})$, where $K(\theta) = \lim_{n \to \infty} n^{-1} J_n(\theta)$ is the unit information matrix and r is the number of parameters of the new distribution.

Often with lifetime data and reliability studies, one encounters censoring. In a very realistic random censoring mechanism, each individual i is assumed to have a lifetime X_i and a censoring time C_i, where X_i and C_i are independent random variables. Suppose that the data consist of n independent observations $x_i = \min(X_i, C_i)$ and $\delta_i = I(X_i \leq C_i)$ is such that $\delta_i = 1$ if X_i is a time to event and $\delta_i = 0$ if it is right censored for $i = 1, \ldots, n$. The censored likelihood $L(\theta)$ for the model parameters is

$$L(\theta) \propto \prod_{i=1}^{n} [f(x_i; p, \alpha, \lambda, \boldsymbol{\xi})]^{\delta_i} \, [S(x_i; p, \alpha, \lambda, \boldsymbol{\xi})]^{1-\delta_i},$$

where $f(x; p, \alpha, \lambda, \boldsymbol{\xi})$ is given by (3.3) and $S(x; p, \alpha, \lambda, \boldsymbol{\xi})$ is the survival function evaluated from (3.2).

3.14 Applications

In this section, we use a real data set, collected by Prater (1956) and analyzed by Atkinson (1985), referent to stress among women in Townsville, Queensland, Australia (Data Set 1) and the proportion of crude oil converted to gasoline after distillation and fractionation (Data Set 2). This application aims of the illustrate the potentiality of the EMO family. All the computations were done using the `R` software (R Development Core Team, 2012). For this application, were consider the following distributions: EMOB (a,b,p,λ,α), Kw-WP(a,b,c,λ,β) (Kwmaraswamy Weibull Poisson distribution) proposed by Ramos *et al.*(2015), beta beta-BB(α,β,a,b), Kw-Beta (α,β,a,b), Exp-NH(α,λ,β) proposed by Lemonte(2013), Exp-W(λ,k,a) (MUDHOLKAR; SRIVASTAVA; FREIMER, 1995), Weibull(λ,k), Chen(λ,β), flexible Weibull distribution-FW(α,β) (BEBBINGTON; LAI; ZITIKIS, 2007) and Gamma(α,β). A descriptive analysis of the data is presented in Table 3.3. The densities of these distributions are given by

- Kw-WP: $f(x) = \frac{\lambda abc\beta^c x^{c-1}[1-\mathrm{e}^{-(\beta x)^c}]^{a-1}\exp[\lambda\{1-[1-\mathrm{e}^{-(\beta x)^c}]^a\}^b-(\beta x)^c]}{(\mathrm{e}^{\lambda}-1)\{1-[1-\mathrm{e}^{-(\beta x)^c}]^a\}^{1-b}}$;
- BB: $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}[I_x(\alpha,\beta)]^{a-1}[1-I_x(\alpha,\beta)]^{b-1}}{B(\alpha,\beta)B(a,b)}$;
- Kw-B: $f(x) = \frac{a\,b\,x^{\alpha-1}(1-x)^{\beta-1}[I_x(\alpha,\beta)]^{a-1}\{1-[I_x(\alpha,\beta)]^a\}^{b-1}}{B(\alpha,\beta)}$;
- Exp-NH: $f(x) = \alpha\lambda\beta(1+\lambda x)^{\alpha-1}\exp[1-(1+\lambda x)^{\alpha}]\{1-\exp[1-(1+\lambda x)^{\alpha}]\}^{\beta-1}$;
- Exp-W: $f(x) = k\,a\lambda(\lambda x)^{k-1}\exp[-(\lambda x)^k]\{1-\exp[-(\lambda x)^k]\}^{a-1}$;
- W: $f(x) = k\lambda^k\,x^{k-1}\exp[-(\lambda x)^k]$;
- Chen: $f(x) = \beta\lambda\,x^{\beta-1}\mathrm{e}^{\lambda(1-\mathrm{e}^{x^{\beta}})+x^{\beta}}$;
- FW: $f(x) = \left(\alpha + \frac{\beta}{x^2}\right)\mathrm{e}^{\alpha x-\beta/x}\exp[-\mathrm{e}^{-\alpha x-\beta/x}]$;
- Gamma: $f(x) = \frac{x^{\alpha-1}\mathrm{e}^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}$.

Table 3.3: Descriptive statistics.

Statistics	Real data sets	
	Data Set 1	Data Set 2
Mean	0.2642	0.1966
Median	0.2500	0.1780
Mode	0.2500	0.1500
Variance	0.0376	0.0115
Skewness	0.9712	0.3867
Kurtosis	0.8272	-0.6561
Maximum	0.8500	0.4570
Minimum	0.0100	0.0280
n	166	32

Source: Author's elaboration.

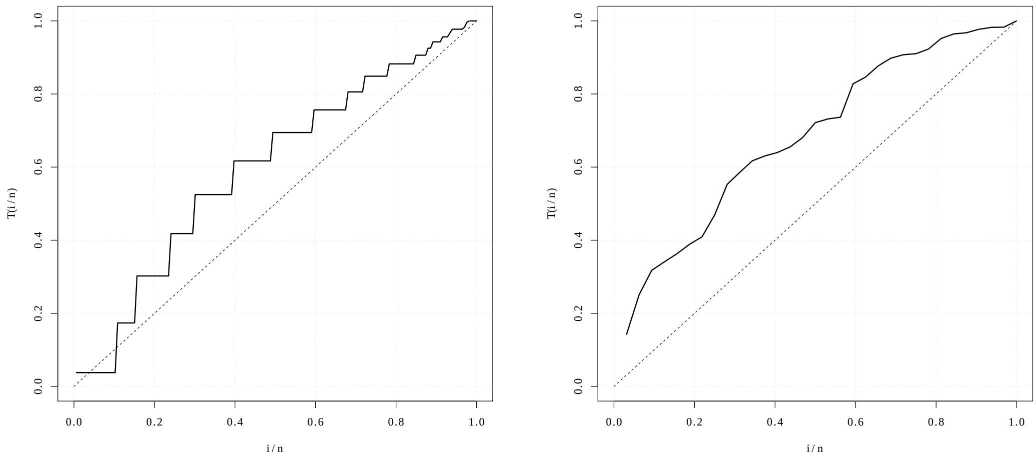
It is possible to obtain qualitative information about the hrf by means of plot analysis when we have the data are censored or uncensored. We emphasize that the data sets here are uncensored. For this type of data, the total time in test (TTT) plot proposed by Aarset (1987) may be used. Let T be a random variable with non-negative values that represents the survival time. The TTT curve is constructed by plotting the statistic $G(r/n) = \left[\sum_{i=1}^{r} T_{i:n} + (n+r)T_{r:n}\right] / \left(\sum_{i=1}^{n} T_{i:n}\right)$ versus r/n ($r = 1, \ldots, n$), where the values $T_{i:n}$ are the order statistics of the sample, for $i = 1, \ldots, n$. The plots can be easily obtained using the `TTT` function of the `AdequacyModel` package from the `R` software. More details about this package are available from `help(TTT)`. The TTT plots for the dataset in this application are shown in Figure 3.8. For both plots, the TTT curve is concave, which, according to Aarset (1987), provides evidence that a monotonic increasing hrf is adequate.

The Figure 3.9 displays the fitted densities to the current data obtained in a nonparametric manner using the gaussian kernel density estimation, defined as follows. Let $X_1, \ldots, X_n$ be a random vector of random variables independent and identically distributed where each variable follows an unknown distribution, denoted by f. The kernel density estimator is given by the following expression

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - x_i) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right), \tag{3.34}$$

where $K(\cdot)$ is the symmetrical kernel function and $\int_{-\infty}^{\infty} K(x)dx = 1$. Furthermore, $h > 0$ is known in literature as bandwith, which is a smoothing parameter. It is possible to find in literature numerous kernel function, as the normal standard distribution, for example. Silverman (1986) demonstrated that for the K standard normal, a reasonable bandwith is given by $h = \sqrt[5]{(4\hat{\sigma}^5/3n)} \approx 1.06\hat{\sigma}/\sqrt[5]{n}$, where $\hat{\sigma}$ is defined by the standard deviation of the sample.

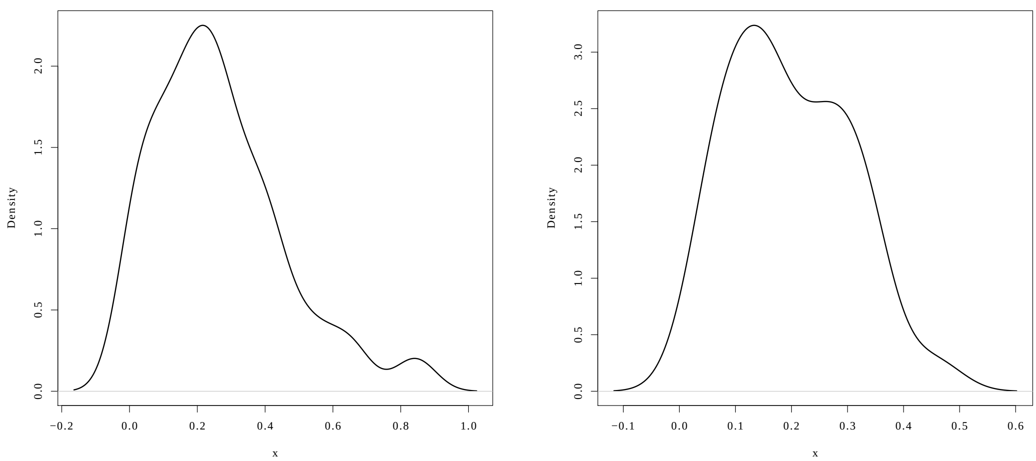
Figure 3.8: The TTT plot for: (a) Data Set 1 (b) Data Set 2.



(a) (b)

Source: Author's elaboration.

Figure 3.9: Gaussian kernel density estimation for: (a) Data Set 1 (b) Data Set 2.

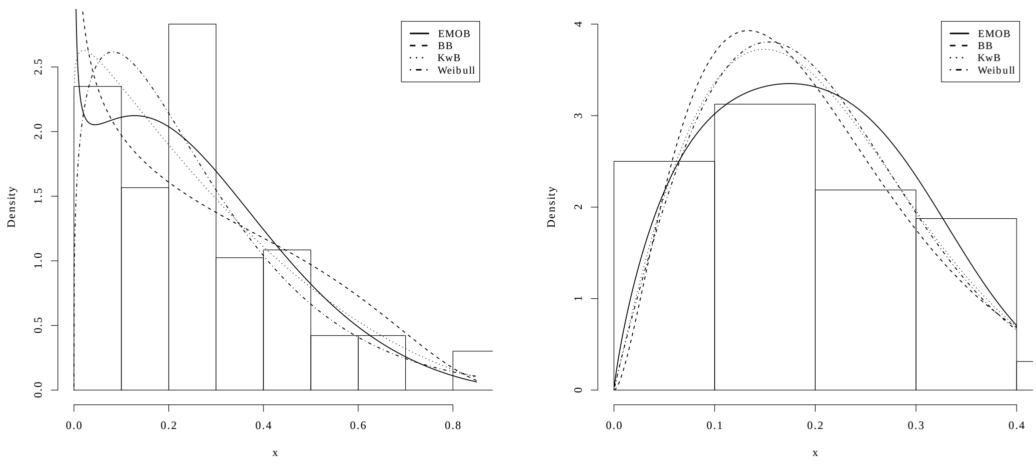


(a) (b)

Source: Author's elaboration.

The plots displayed in Figure 3.10 indicates that the EMOB distribution provides the best fit compared with the other fitted distributions. We note the good adequacy of the fitted EMOB distribution in Figure 3.11. In this application, we use the package `AdequacyModel`. This pack-

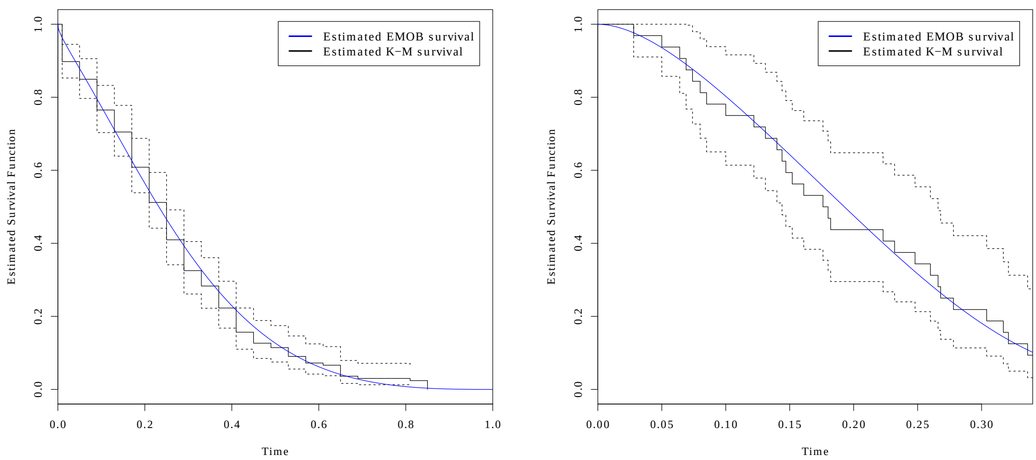
Figure 3.10: Estimates of the pdf for: (a) Data Set 1 (b) Data Set 2.



(a) (b)

Source: Author's elaboration.

Figure 3.11: Estimated K-M survival (K-M estimates) compared with the EMOB survival estimates for: (a) Data Set 1 (b) Data Set 2. The confidence intervals are 95%.



(a) (b)

Source: Author's elaboration.

age is intended to provide a computational support to work with probability distributions, mainly distributions aimed to survival analysis. This package was used to calculate some fitness statistics adjustment such as AIC (Akaike Information Criterion), CAIC (Consistent Akaikes Information Criterion), BIC (Bayesian Information Criterion), HQIC (Hannan-Quinn information criterion), KS (Test of Kolmogorov-Smirnov), A^* (statistic of Anderson-Darling) and W^* (statistic of Cramér-von Mises), which are described by Chen (1995), based on the results presented by Stephens (1986). When we want to test if one random sample, denoted by $x_1, x_2, \ldots, x_n$ with empirical distribution function $F_n(x)$, comes a especial distribution, we use these statistics. The Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics are given by the following expressions:

$$W^* = W^2\left(1 + \frac{0.5}{n}\right) \quad \text{and} \quad A^* = A^2\left(1 + \frac{0.75}{n} + \frac{2.25}{n}\right).$$

respectively. In these expressions, we have that $F_n(x)$ is the empirical distribution function, $F(x; \hat{\theta}_n)$ is the postulated distribution function evaluated at the MLE of θ, i.e. $\hat{\theta}_n$. Lower values of W^* and A^* provide evidence that $F(x; \hat{\theta}_n)$ generates the sample. More details about these statistics are given by Chen (1995). The `goodness.fit` function is used to calculate these statistics. More details can be obtained using the command `help(goodness.fit)`. The Table 3.4 shows the goodness-of-fit statistics (rouding to the fourth decimal place) for the datased used in this application. The results showed that the EMOB (a,b,p,λ,α) distribution presented better results for the KS, A^* and W^* statistics when compared with the other distributions used in this application.

In this study, the MLEs in Table 3.5 were obtained by global search heuristic method called Particle Swarm optmization - PSO proposed by Eberhart (1995). One of the advantages of using the PSO method in addition to being a robust optimization method is no need to provide initial guesses. However, this is a computationally intensive method. The Appendix 3.16 shows the function `pso` Implemented in language `R`. At the end of the code there is a small example of how to use the function for minimize an objective function. The obtaining of the errors of maximum likelihood estimates can be reached by the method bootstrap. By being computationally intensive, due to the use of the PSO method, the errors were not obtained.

3.15 Concluding remarks

We derive general mathematical properties of a new generator of continuous distributions with three extra shape parameters. We present some special models of the new EMO family of distributions. We investigate the shapes and derive explicit expressions for the ordinary and incomplete moments, quantile and generating functions, probability weighted moments, Bonferroni and Lorenz curves, Shannon and Rényi entropies and order statistics, which hold for any baseline model. The estimation of the model parameters is approached by the method of maximum likelihood and were obtained by global search heuristic method called Particle Swarm Optmization-PSO. Ultimately, we fit some EMO-G distributions to two real data sets to demonstrate the potentiality of this family. We hope this generalization may attract wider applications in statistics.

Table 3.4: Goodness-of-fit statistics for the data: (I) stress among women in Townsville, Queensland, Australia (II) proportion of crude oil converted to gasoline after distillation and fractionation.

Data Set	**Distribution**	**A***	**W***
	EMOB (a,b,p,λ,α)	**0.1554**	**0.0786**
	Kw-WP(a,b,c,λ,β)	0.4974	0.1305
	BB(α,β,a,b)	0.5676	9.7582
	Kw-B(α,β,a,b)	0.2750	0.1103
I	Exp-NH(α,λ,β)	0.2534	0.1153
	Exp-W(λ,k)	0.2206	0.1044
	Weibull(λ,k)	0.3296	0.1136
	Chen(λ,β)	0.2594	0.1073
	FW(α,β)	1.3931	0.3885
	Gamma(α,β)	0.4347	0.1986
	EMOB (a,b,p,λ,α)	**0.0348**	**0.0983**
	Kw-WP(a,b,c,λ,β)	0.0489	0.0993
	BB(α,β,a,b)	0.2473	0.9840
	Kw-B(α,β,a,b)	0.0381	0.0840
II	Exp-NH(α,λ,β)	0.0407	0.0888
	Exp-W(λ,k)	- 0.0416	0.0931
	Weibull(λ,k)	0.0413	0.0875
	Chen(λ,β)	0.0396	0.0836
	FW(α,β)	0.0430	0.1130
	Gamma(α,β)	0.0518	0.1143

Source: Author's elaboration.

Table 3.5: MLEs for: (I) Data Set 1 (II) Data Set 2.

Data Set	Distribution	Maximum Likelihood Estimates - MLE				
I	EMOB (a,b,p,λ,α)	0.2454	0.7793	-19.8703	3.4036	1.4276
	Kw-WP(a,b,c,λ,β)	12.3010	20.1431	0.1647	24.6569	2.3195
	BB(α,β,a,b)	0.0874	1.6288	9.0398	3.5456	
	Kw-B(α,β,a,b)	1.4193	0.5955	0.7365	4.4900	
	Exp-NH(α,λ,β)	2.2323	1.3448	1.0907		
	Exp-W(λ,k)	0.5651	2.5433	1.8076		
	Weibull(λ,k)	0.2826	1.2659			
	Chen(λ,β)	3.4689	1.1160			
	FW(α,β)	2.0968	0.0400			
	Gamma(α,β)	0.2897	0.9014			
II	EMOB (a,b,p,λ,α)	4.6421	10.7513	-1.2077	1.9915	0.3989
	Kw-WP(a,b,c,λ,β)	15.2365	8.3171	0.2446	24.9571	10.5076
	BB(α,β,a,b)	5.9557	2.2541	0.4675	3.296	
	Kw-B(α,β,a,b)	0.2179	2.3679	9.2614	10.127	
	Exp-NH(α,λ,β)	3.8235	1.2805	2.0247		
	Exp-W(λ,k)	1.0344	4.6087	1.9107		
	Weibull(λ,k)	0.222	1.9526			
	Chen(λ,β)	16.5625	1.8923			
	FW(α,β)	3.9236	0.1688			
	Gamma(α,β)	0.0997	2.0642			

Source: Author's elaboration.

3.16 Code in R language for PSO method

```
    $pso <- function(func,S=150,lim_inf,lim_sup,e=0.0001,data=NULL,N=100){
b_lo = min(lim_inf)
b_up = max(lim_sup)
integer_max = .Machine integer.max
if(length(lim_sup)!=length(lim_inf)){
   stop("The␣vectors␣lim_inf␣and␣lim_sup␣must␣have␣the␣same␣dimension.")
}
dimension = length(lim_sup)
swarm_xi = swarm_pi = swarm_vi = matrix(NA,nrow=S,ncol=dimension)

# The best position of the particles.
g = runif(n=dimension,min=lim_inf,max=lim_sup)

# Objective function calculated in g.
f_g = func(par=as.vector(g),x=as.vector(data))

if(NaN%in%f_g==TRUE || Inf%in%abs(f_g)==TRUE){
   while(NaN%in%f_g==TRUE || Inf%in%abs(f_g)==TRUE){
      g = runif(n=dimension,min=lim_inf,max=lim_sup)
      f_g = func(par=g,x=as.vector(data))
   }
}

# Here begins initialization of the algorithm.
x_i = mapply(runif,n=S,min=lim_inf,max=lim_sup)

# Initializing the best position of particularities i to initial position.

swarm_pi = swarm_xi = x_i
f_pi = apply(X=x_i,MARGIN=1,FUN=func,x=as.vector(data))

is.integer0 <- function(x){
   is.integer(x) && length(x)==0L
}

if(NaN%in%f_pi==TRUE || Inf%in%abs(f_pi)){
   while(NaN%in%f_pi==TRUE || Inf%in%abs(f_pi)){
      id_inf_fpi = which(abs(f_pi)==Inf)
      if(is.integer0(id_inf_fpi)!=TRUE){
         f_pi[id_inf_fpi] = integer_max
      }
      id_nan_fpi = which(f_pi==NaN)
      if(is.integer0(id_nan_fpi)!=TRUE){
         x_i[id_nan_fpi,] = mapply(runif,n=length(id_nan_fpi),min=lim_inf,
         max=lim_sup)
         swarm_pi = swarm_xi = x_i
      f_pi = apply(X=x_i,MARGIN=1,FUN=func,x=as.vector(data))
```

```
      }
   }
}

minimo_fpi = min(f_pi)
if(minimo_fpi < f_g) g = x_i[which.min(f_pi),]

# Initializing the speeds of the particles.

swarm_vi = mapply(runif,n=S,min=-abs(rep(abs(b_up-b_lo),dimension)),
max=abs(rep(abs(b_up-b_lo),dimension)))

# Here ends the initialization of the algorithm

omega = 0.5
phi_p = 0.5
phi_g = 0.5

m=1
vector_f_g <- vector()

while(is.na(var(vector_f_g)) || m<50 ||
   var(vector_f_g[length(vector_f_g):(length(vector_f_g)-10)])>e){
   # r_p and r_g are randomized numbers in (0.1).
   r_p = runif(n=dimension,min=0,max=1)
   r_g = runif(n=dimension,min=0,max=1)

   # Updating the vector speed.
   swarm_vi = omega*swarm_vi+phi_p*r_p*(swarm_pi-swarm_xi)+
   phi_g*r_g*(g-swarm_xi)

   # Updating the position of each particle.
   swarm_xi = swarm_xi+swarm_vi

   myoptim = function(...) tryCatch(optim(...), error = function(e) NA)

   f_xi = apply(X=swarm_xi,MARGIN=1,FUN=func,x=as.vector(data))
   f_pi = apply(X=swarm_pi,MARGIN=1,FUN=func,x=as.vector(data))
   f_g =  func(par=g,x=as.vector(data))

   if(NaN%in%f_xi==TRUE || NaN%in%f_pi==TRUE){
      while(NaN%in%f_xi==TRUE){
         id_comb = c(which(is.na(f_xi)==TRUE),which(is.na(f_pi)==TRUE))
         if(is.integer0(id_comb)!=TRUE){
            new_xi = mapply(runif,n=length(id_comb),min=lim_inf,
            max=lim_sup)
            swarm_pi[id_comb,]=swarm_xi[id_comb,] = new_xi
            if(length(id_comb)>1){
               if_xi[id_comb] = apply(X=swarm_xi[id_comb,],MARGIN=1,
               FUN=func,x=as.vector(data))
```

```
                f_pi[id_comb] = apply(X=swarm_pi[id_comb,],MARGIN=1,FUN=func,
                 x=as.vector(data))
             }else{
                f_xi[id_comb] = func(par=new_xi,x=as.vector(data))
             }
          }
       }
   }

   if(Inf%in%abs(f_xi)==TRUE){
      f_xi[which(is.infinite(f_xi))]=integer_max
   }
   if(Inf%in%abs(f_pi)==TRUE){
      f_pi[which(is.infinite(f_pi))]=integer_max
   }

   # There are values below the lower limit of restrictions?
   id_test_inf=
   which(apply(swarm_xi<t(matrix(rep(lim_inf,S),dimension,S)),1,sum)>=1)
   id_test_sup=
   which(apply(swarm_xi>t(matrix(rep(lim_sup,S),dimension,S)),1,sum)>=1)

   if(is.integer0(id_test_inf)!=TRUE){
      swarm_pi[id_test_inf,] = swarm_xi[id_test_inf,] =
      mapply(runif,n=length(id_test_inf),
      min=lim_inf,max=lim_sup)
   }

   if(is.integer0(id_test_sup)!=TRUE){
      swarm_pi[id_test_sup,] = swarm_xi[id_test_sup,] =
      mapply(runif,n=length(id_test_sup),
      min=lim_inf,max=lim_sup)
   }

   if(is.integer0(which((f_xi<=f_pi)==TRUE))){
      swarm_pi[which((f_xi<=f_pi)),] = swarm_pi[which((f_xi<=f_pi)),]
   }

   if(f_xi[which.min(f_xi)] <= f_pi[which.min(f_pi)]){
      swarm_pi[which.min(f_pi),] = swarm_xi[which.min(f_xi),]
      if(f_pi[which.min(f_pi)] < f_g) g = swarm_pi[which.min(f_pi),]
   } # Here ends the block if.

   vector_f_g[m] = f_g
   m = m+1
   if(m>N){
     break
  }

} # Here ends the block while.
```

```
    f_x = apply(X=swarm_xi,MARGIN=1,FUN=func,x=as.vector(data))
    list(par_pso=g,f_pso=vector_f_g)

} # Here ends the function.

# Example of using the PSO function. We are looking to minimize easom
# function in that -10<=x1<=10 and -10<=x2<=10.
easom <- function(par,x){
    x1 = par[1]
    x2 = par[2]
    -cos(x1)*cos(x2)*exp(-((x1-pi)^2 + (x2-pi)^2))
}
set.seed(0)
# Using the PSO function
# S refers to the number of particles considered.
pso(func=easom,S=350,lim_inf=c(-10,-10),lim_sup=c(10,10))
```

Chapter 4

A New Family of Distributions for Survival Analysis

Resumo

Análise estatística de dados de tempo de vida é um importante tópico em engenharia, biomedicina, ciências sociais e outros e há uma necessidade de estender as formas das clássicas distribuições. A distribuição Nadarajah-Haghighi é uma generalização da distribuição exponencial como alternativa às distribuições gama e Weibull. Baseada nessa distribuição, nós propomos uma nova classe estendida denominada NH-G família de distribuições e estudamos algumas propriedades matemáticas. Alguns modelos especiais da nova família são apresentados. As propriedades encontradas valem para qualquer distribuição nesta nova família. Adicionalmente, obtemos explícitas expressões para a função quantílica, momentos ordinários e incompletos, função geratriz de momentos e estatísticas de ordem. Nós exploramos a estimação dos parâmetros do modelo por máxima verossimilhança e ilustramos a potencialidade da nova família estendida com duas aplicações a dados reais uma simulação de Monte Carlo é apresentada para avaliar as estimativas.

Keywords: Distribuições Generalizadas, máxima verossimilhança, momentos, simulação de Monte Carlo, distribuição Nadarajah-Haghighi .

Abstract

Statistical analysis of lifetime data is an important topic in reliability engineering, biomedical, social sciences and others and there is a need for extended forms of the classical distributions. The Nadarajah-Haghighi distribution is a generalization of the exponential distribution as an alternative to the gamma and Weibull distributions. Based on this distribution, we propose a new wider Nadarajah-Haghighi-G family of distributions and we study some mathematical properties. Some special models in the new family are discussed. The properties derived hold to any distribution in this family. Further, We obtain general explicit expressions for the quantile function, ordinary and incomplete moments, generating function and order statistics. We discuss the estimation of the model parameters by maximum likelihood and illustrate the potentiality of the extended family with two applications to real data and a small Monte Carlo simulation is conducted to evaluate these estimates.

Keywords: Generalized distribution, lifetime, generating function, hazard function, moment, maximum likelihood estimation, Monte Carlo simulation, Nadarajah-Haghighi distribution.

4.1 Introduction

In many applied areas like lifetime analysis, finance, actuary and biology there is a need for extended forms of the classical distributions. A new generalization of the exponential distribution as an alternative to the gamma and Weibull distributions was recently proposed by Nadarajah and Haghighi (2011). The cumulative function is given by

$$G(x) = 1 - \exp[1 - (1 + \lambda x)^{\alpha}], \quad x > 0, \tag{4.1}$$

where λ and $\alpha > 0$. The probability density function (pdf) and the hazard rate function (hrf) corresponding to (4.1) are given by

$$g(x) = \alpha\,\lambda(1 + \lambda x)^{\alpha-1}\exp[1 - (1 + \lambda x)^{\alpha}] \tag{4.2}$$

and

$$h(x) = \alpha\,\lambda(1 + \lambda x)^{\alpha-1},$$

respectively. New distributions were developed based on new techniques by adding parameters to existing distributions for building classes of more flexible distributions to model real data. For example, a new family of distributions, denoted by exponentiated family, is rather simple and is constructed by raising the cumulative function to an arbitrary power positive. The properties of exponentiated distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1995) for exponentiated Weibull, Gupta; Gupta; Gupta (1998) for exponentiated Pareto, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah (2005) for exponentiated Gumbel, Lemonte (2013) for exponentiated Nadarajah-Haghighi, among several others. Furthermore, several ways of generating new distributions from classic ones were developed and discussed in the past few years. For example, Jones (2004) studied a family of distributions that arises naturally from the distribution of the order statistics, the beta-generated family was proposed by Eugene; Lee; Famoye (2002), Zografos and Balakrishnan (2009) introduced the gamma-generated family of distributions, Cordeiro and de Castro (2011) introduced the so-called family of Kumaraswamy generalized distributions and Bourguignon; Silva; Cordeiro (2014) developed the Weibull-G family of distributions.

Recently, Alzaatreh; Lee; Famoye (2013) proposed a new technique to derive families of distributions by using any pdf as a generator. This new generator was named as T-X family of distributions and the its cdf is defined as

$$F(x) = \int_{a}^{W[G(x)]} r(t)dt \tag{4.3}$$

where $G(x)$ is the cdf of a random variable X, $r(t)$ is a pdf of the random variable T defined on $[a, b]$ and $W[G(x)]$ is a function of $G(x)$ so that satisfies the following conditions:

- $W[G(x)] \in [a, b]$;
- $W[G(x)]$ is differentiable and monotonically non-decreasing;
- $W[G(x)] \to a$ as $x \to -\infty$ and $W[G(x)] \to b$ as $x \to \infty$.

Different $W[G(x)]$ will give a new family of distributions. For example, we can define $W[G(x)]$ as $-\log[1 - G(x)]$, $G(x)/[1 - G(x)]$ or $-\log[G(x)]$, when the support of T is $[0, \infty]$. Based on this paper, by using $W[G(x)] = -\log[1 - G(x)]$ we propose a class of distributions called the new Nadarajah-Haghighi-G ("NH-G" for short) family, where for each baseline distribution G we have a different distribution F. The main aim of this paper is to study a new family of distributions, with the hope it yields a better fit in certain practical situations. Additionally, we provide a comprehensive account of the mathematical properties of the proposed family of distributions.

This paper is organized as follows. In Section 4.2, we define the NH-G family of distributions. Section 4.3 provides some special distributions obtained by the NH generator. In Section 4.4, some general mathematical properties of the family are discussed. The formulas derived are manageable by using modern computer resources with analytic and numerical capabilities. In Section 4.5, the estimation of the model parameters is performed by the method of maximum likelihood. In Section 4.6, two illustrative applications based on real data are investigated and and a small simulation study are addressed in Sections 4.7. Finally, concluding remarks are presented in Section 4.8.

4.2 The new family

Consider a continuous distribution $G(x)$ with density $g(x)$. Based in equation (4.3), let $a = 0$ and $W[G(x)] = -\log[1 - G(x)]$. By using $r(t) = \alpha\,\lambda(1 + \lambda t)^{\alpha-1} \exp[1 - (1 + \lambda t)^{\alpha}]$ (for $t > 0$), given by equation 4.2, we define the cdf of the NH-G family as

$$\begin{aligned} F(x) &= \int_{0}^{-\log[1-(G(x)]} \alpha\,\lambda(1 + \lambda\, t)^{\alpha-1} \exp[1 - (1 + \lambda\, t)^{\alpha}]dt \\ &= 1 - \exp\{1 - [1 - \lambda \log[1 - G(x)]]^{\alpha}\}, \qquad x > 0; \alpha, \lambda > 0. \end{aligned} \tag{4.4}$$

The family pdf reduces to

$$f(x)=\frac{\alpha\lambda\, g(x)\{1-\lambda\log[1-G(x)]\}^{\alpha-1}\exp\{1-[1-\lambda\log[1-G(x)]]^{\alpha}\}}{1-G(x)}. \tag{4.5}$$

Hereafter, a random variable X with pdf (4.5) is denoted by $X\sim$ NH-G($\alpha,\lambda,\boldsymbol{\xi}$), where $\boldsymbol{\xi}$ is the parameter vector of $G(x)$.

The NH-G family of distribution can be simulated from (4.4). Then, if U has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation

$$X=G^{-1}\left\{1-\exp\left[\frac{1-[1-\log(1-u)]^{1/\alpha}}{\lambda}\right]\right\} \tag{4.6}$$

has the NH-G($\alpha,\lambda,\boldsymbol{\xi}$) distribution.

4.3 Examples of the NH-G family

In this section, we provide some examples of the NH-G distributions. The special cases are defined by taking $G(x)$ and $g(x)$ to be the cdf and pdf of the known distributions. These sub-models generalize several important existing distributions in the literature.

4.3.1 NH-Fréchet Distribution

As the first example, take the the parent distribution as standard Fréchet with pdf and cdf given by $g(x)=\delta\sigma^{\delta}\exp[-(\frac{\sigma}{x})^{\delta}]/x^{\delta+1}$ and $G(x)=\exp[-(\frac{\sigma}{x})^{\delta}]$, respectively, for $x,\sigma,\delta>0$. Then, the NH-Fréchet (NH-F) cdf becomes

$$F(x)=1-\exp\left\{1-\left[1-\lambda\log\left[1-\exp\left[-\left(\frac{\sigma}{x}\right)^{\delta}\right]\right]\right]^{\alpha}\right\}.$$

The corresponding pdf is

$$f(x)=\frac{\alpha\lambda\delta\sigma^{\delta}\mathrm{e}^{-(\frac{\sigma}{x})^{\delta}}x^{-(\delta+1)}\{1-\lambda\log[1-\mathrm{e}^{-(\frac{\sigma}{x})^{\delta}}]\}^{\alpha-1}\exp\{1-[1-\lambda\log[1-\mathrm{e}^{-(\frac{\sigma}{x})^{\delta}}]]^{\alpha}\}}{1-\mathrm{e}^{-(\frac{\sigma}{x})^{\delta}}}.$$

4.3.2 NH-Burr XII Distribution

As second example, let us consider the parent Burr XII distribution with pdf and cdf given by $g(x)=cks^{-c}x^{c-1}[1+(x/s)^{c}]^{-k-1}, s,k,c>0$ and $G(x)=1-[1+(x/s)^{c}]^{-k}$, respectively. Then, for $x>0$, the NH-Burr XII (NH-BXII) distribution has cdf and pdf given by

$$F(x)=1-\exp\{1-[1-\lambda\log[(1+(x/s)^{c})^{-k}]]^{\alpha}\}.$$

and

$$f(x)=\alpha\lambda cks^{-c}x^{c-1}[1+(x/s)^{c}]^{-1}\{1-\lambda\log[(1+(x/s)^{c})^{-k}]\}^{\alpha-1}\exp\{1-[1-\lambda\log[(1+(x/s)^{c})^{-k}]]^{\alpha}\},$$

respectively. Note that when $c=1$ and $k=1$, we obtain as a special case the NH-Lomax (NH-L) distribution.

4.3.3 NH-Gompertz Distribution

The third example considers the Gompertz distribution. This distribution is a generalization of the exponential model and its distribution, with parameters $\theta > 0$ and $\gamma > 0$, has the cdf and pdf given by $g(x) = \theta e^{\gamma x} e^{-\frac{\theta}{\gamma}(e^{\gamma x}-1)}$ and $G(x) = 1 - e^{-\frac{\theta}{\gamma}(e^{\gamma x}-1)}$, respectively.

For $x > 0$, the NH-Gompertz (NH-Go) distribution has cdf expressed by

$$F(x) = 1 - \exp\left\{1 - \left[1 + \frac{\lambda\theta\left(e^{\gamma x}-1\right)}{\gamma}\right]^{\alpha}\right\},$$

and pdf determined by

$$f(x) = \alpha\lambda\theta e^{\gamma x}\left[1 + \frac{\lambda\theta\left(e^{\gamma x}-1\right)}{\gamma}\right]^{\alpha-1} \exp\left\{1 - \left[1 + \frac{\lambda\theta\left(e^{\gamma x}-1\right)}{\gamma}\right]^{\alpha}\right\}.$$

4.3.4 NH-Normal Distribution

The last example refers to the normal distribution. The NH-Normal (NH-N) distribution is obtained by taking $G(\cdot)$ and $g(\cdot)$ to be the cdf and pdf of the normal $N(\mu;\sigma^2)$ distribution. Then, the NH-N distribution has cdf given by

$$F(x) = 1 - \exp\left\{1 - \left[1 - \lambda\log[(1 - \Phi[(x-\mu)/\sigma]]\right]^{\alpha}\right\}.$$

The corresponding pdf is

$$f(x) = \frac{\alpha\lambda\phi\left(\frac{x-\mu}{\sigma}\right)\left\{1 - \lambda\log\left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha-1} \exp\left\{1 - \left[1 - \lambda\log\left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]\right]^{\alpha}\right\}}{1 - \Phi\left(\frac{x-\mu}{\sigma}\right)}.$$

For $\mu = 0$ and $\sigma = 1$, we obtain the standard NH-N distribution. Figure 4.1 illustrates possible shapes of the density functions for some NH-G distributions.

4.4 Mathematical Properties

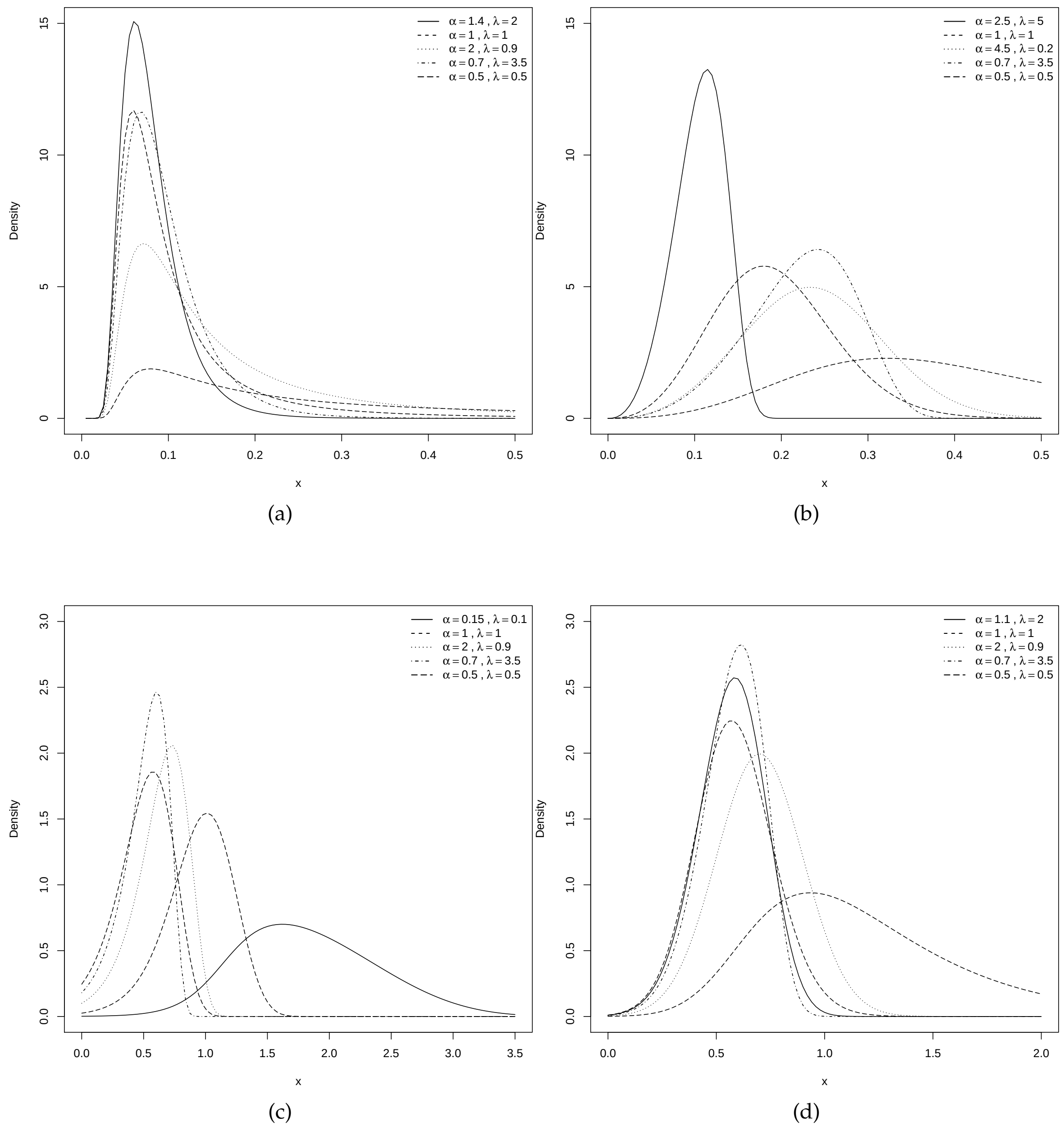
Although of the fact that the NH-G cdf and pdf have mathematical functions that are widely available in modern statistical packages such as Maple and Mathematica, here we provide some mathematical properties of X.

4.4.1 A mixture representation

Frequently analytical and numerical derivations take advantage of power series for the pdf. By using the power series for the exponential function and the generalized binomial expansion we obtain the NH-G density function can be expressed as an infinite linear combination of exponentiated-G (exp-G for short) density functions. Then, the pdf of X can be expressed as

$$f(x) = \sum_{m=0}^{\infty} w_m\, h_{m+1}(x;\boldsymbol{\xi}), \tag{4.7}$$

Figure 4.1: Plots of the density function for some parameter values. (a) NH-F($\alpha; \lambda; 1.5; 0.1$), (b) NH-BXII($\alpha; \lambda; 3.5; 10; 0.5$), (c) NH-Go($\alpha; \lambda; 0.1; 5.5$) and (d) NH-N($\alpha; \lambda; 0.7; 0.2$) density functions.



Source: Author's elaboration.

where

$$w_m = \sum_{i,j=0}^{\infty} \sum_{l=0}^{\infty} \frac{e\alpha\lambda^{j+1}(-1)^{i+l+m}}{(m+1)i!} \binom{l-1}{m} \binom{\alpha(i+1)-1}{j} \left[\binom{j}{l} + \sum_{k=0}^{\infty} p_k(j) \binom{k+j+1}{l} \right], \quad (4.8)$$

$p_0(c) = c/2$, $p_1(c) = c\,(3c+5)/24$, $p_2(c) = c\,(c^2+5c+6)/48$,$p_3(c) = c\,(15c^3+150c^2+485c+502)/5760$, etc, and $h_m = m\,g(x)G(x)^{m-1}$ is the exp-G density function with power parameter m.

Note that if we use $W[G(x)] = -\log[G(x)]$ or $W[G(x)] = G(x)/[1-G(x)]$ in Equation (4.4), we obtain the same mixture, but with distincts coefficients w_m given by

$$w_m = \sum_{i,j=0}^{\infty} \frac{e\alpha\lambda^{j+1}(-1)^{i+m}}{i!m} \binom{\alpha(i+1)-1}{j} \left[\binom{j}{m} + \sum_{k=0}^{\infty} p_k(j) \binom{j+k+1}{m} \right]$$

and

$$w_m = \sum_{i,j=0}^{\infty} \binom{\alpha(i+1)-1}{j} \frac{e\alpha\lambda^{j+1}(-1)^{i}\Gamma(j+m+2)}{m!i!(j+m+1)\Gamma(j+2)},$$

respectively.

4.4.2 Moments

Let Y_m be a random variable having the exp-G density function $h_m(x)$. The formula for the nth moment of X can be obtained from (4.7) and the monotone convergence theorem as

$$E(X^n) = \sum_{m=0}^{\infty} w_m\, E(Y_{m+1}^n). \quad (4.9)$$

Nadarajah and Kotz (2006) determined closed-form expressions for moments of several exponentiated distributions that can be used to produce the NH-G moments.

An alternative formula for $E(X^n)$ follows from (4.9) in terms of $Q_G(u) = G^{-1}(u)$ as

$$E(X^n) = \sum_{m=0}^{\infty} (m+1) w_m\, \tau_{n,m},$$

where $\tau_{n,m} = \int_0^{\infty} x^n G(x)^m g(x) dx = \int_0^1 Q_G(u)^n u^m du$ is the $(n;m)$th Probability Weighted Moment (PWM) of Y. The PWM for some distributions such as normal, beta, gamma and others were obtained by Cordeiro and Nadarajah (2011b), which can be used to determine the NH-G moments.

4.4.3 Incomplete Moments

For empirical purposes, the shapes of many distributions can be usefully described by what we call the incomplete moment, namely $m_n(y) = \int_{-\infty}^{y} x^n f(x) dx$, which plays an important role for measuring inequality, for example, mean deviations and income quantiles and Lorenz and

Bonferroni curves. Here, we provide two alternative ways to compute the nth incomplete moment. First, it follows from Equation (4.7) and the monotone convergence theorem by

$$m_s(y) = \sum_{m=0}^{\infty} w_m \, J_{m+1}(y), \tag{4.10}$$

where $J_{m+1}(y) = \int_{-\infty}^{y} x^s h_{m+1}(x)dx$. Then, the sth incomplete moment of X depend only on the sth incomplete moment of the exp-G distribution.

A second general formula for $m_s(y)$ can be derived by setting $u = G(x)$ in Equation (4.7)

$$m_s(y) = \sum_{m=0}^{\infty} (m+1) w_m \, T_m(y), \tag{4.11}$$

where $T_m(y) = \int_0^{G(y)} Q_G(u)^s u^m du$. In a similar way, the incomplete moments of the NH-G distribution can be computed from Equation (4.11). Equations (4.10) and (4.11) are the main results of this section.

4.4.4 Generating Function

Let $M_X(t) = E(e^{tX})$ be the mgf of X. Then, using (4.7) and the monotone convergence theorem we can write

$$M_X(t) = \sum_{m=0}^{\infty} w_m \, M_{m+1}(t), \tag{4.12}$$

where $M_{m+1}(t)$ is the mgf of Y_m. Hence, $M_X(t)$ can be determined from the generating function of the exp-G distribution.

Alternatively, we can compute $M_X(t)$ from (4.12) as

$$M_X(t) = \sum_{m=0}^{\infty} (m+1) w_m \, \rho(t, m), \tag{4.13}$$

where

$$\rho(t, m) = \int_{-\infty}^{\infty} e^{tx} \, G(x)^m \, g(x) dx = \int_0^1 \exp\left[t Q_{G(u)}\right] u^m du.$$

Equations 4.12 and 4.13 are the main results of this section.

4.4.5 Order Statistics

Order statistics are among the most fundamental tools in non-parametric statistics and inference. They enter in the problems of estimation and hypothesis tests in a variety of ways. Therefore, we now discuss some properties of the order statistics for the proposed class of distributions. The pdf $f_{i:n}(x)$ of the ith order statistic for a random sample $X_1, X_2, \cdots, Xn$ from the NH-G distribution is given by

$$f_{i:n}(x) = K f(x) F^{i-1}(x) [1 - F(x)]^{n-i} \tag{4.14}$$

where $K = n!/[(i-1)!\,(n-i)!]$. Then,

$$f_{i:n}(x) = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x) F^{i+j-1}(x). \tag{4.15}$$

Integrating the mixture (4.7) and using the monotone convergence theorem, the cdf of X can be expressed as

$$F(x) = \sum_{m=0}^{\infty} w_m H_{m+1}(x), \tag{4.16}$$

where $H_{m+1}(x) = G(x)^{m+1}$ is the exp-G cdf with power parameter $m+1$. The pdf of $X_{i:n}$ can be expressed from (4.7) and (4.16) as

$$f_{i:n}(x) = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[\sum_{r=0}^{\infty} w_r\,(r+1)\,G(x)^r\,g(x)\right] \left[\sum_{m=0}^{\infty} w_m\,G(x)^{m+1}\right]^{i+j-1}. \tag{4.17}$$

Using the power series raised to a positive integer, we can write

$$\left[\sum_{m=0}^{\infty} w_m\,G(x)^{m+1}\right]^{i+j-1} = \sum_{m=0}^{\infty} f_{i+j-1,m}\,G(x)^{i+j+m-1},$$

where $f_{i+j-1,0} = w_0^{i+j-1}$ and (for $m \geq 1$)

$$f_{i+j-1,m} = (m\,w_0)^{-1} \sum_{p=1}^{m} [p(i+j)-m]\,w_p\,f_{i+j-1,m-p}.$$

Hence,

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{r,m=0}^{\infty} c_{j,r,m}\,h_{i+j+r+m}(x), \tag{4.18}$$

where

$$c_{j,r,m} = \frac{(-1)^j\,n!}{(i-1)!\,(n-i-j)!\,j!}\;\frac{(r+1)\,w_r\,f_{i+j-1,m}}{(i+j+r+m)}.$$

Equation (4.18) is the main result of this section. It reveals that the pdf of the NH-G order statistics can be expressed as a triple linear combination of exp-G density functions. So, several mathematical quantities of $X_{i:n}$ (order statistics like ordinary, incomplete and factorial moments, mgf, mean deviations, among others) can be obtained from those quantities of exp-G distributions.

4.5 Maximum Likelihood Estimation

Here, we determine the maximum likelihood estimates (MLEs) of the parameters of the new family of distributions from complete samples only. Let $x_1, \cdots, x_n$ be observed values from the NH-G distribution with parameters α, λ and $\boldsymbol{\xi}$. Let $\Theta = (\alpha; \lambda; \boldsymbol{\xi})^\top$ be the $p \times 1$ parameter vector. The total log-likelihood function for Θ is given by

$$\begin{aligned} l(\Theta) &= n + n\log(\alpha) + n\log(\lambda) + \sum_{i=1}^{n} \log[g(x_i;\boldsymbol{\xi})] - \sum_{i=1}^{n} \log[1 - G(x_i;\boldsymbol{\xi})] \\ &+ (\alpha - 1)\sum_{i=1}^{n} \log\{1 - \lambda \log[1 - G(x_i;\boldsymbol{\xi})]\} - \sum_{i=1}^{n} \{1 - \lambda \log[1 - G(x_i;\boldsymbol{\xi})]\}^{\alpha}. \quad (4.19) \end{aligned}$$

The log-likelihood function can be maximized either directly by using the `Ox` program (subroutine `MaxBFGS`) (DOORNIK, 2007) or the `SAS (PROC NLMIXED)` or by solving the nonlinear likelihood equations obtained by differentiating (4.19). The components of the score function $U_n(\theta) = (U_\alpha = \partial \ell_n/\partial\alpha, U_\lambda = \partial \ell_n/\partial\lambda, U_{\boldsymbol{\xi}} = \partial \ell_n/\partial\boldsymbol{\xi})^\top$ are

$$\begin{aligned} U_\alpha &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log\{1 - \lambda\log[1 - G(x_i;\boldsymbol{\xi})]\} \\ &- \sum_{i=1}^{n} \{1 - \lambda\log[1 - G(x_i;\boldsymbol{\xi})]\}^{\alpha} \log\{1 - \lambda\log[1 - G(x_i;\boldsymbol{\xi})]\}, \\ U_\lambda &= \frac{n}{\lambda} - (\alpha - 1)\sum_{i=1}^{n} \frac{\log[1 - G(x_i;\boldsymbol{\xi})]}{1 - \lambda\log[1 - G(x_i;\boldsymbol{\xi})]} \\ &+ \alpha\sum_{i=1}^{n} \{1 - \lambda\log[1 - G(x_i;\boldsymbol{\xi})]\}^{\alpha-1} \log[1 - G(x_i;\boldsymbol{\xi})], \\ U_{\boldsymbol{\xi}} &= \sum_{i=1}^{n} \frac{\dot{g}(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})} + \lambda(\alpha - 1)\sum_{i=1}^{n} \frac{\dot{G}(x_i;\boldsymbol{\xi})[1 - G(x_i;\boldsymbol{\xi})]^{-1}}{\{1 - \lambda\log[1 - G(x_i;\boldsymbol{\xi})]\}} \\ &- \alpha\lambda\sum_{i=1}^{n} \frac{\{1 - \lambda\log[1 - G(x_i;\boldsymbol{\xi})]\}^{\alpha-1}\dot{G}(x_i;\boldsymbol{\xi})}{1 - G(x_i;\boldsymbol{\xi})} + \sum_{i=1}^{n} \frac{\dot{G}(x_i;\boldsymbol{\xi})}{1 - G(x_i;\boldsymbol{\xi})}, \end{aligned}$$

where $\dot{g}(x_i;\boldsymbol{\xi}) = \partial g(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k$ and $\dot{G}(x_i;\boldsymbol{\xi}) = \partial G(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k$, for $k = 1, \cdots, p$.

Note that for interval estimation on the model parameters, we require the observed information matrix

$$J_n(\Theta) = -\begin{pmatrix} U_{\alpha\alpha} & U_{\alpha\lambda} & | & U_{\alpha\boldsymbol{\xi}}^\top \\ U_{\lambda\alpha} & U_{\lambda\lambda} & | & U_{\lambda\boldsymbol{\xi}}^\top \\ -- & -- & -- & -- \\ U_{\boldsymbol{\xi}\alpha} & U_{\boldsymbol{\xi}\lambda} & | & U_{\boldsymbol{\xi}\boldsymbol{\xi}} \end{pmatrix},$$

whose elements can be computed numerically in softwares such as `MAPLE` (GARVAN, 2002), `MATLAB` (SIGMON; DAVIS, 2002) and `MATHEMATICA` (WOLFRAM, 2003). These symbolic software have currently the ability to deal with analytic expressions of formidable size and complexity. Let $\hat{\theta}$ be the MLE of θ. Under standard conditions (COX; HINKLEY, 1974) that are fulfilled for the proposed model whenever the parameters are in the interior of the parameter space, we can approximate the distribution of $\sqrt{n}(\hat{\theta} - \theta)$ by the multivariate normal $N_p(0, K(\theta)^{-1})$, where $K(\theta) = \lim_{n\to\infty} J_n(\theta)$ is the unit information matrix and p is the number of parameters of the new distribution.

4.6 Application

In this section, we illustrate the applicability of the NH-G distributions by considering two different datasets used by different researchers. The first set consists of 63 observations of the strengths of 1.5cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55,1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67,1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89,2.00, 2.01, 2.24. These data have also been analyzed by Smith and Naylor (1987). For these data, we fit the NH-Gompertz distribution. Its fit is also compared with the widely known beta Weibull (BW) (FAMOYE; LEE; OLUMOLADE, 2005), Kumaraswamy Weibull (KW) (CORDEIRO; EDWIN; NADARAJAH, 2010) and beta Gompertz (BGo) (JAFARI; TAHMASEBI; ALIZADEH, 2014) models with corresponding densities:

- BW: $f(x) = \frac{c\lambda^c}{B(a,b)} x^{c-1} \mathrm{e}^{-b(\lambda x)^c} [1 - \mathrm{e}^{-(\lambda x)^c}]^{a-1}$;
- KW: $f(x) = a\, b\, c\, \lambda x^{c-1} \mathrm{e}^{-(\lambda x)^c} \{1 - [1 - \mathrm{e}^{-(\lambda x)^c}]^a\}^{b-1}$;
- BGo: $f(x) = \frac{\theta \mathrm{e}^{\gamma x} \mathrm{e}^{\frac{-b\theta}{\gamma}(\mathrm{e}^{\gamma x} - 1)}}{B(a,b)} [(1 - \mathrm{e}^{\frac{-\theta}{\gamma}(\mathrm{e}^{\gamma x} - 1)}]^{a-1}$.

The second data set consists of 100 observations of breaking stress of carbon fibres (in Gba) given by Nichols and Padgett (2006). The data are: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.40, 3.15, 2.67,3.31, 2.81, 0.98, 5.56, 5.08, 0.39, 1.57, 3.19, 4.90, 2.93, 2.85, 2.77, 2.76, 1.73, 2.48, 3.68, 1.08, 3.22, 3.75, 3.22, 2.56, 2.17, 4.91, 1.59, 1.18, 2.48, 2.03, 1.69, 2.43, 3.39, 3.56, 2.83, 3.68, 2.00, 3.51, 0.85, 1.61, 3.28, 2.95, 2.81, 3.15, 1.92, 1.84, 1.22, 2.17, 1.61, 2.12, 3.09, 2.97, 4.20, 2.35, 1.41, 1.59, 1.12, 1.69, 2.79, 1.89, 1.87, 3.39, 3.33, 2.55, 3.68, 3.19, 1.71, 1.25, 4.70, 2.88, 2.96, 2.55, 2.59, 2.97, 1.57, 2.17, 4.38, 2.03, 2.82, 2.53, 3.31, 2.38, 1.36, 0.81, 1.17, 1.84, 1.80, 2.05, 3.65. For these data, we fit the NH-Fr distribution defined in the section 4.3 and compare it with the beta Fréchet (BFr) (NADARAJAH; GUPTA, 2004), Kumaraswamy Fréchet (KFr) (MEAD; ABD-ELTAWAB, 2014) and Fréchet models with corresponding densities:

- BFr: $f(x) = \frac{\beta\alpha^\beta}{B(a,b)} x^{-\beta-1} \mathrm{e}^{-a\left(\frac{\alpha}{x}\right)^\beta} [1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^\beta}]^{b-1}$;
- KFr: $f(x) = ab\beta\alpha^\beta x^{-\beta-1} \mathrm{e}^{-a\left(\frac{\alpha}{x}\right)^\beta} [1 - \mathrm{e}^{-a\left(\frac{\alpha}{x}\right)^\beta}]^{b-1}$;
- Fr: $f(x) = \beta\alpha^\beta x^{-\beta-1} \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^\beta}$.

The required numerical evaluations are carried out using the AdequacyModel package of the R software. The MLEs of the model parameters for the fitted distributions and the Cramér-von Mises (W^*) and Arderson-Darling (A^*) statistics are given in Tables 4.1. These test statistics are described by Chen and Balakrishnan (1995). They are used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit. These statistics indicate that the NH-Go and NH-Fr distributions are the best models to the first and second data sets, respectively. More information is provided by a visual comparison of the histogram of the data and the fitted densities. The plots of the fitted density functions are displayed in Figures 4.2 and 4.3. These plots indicate that the new distribution provides a good fit to these data and is a very competitive model.

Table 4.1: MLEs (standard errors in parentheses) and A* and W* statistics.

Data Set	Distribution	Estimates				A*	W*
1(n=63)	NH-Go($\alpha, \lambda, \theta, \gamma$)	0.6553	0.0028	3.0742	4.1341	**0.8811**	**0.1561**
		(0.2042	0.0006	2.1757	0.6901)		
	BW(a, b, c, λ)	0.7711	1.9773	6.5542	0.5304	1.1975	0.2170
		(0.2915	2.2727	1.5996	0.0974)		
	KW(a, b, c, λ)	0.9983	1.8454	5.5443	0.5487	1.3570	0.2472
		(0.4651	1.6403	1.8130	0.0754)		
	BGo(a, b, θ, γ)	4.4042	6.8651	0.0860	1.4513	1.4565	0.2655
		(2.7398	6.1621	0.0734	0.6700)		
1(n=100)	NH-Fr(α, λ, a, b)	8.6952	0.7845	0.7876	8.2523	**0.6393**	**0.1173**
		(4.5775	0.5761	0.1965	5.4522)		
	BFr(α, β, a, b)	9.9856	0.9804	0.5146	13.6431	1.2690	0.2364
		(2.3854	0.1827	0.2893	5.2818)		
	KFr(α, β, a, b)	0.5395	0.8604	10.3147	11.9438	1.5448	0.2867
		(0.2177	0.1176	3.5310	5.5298)		
	Fr(α, β)	1.8941	1.7660			4.3032	0.7548
		(0.1141	0.1118)				

Source: Author's elaboration.

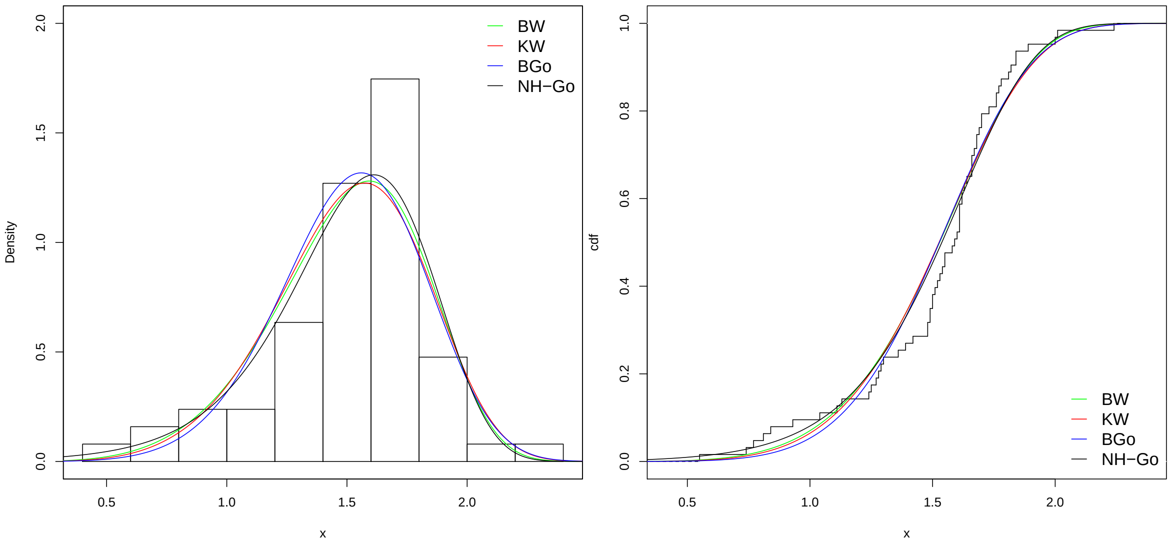
4.7 Simulation

In this section, we conduct Monte Carlo simulation studies to assess on the finite sample behavior of the MLEs of the NH-exponentiated exponential (NH-EE) distribution with parameters, α, λ, β and δ. Its pdf is given by

$$f(x) = \frac{\alpha\lambda\beta\delta e^{-\delta x}(1-e^{-\delta x})^{\beta-1}\exp\{1-[1-\lambda\log[1-(1-e^{-\delta x})^{\beta}]]^{\alpha}\}}{[1-(1-e^{-\delta x})^{\beta}]\{1-\lambda\log[1-(1-e^{-\delta x})^{\beta}]\}^{1-\alpha}}.$$

All results were obtained from 5000 Monte Carlo replications and the simulations were carried out the `R` programming language. In each replication, a random sample of size n is drawn from the NH-EE(α, λ, a, b) distribution and the BFGS method has been used by the authors for maximizing the total log-likelihood function $l(\theta)$. The NH-EE random number generation was performed using the inversion method. The true parameter values used in the data generating processes are $\alpha = 1.5, \lambda = 2, a = 0.5$ and $b = 2.5$. The Table 4.2 reports the empirical means, bias the mean squared errors (MSE) of the corresponding estimators for sample sizes $n = 25, 50, 100, 200$ and 400. From these figures in this table, we note that, as the sample size increases, the empirical biases and mean squared errors decrease in all the cases analyzed, as expected.

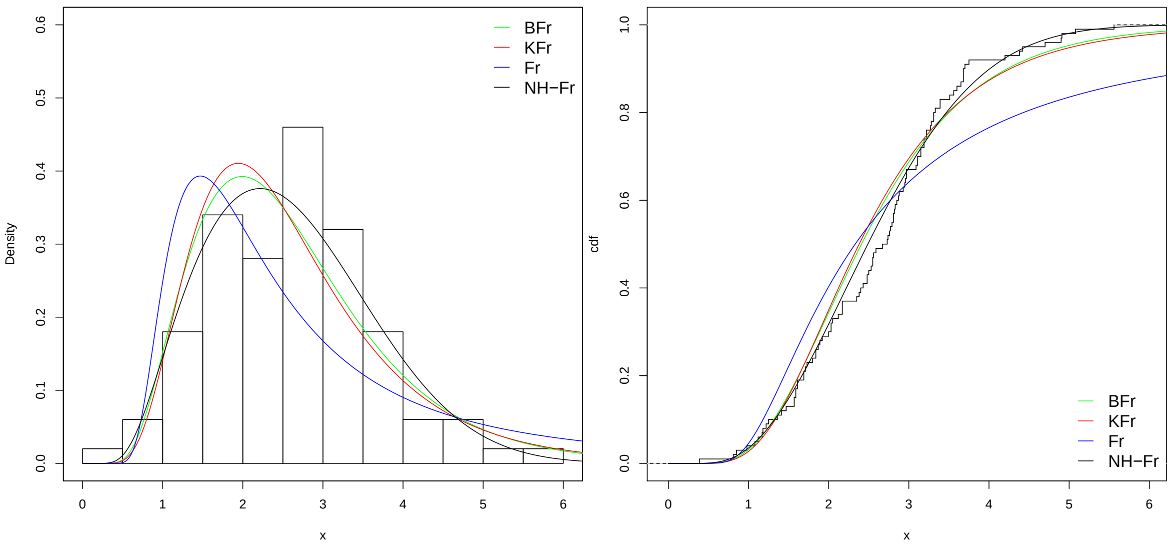
Figure 4.2: Plots of the estimated pdf and cdf of the NH-Go, BW, KW and BGo models.



(a) (b)

Source: Author's elaboration.

Figure 4.3: Plots of the estimated pdf and cdf of the NH-Fr, BFr, KFr and Fr models.



(a) (b)

Source: Author's elaboration.

Table 4.2: Empirical means, bias and mean squared errors

n	Parameter	Mean	Bias	MSE
25	α	1.7336	0.2336	0.4686
	λ	2.0916	0.0916	0.0452
	a	0.5719	0.0719	0.0953
	b	2.9086	0.4086	0.9877
50	α	1.6311	0.1311	0.2613
	λ	2.0475	0.0475	0.0217
	a	0.5439	0.0439	0.0530
	b	2.7287	0.2287	0.5294
100	α	1.5545	0.0545	0.1380
	λ	2.0261	0.0261	0.0084
	a	0.5332	0.0332	0.0295
	b	2.6437	0.1437	0.2532
200	α	1.5445	0.0445	0.0837
	λ	2.0134	0.0134	0.0048
	a	0.5137	0.0137	0.0146
	b	2.5678	0.0678	0.1168
400	α	1.5106	0.0106	0.0385
	λ	2.0070	0.0070	0.0015
	a	0.5113	0.0113	0.0073
	b	2.5420	0.0420	0.0523

Source: Author's elaboration.

4.8 Concluding Remarks

We propose another new class of distributions based on the distribution introduced by Nadarajah and Haghighi (2011). We derive general mathematical properties of a new wider NH-G family of distributions.This generator can extend several widely known distributions. The NH-G density function can be expressed as a mixture of exponentiated-G density functions. This mixture representation is important to derive several structural properties of this family in full generality. Some of them are provided such as the ordinary and incomplete moments, quantile function and order statistics. For each baseline distribution G, our results can be easily adapted to obtain its main structural properties. The estimation of the model parameters is approached by the method of maximum likelihood and we fit some NH-G distributions to two real data sets to demonstrate the potentiality of this family. We hope this generalization may attract wider applications in statistics.

References

AARSET, M. V. How to identify a bathtub hazard rate. **IEEE Transactions Reliability**, v.36, 106-108, 1987.

ADAMIDIS, K.; LOUKAS, S. A lifetime distribution with decreasing failure rate. **Statistics Probability Latters**, v.39, 35-42, 1998.

AKINSETE, A.; FAMOYE, F.; LEE, C. The beta-Pareto distribution. **Statistics** v. 42, n. 6, 547-563, 2008.

ALZAATREH, A.; LEE, C.; FAMOYE, F. A new method for generating families of continuous distributions. **METRON**, v. 81, 63-79, 2013.

ALZAGHAL, A.; FAMOYE, F.; LEE, C.. Exponentiated T-X Family of Distributions with Some Applications. **International Journal of Statistics and Probability**, v. 2, n. 3, 31-49, 2013.

ATKINSON, A. C. Plots, transformations, and regression: an introduction to graphical methods of diagnostic regression analysis. **Clarendon Press Oxford**, 1985.

BARRETO-SOUZA, W.; MORAIS, A.L.; CORDEIRO, G.M. The Weibull-geometric distribution. **Journal of Statistical Computation and Simulation**, v. 81, 645-657, 2011.

BARRETO-SOUZA, W.; SANTOS, A.H.S.; CORDEIRO, G.M. The beta generalized exponential distribution. **Journal of Statistical Computation and Simulation**, v. 80, n. 2, 159-172, 2010.

BEBBINGTON, M.; LAI, C.D.; ZITIKIS, R.. A flexible weibull extension. **Reliability Engineering & System Safety**, v. 92, 719-726, 2007.

BOURGUIGNON, M.; SILVA, R.B.; CORDEIRO, G.M. The Weibull-G family of probability distributions. **Journal of Data Science**, v. 12, 53-68, 2014.

CHEN, G.; BALAKRISHNAN, N. A general purpose approximate goodness-of-fit test. **Journal of Quality Technology**, v. 27, 154-161, 1995.

CORDEIRO, G.M.; EDWIN, M.M.; NADARAJAH, S. The Kumaraswamy Weibull distribution with application to failure data. **Journal of the Franklin Institute**, v. 347, 1399-1429, 2010.

CORDEIRO, G. M.; DE CASTRO, M. A new family of generalized distributions. **J. Stat. Comput. Simulation**, v. 81, n. 7, 883-898, 2011.

CORDEIRO, G.M.; NADARAJAH, S. Closed-form expressions for moments of a class of beta generalized distributions. **Brazilian Journal of Probability and Statistics**, v. 25, 14-33, 2011b.

CORDEIRO, G. M.; LEMONTE, A. J. The Beta-Birnbaum-Saunders distribution: An improved distribution for fatigue life modeling. **Computational Statistics and Data Analysis**, v. 55, 1445-1461, 2011.

CORDEIRO, G.M.; ORTEGA, E.M.M.; CUNHA, D.C.C. The Exponentiated Generalized Class of Distributions. **Journal of Data Science**, v.11, 1-27, 2013.
COX, D.R.; HINKLEY, D.V. Theoretical Statistics. Chapman & Hall, London, 1974.

DOORNIK, J. A. Ox 6: **Object-oriented matrix programming language**. 5th ed. London: Timberlake Consultants, 2009.

EBERHART, R. C.; KENNEDY, James. A new optimizer using particle swarm theory. **Proceedings of the sixth international symposium on micro machine and human science**, v. 1, 39-43. New York, NY, 1995.

EUGENE, N.; LEE, C.; FAMOYE, F. Beta-normal distribution and its applications. **Communications in Statistics - Theory and Methods**, v. 31, 497-512, 2002.

FAMOYE, F.; LEE, C.; OLUMOLADE, O. The beta-Weibull distribution. **Journal of Statistical Theory and Applications**, v. 4, n. 2, 121-136, 2005.

GARVAN, F. **The Maple Book**. Chapman & Hall, CRC, London, 2002.

GRADSHTEYN, I. S.; RYZHIK, I. M. **Table of Integrals, Series, and Products**. Edited by Alan Jeffrey and Daniel Zwillinger, 7th ed., Academic Press, New York, 2007.

GUPTA, R.C.; GUPTA, P.L.; GUPTA R.D. Modeling failure time data by Lehman alternatives. **Communications in Statistics - Theory and Methods**, v.27, 887-904, 1998.

GUPTA, R.C.; GUPTA, R.D. Proportional reversed hazard rate model and its applications. **Journal of Statistical Planning and Inference**, v. 137, 3525-3536, 2007.

GUPTA, R.D.; KUNDU, D. Generalized exponential distributions. **Australian and New Zealand Journal of Statistics**, v. 41, 173-188, 1999.

GUPTA, R.D.; KUNDU, D. Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions. **Biometrical Journal**, v. 43, 117-130, 2001.

JAFARI, A.A.; TAHMASEBI, S.; ALIZADEH, M. The Beta-Gompertz Distribution. **Revista Colombiana de EstadÃstica**, v. 37, n. 1, 139-156, 2014.

JONES, M.C. **Families of distributions arising from distributions of order statistics (with discussion)**. Test, v. 13, 1-43, 2004.

KENNEY, J. F.; KEEPING, E. S. **Mathematics of Statistics**, 3rd ed., Part 1. New Jersey, 1962.

LEE, E.T. **Statistical Methods for Survival Data Analysis**. Wiley: New York, 1992.

LEE, E.T.; Wang, J.W. **Statistical Methods for Survival Data Analysis**, 3rd ed., Wiley, New York, 2003.

LEE, C.; FAMOYE, F.; OLUMOLADE, O. Beta-Weibull distribution: Some properties and applications to censored data. **Journal of Modern Applied Statistical Methods**, v. 6, n. 1, 173-186, 2007.

LEMONTE, A. A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. **Computational Statistics and Data Analysis**, v. 62, 149-170, 2013.

MARSHALL, A.W.; OLKIN, I. A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. **Biometrika**, v. 84, 641-652, 1997.

MAHMOUDI, E.; SHIRAN, M. Exponentiated weibull-geometric distribution and its applications, **arXiv:1206.4008vl** [stat.ME], 2012.

MEAD, M. E.; ABD-ELTAWAB A. R. A note on Kumaraswamy Fréchet distribution. **Australian Journal of Basic and Applied Sciences**, v. 8, 294-300, 2014.

MOORS, J. J. A. A quantile alternative for kurtosis. **Journal of the Royal Statistical Society (Series D)**, v. 37, 25-32, 1988.

MUDHOLKAR, G.S.; HUTSON, A.D. The exponentiated Weibull family: some properties and a flood data application. **Communications in Statistics, Theory and Methods**, v. 25, 3059-3083, 1996.

MUDHOLKAR, G.S.; SRIVASTAVA, D.K.; FREIMER, M. The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. **Technometrics**, v.37, 436-445, 1995.

NADARAJAH, S. The exponentiated Gumbel distribution with climate application. **Environmetrics**, v. 17, 13-23, 2005.

NADARAJAH, S.; GUPTA, A. K. The beta Fréchet distribution. **Far East Journal of Theoretical Statistics**, v. 14, 15-24, 2004.

NADARAJAH, S.; KOTZ, S. The exponentiated type distributions. **Acta Applicandae Mathematicae**, v. 92, 97-111, 2006.

NADARAJAH, S.; KOTZ, S. The beta Gumbel distribution. **Mathematical Problems in Engineering**, v. 10, 323-332, 2004.

NADARAJAH, S.; KOTZ, S. The beta exponential distribution. **Reliability Engineering System Safety**, v. 91, 689-697, 2006.

NADARAJAH, S.; HAGHIGHI, F.. An extension of the exponential distribution. **Statistics: A Journal of Theoretical and Applied Statistics**, v. 45, 543-558, 2011.

NICHOLS, M. D.; PADGETT, W. J. A bootstrap control chart for Weibull percentiles. **Quality and Reliability Engineering International**, v. 22, 141-151, 2006.

PESCIM, R.R. et al. The beta generalized half-normal distribution. **Computational Statistics and Data Analysis**, v. 54, 945-957, 2010.

PRATER, NH. Estimate gasoline yields from crudes. **Petroleum Refiner**, v. 35, 236-238, 1956.

RAMOS, M. W. A. et al. The kumaraswamy-G Poisson family of distributions. **Journal of Statistical Theory and Applications**, v. 14, 222-239, 2015.

RÉNYI, A. On measures of information and entropy. **Proceedings of the fourth Berkeley Symposium on Mathematics, Statistics and Probability**, 547-561, 1961.

RIGBY, R. A.; STASINOPOULOS D. M. Generalized additive models for location, scale and sahpe. **Journal of the Royal Statistical Society: Series C (Applied Statistics)**, v. 54, 507-554, 2005.

RISTIC, M.M.; BALAKRISHNAN, N. The gamma exponentiated exponential distribution. **Journal of Statistical Computation and Simulation**, v. 82, 1191-1206, 2012.

SHANNON, C.E. Prediction and entropy of printed English. **The Bell System Technical Journal**, v. 30, 50-64, 1951.

SIGMON, K.; DAVIS, T.A. `MATLAB` **Primer**, 6th ed. Chapman & Hall, CRC, 2002.

SILVA, R.B.; BARRETO-SOUZA, W.; CORDEIRO, G.M. A new distribution with decreasing, increasing and upside-down bathtub failure rate. **Computational Statistics and Data Analysis**, v. 54, 935-944, 2010.

SILVERMAN, B. W. Density estimation for statistics and data analysis, **CRC press**, v. 26, 1986.

SMITH, R. L.; NAYLOR, J. C. A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. **Applied Statistics**, v. 36, 358-369, 1987.

STEPHENS, Michael A. Tests based on EDF statistics. **Goodness-of-fit Techniques**, v. 68, 97-193, 1986.

ZOGRAFOS, K.; BALAKRISHNAN, N. On families of beta and generalized gamma-generated distributions and associated inference. **Statistical Methodology**, v. 6, 344-362, 2009.

WOLFRAM, S. **The Mathematica Book**, 5th ed. Cambridge University Press, 2003.